# LEHN'S CONJECTURE FOR SEGRE CLASSES ON HILBERT SCHEMES OF POINTS OF SURFACES AND GENERALIZATIONS

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Contents



## 1. Lecture 1 : Hilbert schemes of surfaces in families

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**(1)** Let $p : S \to M$ be a flat family of nonsingular projective surfaces over any base $M$.

**(a) Examples to keep in mind**

- $M$ is a point and $S$ is a single surface,
- $p : S \to M$ is a $\mathbb{P}^2$ bundle over $M$,
- $p : S \to M$ is the universal family of quasi-pol K3 surface with quasi-polarization $L$.

**(b)** Let $h : S[n] \to M$ be the relative Hilbert schemes of $n$ points in the fibers of $p$. The morphism $h$ is smooth of relative dimension $2n$. Let $L \to S$ be a line bundle on the family of surfaces $p : S \to M$. For example, the K3 family of $(a)$ above has a canonical choice $L$ (up to normalization).

**(c)** Given $p : S \to M$ and $L \to S$, we can construct the tautological rank $n$ vector bundle $L[n] \to S[n]$ as usual with fiber $H^0(Z, L)$ over the length $n$ subscheme $Z$ of a fiber of $p$.

Let $c(L[n])$ be the total Chern class in $A^*(S[n])$ and let

$$Seg(L[n]) = \frac{1}{c(L[n])}$$

**Our main question is the following:** How can we compute $h_*(Seg(L[n])$ in $A^*(M)$?

Of course, we can also start with vector bundle $B \to S$ and ask to compute $h_*(Seg(B[n])$ in $A^*(M)$. The most natural formulation of the problem is to start with an element of $E$ of $K^*(S)$ and ask to compute $h_*(Seg(E[n])$ in $A^*(M)$. The K-theoretic formulation captures also $h_*(c(E[n])$ in $A^*(M)$ since $c(E[n]) = Seg(-E[n])$.

**(2) Why do we want to compute $h_*(Seg(E[n])$ ?**

**(a)** When $M$ is a point, the answer is an integral over $S[n]$

$$\int_{S[n]} Seg(E[n])$$

and has been studied via the Heisenberg algebra (Nakajima's theory of the cohomology of $S[n]$) by M. Lehn in 1999. He discovered there a nontrivial conjectural formula for the answer when $E = L$ is a line bundle which has been now proven in the past few years by [MOP] for K3 surfaces and [Voisin] in general.

- In case, $S$ is a K3 surface, we have complete results for all $E$ in $K^*(S)$ by [MOP].
- In case $S$ is arbitrary and $E \in K^*(S)$ is of rank $-2, -1, 0, 1$, we also have complete results (needs [MOP], A. Mellit, [Voisin]).
- Rank 2 is almost complete.

The geometric inputs have come from 3 very different directions

- The virtual class of Quot scheme on $S$ by [MOP],
- Ideas of Reider/Lazarsfeld applied by Voisin
- Localization partition sums studied by Mellit.

For general $E$ the question, even for $M$ is a point, is open.

(b) The first result on these questions after Lehn's paper of 1999 was [MOP] in 2015. There the question of calculating $h_*(Seg(L[n])$ in $A^*(M)$ for the universal family $p : S \to M$ of quasi-polarized K3 surfaces arose naturally in the study of tautological relations on the moduli of K3 surfaces. To explain the relevance, we first ask: what are tautological classes on $M$ ? Here, we start with any family $p : S \to M$ and and a line bundle $L \to S$. Let

$$\kappa[a, b, c] := p_*(c_1(S)^a c_2(S)^b c_1(L)^c) \in A^*(M)$$

Here $c_1(S)$ and $c_2(S)$ denote the Chern classes of the relative tangent bundle of the family $p : S \to M$.

A tautological class on the base $M$ is a polynomial in the classes

$$\kappa[a, b, c]_{a,b,c\geq 0}$$

**Theorem 1.1.** *(Formulation already implicit in [MOP])*
*$h_*(L[n])$ in $A^*(M)$ is given by universal polynomials in the classes $\kappa[a, b, c]$.*

The Quot scheme method of [MOP] required the calculation of $h_*(L[n])$ in $A^*(M)$ for the moduli space of K3 surfaces as a step in writing relations among the $\kappa[a, b, c]$ classes on $M$.

**Comments on Theorem 1:**

(i) The result shows $h_*(E[n])$ is a tautological class.

