Digital Signal Processing

Computation of DFT

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Discrete Time Fourier Transform—A Review

DFT: Uniform Sample of DTFT

$$X[k] = X(e^{j\omega})\big|_{\omega=(2\pi/N)k} = \sum_{k=0}^{N-1} x[n]W_N^{kn} \qquad W_N = e^{j2\pi/N}$$

$$= \sum_{k=0}^{N-1} \left(\Re\{x[n]\}\Re\{W_N^{kn}\} - \Im\{x[n]\}\Im\{W_N^{kn}\}\right)$$

$$+ j\sum_{k=0}^{N-1} \left(\Re\{x[n]\}\Im\{W_N^{kn}\} - \Im\{x[n]\}\Re\{W_N^{kn}\}\right)$$

Computation Load: $\mathcal{O}(N^2)$

- N^2 complex multiplication
- N^2 complex addition

Fast Fourier Transform

Properties of W_N

Symmetry

$$W_N^{k(N-n)} = W_N^{kN-kn} = W_N^{-kn} = (W_N^{kn})^*$$

Periodic

$$W_N^{kn} = W_N^{k(N+n)} = W_N^{(k+N)n}$$

Exploit the properties for computational saving

$$x[n]W_N^{kn} + x[N-n]W_N^{k(N-n)} = x[n]W_N^{kn} + x[N-n]W_N^{-kn}$$

Take the real part as an example

Saving by a Factor 2

$$\begin{aligned}\Re\{x[n]W_N^{kn} + x[N-n]W_N^{-kn}\} &= (\Re\{x[n]\} + \Re\{x[N-n]\})\Re\{W_N^{kn}\} \\ &- (\Im\{x[n]\} - \Im\{x[N-n]\})\Im\{W_N^{kn}\}\end{aligned}$$

Fast Fourier Transform—Decimation in Time

Expressing DFT with even and odd input samples

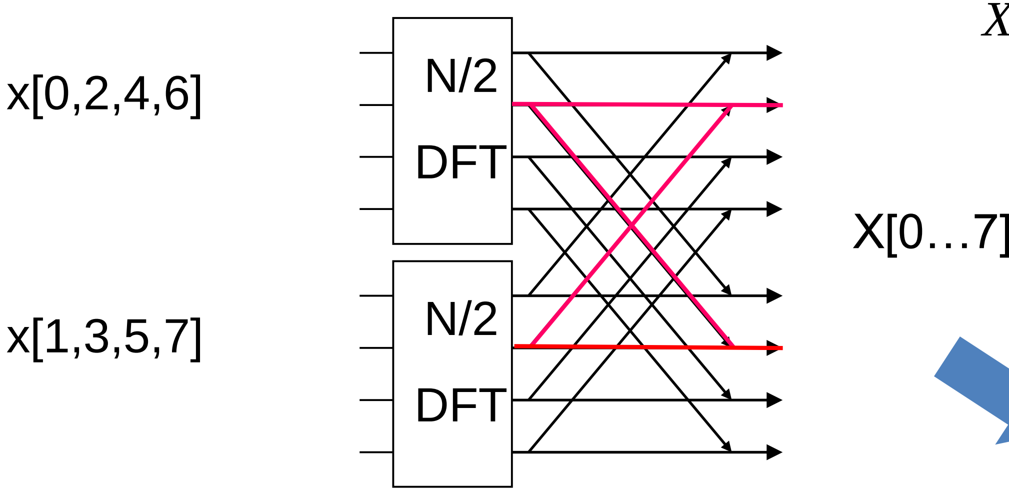
$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{nk} \\
&= \sum_{n\ even} x[n] W_N^{nk} + \sum_{n\ odd} x[n] W_N^{nk} \\
&= \sum_{r=0}^{\frac{N}{2}-1} x[2r] (W_N^2)^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] (W_N^2)^{rk} \\
&= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{N/2}^{rk}
\end{aligned}
$$

