NOTES ON PROJECTIVE MODEL OF K3 SURFACE

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ABSTRACT. This is a reading note on the paper of B.Saint-Donat.

CONTENTS

1. LINEAR SYSTEM ON K3

In the note,S will always be a K3 over $\mathbb{C}$.This section contains several technique lemmas in [SD3]

Theorem 1.1. *Let $C \subset S$ be an irreducible curve with $C^2 > 0$,Then $|\mathcal{O}_S(C)|$ is base point free.*

Proof. It's sufficient to show

$$H^1(S, I_x \otimes \mathcal{O}_S(C)) = 0$$

Take blowups $\pi : \widetilde{S} \to S$ centered at x with E_x as exceptional divisor. then

$$\pi_*(\mathcal{O}_{\widetilde{S}}(-E_x)) = I_x$$

By projection formula,

$$\pi_*(\pi^*(\mathcal{O}_S(C)) \otimes \mathcal{O}_{\widetilde{S}}(-E_x)) = \mathcal{O}_S(C)) \otimes I_x$$

By Riemann-Roch and Lemma 1.2 and 1.4,

$$dim H^0(\mathcal{O}_S(C)) = 2 + \frac{1}{2}C^2 \geq 3$$

,thus we can choose nontrival section $s \in H^0(S, I_x \otimes I_x \otimes \mathcal{O}_S(C))$ In this way,

$$\pi^* S \in H^0(\pi^*(\mathcal{O}_S(C)) \otimes \mathcal{O}_{\widetilde{S}}(-2E_x))$$

will be a non trival section.By lemma1.2 and 1.4,

$$0 = H^1(\widetilde{S}, \pi^*(\mathcal{O}_S(-C)) \otimes \mathcal{O}_{\widetilde{S}}(2E_x) = H^1(\widetilde{S}, \pi^*(\mathcal{O}_S(C)) \otimes \mathcal{O}_{\widetilde{S}}(-E_x) \tag{1.1}$$

$$= H^1(S, \pi_*(\pi^*(\mathcal{O}_S(C)) \otimes \mathcal{O}_{\widetilde{S}}(-2E_x)) = H^1(S, \mathcal{O}_S(C)) \otimes I_x) \tag{1.2}$$

where (1.1) follows serre duality and (1.2) comes from projection formula and Leray spectral sequence. □

Lemma 1.2. *(C.P.Ramanujum) Let D be a $1-$connected effective divisor on S ,then $H^0(D, \mathcal{O}_D) \simeq \mathbb{C}$.*

Proof. The proof follows[R4]

□

Lemma 1.3. *Let X be nonsingular surface and $\pi : \widetilde{X} \to X$ blowup centered at $x \in X.E := \pi^{-1}(x)$ exceptional curve and $C \in Pic(X)$ with each divisor in $|C|$ is $2-$connected. then each $D \in |\pi^*\mathcal{O}_X \otimes \mathcal{O}_{\widetilde{X}}(-2E)|$ is $1-$connected .*

Proof. □

Lemma 1.4. *Let $C \subset S$ be an irreducible curve,then any $D \in |C|$ is 2-connected.*

Proof. suppose $C = D_1 + D_2$ with D_1, D_2 effective.Set

$$r_i = \frac{C.D_i}{C^2}, \Gamma_i = r_iC - D_i,\ i = 1, 2$$

then

$$r_1 + r_2 = 1, r_i \geq 0$$
$$\Gamma_1 + \Gamma_2 = 0, \Gamma_i.c = 0$$

By Hodge index theorem,

$$\Gamma_i^2 \leq 0$$

with $=$ iff $\Gamma_i = 0$. Thus

$$D_1.D_2 = (r_1C - \Gamma_1)(r_2C - \Gamma_2) = r_1r_2C^2 - \Gamma_1^2 \geq r_1r_2C^2$$

Now we need to take analysis in detail.

