Digital Signal Processing

Finite Length Discrete Transform (DFT)

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Outline

- Discrete Fourier Series (DFS)
- Discrete Fourier Transform (DFT)
- Relation between DTFT and DFT
- Circular Convolution
- Linear Convolution Using DFT

Discrete Fourier Series

Discrete Fourier Series

Periodic Sequence $\tilde{x}[n] = \tilde{x}[n + rN]$ We have

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

Fourier Series

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$$

To show it

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} = \sum_{n=0}^{N-1} \frac{1}{N}\sum_{r=0}^{N-1} \tilde{X}[r] e^{j(2\pi/N)rn} e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{N-1} \tilde{X}[r] \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)(r-k)n}$$

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)(r-k)n} = 1 \text{ if } k - r = mN$$

$$= \sum_{r=0}^{N-1} \tilde{X}[r] \quad \text{if } k - r = mN$$

DFS is periodic

$$\tilde{X}[k+N] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)(k+N)n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn} e^{j2\pi} = \tilde{X}[k]$$

Discrete Fourier Series

Example: DFS of $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$

DFS $\quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1 \qquad W_N^{kn} = e^{-j(2\pi/N)kn}$

We can write $\quad \tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$

Example: DFS of $\tilde{x}[n] = 1$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} = \sum_{r=-\infty}^{\infty} N\delta[k - rN]$$

Discrete Fourier Series

Properties of DFS :

Linearity

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \xleftrightarrow{DFS} a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

Shift

$$\tilde{x}[n-m] \xleftrightarrow{DFS} W_N^{km}\tilde{X}[k]$$

$$W_N^{-nl}\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k-l]$$

Duality

$$\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k] \Rightarrow \tilde{X}[n] \xleftrightarrow{DFS} N\tilde{x}[-k]$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \Rightarrow N\tilde{x}[-n] = \sum_{k=0}^{N-1}\tilde{X}[k]W_N^{kn}$$

DFS

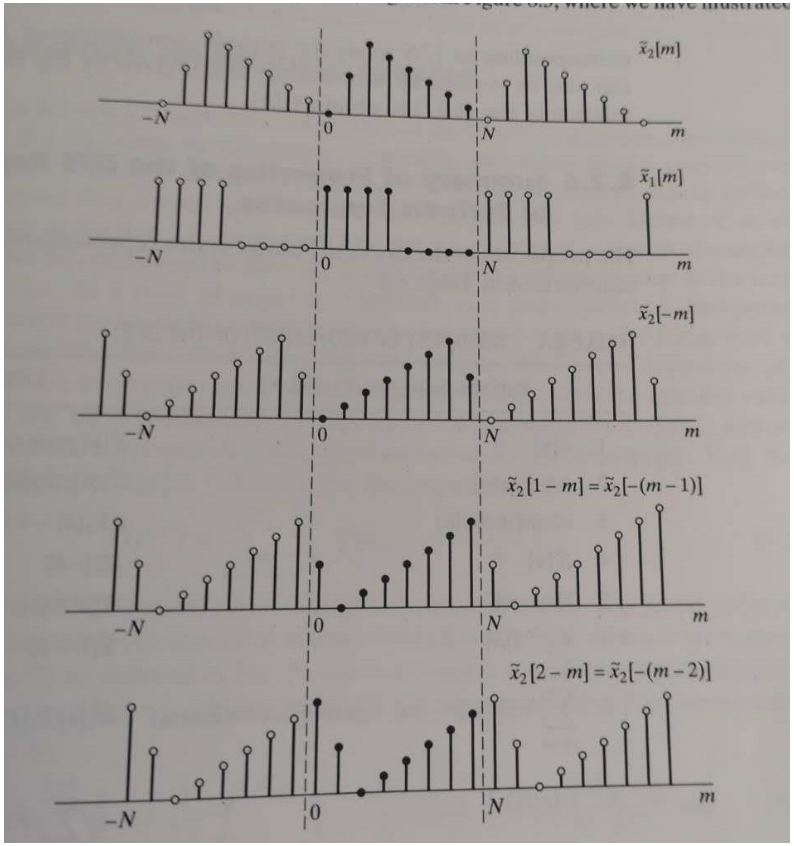
Discrete Fourier Series

Periodic Convolution

$\tilde{x}_1[n]$ $\tilde{x}_2[n]$ With Period of N

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

- Summation only in one period interval
- Computation needs:
  - Flip
  - Circular shift
  - Sum



