Chapter 6
Z-Transform

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§ 6 *z*-Transform

6.1 Definition of ZT

$$x[n] \xleftrightarrow{Z} X(z), \left|z\right| : (\gamma_1, \gamma_2)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (6.1)$$

where *z* = Re(*z*)+*j*Im(*z*) is a complex variable.
If we let $z = re^{j\omega}$ **then we get**

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

where

$$|z| = r$$

ZT is existed when :

$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

The Region Of Convergence (ROC) of ZT is: the range of $|z| = r$ chosen.

Example 6.1: **The ZT of following sequences are wanted.**

$$x[n] = a^n \mu[n]$$

Solution

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} [or \frac{z}{z-a}], |z| : (|a|, \infty)$$

Specially, a = 1,

$$X(z) = \frac{1}{1 - z^{-1}} [or \frac{z}{z-1}], |z| : (1, \infty)$$

Table 6.1: Commonly Used z-transform

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

Prove it by yourself!

6.2 Rational *z*-Transform

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent *z*-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1} :

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

- **The degree *P*(*z*) is *M* and the degree of *D*(*z*) is *N***
- **An alternate representation of a rational *z*-transform is as a ratio of two polynomials in *z*:**

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

- **A rational *z*-transforms can be alternately written in factored form as**

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

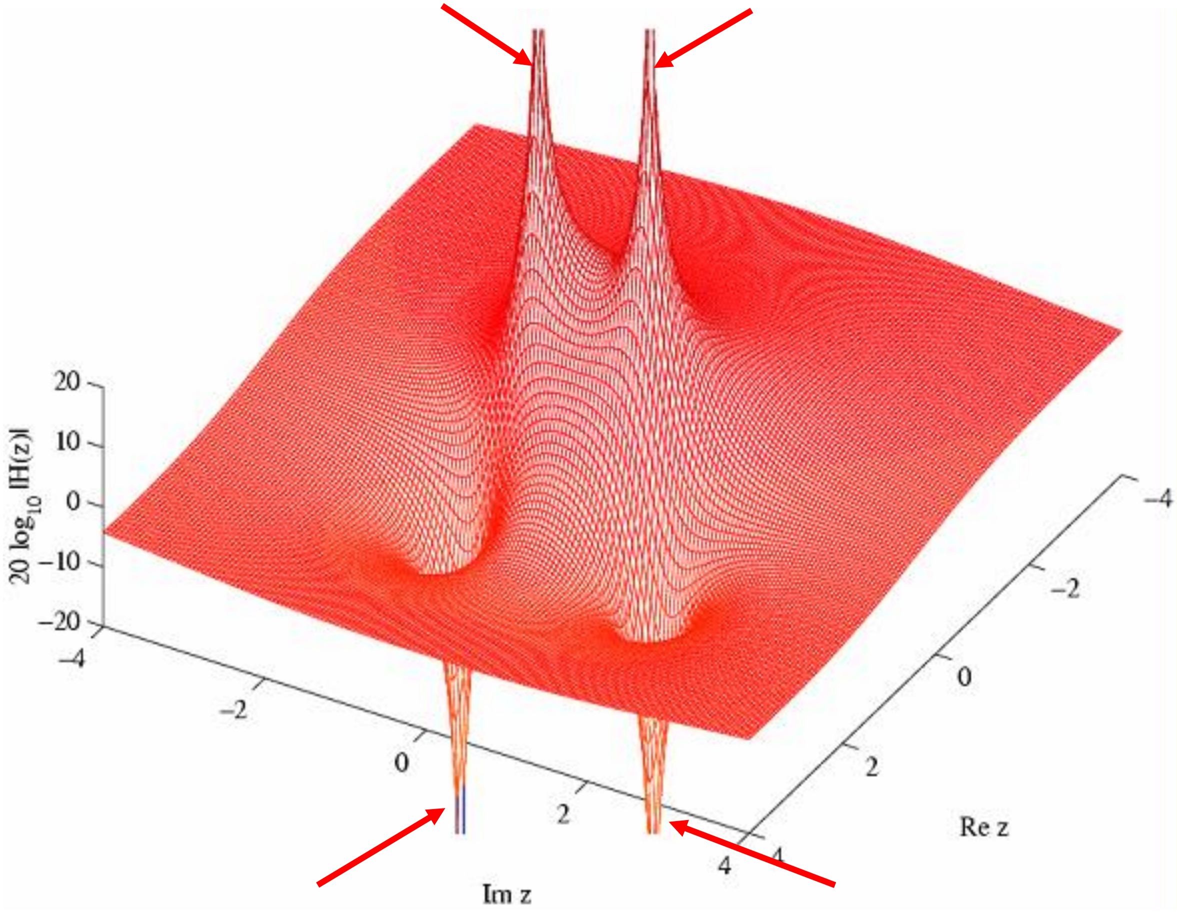
ξ_l : zeros of $G(z)$; λ_l : poles of $G(z)$

- **A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for**

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

```
>> roots([1 -2.4 2.88])

ans =

   1.2000 + 1.2000i
   1.2000 - 1.2000i

>> roots([1 -0.8 0.64])

ans =

   0.4000 + 0.6928i
   0.4000 - 0.6928i
```

Rational z-Transform



Poles z = 0.4±j0.6928; Zeros z = 1.2±j1.2

Which one is pole? Which one is zero?

6.3 ROC of a Rational z-transform

The Region Of Convergence of ZT is: the range of $|z| = r$ chosen, so that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

be convergent. Normally, it is concerned with the poles of a rational ZT.

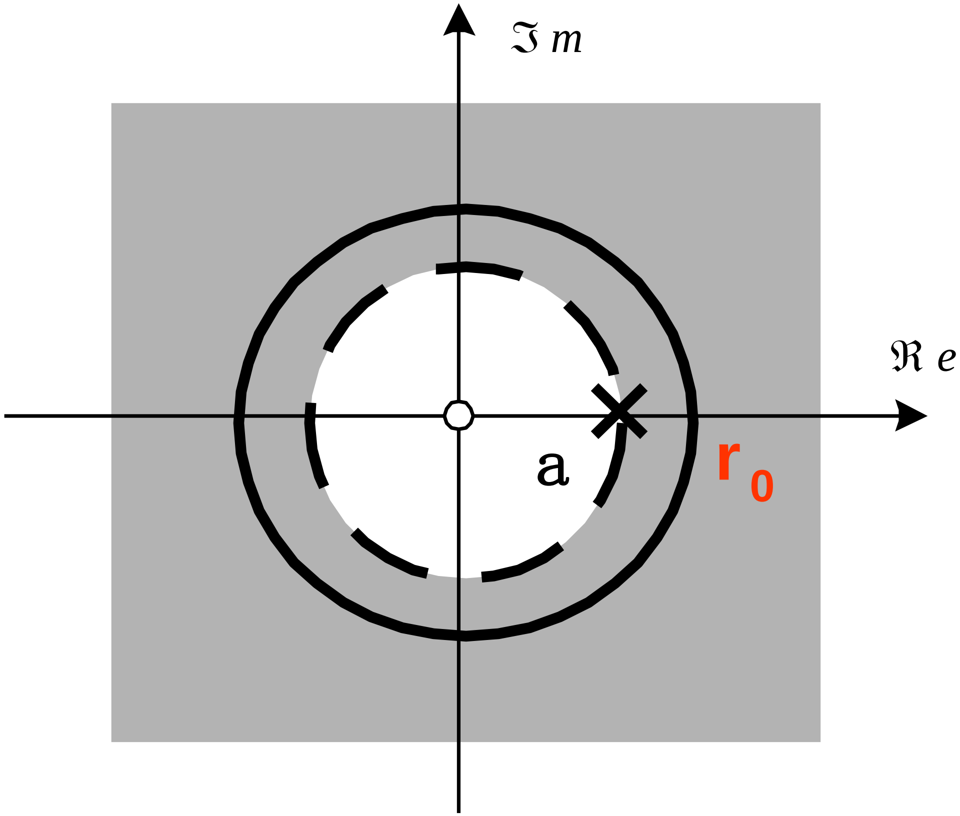
The ZT's ROC of a sequence:

