Finite-Length Discrete Transforms

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§ 5 Finite-Length Discrete Transforms

5.1 DFS (Discrete Fourier Series)

1. Orthogonal Sequences

Basic sequences set $\psi[k, n],\ n:[0, N-1]$

$(k = 0,\ 1,\ 2,\ \ldots,\ N-1)$

If

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi[k, n]\psi^*[\ell, n] = \begin{cases} 0, & \ell \neq k \\ 1, & \ell = k \end{cases} \qquad \textbf{(5.3)}$$

They are orthogonal to each other.

How to describe Eq. (5.3) ? $\delta[k-\ell]$

2. Orthogonal Transforms

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\psi[k,n], 0 \le n \le N-1 \qquad \textbf{(5.2)}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k,n], 0 \le k \le N-1 \qquad \textbf{(5.1)}$$

The proof is in the textbook (Section 5.1).

Eq. (5.2) means that $x[n]$ is decomposed to the combination of $\psi[k, n]$.

3. *Parseval's Relation* of orthogonal transforms

$$\sum_{n=0}^{N-1} \left| x[n] \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| \chi[k] \right|^2, 0 \le k \le N-1$$

4. DFS (Problem 5.2)

A periodic sequence $\tilde{x}[n] = \tilde{x}[n + rN]$ **(*r* is integer, *N* is period) can be represented as a *Fourier Series.***

Just like continuous periodic signal, the orthogonal functions set is

$$\{\psi[k,n] = e^{j\frac{2\pi}{N}kn}\},\ k = 0,1,\ldots,N-1$$

That means

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \quad \textbf{...(A)}$$

$\tilde{X}[k]$ **is the coefficient of the *k*th harmonic. It can be computed by using formula:**

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \qquad \ldots\textbf{(B)}$$

$$W_N = e^{-j\frac{2\pi}{N}} \quad \textbf{(kernel)}$$

Note:

① $\psi[k, n]$ has period N in k, $\psi[0, n] = 1$ is DC component, $\psi[k, n]$ is the *k*th harmonic.

②

$$\sum_{n=0}^{N-1} \psi[k,n]\psi^*[\ell,n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-\ell)n} = \frac{1-e^{j\frac{2\pi}{N}(k-\ell)N}}{1-e^{j\frac{2\pi}{N}(k-\ell)}}$$

$$= \begin{cases} 0, & \ell \neq k \\ N, & \ell = k \end{cases}$$

How to derive this step? (P190)

③

$$\psi[k+N,n] = \psi[k,n] \qquad \{\psi[k,n] = e^{j(\frac{2\pi}{N})kn}\}$$

$$\downarrow$$

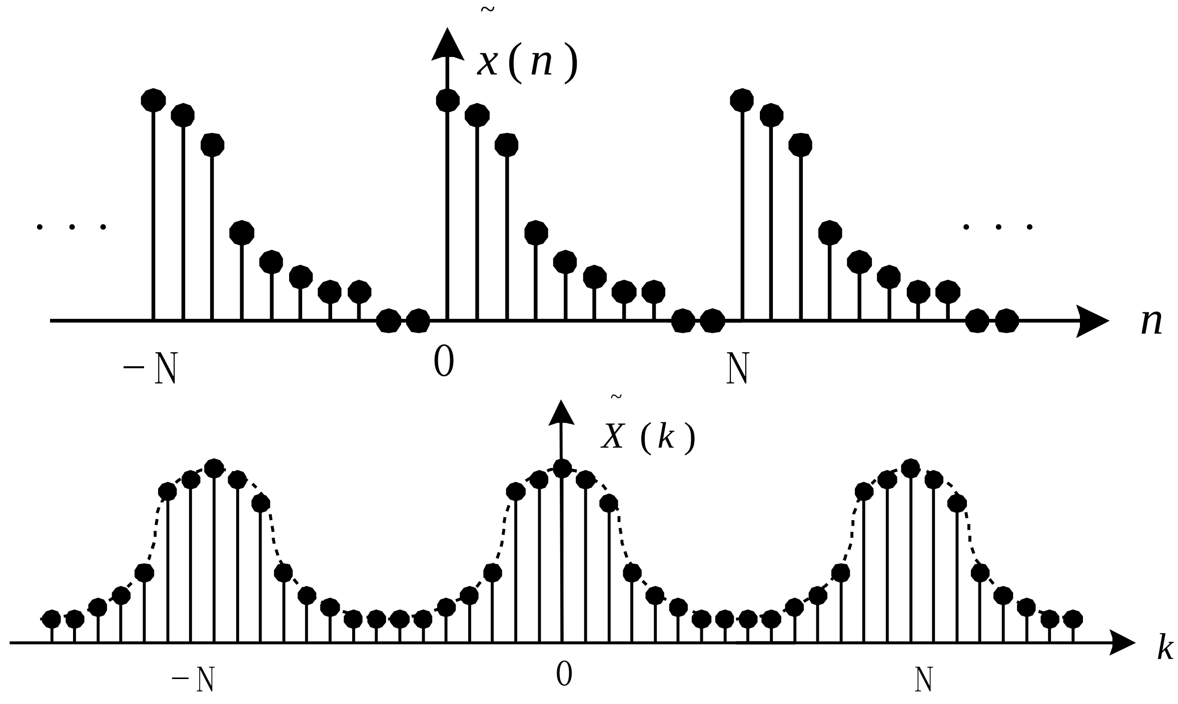
$$\tilde{X}[k+N] = \tilde{X}[k]$$

④The Eq. (A) and Eq. (B) are expressed as

$$\tilde{x}[n] \Leftrightarrow \tilde{X}[k]$$

or

$$\tilde{X}[k] = DFS\{\tilde{x}[n]\} \qquad \tilde{x}[n] = IDFS\{\tilde{X}[k]\}$$



Example: compute DFS for the following sequence:

$$\tilde{x}[n] = \cos\frac{\pi}{6}n$$

Solution 1:

$$\tilde{x}[n] = \frac{1}{2}e^{j\frac{2\pi}{12}n} + \frac{1}{2}e^{-j\frac{2\pi}{12}n} = \frac{1}{2}e^{j\frac{2\pi}{12}n} + \frac{1}{2}e^{j\frac{2\pi}{12}(11)n}$$

(N=12). Comparing with Eq. (A), we get:

When k = 1+12r, and k = 11+12r,

$$\tilde{X}[k] = N/2 = 6, \text{ or}$$

$$\tilde{X}[k] = 6\sum_{r=-\infty}^{\infty}\left\{\delta[k-1-12r] + \delta[k-11-12r]\right\}$$

Solution 2: Using Eq. (B), we get

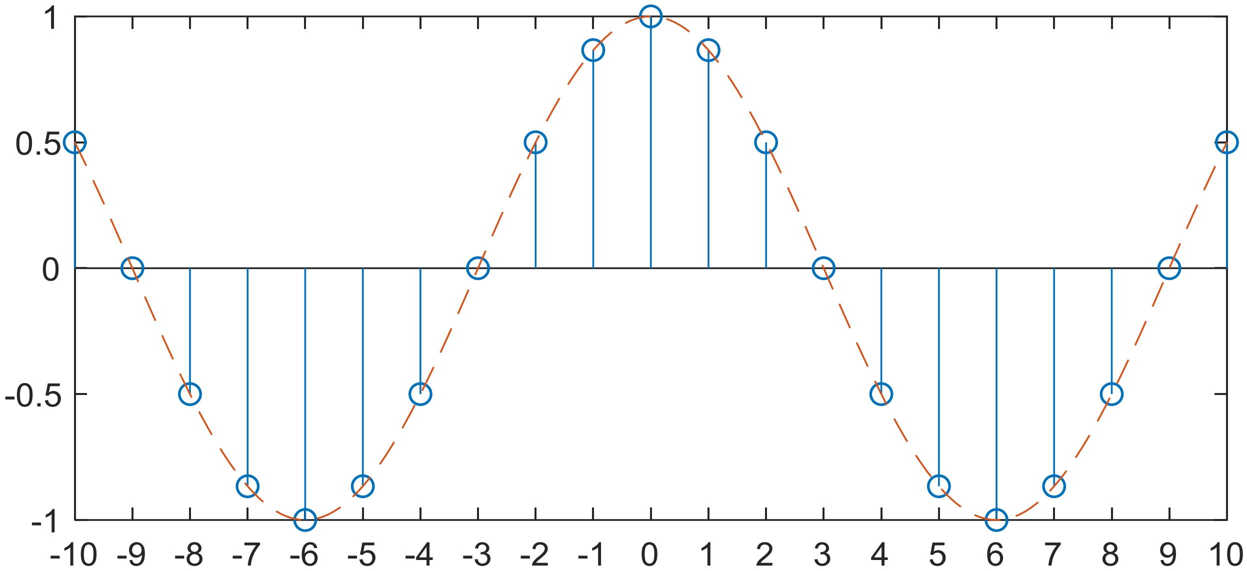
$$\tilde{X}[k] = \sum_{n=0}^{11} [\frac{1}{2} e^{j\frac{2\pi}{12}n} e^{-j\frac{2\pi}{12}kn} + \frac{1}{2} e^{-j\frac{2\pi}{12}n} e^{-j\frac{2\pi}{12}kn}]$$

$$= \frac{1}{2}\sum_{n=0}^{11} e^{-j\frac{2\pi}{12}(k-1)n} + \frac{1}{2}\sum_{n=0}^{11} e^{-j\frac{2\pi}{12}(k-11)n}$$

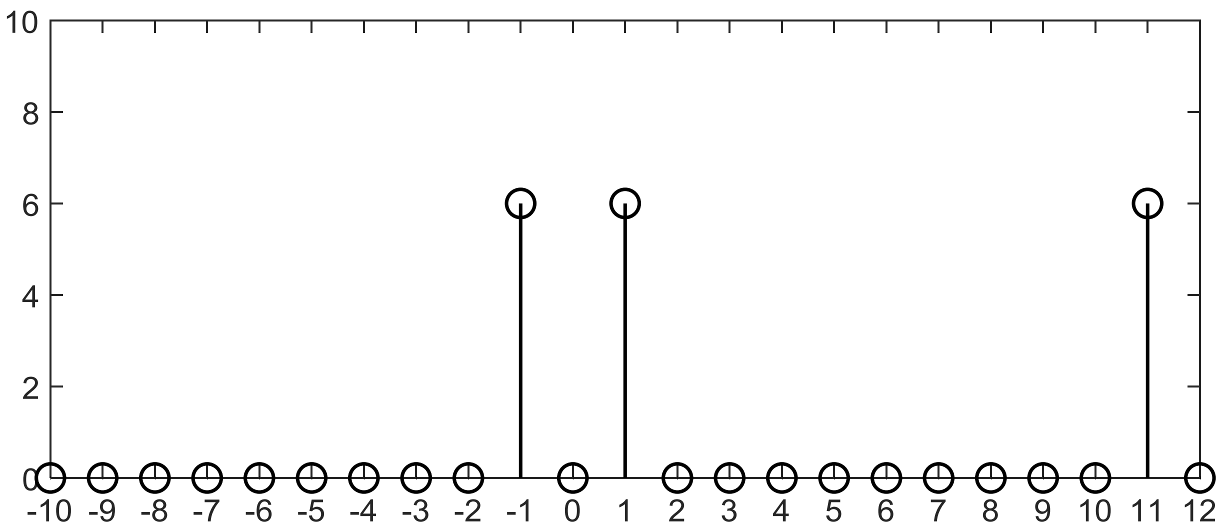
Eq. 5.10

Eq. 5.10

$$\tilde{X}[k] = \frac{1}{2} \times \frac{1 - e^{-j\frac{2\pi}{12}(k-1)\times 12}}{1 - e^{-j\frac{2\pi}{12}(k-1)}} + \frac{1}{2} \times \frac{1 - e^{-j\frac{2\pi}{12}(k-11)\times 12}}{1 - e^{-j\frac{2\pi}{12}(k-11)}} = \begin{cases} 6, k = 1 + 12r \\ 6, k = 11 + 12r \\ 0, others \end{cases}$$



$$\tilde{x}[n] = \cos\left(\frac{\pi}{6}n\right)$$



$$\tilde{X}[k] = \begin{cases} 6, & k = 1 + 12r \\ 6, & k = 11 + 12r \\ 0, & \text{otherwise} \end{cases}$$

5.The Properties of DFS

(1) Linearity

$$a\,\tilde{x}[n]+b\,\tilde{y}[n]\Leftrightarrow a\,\tilde{X}[k]+b\,\tilde{Y}[k]$$

(2) Shift

$$\tilde{x}[n+m]\Leftrightarrow W_N^{-mk}\,\tilde{X}[k]$$ **(Time)**

$$W_N^{nl}\,\tilde{x}[n]\Leftrightarrow \tilde{X}[k+l]$$ **(Frequency)**

(3) Periodic Convolution

(i) $$\tilde{f}[n]=\tilde{x}[n]*\tilde{y}[n]\Leftrightarrow \tilde{F}[k]=\tilde{X}[k]\tilde{Y}[k]$$

(ii) $$\tilde{f}[n]=\tilde{x}[n]\,\tilde{y}[n]\Leftrightarrow \tilde{F}[k]=\frac{1}{N}\tilde{X}[k]*\tilde{Y}[k]$$

where

$$\tilde{f}[n] = \tilde{x}[n] \circledast \tilde{y}[n] = \sum_{l=0}^{N-1} \tilde{x}[l]\, \tilde{y}[n-l]$$

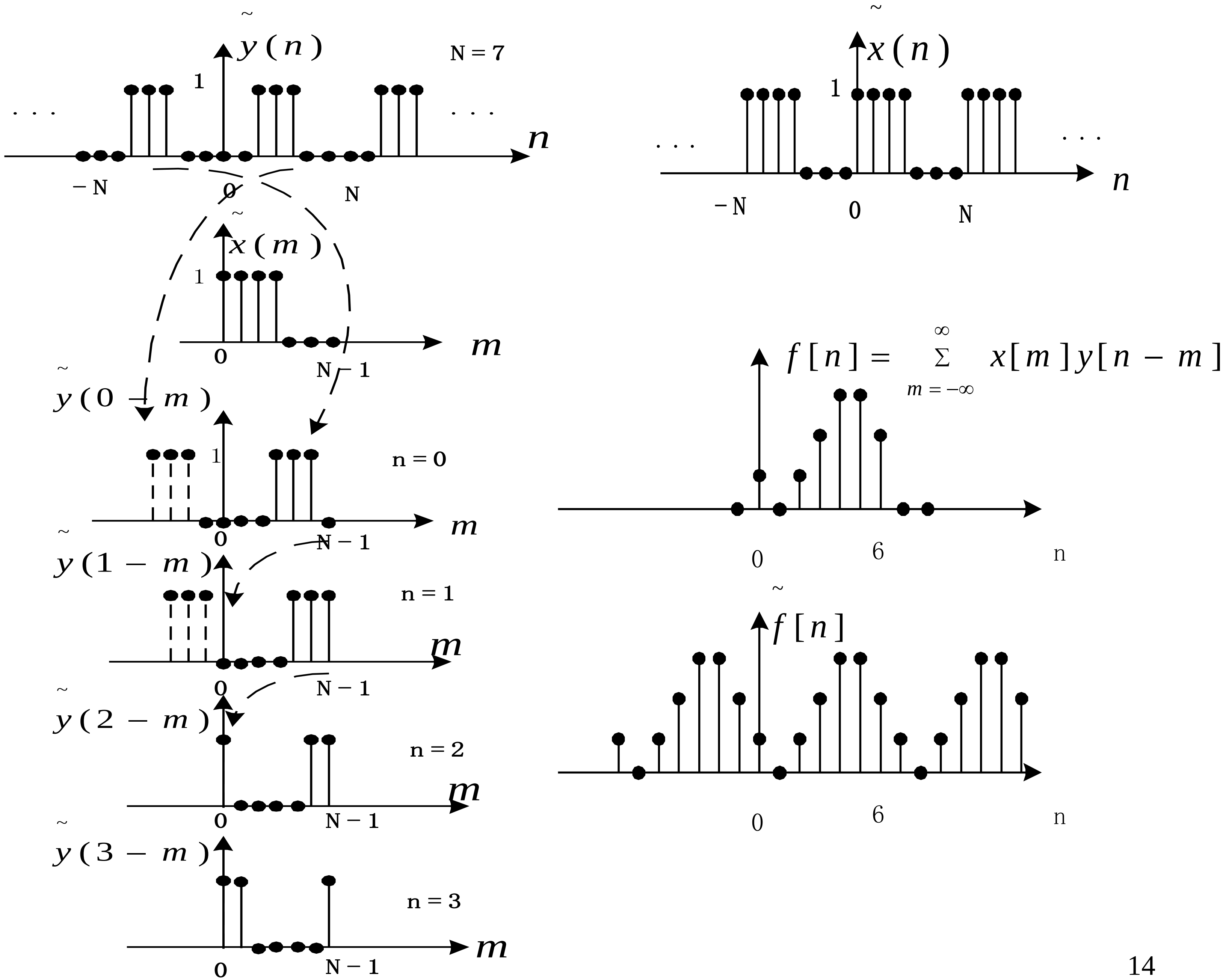
is called as Periodic Convolution.