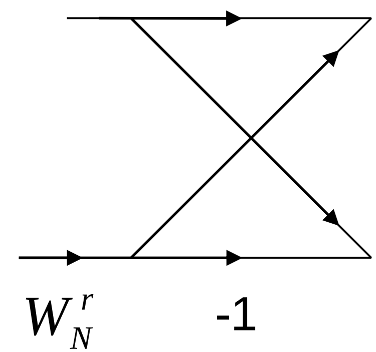
$$
W_N^2 = e^{-j2\pi \cdot 2/N} = e^{-j2\pi/(N/2)} = W_{N/2}
$$

$$
X[k] = \underbrace{G[k]}_{\substack{\text{N/2 DFT} \\ \text{of even samples}}} + W_N^k \cdot \underbrace{H[k]}_{\substack{\text{N/2 DFT} \\ \text{of odd samples}}}
$$

Decompose N point DFT into 2 N/2 DFTs

$$X[k] = \underbrace{G[k]}_{\substack{\text{N/2 DFT} \\ \text{of even samples}}} + W_N^k \cdot \underbrace{H[k]}_{\substack{\text{N/2 DFT} \\ \text{of odd samples}}}$$





"butterfly"

Exploit the periodicity of G[k] and H[k]

Example

$$X[5] = G[5] + W_8^5 H[5]$$
$$= G[4+1] + W_8^{4+1} H[4+1]$$
$$= G[1] - W_8^1 H[1]$$

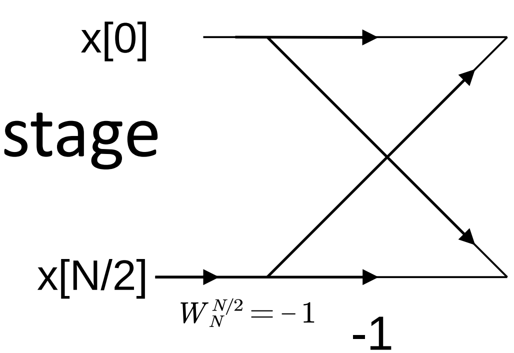
$$W_N^{N/2+r} = e^{j\pi} e^{-j2\pi r/N} = -e^{-j2\pi r/N} = -W_N^r$$

Fast Fourier Transform—Decimation in Time

- Computation Load: from $\mathcal{O}(N^2)$ to $\mathcal{O}(2(N/2)^2)$
  - Each N/2 DFT: $(N/2)^2$ complex multiplication/adding
  - Combine 2 DFTs, N complex adding

$$2(N/2)^2 + N < N^2 \quad \text{for } N > 2$$

- Continue breaking DFTs until reach 2-point DFT
  - Each stage is a reduced number of DFT
  - Number of stages is: $\log_2(N)$
  - N complex multiplication, adding per stage