(1) right-sided (causal) sequence, $|z| : (\gamma_1, \infty)$

(2) left-sided (anti-causal) sequence, $|z| : (0, \gamma_2)$

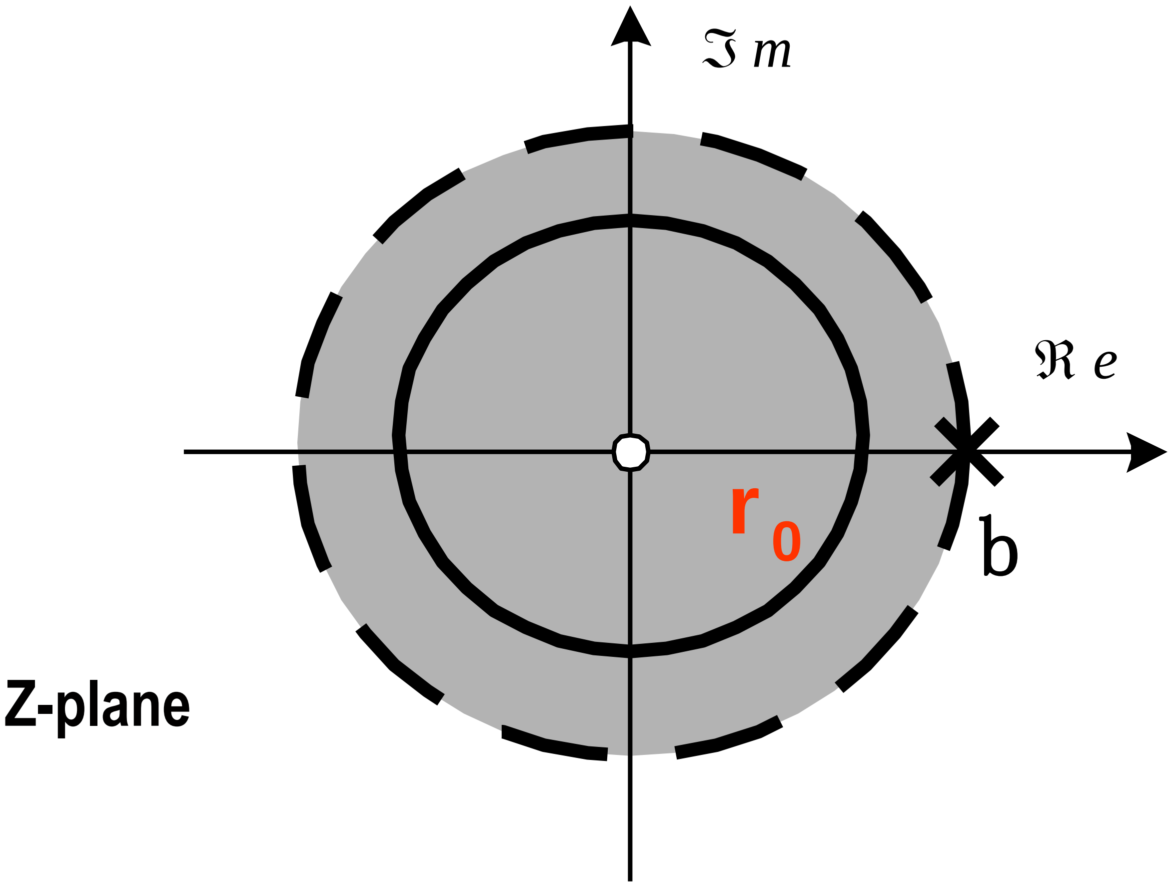
(3) two-sided sequence, $|z| : (\gamma_1, \gamma_2)$

(4) finite length sequence,
(the entire Z-plane) $|z| : (0, \infty)$



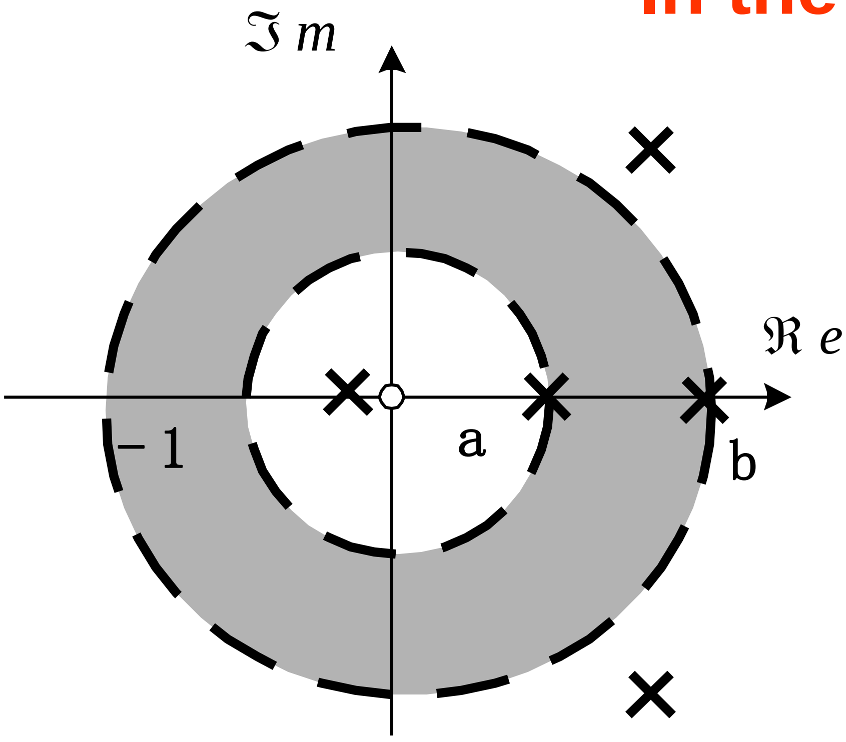
Z-plane

The ROC of a right-sided signal



The ROC of a left-sided signal

There is no any pole
In the ROC.



Z-plane

The ROC of a two-sided signal

Example: **The ZT of following sequences are wanted.**

$$x_1[n] = a^n \mu[n]; \; x_2[n] = a^n \mu[-n-1]; \; x_3[n] = a^{|n|}$$

$$x_4[n] = \delta[n]$$

Solution

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} [or \; \frac{z}{z-a}], \; |z| : (|a|, \infty)$$

$$X_2(z) = \sum_{n=-1}^{-\infty} a^n z^{-n} = \frac{a^{-1} z}{1 - a^{-1} z}, |z| : (0, |a|)$$

$$X_3(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-1}^{-\infty} a^{-n} z^{-n}$$

$$= \frac{1}{1 - az^{-1}} + \frac{az}{1 - az}, |z| : (|a|, |a^{-1}|)$$

when $|a| < 1$ **,** $X_3(z)$ **is existed.**

$$X_4(z) = \sum_{n=0}^{\infty} x_4[n] z^{-n} = \sum_{n=0}^{\infty} \delta[n] z^{-n} = 1,$$

$$ROC : whole \quad z - plane$$

Note:

(1) The *z*-transform is a form of a *Laurent series* and is an analytic function at every point in the ROC.

(2) Obviously, if ROC contain

$$r = 1 \quad or \quad |z| = 1$$

the *unit circle* in the *z*-plane.

ZT → DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

6.4 The Inverse Z-Transform

1. Definition of Inverse ZT

From

$$X(z) = X(re^{j\omega}) = DTFT\{r^{-n}x[n]\}$$

$$\therefore r^{-n}x[n] = IDTFT\{X(re^{j\omega})\}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{jn\omega}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re)^{jn\omega}d\omega$$

Let $z = re^{j\omega}, \quad d\omega = \frac{1}{jz} dz$

So

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Integration around a counterclockwise closed circular contour centered at the origin and with radius r.

2. The calculation for inverse Z-Transform

(1) Integration of complex function by equation.

(2) Compute by Partial Fraction Expansion .

(3) Long Division

Example 1

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}, \quad |z| > \frac{1}{3}$$

Partial Fraction Expansion

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})}, |z| > \frac{1}{3}$$

Slide 4

$$x_1[n] = (\frac{1}{4})^n u[n] \xleftrightarrow{ZT} \frac{1}{(1-\frac{1}{4}z^{-1})}, |z| > \frac{1}{4}$$

$$x_2[n] = 2(\frac{1}{3})^n u[n] \xleftrightarrow{ZT} \frac{2}{(1-\frac{1}{3}z^{-1})}, |z| > \frac{1}{3}$$

$$x[n] = x_1[n] + x_2[n] = (\frac{1}{4})^n u[n] + 2(\frac{1}{3})^n u[n]$$

Example 2 **ROC** $\frac{1}{4} < |z| < \frac{1}{3}$

in example 1

$$x_1[n] = (\frac{1}{4})^n u[n] \xleftrightarrow{ZT} \frac{1}{(1 - \frac{1}{4} z^{-1})}, |z| > \frac{1}{4}$$

$$x_2[n] \xleftrightarrow{ZT} \frac{2}{(1 - \frac{1}{3} z^{-1})}, |z| < \frac{1}{3}$$

$$x_2[n] = -2(\frac{1}{3})^n u[-n-1]$$

$$x[n] = x_1[n] + x_2[n]$$

$$= (\frac{1}{4})^n u[n] - 2(\frac{1}{3})^n u[-n-1]$$

Normally, Partial Fraction Expansion of rational X(z)

$$X(z) = \sum_{i=1}^{N} \frac{A_i}{1 - a_i z^{-1}}$$

Example 3

$$X(z) = 4z^{2} + 2 + 3z^{-1}, 0 < |z| < \infty$$

From the definition of ZT, we get:

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & otherwise \end{cases}$$

or

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

$$\delta[n + n_0] \xleftrightarrow{ZT} z^{n_0}, \quad 0 \le |z| < \infty$$

Example 4

Consider $X(z) = \dfrac{1}{1 - az^{-1}}$ **ROC₁:** $|z| > |a|$

long division

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2 z^{-2} + \ldots \\
\hline
1 - az^{-1} & \mathbf{1} \\
 & 1 - az^{-1} \\
\hline
 & \quad az^{-1} \\
 & \quad az^{-1} - a^2 z^{-2} \\
\hline
 & \quad\quad a^2 z^{-2}
\end{array}
$$

$$x[n] = \{1, a, a^2, \ldots\}$$

$\uparrow$

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \ldots$$

ROC₂: $|z| < |a|$

long division

$$-a^{-1}z - a^{-2}z^{2} - \ldots$$

$$-az^{-1} + 1 \,\overline{)\; \mathbf{1}}$$

$$\underline{1 - a^{-1}z}$$

$$a^{-1}z$$

$$\underline{az^{-1} - a^{2}z^{2}}$$

$$a^{2}z^{2}$$

$$x[n] = \{\ldots, -a^{2}, -a, 0\}$$

$$\frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^{2} - \ldots$$

3. ZT Using MATLAB

(i) $$G(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}} = \frac{p_0}{d_0} \frac{\prod_{i=1}^{M} (1 - \xi_i z^{-1})}{\prod_{i=1}^{N} (1 - \lambda_i z^{-1})}$$

$$\mathbf{k} = \frac{p_0}{d_0}$$

% Find zeros and poles

[z,p,k] = tf2zp(num,den)

% Inverse process

[num,den] = zp2tf (z,p,k)

% Pole-zero plot

zplane(zeros, poles)

zplane(num, den)

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

```
%Find second-order factors
sos = zp2sos(z,p,k)
```

