Digital Signal Processing

Discrete Time Signal in Frequency Domain

Wenhui Xiong

NCL

UESTC

whxiong@uestc.edu.cn

Outline

- Continuous-Time Fourier Transform
- Discrete Time Fourier Transform
- Band Limited Discrete Time Signal

Outline

- Continuous-Time Fourier Transform
- Discrete Time Fourier Transform
- Band Limited Discrete Time Signal

Continuous Time Fourier Transform(review)

$$f(t) \xleftrightarrow{CTFT} F(j\Omega)$$

Properties

- Linear $$af_1(t) + bf_2(t) \longleftrightarrow aF_1(j\Omega) + bF_2(j\Omega)$$
- Time shift $$f(t-t_0) \longleftrightarrow F(j\Omega)e^{-j\Omega t_0}$$
- Freq. shift $$e^{j\Omega_0 t} f(t) \longleftrightarrow F(j(\Omega-\Omega_0))$$
- Parseval $$\mathbb{E} = \int |f(t)|^2 dt = \frac{1}{2\pi}\int |F(j\Omega)|^2 d\Omega$$
- Convolution $$f_1(t) \circledast f_2(t) \longleftrightarrow F_1(j\Omega) F_2(j\Omega)$$

Outline

- Continuous-Time Fourier Transform
- Discrete Time Fourier Transform
- Band Limited Discrete Time Signal

Discrete Time Fourier Transform(DTFT)

Definition

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

Analyze

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Syn.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jwn}\,d\omega$$

Proof

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{l=-\infty}^{\infty} x[l]e^{-j\omega l} e^{jwn}\,d\omega = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi} x[l]e^{-j\omega l}e^{jwn}\,d\omega$$

$$= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_{-\pi}^{\pi} x[l]e^{jw(n-l)}\,d\omega = \sum_{l=-\infty}^{\infty} x[l]\frac{\sin\pi(n-l)}{\pi(n-l)} = x[n]$$

$$\frac{\sin\pi(n-l)}{\pi(n-l)} = \begin{cases} 1 & n = l \\ 0 & n \neq l \end{cases}$$

Discrete Time Fourier Transform(DTFT)

Example

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

$$x[n] = a^n u[n]$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

Discrete Time Fourier Transform(DTFT)

Convergence: Uniform Convergence

If $|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$

Absolute Summable $\Rightarrow$ **Uniform Convergence**

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete Time Fourier Transform(DTFT)

Convergence: Mean Square Convergence

Not absolute Summable

↓

MSE Convergence

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

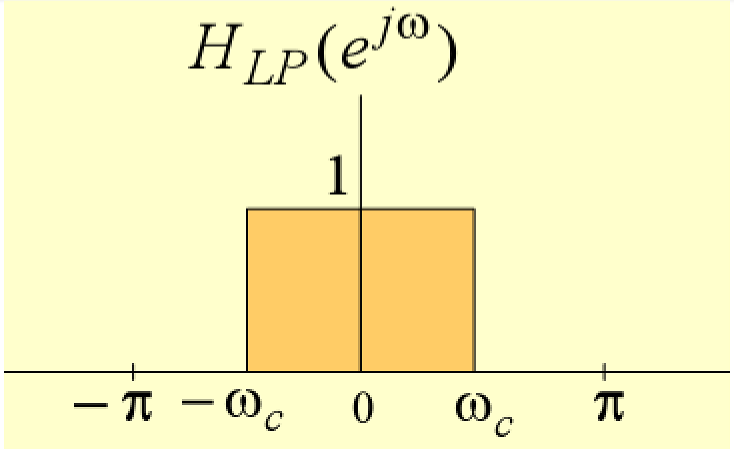
$$\lim_{K\to\infty}\int_{-\pi}^{\pi} |X_K(e^{j\omega}) - X(e^{j\omega})|^2 d\omega = 0$$

- the absolute may not go to zero as K goes to ∞
- the DTFT is no longer bounded

Discrete Time Fourier Transform(DTFT)

Example : Low Pass Filter

$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$



$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\frac{1}{jn}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