It has the different computation from Linear Convolution

$$x[n] \circledast y[n] = \sum_{l=-\infty}^{\infty} x[l]\, y[n-l]$$

The summation of Periodic Convolution is computed only in one period interval.

The computation can be illustrated as following Figure.



$\tilde{y}(n)$
N = 7
1
−N
0
N
n
$\tilde{x}(n)$
1
−N
0
N
n
$\tilde{x}(m)$
1
0
N − 1
m
$\tilde{y}(0-m)$
1
n = 0
0
N − 1
m
$\tilde{y}(1-m)$
n = 1
0
N − 1
m
$\tilde{y}(2-m)$
n = 2
0
N − 1
m
$\tilde{y}(3-m)$
n = 3
0
N − 1
m
$f[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m]$
0
6
n
$\tilde{f}[n]$
0
6
n

5.2 DFT

5.2.1 The Definition of DFT

The relation, between a finite-length sequence $x[n]$ defined for $0 \le n \le N-1$,and its uniformly sampled values

$$X[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}}, \ 0 \le k \le N-1$$

on the ω-axis between $0 \le \omega \le 2\pi$, $X[k]$ is called as DFT.

$$X[k] = X(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad k:[0, N-1] \qquad \textbf{...(5.13)}$$

$$\{\psi[k,n] = e^{j(\frac{2\pi}{N})kn}\} \quad (k = 0,1,2,..., N-1)$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi}{N}kn}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$$

$$n:[0,N-1] \qquad \textbf{...(5.14)}$$

The Eq. (5.13) and Eq. (5.14) are represented as

$$x[n]\Leftrightarrow X[k]$$

or

$$X[k]=DFT\{x[n]\} \qquad x[n]=IDFT\{X[k]\}$$

Called as *N*-point DFT.

It should be noted that $X[k]$ **is also a finite-length sequence.**

Example: Compute *N*-point (*N*=12) DFT of the following sequence:

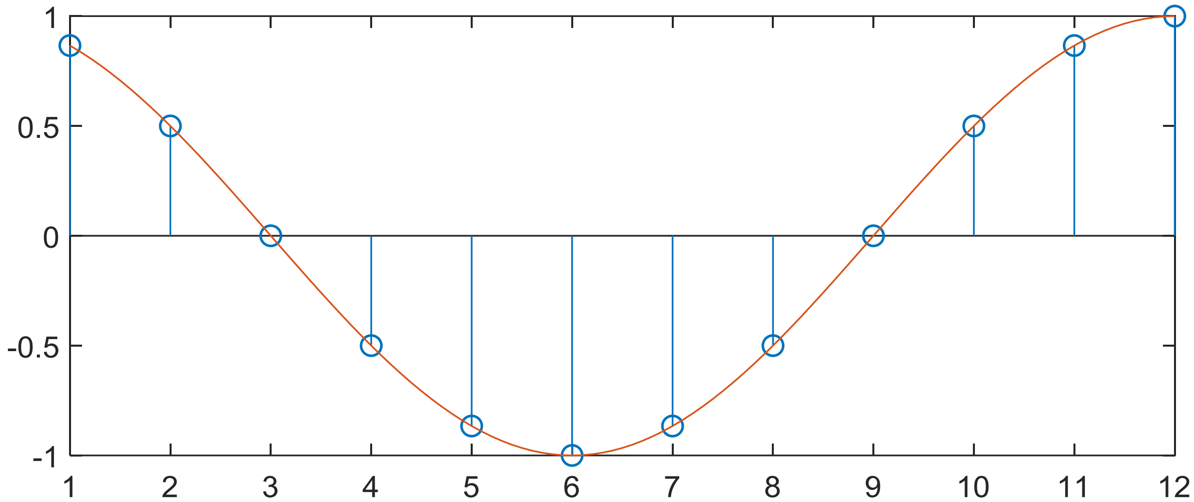
$$x[n] = \cos\frac{\pi}{6} n R_N[n],\ R_N[n] = \begin{cases} 1,\ 0 \le n \le N-1 \\ 0,\ \text{otherwise} \end{cases}$$

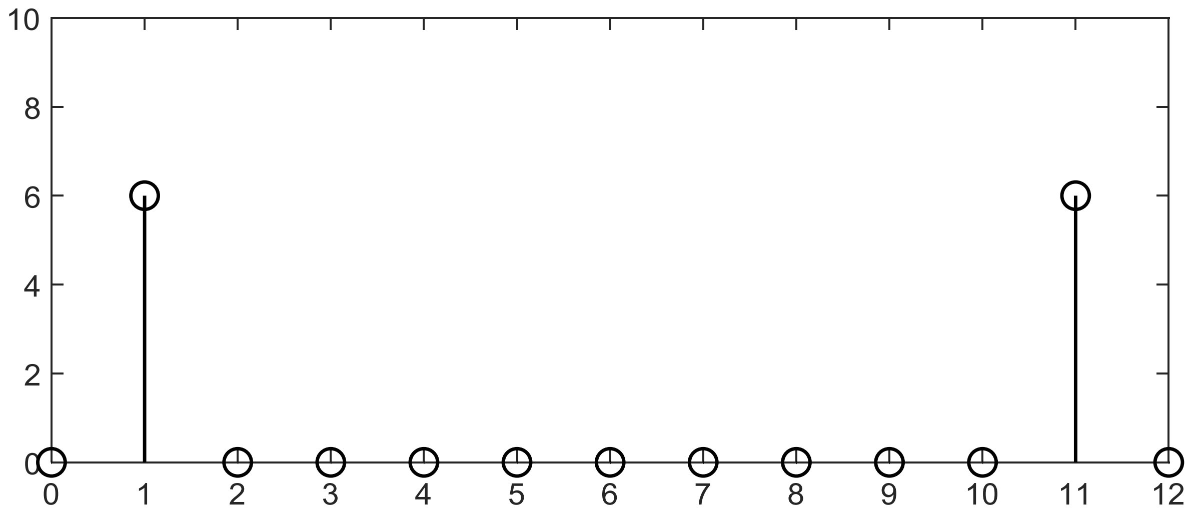
Solution:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{11} \frac{1}{2}\left[e^{j\frac{2\pi}{12}n} + e^{-j\frac{2\pi}{12}n}\right] e^{-j\frac{2\pi}{12}kn}$$

$$= \begin{cases} 6, k = 1,\ 11 \\ 0, others \end{cases} \qquad 0 \le k \le (N-1=11)$$

$$x[n] = \cos\frac{\pi}{6} n R_N[n],\ N = 12$$





$$X[k] = \begin{cases} 6, & k = 1, 11 \\ 0, & \text{otherwise} \end{cases}$$

Example: **Consider the length-*N* sequence defined for $0 \le n \le N-1$**

$$g[n] = \cos\left(\frac{2\pi rn}{N}\right), \quad 0 \le r \le N-1$$

Using a trigonometric identity we can write

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rN} + W_N^{rN}\right)$$

The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \le k \le N-1$$

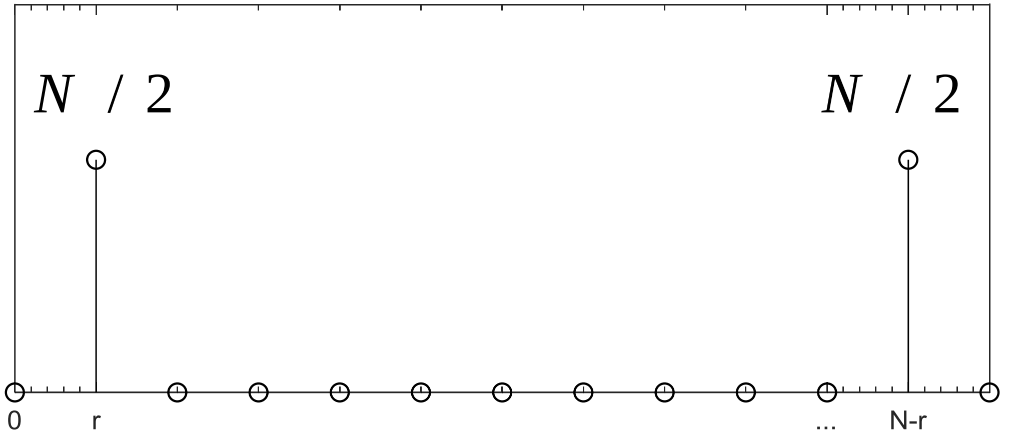
Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-r)n} = \begin{cases} N, & \text{for } k-r = lN,\ l \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

We get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N-r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k \le N-1$$



5.2.2 Matrix Relations of DFT

$$\mathbf{X} = D_N \mathbf{x}$$

$$\mathbf{x} = [x[0]\ x[1] \ldots x[N-1]]^T$$

$$\mathbf{X} = [X[0]\ X[1] \ldots X[N-1]]^T$$

$$D_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)\times(N-1)} \end{bmatrix}$$

$$\mathbf{x} = D_N^{-1} \mathbf{X}$$

5.2.3 DFT Computation Using MATLAB

Example 5.3

$$u[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

U=fft(u, M) **(*M*=*N*=8)**

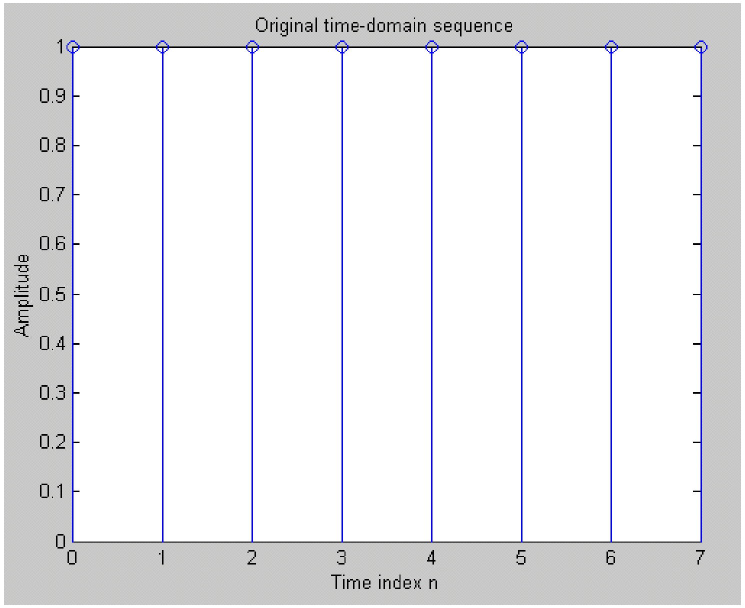


Figure 5.1 (a)

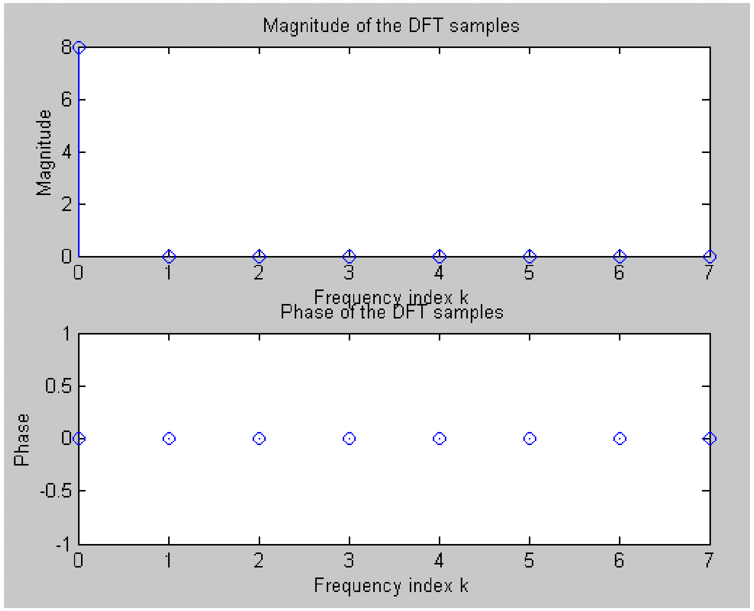
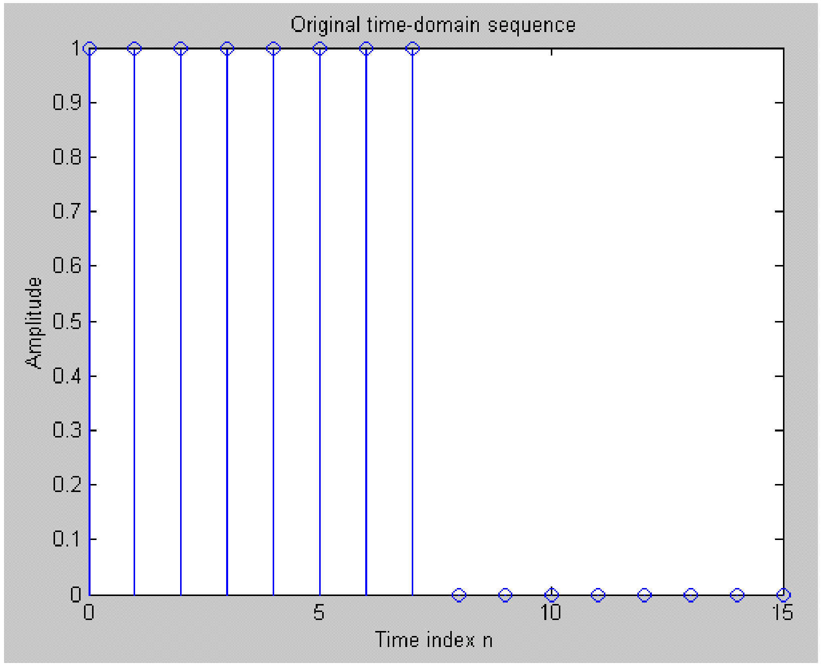


Figure 5.1 (b)

U=fft(u, M) (*N*=8, *M*=16)