Discrete Fourier Series

Periodic Convolution

$$\tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k] \qquad \tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_2[k]$$

Period of N

What is DFS for

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}_3[n] W_N^{kn} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] W_N^{kn}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} = \sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{X}_2[k] W_N^{km} = \tilde{X}_1[k]\tilde{X}_2[k]$$

$$\boxed{\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]}$$

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Discrete Fourier Transform

Fourier Transform of Periodic signal

Using DFS $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$ $\quad W_N^{-kn} \overset{DTFT}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi\delta\left(w - \frac{2\pi}{N}k\right)$

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta\left(w - \frac{2\pi}{N}k\right)$$

Example: DFT of a periodic impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty}\delta[n - rN]$$

$\tilde{P}[k] = 1 \text{ for all } k$ $\Rightarrow$ $\tilde{P}(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(w - \frac{2\pi}{N}k\right)$

Discrete Fourier Transform

Periodic signal can be expressed as

$$\tilde{x}[n] = x[n] \circledast \tilde{p}[n] = x[n] \circledast \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

x[n] is the finite length signal defined in one period $0 \le n \le N-1$

Property of DTFT $\tilde{X}(e^{j\omega}) = X(e^{j\omega})\tilde{P}(e^{j\omega})$

$$= \frac{2\pi}{N} X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \delta\left(w - \frac{2\pi k}{N}\right)$$

$$= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X(e^{j(2\pi/N)k})\delta\left(w - \frac{2\pi k}{N}\right)$$

Compare with $\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \tilde{X}[k]\delta\left(w - \frac{2\pi}{N}k\right)$

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\big|_{\omega = (2\pi/N)k}$$

DFS is the uniform samples of the DTFT of finite length Sequence

Discrete Fourier Transform

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

Periodic sequence and finite length sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN] = x[(n)_N]$$

DFS: Discrete, Finite length, Periodic

$$\tilde{X}[k] = X[k-rN] = X[(k)_N]$$

For one period

$$X[k] = \sum_{k=0}^{N-1} x[n]W_N^{kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$

$\boxed{\text{DFT}}$

Properties of Discrete Fourier Transform

Linearity

$$a x_1[n] + b x_2[n] \overset{DFT}{\longleftrightarrow} a X_1[k] + b X_2[k]$$

Duality

$$x[n] \overset{DFT}{\longleftrightarrow} X[k] \quad \Rightarrow \quad X[n] \overset{DFT}{\longleftrightarrow} N x[((-k))_N]$$

Shift

$$x[((n-m))_N] \overset{DFT}{\longleftrightarrow} W_N^{km} X[k]$$

Define the periodic sequence

$$\tilde{x}_1[n] = x[((n))_N] \qquad \tilde{x}_1[n] \overset{DFS}{\longleftrightarrow} \tilde{X}_1[k]$$

$$\tilde{x}_2[n] = \tilde{x}_1[n-m] = x[((n-m))_N] \qquad \tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} W_N^{km} \tilde{X}_1[k]$$

For one period $x[((n-m))_N] \overset{DFT}{\longleftrightarrow} W_N^{km} X[k]$

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Relation Between DTFT and DFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}k} = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}$$

DFT is the uniform sample of DTFT

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n} \quad \Rightarrow \quad X[k] = X(z)\Big|_{z=e^{-j\frac{2\pi}{N}k}}$$

DFT is the uniform sample of X(z) on unit circle

Relation Between DTFT and DFT

DTFT from DFT

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1} W_N^{-kn}e^{-j\omega n}$$

Change the order of summation

$$\sum_{n=0}^{N-1} W_N^{-kn}e^{-j\omega n} = \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j[\omega-(2\pi k)/N]}} = \Phi\left(\omega-\frac{2\pi k}{N}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\Phi\left(\omega-\frac{2\pi k}{N}\right)$$

$$\Phi(\omega) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)}e^{-j\omega[(N-1)/2]}$$

$\boxed{\text{Frequency Domain interpolation}}$

Relation Between DTFT and DFT

Frequency Domain Sampling

$$Y[k] = X(e^{j\omega})\big|_{\omega = 2\pi k/N} = \sum_{l=\infty}^{\infty} x[l] e^{-j\frac{2\pi}{N}kl}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=\infty}^{\infty} x[l] e^{-j\frac{2\pi}{N}kl} W_N^{-kn}$$

$$= \sum_{l=-\infty}^{\infty} x[l] \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-kl} W_N^{-kn}$$