case1:$r_1 = 0$,ie,$D_1.C = 0 \Rightarrow$

$$D_1.D_2 = -D_1^2 = -\Gamma_1^2 \geq 2$$

since $D_1 \neq 0$

case2:$D_1.C = 1, D_2.C = 1$.then

$$D_1.D_2 = \frac{1}{2} - \Gamma_1^2$$

case3:$r_1 \geq \frac{2}{C^2}$

□

Lemma 1.5. *Let D be effective divisor on S. then*

$$h^1(\mathcal{O}_S(-D)) = dimH^0(D, \mathcal{O}_D) - 1$$

in particular,if D is $1-$connected,then $h^1(\mathcal{O}_S(-D)) = 0$

Proof. consider the following exact sequence and take its cohomology:

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

□

Lemma 1.6. *Let $L \in Pic(S)$ be an invertible sheaf without fixed components and $|L| \neq \phi$,then*

(1) $L^2 > 0$ *,then generic member of is an irreducible curve of* $p_a = 1 + \frac{1}{2}L^2$ *and* $h^1(L) = 0$
(2) $L^2 = 0$,*then* $L \simeq \mathcal{O}_S(kE)$ *where integer* $k > 0$ *and* E *is irreducible curve of* $p_a(E) = 1$. *and* $h^1(L) = k - 1$

Proof. □

Lemma 1.7. *Let $C \subset S$ be an irreducible curve and $C^2 > 0$.$D \geq 0 \in Div(S)$ st $D^2 \geq 0$ and $|D - C| = \phi$, then $D.C > 1$.in particular,for $D = E$ irreducible curve with $E^2 = 0$,$E.C > 1$*

Proof. (1) If C is a component of D,$|D - C| = \phi$ implies $D = C$ since D is effective, then

$$D.C = C^2 \geq 2$$

(2) If C is not a component of D, then

$$D.C \geq 0$$

.

(3) If $D.C = 0$,then by Hodge index theorem,

$$D \leq 0$$

with " $=$ " iff $D = 0 \Rightarrow D = 0$ since $D^2 \geq 0 \Rightarrow |C| = |D-C| = \phi$, contradiction !

(4) If $D.C = 1$, suppose $D = \sum_{i=1}^{N} n_i D_i$ where D_i is irreducible component and $n_i > 0 \Rightarrow$ only possibility: $n_1 = 1, D_1.C = 1, D_i > C = 0 for\ i > 1$. $\Rightarrow (D - C)^2 = D^2 + C^2 - 2 \geq 0$ since $C^2 > 0$ By Riemann-Roch,$h^0(D - C) + h^0(C - D) = h^1(D - C) + \frac{1}{2}(D - C)^2 + 2 \geq 2$,thus we get contradiction since $|D - C| = \phi$.

□

Remark 1.8. (Basic facts about fixed components of a linear system)
Let $D \in Div(S)$ be an effective divisor with $D^2 > 0$,we can write

$$D \sim D' + \Delta$$

then fixed part $dim|\Delta| = 0$ and moving part D' has no fixed component.and $dim|D| = dim|D'|$ Using

$$0 \to \mathcal{O}_S(-\Delta) \to \mathcal{O}_S \to \mathcal{O}_\Delta \to 0,$$

we have

$$h^0(\Delta, \mathcal{O}_\Delta) = h^1(S, \mathcal{O}_S(-\Delta)) + 1 = N$$

where N is the number of connected components and

$$\Delta = \sum_i \Delta_i, \Delta_i^2 = -2, \Delta_i.\Delta_j = 0.$$

By lemma 1.6, there are only two cases:

Case 1. D' irreducible. If $D'.\Delta_i > 0$, then $D'.\Delta_i = 0$

Conversely,if Γ is a connected reduced curve with $\Gamma^2 = -2$, then Γ is fixed in $|D' + \Gamma|$ iff $\Gamma.D' = 0,\ or\ 1$

Case 2. $D' = kE$ where E is irreducible of $p_a(E) = 1$,$k > 1$.

then $\exists\ !\ \Delta_j$ st: $\Delta_j.E > 0$ and such case $\Delta_j > E = 1$ will occur.