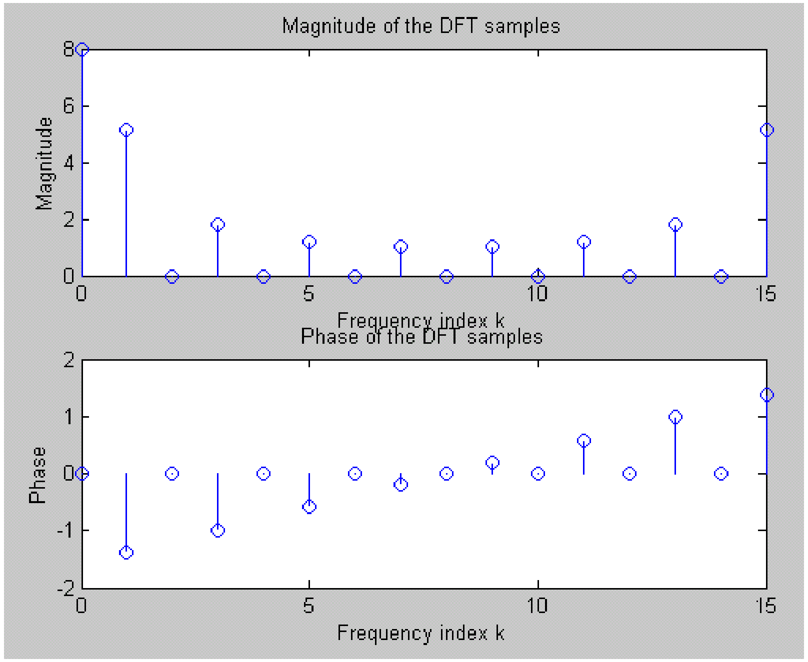
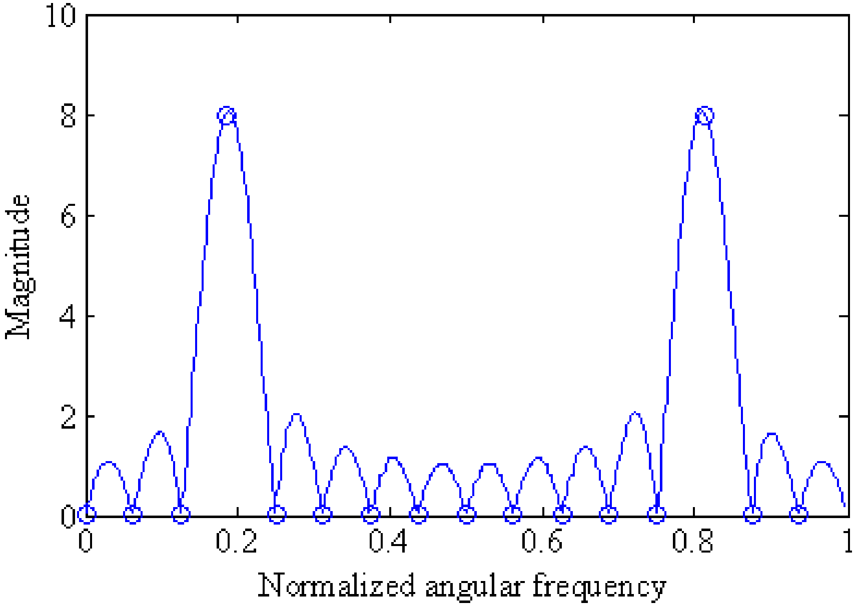


Figure 5.1 (b)

- **<u>Example</u> - Program 5_3.m can be used to compute the DFT and the DTFT of the sequence**

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

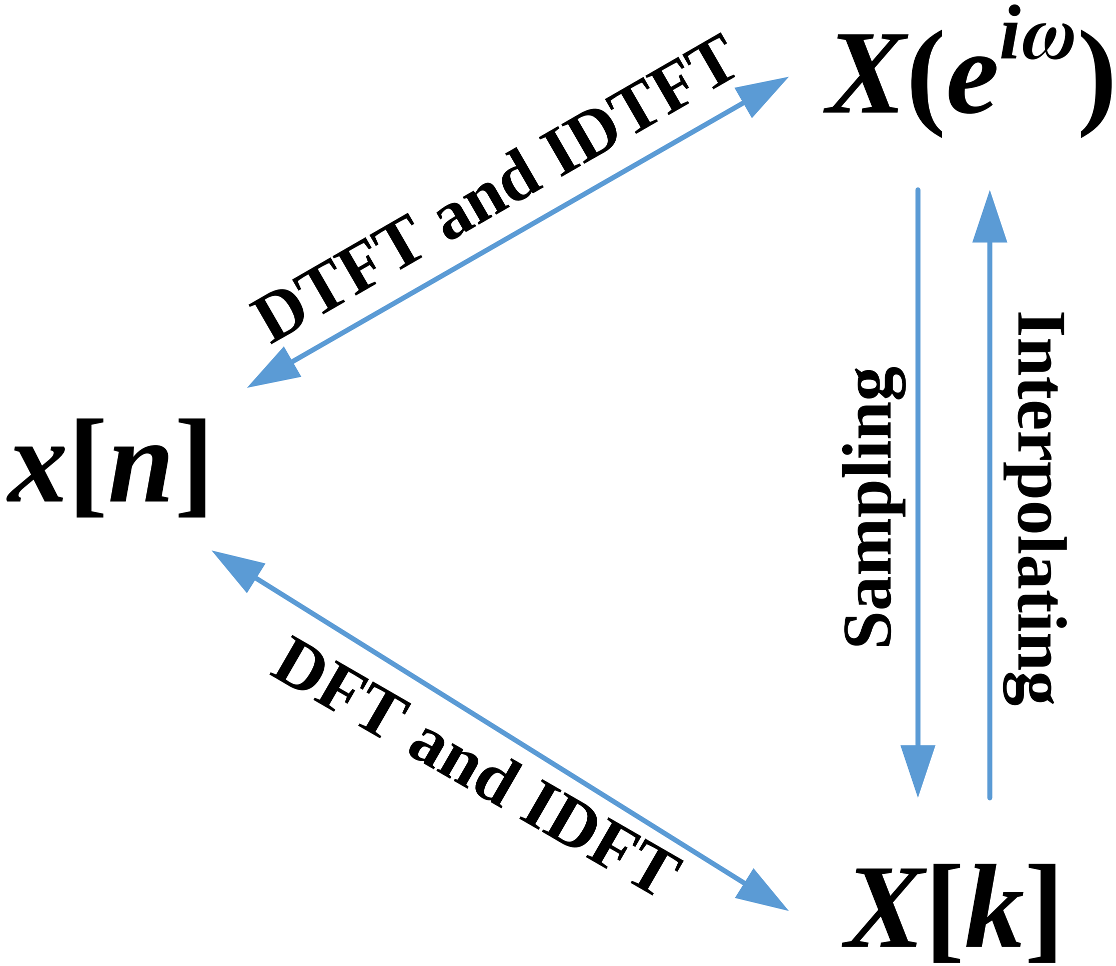
as shown below



○ indicates DFT samples

Figure 5.3

5.3 Relation between DFT and DTFT



5.3 Relation between DFT and DTFT

5.3.1 Relation with DTFT

A finite-length sequence *x*[*n*] has DTFT

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-jn\omega}$$

Period: 2π

It has *N*-point-DFT

$$X[k] = X(e^{j\omega})\Big|_{\boxed{\omega = 2\pi k/N}} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N},$$

$$0 \le k \le N-1$$

5.3.2 Numerical Computation of the DTFT Using the DFT

- **A practical approach to the numerical computation of the DTFT of a finite-length sequence $x[n]$**
- **We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k/M$, $0 \le k \le M-1$, where $M >> N$:**

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$$

Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n]e^{-j2\pi kn/M}$$

- **Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$.**
- **The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2.**
- **The function freqz employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$.**

5.3.3 DTFT from DFT by interpolation

- **The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points $\omega = \omega_k = 2\pi k/N$, $0 \leq k \leq N-1$.**
- **Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$.**

- **Thus**

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi k}{N}\right) n}}_{S}$$

- **To develop a compact expression for the sum *S*, let *r* = $e^{-j(\omega - 2\pi k/N)}$**

$$\text{Then} \quad S = \sum_{n=0}^{N-1} r^n$$

From the above

$$rS = \sum_{n=1}^{N} r^n = 1 + \sum_{n=1}^{N} r^n + r^N - 1$$

$$= \sum_{n=0}^{N-1} r^n + r^N - 1 = S + r^N - 1$$

- **Or, equivalently,** $S - rS = (1-r)S = 1-r^N$

Hence

$$S = \frac{1-r^N}{1-r} = \frac{1-e^{-j(\omega N - 2\pi k)}}{1-e^{-j[\omega-(2\pi k/N)]}}$$

$$= \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

Therefore,

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Phi(\omega - \frac{2\pi k}{N}), \text{ where } \Phi(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right)}{N\sin\left(\frac{\omega}{2}\right)} e^{-j\omega[(N-1)/2]}$$

DTFT **DFT**

Thus, $X(e^{j\omega})$ is interpolated.

5.3.4 Sampling the DTFT

- **Consider a sequence $x[n]$ with DTFT $X(e^{j\omega})$**
- **We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k/N$, $0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$.**
- **These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$.**

- **Now**

$$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}$$

Thus $Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$

$$= \sum_{l=-\infty}^{\infty} x[l]e^{-j2\pi kl/N} = \sum_{l=-\infty}^{\infty} x[l]W_N^{kl}$$

An IDFT of $Y[k]$ yields

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]W_N^{-kn}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=-\infty}^{\infty} x[l]W_N^{kl}W_N^{-kn}$$

$$= \sum_{l=-\infty}^{\infty} x[l][\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-l)}]$$

Making use of the identity

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-l)} = \begin{cases} 1, & for \quad l = n + mN \\ 0, & otherwise \end{cases}$$

$$\sum_{m=-\infty}^{+\infty} \delta[l-(n+mN)]$$

- **We arrive at the desired relation**

$$y[n] = \sum_{m=-\infty}^{\infty} x[n + mN], \qquad 0 \le n \le N - 1$$

Why?

Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants.

Observing the sum only for the interval

$$\mathbf{0 \le n \le N-1}$$

- **To apply**

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros.

Thus if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$.

(Example: M=1, N=2)

- **If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$**

(Example: M=6, N=4)

- ***Example*: Let $\{x[n]\}=\{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$**

$\uparrow$

By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \leq k \leq 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

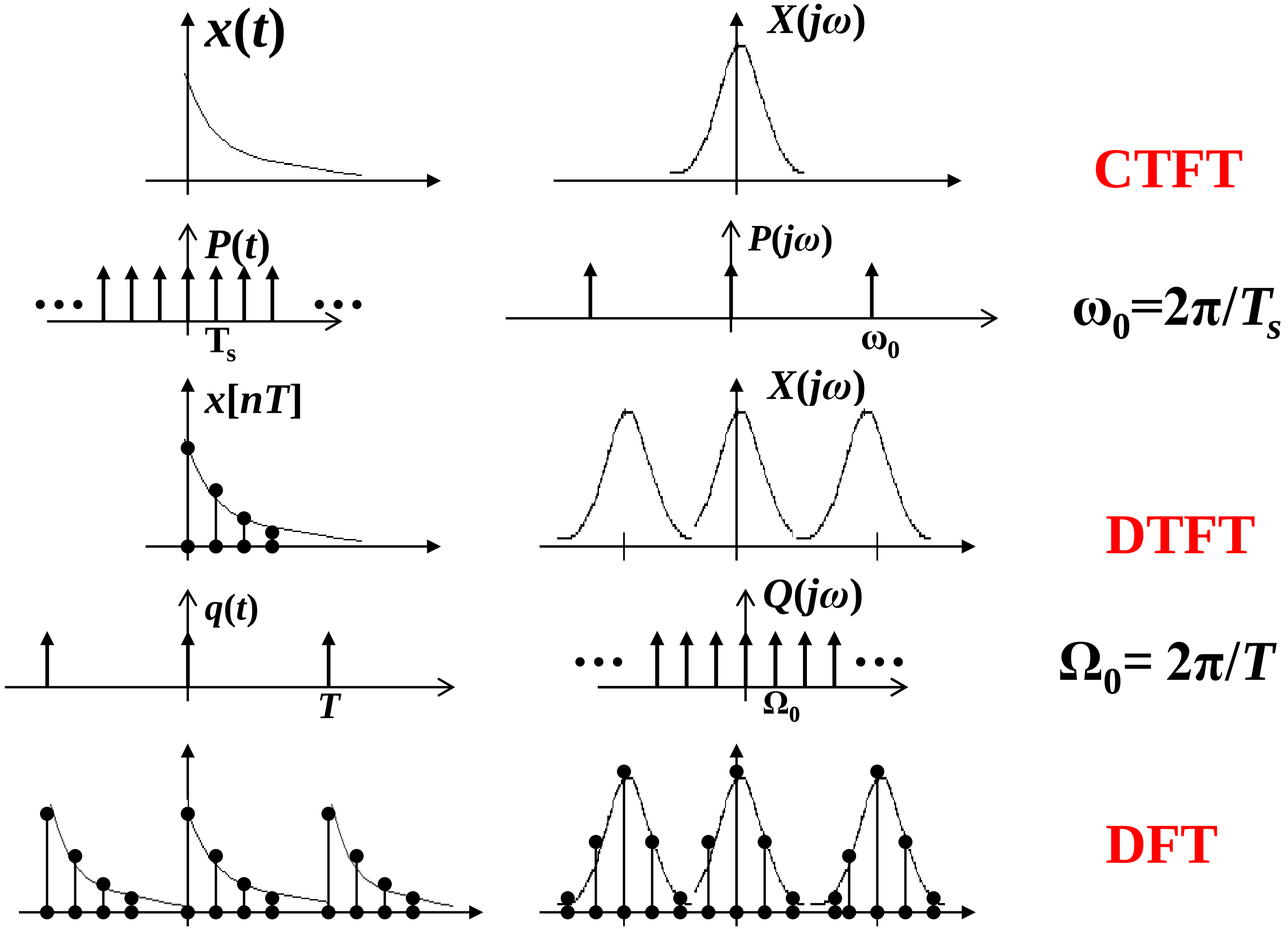
- $y[n] = x[n] + x[n+4] + x[n-4], \; 0 \leq n \leq 3$
- **i. e.**

$$\{y[n]\} = \{4 \quad 6 \quad 2 \quad 3\}$$

$$\uparrow$$

$\{x[n]\}$ cannot be recovered from $\{y[n]\}$

CTFT/DTFT/DFT



5.4 Operations on Finite-length Sequences

5.4.1 Circular Shift of a Sequence

- **This property is analogous to the time-shifting property of the DTFT , but with a subtle difference**
- **Consider length-N sequences defined for**

$$0 \leq n \leq N-1$$

- **Sample values of such sequences are equal to zero for values of $n < 0$ and**

$n \geq N$

- **If $x[n]$ is such a sequence, then for any arbitrary integer n_0 , the shifted sequence**

$$x_1[n] = x[n - n_0]$$

is no longer defined for the range $0 \le n \le N-1$

- **We thus need to define another type of a shift that will always keep the shifted sequence in the range $0 \le n \le N-1$**

