# COHOMOLOGY OF MODULI SPACE OF CUBIC FOURFOLDS I

FEI SI

Abstract. In this paper we calculate the cohomology of moduli space of cubic fourfolds with ADE type singularities relying on Kirwan's blowup and Radu. Laza's GIT descriptions.

Contents



## 1. Introduction

The study of cubic fourfolds is a classical topic in algebraic geometry and have attracted lots of attention in various aspects. In this paper we deal with the topology of the moduli space $\mathcal{M}$ of cubic fourfolds with ADE singularities at worst.

The works of C. Voisin [31] [32], B. Hassett[9], R. Laza [19] and E. Looijenga [23] establish the global Torelli theorem for the cubic fourfolds completely, including the image of period maps. The theorem says the moduli space of cubic fourfolds with ADE singularities is the complement $\mathcal{D}/\Gamma - \mathcal{H}_\infty$ of a Heegner divisor $\mathcal{H}_\infty$ in the shimura variety $\mathcal{D}/\Gamma$ (see 5.2). This provides compactification of $\mathcal{M}$ from arithmetic side, eg, Baily-Borel, Lojiegena, Toriodal... and also shows $\mathcal{M}$ is quasi-projective.

The cohomology of shimura variety is a central topic deeply related to representation theory (see [5] ). Usually, it is very hard to compute its cohomology.

But in our case, the Torelli theorem provides a birational map $p : \overline{\mathcal{M}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$ between geometric invariant theory (GIT) moduli space $\overline{\mathcal{M}}$ of cubic fourfolds and the Baly-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ of shimura variety $\mathcal{D}/\Gamma$. Moreover, the birational map is explicitly resolved via Kirwan's partial desingularization (see [19]),

$$\begin{tikzcd} & \hat{\mathcal{M}} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}} \arrow[rr, dashed, "p"] & & \overline{\mathcal{D}/\Gamma}^{BB} \end{tikzcd}$$

For the GIT quotient space, F. C. Kirwan developed a systematic methods to compute its cohomology, see [17], [16] and section 2 for brief introduction. Kirwan's method is inspired by works of Atiyah and Bott [1] on topology of quotient space of connections space by gauge group, which is a infinite dimenional version of GIT. Also, Kirwan studied how the cohomology changed after the blowups. This provides a strategy to attach the cohomology: first, one can obtain cohomology of GIT quotient space; then by keeping track of changes of cohomology for each step of Kirwan's partial desingularization and each step we are able to compute the cohomology

This strategy has been worked out for moduli space of degree 2 K3 surface (see [13], [14]), moduli space of cubic threefold (see [6]). Even the boundary stratas of GIT moduli space of cubic fourfolds is much complicated than the above two case, the good news is that the GIT moduli space of degree 2 K3 surface will appear as an exceptional divisor in Kirwan's desingularization, then Kirwan-Lee's some computations in [14] help a lot.

For a topological space $Y$, we denote

$$P_t(Y) := \sum b_i(X)t^i, \quad IP_t(Y) := \sum \dim IH^i(X)t^i$$

the Poincare polynomial of singular cohomology and intersection cohomology ( with respect to middle perversity).

Let $\widetilde{\mathcal{M}}$ be the partial desingularization of $\overline{\mathcal{M}}$. The first main results in our paper is

**Theorem 1.1.** *The Poincare polynomial of $\widetilde{\mathcal{M}}$ is given by*

$$\begin{aligned} P_t(\widetilde{\mathcal{M}}) =& 1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} + 193t^{14} \\ &+ 236t^{16} + 280t^{18} + 324t^{20} + 280t^{22} + 236t^{24} + 193t^{26} \\ &+ 152t^{28} + 115t^{30} + 81t^{32} + 51t^{34} + 26t^{36} + 9t^{38} + t^{40} \end{aligned} \tag{1.1}$$

Using resolution of period maps, we also compute intersection cohomology of Baily-Borel compactification.

**Theorem 1.2.** *The intersection cohomology poincare polynomial of* $\overline{\mathcal{D}/\Gamma}^{BB}$ *is given by*

$$\begin{aligned}\mathrm{IP}_t(\overline{\mathcal{D}/\Gamma}^{BB}) = 1 + 2t^2 + 4t^4 + 9t^6 + 16t^8 + 26t^{10} + 38t^{12} \\ +50t^{14} + 65t^{16} + 82t^{18} + 112t^{20} + 82t^{22} + 65t^{24} + 50t^{26} \\ +38t^{28} + 26t^{30} + 16t^{32} + 9t^{34} + 4t^{36} + 2t^{38} + t^{40}\end{aligned} \tag{1.2}$$

It is an interesting topic to study generators of the intersection cohomology in each degree and ask whether these generators are generated by special cycles (for example, see [29]). Actually, this is one of our motivation to calculate the cohomology. Our results provide the starting point. Parallel to the moduli space of quasi-polarised K3 surface of fixed degree (see [4], [27] [3]), one can define the tautological ring and study its relation to the cohomology ring, the results in our paper also provide information in this direction.

Recently, K-moduli space is a very active subject (see [33] [30] for a survey). By the recent results of Liu [20], the GIT moduli space is isomorphic to the K-moduli space of cubic 4-folds, that is, the space of isomorphic classes of cubic folds admitting Kahler-Einstein metrics. So our computations also provide cohomological results on the K-moduli space.

**Remark 1.3.** we are most interested in cohomology of the open part $\mathcal{D}/\Gamma$ and the complement of Heegener divisor $\mathcal{D}/\Gamma - \mathcal{H}_\infty$. we plan to investigate the problem in the forthcoming paper.

**Outline.** The paper is organised as follows: In section 2, we review the basic properties of equivariant cohomology and intersection cohomology theory, and introduce Kirwan's partial desingularization package. section 3, we describe the GIT stratas of moduli space of cubic 4-folds and discuss their geometries relying on the work of R. Laza. In section 4, we use Kirwan's methods to compute the cohomology of the partial resolution $\widetilde{\mathcal{M}}$. In section 5, we use the decomposition theorems to compute the intersection cohomology of Baily-Borel compatification $\overline{\mathcal{D}/\Gamma}^{BB}$ of moduli space of cubic 4-folds.

**Conventions.**

(1) $\mathcal{M}$ moduli space of cubic fourfolds with ADE singularities;
(2) $\overline{\mathcal{M}}$ GIT comaptification of $\mathcal{M}$;
(3) $\widetilde{\mathcal{M}}$ Kirwan's desingularization space;
(4) $\mathbb{C}[x_1, x_2, ..., x_n]_d$ means degree $d$ homogenous polynomials in $n+1$ variables;
(5) $\alpha, \mu, \gamma, \delta, \cdots$ strata of boundaries;
(6) $Z_\alpha, Z_\mu, Z_\gamma \cdots$ parametrizing space of strata $\alpha, \mu, \gamma, \delta, \cdots$;
(7) $E_\alpha, E_\mu, E_\gamma \cdots$ the exceptional divisor of Kirwan blowups;
(8) $N(R)$ normalier subgroup of a subgroup $R$ in group $G$;
(9) stab($\beta$) the stabilier subgroup a vector $\beta$ in Lie algebra by adjoint action;
(10) All cohomology theory $H^*$, $\mathrm{IH}^*, \cdots$ will take $\mathbb{Q}$-coefficients.

## 2. Basic tools

2.1. **Equivariant Cohomology.** For a topological space $X$ with a group action $G$, its equivariant cohomology (here we use singular cohomology) measures the group action. The $i$-th equivariant cohomology $H^i_G(X,\mathbb{Q})$ is defined by the ordinary cohomology $H^i(EG\times_G X,\mathbb{Q})$ where $EG\to BG$ is the universal $G$-space. We list the properties we need below and one may refer to [10] for more details on equivariant cohomology theory.

**Theorem 2.1.** *Let $G$ be a group acting on a complex variety $X$.*
*(1) If the quotient $X/G$ has only quotient singularities, then*

$$H^i_G(X,\mathbb{Q}) = H^i(X/G,\mathbb{Q})$$

*(2) If the quotient space $X/G$ is contractible, then*

$$H^i_G(X) = H^i(BG)$$

2.2. **Intersection cohomology.** We refer to [15] for definitions and details on Intersection cohomology. We only state the blowup formula

**Theorem 2.2** (Kirwan)**.** *Let $X$ be a smooth variety with action $G$ and $Z$ is a smooth $G$-subvariety with reductive stablizer subgroup $R$. Let $\widetilde{X} := Bl_Z(X)\to X$ be the blow up of $X$ along $Z$, then*

$$\begin{aligned} \dim IH^i(X/G) &= \dim \mathrm{IH}^i(\widetilde{X}/G) \\ &- \sum_{p+q=i} \dim(H^p(Z/N_0))\otimes \mathrm{IH}^{\lambda(q)}(\mathbb{P}/R))^{\pi_0(N)} \end{aligned} \tag{2.1}$$

*where* $\lambda(q) = \begin{cases} q-2 & \textit{if } q\le \dim \mathbb{P}/R, \\ q & \textit{if others.}\end{cases}$ *and $\mathbb{P}$ is the projection of a normal vector of any point in $Z$.*

2.3. **Kirwan's desingularization package.** Assume $X$ is a smooth projective variety over $\mathbb{C}$ with a reductive group $G$ action on $X$. Denote $Z_R$ the locus where the stabilizer group is $R$ (unique up to a conjugate). Suppose there are only finitely many such locus

$$\{\ Z_{R_1},...,Z_{R_r}:\ \dim R_1\ge ...\ge \dim R_r\ \}$$

and all the stabilizer groups $R_i$ are reductive subgroup of $G$ and all $Z_R$ are smooth, then Kirwan took successive blowups along these locus (see [16],[11])

$$\widetilde{X} = Bl_{\widetilde{Z_{R_r}}} \to \cdots \to Bl_{Z_{R_1}}X \to X$$

where $\widetilde{Z_{R_r}}$ is the strict transformations of $Z_{R_r}$ and showed the $G$-action can be lifted to $\widetilde{X}$. Moreover, it commutes with the GIT quotient

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \arrow[d] \\ \widetilde{X}/\!\!/G \arrow[r] & X/\!\!/G \end{tikzcd} \tag{2.2}$$

In this way, after finite steps, Kirwan obtained a partial resolution $\widetilde{X}/\!\!/G$ of $X/\!\!/G$, which has only quotient singularities at worst.