(ii) Universal polynomial here means that the polynomial does not depend upon $p : S \to M$ or $E \to S$ Theorem 1 stated for any $E \in K^*(S)$ takes the form: $h_*(E[n]) \in A^*(M)$ is given by universal polynomials in the classes $\kappa[a, b, C]$ depending only on the rank of $E$. Here $C$ is vector indexed by all the Chern classes of $E$.

(iii) While a proof has not (yet) been written, the method is standard. Use the geometric arguments of Lehn's 1999 paper (studied there when $M$ is a point) and apply them to the whole family $h : S[n] \to M$.

**(c)** A parallel exists in the more developed study of tautological relations in the moduli space of curves. Let

$$p : C \to M_g$$

be the universal curve over the moduli space of nonsingular genus $g$ curves. Let $h : C[n] \to M_g$ be the universal Hilbert scheme of points. Push-forward calculations of, for example, $h_*(w_C[n]) \in A^*(M_g)$ play an import role in the study of tautological relations. The $h_*$ calculation for $C[n]$ is much easier than for $S[n]$.

A fourth example (in addition to the three of part (1.a) for a family of surfaces is constructed from curves. Let $C_1 \to M_{g_1}$ and $C_2 \to M_{g_2}$ the universal families over the moduli spaces of nonsingular curves of genus $g_1$ and $g_2$. Let

$$p : S = C_1 \times C_2 \to M_{g_1} \times M_{g_2} = M$$

Then Theorem 1 says, for example, that $h_*(E[n])$ is tautological on $M_{g_1} \times M_{g_2}$, where $E$ is obtained from the Hodge bundles and relative dualizing sheaves of the two moduli of curves.

(3) The toric case

(a) Theorem 1 tells us where the answer to our main question lives. But how can we calculate?

Answer: Use toric geometry.

The simplest case is to let the surface be $S = \mathbb{C}^2$ the complex plane. Of course $S$ is not compact. To compensate for the non-compactness, we must study the geometry equivariantly with respect to a torus action. Let the toruc $T = \mathbb{C}^* \times \mathbb{C}^*$ act on $\mathbb{C}^2$ with tangent weights $s,\ t$ at the origin. Let $T$ also act on the trivial line bundle $L = \mathcal{O}_{\mathbb{C}^2} \to \mathbb{C}^2$ equivariantly with weight $w$. A localization vertex for the Segre class push-forward is the following:

$$Z(q, s, t, w) := \sum_{n=0}^{\infty} q^n \int_{S[n]} Seg(L[n]) \tag{1.1}$$

This generating is defined by Atiyah-Bott localization with respect to $T$ and is entirely explicit. When T act on $\mathbb{C}^2[n]$, the $T$-fixed points are given by monomial ideals which are viewed as partitions in the plane. So we write

$$Z(q, s, t, w) = \sum_{\sigma} q^{|\sigma|} Res_\sigma Seg(L[n]) \tag{1.2}$$

where the sum is over all partitions $\sigma$ (including the empty partition which contributes a leading 1 to the sum). The localization residue is

$$Res_\sigma Seg(L[n]) = \frac{1}{e(Tan_\sigma) c(O[n] \otimes w)} \tag{1.3}$$

where $Tan_\sigma$ is the standard $T$-representation on the tangent space of $\mathbb{C}^2[n]$ at the $T$-fixed point $\sigma$ (given by arm and leg lengths) and $O[n]$ is the $T$-representation the structure sheaf of the associated scheme.

(b) Since $Z$ starts with 1, we can take the *log*,

$$F = Log(Z). \tag{1.4}$$

**Theorem 1.2.** *The logarithm $F$ has the follow basic structure*

$$F = \frac{F_0}{st} + \frac{F_1}{st} + \frac{F_2}{st} + ... \tag{1.5}$$

*where $F_0(q)$ is just a series in $q$, $F_1(q, s, t, w)$ is of degree 1 in $s, t, w$, $F_2(q, s, t, w)$ is of degree 2 in $s, t, w$, ... and the $F_i$ are of course symmetric in $s, t$.*

So $E_0(q)$ is really just 1 $q$-function;
$E_1(q)$ is 2 $q$-functions (coeffs of $s + t$ and $w$);
$E_2(q)$ is 4 functions (coeffs of $(s + t)^2$, $st$, $(s + t)w$, $w^2$), etc.

There are 2 proofs of Theorem 1.2

- Mellit has a proof which directly depends upon analysis of the localization sum defining $Z$.
- There is a second proof by geometry of log Hilbert scheme which I will explain.