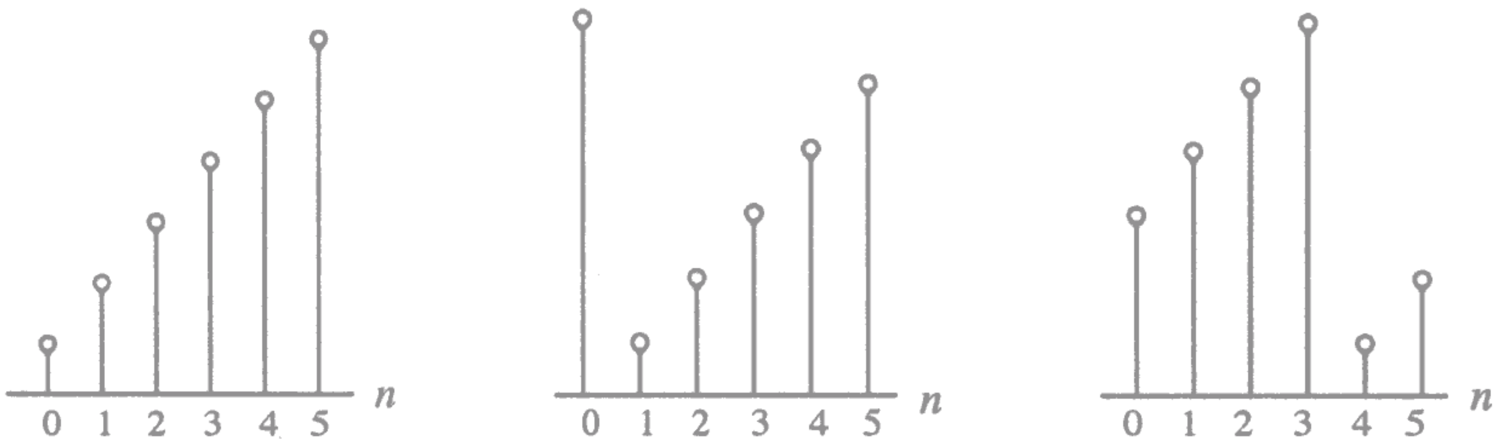
- **The desired shift, called the circular shift, is defined using a modulo operation:**

$$x_c[n] = x[\langle n - n_o \rangle_N]$$

For $n_0 > 0$ (right circular shift), the above equation implies

$$x_c[n] = \begin{cases} x[n - n_o], & \text{for} \quad n_o \le n \le N - 1 \\ x[N - n_o + n], & \text{for} \quad 0 \le n < n_o \end{cases}$$

- **Illustration of the concept of a circular shift**



$x[n]$ $\quad$ $x\left[\langle n-1\rangle_6\right] = x\left[\langle n+5\rangle_6\right]$ $\quad$ $x\left[\langle n-4\rangle_6\right] = x\left[\langle n+2\rangle_6\right]$

- **As can be seen from the previous figure, a right circular shift by n_0 is equivalent to a left circular shift by $N-n_0$ sample periods**
- **A circular shift by an integer number greater than N is equivalent to a circular shift by $\langle n_0 \rangle_N$**

$$x[<-2>_6] = x[<4>_6]$$

$$<-2>_6 = 4 \qquad \mathbf{(-2 = -1 \times 6 + 4)}$$

$x[n]$ has length N, n: $[0, N-1]$.

Periodization: $$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

$$\tilde{x}[n] = x[\langle n \rangle_N]$$

$[\langle n \rangle_N]$ means module N operation on n.

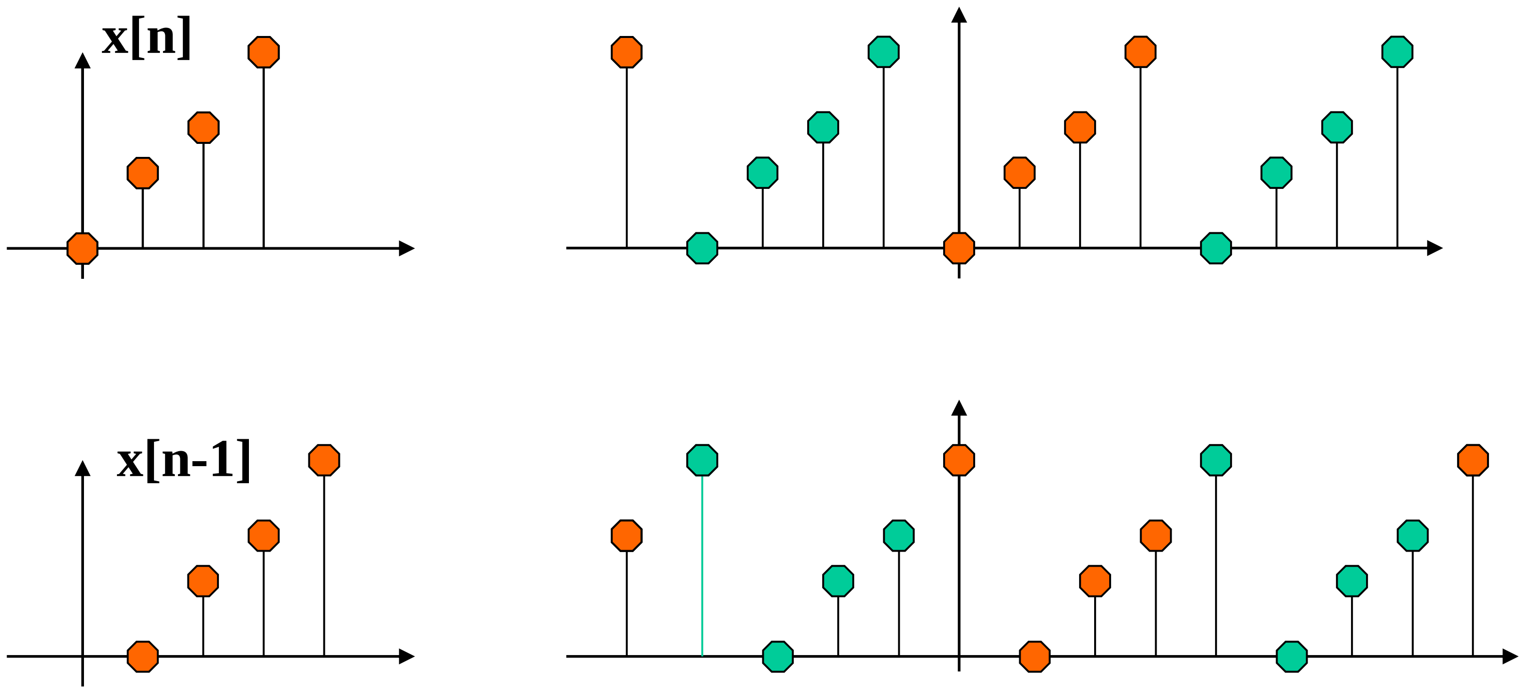
Example: Supposing period is $N = 6$, then

$\tilde{x}[19] = \tilde{x}[1]$ **, or** $x[\langle 19 \rangle_6] = x[\langle 1 \rangle_6]$ **.**

For $\langle 19 \rangle_6 = 1$ **(19=3×6+1)**

Non-circular shift

circular shift



5.4.2 Circular Convolution

- **This operation is analogous to linear convolution, but with a subtle difference**
- **Consider two length-N sequences, $g[n]$ and $h[n]$, (n:[0, $N-1$]) respectively**
- **Their linear convolution results in a length-($2N-1$) sequence $y_L[n]$ given by**

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \; 0 \le n \le 2N-2$$

$$y_L[n] = 0, \;\; for \; n < 0, or \quad n > 2N-2$$

- **Circular convolution, is defined by**

$$y_C[n] = \sum_{m=0}^{N-1} g[m]\,h[\langle n-m \rangle_N], \qquad 0 \le n \le N-1$$

- **$y_C[n]$---a length-N sequence. we need to define a circular time-reversal, and then apply a circular time-shift.**

- **Since the operation defined involves two length-*N* sequences, it is often referred to as an *N*-point circular convolution, denoted as**

$$y[n] = g[n] \circledN h[n]$$

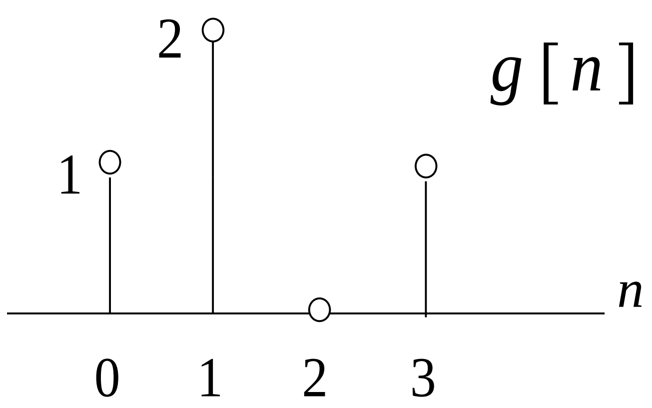
- **The circular convolution is commutative, i.e.**

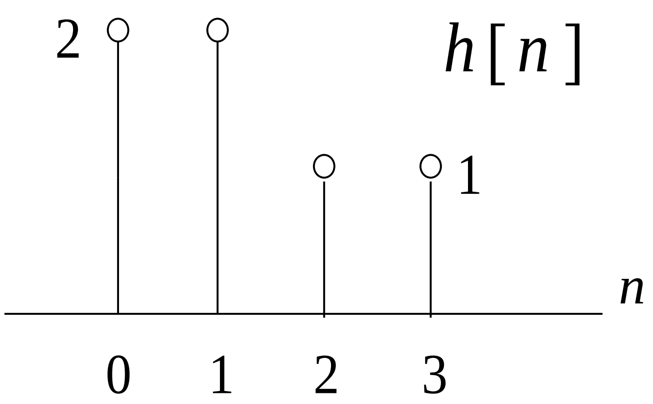
$$g[n] \circledN h[n] = h[n] \circledN g[n]$$

- **<u>Example</u> – Consider the two length-4 sequences:**

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

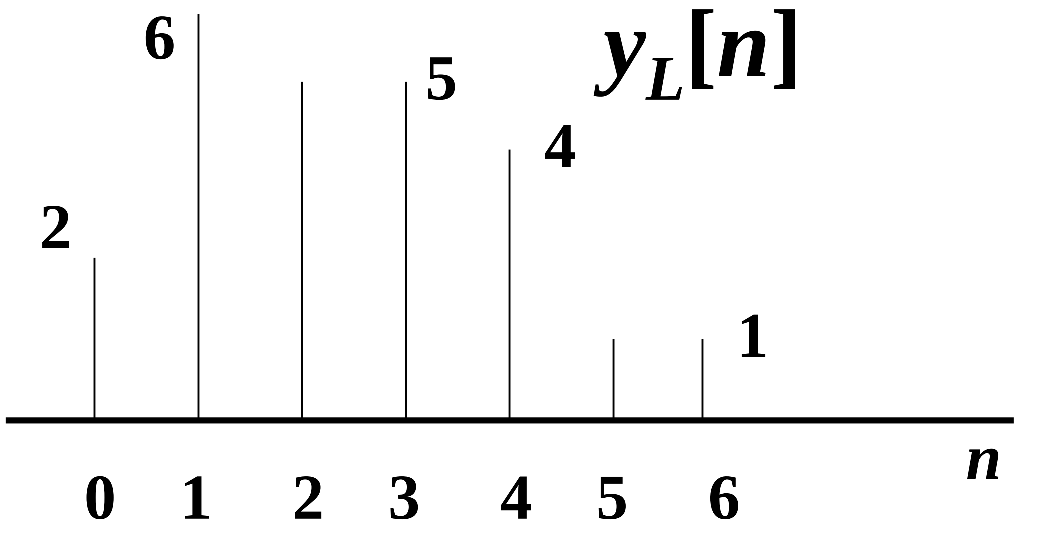
as sketched below





$y_L[n]$ ---- linear convolution of $g[n]$ and $h[n]$

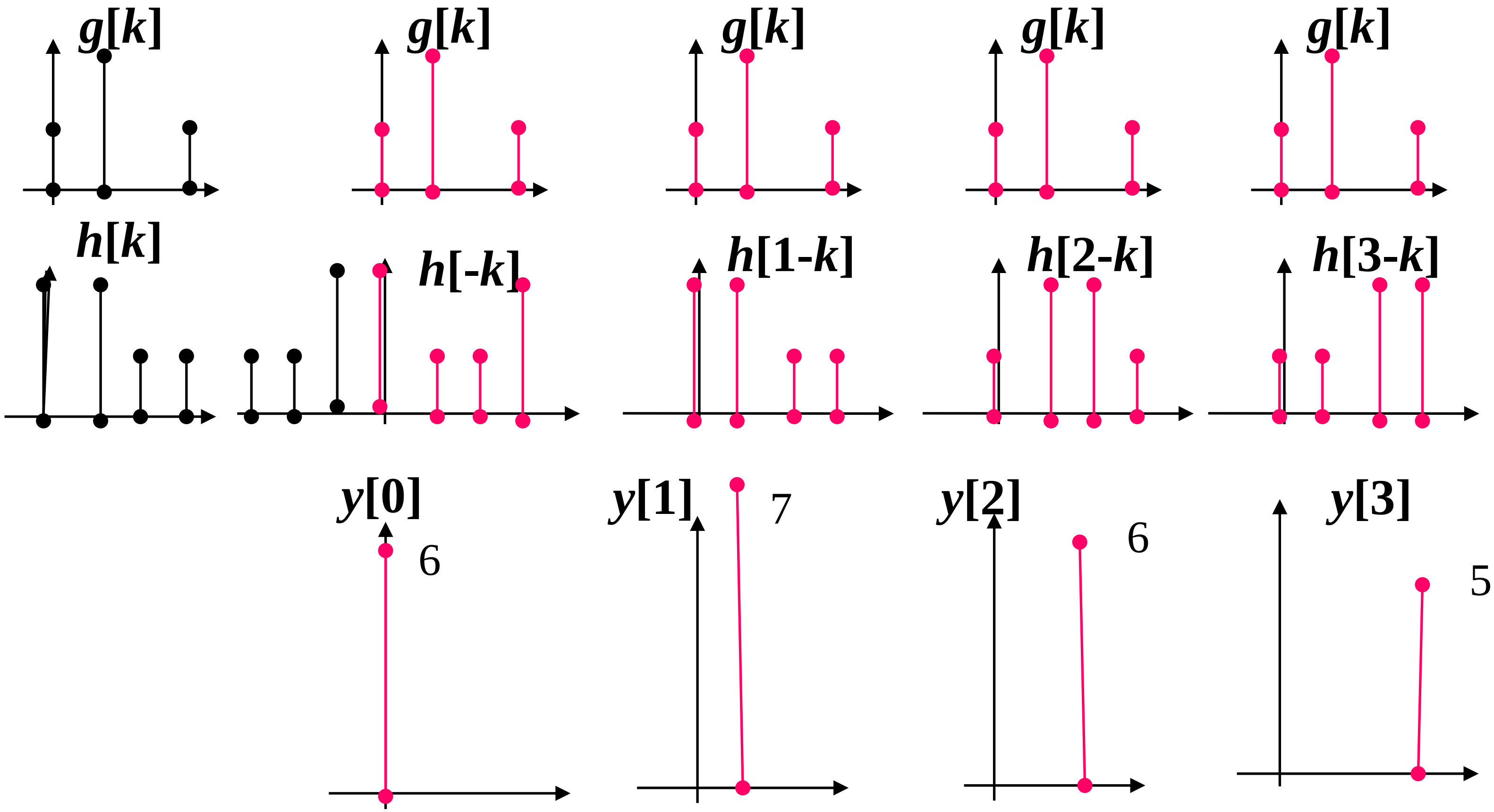
$$y_L[n] = g[n] * h[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m]$$