(ii) Partial-fraction expansion using MATLAB

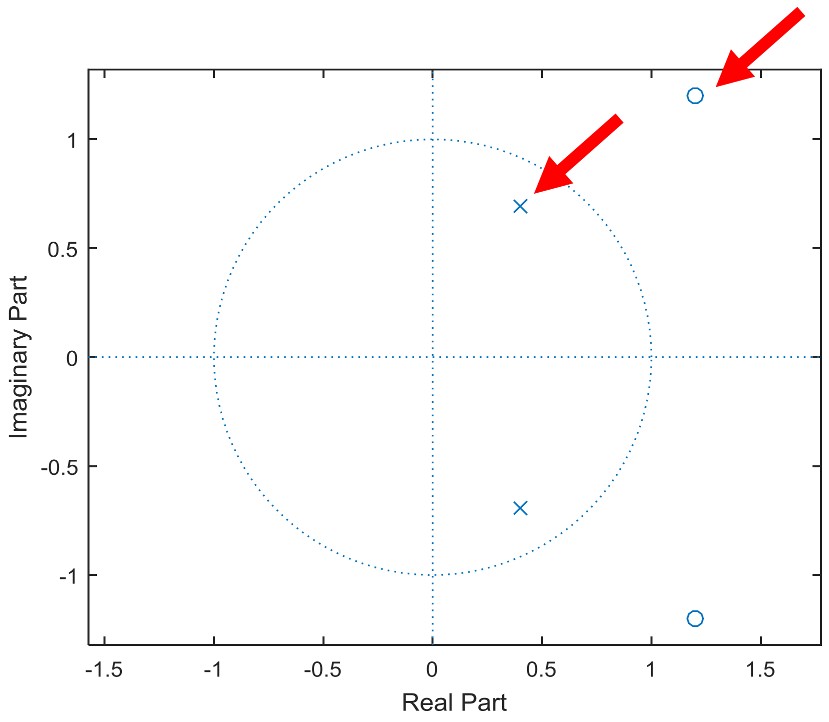
```
%partial-fraction expansion
[r,p,k] = residuez(num,den)
% reverse operation
[num,den] = residuez(r,p,k)
```

(iii) Inverse ZT using MATLAB

```
[h,t]=impz(num,den)
%x=[1 zeros(L-1)]
y = filter(num,den ,x)
```

Example on Slide 8

```
>> num = [1 -2.4 2.88]

num =

    1.0000   -2.4000    2.8800

>> den = [1 -0.8 0.64]

den =

    1.0000   -0.8000    0.6400

>> zplane(num, den)
```



```
>> [r,p,k] = residuez(num,den)

r =

    2.0000
    1.0000


p =

    0.3333
    0.2500


k =

     []
```

$$X(z) = \frac{1}{(1 - \frac{1}{4} z^{-1})} + \frac{2}{(1 - \frac{1}{3} z^{-1})}, \quad |z| > \frac{1}{3}$$

$$\frac{B(z)}{A(z)} = \frac{r(1)}{1 - p(1) z^{-1}} + \cdots + \frac{r(n)}{1 - p(n) z^{-1}} + k(1) + k(2) z^{-1} + \cdots + k(m-n+1) z^{-(m-n)}$$

6.5 The properties Of ZT

Table 6.2: z-Transform Properties

p.321

Property	Sequence	z -Transform	ROC
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g-}R_{h-} < |z| < R_{g+}R_{h+}$.

Time Shifting

If $x[n] \xleftrightarrow{ZT} X(z)$, R

Then $x[n-n_0] \xleftrightarrow{ZT} z^{-n_0} X(z)$, R

Example From:

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

We can get

$$a^{n-1} u[n-1] \xleftrightarrow{ZT} \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}, |z| > |a|$$

Time Reversal

If $x[n] \xleftrightarrow{ZT} X(z)$, **R**

Then $x[-n] \xleftrightarrow{ZT} X(\frac{1}{z})$, **1/R**

Example From:

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1 - az^{-1}} \quad , |z| > |a|$$

We can get

$$a^{-n} u[-n] \xleftrightarrow{ZT} \frac{1}{1 - az} = \frac{-(1/az)}{1 - a^{-1}z^{-1}} \quad , |z| < |a|^{-1}$$

Furthermore,

$$-a^n u[-n-1] \xleftrightarrow{ZT} \frac{1}{1 - az^{-1}} \quad , |z| < |a|^{-1}$$

Time Expansion

If $x[n] \xleftrightarrow{ZT} X(z)$, R

Then, $x_{(k)}[n] \xleftrightarrow{ZT} X(z^k)$, $R^{1/k}$

Conjugate

If $x[n] \xleftrightarrow{ZT} X(z)$, R

Then, $x^*[n] \xleftrightarrow{ZT} X^*(z^*)$, R

Scaling in z-Domain

If $x[n] \xleftrightarrow{ZT} X(z),$ R

Then $z_0^n x[n] \xleftrightarrow{ZT} X(z/z_0),$ |z₀| R

Specially, $e^{j\omega_0 n} x[n] \xleftrightarrow{ZT} X(e^{-j\omega_0} z),$ R

Example $\omega_0 = \pi$

$$(-1)^n x[n] \xleftrightarrow{ZT} X(-z), \quad \text{R}$$

Differentiation in the z-Domain

If $x[n] \xleftrightarrow{ZT} X(z),$ **R**

Then, $nx[n] \xleftrightarrow{ZT} -z\dfrac{d}{dz}X(z),$ **R**

Example From:

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}, |z| > |a|$$

We can get

$$na^n u[n] \xleftrightarrow{ZT} -z\frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right] = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

Example $$X(z) = \log(1 - az^{-1}), |z| > |a|$$

Then, $$nx[n] \xleftrightarrow{ZT} -z\frac{d}{dz}X(z)$$

$$-aa^{n-1}u[n-1] \xleftrightarrow{ZT} = -z\frac{az^{-2}}{(1-az^{-1})}, \quad , |z| > |a|$$

We can get $$nx[n] = -aa^{n-1}u[n-1]$$

$$x[n] = -\frac{1}{n}a^{n}u[n-1]$$

Note: ZT and DTFT and DFT

1.Relationships

A finite-length sequence

$$x[n];\ 0 \le n \le N-1,$$

ZT

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n}$$

DTFT

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

when $z = z_k = e^{j\frac{2\pi}{N}k} = W_N^{-k}$

DFT

$$X(z_k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n] W_N^{kn} = DFT\,[x[n]]$$

That means:

The ZT on the unit circle in Z-plane is the DTFT $X(e^{j\omega})$ of $x[n]$. The samples on the unit circle in *Z-plane*, $X(z_k)$, are the DFT

$X[k] = X(e^{j\frac{2\pi}{N}k})$ **of** $x[n]$.

Example $x[n] = R_N[n]$ **, calculate**

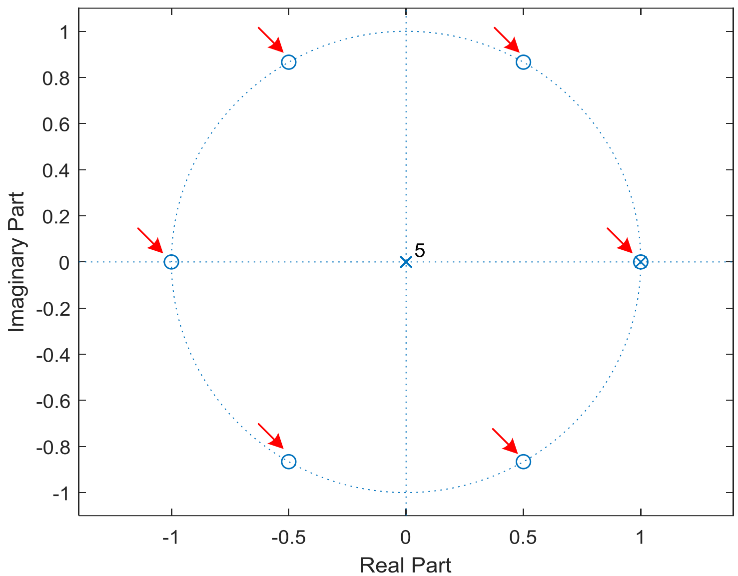
$$X(z);\ X(e^{j\omega});\ X[k]$$

Solution

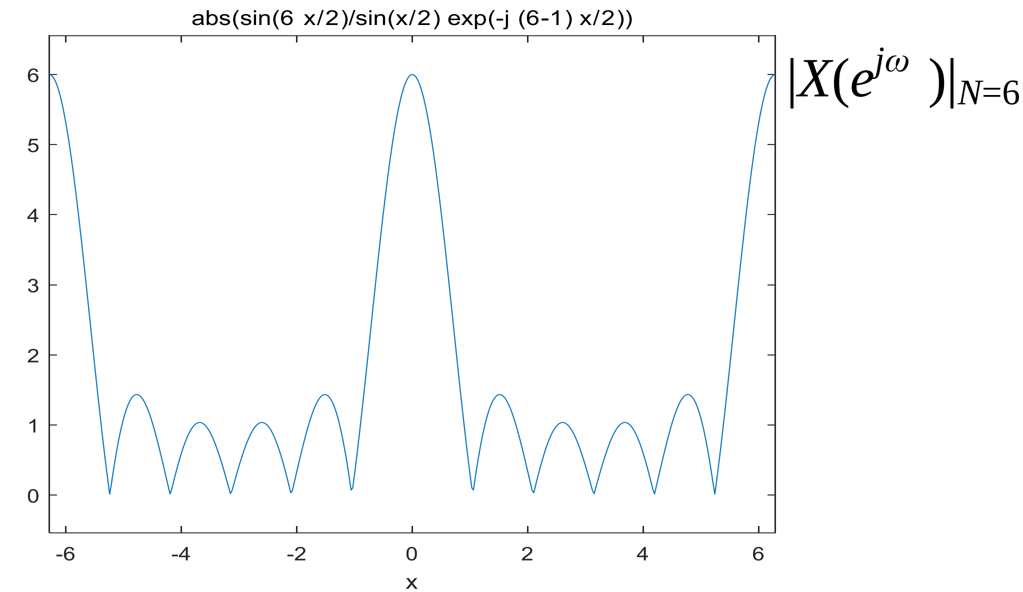
$$X(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}}$$

$$X(e^{j\omega}) = \frac{1-e^{-jN\omega}}{1-e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{e^{-jN\omega/2}\left(e^{jN\omega/2} - e^{-jN\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = \frac{\sin(N\omega/2)}{\sin(\omega/2)} e^{-j\frac{(N-1)}{2}\omega}$$