Finally, I note that $Z$ here is not unrelated to the Nekrasov partition function — the new aspect here is the inclusion of the Segre class.

(c) How does Theorem 1.2 provide a calculation of $h_*(Seg(L[n])$ in $A^*(M)$ for all families $p : S \to M$ and line bundles $L \to S$ ?
The answer is given in a few steps.

(1) Take the expansion of $F = log(Z)$ in Theorem 1.2 and throw away $F_0$ and $F_1$:
$$F\# = \frac{F_2}{st} + \frac{F_3}{st} + \frac{F_4}{st} + ... \tag{1.6}$$
(2) Rewrite each term with $\kappa$ classes. Start with $F_2$ which is
$$*\frac{(s + t)^2}{st} + *\frac{st}{st} + *\frac{(s + t)w}{st} + *\frac{w^2}{st}$$
where the 4 instances of $*$ denote 4 series in $q$ (all without constant term). Write this as
$$G_2 = *\kappa[2, 0, 0] + *\kappa[0, 1, 0] + *\kappa[1, 0, 1] + *\kappa[0, 0, 2] \tag{1.7}$$
where the $q$-series are the same as for $F_2$. Similarly, write $G_3, G_4, ...$ starting with $F_3, F_4, ...$ by
$$(s + t) \Rightarrow c_1(S)$$
$$st \Rightarrow c_2(S)$$
$$w \Rightarrow c_1(L)$$
(3) Let $G = G_2 + G_3 + G_4 + ...$ where the right hand side is now a polynomial in the kappa classes $\kappa[a, b, c]_{a,b,c\geq 0}$.
Actually, only the $\kappa$ classes of *codim* $\geq 0$ in $M$ appear (since we have thrown out the negative dimensional kappas).
(4) Let $p : S \to M$ be a family of nonsingular project surfaces, and let $L \to S$ be a line bundle.

**Theorem 1.3.** *We have the equality*

$$Exp(G) = \sum_{n=0}^{\infty} h_*(Seg(L[n]))$$

**Comments.** The proof of Theorem 1.3 uses

- Theorem 1 — so we need only to find the universal formula
- Theorem 2 — which gives the specialization to the projective toric case and a bit more.

The more is because projective toric surfaces, unfortunately do not fully explore the tautological classes. To see this, consider the following exercise.

Let $S$ be a projective toric surface with $T$-action. Prove the following formula

$$\int_S c_1 c_2 = 0$$

in $T$-equivariant cohomology.

As a result, $\kappa[1,1,0]$ will always be zero in the projective toric case (and its contribution will be lost). A nice way to repair the defect in projective toric geometry is to consider toric surface relative to toric divisor (log toric situation). Then the issue can be overcome.

(v) The localization strategy, including Theorems 1.2 and 1.3, are valid for $K$-theory classes of all ranks (not just the line bundle case discussed above). The statements and methods are the same.

(d) The initial $F_0$ term has a very nice formula.

**Theorem 1.4.** *(Mellit) Consider the rank $= r$ case, so*

$$Z(q,s,t,w_1,...,w_r) = \sum_\sigma q^{|\sigma|} \frac{1}{e(Tan_\sigma)c(O[n]\otimes w_1 + ... + O[n]\otimes w_r)} \tag{1.8}$$

*then the leading $\frac{F_0}{st}$ term of $F = logZ$ is*

$$F_0 = q + \sum_{n\geq 2} q^n (-1)^{(n+1)} r(r+1)^{(n-1} \frac{\binom{(r+2)n-3}{n-2}}{n^2(n-1)} \tag{1.9}$$

Mellit also has results on $F_1$. Both $F_0$ and $F_1$ are terms corresponding to "negative degree" kappa classes and do not play a direct role in the original question (they are removed in step (c.1) above).

**(e)** Consider the rank 0 case (which is not at all trivial). But a special case of rank 0, when $E$ is actually 0, is easy to understand. Then the parition function is

$$Z(q,s,t) = \sum_\sigma q^{|\sigma|} \frac{1}{e(Tan_\sigma)} \tag{1.10}$$

Recall the tangent representation $Tan_\sigma$ is

$$\sum_{Boxes\ b\ of\ \sigma} (Leg(b)+1)s - Arm(b)t) + \sum_{Boxes\ b\ of\ \sigma} -Leg(b)s + (Arm(b)+1)t \tag{1.11}$$

While the sum defining $Z$ looks complicated, the answer is simple $Z(q,s,t) = Exp(\frac{q}{st})$ .