Next slide shows the 4-point circular convolution of $g[n]$ and $h[n]$

$$g[n] = \delta[n] + 2\delta[n-1] + \delta[n-3]$$

$$h[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \delta[n-3]$$



$$y[n] = 6\delta[n] + 7\delta[n-1] + 6\delta[n-2] + 5\delta[n-3]$$

- **The result is a length-4 sequence $y_C[n]$ given by**

$$y_C[n] = g[n] \,④\, h[n] = \sum_{m=0}^{3} g[m]\,h[\langle n-m\rangle_4],$$
$$0 \le n \le 3$$

From the above we observe

$$y_C[0] = \sum_{m=0}^{3} g[m]\,h[\langle -m\rangle_4]$$
$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$
$$= (1\times 2) + (2\times 1) + (0\times 1) + (1\times 2) = 6$$

- **Likewise**

$$y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m\rangle_4]$$

$$= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= (1\times 2) + (2\times 2) + (0\times 1) + (1\times 1) = 7$$

$$y_C[2] = \sum_{m=0}^{3} g[m]h[\langle 2-m\rangle_4]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]$$

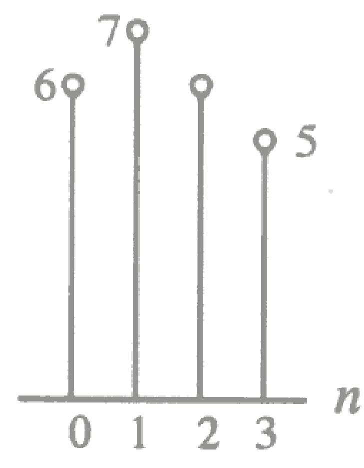
$$= (1\times 1) + (2\times 2) + (0\times 2) + (1\times 1) = 6$$

$$y_C[3] = \sum_{m=0}^{3} g[m]\,h[\langle 3-m \rangle_4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

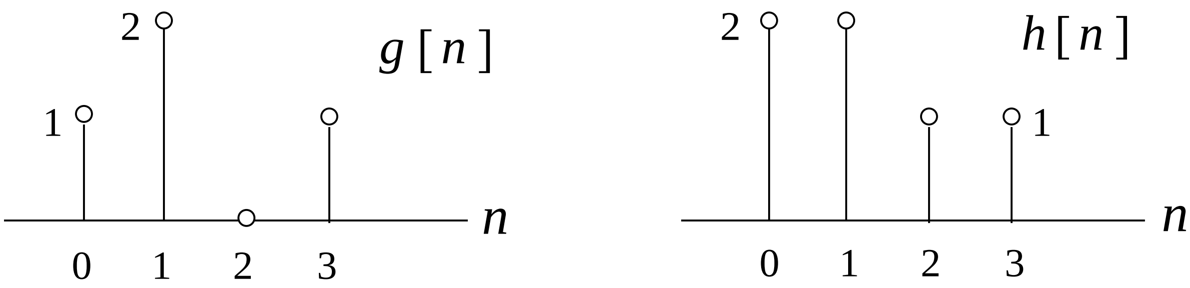
$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

- **The circular convolution can also be computed using a DFT-based approach**



$y_C[n]$

- **Example:** **Consider the two length-4 sequences repeated below for convenience:**



The 4-point DFT $G[k]$ of $g[n]$ is given by

$$
\begin{aligned}
G[k] &= g[0] + g[1]e^{-j2\pi k/4} \\
&\quad + g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4} \\
&= 1 + 2e^{-j\pi k/2} + e^{-j3\pi k/2}, \quad 0 \le k \le 3
\end{aligned}
$$

- **Therefore**

$$G[0] = 1 + 2 + 1 = 4,$$

$$G[1] = 1 - j2 + j = 1 - j,$$

$$G[2] = 1 - 2 - 1 = -2,$$

$$G[3] = 1 + j2 - j = 1 + j$$

Likewise,

$$H[k] = h[0] + h[1]e^{-j2\pi k/4}$$

$$+ h[2]e^{-j4\pi k/4} + h[3]e^{-j6\pi k/4}$$

$$= 2 + 2e^{-j\pi k/2} + e^{-j\pi k} + e^{-j3\pi k/2}, \quad 0 \le k \le 3$$

- **Hence,**

$$H[0] = 2 + 2 + 1 + 1 = 6,$$

$$H[1] = 2 - j2 - 1 + j = 1 - j,$$

$$H[2] = 2 - 2 + 1 - 1 = 0,$$

$$H[3] = 2 + j2 - 1 - j = 1 + j$$

The two 4-point DFTs can also be computed using the matrix relation given earlier

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = D_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = D_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

D_4 is the 4-point DFT matrix

- **If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then we observe**

$$Y_C[k]=G[k]H[k], \qquad 0 \le k \le 3$$

Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

- **A 4-point IDFT of $Y_C[k]$ yields**

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4} \mathrm{D}_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

- **Example: Now let us extended the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.**

$$g_e[n] = \begin{cases} g[n], & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \leq n \leq 3 \\ 0, & 4 \leq n \leq 6 \end{cases}$$

- **We next determine the 7-point circular convolution of $y_c[n]=g_e[n]$ ⑦ $h_e[n]$:**

$$y_C[n] = \sum_{m=0}^{6} g_e[m] h_e[\langle n-m \rangle_7], \; 0 \le n \le 6$$

From the above

$$y_c[0]=g_e[0]h_e[0]+ g_e[0]h_e[0]+ g_e[0]h_e[0]+ g_e[0]h_e[0]+ g_e[0]h_e[0]+ g_e[0]h_e[0]$$

$$=g[0]h[0]=1\times2=2$$

- **Continuing the process we arrive at**

$$y_c[1] = g[0]h[1] + g[1]h[0] = (1\times2)+(2 \times 2)=6$$

$$y_c[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$
$$=(1\times1)+(2\times2)+(0\times2)=5$$

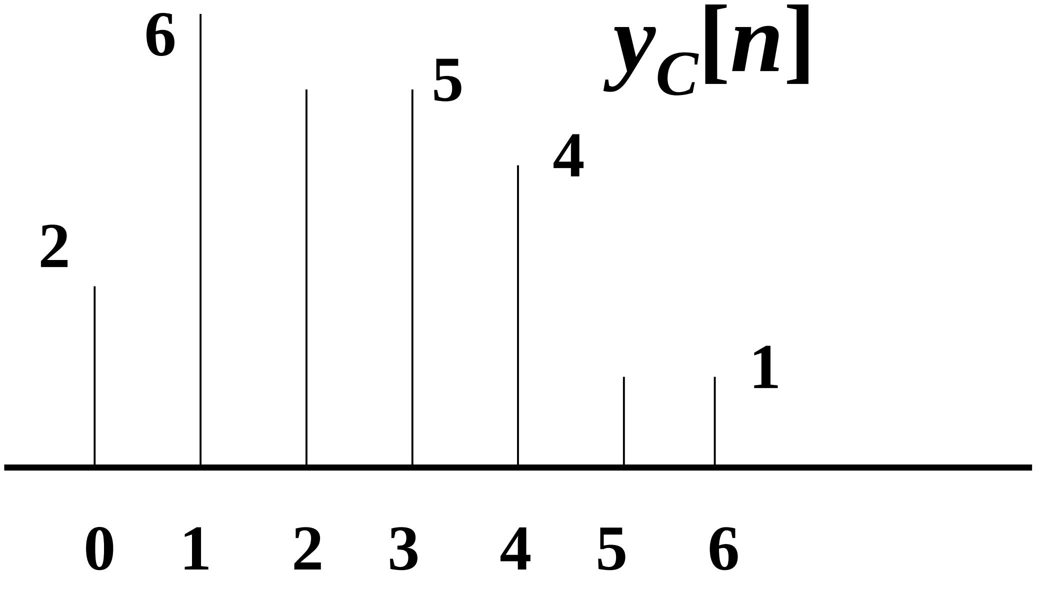
$$y_c[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$
$$=(1\times1)+(2\times1)+(0\times2)+(1\times2)= 5$$

$$y_c[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$
$$=(2\times1)+(0\times1)+(1\times2)= 4$$

$$y_c[5] = g[2]h[3] + g[3]h[2]=(0\times1)+(1\times1)=1$$

$$y_c[6] = g[3]h[3]=(1\times1) =1$$

- **As can be seen from the above that $y_c[n]$ is precisely the sequence $y_L[n]$ obtained by a linear convolution of $g[n]$ and $h[n]$**



- **The *N*-point circular convolution can be written in matrix form as**

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

•**Note: The elements of each diagonal of the *N*×*N* matrix are equal.**

•**Such a matrix is called a circulant matrix.**

Note: The circular convolution can be Implemented in tabular method listed in textbook P249-250. Read it by yourself!

5.4.3 Circular Convolution and Linear Convolution (appedned)

- **Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively**
- **Then $y_L[n] = g[n] \circledast h[n]$**
- **Length : $L = N+M-1$**

Appending each with three zero-valued samples, i.e.

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

Then $y_L[n]=g[n] \circledast h[n]=y_C[n]=g_e[n] \textcircled{L} h_e[n]$

Prove it by yourself!

5.5 Classifications of Finite-Length Sequences

5.5.1 Based on Conjugate Symmetry

If $x[n]$ is a causal N-point (n:[0, $N-1$]) sequence,

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \frac{1}{2}\left\{x[n] + x^*\left[\left\langle -n \right\rangle_N\right]\right\}, \quad n:[0, N-1]$$

(circular conjugate-symmetric part)

$$x_{ca}[n] = \frac{1}{2}\left(x[n] - x^*[\langle -n \rangle_N]\right), \quad n : [0, N-1]$$

(circular conjugate-antisymmetric part)

Example 5.10 **A length-4 complex sequence (n: [0, 3])**

$\{u[n]\} = \{1+j4, -2+j3, 4-j2, -5-j6\}$

$\{u^*[n]\} = \{1-j4, -2-j3, 4+j2, -5+j6\}$

$\{u^*[\langle -n \rangle_4]\} = \{1-j4, -5+j6, 4+j2, -2-j3\}$

Note: $u^*[\langle -0 \rangle_4] = u^*[0] = 1-j4$

$\{u_{cs}[n]\}$?
$\{u_{ca}[n]\}$?

$\{u_{cs}[n]\} = \{1, -3.5+j4.5, 4, -3.5-j4.5\}$

$\{u_{ca}[n]\} = \{j4, 1.5-j1.5, -j2, -1.5-j1.5\}$

Note: $u_{cs}[0]$ **is real number,**
$u_{ca}[0]$ **is pure imaginary.**

5.5.2 Based on Geometric Symmetry

If $x[n]$ is a causal N-point sequence, n:[0, N-1]

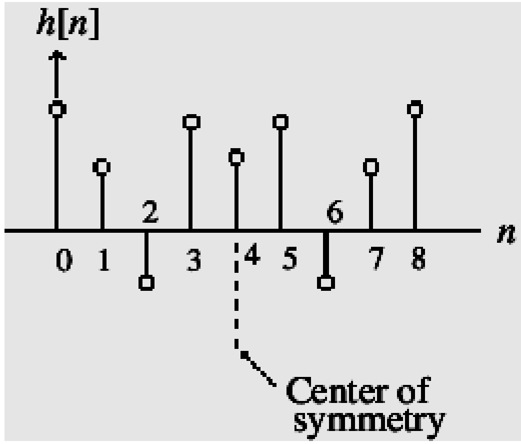
$$x[n] = x[N-1-n]$$

Called as symmetric.

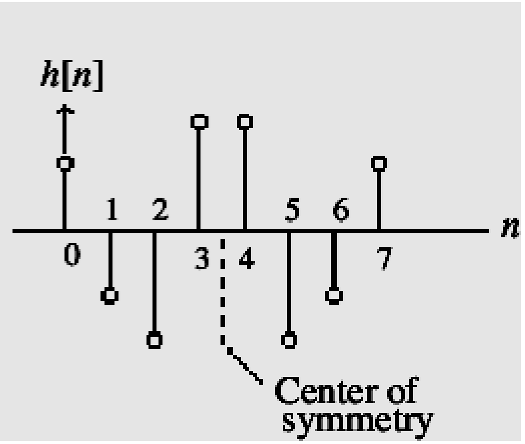
If $x[n] = -x[N-1-n]$

Called as antisymmetric.

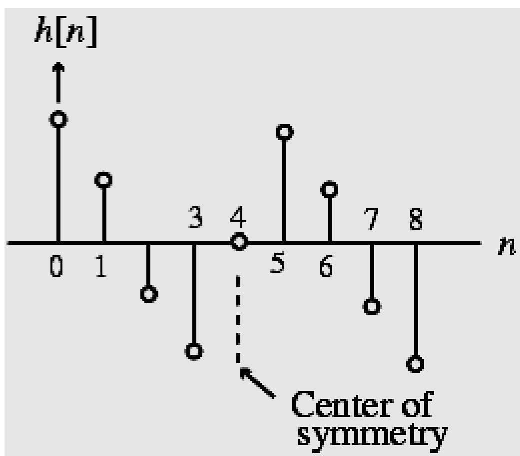
There are 4 types of such sequences.



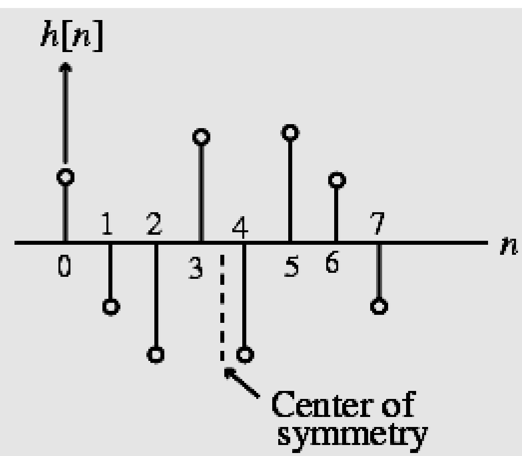
Type 1: $N = 9$



Type 2: $N = 8$



Type 3: $N = 9$



Type 4: $N = 8$

Type 1: $x[n] = x[N-1-n]$ with Odd Length

Example:
***N*=3**

$$X(e^{j\omega}) = e^{-j(N-1)\omega/2}\{x[\frac{N-1}{2}] +$$

$$2\sum_{n=1}^{(N-1)/2} x[\frac{N-1}{2} - n]\cos[\omega n]\}$$

?

$$X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = e^{-j(N-1)\pi k/N}\{x[\frac{N-1}{2}] +$$

$$2\sum_{n=1}^{(N-1)/2} x[\frac{N-1}{2} - n]\cos\left[\frac{2\pi kn}{N}\right]\}$$

Type 2: $x[n] = x[N-1-n]$ **with Even Length**

$$X(e^{j\omega}) = e^{-j(N-1)\omega/2}\{$$

$$2\sum_{n=1}^{N/2} x[\frac{N}{2}-n]\cos\left[\omega(n-\frac{1}{2})\right]\}$$

$$X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = e^{-j(N-1)\pi k/N}\{$$

$$2\sum_{n=1}^{N/2} x[\frac{N}{2}-n]\cos\left[\frac{\pi k(2n-1)}{N}\right]\}$$

Type 3: $x[n] = -x[N-1-n]$ with Odd Length

$$X(e^{j\omega}) = je^{-j(N-1)\omega/2}\{$$

$$2\sum_{n=1}^{(N-1)/2} x[\frac{N-1}{2} - n]\sin[\omega n]\}$$

$$X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = je^{-j(N-1)\pi k/N}\{$$

$$2\sum_{n=1}^{(N-1)/2} x[\frac{N-1}{2} - n]\sin\left[\frac{2\pi kn}{N}\right]\}$$

Type 4: $x[n] = -x[N-1-n]$ with Even Length

$$X(e^{j\omega}) = je^{-j(N-1)\omega/2}\{$$

$$2\sum_{n=1}^{N/2} x[\frac{N}{2}-n]\sin\left[\omega(n-\frac{1}{2})\right]\}$$

$$X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = je^{-j(N-1)\pi k/N}\{$$

$$2\sum_{n=1}^{N/2} x[\frac{N}{2}-n]\sin\left[\frac{\pi k(2n-1)}{N}\right]\}$$

5.6 DFT Properties (5.6—5.7)

- **Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications**
- **A summary of the DFT properties are given in tables in the following slides**

Table5.1: Symmetry properties of the DFT of a complex sequence

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[<-k>_N]$
$x^*[<-n>_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{cs}[k] = \{X[<k>_N]+X^*[<-k>_N]\}/2$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}[k] = \{X[<k>_N]-X^*[<-k>_N]\}/2$
$x_{cs}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{ca}[n]$	$j\mathrm{Im}\{X[k]\}$

Note: $x[n]$ is a complex sequence. $x_{cs}[n]$ and $x_{ca}[n]$ are the circular conjugate-symmetric and antisymmetric parts of $x[n]$, respectively.