Conversely,if Γ is a connected reduced curve with $\Gamma^2 = -2$,then Γ is fixed in $|kE + \Gamma|$ iff $\Gamma.E = 0, or\ 1$

Theorem 1.9. *$L \in Pic(S)$ ia ample,then L is one of the following:*

(1) *$L \cong \mathcal{O}_S(C)$ where C is irreducible curve of $p_a(C) > 1$*
(2) *$L \cong \mathcal{O}_S(kE + B)$ where E is an irreducible curve of $p_a(E) = 1$, B ia a rational curve st:$B.E = 1,\ k \geq 3$*

Proof. By ampleness,$L^2 > 0$.

If L base point free,then it's case (1) by lemma1.6.

If not,suppose $L = D + \Delta$ where D ia mobile part and Δ is fixed part.By lemma1.6,either D is irreducible curve or $D = kE$ with E is irreducible curve of $p_a(E) = 1$ and $k \geq 2$. If D is irreducible, then $h^1(\mathcal{O}(D)) = 0$ by lemma1.5,thus

$$h^0(\mathcal{O}(D)) = \frac{1}{2}D^2 + 2 = h^0(L) = \frac{1}{2}L^2 + 2 + h^1(L) \geq \frac{1}{2}L^2 + 2$$

$\Rightarrow\ 2D.\Delta + \Delta^2 \leq 0$. however by Nakai's

$$L.\Delta = D.\Delta + \Delta^2 > 0$$

sice L ia ample and $\Delta > 0$.This cause a contradiction ! So $D = kE$,by remark1.8,$E.\Delta = 1$.By $L.\Delta = k - 2 > 0$,we have $k > 2$.This is case (ii). □

Theorem 1.10. *$L \in Pic(S)$ ia ample,then L^n is generated by sections for $n \geq 2$ and very ample for $n \geq 3$*

Morrison's discussion: Following ideas in Reider's paper4, D.Morrison 2 gave a detail discussion of linear system of K3,We summarise as follows: The key to Rieder's method is Bogolomov property.

Definition 1.11. 1.A rank=2 vector bundle E has Strong Bogomolov property (SBP)if $\exists\ M, N \in Pic(X)$ and a 0-cycle $A \in A^2(X)$ st:

$$0 \to M \to E \to \mathcal{I}_A \oslash N \to 0 \tag{1.3}$$

$$h^0((M \otimes N^{-1})^{\otimes k}) > 0, k > 0 \tag{1.4}$$

2..A rank=2 vector bundle E has weak Bogomolov property (WBP) if $L := c_1(E)$ is nef and $\exists\ M, N \in Pic(X)$ nd a 0-cycle $A \in A^2(X)$ st:

$$0 \to M \to E \to \mathcal{I}_A \oslash N \to 0 \tag{1.5}$$

$$L.M \otimes N^{-1} \geq 0 \tag{1.6}$$

Remark:clearly,$h^0((M \otimes N^{-1})^{\otimes k}) > 0 \Rightarrow$
$\exists kD \in |M \otimes N^{-1})^{\otimes k}|$ st:$L.kD \geq 0 \Rightarrow$
$L.M \otimes N^{-1} = L.D \geq 0$.so SBP implies WBP if L nef.