**Proof (Using the $GW/DT$ correspondence of MNOP)**

Consider $\mathbb{C}^2 \times \mathbb{P}^1$. Then the DT theory of ideal sheaves in curve class $d[\mathbb{P}^1]$ starts with

$$Z_{\mathbb{C}^2\times\mathbb{P}^1,DT} = q^d Coe(Z, q^d) + ...$$

since the leading term corresponds to the Hilbert scheme of $d$ points of $\mathbb{C}^2$. The $GW/DT$ correspondence here takes the form

$$(-q)^d Z_{DT} = (-iu)^{2d} Z_{GW}$$

where $Z_{\mathbb{C}^2\times\mathbb{P}^1,GW}$ is the Gromov-Witten partition function of $\mathbb{C}^2\times\mathbb{P}^1$ is curve class $d[\mathbb{P}^1]$.

For simple dimension reasons, the connected domain components of maps which can contribute to $Z_{GW}$ are of genus 0 and map with degree 1 to $\mathbb{P}^1$. Hence,

$$Z_{GW} = \frac{u^{-2d}}{d!}$$

is the entire partition function. Using the $GW/DT$ correspondence,

$$(-iu)^{2d} Z_{GW} = (-1)^d \frac{1}{d!}(st)^{-d} = (-1)^d Coeff(Z, q^d) + higher\ order \qquad (1.12)$$

We conclude

(1) $Coeff(Z, q^d) = \frac{1}{d!}(st)^{-d}$
(2) There are no higher order terms.

The claim $Z = Exp(\frac{q}{st})$ follows from (1). Conclusion (2) is an interesting extra property.

## 2. Lecture 2 : Hilbert schemes of log surfaces

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(1) Let $S$ be a nonsingular projective surface and let $D \subset S$ be a nonsingular projective curve. Let $S/D[n]$ be the Hilbert scheme of points of a surface S relative to $D$.

- For the usual Hilbert scheme of point $S[n]$, the supports move freely in S. For $S/D[n]$ the space bubbles when the supports head to the relative divisor $D$.
- While $S[n]$ is very familiar, the log geometry $S/D[n]$ has been much less studied
- For those familiar with relative DT, stable pairs, etc. $S/D[n]$ is a special case of the construction
- For computation of the Betti numbers of $S/D[n]$, see the Princeton Ph.D. thesis of I. Setayesh

(2) The main questions and most of the discussion of Lecture 1 can be carried out for the relative geometry $S/D$.

An important example is $S = \mathbb{P}^1 \times \mathbb{C}$ with $D = \infty \times \mathbb{C}$. Then $S/D$ carries a $T = \mathbb{C}^* \times \mathbb{C}^*$ action extending the torus action on $\mathbb{C}^2 = S - D$. Here, $T$ acts with weight $s, t$ at the origin $(0, 0)$ and with weights $-s, t$ at $(\infty, 0)$.

Let $T$ also act on the trivial line bundle $L = \mathcal{O}_S \to S$ equivariantly with weight $w$. We will now study the partition function

$$Z_{S/D}(q,s,t,w) = \sum_{n=0}^{\infty} q^n \int_{S/D[n]} Seg(L[n]) \tag{2.1}$$

which nicely factors as $Z_{S/D} = Z_{vert} \cdot Z_{rel}$ where $Z_{vert}$ is the localization vertex at $(0,0)$ from Lecture 1 and $Z_{rel}$ is the localization vertex associated to the rubber at $(\infty, 0)$.

Why is the relative geometry $S/D$ here better than the the naked vertex $Z_{vert}$ from Lecture 1?

Answer: $S/D$ is compact in the $s$ direction related to the first $\mathbb{C}^*$ factor of $T$, the $q$-coefficients of $Z_{S/D}$ are polynomials in $s$, while the $q$-coefficients of $Z_{vert}$ have denominators in $s, t$, To use the polynomality of $Z_{S/D}$, we must of study the series $Z_{rel}$ via the rubber calculus.