Circular Conjugate-Symmetry

Example: N point complex sequence $x[n] \leftrightarrow$ $X[k]$, $x^*[n] \Leftrightarrow X^*[<-k>_N]$

then

Could you deduce $X[N-k] = X[<-k>_N]$ **?**

Proof:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \qquad 0 \le k \le N-1$$

$$X^*[k] = \sum_{n=0}^{N-1} x^*[n] e^{j2\pi kn/N} \qquad 0 \le k \le N-1$$

$$\because\ X^*[<-k>_N] = X^*[N-k]$$

$$= \sum_{n=0}^{N-1} x^*[n] e^{j2\pi(N-k)n/N} = \sum_{n=0}^{N-1} x^*[n] e^{-j2\pi kn/N}$$

$$\therefore\ X^*[<-k>_N] = \sum_{n=0}^{N-1} x^*[n] e^{-j2\pi kn/N}$$

$$x^*[n] \Leftrightarrow X^*[<-k>_N]$$

IF

$$x[n] = x_{re}[n] + jx_{im}[n] \Leftrightarrow X[k] = X_{cs}[k] + X_{ca}[k]$$

Where

$$X_{cs}[k] = \frac{1}{2}[X[<k>_N] + X^*[<-k>_N]]$$

$$X_{ca}[k] = \frac{1}{2}[X[<k>_N] - X^*[<-k>_N]]$$

Then,

$$x_{re}[n] \Leftrightarrow X_{cs}[k] \qquad jx_{im}[n] \Leftrightarrow X_{ca}[k]$$

Table 5.2: DFT Properties: Symmetry Relations of a real sequence

Length-N Sequence	N-point DFT
Real $x[n]$	$X[k] = \mathrm{Re}\{X[k]\} + j\mathrm{Im}\{X[k]\}$
$x_{\mathrm{ev}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{od}}[n]$	$j\mathrm{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$
	$\mathrm{Re}X[k] = \mathrm{Re}X[\langle -k\rangle_N]$
	$\mathrm{Im}X[k] = -\mathrm{Im}X[\langle -k\rangle_N]$
	$\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$
	$\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x[n]$ is a real sequence. $x_{\mathrm{ev}}[n]$ and $x_{\mathrm{od}}[n]$ are the circular even and odd parts of $x[n]$

If a N point real sequence $x[n] \leftrightarrow X[k]$,

then

$$X[k] = X^*[<-k>_N]$$

circular even and odd parts of $x[n]$

$$x_{ev}[n] = \frac{1}{2}\{x[n] + x[<-n>_N]\} = \frac{1}{2}[x[n] + x[N-n]]$$

$$x_{od}[n] = \frac{1}{2}\{x[n] - x[<-n>_N]\} = \frac{1}{2}[x[n] - x[N-n]]$$

Example: **A length-14 real sequence $x[n]$ has 14-point DFT $X[k]$. If $X[2]=-1+j3$, compute $X[12]$?**

Then (circular Conjugate-Symmetry)

$$X[12] = X^*[2] = -1 - j3$$

$$X[12] = X^*[< -12 >_{14}] = X^*[2]$$

Which property do we use?

Example (5.9.1) **Calculate the N-point DFT of two N-point real sequences by using a N-point complex DFT once.**

Solution **Supposing $x_1[n]$, $x_2[n]$ are real sequences, their N-point DFT are**

$$DFT[x_1[n]] = X_1[k] \qquad DFT[x_2[n]] = X_2[k]$$

we create a N-point complex sequence

$$w[n] = x_1[n] + jx_2[n]$$

$$W[k] = X_1[k] + jX_2[k]$$

$$DFT[w[n]] = DFT[x_1[n] + jx_2[n]]$$

$$= DFT[x_1[n]] + jDFT[x_2[n]]$$

Obviously, $X_1[k] = W_{cs}[k]$ **and** $jX_2[k] = W_{ca}[k]$

$$X_1[k] = \frac{1}{2}\{W[<k>_N] + W^*[<-k>_N]\}$$

$$X_2[k] = \frac{1}{2}\{W[<k>_N] - W^*[<-k>_N]\}$$

Which property do we use?

Table 5.3: DFT Properties (Appended)

Type of Property	length-N sequence	N-point DFT
	$g[n]$	$G[k]$
	$h[n]$	$H[k]$
Linearity	$ag[n]+bh[n]$	$aG[k]+bH[k]$
Circular Time-shifting	$g[\langle n-n_0\rangle_N]$	$W_N^{kn_0}G[k]$
Frequency-shifting	$W_N^{-kn_0}g[n]$	$G[\langle k-k_0\rangle_N]$
Duality	$G[n]$	$g\langle -k\rangle_N$
Circular Convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	

5.7 Linear Convolution Using the DFT (5.10)

5.7.1 Linear Convolution of Two Finite-Length Sequences

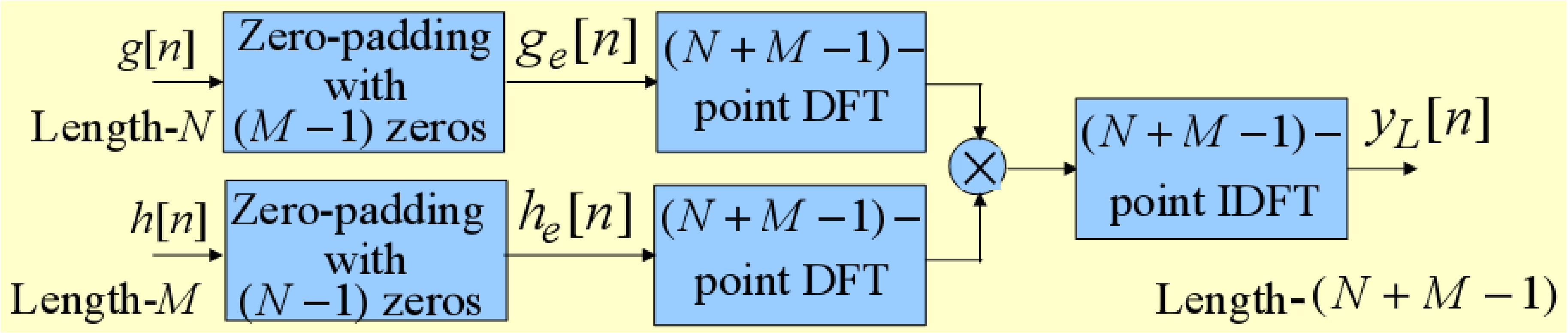
- **Linear convolution is a key operation in many signal processing applications**
- **Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT**

- **Let *g*[*n*] and *h*[*n*] be two finite-length sequences of length *N* and *M*, respectively**
- **Denote $L = N+M-1$**
- **Define two length-*L* sequences**

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

- **Then $y_L[n] = g[n] \circledast h[n] = y_C[n] = g_e[n] \text{Ⓛ} h_e[n]$**
- **The corresponding implementation scheme is illustrated below**



- **We next consider the DFT-based implementation of**

$$y[n] = \sum_{\ell=0}^{M-1} h[\ell]\, x[n-\ell] = h[n] \circledast x[n]$$

where *h*[*n*] is a finite-length sequence of length *M* and *x*[*n*] is an infinite length (or a finite length sequence of length much greater than *M*).

L-point Circular Convolution

$$y_l[n] = [\sum_{m=0}^{L-1} x[m]h[< n-m >_L]R_L[n] = x[n] \textcircled{L} h[n]$$

When $L \geq N + M - 1$ **,**

$$y_l[n] = \begin{cases} y[n], 0 \leq n \leq N + M - 2 \\ 0, N + M - 1 \leq n \leq L - 1 \end{cases}$$

(zero-padding).

In fact,

$$y_l[n] = \tilde{y}_l[n]R_L[n] \qquad \tilde{y}_l[n] = \sum_{r=-\infty}^{\infty} y[n + rL]$$

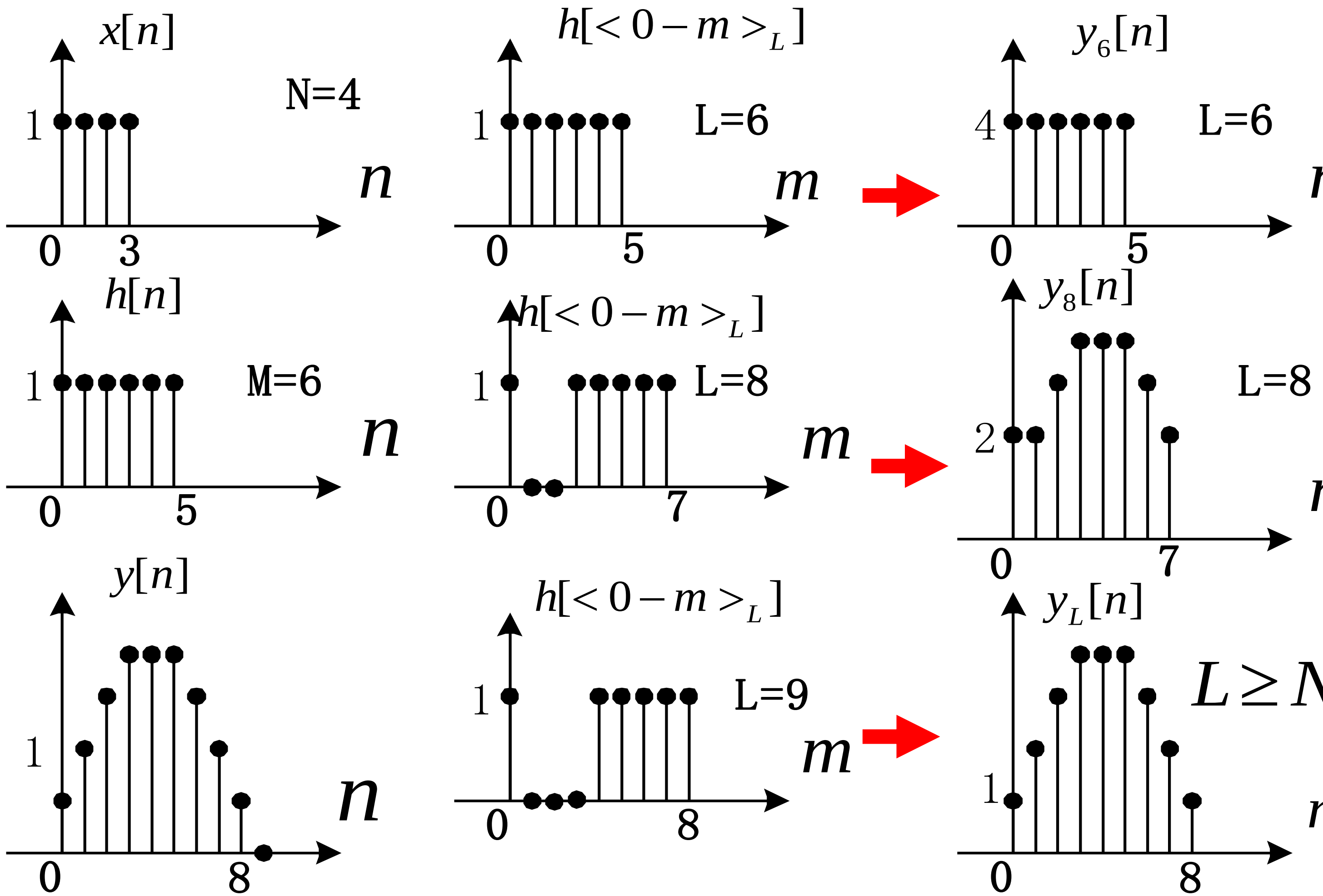
This is the requirement if linear convolution is computed by using DFT.

when $L < N + M - 1$ **,** $y_l[n] = \tilde{y}_l[n] R_L[n]$ **.**
There is overlap.

Example $x[n] = R_4[n]$ $h[n] = R_6[n]$

sketch

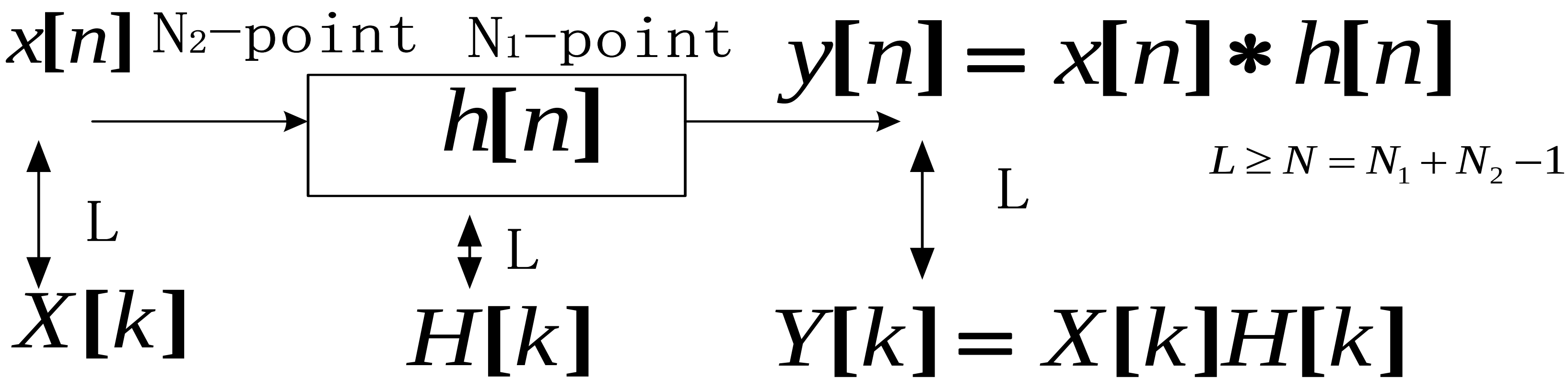
$$y[n], y_6[n], y_8[n], y_l[n] \; L \geq 9$$


$x[n]$
N=4
1
0
3
n
$h[n]$
M=6
1
0
5
n
$y[n]$
1
0
8
n
$h[<0-m>_L]$
L=6
1
0
5
m
$h[<0-m>_L]$
L=8
1
0
7
m
$h[<0-m>_L]$
L=9
1
0
8
m
$y_6[n]$
L=6
4
0
5
n
$y_8[n]$
L=8
2
0
7
n
$y_L[n]$
1
0
8
n

$$L \geq N + M - 1 = 9$$

5.8.3 Linear Convolution Using the FFT(5.10)

1. Linear Convolution of two finite-length sequences(5.10.1)



Why linear convolution is computed by using DFT?