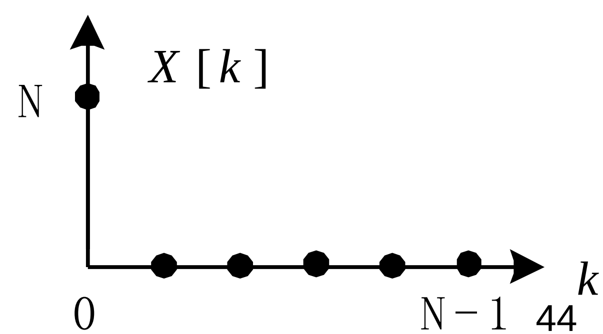


Samples in Z-plane (N=6)

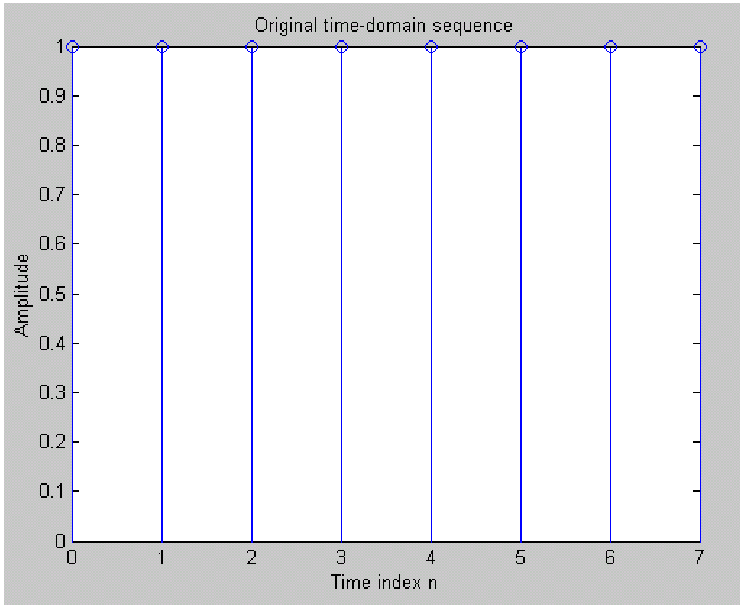


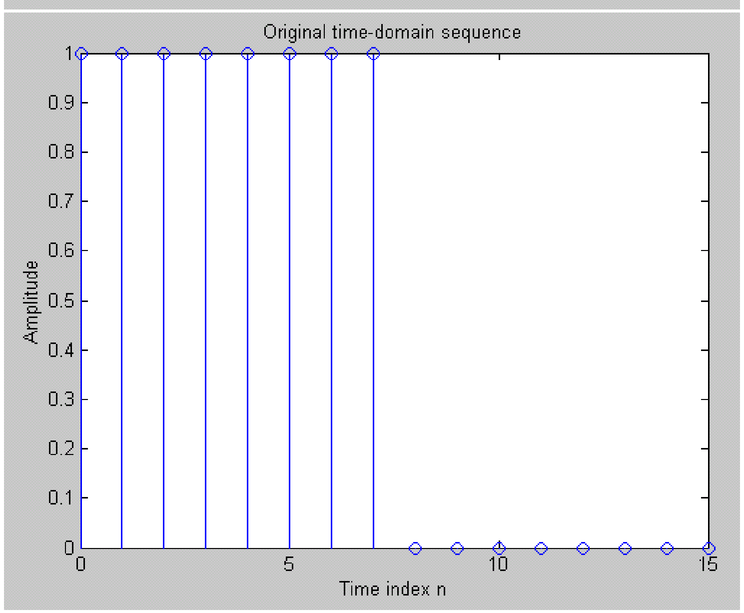
$$X[k] = X(e^{j\frac{2\pi}{N}k}) = \frac{\sin(k\pi)}{\sin(\frac{k\pi}{N})} e^{-j\frac{((N-1)\pi k)}{N}}$$

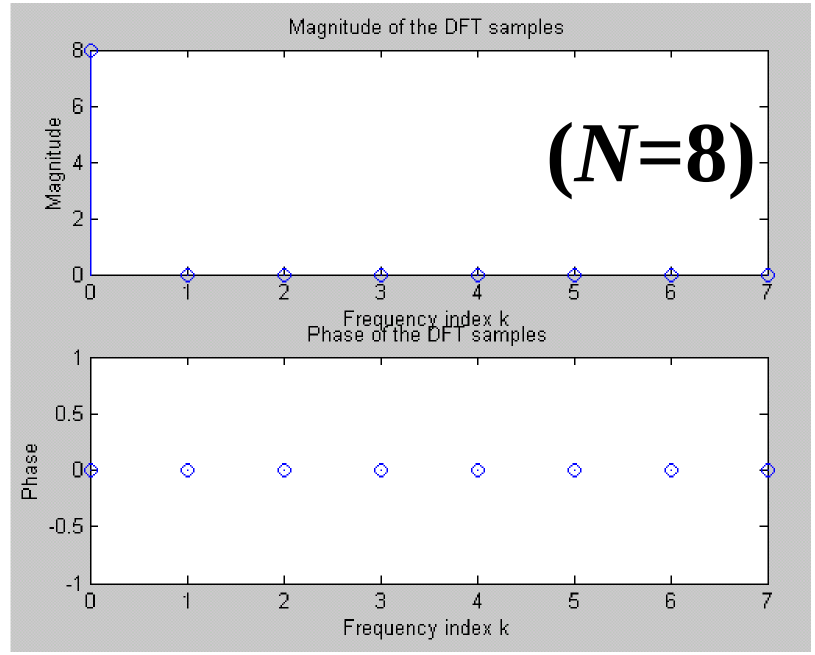
$$= \begin{cases} N, k = 0 \\ 0, k = 1 \sim N-1 \end{cases}$$