See [MNOP2] for a parallel study in the case of DT theory of 3-folds

(3) Rubber calculus

The $T$ fixed locus of $S/D[n]$ with all points over $\infty$ in $\mathbb{P}^1$ is not isolated. Rather, the $T$-fixed locus is the rubber moduli space $R[n]$ of dimension $2n-1$. The rubber space $R[n]$ is the Hilbert scheme of points of $\mathbb{P}^1 \times \mathbb{C}$ relative to both $0 \times \mathbb{C}$ and $\infty \times \mathbb{C}$ (with bubbling when the the supports meet either) up to the $\mathbb{C}^*$-scale. It is the $\mathbb{C}^*$-scale equivalence which brings the dimension down to $2n-1$. Localization gives the formula

$$Z_{rel} = 1 + \sum_{n=1}^{\infty} q^n \frac{\int_{R[n]} Seg(L[n])}{(-s - \Psi)} \tag{2.2}$$

where $L[n]$ is the tautological bundle on the rubber and $-\Psi$ is the tangent line associated to the relative divisor.

We can expand $Z_{rel}$ as a series in $s$:

$$Z_{rel} = 1 + \frac{Z_1}{-s} + \frac{Z_2}{(-s)^2} + \frac{Z_3}{(-s)^3} + .... \tag{2.3}$$

The crucial equation of the rubber calculus is

$$DZ_k = Z_{k-1} DZ_1 \quad \forall\ k \geq 2 \tag{2.4}$$

Some comments

(1) The operator D is defined as $D = q\frac{d}{dq}$.
(2) The equation comes from producing a section of the cotangent line $\Psi$.

The solution to the rubber differential equation is

$$Z_k = \frac{Z_1^k}{k!}$$

so we have

$$Z_{rel} = exp(-\frac{Z_1}{s})$$

And what is $Z_1$ ?

By definition, $Z_1$ is exactly the integral of $Seg(L[n])$ over $R[n]$,

$$Z_1 = \sum_{n=1}^{\infty} q^n \int_{R[n]} Seg(L[n]) \tag{2.5}$$

Since the first $\mathbb{C}^*$ factor of $T$ does not act on $R[n]$, $Z_1$ has no $s$-dependence ($Z_1$ depends only on $t, w$). Since $w$ acts only on the line bundle $L$, $Z_1$ is polynomial in $w$.

Conclusion:

$$LogZ_{rel} = -\frac{Z_1}{s}$$

and has exactly simple poles in $s$ and no poles in $w$.

**(4) Vertex analysis**

By geometry, the $q$-coefficients of $Z_{S/D}$ are polynomial in $s$
$\Rightarrow$ the q-coefficients of $LogZ_{S/D}$ are polynomial in $s$ since

$$LogZ_{S/D} = LogZ_{vert} + LogZ_{rel}$$

, the q-coefficients of $LogZ_{vert}$ must have at most simple poles in $s$ (and no poles in $w$). since $LogZ_{vert}$ is symmetric in $s, t$, the q-coefficients of $LogZ_{vert}$ must have at most simple poles in $t$.

We can write

$$Log(Z_{vert}) = \frac{F_0}{st} + \frac{F_1}{st} + \frac{F_2}{st} + ... \tag{2.6}$$

The geometric proof of Theorem 2 is now complete.

**(5) Further comments**.

(a) We can calculate $\frac{Z_1}{s}$ as exactly the polar part of $LogZ_{vert}$. Since we can calculate $LogZ_{vert}$ by localization, we have a method of computing

$$\int_{R[n]} Seg(L[n])$$

(b) Since $LogZ_{vert}$ has at most simple poles in $t$, $\int_{R[n]} Seg(L[n])$ must also have at most simple poles in $t$. We conclude

- $\int_{R[n]} Seg_{<2n-2}(L[n]) = 0$
- $$\sum_{n=1}^{\infty} q^n \int_{R[n]} Seg_{2n-2}(L[n]) = F_0$$

  where $F_0$ is determined in Theorem 4 of Lecture 1,

An interesting question is to find a direct geometric approach to (i), (ii), and the computation of the higher Segre classes over rubber.

**(6) Compact surfaces**

Let $S$ be a nonsingular projective toric surface with a $T$-action. Using localization, Theorem 2 immediately implies Theorem 3. In fact, using further localization analysis, we can prove the corresponding statement for relative geometries in the toric case.

Let $S$ be a nonsingular projective toric surface, and let $D$ be a nonsingular toric divisor. Then, Theorem 3 holds for the $T$-equivariant theory of $S/D$ with the rules

$$(s+t) \Rightarrow c_1(S/D)$$
$$st \Rightarrow c_2(S/D)$$
$$w \Rightarrow c_1(L)$$

Here $c_1(S/D)$ and $c_2(S/D)$ are the Chern classes of log tangent bundle of $S/D$ — dual to the bundle of holomorphic differentials $\Omega^1_S(D)$ with log poles along $D$.