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-kl} W_N^{-kn} = 1 \quad \text{for } l = n + mN$$

$$= \sum_{l=-\infty}^{\infty} x[n + lN]$$

Time Domain Shift: Alias if x[n] is longer than N

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Properties of Discrete Fourier Transform

Circular Convolution

$$x_3[n] = x_1[n] \circledcirc x_2[n]$$

$$= \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

$$X_3[k] = \sum_{n=0}^{N-1} x_3[n] W_N^{kn} \quad = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N] W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[((n-m))_N] W_N^{kn} \quad = \sum_{m=0}^{N-1} x_1[m] W_N^{km} X_2[k]$$

$$= X_1[k] X_2[k]$$

$$\boxed{x_1[n] \circledcirc x_2[n] \overset{DFT}{\longleftrightarrow} X_1[k] X_2[k]}$$

One period of DFS

Linear Convolution Using DFT

- Let *g*[*n*] & *h*[*n*] be two finite-length sequences of length *N* and *M*

$$y_L[n] = g[n] \circledast h[n]$$

- Y_L[n] of length L=N+M-1

Define $g_e[n] = \begin{cases} g[n], 0 \le n \le N-1 \\ 0, N \le n \le L-1 \end{cases}$ $h_e[n] = \begin{cases} h[n], 0 \le n \le M-1 \\ 0, M \le n \le L-1 \end{cases}$

$$y_L[n] = g[n] \circledast h[n] = g_e[n] \circledcirc h_e[n]$$

$$Y_L[k] = G_e[k] H_e[k]$$ Of length L

Linear Convolution Using DFT

Finite Length convolve with infinite length sequence

$$y[n] = h[n] \circledast x[n]$$

h[n] of length *M*

Break x

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mN]$$

X_m[n] of length *N*

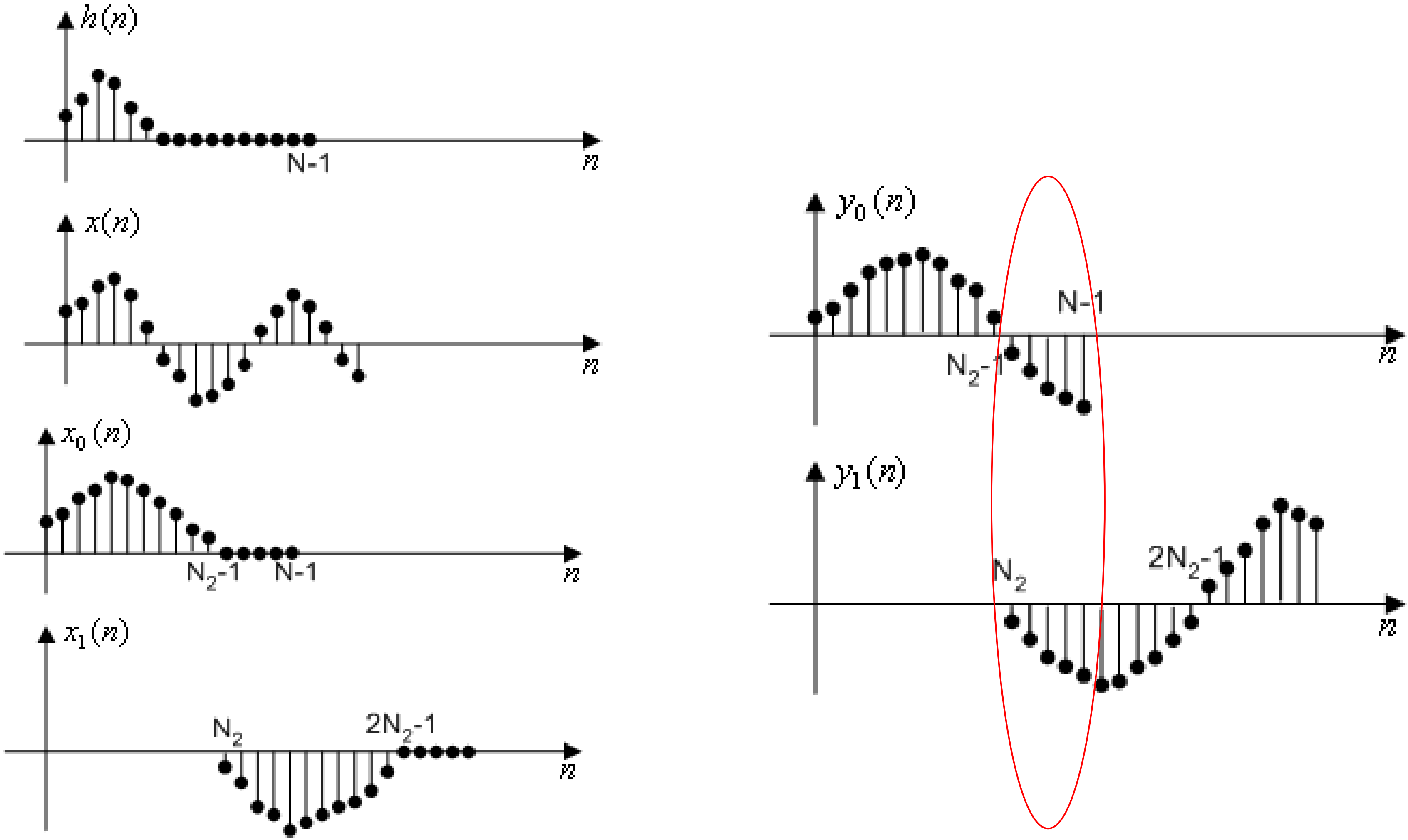
$$y[n] = h[n] \circledast x[n] = h[n] \circledast \sum_{m=0}^{\infty} x_m[n - mN]$$