To study the cohomology of $\widetilde{X}/\!\!/G$, Kirwan developed several useful cohomological formulas:

(1) cohomology formula for GIT quotient:

For a reductive group $G$ acting on a smooth complex variety $X$ with $G$ linearlised polarization $L$ (even a symplectic manifold), we can choose a $G$-equivalent embedding $X \hookrightarrow \mathbb{P}^N$ via $L$. Let $T \subset G$ be a maximal torus and $\mathfrak{t}$ be its Lie algebra, fix a positive Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$, then define the index set $\mathcal{B}_0$ consists of $\beta \in \mathfrak{t}_+$ such that $\beta$ is the closest point to the origin 0 of the nonempty convex hull $con(\alpha_1, ..., \alpha_m)$ generated by some weights $\alpha_1, ..., \alpha_m$. Fix a a norm $||$ on $\mathfrak{t}$ (eg, the one induced by Killing form), set

$$Z_\beta = \{[x_0, \cdots, x_N] \in X : \ x_i = 0, if \ \alpha_i.\beta \neq |\beta|^2\}$$

$$Y_\beta = \{[x_0, \cdots, x_N] \in X : \ x_j = 0, if \ \alpha_j.\beta = |\beta|^2 \ \& \ \exists \ \alpha_i.\beta \neq |\beta|^2\}$$

then there is a natural retraction map

$$p_\beta : \ Y_\beta \to Z_\beta$$

Denote $X^{ss}$ the semistable locus of $X$ in the sense of Mumford's GIT and $Z_\beta^{ss}$ the locus of semistable points in $Z_\beta$ and let

$$Y_\beta^{ss} := p_\beta^{-1}(Z_\beta^{ss}), \quad S_\beta := G \cdot Y_\beta^{ss}$$

then combing the theory of moment maps and relation of symplectic reduction and geometric invariant theory, it is shown in [17] that $\{S_\beta\}_{\beta \in \mathcal{B}_0}$ gives $X$ a $G$-equivarant perfect Morse stratification . In particular, for $\beta = 0$, $S_0 = X^{ss}$. Using such stratification, Kirwan obtained the formula for Poncare's polynomials,

$$P_t^G(X^{ss}) = P_t(X)P_t(BG) - \sum_{0 \neq \beta \in \mathcal{B}_0} t^{2\,\mathrm{codim}(S_\beta)} P_t^{stab(\beta)}(Z_\beta^{ss}) \tag{2.3}$$

Moreover, there is a natural identification

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$$

where $P_\beta \leq G$ is the parabolic subgroup associated to $\beta$.

In this way, we have also have a dimension formula

$$\dim S_\beta = \dim G + \dim Y_\beta^{ss} - \dim P_\beta \tag{2.4}$$

(2) cohomology formula for blowups:

Assume $X$ a smooth projective variety with action $G$ and $R$ is a reductive subgroup of $G$ w.r.t the locus $Z_R$. If we take a blow up

$$\pi : \ \widetilde{X} \to X^{ss}$$

along the smooth center $G \cdot Z_R^{ss}$. Let $N(R)$ be the normalizer subgroup of $R$ in $G$ and $d_R$ be the complex codimension of $Z_R$ in $X$. then Kirwan's blowup formula [13] gives

$$\begin{aligned} P_t^G(\widetilde{X}^{ss}) = P_t^G(X^{ss}) + (t^2 + ... + t^{2d_R}) P_t^{N(R)}(Z_R^{ss}) \\ - \sum_{\beta \in \mathcal{B}_{0,\rho}} t^{2\,\mathrm{codim}(S_\beta)} P_t^{stab(\beta) \cap N(R)}(Z_{\beta,\rho}^{ss}) \end{aligned} \tag{2.5}$$

Here $\mathcal{B}_{0,\rho}$ is the index set obtained as in 1 with respect to the normal representation

$$\rho : R \to \mathrm{Aut}(\mathbb{P}\mathcal{N}),$$

where $\mathcal{N}$ is the normal vector space of a point in $Z_R^{ss}$ and $Z_{\beta,\rho}^{ss} \subset \mathbb{P}\mathcal{N}$ is the associated semi-stable strata given by weight $\beta$.

Suppose there are series of blowups

$$\cdots \to X_2 \to X_1 \to X_0 = X.$$

We write the correction terms contributed in the $i$-th blowup as follows:

$$A_i(t) := (t^2 + ... + t^{2d_R})P_t^{N(R_\omega)}(Z_R^{ss}) \\ - \sum_{\beta \in \mathcal{B}_{0,\rho}} t^{2\,\mathrm{codim}(S_\beta)} P_t^{\mathrm{stab}(\beta)\cap N(R_\omega)}(Z_{\beta,\rho}^{ss}) \tag{2.6}$$

(3) Equivariant cohomology for fibration :
Let $F \hookrightarrow X \to B$ be a $G$ equivariant topological fibration on $X$ over base space $B$ and $F$ is the fiber, then we have the spectral sequence:

$$H_G^p(B, H^q(F, \mathbb{Q})) \Rightarrow H_G^{p+q}(X, \mathbb{Q}).$$

This implies

$$P_t^G(X) = P_t^G(B) \cdot P_t(F). \tag{2.7}$$

## 3. GIT of cubic fourfold and its Kirwan's dedingularization

Laza's work [18] on the GIT stabilites of cubic fourfolds is important to the computation. In this section we give a summary of main results in [18] and take a further detailed study of geometry of GIT boundaries, which is prepared for computations in the next two sections.

**Proposition 3.1.** *(R. Laza, Prop 2.6 [18])*

*(1) A strictly semi-stable cubic fourfold with minimal orbit have defining equation of the following type:*
- $\alpha$: $x_0q_1(x_2,..,x_5) + x_1q_2(x_2,..,x_5)$
- $\mu$: $ax_0x_4^2 + x_0x_5l_1(x_2,x_3) + bx_1^2x_5 + x_1x_4l_2(x_2,x_3) + c(x_2,x_3)$
- $\gamma$: $x_0q(x_3,..,x_5) + x_1^2l_1(x_3,...,x_5) - 2x_1x_2l_2(x_3,...,x_5) + x_2^2l_3(x_3,...,x_5)$
- $\delta$: $x_0q(x_4,x_5) + f(x_1,x_2,x_3)$

*where l, q, f means linear, quadratic and cubic equation respectively. ie,*

$$\overline{\mathcal{M}} - \mathcal{M}^s = \alpha \cup \mu \cup \gamma \cup \delta.$$

*(2) A stable cubic fourfold with worse than ADE singularities is of type $\epsilon$, $\phi$.*

we still use $\alpha, \delta, \cdots \mu$ to denote the locus in $\overline{\mathcal{M}}$ parametrizing $S$-equivalence classes of strictly semistable cubic fourfolds (in the sense of geometric invariant theory [25] ), which are boundaries strata, ie,

$$\overline{\mathcal{M}} - \mathcal{M} = \alpha \cup \cdots \cup \mu$$

By the analysis of the degeneration, Radu. Laza also gives the incidence relations of the these boundary divisors, see 1

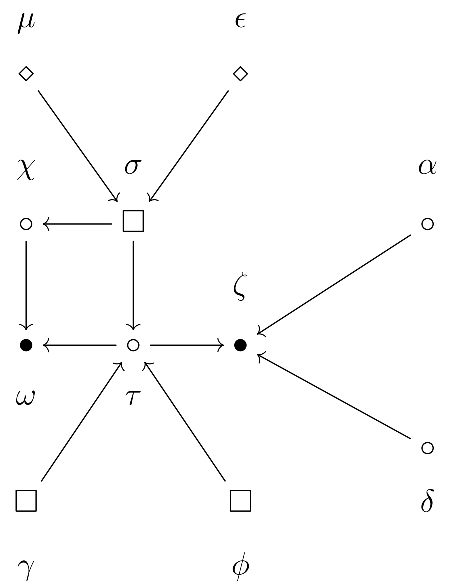


FIGURE 1. The incidence of boundary components of $\mathcal{M}$ in $\overline{\mathcal{M}}$

**Theorem 3.2.** *(R. Laza, [18]) The reductive subgroup of $G = SL(6, \mathbb{C})$ which are stabilizers of general points the GIT boundary strata are one of the following (up to a conjugate)*