**(7) Sketch of Proof of Theorem 3 in all cases.**

(a) We need Theorem 1 for families of log surfaces. The statement is unchanged (with the relative tangent bundle of the family replaced by the relative log tangent bundle of the log family).

(b) Using (a), we can determine the universal polynomials of Theorem 1 using families of log surfaces.

(c) While Theorem 3 for the $T$-equivariant theory of nonsingular projective toric surfaces was not enough to determine the universal polynomials, Theorem 3 for the $T$-equivariant theory of nonsingular projective log toric surfaces is enough.

**(8) A fixed surface** $S$

Let $S$ be a fixed nonsingular projective surface with a nonsingular divisor $D$. Via Theorem 3, we have

$$Z_S(q) = Exp(A_1c_1^2 + A_2c_2 + A_3c_1.L + A_4L^2) \tag{2.7}$$

So the Segre integral question amounts exactly to determining the 4 q-series $A_1, A_2, A_3, A_4$. Also

$$Z_{S/D}(q) = Exp(A_1c_1(S/D)^2 + A_2c_2(S/D) + A_3c_1(S/D).L + A_4L^2) \tag{2.8}$$

for the same functions.

What are these functions? We will see in Lecture 3.

## 3. Lecture 3 : Formulas and proof for K3

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(1) Our original question was the following:

Let $p : S \to M$ be a family of nonsingular projective surfaces over a base $M$ , and let $E$ be a $K$-theory class on the total space $S$.

**Question:** What is $h_*(Seg(E[n])) \in A^*(M)$ ?

An answer was given in term of universal formulae in tautological classes via the associated localization vertex. We ask here for much more explicit answers. There are 3 measures of difficulty for the question

- **1:** Codimension in $M$ — the higher the codimension, the more complicated the formula. Codimension 0 concerns integration over a Hilbert scheme of points of a fixed surface and is the simplest

**2:** Rank of the $K$-theory class $E$ — ranks equal to 0 or near 0 are the simplest.

**3:** Complexity of the surfaces in the family — Abelian surfaces tend to be simplest, followed by K3 surfaces. Toric surfaces are of no advantage for the formulas (but toric surfaces can help in proofs).

(2) The complete answer to the question in codimension 0, all rank $r$, for K3 surfaces is known via [MOP].

**a:** Let $S$ be a K3 surface and let $E$ in $K^*(S)$ be of rank $r$

$$\sum_{n=0}^{\infty} q^n \int_{S[n]} Seg_{2n}(E[n]) = A(q)^{(c_1(E)^2 - c_2(E))} B(q)^{(\chi(c_1(E)))} C(q) \tag{3.1}$$

where

$$\chi(c_1(E)) = \frac{1}{2}c_1(E)^2 + \frac{1}{2}c_1(E)c_1 + \frac{1}{21}(c_1^2 + c_2) = \frac{1}{2}c_1(E)^2 + 2$$

$$A = (1 + (r+1)x)^{(r+1)}(1 + (r+2)x)^{(-r)}$$

$$B = (1 + (r+1)x)^{(-r-2)}(1 + (r+2)x)^{(r+1)}$$

$$C = (1 + (r+1)x)^{((r+2)^2)}(1 + (r+2)x)^{(-(r+1)^2)}(1 + (r+2)(r+1)x)^{(-1)}$$

$$q = x(1 + (r+1)x)^{(r+1)}$$

**b:** If we write the solution in the language of Lectures 1 and 2, we must take the logarithm:

$$c_1(E)^2 \ Coeff \ of \ F_2 = LogA + \frac{1}{2}LogB$$

$$c_2(E) \ Coeff \ of \ F_2 = -LogA$$

$$(st) \ Coeff \ of \ F_2 = \frac{1}{12}LogB + \frac{1}{24}LogC$$

The K3 results give no information on the $(s+t)^2$ Coeff of $F_2$ and the $(s+t)c_1(E)$ Coeff of $F_2$