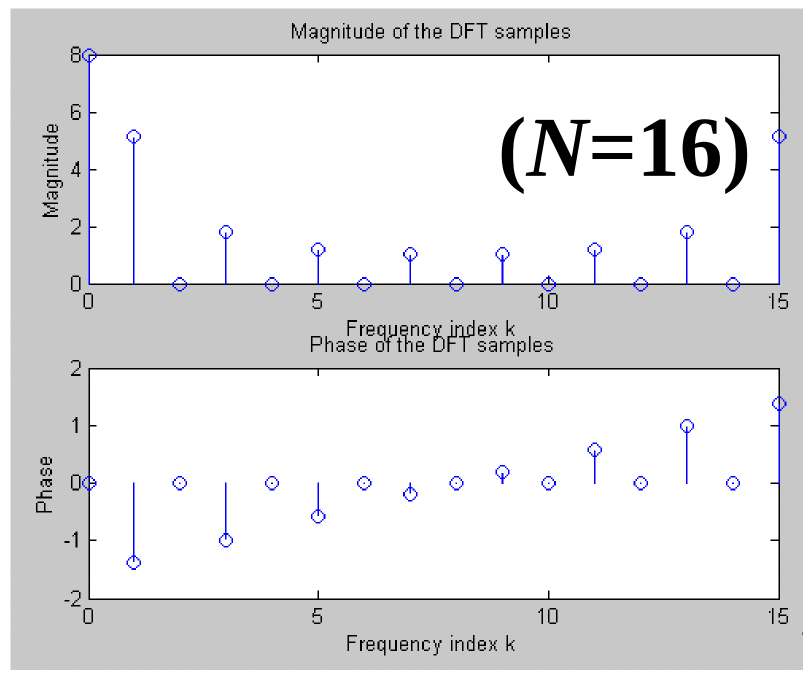


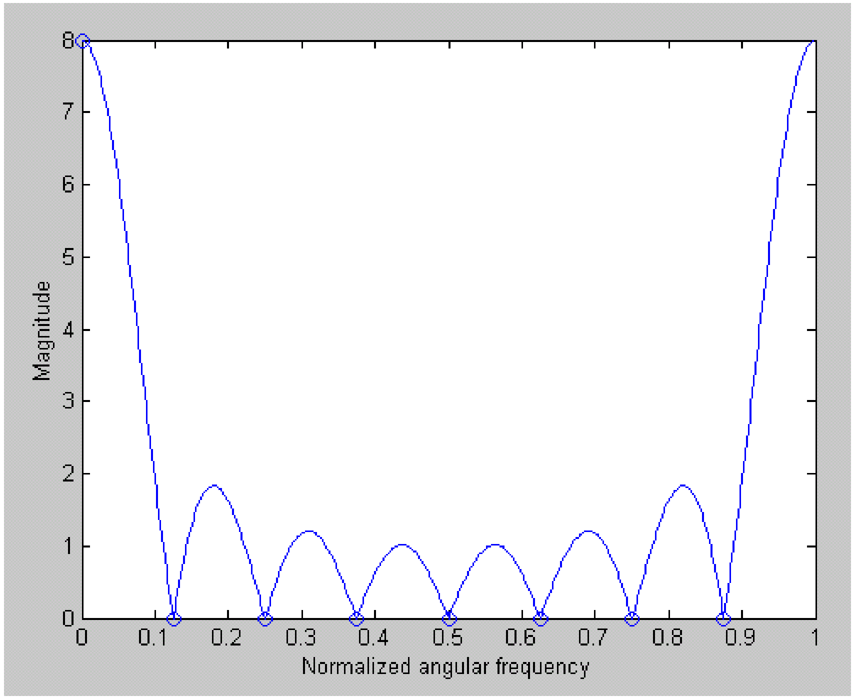
Example

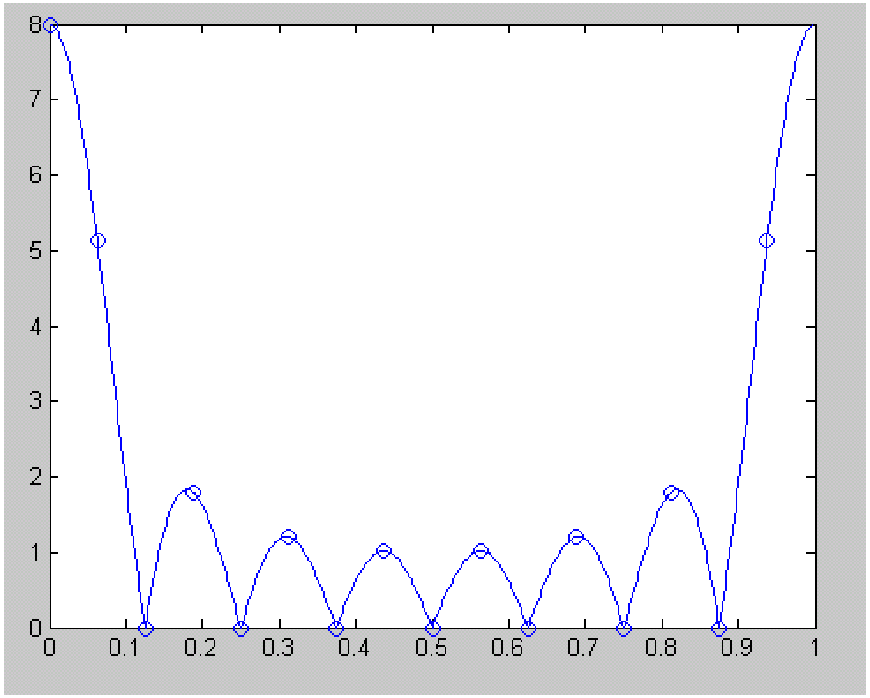

Original time-domain sequence
Amplitude
Time index n


Original time-domain sequence
Amplitude
Time index n


Magnitude of the DFT samples
Magnitude
(N=8)
Frequency index k
Phase of the DFT samples
Phase
Frequency index k


Magnitude of the DFT samples
Magnitude
(N=16)
Frequency index k
Phase of the DFT samples
Phase
Frequency index k





DTFT and DFT (*N*=8, *N*=16)

2. Interpolation In *Z*-Domain

Using N-point samples $X[k]=X(z_k)$, we can obtain $X(z)$ of N-point length sequence $x[n]$.

$$X(z)=\sum_{k=0}^{N-1}X[k]\Phi_k(z)$$

where $$\Phi_k(z)=\frac{1}{N}\frac{1-z^{-N}}{1-W_N^{-k}z^{-1}}$$

are called as interpolation functions.

Proof:

$$X(z)=\sum_{n=0}^{N-1}x[n]z^{-n}=\sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}\right]z^{-n}$$

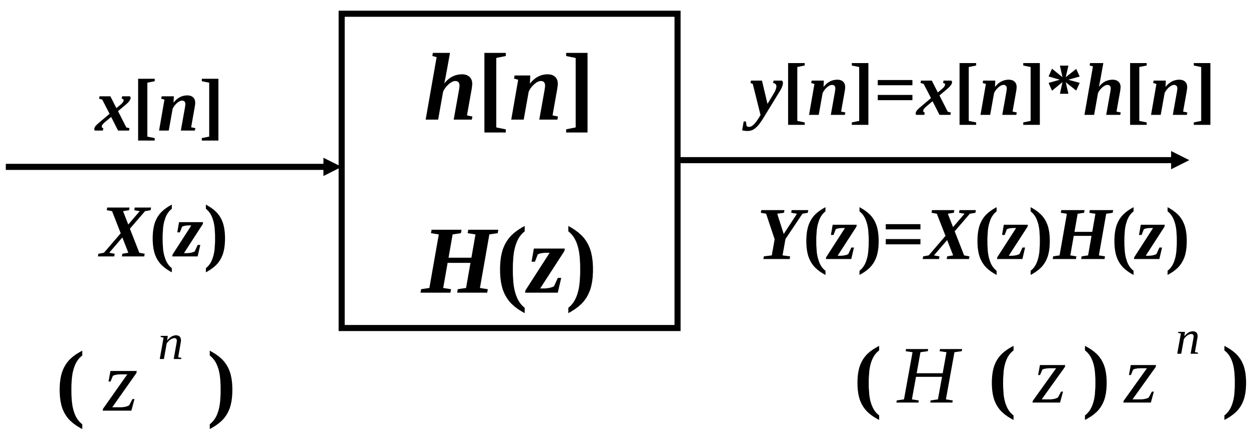
$$X(z) = \sum_{k=0}^{N-1} X[k]\left[\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-kn} z^{-n}\right]$$

$$= \sum_{k=0}^{N-1} X[k]\frac{1}{N}\frac{1-z^{-N}}{1-W_N^{-k}z^{-1}}$$

Interpolation Functions

6.6 Analysis and Characterization of LTI systems Using ZT (6.7)

Consider a LTI system:



1. Causality

(1) A causal system $\Longrightarrow H(z)$, **ROC:** $(|z| > r_1)$

$(h[n] = 0, n < 0.)$ **exterior of a circle**

(including infinity)

(2) For rational $H(z) = \dfrac{N(z)}{D(z)}$,

A causal system $\Longleftrightarrow$ **(a) ROC:** $(|z| > r_1)$

exterior of a circle outside the outmost pole (r_1)

(b) The order of the numerator $N(z)$ cannot be greater than the order of the denominator $D(z)$.

Example 1

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

Even ROC: $|z| > r_1$

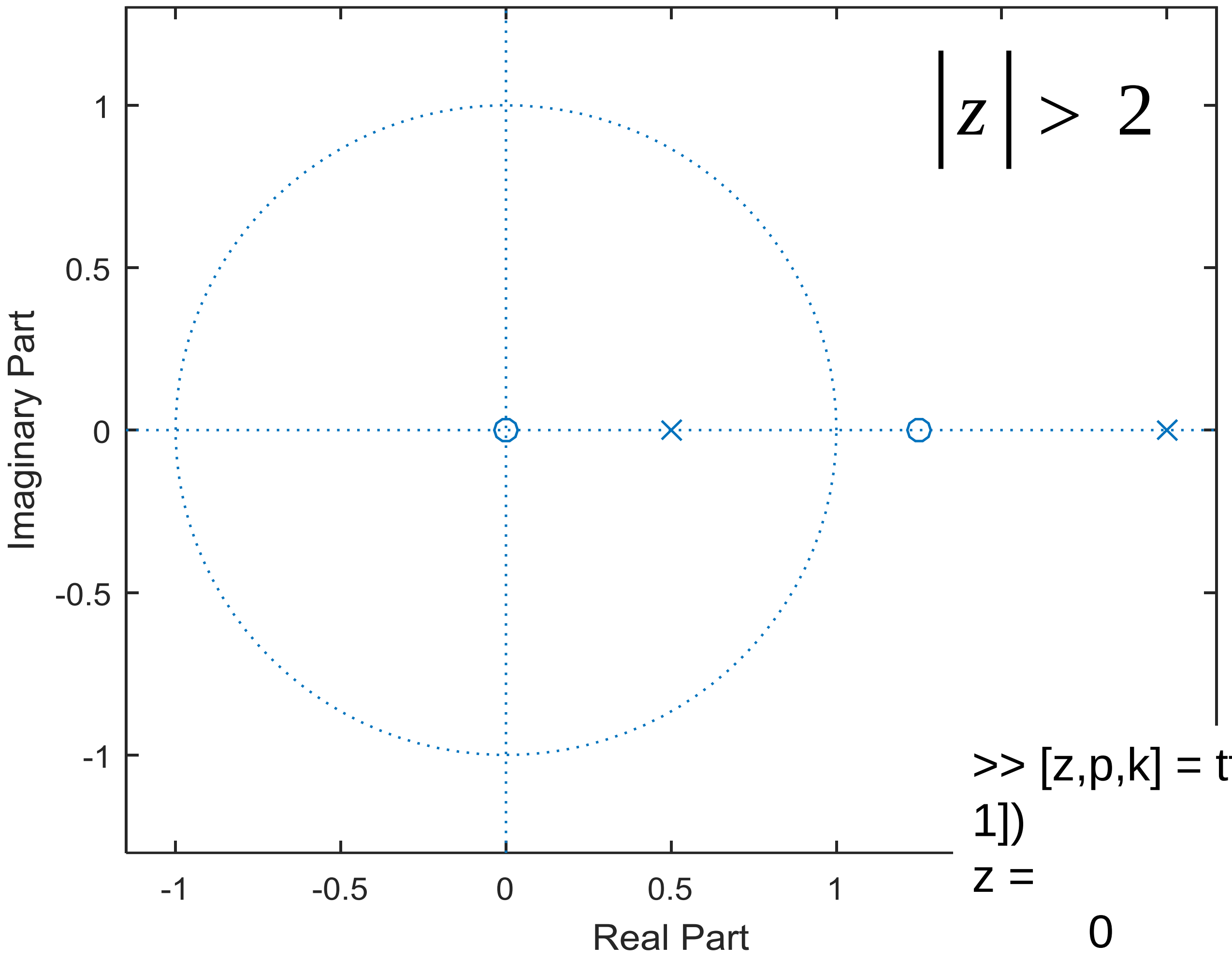
not causal system

Example 2

$$H(z) = \frac{2 - 2.5z^{-1}}{(1 - 0.5z^{-1})(1 - 2z^{-1})}$$

$$= \frac{2z^2 - 2.5z}{z^2 - 2.5z + 1}, \quad |z| > 2$$

causal system




```
>> [z,p,k] = tf2zp([2 -2.5 0],[1 -2.5 1])
z =
        0
   1.2500
p =
    2.0000
    0.5000
k =
     2
```