- $R_\omega \cong SL(3, \mathbb{C})$
- $R_\zeta \cong (\mathbb{C}^*)^4$
- $R_\chi \cong SL(2, \mathbb{C})$
- $R_\delta \cong SO(2)(\mathbb{C}) \times \{\mathrm{diag}(t^{-2}, t, t, 1, 1, 1) : t \in \mathbb{C}^*\}$
- $R_\tau \cong \{\mathrm{diag}(t^2, t, 1, t^{-1}, t^{-2}, 1) : t \in \mathbb{C}^*\} \times \{\mathrm{diag}(t^4, t, t, t^{-2}, t^{-2}, t^{-2}) : t \in \mathbb{C}^*\}$
- $R_\alpha \cong \{\mathrm{diag}(t^2, t^2, t^{-1}, t^{-1}, t^{-1}, t^{-1}) : t \in \mathbb{C}^*\}$
- $R_\gamma \cong \{\mathrm{diag}(t^4, t, t, t^{-2}, t^{-2}, t^{-2}) : t \in \mathbb{C}^*\}$
- $R_\mu \cong \{\mathrm{diag}(t^2, t, 1, 1, t^{-1}, t^{-2}) : t \in \mathbb{C}^*\}$

*where $\chi$ is a curve that parametrize cubic fourfold with equations of the form*

$$bx_5^3 + \det \begin{pmatrix} x_0 & x_1 & x_2 + 2ax_5 \\ x_1 & x_2 - ax_5 & x_3 \\ x_2 + 2ax_5 & x_3 & x_4 \end{pmatrix} = 0.$$

*here $a, b \in \mathbb{C}$. $\tau$ is a curve that parametrize cubic fourfold with equation of the form*

$$\det \begin{pmatrix} x_0 & x_1 & ax_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0$$

*and $\zeta$ is the point representing $x_0x_4x_5 + x_1x_2x_3$, $\omega$ is point representing*

$$-\det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0$$

*and the inclusion relation is given by the figure 3.*

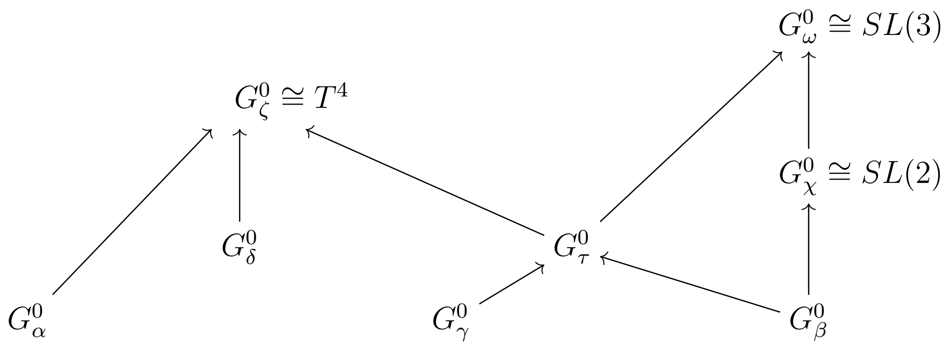


FIGURE 2. The stabilizers of general points in the boundary strata

**Proposition 3.3.** *The GIT boundaries $\overline{\mathcal{M}} - \mathcal{M}$ are the following stratas :*
*(1) 1-dimensional strata:* $\alpha \cong \mathbb{P}^1$, $\delta \cong \mathbb{P}^1, \tau \cong \mathbb{P}^1$, $\chi \cong \mathbb{P}^1$.
*(2) 2-dimensional strata:* $\gamma$ *is* $\mathbb{P}^1 \times \mathbb{C}$ , $\phi \cong \mathbb{P}^1 \times \mathbb{P}^1$.
*(3) 3-dimensional strata:* $\mu \cong \mathbb{P}(1,3,6,8)$, $\varepsilon \cong \mathbb{P}^1 \times \mathbb{P}(1,2,3)$.

*Proof.* $\chi \cong \mathbb{P}^1$ is shown in [19].

By lemma 4.5 in [18], we can write the defining equation of $\delta$ as

$$x_0x_4x_5 + f(x_1, x_2, x_3)$$

and it's semi-stable iff the cubic $f(x_1, x_2, x_3)$ has node at worst. By Luna's slice theorem, the GIT quotient is isomorphic to GIT of plane cubics, thus

$$\delta \cong \mathbb{P}|\mathcal{O}_{\mathbb{P}^2}(3)|/\!\!/SL(3) \cong \mathbb{P}^1$$

For each element, say $F = x_0q_1 + x_1q_2$ in $\alpha$, it can be viewed as a pencil in $\mathbb{P}^3$, by [24] , We know

$$\alpha \cong Proj((Sym^4(\mathbb{C}))^{SL_2}) = Proj(\mathbb{C}[g_2, g_3]) \cong \mathbb{P}^1$$

For $\gamma$, note that by lemma 4.6 in [18], a cubic fourfold in locus $\gamma$ has normal form

$$x_0(x_3^2 - x_4x_5) + x_1^2x_4 - 2x_1x_2l(x_3, x_4, x_5) + x_2^2x_5$$

and its stablizer group $\mathbb{C}^* = \{\ diag(1, t^{-1}, t, 1, t^2, t^{-2}) :\ t \in \mathbb{C}^*\ \}$. By Luna's slice theorem, this reduce GIT to $\mathbb{C}^*$ action on $\mathbb{A}^3(a, b, c)$ where $a, b, c$ are coefficients in the normal form of $l(x_3, x_4, x_5) = ax_3 + bx_4 + cx_5$, then

$$\gamma \cong \mathbb{C} \times \mathbb{P}^1$$

For $\beta$, we first use action $GL(2)$ on $x_2, x_3$ to reduce to consider the torus action

$$\mathrm{diag}\{\ (t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}, t^{a_4}, t^{a_5}) :\ a_2 = a_3,\ \sum a_i = 0\ \}$$

on the space

$$\mathbb{P}V = \mathbb{P}^6(y_0, \cdots, y_6)$$

where $V$ is the vector space spanned by the monomials

$$\{x_1x_3x_4,\ x_0x_4^2, x_5x_1^2,\ xx_2^3,\ x_2x_3^2,\ x_3x_2^2, x_3^3\}.$$

then we compute its invariant ring has minimal generator

$$y_0,\ y_1y_2y_5,\ (y_1y_2)^2y_4y_6,\ (y_1y_2)^3y_3y_6$$

then we obtain $\beta \cong \mathbb{P}(1,3,6,8)$.

From [18] we know $\varepsilon$ parameterize the cubic 4-folds singular along an irreducible rational normal curve of degree 4,

By proposition 6.6 in [18], the parameter space for such 4-folds is product of $sym^4(\mathbb{P}^1)$ and affine space $\mathbb{A}(a,b,c)$ with natural action $SL(2)$ on $\mathbb{P}^1$ and $\mathbb{C}^*$ on $x_5$ in the equation above, thus

$$\varepsilon \cong \mathbb{P}^4 /\!\!/ SL(2) \times \mathbb{P}(1,2,3) \cong \mathbb{P}^1 \times \mathbb{P}(1,2,3)$$

□

## 4. Cohomology of partial desingularization $\widetilde{\mathcal{M}}$

In this section, We follow Kirwan's strategy (see [13]) on computation: Note that $\widetilde{\mathcal{M}}$ has quotient singularities at worst and thus the equivariant cohomology equals to the usual cohomology by 2.1. By duality of intersection cohomology, we will only consider the cohomology term $P_t(\widetilde{\mathcal{M}})$ of degree lower than 20. Throughout the section, $X = \mathbb{P}^{55}$ and $G = SL(6,\mathbb{C})$.

### 4.1. Computations for blowups.

4.1.1. *Computation of $P_t^G(X^{ss})$.*

As in section 2, we have the formula

$$P_t^G(X^{ss}) = P_t(X)P_t(BG) - \sum t^{2co\dim(S_\beta)} P_t^{stab(\beta)}(Z_\beta^{ss})$$

where index set $\mathcal{B}_0$ consists of all closest points $\beta$ lying in a positive Weyl chamber $\tau_+$ to origin 0 in convex hull $con(\alpha_1, ..., \alpha_5)$ generated by some weights $\alpha_1, ..., \alpha_5$. Let

$$T := \{diag(e^{\sqrt{-1}\theta_0}, e^{\sqrt{-1}\theta_2}, ..., e^{\sqrt{-1}\theta_5}) : \sum \theta_i = 0\}$$

be maximal torus in $SU(6,\mathbb{C})$ and its lie algebra

$$\mathfrak{t} = \{diag(\sqrt{-1}\theta_0, \sqrt{-1}\theta_1, ..., \sqrt{-1}\theta_5) : \sum \theta_i = 0\}$$

Each 1-parameter subgroup is of the form

$$\lambda(t) = \{diag(t^{r_0}, t^{r_1}, t^{r_2}, t^{r_3}, t^{r_4}, t^{r_5}) : \sum r_i = 0\}$$

We have the weights

$$W = \{(i_1 + i - 3, i_2 + i - 3, i_3 + i - 3, i + i_4 + i_5 - 3, i + i_5 - 3) : 0 \le \sum i_j \le 3\}$$

where $i = i_1 + i_2 + i_3 + i_4 + i_5$. Choosing a positive root system

$$\Phi_+ = \{(1,-1,0,0,0), (0,1,-1,0,0),$$
$$(0,0,1,-1,0), (0,0,0,1,-1), (0,0,0,0,0,3)\}$$

we obtain the positive weyl chamber in $\mathfrak{t}$

$$\mathfrak{t}_+ = (\theta_1, .., \theta_5) : \theta_1 \ge ... \ge \theta_5 \ge 0\}$$

| $\beta$ | Monomials | stable($\beta$) |
|---|---|---|
| $\frac{1}{4}$(1,1,1,1) | $\mathbb{C}[x_1,x_2,x_3,x_4]_3$ | $\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}$ |
| (0.9,0.8,0.7,0.6) | $\{x_1x_2^2, x_1x_3^2, x_1x_2x_4\}$ | $T^4$ |
| (1,0,0,0) | $x_1 \cdot \mathbb{C}[x_2,x_3,x_4]_2 \oplus x_0x_1^2$ | $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & A \end{pmatrix}$ |
| (1,1,$\frac{1}{2}$, $\frac{1}{2}$) | $\{x_1^2, x_2^2, x_1x_2\} \cdot \{x_3, x_4\}$ | $\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}$ |
| (1.08,0.84,0.72,0.36) | $\{x_2^3, x_1x_3^2, x_4x_1^2\}$ | $T^4$ |
| (1,1,1,0) | $\mathbb{C}[x_1,x_2,x_3]_3$ | $SL(3,\mathbb{C}) \times T^2$ |
| (1.2,0.9,0.6,0.3) | $\{x_2^3, x_1x_2x_3, x_4x_1^2\}$ | $T^4$ |
| (1.25,0.75,0.75,0.25) | $\{x_1^2x_4, x_1x_2x_3, x_1x_2^2, x_1x_3^2\}$ | $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$ |

TABLE 1. Unstable stratification.