**c:** Voisin's proof in codimension 0, rank $r = 1$, for K3 surfaces is elegant. I give her complete argument here.

- Let $(S, L)$ be a K3 surface with a line bundle $L$ with $L^2 = 2l$. Using the general structure result (see Theorem 3 of Lecture 1),

  $$\sum_{n=0}^{\infty} q^n \int_{S[n]} Seg_{2n}(L[n]) = A(q)^{24} B(q)^{2l}$$

  for $q$-series $A$ and $B$. By calculating the $n = 1$ case by hand,

  $$B(q) = 1 + q + \ldots$$

  We therefore see that $\int_{S[n]} Seg_{2n}(L[n])$ is a polynomial of degree $n$ in $l$ with leading coefficient $\frac{2^n}{n!}$.
  The result of [MOP] is that

  $$\int_{S[n]} Seg_{2n}(L[n]) = 2^n \binom{l - 2n + 2}{n}$$

which is in fact a polynomial of degree $n$ in $l$ with the correct leading coefficient. In order to prove the [MOP] evaluation, Voisin needs only to prove the vanishing for $n > 0$

$$\int_{S[n]} Seg_{2n}(L[n]) = 0$$

for the $n$ values $l = 2n-2, 2n-1, ..., 3n-3$, since the degree $n$ polynomial is uniquely determined by these $n$ roots and the leading coefficient.

- We switch notation now to $L^2 = 2g-2$, so $g = l+1$ matches the notation of part (3.a) above. Let $(S, L)$ further satisfy three more conditions
(i) $L$ is primitive,
(ii) $Pic(S)$ is rank 1 generated by $L$,
(iii) $g > 0$.
By (iii), $L^{\perp}$ has no sections.
- Let $\mathbb{P}(L[n]^*) \to S[n]$ be the projective bundle, and Let $\mathcal{O}_{\mathbb{P}}(1) \to P(L[n]^*)$ be the tautological bundle of the polarization. An elementary exercise shows

$$H^0(\mathbb{P}(L[n]^*), O_{\mathbb{P}}(1)) = H^0(S[n], L[n]) = H^0(S, L)$$

and these spaces are of dimension $g+1$.
(i) We obtain a rational map

$$w : \mathbb{P}(L[n]^*) \to \mathbb{P}^g$$

via the complete space of sections of $\mathcal{O}_{\mathbb{P}}(1)$.
(ii) The Segre class is recovered via the intersection product

$$\int_{\mathbb{P}(L[n]^*)} c_1(\mathcal{O}_{\mathbb{P}}(1))^{3n-1} = \int_{S[n]} Seg_{2n}(L[n])$$

(iii) Main Claim: if $g > 2n-2$, then $L[n]$ is generated by global sections. Let us assume the Main Claim for now to see how the vanishing is implied.
If$g > 2n-2 \Rightarrow L[n]$ is generated by global sections $w : \mathbb{P}(L[n]^*) \to \mathbb{P}^g$ is an actual morphism.
if $3n-1 > g$, then $\int_{S[n]} Seg_{2n}(L[n]) = 0$ by (ii). So $\int_{S[n]} Seg_{2n}(L[n]) = 0$ for precisely the range $g = 2n-1, 2n, ..., 3n-2$ which is exactly the [MOP] vanishing range.
- Proof of the Main Claim (closely related to older results of Lazarsfeld). Let $g > 2n-2$ and assume $L[n]$ is not globally generated. We construct a contradiction. Let $[Z]$ in $S[n]$ be a witness to the failure of global generation. Then,

$$H^0(S, L) \to H^0(S, L|_Z)$$

is not surjective. Hence $H^1(S, L \otimes I_Z)$ is not zero. Using Serre duality, $Ext^1(I_Z, L^*)$ is also not zero, so we have a non-split extension

$$0 \to L^* \to E \to I_Z \to 0$$

We first note that $Hom(E, L^*) = 0$. If $h \in Hom(E, L^*)$ then the composition $L^* \to E \to L^*$ must be zero (or else the extension splits), so $h$ induces a map $I_Z \to L^*$.
But the latter would give a section of $L^*$ (by Hartogs) which is impossible, so $h = 0$. The contradiction will come by constructing a non-trival map $E \to L^*$. We calculate $ch_i(E, E)$ by Riemann-Roch to be $2g + 6 - 4n = 2(g - 2n + 2) + 2 > 2$.
Therefore, there exists a non-trivial element $f \in Hom(E, E)$ which is not proportional to the identity $Id : E \to E$. By a standard eigenvalue argument, we may assume the sheaf $Im(f) \subset E$ has generic rank equal to 1. Let $F =$ Saturation of $Im(f)$ in $E$. Then we obtain a second exact sequence

$$0 \to F \to E \to Q \to 0$$

where both $F$ and $Q$ are torsion free of generic rank 1.
Consider first $F$. Since $F$ is torsion free of rank 1, we have an injection $F \to F^{**}$ where $F^{**} = L^k$ is locally free (remember $L$ generates $Pic(S)$). Since