$$= \sum_{m=0}^{\infty} h[n] \circledast x_m[n - mN] = \sum_{m=0}^{\infty} y_m[n - mN]$$

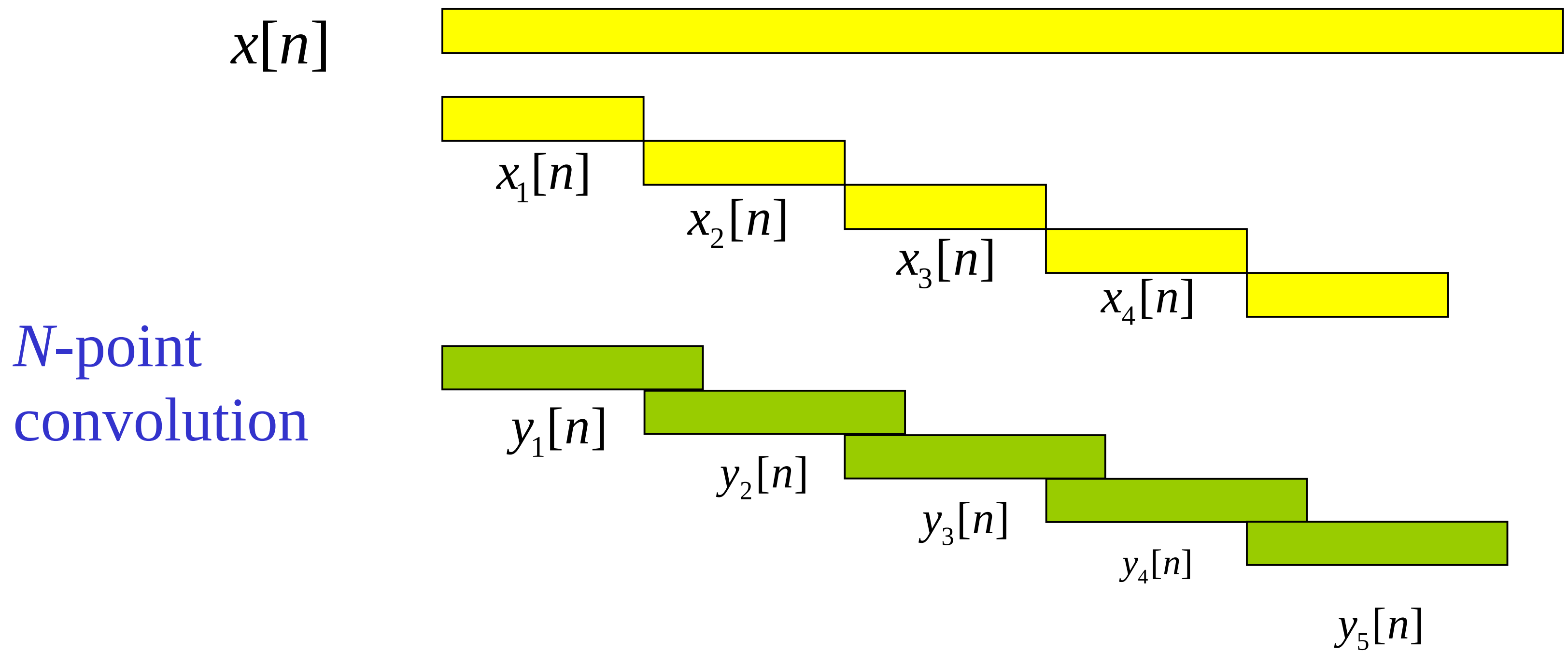
y_m[n] of length *N+M-1* → **Overlap and Add**