2. Stability

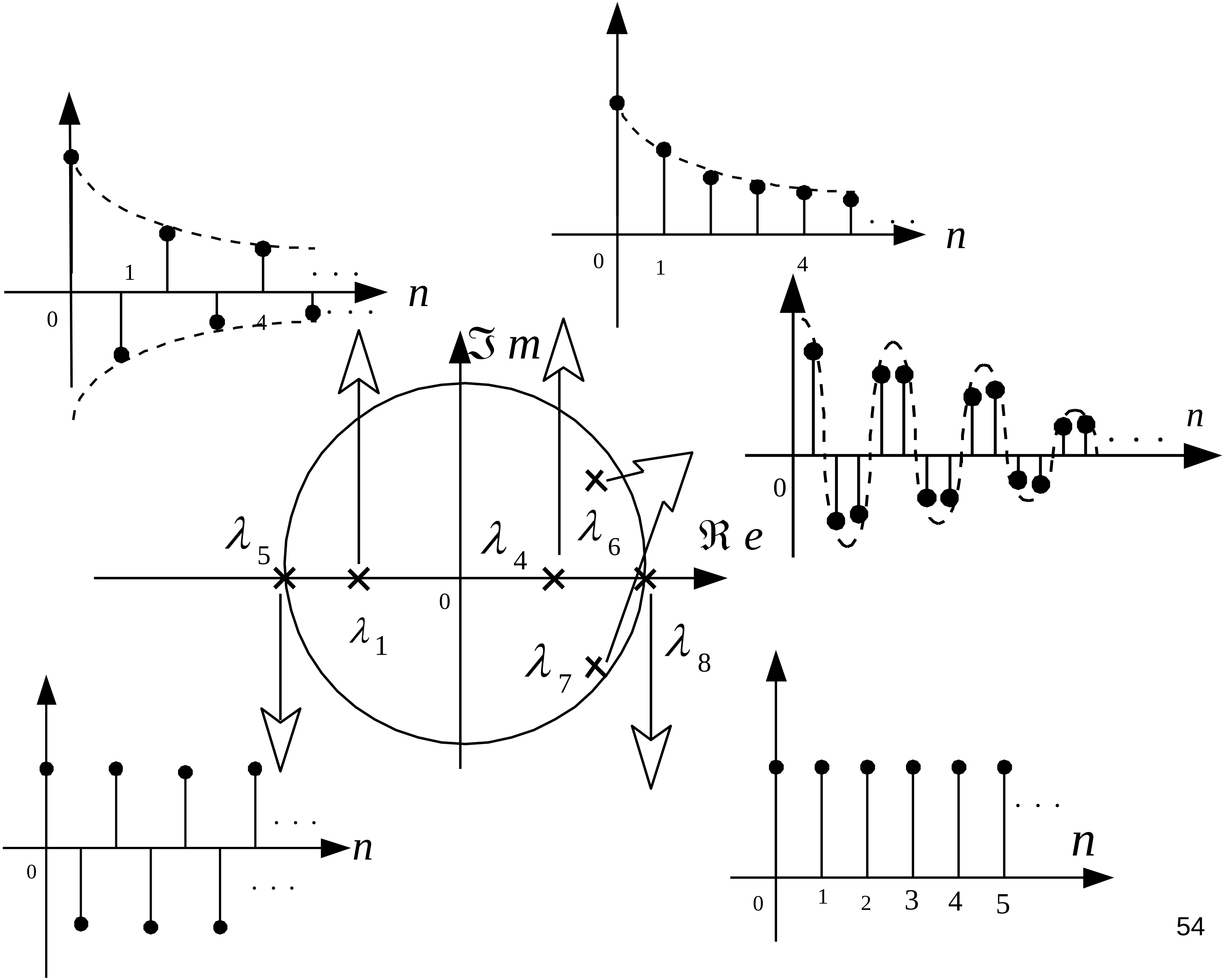
(1) A stable system $\Leftrightarrow H(z)$, **ROC: Includes |z|=1**

(the unit circle)

(2) A causal stable system with rational

$$H(s) = \frac{N(z)}{D(z)}, \quad \Leftrightarrow$$

All poles lies inside the unit circle of z-plane

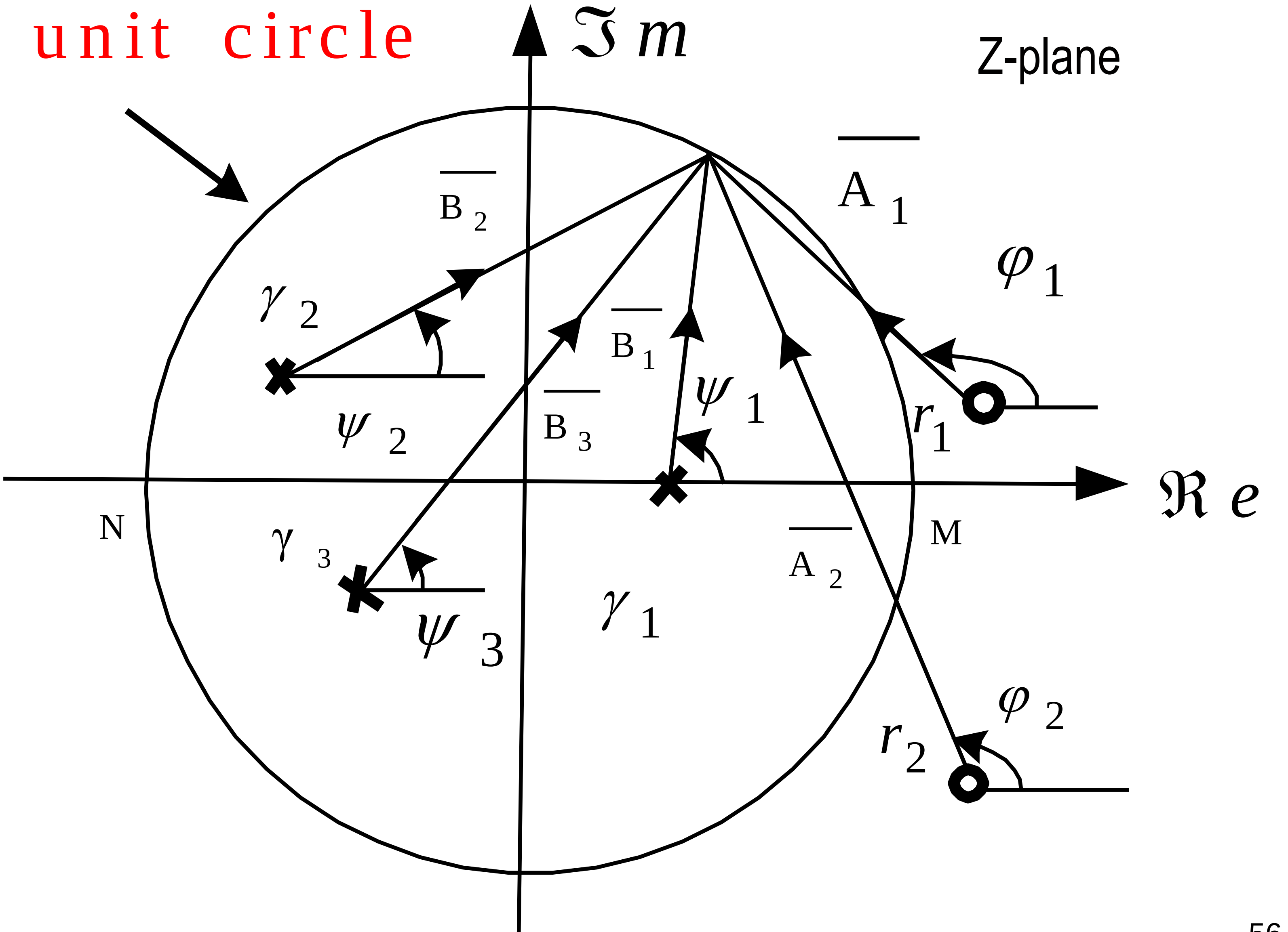


$\Im m$
$\Re e$
λ_1
λ_4
λ_5
λ_6
λ_7
λ_8
0
n
0
1
4
n
0
1
4
n
0
n
0
n
0
1
2
3
4
5

3. Pole-Zero Plot of $H(z)$ and Evaluation of Frequency Response $H(e^{j\omega})$

$$H(z) = \frac{P(z)}{D(z)} \qquad H(z) = z^{N-M} \frac{p_0 \prod_{i=1}^{M} (z - \gamma_i)}{d_0 \prod_{i=1}^{N} (z - \lambda_i)}$$

