COHOMOLOGY OF MODULI SPACE OF CUBIC FOURFOLDS I

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Abstract. In this paper we calculate the cohomology of moduli space of cubic fourfolds with ADE type singularities relying on Kirwan's blowup and Laza's GIT descriptions. More precisely, We obtain the betti numbers of Kirwan's resoluton of the moduli space. With the help of decomposition theorem, we also obtain the betti numbers of the intersection cohomology of Baily-Borel compactification of the moduli space.

Contents

1. Introduction

The study of cubic fourfolds is a classical topic in algebraic geometry and has attracted lots of attention in various aspects. In this paper we deal with the topology of the moduli space $\mathcal{M}$ of cubic fourfolds with ADE singularities at worst.

The works of C. Voisin [37] [38], B. Hassett[13], R. Laza [24] and E. Looijenga [28] establish the global Torelli theorem for the cubic fourfolds completely, including the description of image of period maps. The theorem says the moduli space of cubic fourfolds with ADE singularities is isomorphic to the complement $\mathcal{D}/\Gamma - \mathcal{H}_\infty$ of a Heegner divisor $\mathcal{H}_\infty$ in the shimura variety $\mathcal{D}/\Gamma$ (see 5.2). This provides many compactifications of $\mathcal{M}$ from the arithmetic side, e.g. Baily-Borel

[7], Loojiegena [27], toriodal [1]... these compactifications imply that $\mathcal{M}$ is a quasi-projective variety.

The cohomology of a shimura variety is a central topic deeply related to the representation theory (see [7]). Usually, it is very hard to compute its cohomology. But in our situation for $\mathcal{M}$, the Torelli theorem provides a birational map $p : \overline{\mathcal{M}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$ between its geometric invariant theory (GIT) compactified moduli space $\overline{\mathcal{M}}$ and the Baly-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ of shimura variety $\mathcal{D}/\Gamma$. Moreover, the birational map is explicitly resolved via Kirwan's partial desingularization (see [24]), that is, there is a diagram

$$\begin{tikzcd} & \widehat{\mathcal{M}} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}} \arrow[rr, dashed, "p"] & & \overline{\mathcal{D}/\Gamma}^{BB} \end{tikzcd}$$

where $\widehat{\mathcal{M}}$ is the Kirwan's partial desingularization space of $\overline{\mathcal{M}}$. Thus, once we know the cohomology of GIT quotient space $\overline{\mathcal{M}}$ and $\widehat{\mathcal{M}}$, the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [6] will provide a way to compute the cohomology of baily-Borel compactification of the shimura variety $\mathcal{D}/\Gamma$.

For the GIT quotient space, F. C. Kirwan developed a systematic methods to compute its cohomology, see [22], [21] and section 2 for an brief introduction. Kirwan's method is inspired by the works of Atiyah and Bott [2] on topology of quotient space of connections space under gauge group action, which is a infinite dimenional version of GIT. Also, Kirwan studied how the cohomology changed after the blowups. This provides a strategy to attach the computation problem of cohomology: first, one can obtain the cohomology of GIT quotient space via applying Kirwan's formula; then by keeping track of changes of cohomology for each blowup in Kirwan's partial desingularization, we are able to compute the cohomology of desingularization space.

This strategy has been worked out for the moduli space of degree 2 K3 surface (see [18], [19]), the moduli space of cubic threefolds (see [8]). Even the boundary strata of the GIT moduli space of cubic fourfolds is much more complicated than the above two cases, the good news is that the GIT moduli space of degree 2 K3 surfaces will appear as an exceptional divisor in Kirwan's desingularization, then some computations of Kirwan-Lee in [19] will help to simplify the our computations.

For a topological space Y, we denote by

$$P_t(Y) := \sum b_i(X)t^i, \quad IP_t(Y) := \sum \dim IH^i(X)t^i$$

the Poincare polynomial of singular cohomology and intersection cohomology (with respect to middle perversity).

Let $\widetilde{\mathcal{M}}$ be the partial desingularization of $\overline{\mathcal{M}}$ in the sense of Kirwan. Following the strategy discussed above, we have the first main result in the paper:

Theorem 1.1. *The Poincare polynomial of* $\widetilde{\mathcal{M}}$ *is given by*

$$\begin{aligned} P_t(\widetilde{\mathcal{M}}) =&1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} + 193t^{14} \\ &+ 236t^{16} + 280t^{18} + 324t^{20} + 280t^{22} + 236t^{24} + 193t^{26} \\ &+ 152t^{28} + 115t^{30} + 81t^{32} + 51t^{34} + 26t^{36} + 9t^{38} + t^{40} \end{aligned} \tag{1.1}$$

Using the explicit resolution of of the period maps, we also compute the intersection cohomology of the Baily-Borel compactification.

Theorem 1.2. *The intersection cohomology poincare polynomial of* $\overline{\mathcal{D}/\Gamma}^{BB}$ *is given by*

$$\begin{aligned} \mathrm{IP}_t(\overline{\mathcal{D}/\Gamma}^{BB}) =&1 + 2t^2 + 4t^4 + 9t^6 + 16t^8 + 26t^{10} + 38t^{12} + 50t^{14} \\ &+ 65t^{16} + 82t^{18} + 112t^{20} + 82t^{22} + 65t^{24} + 50t^{26} \\ &+ 38t^{28} + 26t^{30} + 16t^{32} + 9t^{34} + 4t^{36} + 2t^{38} + t^{40} \end{aligned} \tag{1.2}$$

It is an interesting topic to study the generators of the intersection cohomology in each degree and ask whether these generators are generated by special cycles (for example, see [35]). We plan to investigate the problem in the future. Actually, this is one of our motivation to calculate the cohomology. Our results provide the starting point. Parallel to the moduli space of quasi-polarised K3 surface of fixed degree (see [5], [33] [4]), one can define the tautological ring and study its relation to the cohomology ring, the results in our paper also provide information in this direction.

Recently, K-moduli space is a very active subject (see [39] [36] [40]for a survey). By the recent results of Liu [25], the GIT moduli space is isomorphic to the K-moduli space of cubic fourfolds, that is, the space of isomorphic classes of cubic fourfolds admitting Kahler-Einstein metrics. So our computations also provide cohomological results on the K-moduli space.

Remark 1.3. we are most interested in cohomology of the open part $\mathcal{D}/\Gamma$ and the complement of Heegener divisor $\mathcal{D}/\Gamma - \mathcal{H}_\infty$. we plan to investigate the problem in the forthcoming paper.

Outline. The paper is organised as follows: In section 2, we review the basic properties of equivariant cohomology and intersection cohomology theory, and introduce Kirwan's partial desingularization package. In section 3, we describe the GIT stratas of moduli space of cubic 4-folds and discuss their geometries relying on the work of R. Laza. In section 4, we use Kirwan's methods to compute the cohomology of the partial resolution $\widetilde{\mathcal{M}}$. In section 5, we use the decomposition theorems to compute the intersection cohomology of the Baily-Borel compatification $\overline{\mathcal{D}/\Gamma}^{BB}$ of the moduli space of cubic 4-folds.

Conventions.

(1) $\mathcal{M}$ the moduli space of cubic fourfolds with ADE singularities;
(2) $\overline{\mathcal{M}}$ the GIT comaptification of $\mathcal{M}$;
(3) $\widetilde{\mathcal{M}}$ the Kirwan's desingularization space;
(4) $\mathbb{C}[x_1, x_2, ..., x_n]_d$ means degree d homogenous polynomials in $n+1$ variables;

(5) $l(x), q(x), c(x)$ means linear, quadratic and cubic forms in x.
(6) $\alpha, \mu, \gamma, \delta, \cdots$ strata of boundaries;
(7) $Z_\alpha, Z_\mu, Z_\gamma \cdots$ parametrizing space of strata $\alpha, \mu, \gamma, \delta, \cdots$;
(8) $E_\alpha, E_\mu, E_\gamma \cdots$ the exceptional divisor of Kirwan blowups;
(9) $N(R)$ normalier subgroup of a subgroup R in group G;
(10) T^n n-dimensional complex torus;
(11) $\mathrm{stab}(\beta)$ the stabilier subgroup a vector β in Lie algebra by adjoint action;
(12) All cohomology theory H^*, $\mathrm{IH}^*, \cdots$ will take $\mathbb{Q}$-coefficients.
(13) $D_c(X)$ the derived category of constructible sheaves with $\mathbb{Q}$-coefficients.

2. Basic tools

2.1. **Equivariant Cohomology.** For a topological space X with a group action G, its equivariant cohomology (here we use singular cohomology) measures the group action. The i-th equivariant cohomology $H^i_G(X, \mathbb{Q})$ is defined by the ordinary cohomology $H^i(EG \times_G X, \mathbb{Q})$ where $EG \to BG$ is the universal G-space. We list the properties we need below and one may refer to [14] for more details on equivariant cohomology theory.