With the help of computer, we find the only data with *codimension* $< 10$ is the case $\beta = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}) \in \tau_+$. In this case, we have

$$stab(\beta) = \{\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in SL(6,\mathbb{C}) : a \cdot \det A = 1, A \in GL(5,\mathbb{C})\} = \mathbb{C}^* \times SL(5,\mathbb{C})$$

where the first factor acts trivially on $Z_\beta = \mathbb{P}\mathbb{C}[x_1,x_2,x_3,x_4,x_5]_3$, Thus,

$$\begin{aligned} P_t^{stab(\beta)}(Z_\beta^{ss}) =& \frac{1}{(1-t^2)} \cdot P_t^{SL(5,\mathbb{C})}(Z_\beta^{ss}) \\ =& \frac{1}{(1-t^2)} \cdot P_t^{SL(5,\mathbb{C})}((\mathbb{P}\left(\mathbb{C}[x_1,x_2,x_3,x_4,x_5]_3\right)^{ss})) \end{aligned}$$

Using the same methods as in computations of $P_t^G(X^{ss})$, we have the table 4.1.1 below to be used in the formula.

In the table 4.1.1, by choosing suitable 1-parameter subgroup, we find some locus $Z_\beta^{ss} = \phi$ and in this case, $P_t^{stab(\beta)}(Z_\beta^{ss}) = 0$. For example, 1-parameter subgroup

$$\lambda(t) = diag(1, t^3, t^{-1}, t^{-1}, t^{-1})$$

will destabilize fourth data . Case by case check, we have only two nonzero data: 1st and 6th.

For 1st data, $v_1 = \frac{1}{4}(1,1,1,1)$, by the dimension formula 2.4, the codimension of unstable stratum equal to $35 - (24 + 20 - 15) = 6$. By the formula 2.3, we have

$$P_t^{stab(v_1)}(Z_{v_1}^{ss}) = \frac{1}{(1-t^2)} \cdot P_t^{SL(4,\mathbb{C})}(\mathbb{P}\mathbb{C}[x_1,x_2,x_3,x_4]_3) \mod t^{20}$$

The same method give only two unstable data for $SL(4,\mathbb{C}) \curvearrowright \mathbb{P}\mathbb{C}[x_1,x_2,x_3,x_4]_3$:
1. $\beta_1 = (1,1,1)$
we have $codim = 4$ and

$$\begin{aligned} P_t^{stab(\beta_1)}(Z_{\beta_1}^{ss}) &= \frac{1}{(1-t^2)} \cdot P_t^{SL(3,\mathbb{C})}(\mathbb{P}\mathbb{C}[x_1,x_2,x_3]_3) \\ &= \frac{1}{(1-t^2)} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)} \end{aligned} \tag{4.1}$$

2. $\beta_2 = (1,0,0)$
we have $codim = 6$ and

$$P_t^{stab(\beta_2)}(Z_{\beta_2}^{ss}) = \frac{1+t^2+t^4+t^6}{(1-t^2)\cdot(1-t^4)} - \frac{(1+t^2)\cdot t^2}{(1-t^2)^2} = \frac{1}{(1-t^2)} \tag{4.2}$$

Combing these data, we have

$$\begin{aligned} P_t^{stab(v_1)}(Z_{v_1}^{ss}) = \frac{1}{(1-t^2)} \cdot \{ & \frac{1+t^2+...+t^{38}}{(1-t^4)\cdot(1-t^6)\cdot(1-t^8)} \\ & - \frac{t^{12}}{(1-t^2)} - \frac{t^8}{(1-t^2)} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)} \} \end{aligned}$$

For 6th data, $v_2 = (1,1,1,0)$

$$stab(v_2) = (\mathbb{C}^*)^2 \times SL(3,\mathbb{C})$$

then the codimension of removing strata is

$$34 - (24+9-15) = 16 > 10.$$

then

$$\begin{aligned} P_t^{stab(v_2)}(Z_{v_2}^{ss}) &= \frac{1}{(1-t^2)^2} \cdot P_t^{SL(3,\mathbb{C})}(\mathbb{P}\mathbb{C}[x_1,x_2,x_3]_3) \mod t^{20} \\ &= \frac{1}{(1-t^2)^2} \cdot \frac{1+t^2+t^{10}+t^{12}}{(1-t^4)\cdot(1-t^6)} \mod t^{20} \end{aligned}$$

the codimension of unstable stratification of this data is given by

$$\begin{aligned} \operatorname{codim} S_\beta =& 55 - (\dim G + \dim Y_\beta - \dim P_\beta) \\ =& 55 - (35+34-20) = 6. \end{aligned}$$

Here $P_\beta$ is a parabolic subgroup consisting of all up triangle matrix thus has $\dim P_\beta = \frac{(6+1)\cdot 6}{2} - 1 = 20$ and

$$\dim Y_\beta = \#\{\ \alpha \in W : \alpha.\beta \geq \beta.\beta\ \} = 34$$

Thus, all the discussions above put into the formula 2.3 will show

**Proposition 4.1.**

$$\begin{aligned} P_t^G(X^{ss}) &\equiv \frac{1-t^{112}}{\prod\limits_{1\leq i\leq 6}(1-t^{2i})} - t^{12}P_t^{GL(5)}(\mathbb{P}\mathbb{C}[x_0,x_1,x_2,x_3,x_4]_3) \\ &\equiv \frac{1}{\prod\limits_{1\leq i\leq 6}(1-t^{2i})} - \frac{t^{12}}{1-t^2}\frac{1}{\prod\limits_{1\leq i\leq 5}(1-t^{2i})}\ mod\ t^{20} \end{aligned} \tag{4.3}$$

| Blowup locus | stabilizer group( up to finite index) | codimension |
|---|---|---|
| $G\omega$ | $SL(3,\mathbb{C})$ | 27 |
| $G\zeta$ | $T^4$ | 24 |
| $GZ_\chi$ | $SO(2)$ | 21 |
| $GZ_\tau$ | T | 20 |
| $GZ_\delta$ | $\mathbb{C}^* \times \mathbb{C}^*$ | 19 |
| $GZ_\alpha$ | $\mathbb{C}^*$ | 19 |
| $GZ_\gamma$ | $\mathbb{C}^*$ | 18 |
| $GZ_\beta$ | $\mathbb{C}^*$ | 17 |

TABLE 2. List of datas to be blowed up

In the next, we will blowup successively along locus discussed in section 3 3, here we give the list of locus to be blowuped:

4.1.2. *Computation of $P_t^G(X_1^{ss})$.* Take a blow up

$$\pi: \ X_1 \to X^{ss}$$

along $G \cdot Z^{ss}_{R_\omega}$. By dimension counting,

$$d_R + 1 = codim G \cdot Z_R^{ss}$$
$$= 55 - (dim G - dim N(R_\omega)) = 55 - (35 - 8) = 28$$

Moreover, we have

$$P_t^{N(R_\omega)}(Z_R^{ss}) = P_t(BN(R_\omega)) = \frac{1}{(1-t^4)\cdot(1-t^6)}$$

since $Z^{ss}_{R_\omega}$ is just a point.

In [19], by using the fact the cubic 4-fold $\omega$ is the secant variety of verose embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, R. Laza proved

**Theorem 4.2.** *The natural representation of $R_\omega$ on the normal slice $\mathcal{N}_\omega$ is isomorphic to $Sym^6(\mathbb{C}^3)$, where $R_\omega \cong SL(3,\mathbb{C})$ has natural representation $\mathbb{C}^3$. In particular, the exceptional divisor $\mathbb{P}Sym^6(\mathbb{C}^3)/\!\!/R_\omega$ is isomorphic to GIT quotient space of plane sextic curves.*

A very helpful observation is

**Corollary 4.3.** *After 1st blowup, the incidence relations of boundaries on exceptional divisor $\mathbb{P}\mathcal{N}_\omega/\!\!/SL(3,\mathbb{C})$ coincide with that of GIT moduli space of degree 2 K3 surfaces. That is, Let $\chi_1,\ \beta_1,\ \tau_1,\ \beta_1 \subset X_1^{ss}/G$ be the strict transformation of the GIT boundary after the 1st blowup along $G \cdot Z_\omega$, then*

$$\begin{aligned} E^{ss}_\omega/G \cap \chi_1 = pt,\ \ E^{ss}_\omega/G \cap \tau_1 = pt \\ E^{ss}_\omega/G \cap \gamma_1 \cong |\mathcal{O}_{\mathbb{P}^1}(4)|//SL(2) = \mathbb{P}^1 \\ E^{ss}_\omega/G \cap \beta_1 \cong \mathbb{P}^3//\mathbb{C}^* = \mathbb{P}(1,2,3). \end{aligned} \tag{4.4}$$

*Proof.* By [19], the moduli space of plane sextics $|\mathcal{O}_{\mathbb{P}^2}(6)|/\!\!/SL(3)$ is identified with $E_\omega$. The same kirwan's desingularization package for $|\mathcal{O}_{\mathbb{P}^2}(6)|/\!\!/SL(3)$ is done in [17], from the locus to be blownup and the incidence relation 1, we obtain the results. ∎

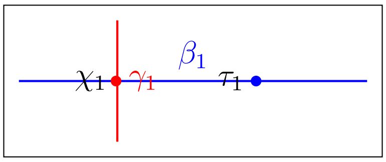


TABLE 3. incidence relation

**Remark 4.4.** This corollary shows in the 2th-8th blowups, the blowup on $E^{ss}_\omega/G \cap \beta_1$ will be the same as that in GIT moduli space of degree 2 K3 surfaces, then Kirwan-Lee's results (see [13] [14]) will help us simplify many computations.