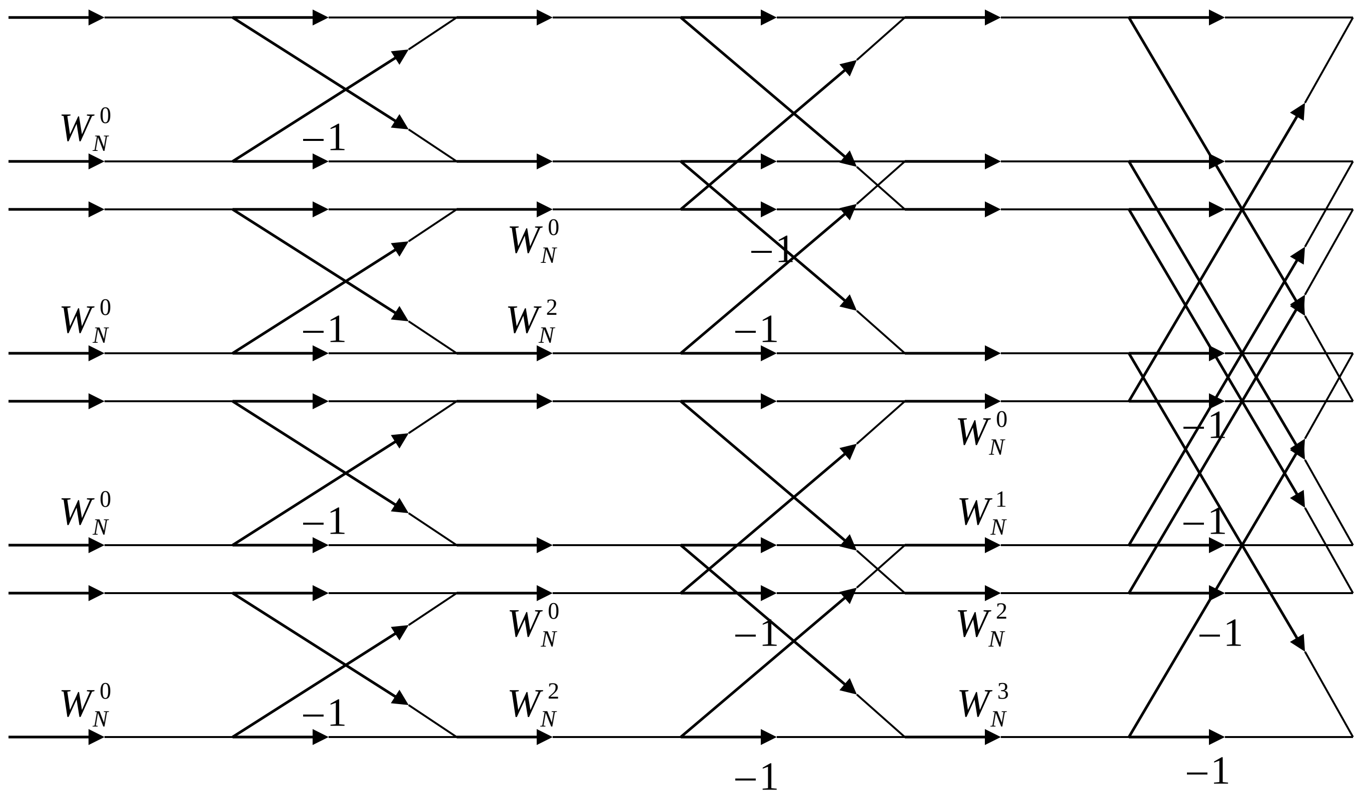


$$\boxed{\text{Total Computation } \mathcal{O}(N \log_2 N)}$$

Fast Fourier Transform—Decimation in Time

8-Point DFT

x[0,4,2,6,1,5,3,7] X[0…7]



Fast Fourier Transform—Decimation in Time

Bit Reverse

- The input/out of DFT in Bit Reverse Order

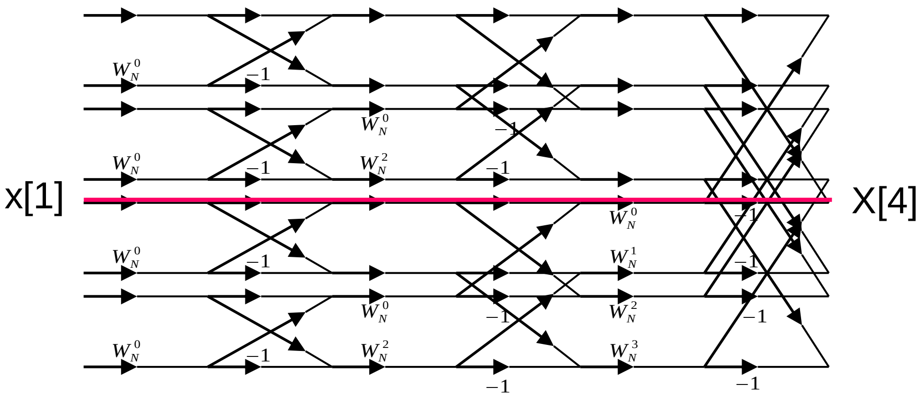
$$X[n_2, n_1, n_0] \leftrightarrow x[n_0, n_1, n_2]$$

  - [n2,n2,n1] is the binary representation of the index

$$4 = (100)_2$$

$$n_0 = 0$$
$$n_1 = 0$$
$$n_2 = 1$$

$$X[4] \leftrightarrow x[1]$$



Fast Fourier Transform—Decimation in Freq.