given $Z \in A^2(X)$ cycle of codim =2 and $L \in Pic(X)$,we can associate a rank 2 vector bundle E by a section

$$s \in Ext^1(\mathcal{O}_X, \mathcal{I}_Z \oslash L) = H^1(X, \mathcal{I}_Z \oslash L)$$

ie,

$$0 \to^{\times s} \mathcal{O}_X \to E \to \mathcal{I}_Z \oslash L \to 0$$

Thus,

$$c_2(E) = deg(Z)$$

Lemma 1.12. *As above,if $L^2 > 0$ and E has WBP w.r.t*

$$0 \to M \to E \to \mathcal{I}_A \oslash N \to 0$$

,then $\exists\ D \geq 0 \in Pic(X)$ st:

$$Z \in D,\ \ N = \mathcal{O}_X(D)$$

and $D = 0$ implies $Z = \phi$

Proof. observe

$$\begin{tikzcd} \mathcal{O}_X \arrow[d, "\times s"'] \arrow[dr, "\mu \circ s"] & \\ E \arrow[r, "\mu"'] & \mathcal{I}_A \oslash N \end{tikzcd}$$

$\Rightarrow \mu \circ s \in Hom_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}_A \oslash N) = H^0(X, \mathcal{I}_A \oslash N)$

claim:$\mu \circ s \neq 0$.Then we set $D := \{\mu \circ s = 0\}$ If $\mu \circ s \neq 0$,then $im(s) \subset \ker(\mu) \Rightarrow im(s) \subset M$,$M \simeq \mathcal{O}_X \Rightarrow$ By WBP,

$$L = c_1(E) = c_1(M) + c_1(N) = c_1(N),\ L.M \otimes N^{-1} = L.N^{-1} = -L^2 \geq 0$$

,contradiction. Note that $Z = \{s = 0\}$,thus $Z \in D$ $\square$

Theorem 1.13. *Let $L \in Pic(S)$ be nef line bundle,then*

(i) *If* $p \in S$ *is a base point of* L *and* $L^2 = 2g-2 > 0$,*then*$\exists D \geq 0 \in Div(S)$ *containing* p *st:*

$$L.D = 1, D^2 = 0$$

(ii) *If* $p, q \in S$ *are not base points of* L *and not separated by*L, $L^2 \geq 4$, *then* $\exists D \geq 0 \in Div(S)$ *containing* p, q *such that only one of the following is possible:*
  (a) $L.D = 0, D^2 = -2$
  (b) $L.D = 1$ *or* $2, D^2 = 0$
  (c) $L.D = 4, D^2 = 2, L^2 = 8, L \approx 2D$

Proof. 1.proof of (i): let $F := L - gD$,then $F^2 = -2$,thus

$$h^0(F) + h^0(-F) = h^1(F) + 1 \geq 1,\ L.F = (2g-2) - g = g - 2$$

thus,F is effective.By

$$0 \to \mathcal{O}_S((n-1)D) \to \mathcal{O}_S(D) \to \mathcal{O}_D(nD) \to 0$$

,then we have

$$h^0(gD) \geq h^0(D) + g - 1 \geq g + 1$$

while $h^0(gD) \leq h^0(L) = g + 1$,thua $|L| = |gD|$

2.proof of (ii): It relies on Bogomolov's weak property, which is the key point of Reider's method.

□

Conversely,We have:

Theorem 1.14. *Let* $L \in Pic(S)$ *be nef and big line bundle with* $L^2 = 2g - 2 > 0$.

(i) *If* $\exists D \geq 0 \in Div(S)$ *containing* p *st:*

$$L.D = 1, D^2 = 0$$

then $|L|$ *has a fixed component* $L - gD$

(ii) *If* $L^2 > 10$ *and* φ_L *is not birational,then* $\exists$ *a pencil* $\{D_t\}$ *st:*

$$D_t^2 = 0, L.D_t = 1, or\ 2$$

(iii) *If* $|L|$ *base point free and* $\exists D \geq 0 \in Div(S)$ *st:*$L.D = 2, D^2 = 0$,*then the morphism has degree* 2 *and each smooth curve* $C \in |L|$ *is hyperelliptic.*

Remark: If L is nef,we can use

2. Classification of the projective model

Throughout the section,L will be always big and base point free,$L^2 = 2l = 2g - 2 > 0$.