Theorem 2.1. *Let G be a group acting on a complex variety X.*
(1) If the quotient X/G has only quotient singularities, then

$$H^i_G(X, \mathbb{Q}) = H^i(X/G, \mathbb{Q})$$

(2) If the quotient space X/G is contractible, then

$$H^i_G(X) = H^i(BG)$$

2.2. **Intersection cohomology.** Intersection cohomology is invented by Goresky-MacPherson [10] [11] to study the singular topological spaces. Let $IC_X \in D_c(X)$ be an intersection complex, then the Intersection cohomology is defined to be the hypercohomology

$$\mathrm{IH}^i(X, \mathbb{Q}) := \mathbb{H}^i(X, IC_X).$$

We refer to [20] for the definition of intersection complex and more details on the theory of Intersection cohomology. We state the blowup formula:

Proposition 2.2 (Kirwan [18] proposition 6.2)**.** *Let X be a smooth variety with action G and Z is a smooth G-subvariety with reductive stablizer subgroup R. Let $\widetilde{X} := Bl_Z(X) \to X$ be the blow up of X along Z, then*

$$\begin{aligned} \dim \mathrm{IH}^i(X/G) = \dim \mathrm{IH}^i(\widetilde{X}/G) \\ - \sum_{p+q=i} \dim(H^p(Z/N_0)) \otimes \mathrm{IH}^{\lambda(q)}(\mathbb{P}/R))^{\pi_0(N)} \end{aligned} \tag{2.1}$$

where $\lambda(q) = \begin{cases} q-2 & \text{if } q \leq \dim \mathbb{P}/R, \\ q & \text{if others.} \end{cases}$ *and $\mathbb{P}$ is the projection of a normal vector space of any point in Z.*

2.3. **Kirwan's desingularization package.** Assume X is a smooth projective variety over $\mathbb{C}$ with a reductive group G action on X. Denote Z_R the locus whose stabilizer group is R. Suppose there are only finitely many such locus

$$\{\ Z_{R_1},...,Z_{R_r}:\ \dim R_1 \geq ... \geq \dim R_r\ \}$$

and all the stabilizer groups R_i are reductive subgroups of G and all Z_R are smooth, then Kirwan took the blowups successively along these locus (see [21],[16])

$$\widetilde{X} = Bl_{\widetilde{Z_{R_r}}} \to \cdots \to Bl_{Z_{R_1}}X \to X$$

where $\widetilde{Z_{R_r}}$ is the strict transformations of Z_{R_r} and showed the G-action can be lifted to $\widetilde{X}$. Moreover, it commutes with the GIT quotient

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \widetilde{X}/\!\!/G & \longrightarrow & X/\!\!/G \end{array} \tag{2.2}$$

In this way, after finite steps, Kirwan obtained a partial resolution $\widetilde{X}/\!\!/G$ of $X/\!\!/G$, which has only quotient singularities at worst.

To study the cohomology of $\widetilde{X}/\!\!/G$, Kirwan developed several useful cohomological formulas:

(1) cohomology formula for GIT quotient:

For a reductive group G acting on a smooth complex variety X with G linearlised polarization L (even a symplectic manifold), we can choose a G-equivalent embedding $X \hookrightarrow \mathbb{P}^N$ via L. Let $T \subset G$ be a maximal torus and $\mathfrak{t}$ be its Lie algebra, fix a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$, then define the index set $\mathcal{B}_0$ consists of $\beta \in \mathfrak{t}_+$ such that β is the closest point to the origin 0 of the nonempty convex hull $con(\alpha_1,...,\alpha_m)$ generated by some weights $\alpha_1,...,\alpha_m$. Fix a a norm $||$ on $\mathfrak{t}$ (eg, the one induced by Killing form), set

$$Z_\beta = \{[x_0,\cdots,x_N] \in X:\ x_i = 0, if\ \alpha_i.\beta \neq |\beta|^2\}$$

$$Y_\beta = \{[x_0,\cdots,x_N] \in X:\ x_j = 0, if\ \alpha_j.\beta = |\beta|^2\ \&\ \exists\ \alpha_i.\beta \neq |\beta|^2\}$$

then there is a natural retraction map

$$p_\beta:\ Y_\beta \to Z_\beta$$

Denote X^{ss} the semistable locus of X with respect to the polsrization L in the sense of Mumford's GIT and Z_β^{ss} the locus of semistable points in Z_β and let

$$Y_\beta^{ss} := p_\beta^{-1}(Z_\beta^{ss}),\quad S_\beta := G\cdot Y_\beta^{ss}$$

then combing the theory of moment maps and relation of symplectic reduction and geometric invariant theory, it is shown in [22] that $\{S_\beta\}_{\beta\in\mathcal{B}_0}$ gives X a G-equivarant perfect Morse stratification . In particular, for $\beta = 0$, $S_0 = X^{ss}$. Using such stratification, Kirwan obtained the formula for Poncare's polynomials,

$$P_t^G(X^{ss}) = P_t(X)P_t(BG) - \sum_{0\neq\beta\in\mathcal{B}_0} t^{2\,\mathrm{codim}(S_\beta)} P_t^{stab(\beta)}(Z_\beta^{ss}). \tag{2.3}$$

We will call the term $\sum_{0\neq\beta\in\mathcal{B}_0} t^{2\operatorname{codim}(S_\beta)}P_t^{stab(\beta)}(Z_\beta^{ss})$ the removing part in the formula 2.3. Moreover, there is a natural identification

$$S_\beta \cong G\times_{P_\beta} Y_\beta^{ss}$$

where $P_\beta \leq G$ is the parabolic subgroup associated to β.

In this way, we have also have a dimension formula

$$\dim S_\beta = \dim G + \dim Y_\beta^{ss} - \dim P_\beta \tag{2.4}$$

(2) cohomology formula for blowups:

Assume X a smooth projective variety with action G and R is a reductive subgroup of G w.r.t the locus Z_R. If we take a blow up

$$\pi:\ \widetilde{X}\to X^{ss}$$

along the smooth center $G\cdot Z_R^{ss}$. Let $N(R)$ be the normalizer subgroup of R in G and d_R be the complex codimension of Z_R in X. then Kirwan's blowup formula [18] gives

$$\begin{aligned}P_t^G(\widetilde{X}^{ss}) = P_t^G(X^{ss}) + (t^2+...+t^{2d_R})P_t^{N(R)}(Z_R^{ss})\\ - \sum_{\beta\in\mathcal{B}_{0,\rho}} t^{2\operatorname{codim}(S_\beta)}P_t^{stab(\beta)\cap N(R)}(Z_{\beta,\rho}^{ss})\end{aligned} \tag{2.5}$$

Here $\mathcal{B}_{0,\rho}$ is the index set obtained as in 1 with respect to the normal representation

$$\rho: R\to \operatorname{Aut}(\mathbb{P}\mathcal{N}),$$

where $\mathcal{N}$ is the normal vector space of a point in Z_R^{ss} and $Z_{\beta,\rho}^{ss}\subset \mathbb{P}\mathcal{N}$ is the associated semi-stable strata given by weight β.

Suppose there are series of blowups

$$\cdots\to X_2\to X_1\to X_0 = X.$$

We write the correction terms contributed in the formula of i-th blowup as follows:

$$\begin{aligned}A_i(t) :=&(t^2+...+t^{2d_R})P_t^{N(R_\omega)}(Z_R^{ss})\\ &- \sum_{\beta\in\mathcal{B}_{0,\rho}} t^{2\operatorname{codim}(S_\beta)}P_t^{\mathrm{stab}(\beta)\cap N(R_\omega)}(Z_{\beta,\rho}^{ss})\end{aligned} \tag{2.6}$$

(3) Equivariant cohomology for fibration :

Let $F\hookrightarrow X\to B$ be a G equivariant topological fibration on X over base space B and F is the fiber, then we have the spectral sequence:

$$H_G^p(B, H^q(F,\mathbb{Q}))\Rightarrow H_G^{p+q}(X,\mathbb{Q}).$$

It is easy to see that the the spectral sequence implies

$$P_t^G(X) = P_t^G(B)\cdot P_t(F). \tag{2.7}$$

3. GIT results of cubic fourfold and Kirwan's dedingularization

Laza's work [23] on the GIT stabilites of cubic fourfolds is important to the computation. In this section we give a summary of main results in [23] and take a further detailed study of geometry of GIT boundaries, which is prepared for computations in the next two sections.

Proposition 3.1 (Laza, Prop 2.6 [23])**.** *A strictly semi-stable cubic fourfold with minimal orbit have defining equation of the following type:*

- α: $x_0q_1(x_2,..,x_5)+x_1q_2(x_2,..,x_5)$
- μ: $ax_0x_4^2+x_0x_5l_1(x_2,x_3)+bx_1^2x_5+x_1x_4l_2(x_2,x_3)+c(x_2,x_3)$
- γ: $x_0q(x_3,..,x_5)+x_1^2l_1(x_3,...,x_5)-2x_1x_2l_2(x_3,...,x_5)+x_2^2l_3(x_3,...,x_5)$
- δ: $x_0q(x_4,x_5)+f(x_1,x_2,x_3)$

where l, q, f means linear, quadratic and cubic equation respectively. Thus, we have

$$\overline{\mathcal{M}}-\mathcal{M}^s=\alpha\cup\mu\cup\gamma\cup\delta.$$

we still use $\alpha,\delta,\cdots\mu$ to denote the locus in $\overline{\mathcal{M}}$ parametrizing S-equivalence classes of strictly semistable cubic fourfolds (in the sense of geometric invariant theory [31]), which are boundaries strata, ie,

$$\overline{\mathcal{M}}-\mathcal{M}=\alpha\cup\cdots\cup\mu$$

By the analysis of the degeneration, Radu. Laza also gives the incidence relations of the these boundary divisors, see 1

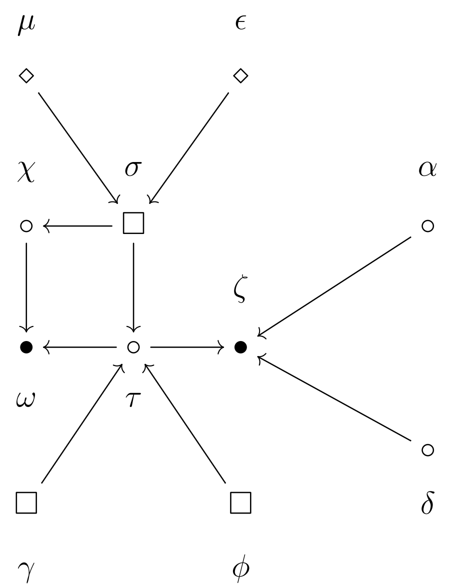

Figure 1. The incidence of boundary components of $\mathcal{M}$ in $\overline{\mathcal{M}}$