Linear Convolution Using DFT

Finite Length convolve with infinite length sequence



Linear Convolution Using DFT



Overlap- add

Linear Convolution Using DFT

Finite Length convolve with infinite length sequence

$$y[n] = h[n] \circledast x[n]$$

h[n] of length *M*

Break x $x[n] = \sum_{m=0}^{\infty} x_m[n - m(N - M + 1)]$

X_m[n] of length *N* *and overlaps*

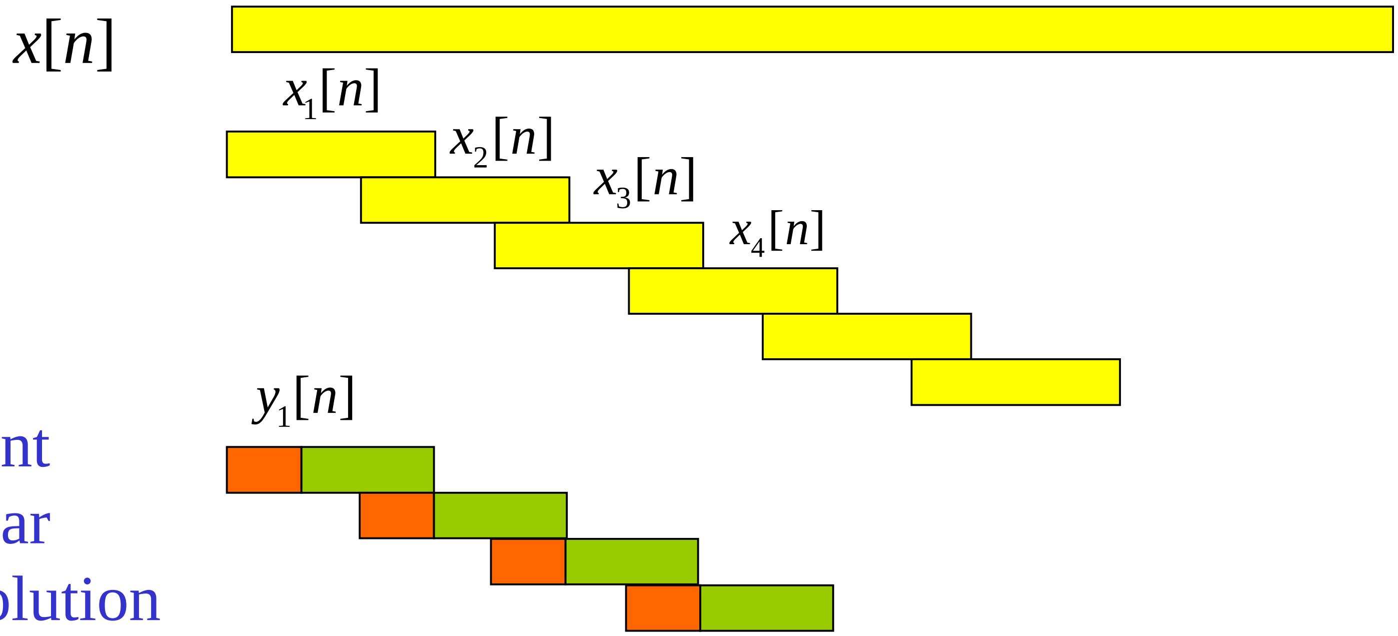
$$y[n] = h[n] \circledast x[n] = h[n] \circledast \sum_{m=0}^{\infty} x_m[n - m(N - M + 1)]$$

$$= \sum_{m=0}^{\infty} y_m[n - mN]$$

$$y_m[n] = h[n] \circledcirc x_m[n]$$

y_m[n] of length *N+M-1* → **Overlap and Save**

Linear Convolution Using DFT



N-point circular convolution

Overlap-save