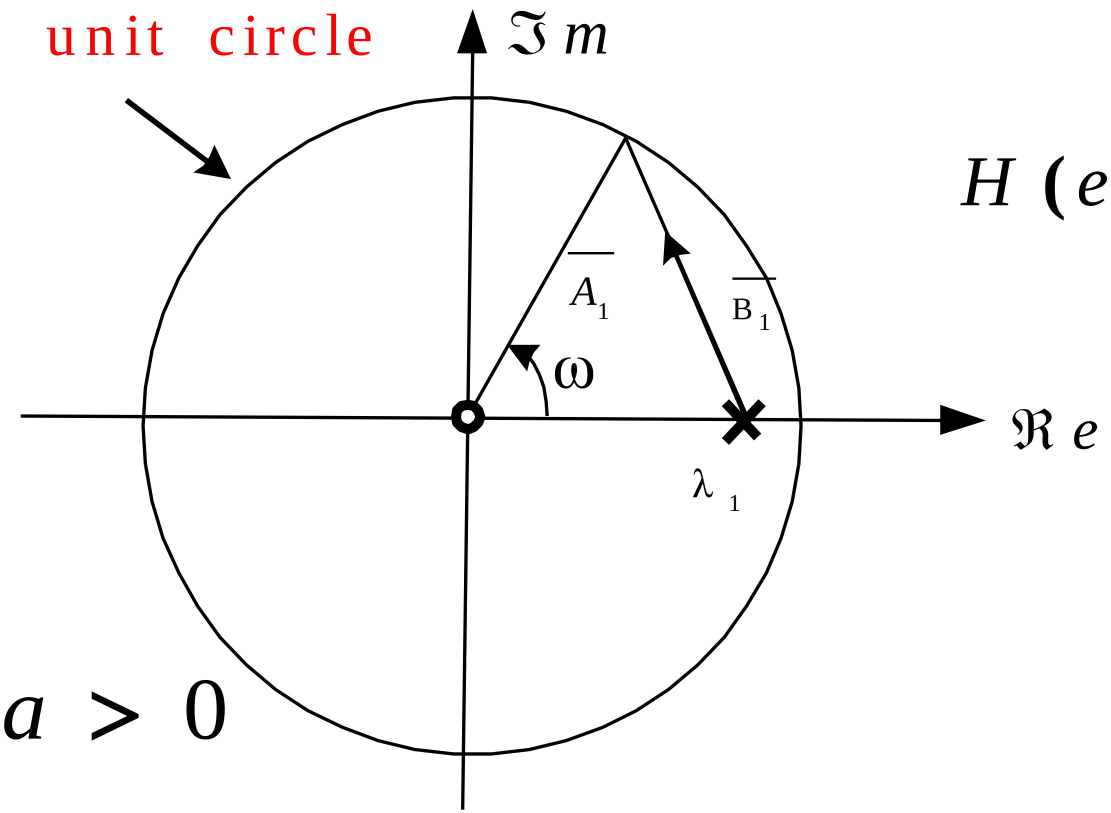
$$H(e^{j\omega}) = e^{j\omega(N-M)} \frac{p_0 \prod_{i=1}^{M} (e^{j\omega} - \gamma_i)}{d_0 \prod_{i=1}^{N} (e^{j\omega} - \lambda_i)}$$


unit circle
$\Im m$
Z-plane
$\overline{A}_1$
$\overline{B}_2$
φ_1
γ_2
$\overline{B}_1$
$\overline{B}_3$
ψ_1
r_1
ψ_2
$\Re e$
N
M
γ_3
$\overline{A}_2$
γ_1
ψ_3
φ_2
r_2

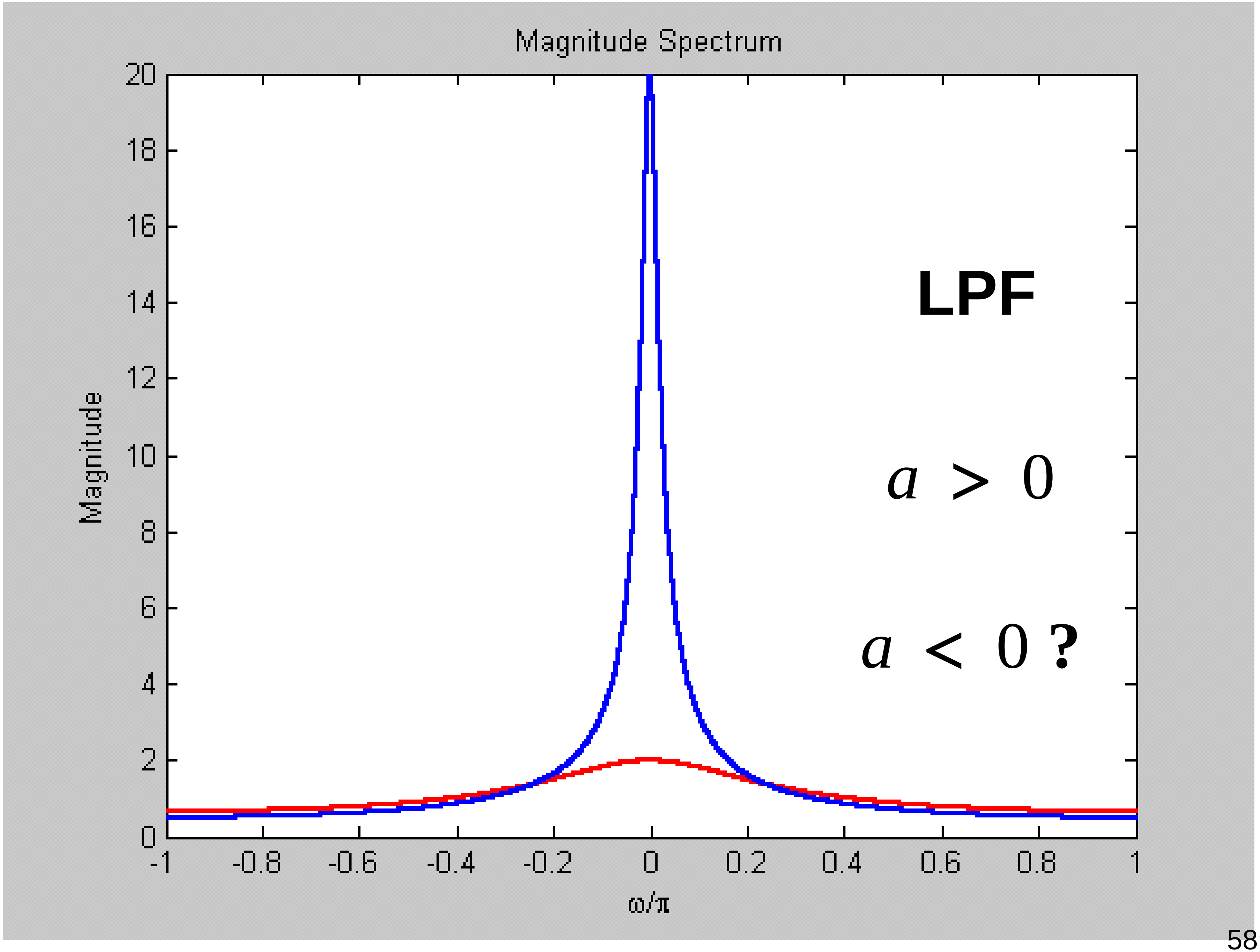
Example   first order systems

$$h[n] = a^n u[n] \qquad H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

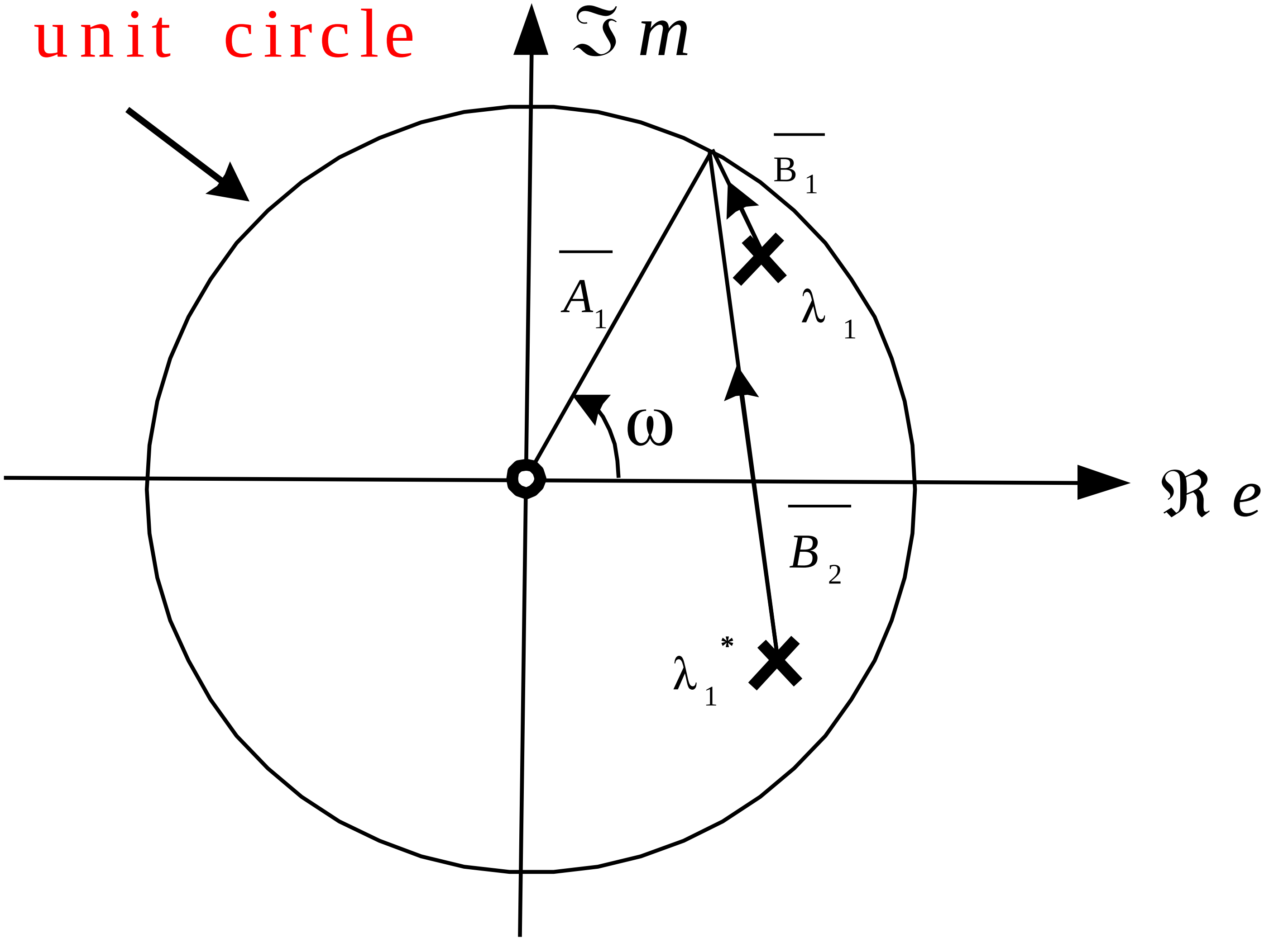
$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}$$