Theorem 3.2 (Laza, [23])**.** *The reductive subgroup of $G=SL(6,\mathbb{C})$ which are stabilizers of general points the GIT boundary strata are one of the following (up to a conjugate)*

- $R_\omega \cong SL(3,\mathbb{C})$
- $R_\zeta \cong (\mathbb{C}^*)^4$
- $R_\chi \cong SL(2,\mathbb{C})$
- $R_\delta \cong SO(2)(\mathbb{C}) \times \{\mathrm{diag}(t^{-2}, t, t, 1, 1, 1) : t \in \mathbb{C}^*\}$
- $R_\tau \cong \{\mathrm{diag}(t^2, t, 1, t^{-1}, t^{-2}, 1) : t \in \mathbb{C}^*\} \times \{\mathrm{diag}(t^4, t, t, t^{-2}, t^{-2}, t^{-2}) : t \in \mathbb{C}^*\}$
- $R_\alpha \cong \{\mathrm{diag}(t^2, t^2, t^{-1}, t^{-1}, t^{-1}, t^{-1}) : t \in \mathbb{C}^*\}$
- $R_\gamma \cong \{\mathrm{diag}(t^4, t, t, t^{-2}, t^{-2}, t^{-2}) : t \in \mathbb{C}^*\}$
- $R_\mu \cong \{\mathrm{diag}(t^2, t, 1, 1, t^{-1}, t^{-2}) : t \in \mathbb{C}^*\}$

where χ is a curve that parametrize cubic fourfold with equations of the form

$$bx_5^3 + \det \begin{pmatrix} x_0 & x_1 & x_2 + 2ax_5 \\ x_1 & x_2 - ax_5 & x_3 \\ x_2 + 2ax_5 & x_3 & x_4 \end{pmatrix} = 0.$$

here $a, b \in \mathbb{C}$. τ is a curve that parametrize cubic fourfold with equation of the form

$$\det \begin{pmatrix} x_0 & x_1 & ax_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0$$

and ζ is the point representing $x_0x_4x_5 + x_1x_2x_3$, ω is point representing

$$-\det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0$$

and the inclusion relation is given by the figure 3.

$$\begin{tikzcd}
 & & & & G^0_\omega \cong SL(3) \\
 & G^0_\zeta \cong T^4 & & & \\
 & & & & G^0_\chi \cong SL(2) \arrow[uu] \\
 & G^0_\delta \arrow[uu] & & G^0_\tau \arrow[uull] \arrow[uuur] & \\
G^0_\alpha \arrow[uuur] & & G^0_\gamma \arrow[ur] & & G^0_\beta \arrow[uu] \arrow[ul]
\end{tikzcd}$$

FIGURE 2. The stabilizers of general points in the boundary strata

Proposition 3.3. *The GIT boundaries $\overline{\mathcal{M}} - \mathcal{M}$ are the following stratas :*
(1) 1-dimensional strata: $\alpha \cong \mathbb{P}^1$, $\delta \cong \mathbb{P}^1, \tau \cong \mathbb{P}^1$, $\chi \cong \mathbb{P}^1$.
(2) 2-dimensional strata: γ is $\mathbb{P}^1 \times \mathbb{C}$, $\phi \cong \mathbb{P}^1 \times \mathbb{P}^1$.
(3) 3-dimensional strata: $\mu \cong \mathbb{P}(1,3,6,8)$, $\varepsilon \cong \mathbb{P}^1 \times \mathbb{P}(1,2,3)$.

Proof. $\chi \cong \mathbb{P}^1$ is shown in [24].

By lemma 4.5 in [23], we can write the defining equation of δ as

$$x_0x_4x_5 + f(x_1, x_2, x_3)$$

and it's semi-stable iff the cubic $f(x_1, x_2, x_3)$ has node at worst. By Luna's slice theorem, the GIT quotient is isomorphic to GIT of plane cubics, thus

$$\delta \cong \mathbb{P}|\mathcal{O}_{\mathbb{P}^2}(3)|/\!\!/SL(3) \cong \mathbb{P}^1$$

For each element, say $F = x_0q_1 + x_1q_2$ in α, it can be viewed as a pencil in $\mathbb{P}^3$, by [30] , We know

$$\alpha \cong Proj((Sym^4(\mathbb{C}))^{SL_2}) = Proj(\mathbb{C}[g_2, g_3]) \cong \mathbb{P}^1$$

For γ, note that by lemma 4.6 in [23], a cubic fourfold in locus γ has normal form

$$x_0(x_3^2 - x_4x_5) + x_1^2x_4 - 2x_1x_2l(x_3, x_4, x_5) + x_2^2x_5$$

and its stablizer group $\mathbb{C}^* = \{\ diag(1, t^{-1}, t, 1, t^2, t^{-2}) :\ t \in \mathbb{C}^*\ \}$. By Luna's slice theorem, this reduce GIT to $\mathbb{C}^*$ action on $\mathbb{A}^3(a, b, c)$ where a, b, c are coefficients in the normal form of $l(x_3, x_4, x_5) = ax_3 + bx_4 + cx_5$, then

$$\gamma \cong \mathbb{C} \times \mathbb{P}^1$$

For β, we first use the action $GL(2)$ on x_2, x_3 to reduce the problem to consider the torus action

$$\mathrm{diag}\{\ (t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}, t^{a_4}, t^{a_5}) :\ a_2 = a_3,\ \sum a_i = 0\ \}$$

on the space

$$\mathbb{P}V = \mathbb{P}^6(y_0, \cdots, y_6)$$

where V is the vector space spanned by the monomials

$$\{x_1x_3x_4,\ x_0x_4^2, x_5x_1^2,\ xx_2^3,\ x_2x_3^2,\ x_3x_2^2, x_3^3\}.$$

then we compute its invariant ring which has the minimal generators

$$y_0,\ y_1y_2y_5,\ (y_1y_2)^2y_4y_6,\ (y_1y_2)^3y_3y_6$$

then we obtain $\beta \cong \mathbb{P}(1, 3, 6, 8)$.

From [23] we know ε parameterize the cubic 4-folds singular along an irreducible rational normal curve of degree 4,

By proposition 6.6 in [23], the parameter space for such 4-folds is the product of $sym^4(\mathbb{P}^1)$ and affine space $\mathbb{A}(a, b, c)$ with natural action $SL(2)$ on $\mathbb{P}^1$ and $\mathbb{C}^*$ on x_5 in the equation above, thus

$$\varepsilon \cong \mathbb{P}^4/\!\!/SL(2) \times \mathbb{P}(1, 2, 3) \cong \mathbb{P}^1 \times \mathbb{P}(1, 2, 3)$$

$\square$

4. Cohomology of partial desingularization $\widetilde{\mathcal{M}}$

In this section, We follow Kirwan's strategy (see [18]) on computation: Note that $\widetilde{\mathcal{M}}$ has quotient singularities at worst and thus the equivariant cohomology equals to the usual cohomology by 2.1. By duality of intersection cohomology, we will only consider the cohomology term $P_t(\widetilde{\mathcal{M}})$ of degree lower than 20. Throughout the section, $X = \mathbb{P}^{55}$ and $G = SL(6, \mathbb{C})$.

4.1. **Computations for blowups.**

4.1.1. *Computation of $P_t^G(X^{ss})$.* According to the formula 2.3, we need to describe the index set $\mathcal{B}_0$. It consists of all closest points β lying in a positive Weyl chamber τ_+ to origin 0 in convex hull $con(\alpha_1, ..., \alpha_5)$ generated by some weights $\alpha_1, ..., \alpha_5$. Let T be a maximal torus in $SU(6, \mathbb{C})$ and its lie algebra is

$$\mathfrak{t} = \{\mathrm{diag}(\sqrt{-1}\theta_0, \sqrt{-1}\theta_1, ..., \sqrt{-1}\theta_5) : \sum \theta_i = 0\}.$$

Each 1-parameter subgroup is of the form

$$\lambda(t) = \{\mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2}, t^{r_3}, t^{r_4}, t^{r_5}) : \sum r_i = 0\}.$$

So we have the weights

$$W = \{(i_1 + i - 3, i_2 + i - 3, i_3 + i - 3, i + i_4 + i_5 - 3, i + i_5 - 3) : 0 \le \sum i_j \le 3\}$$

where $i = i_1 + i_2 + i_3 + i_4 + i_5$. By choosing a positive root system

$$\Phi_+ = \{(1, -1, 0, 0, 0), (0, 1, -1, 0, 0),$$

$$(0, 0, 1, -1, 0), (0, 0, 0, 1, -1), (0, 0, 0, 0, 0, 3)\},$$

we obtain the positive weyl chamber in $\mathfrak{t}$ as follows

$$\mathfrak{t}_+ = (\theta_1, .., \theta_5) : \theta_1 \ge ... \ge \theta_5 \ge 0\}.$$

With the help of computer, we find the only data with codimension < 10 is the case $\beta = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}) \in \tau_+$. In order to get $P_t^G(X^{ss})$, we need to compute the removing part. In this case, we have

$$\begin{aligned} stab(\beta) =& \{\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in SL(6, \mathbb{C}) : a \cdot \det A = 1, A \in GL(5, \mathbb{C})\} \\ =& \mathbb{C}^* \times SL(5, \mathbb{C}) \end{aligned}$$

where the first factor acts trivially on $Z_\beta = \mathbb{P}\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3$. Thus,

$$\begin{aligned} P_t^{stab(\beta)}(Z_\beta^{ss}) =& \frac{1}{(1-t^2)} \cdot P_t^{SL(5,\mathbb{C})}(Z_\beta^{ss}) \\ =& \frac{1}{(1-t^2)} \cdot P_t^{SL(5,\mathbb{C})}(\mathbb{P}(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3)^{ss}) \end{aligned}$$

To compute $P_t^{SL(5,\mathbb{C})}(\mathbb{P}(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3)^{ss})$, we will use Kirwan's formula 2.3 once more since these terms can be viewed as the cohomology of GIT quotient space. As before, we have unstable data given in the table 4.1.1 below with help of computer.