Following the computation in [12] for K3 surface and 2.6, we have

$$\begin{aligned} A_1 =& \frac{(t^2 - t^{56})}{(1-t^2)\cdot(1-t^4)\cdot(1-t^6)} \\ & - \left(\frac{t^{50}-t^{56}}{(1-t^2)\cdot(1-t^4)(1-t^6)} + \frac{t^{20}-t^{28}}{(1-t^2)^3}\right) \\ \equiv & \frac{t^2}{(1-t^2)\cdot(1-t^4)\cdot(1-t^6)} - \frac{t^{20}}{(1-t^2)^3} \quad \mod \ t^{20} \end{aligned} \tag{4.5}$$

4.1.3. *Computation of $P^G_t(X^{ss}_2)$.* Thanks to the disjointness of orbit $G\omega$ and $G\zeta$, the second blowup do not need to consider the effect of first blowup. So we take the second blowup

$$\pi:\ X_2 \to X^{ss}_1 \tag{4.6}$$

along $G\cdot Z^{ss}_{R_\zeta}$. It's easy to see the normalization group of $R_\zeta$ in $G = SL(6,\mathbb{C})$ is given by the extension

$$0 \to T^5 \to N(R_\zeta) \to S_6 \to 0.$$

here the symmetric group $S_6$ is the Weyl group of $R_\zeta$. Then

$$d_{R_\zeta} + 1 = codim(G\cdot Z^{ss}_{R_\zeta})$$

$$= 55 - (\dim G - \dim N(R_\zeta)) = 25$$

This gives

$$\begin{aligned} &(t^2 + \ldots + t^{2d_R})P^{N(R_\zeta)}_t(Z^{ss}_{R_\zeta}) \\ =& (t^2 + \ldots + t^{48})P_t(BN(R_\zeta)) \\ =& \frac{(t^2 - t^{48})}{(1-t^2)\ldots(1-t^{12})} \end{aligned} \tag{4.7}$$

since $G\cdot Z^{ss}_{R_\zeta} = G\times_{N(R_\zeta)} Z^{ss}_{R_\zeta}$ and $Z^{ss}_{R_\zeta}$ is just a point.

Actually, following a lemma of section 4.2 in [13], we can compute the normal space $\mathcal{N}_\zeta$ at $\zeta$

$$\mathcal{N}_\zeta = \mathbb{C}x_0^3 \oplus \ldots \oplus \mathbb{C}x_5^3 \oplus \{x_0^2, x_4^2, x_5^2\}\mathbb{C}[x_1,x_2,x_3]_1$$

$$\oplus\{x_1^2,x_2^2,x_3^2\}\mathbb{C}[x_0,x_4,x_5]_1 \oplus \mathbb{C}\{x_0x_4x_5 + x_1x_2x_3\}$$

So the intersection of exceptional divisor $E_2/N(R)$ with proper transformation of $\alpha,\delta,\tau$ with 3 distinct points.

Note that $R_\zeta = \{diag(a,b,c,d,c^{-1}d^{-1},a^{-1}b^{-1}) : a,b,c,d \in \mathbb{C}^*\}$ acts trivially on $x_0x_4x_5 + x_1x_2x_3$. By Kirwan, the unstable data is identified with the unstable data of natural action $R_\zeta \curvearrowright \mathbb{P}\mathcal{N}_\zeta$. So we only consider this action. Each 1-parameter subgroup can be written as $diag = \{t^{a_0}, ..., t^{a_5}\}$ and the weight is of the form

$$W = \{\ a \cdot I :\ x^I \in \mathcal{N}_\zeta\ \}$$

where $x^I = x_0^{i_0} \cdots x_5^{i_5}$ with $i_0 + \cdots + i_5 = 3$ and $a \cdot I$.

Note that w.r.t formula 2.6, the codimension is

$$codim = 24 - \#\{\ a \cdot I \in W :\ a \cdot I > 0\} \geq 11 \tag{4.8}$$

Thus, we obtain

$$A_2 \equiv \frac{t^2}{(1-t^2)...(1-t^{12})} mod\ t^{20} \tag{4.9}$$

4.1.4. *Computation of $P_t^G(X_3^{ss})$.* Take the third blowups

$$\pi :\ X_3 \to X_2^{ss}$$

along $G \cdot \widehat{Z}_{R_\chi}^{ss}$ where $\widehat{Z}_{R_\chi}^{ss}$ is the strict transform of $Z_{R_\chi}^{ss}$ under composition of previous blowups, since $\chi$ contains point $\omega$ and $\zeta$.

we have

$$codim(G \cdot \widehat{Z}_{R_\chi}^{ss}) = 55 - (\dim G + \dim \widehat{Z}_{R_\chi}^{ss} - \dim N(R_\chi)) = 23$$

By 3.3, we have

$$\widehat{Z}_{R_\chi} /\!\!/ N(R_\chi) \cong \mathbb{P}(1:3) \cong \mathbb{P}^1$$

This gives

$$\begin{aligned}(t^2 + ... + t^{2d_R}) \cdot P_t^{N(R_\chi)}(\widehat{Z}_{R_\chi}^{ss}) & \\ &= \frac{t^2 - t^{46}}{1-t^2} \cdot P_t(BR_\chi) \cdot P_t^{N(R_\chi)/R_\chi}(\widehat{Z}_{R_\chi}^{ss}) \\ &= (\frac{t^2 - t^{46}}{1-t^2}) \cdot \frac{1}{1-t^4} \cdot (1+t^2)\end{aligned} \tag{4.10}$$

since the action $N(R_\chi)$ on $\widehat{Z}_{R_\chi}$ is isomorphic to the action $N(R_\chi)/R_\chi$ on $\widehat{Z}_{R_\chi}$.

In the same paper [19], Laza showed the normal representation $\rho : R_\chi \curvearrowright \mathcal{N}_\chi$ can be identified as

$$SL(2) \curvearrowright H^0(\mathcal{O}_{\mathbb{P}^1}(12)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(8))$$

This gives

$$\sum t^{2co\dim(S_\beta)} P_t^{stab(\beta)\cap N(R)}(Z_{\beta,\rho}^{ss}) =$$

$$\frac{t^{24} + t^{26} + t^{28} + t^{30}(1+t^2) + t^{30} + t^{34} + t^{36} + t^{38} + t^{40}(1+t^2)}{1-t^2}$$

$$= \frac{t^{24} - t^{44}}{(1-t^2)^2}$$

Putting these together, we obtain

$$\begin{aligned} A_3 =&(\frac{t^2-t^{46}}{1-t^2})\cdot\frac{(1+t^2)}{1-t^4}-\frac{t^{24}-t^{44}}{(1-t^2)^2} \\ \equiv&\ \frac{t^2}{(1-t^2)^2}\ mod\ t^{20} \end{aligned} \tag{4.11}$$

4.1.5. *To compute* $P_t^G(X_4^{ss})$. Take 4th blowup $\pi:\ X_4\to X_3^{ss}$ along $G\cdot\widehat{Z}^{ss}_{R_\tau}$, $N(R_\tau)=R_\tau\leq N(R_\omega)=SL(3)$ then

$$codim(G\cdot\widehat{Z}^{ss}_{R_\tau})=55-(dimG+dimZ_{R_\tau}-dimN(R_\tau))=21$$

we can identify the normal representation $R_\tau\curvearrowright\mathcal{N}_\tau\cong R_1\curvearrowright\mathcal{N}_1$ where the normal representation $R_1\curvearrowright\mathcal{N}_1$ second blowup in [13] by corollary 4.3. then from their table 2 in [13], we have

$$\begin{aligned} A_4 =&\frac{t^2-t^{42}}{1-t^2}\cdot P_t(BR_\tau)\cdot(1+t^2)-\frac{t^{18}+t^{20}}{1-t^2}\ mod\ t^{20} \\ \equiv&\frac{t^2(1-t^2)}{(1-t^2)^3}-\frac{t^{18}+t^{20}}{1-t^2}\ mod\ t^{20} \end{aligned} \tag{4.12}$$

where the multiplication term $1+t^2$ is due to the geometry of locus $\widetilde{Z_\tau}^{ss}/(N(R_\tau)/R_\tau)\cong\mathbb{P}^1$ by proposition 3.3.