2.1. **Hyperelliptic.**

Theorem 2.1. *If $L^2 \geq 4$, then $|L|$ is hyperelliptic only when*
(i) $\exists E$ *irreducible curve st:*$E^2 = 0,\ E.L = 2$
(2) $\exists B$ *irreducible curve st:*$B^2 = 2,\ L \simeq \mathcal{O}_S(2B)$

Proof. The proof follows lemma2.2.
By lemma 1.6,take $C \in |L|$ irreducible. By lemma 1.7,for each irreducible curve of $p_a(E) = 1$,

$$L.E = C.E \geq 2$$

If there is a such curve st:$L.E = C.E = 2$, this is the case(i).
Otherwise,

$$C.E \geq 3$$

for each irreducible curve of $p_a(E) = 1$,this satisfy conditions in lemma2.2.since C is hyperelliptic and $\varphi_L = \varphi_C$ is a degree 2 map,we can choose carefully a open subset $U \subset S - \mathcal{E}_C$ st:$x \neq y \in U$ with property

$$\varphi_C(x) = \varphi_C(y),\ \varphi_C(U)$$

where $\mathcal{E}_C$ is union of curves Δ st:$C.\Delta$.
Take blowups $\pi : \widetilde{S} \to S$ centered at x, y with E_x, E_y as exceptional divisors. consider

$$\Gamma \in |\pi^*(\mathcal{O}_S(C)) \otimes \mathcal{O}_{\widetilde{S}}(-2E_x) \otimes \mathcal{O}_{\widetilde{S}}(-2E_y)|$$

Then Γ is not 1-conncted.If not,by lemma1.4,

$$H^1(S, I_x \otimes I_y \otimes \mathcal{O}_S(C)) = 0$$

,which contradicts to $\varphi_C(x) = \varphi_C(y)$.
$\pi^*(D) = \Gamma + 2E_x + 2E_y$,suppose $\Gamma = \Gamma_1 + \Gamma_2$ with $\Gamma_1, \Gamma_2 > 0$. set $\Delta_i = \Gamma_i + (\Gamma_i.E_x)E_x + (\Gamma_i.E_y)E_y,\ i = 1, 2$
then $\Delta_i \geq 0$ so we can choose $D_i \in Pic(S)$ st: $\pi^*(D_i) = \Delta_i$ so

$$D = D_1 + D_2$$

$$D_1.D_2 = \Delta_1.\Delta_2 = \Gamma_1.\Gamma_2 + (\Gamma_1.E_x)(\Gamma_2.E_x) + (\Gamma_1.E_y)(\Gamma_2.E_y)$$
$$\Gamma_1.E_x + \Gamma_2.E_x = \Gamma.E_x = (C - 2E_x - 2E_y).E_x = 2$$
$$\Gamma_1.E_y + \Gamma_2.E_y = \Gamma.E_x = (C - 2E_x - 2E_y).E_y = 2$$
$$\Gamma_1.E_x,\ \Gamma_2.E_x,\ \Gamma_1.E_y,\ \Gamma_2.E_y \geq 0$$

We know Γ is not 1-connected and thus can choose $\Gamma_1, \Gamma_2 > 0$ st: $\Gamma_1.\Gamma_2 = 0$, thus

$$D_1.D_2 = (\Gamma_1.E_x)(\Gamma_2.E_x) + (\Gamma_1.E_y)(\Gamma_2.E_y) \leq 2$$

By lemma1.4,above inequality is equality and only= iff

$$\Gamma_1.E_x = \Gamma_2.E_x = \Gamma_1.E_y = \Gamma_2.E_y = 1$$

Now apply lemma2.2,we have

$$D_1 \sim D_2 \sim B$$

where B is an irreducible curve with $p_a(B) = 2$ since $x, y \in U - \mathcal{E}_C$ will imply

$$D.D_1 = C.D_1 > 0$$

which excludes case (i) in lemma2.2. □

Lemma 2.2. *$C \subset S$ irreducible curve with $C^2 \geq 4$ and $E.C \geq 3$ for each irreducible curve of $p_a(E) = 1$.If $D = D_1 + D_2$ with $D \in |C|,\ D_1, D_2 \geq 0$,then*

$$D_1.D_2 \geq 3$$

except:

(i) *$D_1.D_2 = 2$ and $D_1^2 = -2$*
(ii *$D_1 \sim D_2 \sim B$ where B is an irreducible curve with $p_a(B) = 2$*

Explicit description

• (1) in Theorem2.1 $\varphi_L(S)$ is a rational normal scroll of degree g-1 in $\mathbb{P}^g$
• (2) in Theorem2.1

$$\begin{tikzcd} S \arrow[r, "\varphi_L"] \arrow[d, "\varphi_B"'] & \mathbb{P}^5 \\ \mathbb{P}^2 \arrow[ur, "\nu"'] & \end{tikzcd}$$

where $\nu : \mathbb{P}^2 \to \mathbb{P}^5$ is veronese embedding.

2.2. **Nonhyperelliptic.**

Theorem 2.3. *Let $L \in Pic(S)$ be an invertible sheaf without fixed component and $\varphi_L$is birational morphism.then*

(i) *generic curve $C \in |L|$ is nonsingular*
(ii) *the natura map*

$$Sym(H^0(S, L)) \to \bigoplus_{n \geq 0} H^0(S, L^n)$$

is surjective

(iii) *Let $\overline{S} = \varphi_L(S)_{red}$,then $\varphi_L : S \to \overline{S}$ is a map contracting (-2)curve to rational double points.*

Proof. we only give proof of (ii). the proof based on induction of n and a theorem M.Noether(see[ACGH]1). (M.Noether) If C is nonsingular curve and not hyperelliptic,then for any integer $k > 0$,the natural map

$$Sym^k(H^0(C, \omega_C) \to H^0(C, \omega_C^k)$$

is surjective
n=1,it's trivial.
Now suppose $H^0(S, L^k)$ is generated by element in $(H^0(S, L))$ for $k < n$.

choose generic curve $C \in |L|$,it's smooth by(1).By taking tensor L^{n+1} to $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$,we have

$$0 \to L^n \to L^{n+1} \to \omega_C^{n+1} \to 0$$

where last term follows from adjuction formula $K_S + C|_C = C|_C = K_C$ $\Rightarrow$

$$0 \xrightarrow{\mu} H^0(S, L^n) \xrightarrow{\nu} H^0(S, L^{n+1}) \to H^0(C, \omega_C^{n+1}) \to H^1(S, L^n) = 0$$

Suppose $H^0(S, L) = span\{s_0, ...s_g\}$.take $f \in H^0(S, L^{n+1})$,when restriction on C,by theorem of M.Noether and $H^0(S, L) \to H^0(C, \omega_C) \to 0$, $\exists$ a polynomial P of degree $n+1$ st:

$$f|_C = P(s_0, ...s_g)|_C$$

then

$$f - P(s_0, ...s_g) ker(\nu) \cong im(\mu)$$

by the above exact sequence. By induction hypothesis,

$$f - P(s_0, ...s_g) = s \cdot Q((s_0, ...s_g)$$

where Q is a polynomial of degree n and $s \in H^0(S, L)$.This completes proof. □

Remark: It's interesting to know the kernel

$$I := ker(Sym(H^0(S, L)) \to \bigoplus H^0(S, L^n))$$

The following result give an answer.

Theorem 2.4. *As in theorem2.2,If $L^2 \geq 8$, then I is generated by elements of degree 2 and 3. I is generated by degree 2 elements except in following cases:*

(1) *$\exists E$ irreducible curve st:*

$$E^2 = 0, \ E.L = 3$$

(2) *$L \cong \mathcal{O}_S(2B + \Gamma)$ where B is an irreducible curve of $g(B) = 2$ and B is an irreducible rational curve with $B.\Gamma = 1$*

Proof. □

Following 5,we define the Noether-Lefchetz diviso for moduli space K_{2l} as:

$$NL_{h,d} := \{(S, L) \in K_{2l} : \exists \ curve \ C \ st : L.C = d, \ C^2 = 2h - 2\}$$

We summary the results above in view of moduli space

Theorem 2.5.