In the table 4.1.1, by choosing suitable 1-parameter subgroup, we find some locus $Z_\beta^{ss} = \phi$ and in this case, $P_t^{stab(\beta)}(Z_\beta^{ss}) = 0$. For example, 1-parameter subgroup

$$\lambda(t) = \mathrm{diag}(1, t^3, t^{-1}, t^{-1}, t^{-1})$$

will destabilize fourth data . Case by case check, we have only two nonzero data: 1st and 6th.

β	Monomials	stable(β)	
1. $\frac{1}{4}$(1,1,1,1)	$\mathbb{C}[x_1, x_2, x_3, x_4]_3$	$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}$	
2. (0.9,0.8,0.7,0.6)	$\{x_1x_2^2, x_1x_3^2, x_1x_2x_4\}$	T^4	
3. (1,0,0,0)	$x_1 \cdot \mathbb{C}[x_2, x_3, x_4]_2 \oplus x_0x_1^2$	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & A \end{pmatrix}$	
4. (1,1,$\frac{1}{2}$, $\frac{1}{2}$)	$\{x_1^2, x_2^2, x_1x_2\} \cdot \{x_3, x_4\}$	$\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}$	
5. (1.08,0.84,0.72,0.36)	$\{x_2^3, x_1x_3^2, x_4x_1^2\}$	T^4	
6. (1,1,1,0)	$\mathbb{C}[x_1, x_2, x_3]_3$	$SL(3,\mathbb{C}) \times T^2$	
7. (1.2,0.9,0.6,0.3)	$\{x_2^3, x_1x_2x_3, x_4x_1^2\}$	T^4	
8. (1.25,0.75,0.75,0.25)	$\{x_1^2x_4, x_1x_2x_3, x_1x_2^2, x_1x_3^2\}$	$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$	

TABLE 1. Unstable stratification.

(1) For 1st data, $v_1 = \frac{1}{4}(1,1,1,1)$, by the dimension formula 2.4, the codimension of unstable stratum equal to $35-(24+20-15)=6$. By the formula 2.3, we have

$$P_t^{stab(v_1)}(Z_{v_1}^{ss}) = \frac{1}{(1-t^2)} \cdot P_t^{SL(4,\mathbb{C})}(\mathbb{P}\mathbb{C}[x_1,x_2,x_3,x_4]_3) \mod t^{20}$$

The same method give only two unstable data for $SL(4,\mathbb{C}) \curvearrowright \mathbb{P}\mathbb{C}[x_1,x_2,x_3,x_4]_3$:
1. $\beta_1 = (1,1,1)$
we have *codim* $=4$ and

$$\begin{aligned} P_t^{stab(\beta_1)}(Z_{\beta_1}^{ss}) &= \frac{1}{(1-t^2)} \cdot P_t^{SL(3,\mathbb{C})}(\mathbb{P}\mathbb{C}[x_1,x_2,x_3]_3) \\ &= \frac{1}{(1-t^2)} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)} \end{aligned} \tag{4.1}$$

2. $\beta_2 = (1,0,0)$
we have *codim* $=6$ and

$$P_t^{stab(\beta_2)}(Z_{\beta_2}^{ss}) = \frac{1+t^2+t^4+t^6}{(1-t^2)\cdot(1-t^4)} - \frac{(1+t^2)\cdot t^2}{(1-t^2)^2} = \frac{1}{(1-t^2)} \tag{4.2}$$

Combing these data, we have

$$\begin{aligned} P_t^{stab(v_1)}(Z_{v_1}^{ss}) = &\frac{1}{(1-t^2)} \cdot \{\frac{1+t^2+...+t^{38}}{(1-t^4)\cdot(1-t^6)\cdot(1-t^8)} \\ &- \frac{t^{12}}{(1-t^2)} - \frac{t^8}{(1-t^2)} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)}\} \end{aligned}$$

Blowup locus	stabilizer group(up to finite index)	codimension
$G\omega$	$SL(3,\mathbb{C})$	27
$G\zeta$	T^4	24
GZ_χ	$SO(2)$	21
GZ_τ	T	20
GZ_δ	$\mathbb{C}^* \times \mathbb{C}^*$	19
GZ_α	$\mathbb{C}^*$	19
GZ_γ	$\mathbb{C}^*$	18
GZ_β	$\mathbb{C}^*$	17

TABLE 2. List of datas to be blowed up

(2) For 6th data, $v_2 = (1,1,1,0)$

$$stab(v_2) = (\mathbb{C}^*)^2 \times SL(3,\mathbb{C})$$

then the codimension of the removing strata is

$$34 - (24 + 9 - 15) = 16 > 10.$$

then by the formula 2.3

$$\begin{aligned} P_t^{stab(v_2)}(Z_{v_2}^{ss}) =& \frac{1}{(1-t^2)^2} \cdot P_t^{SL(3,\mathbb{C})}(\mathbb{PC}[x_1,x_2,x_3]_3) \mod t^{20} \\ =& \frac{1}{(1-t^2)^2} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)} \mod t^{20}. \end{aligned}$$

Observe the codimension of unstable stratification of this data is given by

$$\begin{aligned} \operatorname{codim} S_\beta =& 55 - (\dim G + \dim Y_\beta - \dim P_\beta) \\ =& 55 - (35 + 34 - 20) = 6. \end{aligned}$$

Here P_β is a parabolic subgroup consisting of all up triangle matrix thus has $\dim P_\beta = \frac{(6+1)\cdot 6}{2} - 1 = 20$ and

$$\dim Y_\beta = \#\{\ \alpha \in W : \alpha.\beta \geq \beta.\beta\ \} = 34$$

Thus, all the discussions above put into the formula 2.3 will show

Proposition 4.1.

$$\begin{aligned} P_t^G(X^{ss}) \equiv& \frac{1-t^{112}}{\prod\limits_{1\leq i\leq 6}(1-t^{2i})} - t^{12} P_t^{GL(5)}(\mathbb{PC}[x_0,x_1,x_2,x_3,x_4]_3) \\ \equiv& \frac{1}{\prod\limits_{1\leq i\leq 6}(1-t^{2i})} - \frac{t^{12}}{1-t^2}\frac{1}{\prod\limits_{1\leq i\leq 5}(1-t^{2i})}\ mod\ t^{20} \end{aligned} \tag{4.3}$$

In the next, we will take the blowups successively along the locus discussed in section 3, here we give the list of locus to be blowuped in the table 2 4.1.1

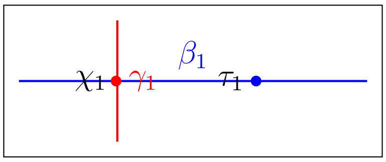

FIGURE 3. incidence relation

4.1.2. *Computation of $P_t^G(X_1^{ss})$.* we take the blow up

$$\pi:\ X_1 \to X^{ss}$$

along $G\cdot Z^{ss}_{R_\omega}$. By dimension counting,

$$d_R + 1 = codim G\cdot Z_R^{ss}$$
$$= 55 - (dim G - dim N(R_\omega)) = 55 - (35-8) = 28$$

Moreover, we have

$$P_t^{N(R_\omega)}(Z_R^{ss}) = P_t(BN(R_\omega)) = \frac{1}{(1-t^4)\cdot(1-t^6)}$$

since $Z^{ss}_{R_\omega}$ is just a point.

In [24], by using the fact the cubic 4-fold ω is the secant variety of verose embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, Laza proved

Proposition 4.2. *The representation of R_ω on the normal slice $\mathcal{N}_\omega$ is isomorphic to $Sym^6(\mathbb{C}^3)$, where $R_\omega \cong SL(3,\mathbb{C})$ has the natural representation on $\mathbb{C}^3$. In particular, the exceptional divisor $\mathbb{P}\mathcal{N}_\omega /\!\!/ R_\omega$ is isomorphic to GIT quotient space of plane sextic curves.*

A very helpful observation is the following

Corollary 4.3. *After the 1st blowup, the incidence relations of boundaries on exceptional divisor $\mathbb{P}\mathcal{N}_\omega/\!\!/SL(3,\mathbb{C})$ coincide with that of GIT moduli space of degree 2 K3 surfaces, See the figure 3 4.1.2.. That is, Let $\chi_1,\ \beta_1,\ \tau_1,\ \beta_1 \subset X_1^{ss}/G$ be the strict transformation of the GIT boundary after the 1st blowup along $G\cdot Z_\omega$, then*

$$\begin{aligned} E_\omega^{ss}/G\cap\chi_1 = pt, \quad E_\omega^{ss}/G\cap\tau_1 = pt \\ E_\omega^{ss}/G\cap\gamma_1 \cong |\mathcal{O}_{\mathbb{P}^1}(4)|//SL(2) = \mathbb{P}^1 \\ E_\omega^{ss}/G\cap\beta_1 \cong \mathbb{P}^3//\mathbb{C}^* = \mathbb{P}(1,2,3). \end{aligned} \tag{4.4}$$

Proof. Under the isomorphism $\mathbb{P}\mathcal{N}_\omega/\!\!/R_\omega \cong \mathbb{P}Sym^6(\mathbb{C}^3)/\!\!/SL(3)$ in proposition 4.2, we can identify the stability on both sides. Since after the first blowup, the rest blowups restricting to the divisor $\mathbb{P}\mathcal{N}_\omega/\!\!/R_\omega$ is laso the partial resolution of $\mathbb{P}\mathcal{N}_\omega/\!\!/R_\omega$ in the sense of Kirwan, then the locus of GIT strictly semistable locus will coincide with that of the GIT moduli space of plane sextics $|\mathcal{O}_{\mathbb{P}^2}(6)|/\!\!/SL(3)$. Such locus have explicitly described in [22]. Then from the incidence relation in figure 1, we obtain the results. $\square$

Remark 4.4. This corollary shows in the 2th-8th blowups, the blowup on $E_\omega^{ss}/G\cap\beta_1$ will be the same as that in GIT moduli space of degree 2 K3 surfaces, then Kirwan-Lee's results (see [18] [19]) will help us simplify many computations.