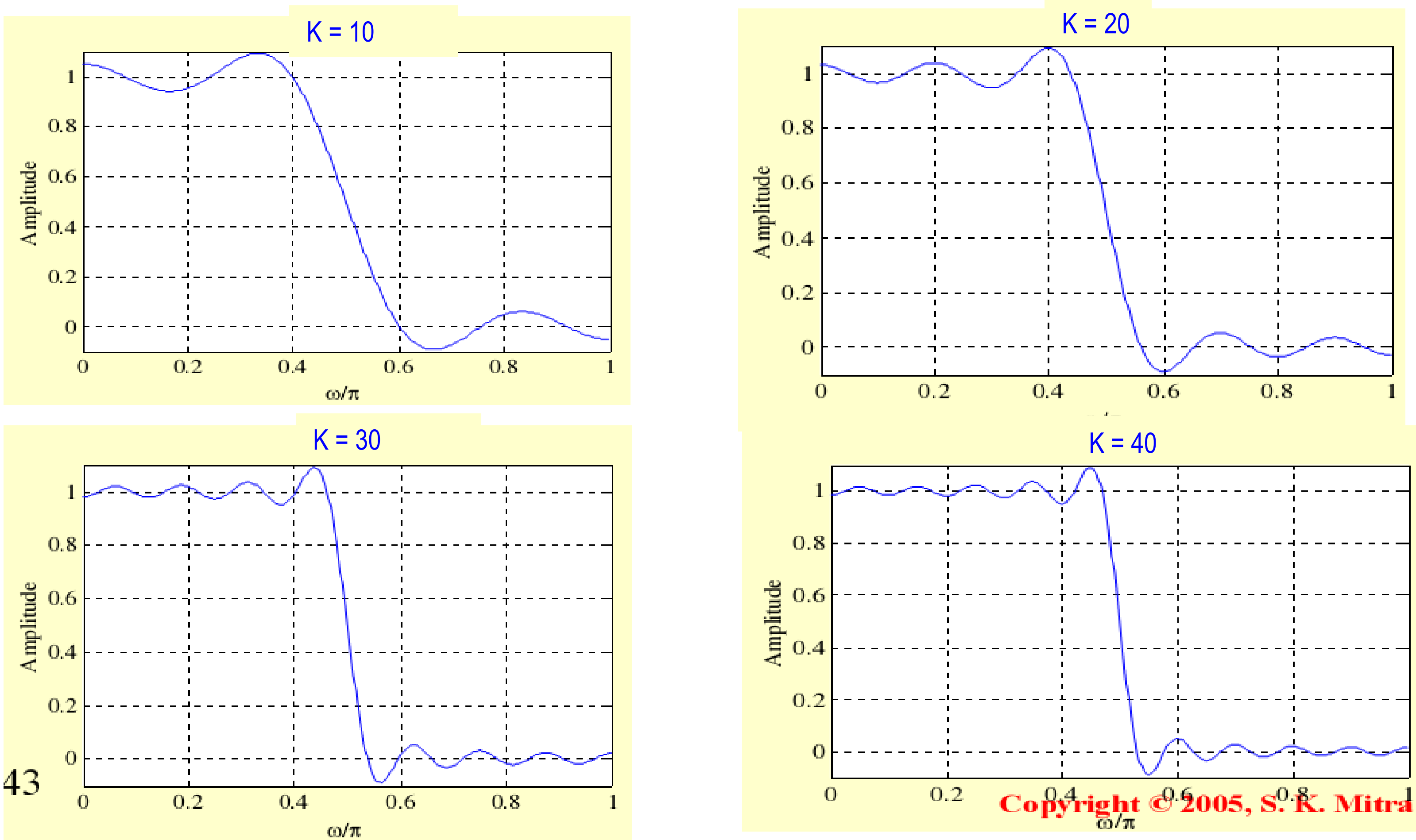
$$= \frac{\sin(\omega_c n)}{\pi n}$$

- $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable --converge in MSE sense

Discrete Time Fourier Transform(DTFT)

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}$$

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



Discrete Time Fourier Transform(DTFT)

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

- Independent of *K*, there are ripples around $w=w_c$
- The number of ripples increases as *K* increases
- same height of the largest ripple for different *k*
- Mean square converge at the discontinuity point: Gibbs phenomenon.

$$\lim_{K \to \infty} \int_{-\pi}^{\pi} |H_{LP,K}(e^{j\omega}) - H_{LP}(e^{j\omega})|^2 d\omega = 0$$

Discrete Time Fourier Transform(DTFT)

Properties: Periodic

$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2kn\pi}$$

$$= X(e^{j\omega})$$

DTFT: continuous function of ω
with period of 2π

Discrete Time Fourier Transform(DTFT)

Properties: Amplitude & Phase

$X(e^{j\omega})$: complex function $X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$

$$X_{\text{re}}(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{j\omega})]$$

$$X_{\text{im}}(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{j\omega})]$$