4.1.6. *To compute* $P_t^G(X_5^{ss})$. We take blowup $\pi:\ X_5\to X_4^{ss}$ along $G\cdot\widehat{Z}^{ss}_{R_\delta}$. Note that here

$$Z_{R_\delta}=\mathbb{P}\{x_0q(x_4,x_5)+c(x_1,...,x_3)\}$$

$$N(R_\delta)=\{diag(a,A,B):a^{-1}=|A|\cdot|B|,A\in GL(3),B\in GL(2)\}$$

Thus,

$$1+d=codimGZ_{R_\delta}=55-(35+12-13)=21$$

since $Z_{R_\delta}$ contains $\zeta=x_0x_4x_5+x_1x_2x_3$, we need to take blow up

$$\widehat{Z}_{R_\delta}\to Z_{R_\delta}$$

along $G_\delta\cdot Z_{R_{\delta\zeta}}$ to compute $P_t^{N(R_\delta)}(\widehat{Z}^{ss}_{R_\delta})$. Note from proposition 3.3, we know the blowup $\widehat{Z}^{ss}_{R_\delta}/N(R)\to Z_{R_\delta}/N(R)=\mathbb{P}^1$ does not change cohomology, ie, $P_t(\widehat{Z}^{ss}_{R_\delta}/N(R))=1+t^2$. Thus

$$A_5=\frac{t^2}{1-t^2}(1+t^2)\frac{1}{(1-t^2)(1-t^4)}-\sum\ mod\ t^{20}$$

where $\sum$ is due to removing unstable strata of representation of $R_\delta$ on the normal vector space of some point in $Z_{R_\delta}$ .

In order to find weight of normal representation, we choose a point $F=x_0x_4x_5+f\in Z_\delta$ distinct to $\zeta$, where $f$ is a generic cubic polynomial in $x_1,x_2,x_3$. For normal representation $R_\delta\curvearrowright\mathcal{N}_F$, we take weight $(2,0,0,0,-1,-1)$ (here we view the weight embedded into Lie algebra of $G$) of the maximal torus of $R_\delta$. By subtracting the weight from $\frac{\partial F}{\partial x_0},...,\frac{\partial F}{\partial x_5}$ and of the form $x_0q+f$ , we have weight of normal space $\mathcal{N}_F$ in following list:

| weight | 6 | 4 | 0 | -3 | -2 | -1 |
|---|---|---|---|---|---|---|
| mul | 1 | 3 | 3 | 2 | 6 | 6 |

Thus, by formula, the removing term $\sum$ is

$$P_t(\widehat{Z}^{ss}_{R_\delta}/N(R))\cdot(t^{2\cdot 8}P_t(\mathbb{P}^5)+t^{2\cdot 6}P_t(\mathbb{P}^1))\ mod\ t^{20}$$

and the correction term for 5-th blowup is

$$A_5\equiv\frac{t^2}{(1-t^2)^3}-\frac{(t^{12}+...+t^{20})(1+t^2)}{1-t^2}\ mod\ t^{20} \tag{4.13}$$

4.1.7. *Computation of $P_t^G(X_6^{ss})$.* take the blowup

$$\pi:\ X_6\to X_5^{ss}$$

along $G\cdot\widehat{Z}^{ss}_{R_\alpha}$ where $\widehat{Z}^{ss}_{R_\alpha}$ is the strict transform of $Z^{ss}_{R_\alpha}$ under previous blowups, since $\alpha$ contains point $\zeta$

$$Z_{R_\alpha}=\mathbb{P}\{(x_0q_0(x_2,...,x_5)+x_1q_1(x_2,...,x_5)\}$$

$$1+d_\alpha=codimG\widehat{Z}_{R_\alpha}=55-(35+19-19)=20$$

$$N(R_\alpha)=\{dig(A,B):det(A)\cdot det(B)=1,A\in GL(2,\mathbb{C}),B\in GL(4,\mathbb{C})\}$$

The blowup

$$\widehat{Z}_{R_\alpha}\to Z^{ss}_{R_\alpha}$$

along $N(R_\alpha)\zeta$ descending to quotients will not change cohomology of quotients as in the case of 5-th blowup and thus gives formula

$$P_t^{N(R_\alpha)}(\widehat{Z}^{ss}_{R_\alpha})=P_t(N(R_\alpha))(1+t^2)=\frac{1+t^2}{1-t^2}$$

since $Z_{R_\alpha}/\!\!/R_\alpha\cong\mathbb{P}^1$.

To determine the normal representation of $R_\alpha$, we choose $F=x_0(x_2^2+x_3^2+x_4^2)+x_1x_5^2\in Z^{ss}_{R_\alpha}$ which is not in the orbit $\zeta$. Then

$$F_{x_0}=x_2^2+x_3^2+x_4^2,\ F_{x_1}=x_5^2,\ F_{x_2}=2x_0x_2$$

$$F_{x_3}=2x_0x_3,\ F_{x_4}=2x_0x_2,\ F_{x_5}=2x_1x_5$$

where $F_{x_i}:=\frac{\partial F}{\partial x_i}$. Thus the tangent space at $F$ is spanned by the monomials $F_{x_0},..F_{x_5}$ and monomials in $Z_{R_\alpha}$. Subtracting from $\mathbb{C}[x_0,x_1,...,x_5]_3$, we obtain the normal vector

$$\mathcal{N}_F=\mathbb{C}[x_0,x_1]_3\oplus span\{x_5V_2,x_0x_5^2,x_1^2x_2,x_1^2x_3,x_1^2x_4,V_3\}$$

Recall the weight of $R_\alpha$ is $(2,2,-1,-1,-1,-1)$, then the weight of normal representation is given by

| weight | -3 | 0 | 3 | 6 |
|---|---|---|---|---|
| mul | 7 | 5 | 4 | 4 |

So the smallest codimension of unstable strata is $19-8=11$ and thus removing term vanishes after $mod\ t^{20}$.

In a summary, the correction term in the 6-th blowup contributes

$$A_6 \equiv \frac{t^2}{1-t^2} \cdot \frac{1+t^2}{1-t^2} \; mod \; t^{20} \tag{4.14}$$

4.1.8. *Computation of* $P_t^G(X_7^{ss})$. Take the blowup

$$\pi: \; X_7 \to X_6^{ss}$$

along $G \cdot \widehat{Z}_{R_\gamma}^{ss}$ where $\widehat{Z}_{R_\gamma}^{ss}$ is the strict transform of $Z_{R_\gamma}^{ss}$ under $X_6^{ss} \to X$. then the codimension of $GZ_{R_\gamma}$ is given by

$$1 + d_\gamma = 55 - (35 + 14 - 13) = 19$$

we have the normalizer subgroup

$$N(R_\gamma) = \{dig(a, A, B) : a \cdot det(A) \cdot det(B) = 1, A \in GL(2, \mathbb{C}), B \in GL(3, \mathbb{C})\}$$

By proposition 3.3, we have $Z_{R_\gamma} /\!\!/ G_\gamma \cong \mathbb{P}^1 \times \mathbb{C}$ and blowup at two points in $Z_{R_\gamma} /\!\!/ G_\gamma$ will give $P_t(\widetilde{Z_{R_\gamma} /\!\!/ G_\gamma}) = 1 + 3t^2$.

Now we consider normal representation

$$R_\gamma \curvearrowright \mathcal{N}_\gamma = \{x_0^3, ..., x_5^3, x_0^2 x_1, ..., x_0^2 x_4,$$

$$x_0 x_1^2, x_0 x_2^2, x_1^2 x_2, x_3^2 x_4, x_3^2 x_5, x_4^2 x_4, x_4^2 x_3\}$$

It ca be identified as the normal representation in the third blowup in [13], then in a summary, the 7th blowup contributes

$$\begin{aligned} A_7 &\equiv (t^2 + ... + t^{2d_\gamma}) P_t^{N(R_\gamma)}(\widetilde{Z_{R_\gamma}}) - \sum unstable \;\; mod \; t^{20} \\ &\equiv \frac{1+3t^2}{1-t^2}(t^2 + t^4 + ... + t^{14}) \; mod \; t^{20} \end{aligned} \tag{4.15}$$

4.1.9. *Computation of* $P_t^G(X_8^{ss})$. Take the blowup

$$\pi: \; X_8 \to X_7^{ss}$$

along $G \cdot \widehat{Z}_{R_\mu}^{ss}$ where $\widehat{Z}_{R_\mu}^{ss}$ is the strict transform of $Z_{R_\mu}^{ss}$ under previous blowups.

$$N(R_\mu) = \{dig(a, b, A, c, d) : a \cdot b \; det(A) \cdot c \cdot d = 1, a, b, c, d \in \mathbb{C}^*, A \in GL(2, \mathbb{C}))\}$$

$$Z_{R_\mu} = \mathbb{P}\{a x_0 x_4^2 + x_0 x_5 l_1(x_2, x_3) + b x_1^2 x_5 + x_1 x_4 l_2(x_2, x_3) + c(x_2, x_3)\}$$

Thus the codimension

$$1 + d_\mu = codimGZ_{R_\beta} = 55 - (35 + 9 - 7) = 18$$

The blowup $\widehat{Z}_{R_\mu} \to Z_{R_\mu}$ along the orbit $N(R_\mu)\zeta$ and $N(R_\mu)\omega$ will decent to blowup along two points in $Z_{R_\mu} /\!\!/ N(R_\mu) \cong \mathbb{P}(1, 3, 6, 8)$. This gives

$$\begin{aligned} P_t^{N(R_\beta)}(\widehat{Z}_{R_\beta}) &= P_t(BR_\beta) \cdot P_t(\widehat{Z}_{R_\beta} /\!\!/ N(R_\beta)) \\ &= \frac{1}{1-t^2} \cdot (P_t(\mathbb{P}(1, 3, 6, 8)) + (t^2 + t^4) + (t^2 + t^4)) \end{aligned} \tag{4.16}$$

The normal representation for $R_\mu$ can be identified in K3 case as done in last blowup of Kirwan-Lee (see 5.3 in [13]), thus the removing term is giving by

$$\frac{t^{18}}{1-t^2} \mod t^{20}$$

In a summary, the correction term will be given by

$$\begin{aligned} A_8(t) \equiv \frac{t^2}{1-t^2} \cdot \frac{1+3t^2+3t^4+t^6}{1-t^2} \\ -\frac{t^{18}\cdot(1+3t^2+3t^4+t^6)}{1-t^2} \mod t^{20} \end{aligned} \tag{4.17}$$

4.2. **Proof of Theorem 1.1.** By previous computations, we have

$$\begin{aligned} P_t(\widetilde{\mathcal{M}}) =& P_t^G(X^{ss}) + \sum_{i=1}^{8} A_i(t) \\ =& 1+9t^2+26t^4+51t^6+81t^8+115t^{10}+152t^{12} \\ &+193t^{14}+236t^{16}+280t^{18}+324t^{20} \mod t^{20} \end{aligned}$$

Then the duality will give the formula.