Following the computation in [17] for K3 surface and 2.6, we have

$$\begin{aligned} A_1 =& \frac{(t^2-t^{56})}{(1-t^2)\cdot(1-t^4)\cdot(1-t^6)} \\ &-\left(\frac{t^{50}-t^{56}}{(1-t^2)\cdot(1-t^4)(1-t^6)}+\frac{t^{20}-t^{28}}{(1-t^2)^3}\right) \\ \equiv& \frac{t^2}{(1-t^2)\cdot(1-t^4)\cdot(1-t^6)}-\frac{t^{20}}{(1-t^2)^3} \quad \mod \ t^{20} \end{aligned} \tag{4.5}$$

4.1.3. *Computation of $P_t^G(X_2^{ss})$.* Thanks to the disjointness of orbit $G\omega$ and $G\zeta$, the second blowup do not need to consider the effect of first blowup. So we take the second blowup

$$\pi: \ X_2 \to X_1^{ss} \tag{4.6}$$

along $G\cdot Z_{R_\zeta}^{ss}$. It's easy to see the normalization group of R_ζ in $G=SL(6,\mathbb{C})$ is given by the extension

$$0\to T^5\to N(R_\zeta)\to S_6\to 0.$$

here the symmetric group S_6 is the Weyl group of R_ζ. Then

$$d_{R_\zeta}+1 = codim(G\cdot Z_{R_\zeta}^{ss})$$
$$=55-(\dim G-\dim N(R_\zeta))=25$$

This gives

$$\begin{aligned} &(t^2+...+t^{2d_R})P_t^{N(R_\zeta)}(Z_{R_\zeta}^{ss}) \\ &=(t^2+...+t^{48})P_t(BN(R_\zeta)) \\ &=\frac{(t^2-t^{48})}{(1-t^2)...(1-t^{12})} \end{aligned} \tag{4.7}$$

since $G\cdot Z_{R_\zeta}^{ss}=G\times_{N(R_\zeta)}Z_{R_\zeta}^{ss}$ and $Z_{R_\zeta}^{ss}$ is just a point.

Actually, following a lemma of section 4.2 in [18], we can compute the normal vector space $\mathcal{N}_\zeta$ at ζ as follows

$$\begin{aligned} \mathcal{N}_\zeta =& \mathbb{C}x_0^3\oplus...\oplus\mathbb{C}x_5^3\oplus\{x_0^2,x_4^2,x_5^2\}\mathbb{C}[x_1,x_2,x_3]_1 \\ &\oplus\{x_1^2,x_2^2,x_3^2\}\mathbb{C}[x_0,x_4,x_5]_1\oplus\mathbb{C}\{x_0x_4x_5+x_1x_2x_3\}. \end{aligned}$$

So the intersections of exceptional divisor $E_2/N(R)$ with proper transformation of α,δ,τ are 3 distinct points.

Note that $R_\zeta=\{\mathrm{diag}(a,b,c,d,c^{-1}d^{-1},a^{-1}b^{-1}) : a,b,c,d\in\mathbb{C}^*\}$ acts trivially on $x_0x_4x_5+x_1x_2x_3$. By Kirwan, the unstable data is identified with the unstable data of natural action $R_\zeta \curvearrowright \mathbb{P}\mathcal{N}_\zeta$. So we only consider this action. Each 1-parameter subgroup can be written as diag $=\{t^{a_0},...,t^{a_5}\}$ and the weight is of the form

$$W=\{\ a\cdot I: \ x^I\in\mathcal{N}_\zeta\ \}$$

where $x^I=x_0^{i_0}\cdots x_5^{i_5}$ with $i_0+\cdots+i_5=3$ and $a\cdot I=a_oi_o+\cdots a_5i_5$.

Note that in the formula 2.6, the codimension is

$$codim = 24-\#\{\ a\cdot I\in W: \ a\cdot I>0\}\geq 11 \tag{4.8}$$

Thus, we obtain

$$A_2 \equiv \frac{t^2}{(1-t^2)...(1-t^{12})} mod\ t^{20} \tag{4.9}$$

4.1.4. *Computation of $P_t^G(X_3^{ss})$.* Take the third blowups

$$\pi:\ X_3 \to X_2^{ss}$$

along $G\cdot \widehat{Z}_{R_\chi}^{ss}$ where $\widehat{Z}_{R_\chi}^{ss}$ is the strict transform of $Z_{R_\chi}^{ss}$ under composition of previous blowups, since χ contains point ω and ζ.

we have

$$codim(G\cdot \widehat{Z}_{R_\chi}^{ss}) = 55-(\dim G+\dim \widehat{Z}_{R_\chi}^{ss}-\dim N(R_\chi)) = 23$$

By 3.3, we have

$$\widehat{Z}_{R_\chi}/\!\!/N(R_\chi) \cong \mathbb{P}(1:3) \cong \mathbb{P}^1$$

This gives

$$\begin{aligned}
&(t^2+...+t^{2d_R})\cdot P_t^{N(R_\chi)}(\widehat{Z}_{R_\chi}^{ss})\\
=&\frac{t^2-t^{46}}{1-t^2}\cdot P_t(BR_\chi)\cdot P_t^{N(R_\chi)/R_\chi}(\widehat{Z}_{R_\chi}^{ss})\\
=&(\frac{t^2-t^{46}}{1-t^2})\cdot\frac{1}{1-t^4}\cdot(1+t^2)
\end{aligned} \tag{4.10}$$

since the action $N(R_\chi)$ on $\widehat{Z}_{R_\chi}$ is isomorphic to the action $N(R_\chi)/R_\chi$ on $\widehat{Z}_{R_\chi}$.

In the same paper [24], Laza showed the normal representation $\rho: R_\chi \curvearrowright \mathcal{N}_\chi$ can be identified as

$$SL(2) \curvearrowright H^0(\mathcal{O}_{\mathbb{P}^1}(12))\oplus H^0(\mathcal{O}_{\mathbb{P}^1}(8))$$

This gives

$$\begin{aligned}
&\sum t^{2co\dim(S_\beta)}P_t^{stab(\beta)\cap N(R)}(Z_{\beta,\rho}^{ss})\\
=&\frac{t^{24}+t^{26}+t^{28}+t^{30}(1+t^2)+t^{30}+t^{34}+t^{36}+t^{38}+t^{40}(1+t^2)}{1-t^2}\\
=&\frac{t^{24}-t^{44}}{(1-t^2)^2}
\end{aligned}$$

Putting these together, we obtain

$$\begin{aligned}
A_3 =&(\frac{t^2-t^{46}}{1-t^2})\cdot\frac{(1+t^2)}{1-t^4}-\frac{t^{24}-t^{44}}{(1-t^2)^2}\\
\equiv&\frac{t^2}{(1-t^2)^2}\ mod\ t^{20}
\end{aligned} \tag{4.11}$$

4.1.5. *To compute $P_t^G(X_4^{ss})$.* Take 4th blowup $\pi:\ X_4 \to X_3^{ss}$ along $G\cdot \widehat{Z}_{R_\tau}^{ss}$, $N(R_\tau) = R_\tau \leq N(R_\omega) = SL(3)$ then

$$codim(G\cdot \widehat{Z}_{R_\tau}^{ss}) = 55-(dimG+dimZ_{R_\tau}-dimN(R_\tau)) = 21$$

we can identify the normal representation $R_\tau \curvearrowright \mathcal{N}_\tau \cong R_1 \curvearrowright \mathcal{N}_1$ where the normal representation $R_1 \curvearrowright \mathcal{N}_1$ second blowup in [18] by corollary 4.3. then from their table 2 in [18], we have

$$\begin{aligned} A_4 =& \frac{t^2 - t^{42}}{1-t^2} \cdot P_t(BR_\tau) \cdot (1+t^2) - \frac{t^{18}+t^{20}}{1-t^2} \; mod \; t^{20} \\ \equiv& \frac{t^2(1-t^2)}{(1-t^2)^3} - \frac{t^{18}+t^{20}}{1-t^2} \; mod \; t^{20} \end{aligned} \tag{4.12}$$

where the multiplication term $1+t^2$ is due to the geometry of locus $\widetilde{Z_\tau}^{ss}/(N(R_\tau)/R_\tau) \cong \mathbb{P}^1$ by proposition 3.3.