$$E \to F \to F^{**} = L^k$$

, we have a non-trivial map $E \to L^k$. Since $Hom(E, L^*) = 0$, we conclude $k \geq 0$. Then $F = I_W \otimes L^{k \geq 0}$.
Next, we use the extension

$$0 \to L^* \to E \to I_Z \to 0$$

Since $F$ sits in $E$ and does not map to $L^*$ (since $k \geq 0$), $F$ must have a non-trival map to $I_Z$. Hence, $k \leq 0$, so $k = 0$.
Putting the above together, we have the extension

$$0 \to I_W \to E \to Q \to 0.$$

Finally, $Q$ is also torsion free of rank 1, so we have an injection $Q \to Q^{*}*$ and a non-trival map $E \to Q \to Q^{**}$.
What is the line bundle $Q^{**}$ ? Since $det(E) = L^*$ by the original extension

$$0 \to L^* \to E \to I_Z \to 0$$

and $det(I_W)$ is trivial, $Q^{**}$ must be $L^*$. So we have constructed a non-trivial map $E \to L^*$ which is a contradiction. QED

(4) The formulas of Lehn's conjecture concern codimension 0, rank $r = 1$, and all $S$.

Let $S$ be a nonsingular projective surface and let $L$ in $K^*(S)$ be of rank 1. Given the series already determined in (2) from the K3 surface specialized to $r = 1$, we need only the Coeffs of $F_2$ corresponding to $(s + t)^2$ and $(s + t)c_1(L)$.

In total we have:

$$c_1(L)^2 \ Coeff \ of \ F_2 = \frac{1}{2} Log(1 + 2x) \tag{3.2}$$

**Comments:**

- The above formulas are strictly more than Lehn's conjecture of 1999 which only considered the case where $L$ is a line bundle on the surface $S$. The 5 series solve the Segre class question for every rank 1 element $L$ of $K^*(S)$. For a line bundle $L$, $c_2(L) = 0$ so the second series does not appear in Lehn's proposal
- The proof use K3 + an extension of Voisin's argument for K3 to the blow-up of K3 in a single point (see her paper).
- Can we find a straight localization proof?

(5) Lehn's conjecture is the complete answer for a fixed surface $S$ and $E$ in $K^*(S)$ of rank 1 We also have complete answers for a fixed surface $S$ and $E$ in $K^*(S)$ of rank $-2, -1$, and 0.

(a) In rank -2, three of the 5 functions come from the K3 result. The remaining 2 functions come from the following simple geometry.

If $E = [-B]$ where $B \to S$ is a rank 2 bundle with a transverse section, then

$$\begin{aligned} &\sum_{n=0}^{\infty} q^n \int_{S[n]} Seg(E[n]) \\ &= \sum_{n=0}^{\infty} q^n \int_{S[n]} c(B[n]) \\ &= (1+q)^{(c_2(B))}. \end{aligned} \tag{3.3}$$

The above geometric calculation holds for all rank 2 bundles $B$ and all $S$, so both $c_1 c_1(E)$ and $c_1^2(E)$ can be probed.

(b) In rank -1, three of the 5 functions come from the K3 result. The remaining 2 functions come from the observation that when $E = [-L]$ where $L \to S$ is a line bundle,

$$\sum_{n=0}^{\infty} q^n \int_{S[n]} Seg(E[n]) = \sum_{n=0}^{\infty} q^n \int_{S[n]} c(L[n]) = 1$$

since the integrals over $S[n]$ vanish for $n > 0$ for dimension reasons. The above geometric calculation holds for all line bundles $L$ and all $S$, so both $c_1(L)$ and $c_1^2$ can be probed.

(c) In rank 0, three of the 5 functions come from the K3 result. One of the remaining 2 functions come from the observation that when $E = [L - L]$, we have

$$\sum_{n=0}^{\infty} q^n \int_{S[n]} Seg(E[n]) = 1$$

for dimension reason. Determining the last function in rank 0 is via Mellit's localization analysis (much less trivial than the geometric constraints discusses above). In rank 2, a complete conjecture was found by MOP (unpublished), but at the moment requires either further geometry or further localization analysis (both approaches are likely to succeed). The last 2 functions in rank 2 are much more complicated than the in the rank -2,-1,0, and 1 cases — and suggest closed forms for all ranks will be difficult to find explicitly. I would hope for some recursive structure instead.

## References


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