## 5. Intersection cohomology of Baily-Borel compactification

In this section, we will compute the intersection cohomology of Baily-Borel compactification $\overline{\mathcal{D}/\Gamma}^{BB}$ based on computation before.

5.1. **Baily-Borel compactification of moduli space of cubic fourfolds.** It is well-known that for a smooth cubic fourfold $X$ its integral middle cohomology $H^4(X,\mathbb{Z})$ has lattice strucure isomorphic to $\Lambda :=< 1 >^{\oplus 21} \oplus < -1 >^{\oplus 3}$. Let $h := c_1(\mathcal{O}_X(1))^2 \in \Lambda$ be the hyperplane class and $\Lambda_0 = E_8^2 \oplus U^2 \oplus A_2 = h^\perp$ be the polarised lattice associated to the smooth cubic fourfold. Denote

$$\mathcal{D} := \{z \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle > 0\}$$

the peroid domain and $\Gamma$the monodromy group. In [2], Baily-Borel gives a compactification of $\mathcal{D}/\Gamma$, whose boundaries corresponds to Type $II, III$ degenerations of cubic fourfolds (see also [22] for refinements ). Following B. Hassett [9], we define

**Definition 5.1.** A cubic fourfold $X$ is called a special cubic fourfold of discriminant $d$ if it contains a surface $T$ which is not homologous to a complete intersection and the classes $h$ and $[T]$ form a saturated rank 2 sublattice of $\Lambda$ with discriminant $d$.

It is also shown in [9] that such locus is a divisor in moduli space. Let $\mathcal{H}_\infty$ be such divisor of discriminant 2.

**Theorem 5.2.** *(Global Torelli, see [19] [23]) The period map*

$$p : \overline{\mathcal{M}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$$

*is a birational map. It is an open immersion over $\mathcal{M}^o$ and can be defined over $\mathcal{M}$, whose image is the complement of Heegner divisor $\mathcal{H}_\infty$.*

5.2. **Intersection cohomology.** Let $\hat{\mathcal{M}}$ be the blowups of $X/\!\!/SL(6)$ only along $\omega$ and $\chi$. Then there is a natural contraction morphism

$$f:\ \widetilde{\mathcal{M}}\ \to\ \hat{\mathcal{M}}$$

From [19], it is known that the period map from GIT compatification to Baily-Borel compatification is resolved by Loojigenga's semi-toric compatification (see for [22] the discussion of Loojigenga's semi-toric compatification):

$$\begin{tikzcd} & \hat{\mathcal{M}} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \overline{\mathcal{M}} \arrow[rr, dashed, "p"] & & \overline{\mathcal{D}/\Gamma}^{BB} \end{tikzcd} \tag{5.1}$$

**Remark 5.3.** By Looijenga's general construction, $p_2$ is composition of a blowup of self-section of the divisor and a small modification. Thus, from the diagram, It is not easy to see

$$\rho(\overline{\mathcal{M}})=1,\quad \rho(\widetilde{\mathcal{M}})=3,\quad \rho(\overline{\mathcal{D}/\Gamma}^{BB})=2$$

We observe the following explicit description

**Proposition 5.4.** *The morphism $p_2$ is the composition of $\phi:\widetilde{\mathcal{M}}\to\overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)}$ and $\nu:\overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)}\to\overline{\mathcal{D}/\Gamma}^{BB}$. Here $\phi$ is the morphism contracting the divisor $E_\chi$ to*

$$E_\chi\cap\widetilde{E}_\omega\cong\mathbb{P}(H^0(C,\mathcal{O}_C(4))\oplus H^0(C,\mathcal{O}_C(6)))/\!\!/SO(3)\subset\overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \tag{5.2}$$

*where $C$ is a smooth plane quadric curve and $\nu$ is a small modification described as follows:*

Table 4. Contraction correspondence $p_2$

| *Exceptional locus in* $\overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)}$ | *Boundaries in* $\overline{\mathcal{D}/\Gamma}^{BB}$ | *fiber* |
|---|---|---|
| $\phi_\infty$ | $A_{17}$ | $\mathbb{P}^1$ |
| $\gamma_\infty$ | $E_7\oplus D_{10}$ | $\mathbb{P}^1$ |
| $\beta_\infty$ | $E_8^{\oplus 2}\oplus A_2$ | $\mathbb{P}^2$ |
| $\epsilon_\infty$ | $A_2\oplus D_{16}$ | $\mathbb{P}^2$ |

*here the locus $\phi_\infty,\cdots,\epsilon_\infty$ is described as lemma 6.9 in* [19].

*Proof.* From section in [19], we know that $p_2$ is a composition of a small modification and a blowup of codimension 2 self-intersection of Heegner divisor $\mathcal{H}_\infty$. Their roots span a sublattice $R\subset\Lambda_0$ with

$$R=\begin{pmatrix} 3 & 2\\ 2 & 2\end{pmatrix}$$

Thus, $R^\perp=U^2\oplus E_8^2$. This implies the self-intersection

$$(\mathcal{H}_\infty\cap\mathcal{H}_\infty)/\Gamma\cong\mathcal{D}_R/\Gamma_R$$

where $\mathcal{D}_R = \mathcal{D} \cap \mathbb{P}(R^\perp \otimes \mathbb{C})$ and $\Gamma_R \leq \Gamma$ is stabilier subgroup of $\mathcal{D}_R$. Note that $R^\perp$ is sublattice with discriminant $= 2$, then $\Gamma_R \leq O(R^\perp)^+$ is subgroup of index 2. By Odaka-Oshima [26], there is isomorphism

$$\mathcal{D}_R/O(R^\perp)^+ \cong \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(8)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(12)))/\!\!/SL(2)$$

Note that $SL(2)$ can be identified as a subgroup of $SO(3)$ of index 2, this shows 5.2.

For small modification, it is described in lemma 6.9 in [19]. □

**Proposition 5.5.** *Let $f: X \to Y$ be a birational morphism of $n$ dimensional irreducible varieties over $\mathbb{C}$ contracting a divisor $E$ to a lower dimensional locus $Z$ and the restriction $f_E$ of $f: E \to Z$ is a topological $\mathbb{P}^m$-bundle, then for $k \leq n$, the intersection cohomology has a decomposition*

$$IH^k(X) = IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^{k-c+j}(Z, \mathbb{Q}) \tag{5.3}$$

*where $c$ is codimension of $Z$ in $X$*

*Proof.* Let $IC_X$ be the intersection complex on $X$ (see [15]), by BBDG, there is a decomposition (non-canonical in general, see [7] )

$$Rf_* IC_X = IC_Y \oplus IC_Z(\mathcal{L}_j)[-i] \tag{5.4}$$

where $\mathcal{L}_j$ are local system on $Z$. Following [8], we can determine these local system: each $\mathcal{L}_j$ is an irreducible summand of $R^j f_{E\,*}\mathbb{Q}_E$, which is rank $= 1$ for $j$ even since each fiber is $\mathbb{P}^m$, thus $\mathcal{L}_j = R^j f_{E\,*}\mathbb{Q}_E$, the shift degree $-i = -j + c$, then taking cohomology of decomposition **??**, we obtain the formula 5.3:

$$IH^k(X) = H^k(Y, IC_Y) \bigoplus_{2\leq j\leq 2m} R^j f_{E\,*}\mathbb{Q}_E[-j+c])$$

$$= IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^k(Z, \mathbb{Q}[-j+c])$$

$$= IH^k(Y) \bigoplus_{2\leq j\leq 2m} H^{k-c+j}(Z, \mathbb{Q})$$

□

Recall an algebraic map $f: X \to Y$ is smeismall if the defect

$$r(f) := \max\{\ i \in \mathbb{Z} :\ {}^p\mathcal{H}^i(Rf_* IC_X[n]) \neq 0\ \}$$

is zero.