4.1.6. *To compute $P_t^G(X_5^{ss})$.* We take the blowup $\pi: \; X_5 \to X_4^{ss}$ along $G \cdot \widehat{Z}_{R_\delta}^{ss}$. Note that here

$$Z_{R_\delta} = \mathbb{P}\{x_0 q(x_4, x_5) + c(x_1, ..., x_3)\}$$

where $\{x_0 q(x_4, x_5) + c(x_1, ..., x_3)\}$ means the vector space spanned by the monomials in a general polynomial of the form $x_0 q(x_4, x_5) + c(x_1, ..., x_3)$. And the normaliser subgroup of such locus is

$$N(R_\delta) = \{\mathrm{diag}(a, A, B) : a^{-1} = |A| \cdot |B|, A \in GL(3), B \in GL(2)\}.$$

Thus, by the dimension counting, we have

$$\mathrm{codim}\, GZ_{R_\delta} = 55 - (35 + 12 - 13) = 21.$$

Observe that Z_{R_δ} contains $\zeta = x_0x_4x_5 + x_1x_2x_3$, so we need to take the blow up

$$\widehat{Z}_{R_\delta} \to Z_{R_\delta}$$

along $G_\delta \cdot Z_{R_{\delta\zeta}}$ to compute $P_t^{N(R_\delta)}(\widehat{Z}_{R_\delta}^{ss})$. Note from proposition 3.3, we know the blowup $\widehat{Z}_{R_\delta}^{ss}/N(R) \to Z_{R_\delta}/N(R) = \mathbb{P}^1$ does not change cohomology, ie, $P_t(\widehat{Z}_{R_\delta}^{ss}/N(R)) = 1 + t^2$. Thus

$$A_5 = \frac{t^2}{1-t^2}(1+t^2)\frac{1}{(1-t^2)(1-t^4)} - \sum \; mod \; t^{20}$$

where $\sum$ is due to removing unstable strata of representation of R_δ on the normal vector space of some point in Z_{R_δ} .

In order to find weight of normal representation, we choose a point $F = x_0x_4x_5 + f \in Z_\delta$ distinct to ζ, where f is a generic cubic polynomial in x_1, x_2, x_3. For normal representation $R_\delta \curvearrowright \mathcal{N}_F$, we take weight $(2, 0, 0, 0, -1, -1)$ (here we view the weight embedded into Lie algebra of G) of the maximal torus of R_δ. By subtracting the weight from $\frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_5}$ and of the form $x_0 q + f$, we have weight of normal space $\mathcal{N}_F$ in following list:

weight	6	4	0	-3	-2	-1
mul	1	3	3	2	6	6

Thus, by formula 2.6, the removing term $\sum$ is

$$P_t(\widehat{Z}_{R_\delta}^{ss}/N(R)) \cdot (t^{2 \cdot 8} P_t(\mathbb{P}^5) + t^{2 \cdot 6} P_t(\mathbb{P}^1)) \; mod \; t^{20}$$

and the correction term for 5-th blowup is

$$A_5 \equiv \frac{t^2}{(1-t^2)^3} - \frac{(t^{12}+...+t^{20})(1+t^2)}{1-t^2}\ mod\ t^{20} \tag{4.13}$$

4.1.7. *Computation of $P_t^G(X_6^{ss})$.* We take the blowup

$$\pi:\ X_6 \to X_5^{ss}$$

along $G\cdot \widehat{Z}_{R_\alpha}^{ss}$ where $\widehat{Z}_{R_\alpha}^{ss}$ is the strict transform of $Z_{R_\alpha}^{ss}$ under previous blowups, since α contains point ζ. It is easy to see

$$Z_{R_\alpha} = \mathbb{P}\{(x_0q_0(x_2,...,x_5)+x_1q_1(x_2,...,x_5)\},$$

$$1+d_\alpha = \operatorname{codim} G\widehat{Z}_{R_\alpha} = 55-(35+19-19) = 20,$$

$$N(R_\alpha) = \{\operatorname{diag}(A,B) : det(A)\cdot det(B) = 1, A\in GL(2,\mathbb{C}), B\in GL(4,\mathbb{C})\}.$$

The blowup

$$\widehat{Z}_{R_\alpha} \to Z_{R_\alpha}^{ss}$$

along $N(R_\alpha)\zeta$ descending to quotients will not change cohomology of quotients as in the case of 5-th blowup and thus we have the formula

$$P_t^{N(R_\alpha)}(\widehat{Z}_{R_\alpha}^{ss}) = P_t(N(R_\alpha))(1+t^2) = \frac{1+t^2}{1-t^2}$$

since $Z_{R_\alpha}/\!\!/R_\alpha \cong \mathbb{P}^1$ where first identity is due to formula 2.7.

To determine the normal representation of R_α, we choose $F = x_0(x_2^2+x_3^2+x_4^2)+x_1x_5^2 \in Z_{R_\alpha}^{ss}$ which is not in the orbit ζ. Then

$$F_{x_0} = x_2^2+x_3^2+x_4^2,\ F_{x_1} = x_5^2,\ F_{x_2} = 2x_0x_2$$

$$F_{x_3} = 2x_0x_3,\ F_{x_4} = 2x_0x_2,\ F_{x_5} = 2x_1x_5$$

where $F_{x_i} := \frac{\partial F}{\partial x_i}$. It is known as before that the tangent space at F is spanned by the monomials $F_{x_0},..F_{x_5}$ and monomials in Z_{R_α}. Subtracting from $\mathbb{C}[x_0,x_1,...,x_5]_3$, we obtain the normal vector

$$\mathcal{N}_F = \mathbb{C}[x_0,x_1]_3 \oplus span_{\mathbb{C}}\{x_5V_2, x_1^2x_2, x_1^2x_3, x_1^2x_4\} \oplus V_3$$

where V_2 is the set of monomials in x_2,x_3,x_4 of degree 2 and V_3 is the vector space of monomials in x_2,x_3,x_4 of degree 3 without $F_{x_0}x_2, F_{x_0}x_3, F_{x_0}x_4$. Recall the weight of R_α is $(2,2,-1,-1,-1,-1)$, then the weight of normal representation is given by

weight	-3	0	3	6
mul	7	5	4	4

So the smallest codimension of unstable strata is $19-8=11$ and thus removing term vanishes after $mod\ t^{20}$.

In a summary, the correction term in the 6-th blowup contributes

$$A_6 \equiv \frac{t^2}{1-t^2}\cdot\frac{1+t^2}{1-t^2}\ mod\ t^{20} \tag{4.14}$$

4.1.8. *Computation of $P_t^G(X_7^{ss})$.* We take the blowup

$$\pi: \ X_7 \to X_6^{ss}$$

along $G \cdot \widehat{Z}_{R_\gamma}^{ss}$ where $\widehat{Z}_{R_\gamma}^{ss}$ is the strict transform of $Z_{R_\gamma}^{ss}$ under $X_6^{ss} \to X$. then the codimension of GZ_{R_γ} is given by

$$1 + d_\gamma = 55 - (35 + 14 - 13) = 19$$

we have the normalizer subgroup

$$N(R_\gamma) = \{dig(a, A, B) : a \cdot det(A) \cdot det(B) = 1, A \in GL(2, \mathbb{C}), B \in GL(3, \mathbb{C})\}$$

By proposition 3.3, we have $Z_{R_\gamma} /\!\!/ G_\gamma \cong \mathbb{P}^1 \times \mathbb{C}$ and blowup at two points in $Z_{R_\gamma} /\!\!/ G_\gamma$ will give $P_t(\widetilde{Z_{R_\gamma} /\!\!/ G_\gamma}) = 1 + 3t^2$.

Now we consider normal representation $R_\gamma \curvearrowright \mathcal{N}_\gamma$. As before, by choosing a suitable element in $\widehat{Z}_{R_\gamma}^{ss}$, we compute its normal vector space $\mathcal{N}_\gamma$, which is a vector space spanned by monomials in the following form

$$\begin{aligned} &\{ \ x_0^3, ..., x_5^3, x_0^2 x_1, ..., x_0^2 x_4, \\ &x_0 x_1^2, x_0 x_2^2, x_1^2 x_2, x_3^2 x_4, x_3^2 x_5, x_4^2 x_4, x_4^2 x_3\}. \end{aligned}$$

It ca be identified as the normal representation in the third blowup in [18], then in a summary, the 7th blowup contributes

$$\begin{aligned} A_7 \equiv &(t^2 + ... + t^{2d_\gamma}) P_t^{N(R_\gamma)}(\widetilde{Z_{R_\gamma}}) - \sum unstable \ \ mod \ t^{20} \\ \equiv &\frac{1 + 3t^2}{1 - t^2}(t^2 + t^4 + ... + t^{14}) \ mod \ t^{20} \end{aligned} \tag{4.15}$$

4.1.9. *Computation of $P_t^G(X_8^{ss})$.* We take the blowup

$$\pi: \ X_8 \to X_7^{ss}$$

along $G \cdot \widehat{Z}_{R_\mu}^{ss}$ where $\widehat{Z}_{R_\mu}^{ss}$ is the strict transform of $Z_{R_\mu}^{ss}$ under previous blowups. The normalizer subgroup is

$$N(R_\mu) = \{\mathrm{diag}(a, b, A, c, d) : abcd \cdot det(A) = 1, a, b, c, d \in \mathbb{C}^*, A \in GL(2, \mathbb{C}))\}.$$

The locus Z_{R_μ} is identified as

$$Z_{R_\mu} = \mathbb{P}\{a x_0 x_4^2 + x_0 x_5 l_1(x_2, x_3) + b x_1^2 x_5 + x_1 x_4 l_2(x_2, x_3) + c(x_2, x_3)\}.$$

Here $\{f\}$ means the vector space spanned by mononials in f.