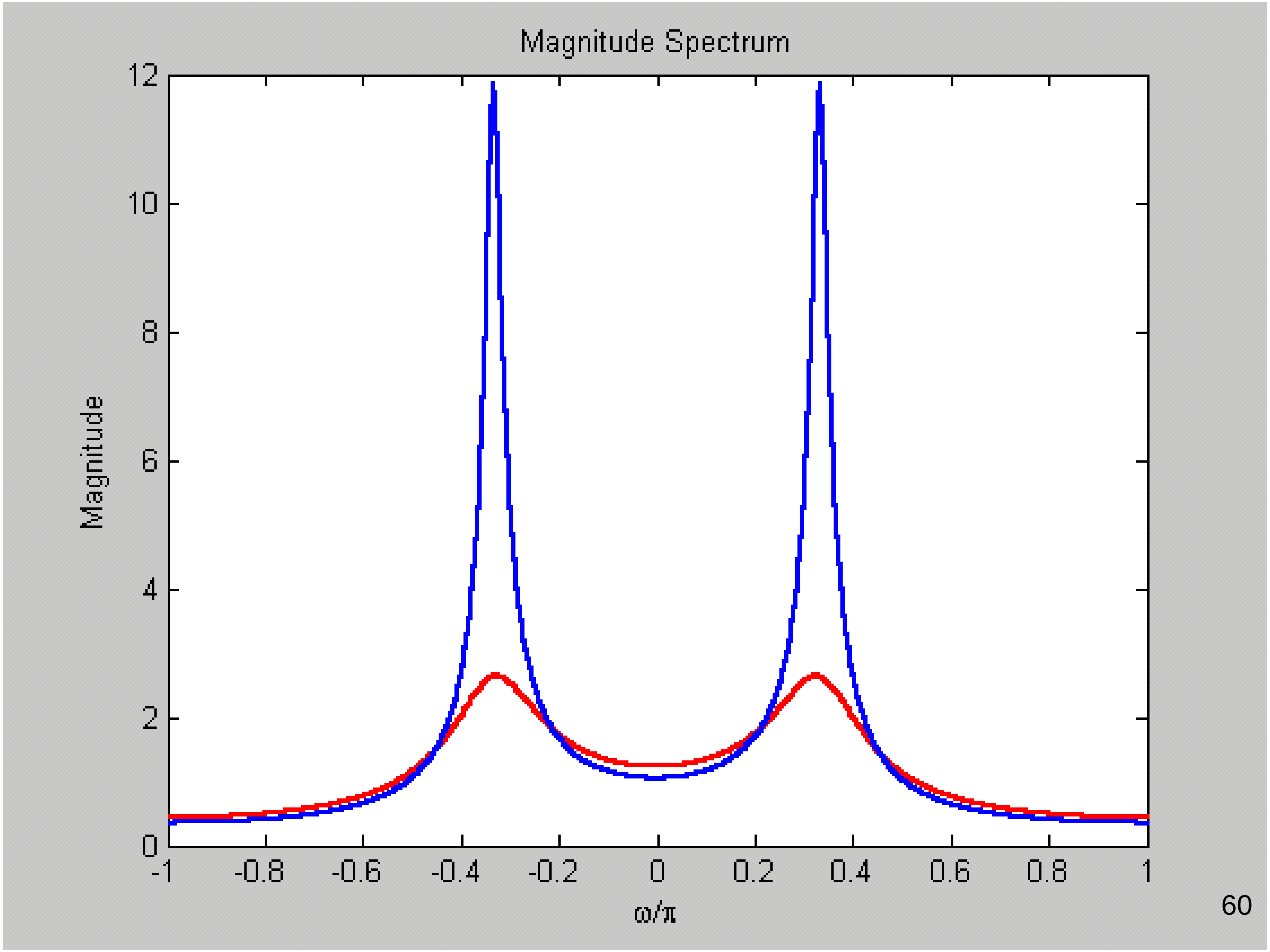


$a > 0$


Magnitude Spectrum
Magnitude
ω/π
20
18
16
14
12
10
8
6
4
2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
LPF
a > 0
a < 0 ?

Example second order systems




Magnitude Spectrum
Magnitude
0
2
4
6
8
10
12
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
ω/π

4. LTI Systems Characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

***Z*-transform:** $$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)} \qquad \text{(rational)}$$

Usually, a practical system is causal and stable.

Example

$$\boxed{y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]}$$

$$H(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

ROC1: $|z| > 1/2$

$$h[n] = (\frac{1}{2})^n u[n] + \frac{1}{3}(\frac{1}{2})^{n-1} u[n-1]$$

ROC2: $|z| < 1/2$

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

Property on Slide 4

5. Computation of the Convolution Sum of Finite-Length Sequences (6.6)

Two Finite-Length Sequences: $0 \le n \le N-1$

(1) Linear Convolution

$$y_L[n] = \sum_{m=0}^{N-1} x[m] \ h[n-m] \quad \xleftarrow{Z}\!\!\longrightarrow \quad Y_L(z) = X(z)H(z)$$

$$0 \le n \le 2N-2$$

Order $2N-2$

(2) Circular Convolution

$$y_C[n] = \sum_{m=0}^{N-1} x[m]h[<n-m>_N] \quad \xleftarrow{Z}\!\!\longrightarrow \quad Y_C(z) = <Y_L(z)>_{(z^{-N}-1)}$$

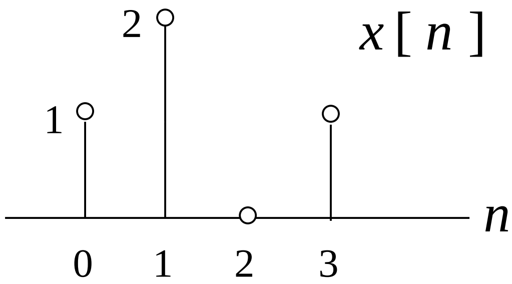
$$0 \le n \le N-1$$

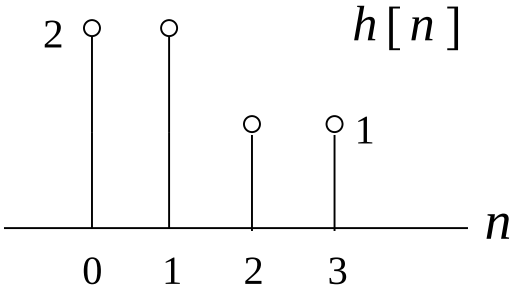
Order $N-1$

$$< Y_L(z) >_{(z^{-N}-1)}$$

is modulo operation by setting $z^{-N} = 1$ **in** $Y_L(z)$

Example





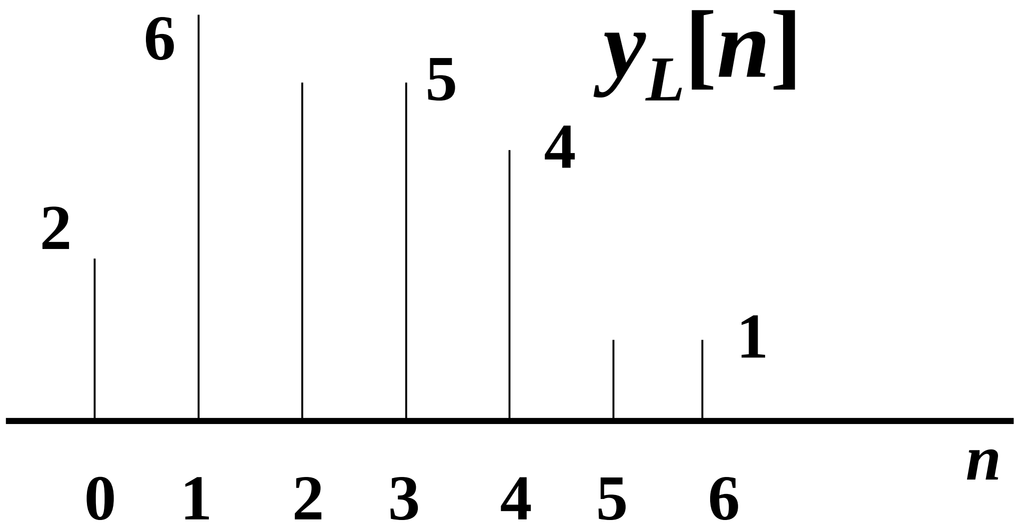
$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$

$$Y_L(z) = X(z)H(z)$$

$$= y_L[0] + y_L[1]\,z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3}$$

$$+ y_L[4]\,z^{-4} + y_L[5]z^{-5} + y_L[6]z^{-6}$$



The 4-point DFT $G[k]$ of $g[n]$ is given by

$$Y_C(z) = \langle Y_L(z) \rangle_{(z^{-N}-1)}$$

$$\boldsymbol{N = 4}$$

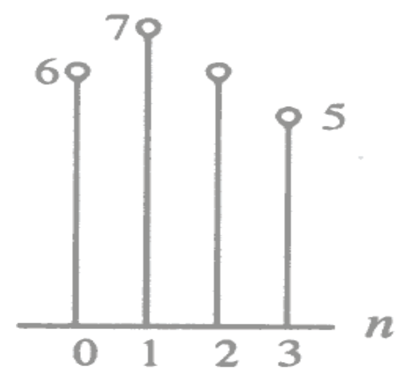
$$\langle Y_L(z) \rangle_{(z^{-N}-1)} = y_L[0] + y_L[1] \quad z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3}$$

$$+ y_L[4] \quad + y_L[5]z^{-1} + y_L[6]z^{-2}$$

$$= (y_L[0] + y_L[4]) + (y_L[1] \quad + y_L[5])z^{-1} +$$

$$(y_L[2] + y_L[6])z^{-2} + y_L[3]z^{-3}$$

$$Y_C(z) \quad = y_C[0] + y_C[1] \quad z^{-1} + y_C[2]z^{-2} + y_C[3]z^{-3}$$



$y_C[n]$

Thanks!

Any questions?