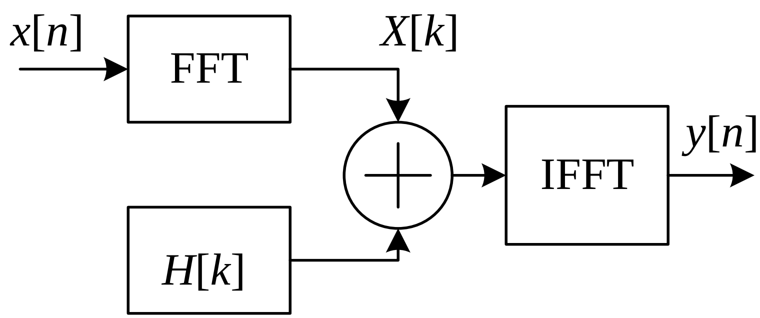
The Complexity of Computation

$h[n]$ $n:[0, N_1 - 1]$ N_1 **-point** $x[n]$ $n:[0, N_2 - 1]$ N_2 **-point**

$$y[n] = x[n] * h[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

$n:[0, N_1 + N_2 - 2]$ $N = N_1 + N_2 - 1, N - po\text{int}$

It needs $N_1 N_2$ **multiplications.**



Using $\bar{y}[n] = \bar{x}[n] \textcircled{L} \bar{h}[n]$ **with *L*-point DFT(*L*>*N*), we can get the following steps and computations.**

1. $H[k] = DFT[\bar{h}[n]]$ $\qquad \frac{L}{2}\log_2 L$

2. $X[k] = DFT[\bar{x}[n]]$ $\qquad \frac{L}{2}\log_2 L$

3. $Y[k] = H[k]X[k]$ $\qquad L$

4. $y[n] = IFFT[Y[k]]$ $\qquad \frac{L}{2}\log_2 L$

The total multiplications: $3 \times \frac{L}{2}\log_2 L + L$

Note: $\bar{x}[n]$ **and** $\bar{h}[n]$ **are** L**-points (zero-padding** $x[n]$ **and** $h[n]$ **). This algorithm is efficient when** N_1 **is about same as** N_2**.**

Example: N_1=N_2= 4, 8, and 100

2. Linear Convolution of two finite-length sequences(5.10.2)

Overlap-Add method

When $x[n]$ is very long (or, *x*[*n*] is infinite length), it can be broken up into a sum of short-length segments.

$$x_i[n] = \begin{cases} x[n], & iN_2 \le n \le (i+1)N_2 - 1 \\ 0, & others \end{cases}$$

Then, $x[n] = \sum_{i=-\infty}^{\infty} x_i[n]$

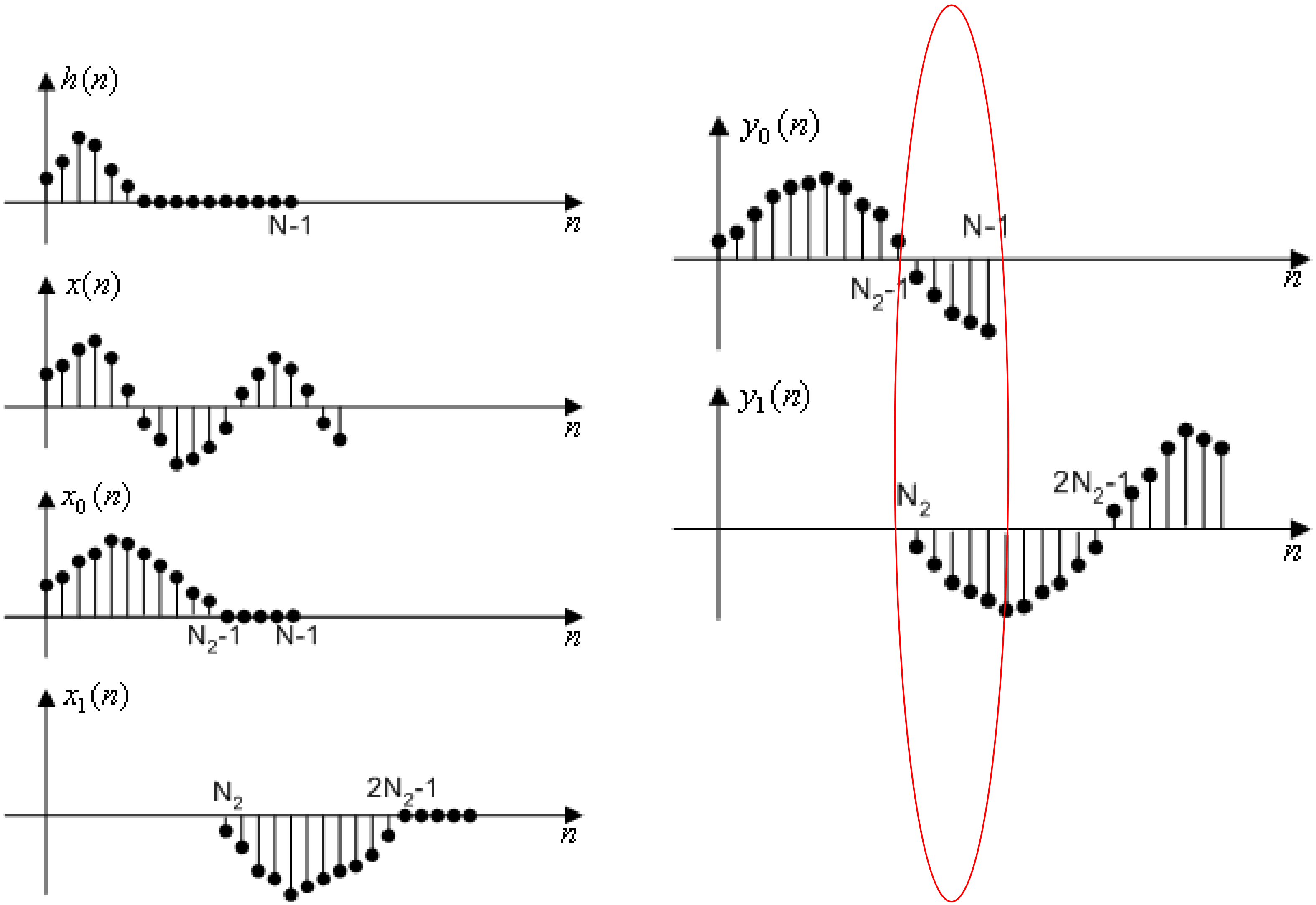
$$y[n] = x[n] \otimes h[n] = \sum_{i=-\infty}^{\infty} x_i[n] \otimes h[n]$$

$$= \sum_{i=-\infty}^{\infty} y_i[n]$$

where $$y_i[n] = x_i[n] \otimes h[n]$$

$$(iN_2 \leq n \leq (i+1)N_2 + N_1 - 2)$$

Using $\bar{y}_i[n] = \bar{x}_i[n] \, \textcircled{L} \, \bar{h}[n]$ **(*L* = *N*),**
We can compute the linear convolution.
But, there is an overlap between two short linear convolutions.


h(n)
N-1
n
x(n)
n
x0(n)
N2-1
N-1
n
x1(n)
N2
2N2-1
n
y0(n)
N-1
N2-1
n
y1(n)
N2
2N2-1
n

Overlap-Save method

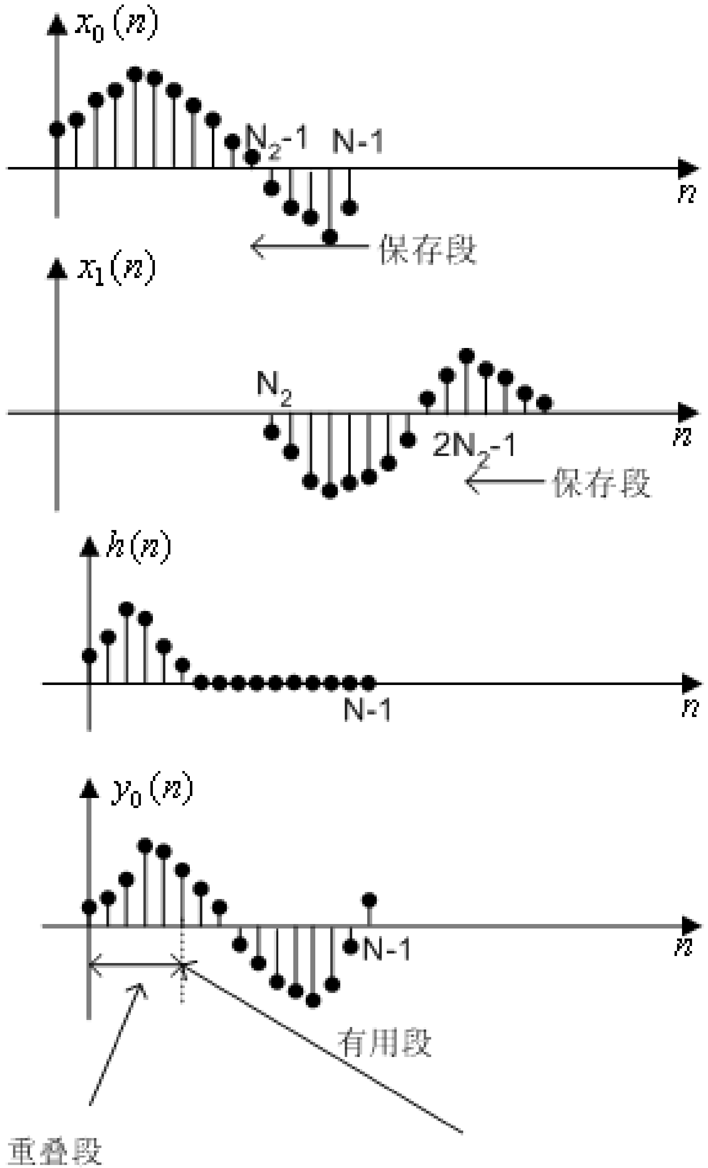
This algorithm is similar to previous method. But, $x_i[n]$ saves the $N - N_2$ samples of next segment without zero-padding ,when

***n*: $N_2 \sim N - 1$**

$$x_i[n] = \begin{cases} x[n + iN_2 - N_1 + 1], 0 \le n \le N - 1 \\ 0, \quad others \end{cases}$$

Then $\bar{y}_i[n] = \bar{x}_i[n] \; L \; \bar{h}[n]$, (*L* = *N*)

There are overlap when $0 \le n \le N_1 - 2$ .We reject the first $N_1 - 1$ samples of $y_i[n]$.

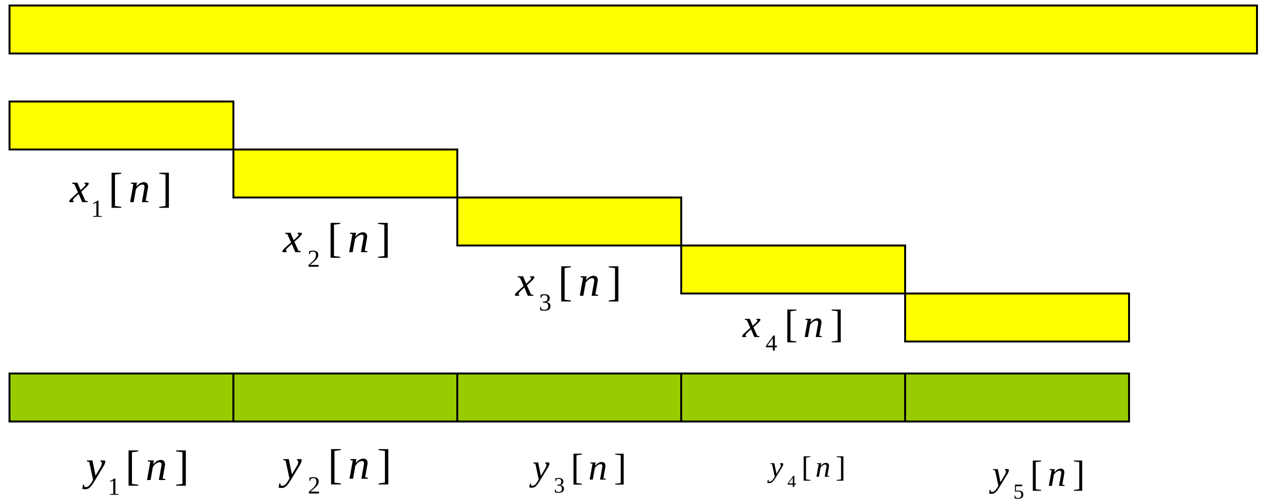


$x_0(n)$
N_2-1
N-1
n
保存段
$x_1(n)$
N_2
n
$2N_2$-1
保存段
$h(n)$
n
N-1
$y_0(n)$
n
N-1
有用段
重叠段