Thus the codimension is

$$1 + d_\mu = \mathrm{codim}\, GZ_{R_\beta} = 55 - (35 + 9 - 7) = 18.$$

The blowup $\widehat{Z}_{R_\mu} \to Z_{R_\mu}$ along the orbit $N(R_\mu)\zeta$ and $N(R_\mu)\omega$ will decent to blowup along two points in $Z_{R_\mu} /\!\!/ N(R_\mu) \cong \mathbb{P}(1, 3, 6, 8)$. This gives

$$\begin{aligned} P_t^{N(R_\beta)}(\widehat{Z}_{R_\beta}) = &P_t(BR_\beta) \cdot P_t(\widehat{Z}_{R_\beta} /\!\!/ N(R_\beta)) \\ = &\frac{1}{1 - t^2} \cdot (P_t(\mathbb{P}(1, 3, 6, 8)) + (t^2 + t^4) + (t^2 + t^4)) \end{aligned} \tag{4.16}$$

The normal representation for R_μ can be identified in K3 case as done in last blowup of Kirwan-Lee (see 5.3 in [18]), thus the removing term is giving by

$$\frac{t^{18}}{1-t^2} \mod t^{20}$$

In a summary, the correction term will be given by

$$\begin{aligned} A_8(t) \equiv & \frac{t^2}{1-t^2} \cdot \frac{1+3t^2+3t^4+t^6}{1-t^2} \\ & - \frac{t^{18}\cdot(1+3t^2+3t^4+t^6)}{1-t^2} \mod t^{20} \end{aligned} \tag{4.17}$$

4.2. **Proof of Theorem 1.1.** By previous computations, we have

$$\begin{aligned} P_t(\widetilde{\mathcal{M}}) = & P_t^G(X^{ss}) + \sum_{i=1}^{8} A_i(t) \\ = & 1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} \\ & + 193t^{14} + 236t^{16} + 280t^{18} + 324t^{20} \mod t^{20} \end{aligned}$$

Then the duality will give the formula.

5. Intersection cohomology of Baily-Borel compactification

In this section, we will compute the intersection cohomology of Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ based on computation before.

5.1. **Baily-Borel compactification of moduli space of cubic fourfolds.** It is well-known that for a smooth cubic fourfold X its integral middle cohomology $H^4(X,\mathbb{Z})$ has lattice strucure isomorphic to $\Lambda := < 1 >^{\oplus 21} \oplus < -1 >^{\oplus 3}$. Let $h := c_1(\mathcal{O}_X(1))^2 \in \Lambda$ be the hyperplane class and $\Lambda_0 = E_8^2 \oplus U^2 \oplus A_2 = h^\perp$ be the polarised lattice associated to the smooth cubic fourfold, which isomorphic to the lattice structure of primitive cohomology $H^4_p(X,\mathbb{Z})$ of X. Denote

$$\mathcal{D} := \{z \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle > 0\}$$

the peroid domain. Let Γ be the monodromy group, then $\Gamma \cong O^*(\Lambda_0)$ where $O^*(\Lambda_0) \leq O^+(\Lambda_0)$ is the index 2 subgroup. By the general result of Baily-Borel in [3], there is a compactification of $\mathcal{D}/\Gamma$, whose boundaries corresponds to Type II, III degenerations of cubic fourfolds (see also [27] for refinements). Such compatification is well-known as Baily-Borel compactification now and we denote by $\overline{\mathcal{D}/\Gamma}^{BB}$. Following B. Hassett [13], we define

Definition 5.1. A cubic fourfold X is called a special cubic fourfold of discriminant d if it contains a surface T which is not homologous to a complete intersection and the classes h and $[T]$ form a saturated rank 2 sublattice of Λ with discriminant d.

It is also shown in [13] that such locus is a divisor in the moduli space. Let $\mathcal{H}_\infty$ be such a divisor of discriminant 2.

Theorem 5.2. *(Global Torelli, see [24] [28]) The period map*

$$p : \overline{\mathcal{M}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$$

is a birational map. It is an open immersion over $\mathcal{M}^o$ and can be defined over $\mathcal{M}$, whose image is the complement of Heegner divisor $\mathcal{H}_\infty$.

Thanks to the Torelli theorem, these divisors defined in 5.1 are also called Heegner divisors if we view $\overline{\mathcal{D}/\Gamma}$ as a Shimura variety. We refer the readers to [29] for the definition and properties of shimura variety.

Remark 5.3. Recently, the property of open immersion of period map on $\mathcal{M}^o$ is also proven by Huybrechts and Rennemo in [15] using Jacobian ring.

5.2. **Intersection cohomology.** Let $\widehat{\mathcal{M}}$ be the blowups of $X/\!\!/SL(6)$ only along ω and χ. Then there is a natural contraction morphism

$$f : \widetilde{\mathcal{M}} \to \widehat{\mathcal{M}}$$

From [24], it is known that the period map from GIT compatification to Baily-Borel compatification is resolved by Loojigenga's semi-toric compatification (see for [27] the discussion of Loojigenga's semi-toric compatification):

$$\begin{tikzcd} & \widehat{\mathcal{M}} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \overline{\mathcal{M}} \arrow[rr, dashed, "p"] & & \overline{\mathcal{D}/\Gamma}^{BB} \end{tikzcd} \tag{5.1}$$

Remark 5.4. By Looijenga's general construction, p_2 is composition of a blowup of self-section of the divisor and a small modification. Thus, from the diagram, It is not easy to see

$$\rho(\overline{\mathcal{M}}) = 1, \quad \rho(\widetilde{\mathcal{M}}) = 3, \quad \rho(\overline{\mathcal{D}/\Gamma}^{BB}) = 2$$

We observe the following explicit description

Proposition 5.5. *The morphism p_2 is the composition of $\phi : \widetilde{\mathcal{M}} \to \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)}$ and $\nu : \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \to \overline{\mathcal{D}/\Gamma}^{BB}$. Here ϕ is the morphism contracting the divisor E_χ to*

$$E_\chi \cap \widetilde{E}_\omega \cong \mathbb{P}(H^0(C, \mathcal{O}_C(4)) \oplus H^0(C, \mathcal{O}_C(6)))/\!\!/SO(3) \subset \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \tag{5.2}$$

where C is a smooth plane quadric curve and ν is a small modification described as follows:

Table 3. Contraction locus of p_2

Exceptional locus in $\overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)}$	*Boundaries in* $\overline{\mathcal{D}/\Gamma}^{BB}$	*fiber*
ϕ_∞	A_{17}	$\mathbb{P}^1$
γ_∞	$E_7 \oplus D_{10}$	$\mathbb{P}^1$
β_∞	$E_8^{\oplus 2} \oplus A_2$	$\mathbb{P}^2$
ϵ_∞	$A_2 \oplus D_{16}$	$\mathbb{P}^2$

Here the locus $\phi_\infty, \cdots, \epsilon_\infty$ is described in lemma 6.9 in [24].

Proof. From section 6 in [24], we know that p_2 is a composition of a small modification in the sense of Looijenga (see [27]) and a blowup of codimension 2 self-intersection of Heegner divisor $\mathcal{H}_\infty$. Their roots span a sublattice $R \subset \Lambda_0$ with

$$R = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

Thus, $R^\perp = U^2 \oplus E_8^2$. This implies the self-intersection

$$(\mathcal{H}_\infty \cap \mathcal{H}_\infty)/\Gamma \cong \mathcal{D}_R/\Gamma_R$$

where $\mathcal{D}_R = \mathcal{D} \cap \mathbb{P}(R^\perp \otimes \mathbb{C})$ and $\Gamma_R \leq \Gamma$ is the stabilier subgroup of $\mathcal{D}_R$. Note that $R^\perp$ is sublattice with discriminant 2, then $\Gamma_R \leq O(R^\perp)^+$ is a subgroup of index 2. By Odaka-Oshima [32], there is isomorphism

$$\mathcal{D}_R/O(R^\perp)^+ \cong \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(8)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(12)))/\!\!/SL(2).$$

Note that $SL(2)$ can be identified as a subgroup of $SO(3)$ with index 2, this shows 5.2.

For the small modification, it is described in lemma 6.9 in [24]. □

Proposition 5.6. *Let $f: X \to Y$ be a birational morphism of n dimensional irreducible varieties over $\mathbb{C}$ contracting a divisor E to a lower dimensional locus Z and the restriction f_E of $f: E \to Z$ is a topological $\mathbb{P}^m$-bundle, then for $k \leq n$, the intersection cohomology has a decomposition*

$$IH^k(X) \cong IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^{k-c+j}(Z, \mathbb{Q}) \tag{5.3}$$

where c is the codimension of Z in X

Proof. Let IC_X be the intersection complex on X. By BBDG's decomposition theorem, there is a decomposition (non-canonical in general, see [9] for more general results)

$$Rf_*IC_X \cong IC_Y \oplus IC_Z(\mathcal{L}_j)[-i] \tag{5.4}$$

where $\mathcal{L}_j$ are the local systems on Z and i is the degree to be shifted. Following [12], we can determine these local system: each $\mathcal{L}_j$ is an irreducible summand of $R^j f_{E\,*}\mathbb{Q}_E$ where f_E is the morphism restrcting on E. $\mathcal{L}_j$ is rank $= 1$ for j even since each fiber is $\mathbb{P}^m$, thus $\mathcal{L}_j = R^j f_{E\,*}\mathbb{Q}_E$ and the shift degree is $-i = -j + c$. then by taking cohomology of the decomposition, we obtain the formula 5.3:

$$\begin{aligned} IH^k(X) =& H^k(Y, IC_Y) \bigoplus_{2\leq j\leq 2m} R^j f_{E\,*}\mathbb{Q}_E[-j+c]) \\ =& IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^k(Z, \mathbb{Q}[-j+c]) \\ =& IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^{k-c+j}(Z, \mathbb{Q}) \end{aligned}$$

□

Recall that an algebraic map $f: X \to Y$ is called smeismall if the defect

$$r(f) := \max\{\, i \in \mathbb{Z} : \ {}^p\mathcal{H}^i(Rf_*IC_X[n]) \neq 0 \,\}$$

is zero.