Alternatively

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)}$$

Amplitude $|X(e^{j\omega})|$

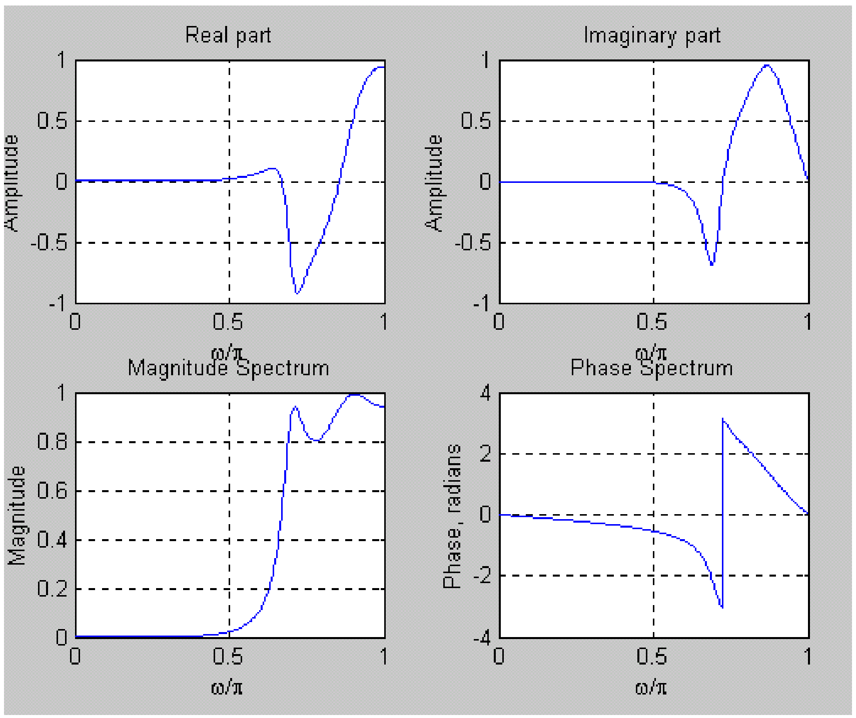
Phase $\theta(\omega)$

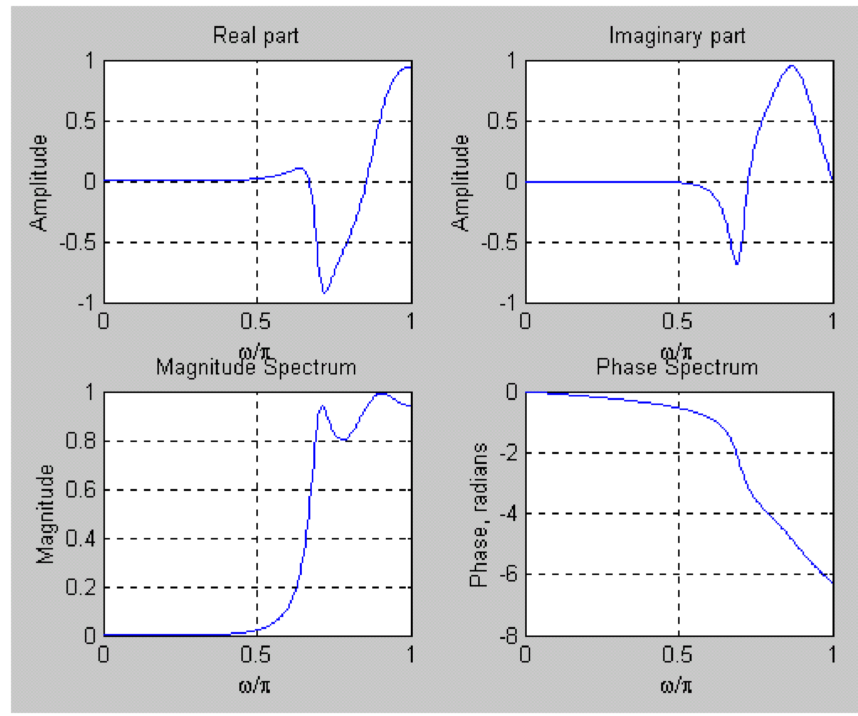
$\Rightarrow$ continuous of ω *with* period of 2π

$$\boxed{\text{Principal value} \quad -\pi \leq \theta(\omega) \leq \pi}$$

Properties: Unwrap Phase

- Principle phase may not be continuous—needs unwrap





unwrapped

wrapped

Discrete Time Fourier Transform(DTFT)