**Proposition 5.6.** *Let $f: X \to Y$ be a semismall birational morphism of $n$ dimensional irreducible varieties over $\mathbb{C}$ such that $Z \subset Y$ is a connected closed subvariety and $f$ is isomorphic outside $Z$ and over $Z$, $f$ is a $\mathbb{P}^m$-bundle, then for $k \leq n$*

$$IH^k(X) = \bigoplus_{0\leq j\leq m} H^{k+2j-n}(Z, \mathbb{Q}) \tag{5.5}$$

*where $H^l(Z, \mathbb{Q}) = 0$ if $l < 0$.*

*Proof.* By semi-small property and semi-simplicity of decomposition theorem, we have

$$\begin{aligned} Rf_*IC_X[n] &= \bigoplus_{-r(f)\le i\le r(f)} {}^p\mathcal{H}^i(Rf_*IC_X[n])[-i] \\ &= \bigoplus_j IC(\overline{Y}_i, \mathcal{L}_{i,j}) \end{aligned} \tag{5.6}$$

where $\mathcal{L}_{i,j}$ is a local system supported on clourse $\overline{Y}_i$ of strata $Y_i$.

In our case, there is a natural stratification $Y_0 = Z, Y_1 = Y - Z$, then we have

$$Rf_*IC_X = IC_Y \bigoplus_{j=2}^{m} \mathbb{Q}_Z[n-2j] \tag{5.7}$$

By taking cohomology, we obtain the cohomology formula. $\square$

**Theorem 5.7.** *The intersection cohomology of $\hat{\mathcal{M}}$ is*

$$\begin{aligned} IP_t(\hat{\mathcal{M}}) &= 1+3t^2+8t^4+17t^6+29t^8+44t^{10}+61t^{12} \\ &+78t^{14}+99t^{16}+121t^{18}+151t^{20}+121t^{22}+99t^{24}+78t^{26} \\ &+61t^{28}+44t^{30}+29t^{32}+17t^{34}+8t^{36}+3t^{38}+t^{40} \end{aligned} \tag{5.8}$$

*Proof.* we will use the blowup formula 2.1 of intersection cohomology.

blow down $E_\mu$: In this case, $\pi_0(N_\mu)$ acts on the fiber trivially since $N_\mu$ is connected, thus we need to shift the polynomial by degree 2 according to the formula 2.1, then

$$\begin{aligned} B_\mu(t) = (1+3t^2+3t^4+t^6)\cdot(&t^2+t^4+2t^6+2t^8+3t^{10}+3t^{12}+4t^{14}+ \\ &4t^{16}+4t^{18}+4t^{20}+3t^{22}+3t^{24}+2t^{26}+2t^{28}+t^{30}+t^{32}) \end{aligned} \tag{5.9}$$

Similarly, for blowing down $E_\gamma$

$$\begin{aligned} B_\gamma(t) = (1+3t^2+t^4)\cdot(&t^2+2t^4+3t^6+4t^8+5t^{10}+6t^{12}+7t^{14}+8t^{16}+ \\ &8t^{18}+8t^{20}+7t^{22}+6t^{24}+5t^{26}+4t^{28}+3t^{30}+2t^{32}+t^{34}) \end{aligned} \tag{5.10}$$

for blowing down $E_\alpha$

$$\begin{aligned} B_\alpha(t) &= (1+t^2)\cdot(t^2+t^4+2t^6+3t^8+4t^{10}+5t^{12}+6t^{14}+7t^{16} \\ &+8t^{18}+8t^{20}+7t^{22}+6t^{24}+5t^{26}+4t^{28}+3t^{30}+2t^{32}++t^{34}+t^{36}) \end{aligned} \tag{5.11}$$

for blowing down $E_\delta$

$$\begin{aligned} B_\delta(t) =&(1+t^2)\cdot(t^2+2t^4+4t^6+6t^8+9t^{10}+12t^{12}+16t^{14} \\ &+19t^{16}+24t^{18}+24t^{20}+19t^{22}+16t^{24} \\ &+12t^{26}+9t^{28}+6t^{30}+4t^{32}+2t^{34}+t^{36}) \end{aligned} \tag{5.12}$$

for blowing down $E_\tau$:

$$\begin{aligned} B_\tau(t) = (1+t^2)\cdot(&t^2+t^4+2t^6+3t^8+4t^{10}+5t^{12}+7t^{14}+8t^{16}+9t^{18} \\ &+9t^{20}+8t^{22}+7t^{24}+5t^{26}+4t^{28}+3t^{30}+2t^{32}+t^{34}+t^{36}) \end{aligned} \tag{5.13}$$

for blowing down $E_\xi$, note in this case, $\pi_0(N_\xi) = S_6$ acts on $H^*(\mathbb{P}N_\xi/R_\xi)$ by permutation of coordinates of $\mathbb{P}N_\xi$, thus,

$$\begin{aligned} IP_t(H^*(\mathbb{P}N_\xi/R_\xi)^{\pi_0(N_\xi)}) \equiv & \\ & P_t(\mathbb{P}N_\xi)P_t((H^*(BR_\xi)^{\pi_0(N_\xi)})) \mod t^{19} \\ & \equiv \frac{1}{\prod_{1\le i\le 6}(1-t^{2i})} \mod t^{19} \end{aligned} \tag{5.14}$$

then using formula 2.1 again, we have

$$\begin{aligned} B_\xi(t) = t^2 + t^4 + 2t^6 + 3t^8 + 5t^{10} + 7t^{12} + 11t^{14} + 14t^{16} + 20t^{18} \\ +26t^{20} + 20t^{22} + 14t^{24} + 11t^{26} + 7t^{28} + 5t^{30} + 3t^{32} + 2t^{34} + t^{36} + t^{38} \end{aligned} \tag{5.15}$$

Put these together then we obtain our formula from

$$\mathrm{IP}_t(\hat{\mathcal{M}}) = P_t(\mathcal{M}) - B_\mu(t) - B_\xi(t) - B_\delta(t)$$

□

**Remark 5.8.** In [6], the authors doubted that the Kirwan resolution of moduli spaces of cubic threefolds is isomorphic to certain toroidal compactification of ball quotient $\mathbb{B}/\Gamma$ for some cone decomposition. the evidence in [6] is that they compute the cohomology of the toroidal compactification and find the betti numbers of the two compactification match perfectly. We want to point out here it is also unknown for the moduli spaces of cubic fourfolds.

**Corollary 5.9.** *The intersecton betti number of* $\overline{\mathcal{D}/\Gamma}^{BB}$ *is*

$$\begin{aligned} IP_t(\overline{\mathcal{D}/\Gamma}^{BB}) = 1 + 2t^2 + 4t^4 + 9t^6 + 16t^8 + 26t^{10} + 38t^{12} \\ +50t^{14} + 65t^{16} + 82t^{18} + 112t^{20} + 82t^{22} + 65t^{24} + 50t^{26} \\ +38t^{28} + 26t^{30} + 16t^{32} + 9t^{34} + 4t^{36} + 2t^{38} + t^{40} \end{aligned} \tag{5.16}$$

*Proof.* First apply formula (5.3) to morpgism $\mu$ in 5.1, we have

$$IP_t(\overline{\mathcal{D}/\Gamma}^{BB}) = IP_t(\hat{\mathcal{M}}) - (1+t^2)(P_t(E_\chi \cap \widetilde{E}_\omega) - 1) \mod t^{20}$$

then we remain to compute the cohomology $E_\chi \cap \widetilde{E}_\omega$. Thanks to 4.3, we identify $E_\chi \cap \widetilde{E}_\omega$ as the exception divisor in the 1st blow up of GIT moduli of degree 2 K3 surfaces. According to [13], after blowing a rational curve $\mathbb{P}^1 \subset E_\chi \cap \widetilde{E}_\omega$, we get a kirwan resolution of $E_\chi \cap \widetilde{E}_\omega$, thus

$$\begin{aligned} P_t(Bl_{\mathbb{P}^1}(E_\chi \cap \widetilde{E}_\omega)) =& \mathrm{IP}_t(Bl_{\mathbb{P}^1}(E_\chi \cap \widetilde{E}_\omega)) \\ =& \mathrm{IP}_t(E_\chi \cap \widetilde{E}_\omega) + \textit{correction term} \\ \equiv& \frac{1}{(1-t^2)\cdot(1-t^4)} + (1+t^2)\cdot(t^2 + 2t^4 + 3t^6 + \\ & 4t^8 + 5t^{10} + 6t^{12} + 7t^{14} + 7t^{16} + 8t^{18}) \mod t^{18} \end{aligned} \tag{5.17}$$

where the correction term is appeared as in formula 2.1.

the usual blowup formula for cohomology gives

$$P_t(E_\chi \cap \widetilde{E}_\omega) = 1 + t^2 + 3t^4 + 5t^6 + 8t^8 + 10t^{10} + 13t^{12} + 15t^{14} + 17t^{16} + 18t^{18} + 17t^{20} + 15t^{22} + 13t^{24} + 10t^{26} + 8t^{28} + 5t^{30} + 3t^{32} + t^{34} + t^{36} \tag{5.18}$$

Last, apply formula 5.5 to morphism $\nu$ and combine the table 5.4, we only need to remove

$$2(t^{18} + t^{20}) + 2(t^{16} + t^{18} + t^{20}) \mod t^{20}$$

All these together imply our formula. □

**Remark 5.10.** Since the Zucker conjecture was established in [21] and [28], our computation also gives all $L^2$-betti numbers of $\overline{\mathcal{D}/\Gamma}^{BB}$.

**Remark 5.11.** To understand the cohomology of locally symmetric space is of particular interest. But at present we cannot get the whole cohomology of $\mathcal{D}/\Gamma$. we hope the methods can be improved to obtain its cohomology (eg, $L^2$ cohomology, singular cohomology, etc).

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Shanghai Research Center For Mathematical Science, Shanghai, 200433, People's Republic of China
*Email address*: `fsi15@fudan.edu.cn`