Proposition 5.7. *Let $f: X \to Y$ be a semismall birational morphism of n dimensional irreducible varieties over $\mathbb{C}$ such that $Z \subset Y$ is a connected closed subvariety and f is isomorphic outside Z and over Z, f is a $\mathbb{P}^m$-bundle, then for $k \leq n$*

$$IH^k(X) = \bigoplus_{0 \leq j \leq m} H^{k+2j-n}(Z, \mathbb{Q}) \tag{5.5}$$

where $H^l(Z, \mathbb{Q}) = 0$ if $l < 0$.

Proof. By semi-small property and semi-simplicity of decomposition theorem, we have

$$\begin{aligned} Rf_* IC_X[n] &= \bigoplus_{-r(f) \leq i \leq r(f)} {}^p\mathcal{H}^i(Rf_* IC_X[n])[-i] \\ &= \bigoplus_j IC(\overline{Y}_i, \mathcal{L}_{i,j}) \end{aligned} \tag{5.6}$$

where $\mathcal{L}_{i,j}$ is a local system supported on clourse $\overline{Y}_i$ of strata Y_i.

In our case, there is a natural stratification $Y_0 = Z, Y_1 = Y - Z$, then we have

$$Rf_* IC_X = IC_Y \bigoplus_{j=2}^{m} \mathbb{Q}_Z[n-2j] \tag{5.7}$$

By taking cohomology, we obtain the result. $\square$

Theorem 5.8. *The intersection cohomology of $\hat{\mathcal{M}}$ is*

$$\begin{aligned} IP_t(\widehat{\mathcal{M}}) =& 1 + 3t^2 + 8t^4 + 17t^6 + 29t^8 + 44t^{10} + 61t^{12} + 78t^{14} \\ &+ 99t^{16} + 121t^{18} + 151t^{20} + 121t^{22} + 99t^{24} + 78t^{26} \\ &+ 61t^{28} + 44t^{30} + 29t^{32} + 17t^{34} + 8t^{36} + 3t^{38} + t^{40} \end{aligned} \tag{5.8}$$

Proof. we will use the blowup formula 2.1 of intersection cohomology reversely. So we need to do step by step computations:

(1) Blow down E_μ: In this case, $\pi_0(N_\mu)$ acts on the fiber trivially since N_μ is connected, thus we need to shift the polynomial by degree 2 according to the formula 2.1, then we get

$$\begin{aligned} B_\mu(t) =& (1 + 3t^2 + 3t^4 + t^6) \cdot (t^2 + t^4 + 2t^6 \\ &+ 2t^8 + 3t^{10} + 3t^{12} + 4t^{14} + 4t^{16} + 4t^{18} \\ &+ 4t^{20} + 3t^{24} + 2t^{26} + 2t^{28} + t^{30} + t^{32}) \end{aligned} \tag{5.9}$$

(2) Blowing down E_γ: it is similar to the case E_μ. we get

$$\begin{aligned} B_\gamma(t) =& (1 + 3t^2 + t^4) \cdot (t^2 + 2t^4 + 3t^6 + 4t^8 \\ &+ 5t^{10} + 6t^{12} + 7t^{14} + 8t^{16} + 8t^{18} + 8t^{20} + 7t^{22} \\ &+ 6t^{24} + 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + t^{34}) \end{aligned} \tag{5.10}$$

(3) Blowing down E_α: It is similar to the case E_μ. we get

$$\begin{aligned} B_\alpha(t) =& (1 + t^2) \cdot (t^2 + t^4 + 2t^6 + 3t^8 + 4t^{10} \\ &+ 5t^{12} + 6t^{14} + 7t^{16} + 8t^{18} + 8t^{20} + 7t^{22} + 6t^{24} \\ &+ 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + +t^{34} + t^{36}). \end{aligned} \tag{5.11}$$

(4) Blowing down E_δ: It is similar to the case E_μ. we get

$$\begin{aligned} B_\delta(t) =&(1+t^2)\cdot(t^2+2t^4+4t^6+6t^8+9t^{10}+12t^{12}+ \\ &16t^{14}+19t^{16}+24t^{18}+24t^{20}+19t^{22}+16t^{24} \\ &+12t^{26}+9t^{28}+6t^{30}+4t^{32}+2t^{34}+t^{36}). \end{aligned} \tag{5.12}$$

(5) Blowing down E_τ: It is similar to the case E_μ. we get

$$\begin{aligned} B_\tau(t) =&(1+t^2)\cdot(t^2+t^4+2t^6+3t^8+4t^{10} \\ &+5t^{12}+7t^{14}+8t^{16}+9t^{20}+8t^{22}+7t^{24} \\ &+5t^{26}+4t^{28}+3t^{30}+2t^{32}+t^{34}+t^{36}). \end{aligned} \tag{5.13}$$

(6) Blowing down E_ξ: note in this case, $\pi_0(N_\xi)=S_6$ acts on $H^*(\mathbb{P}N_\xi/R_\xi)$ by permutation of coordinates of $\mathbb{P}N_\xi$, thus,

$$\begin{aligned} IP_t(H^*(\mathbb{P}N_\xi/R_\xi)^{\pi_0(N_\xi)} &\equiv P_t(\mathbb{P}N_\xi)P_t((H^*(BR_\xi)^{\pi_0(N_\xi)}) \mod t^{19} \\ &\equiv \frac{1}{\prod_{1\le i\le 6}(1-t^{2i})} \mod t^{19} \end{aligned} \tag{5.14}$$

then using formula 2.1 again, we have

$$\begin{aligned} B_\xi(t) =&t^2+t^4+2t^6+3t^8+5t^{10}+7t^{12}+11t^{14}+14t^{16} \\ &+20t^{18}+26t^{20}+20t^{22}+14t^{24}+11t^{26} \\ &+7t^{28}+5t^{30}+3t^{32}+2t^{34}+t^{36}+t^{38} \end{aligned} \tag{5.15}$$

Put these together then we obtain our formula from

$$\mathrm{IP}_t(\widehat{\mathcal{M}})=P_t(\mathcal{M})-B_\mu(t)-B_\xi(t)-B_\delta(t).$$

□

Remark 5.9. In [8], the authors doubted that the Kirwan resolution of moduli spaces of cubic threefolds is isomorphic to certain toroidal compactification of ball quotient $\mathbb{B}/\Gamma$ with respect to some cone decomposition. Their evidence in [8] is that they compute the cohomology of the toroidal compactification and find the betti numbers of the two compactification match perfectly. We want to point out here it is also unknown for the moduli spaces of cubic fourfolds.

Corollary 5.10. *The intersecton betti number of $\overline{\mathcal{D}/\Gamma}^{BB}$ is*

$$\begin{aligned} IP_t(\overline{\mathcal{D}/\Gamma}^{BB}) =& 1+2t^2+4t^4+9t^6+16t^8+26t^{10}+38t^{12} \\ &+50t^{14}+65t^{16}+82t^{18}+112t^{20}+82t^{22}+65t^{24}+50t^{26} \\ &+38t^{28}+26t^{30}+16t^{32}+9t^{34}+4t^{36}+2t^{38}+t^{40} \end{aligned} \tag{5.16}$$

Proof. First apply formula (5.3) to morpgism μ in 5.1, we have

$$IP_t(\overline{\mathcal{D}/\Gamma}^{BB})=IP_t(\widehat{\mathcal{M}})-(1+t^2)(P_t(E_\chi\cap\widetilde{E}_\omega)-1) \mod t^{20}$$

then we remain to compute the cohomology $E_\chi\cap\widetilde{E}_\omega$. Thanks to 4.3, we identify $E_\chi\cap\widetilde{E}_\omega$ as the exception divisor in the 1st blow up of GIT moduli of degree 2

K3 surfaces. According to [18], after blowing a rational curve $\mathbb{P}^1 \subset E_\chi \cap \widetilde{E}_\omega$, we get a kirwan resolution of $E_\chi \cap \widetilde{E}_\omega$, thus

$$\begin{aligned} P_t(Bl_{\mathbb{P}^1}(E_\chi \cap \widetilde{E}_\omega)) =& \mathrm{IP}_t(Bl_{\mathbb{P}^1}(E_\chi \cap \widetilde{E}_\omega)) \\ =& \mathrm{IP}_t(E_\chi \cap \widetilde{E}_\omega) + \textit{correction term} \\ \equiv& \frac{1}{(1-t^2)\cdot(1-t^4)} + (1+t^2)\cdot(t^2+2t^4+3t^6+ \\ & 4t^8+5t^{10}+6t^{12}+7t^{14}+7t^{16}+8t^{18}) \mod t^{18} \end{aligned} \tag{5.17}$$

where the correction term is appeared as in formula 2.1.

the usual blowup formula for cohomology gives

$$\begin{aligned} P_t(E_\chi \cap \widetilde{E}_\omega) =& 1+t^2+3t^4+5t^6+8t^8+10t^{10}+13t^{12} \\ & +15t^{14}+17t^{16}+18t^{18}+17t^{20}+15t^{22} \\ & +13t^{24}+10t^{26}+8t^{28}+5t^{30}+3t^{32}+t^{34}+t^{36} \end{aligned} \tag{5.18}$$

Last, apply formula 5.5 to morphism ν and combine the table 5.5, we only need to remove

$$2(t^{18}+t^{20})+2(t^{16}+t^{18}+t^{20}) \mod t^{20}$$

All these together imply our formula. □

Remark 5.11. Since the Zucker's conjecture was established in [26] and [34], our computation also provides all L^2-betti numbers of $\overline{\mathcal{D}/\Gamma}^{BB}$.

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