Properties: Symmetry Relations

$x[n]$ can be expressed as $x[n] = x_e[n] + x_o[n]$

Even sequence $$x_e[n] = \frac{1}{2}(x[n] + x^*[-n])$$

$$x_e[n] = x_e^*[-n]$$ Conjugate symmetric

Odd sequence $$x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$$

$$x_o[n] = -x_o^*[-n]$$ Conjugate asymmetric

$X(e^{j\omega})$ can be expressed as $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] \qquad X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})]$$

Discrete Time Fourier Transform(DTFT)

Properties: Symmetry Relations

$$x^*[n] \xleftrightarrow{DTFT} X^*(e^{-j\omega}) \qquad x^*[-n] \xleftrightarrow{DTFT} X^*(e^{j\omega})$$

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\text{Im}\{x[n]\}$	$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$	$jX_{\text{im}}(e^{j\omega})$

Note: $X_{\text{cs}}(e^{j\omega})$ and $X_{\text{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\text{cs}}[n]$ and $x_{\text{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Discrete Time Fourier Transform(DTFT)

Properties: Symmetry Relations

Real Sequence

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

Discrete Time Fourier Transform(DTFT)

DTFT of Some Special Sequences

neither absolutely summable nor square summable

Sinusoidal

$$x[n] = A\cos(\omega_0 n + \phi) \qquad \text{for all } n$$

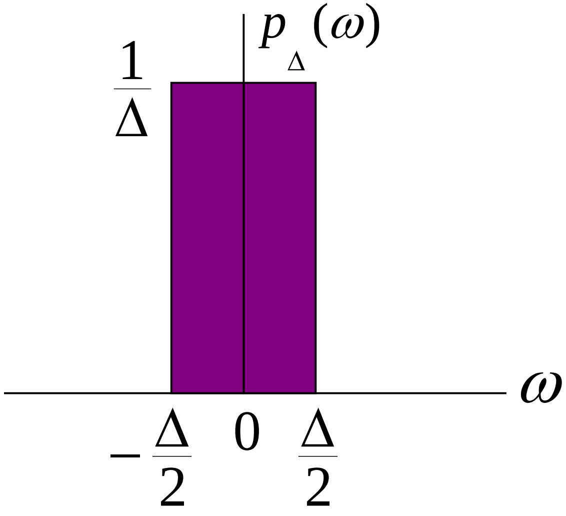
Unit Step

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Exponential sequence

$$x[n] = A\alpha^n$$

$\delta(\omega)$: infinite height, zero width, and unit area



Complex Exponential sequence: Impulse Train

$$x[n] = e^{j\omega_0 n} \qquad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k) e^{j\omega n} d\omega$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \delta(\omega - \omega_0 + 2\pi k) e^{j\omega n} d\omega$$

$$= e^{j\omega_0 n - j2\pi k n} = e^{j\omega_0 n}$$

Discrete Time Fourier Transform(DTFT)

DTFT Pairs

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \ (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (\|\alpha\| < 1)$	$\frac{1}{1-\alpha e^{-j\omega}}$
$(n+1)\alpha^n \mu[n], \ (\|\alpha\| < 1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \ (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \|\omega\| \le \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$

Discrete Time Fourier Transform(DTFT)

DTFT Theorems

- Linear $\alpha g[n] + \beta h[n] \xleftrightarrow{DTFT} \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
- Time reversal $g[-n] \xleftrightarrow{DTFT} G(e^{-j\omega})$
- Time shift $g[n - n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} G(e^{j\omega})$
- Freq. shift $e^{j\omega_0 n} g[n] \xleftrightarrow{DTFT} G\left(e^{-j(\omega - \omega_0)}\right)$
- Convolution $g[n] \circledast h[n] \xleftrightarrow{DTFT} G(e^{-j\omega}) H(e^{-j\omega})$
- Modulation $g[n]h[n] \xleftrightarrow{DTFT} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{-j(\omega - \theta)}) d\omega$
- Parseval $\mathbb{E} = \sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) H^*(e^{j\omega}) d\omega$

Discrete Time Fourier Transform(DTFT)

- Diff in Freq. $$nx[n] \xleftrightarrow{DTFT} j\frac{dX(e^{j\omega})}{d\omega}$$

Example : $y[n] = (n+1)a^n u[n] \quad |a| < 1$

Let $x[n] = a^n u[n] \quad |a| < 1$ $\qquad y[n] = nx[n] + x[n]$

$$x[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega} = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2} + \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - ae^{-j\omega})^2}$$