$x[n]$

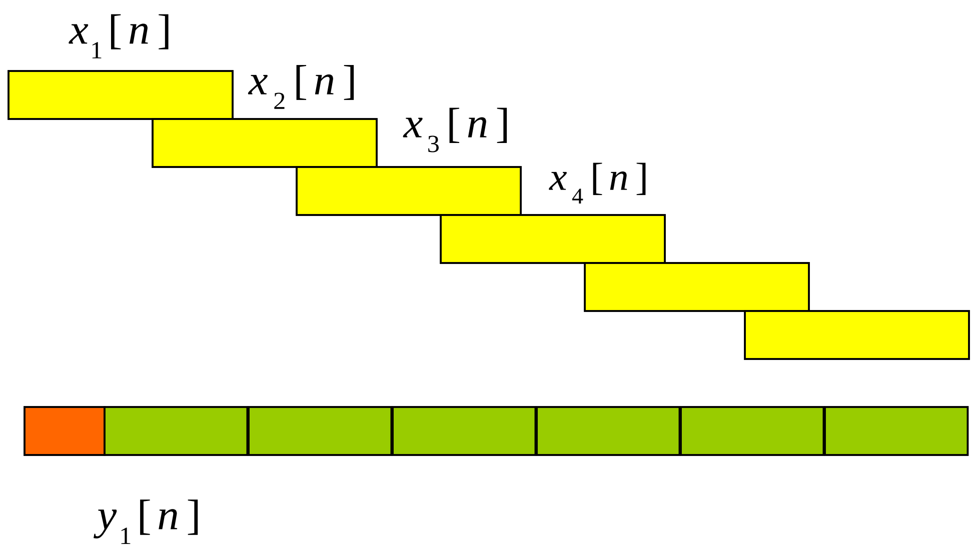
Overlap- add

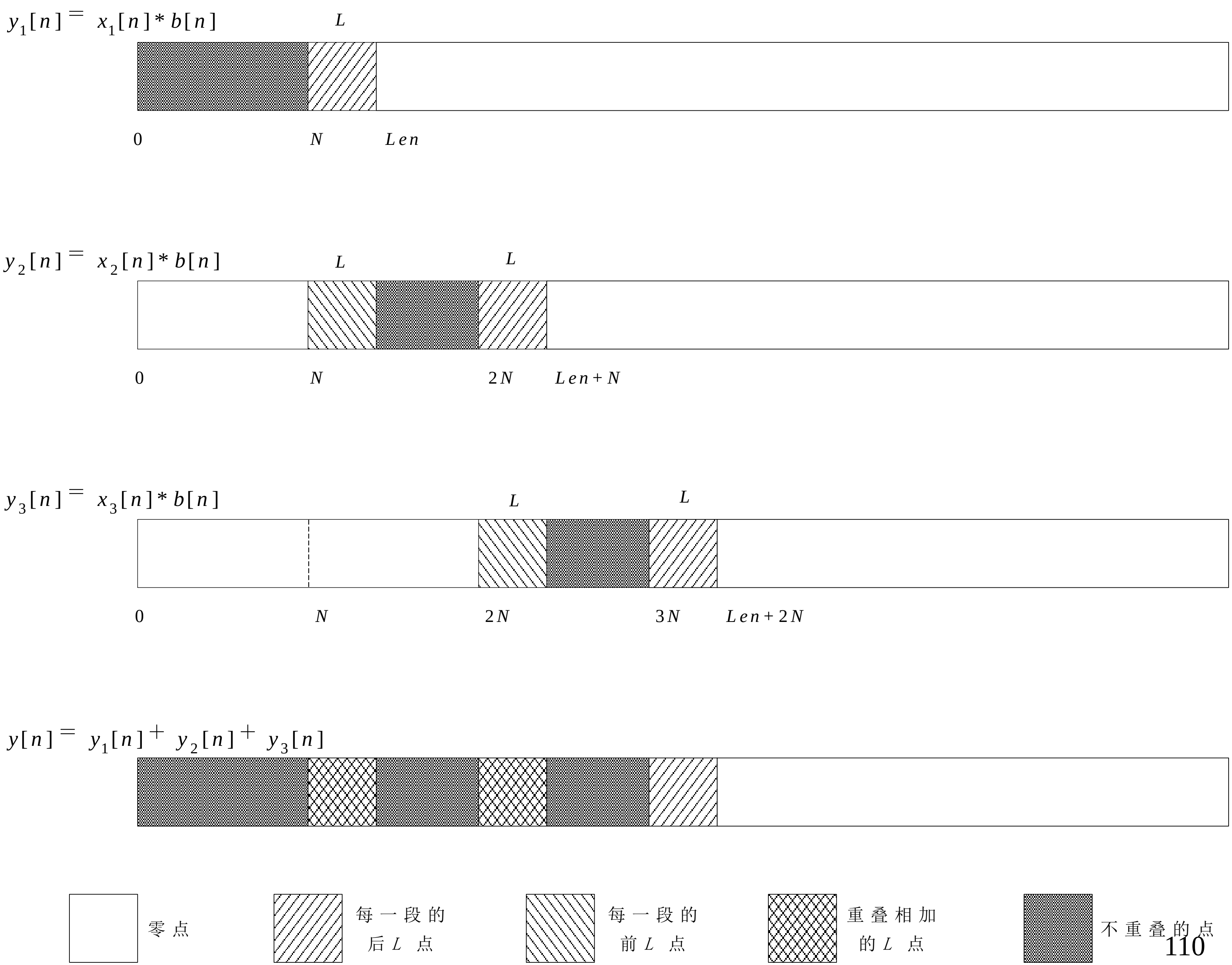
N-point convolution

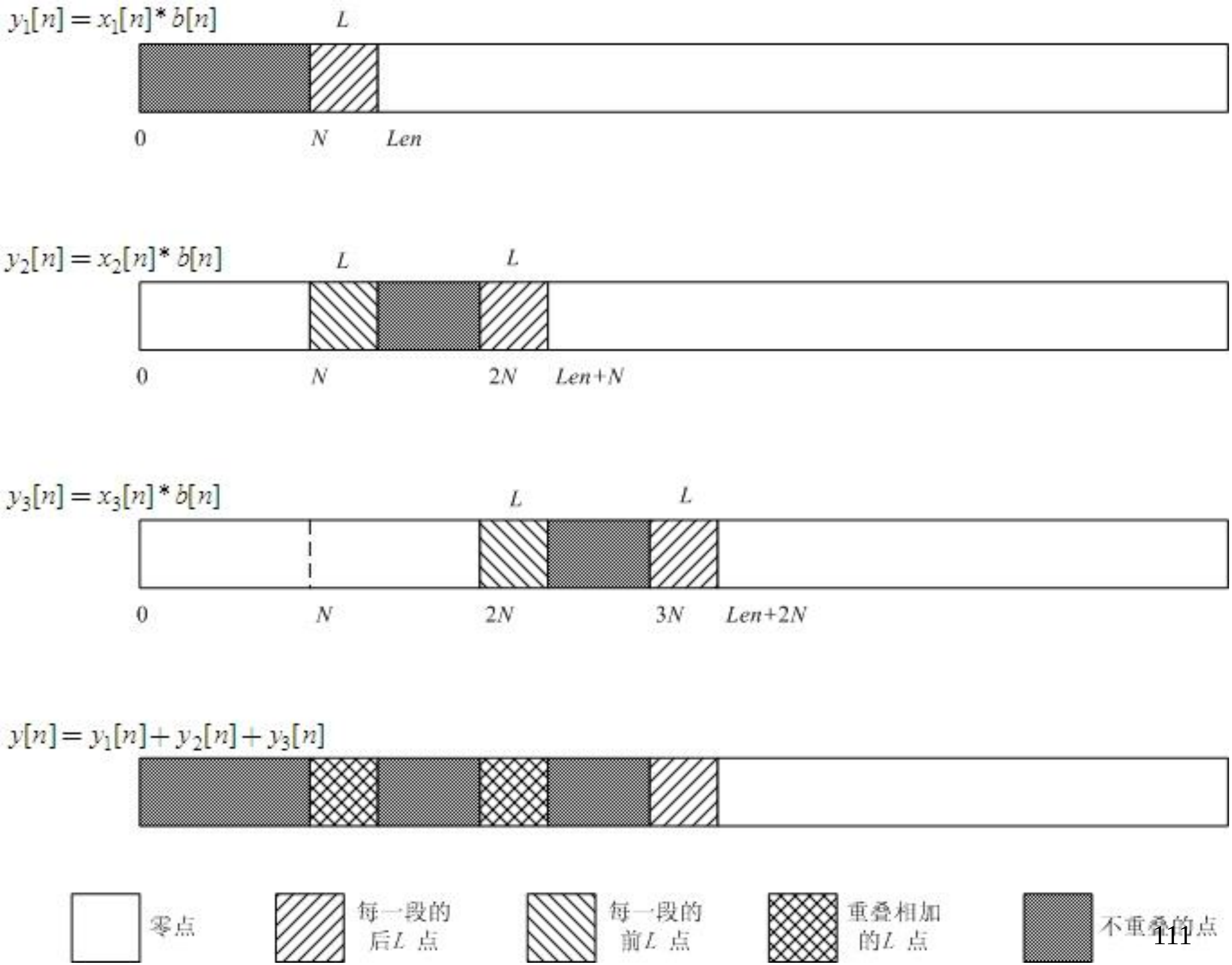


Overlap-save

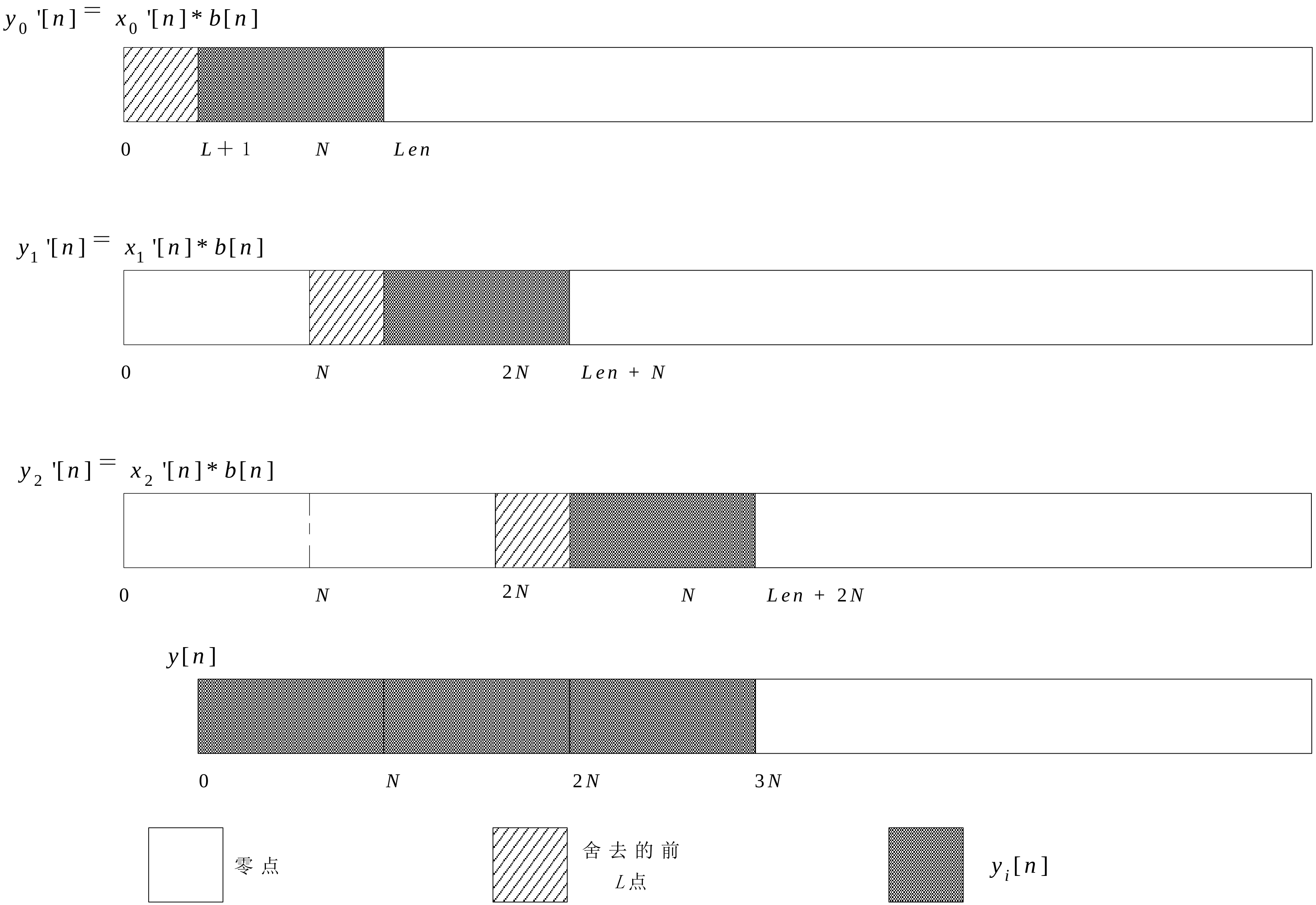
N-point convolution



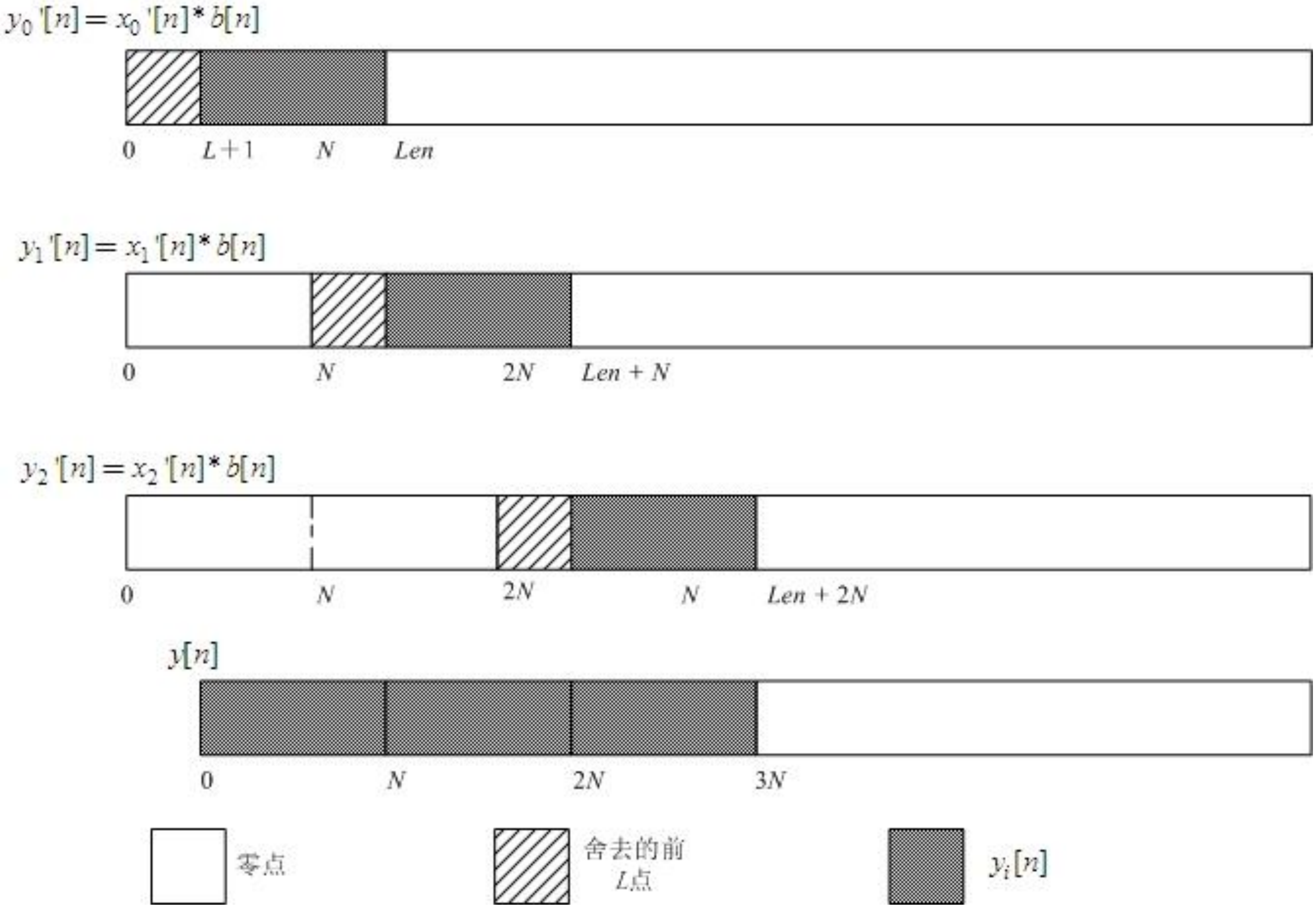

$y_1[n] = x_1[n] * b[n]$
L
0
N
Len
$y_2[n] = x_2[n] * b[n]$
L
L
0
N
$2N$
$Len+N$
$y_3[n] = x_3[n] * b[n]$
L
L
0
N
$2N$
$3N$
$Len+2N$
$y[n] = y_1[n] + y_2[n] + y_3[n]$
零 点
每 一 段 的
后 L 点
每 一 段 的
前 L 点
重 叠 相 加
的 L 点
不 重 叠 的 点



$y_1[n]=x_1[n]*b[n]$
L
0
N
Len
$y_2[n]=x_2[n]*b[n]$
L
L
0
N
2N
Len+N
$y_3[n]=x_3[n]*b[n]$
L
L
0
N
2N
3N
Len+2N
$y[n]=y_1[n]+y_2[n]+y_3[n]$
零点
每一段的后L 点
每一段的前L 点
重叠相加的L 点
不重叠的点



y0 '[n] = x0 '[n] * b[n]
0
L + 1
N
Len
y1 '[n] = x1 '[n] * b[n]
0
N
2N
Len + N
y2 '[n] = x2 '[n] * b[n]
0
N
2N
N
Len + 2N
y[n]
0
N
2N
3N
零 点
舍 去 的 前
L点
yi[n]


y0'[n] = x0'[n] * b[n]
0
L+1
N
Len
y1'[n] = x1'[n] * b[n]
0
N
2N
Len + N
y2'[n] = x2'[n] * b[n]
0
N
2N
N
Len + 2N
y[n]
0
N
2N
3N
零点
舍去的前
L点
yi[n]

Thanks!

Any questions?