3. Examples in low degree

For polarization (S,L), we call $L^2 = 2l$ degree. The section is devoted to give some examples of projective model of low degree.These examples are mainly taken from D.Morrison's lecture note2.

Example 3.1. Let $\pi : X \to \mathbb{P}^2$ be a double cover branched along a smooth curve $C \subset \mathbb{P}^2$of degree $= 6$,then by covering we have

$$K_X = \pi^*(K_{\mathbb{P}^2} \otimes 3H) = \mathcal{O}_X,\ H^1(\mathcal{O}_X) = 0$$

where H is hyperplane section of $\mathbb{P}^2$. Take

$$L = \pi^*(\mathcal{O}_{\mathbb{P}^2}(l))$$

,Then (X,L) is a degree $= 2l$ polarized K3..By projection formula,

$$H^0(X,L) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(l) \odot \mathcal{O}_{\mathbb{P}^2}(l-3))$$

For $l = 1, 2$,L is base point free and hyperelliptic

For $l = 3$, φ_L gives an embedding.

Apply Similar covering techniques,we can construct double covering

$$\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

branched over a smooth bidegree $(2,2)$ curve C,then

$$K_X = \pi^*(K_{\mathbb{P}^1 \otimes \mathbb{P}^1} \otimes \frac{1}{2}C) = \pi^*\mathcal{O}_X(-2,-2) \times \mathcal{O}_X(\frac{1}{2}(4,4)) = 0$$

Example 3.2. $S := \{f_1 = 0, f_2 = 0\} \subset \mathbb{P}^4$, where $f_1, f_2 \in \mathbb{C}[x_0, ..x_4]$ are homogenous polynomial of degree 2,3 respectively.

By Lefchstz hyperplane section theorem,

$$H^1(\mathcal{O}_X) = 0$$

and $K_X = \mathcal{O}_X(2+3-4-1) = \mathcal{O}_X$, thus X is a K3. Let $L = i^*(\mathbb{P}^4(l)), l > 0$ where $i : X \hookrightarrow \mathbb{P}^4$ is natural inclusion.Then

$$L^2 = l^2 deg(X) = 6l^2$$

(X,L) is a polarised K3 of degree $6l^2$.

Example 3.3. (Kummer surface)

$T := \mathbb{C}^2/\Lambda$ where $\Lambda \subset \mathbb{C}^2$ is a lattice of rank 4 with bilinear form induced by the inclusion naturally.

$$\iota : T \to T,\ via\ z + \Lambda \mapsto -z + \Lambda$$

$\Rightarrow Fix(\iota) = 16$ and $Sing(T/\iota) = Fix(\iota)$. Take minimal resolution

$$X \to T/\iota$$

Prop: X is a K3.

Proof. □

Remark: In general, such K3 is not projective.

Example 3.4. Set

$$X := Gr(2,6) \cap \mathbb{P}^8 \hookrightarrow \mathbb{P}^{14}$$

with plucker embedding

Example 3.5. S.Mukai's examples:

4. Cone structures of K3

Definition 4.1.

- The Kahler cone $\mathcal{K}(X) := \{\omega \in H^2(X,\mathbb{R}) : \omega\ is\ Kahler\ form\}$
- The ample cone $Amp(X) := \{\}$
- The effective cone $Eff(X) := \{\alpha \in H^2(X,\mathbb{R}) : \alpha = \sum a_i[C_i],\ C_i \subset X\ Curves\ \}$

Theorem 4.2 (Kovács)**.** *The ample cone of a K3 surface*

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