Decompose the output by even and odd

$$X[2r] = \sum_{n=0}^{N-1} x[n]W_N^{n(2r)} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{n(2r)} + \sum_{n=(N/2)}^{N-1} x[n]W_N^{n(2r)}$$

$$= \sum_{n=0}^{(N/2)-1} x[n]W_N^{n(2r)} + \sum_{n=0}^{(N/2)-1} x[n+N/2]W_N^{(n+N/2)(2r)}$$

$$= \sum_{n=0}^{(N/2)-1} (x[n] + W_N^{Nr} x[n+N/2])W_N^{n(2r)}$$

$$= \sum_{n=0}^{(N/2)-1} (x[n] + x[n+N/2])W_{N/2}^{nr}$$

DFT of x[n]+x[n+N/2]

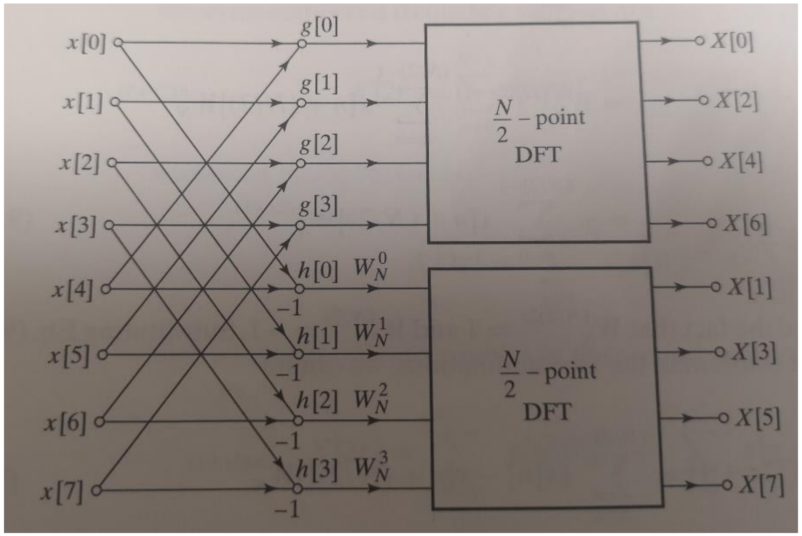
$$X[2r+1] = \sum_{n=0}^{N-1} x[n]W_N^{n(2r+1)} = \sum_{n=0}^{(N/2)-1} (x[n] - x[n+N/2])W_{N/2}^{n(r+1)}$$

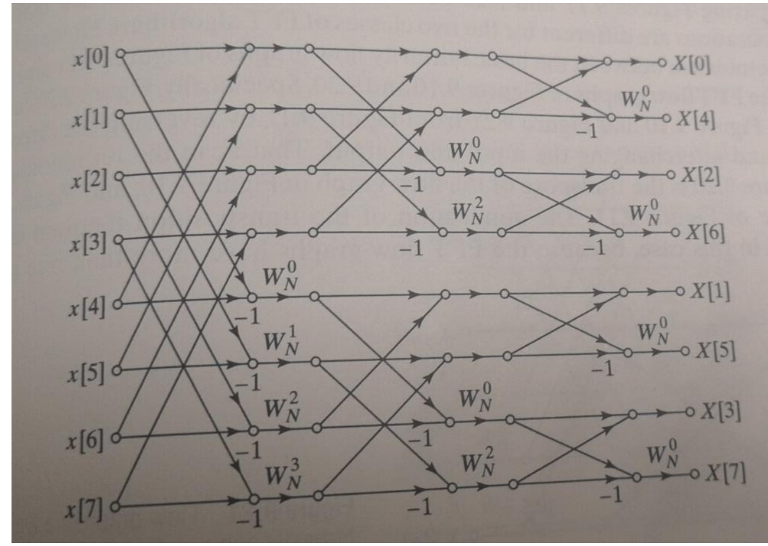
$$= W_{N/2}^{n} \sum_{n=0}^{(N/2)-1} (x[n] - x[n+N/2])W_{N/2}^{nr}$$

DFT of x[n]-x[n+N/2]

Fast Fourier Transform—Decimation in Freq.

Example: 8-point DFT





- Same Computation as Decimation in Time
- Input/output in Bit Revere Order
- Equivalent as in DIT via **Transpose Operation**