Digital Signal Processing

Discrete Time System in Transform Domain

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Discrete Time System

LTI System analysis

LTI System is described by its Impulse response $h[n]$

$$y[n] = h[n] \circledast x[n]$$

Depending on the *length of its impulse response*:

- Finite impulse response (FIR) transfer function
- Infinite impulse response (IIR) transfer function

$$Y(z) = H(z)x(z) \qquad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$H(z)$ System Function

$H(e^{j\omega})$ Frequency Response

$$|H(e^{j\omega})| \angle H(e^{j\omega})$$

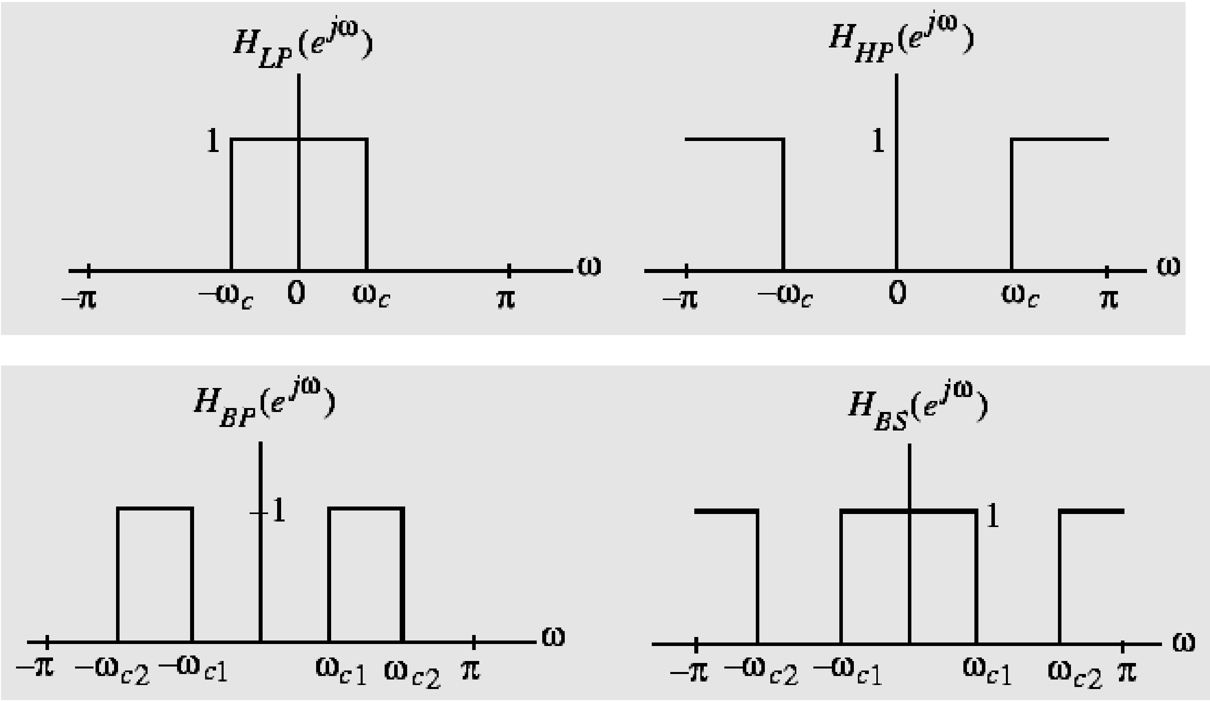
Ideal Filter

LPF $H(e^{j\omega}) = 1 \;\; \text{for} \;\; |\omega| < \omega_c \quad \Rightarrow \quad h_{lp}[n] = \dfrac{\sin \omega_c n}{\pi \omega_c n}$

HPF $H(e^{j\omega}) = 1 \;\; \text{for} \;\; \omega_c < |\omega| < \pi \quad \Rightarrow \quad h_{hp}[n] = \delta[n] - h_{lp}[n]$

- magnitude response equal to 1 in the passband and 0 in the stopband
- 0 phase everywhere

Discrete Time System



Ideal Filter

- Lowpass filter: Passband: $0 \le \omega \le \omega_c$

  Stopband: $\omega_c \le \omega \le \pi$
- Highpass filter: Passband: $\omega_c \le \omega \le \pi$

  Stopband: $0 \le \omega \le \omega_c$
- Bandpass filter:

  Passband: $\omega_{c1} \le \omega \le \omega_{c2}$

  Stopband: $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega < \pi$
- Bandstop filter:

  Stopband: $\omega_{c1} < \omega < \omega_{c2}$

  Passband: $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$

Discrete Time System

Example：ideal LPF

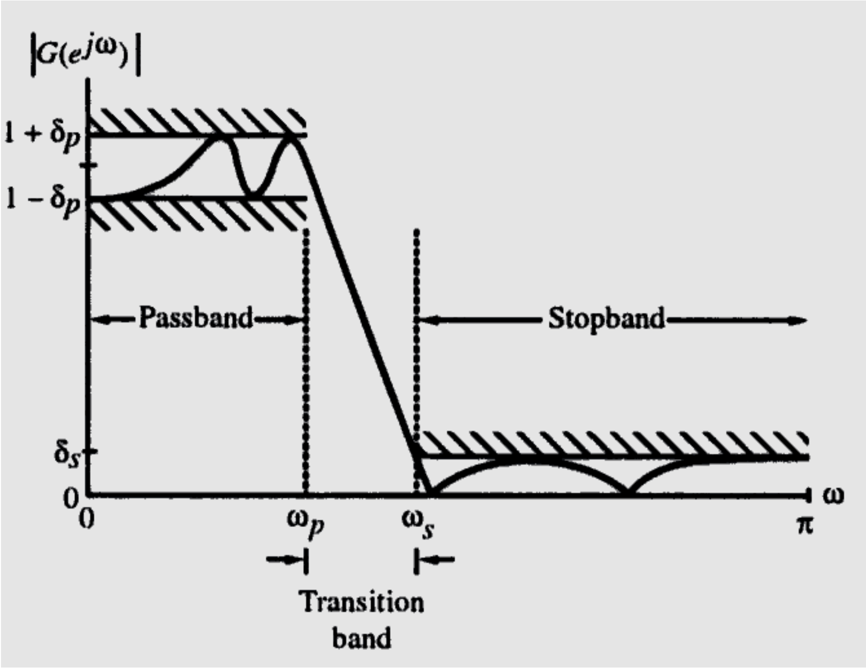
LPF $H(e^{j\omega}) = 1 \text{ for } |\omega| < \omega_c$ $\Rightarrow$ $h_{lp}[n] = \dfrac{\sin \omega_c n}{\pi \omega_c n}$

- not absolutely summable ->not BIBO stable
- infinite length-> not causal

the ideal filters with the ideal "*brick wall*" frequency responses are not realizable

Realizable Filter:

Relax specifications by including *transition band*



- magnitude response decays slowly to 0
- magnitude response varies by a small amount in passband and stopband

Discrete Time System

Phase Distortion and Delay

$$h[n] = \delta[n - n_d] \qquad H(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H(e^{j\omega})| = 1 \qquad \angle H(e^{j\omega}) = -\omega n_d$$

Group delay $$\tau(\omega) = -\frac{d}{d\omega}\angle H(e^{j\omega}) = n_d$$

- Group delay is a function of frequency
- Delay may be different for different frequencies

System Functions

Linear Constant Coefficient Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Z-Transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Downarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=0}^{M} (1 - c_k z^{-1})}{\prod_{k=0}^{N} (1 - d_k z^{-1})}$$

System Functions

Stability of the System

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

Stable if ROC includes unit circle

Causality of the System

$$h[n] = 0 \ \text{for} \ n < n_0$$

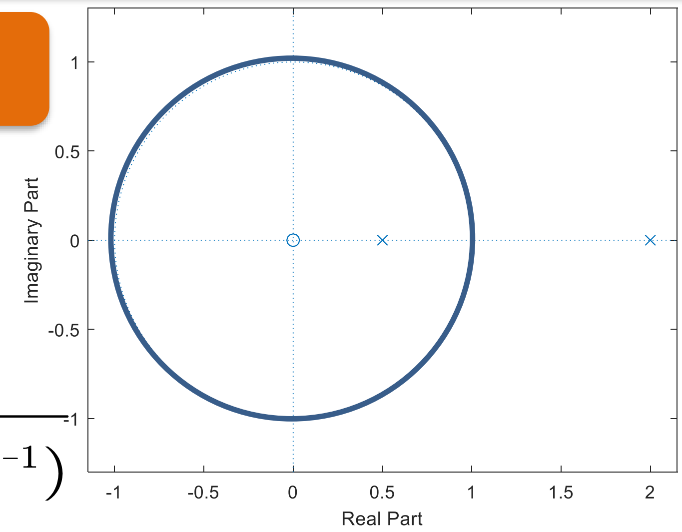
ROC must be outside the outermost pole

System Functions

Example

$$y[n]-\frac{5}{2}y[n-1]+y[n-2]=x[n]$$

$$H(z)=\frac{1}{1-5/2z^{-1}+z^{-2}}=\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}$$



Question: stable? Causal?

- If causal: ROC $|z|>2$, system not stable
- If stable: ROC $1/2<|z|<2$
- If ROC $|z|<1/2$

Inverse System

$$G(z) = H(z)H_i(z) = 1 \Rightarrow H_i(z) = \frac{1}{H(z)}$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=0}^{N}(1 - d_k z^{-1})} \Rightarrow H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=0}^{N}(1 - d_k z^{-1})}{\prod_{k=0}^{M}(1 - c_k z^{-1})}$$

- Poles of Hi(z) are the zeros of H(z)
- Zeros of Hi(z) are the poles of H(z)
- ROC of H(z) and Hi(z) must overlap

System Functions

Example

$$H(z)=\frac{1-0.5z^{-1}}{1-0.9z^{-1}} \quad |z|>0.9 \quad \Rightarrow \quad H_i(z)=\frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

To have overlap ROC, we have $|z|>0.5$

$$h_i[n]=(0.5)^n u[n]-0.9(0.5)^{n-1}u[n-1]$$

Causal and Stable

$$H(z)=\frac{z^{-1}-0.5}{1-0.9z^{-1}} \quad |z|>0.9 \quad \Rightarrow \quad H_i(z)=\frac{2-1.8z^{-1}}{1-2z^{-1}}$$

overlap ROC, we have $|z|>2$ or $|z|<2$

$$h_i[n]=-2(2)^n u[n]+1.8(2)^{n-1}u[n-1] \qquad |z|>2$$

$$h_i[n]=2(2)^n u[-n-1]-1.8(2)^{n-1}u[-n] \qquad |z|<2$$

System Functions

Summary

For Causal system with zeros at $c_k, \; k = 1 \cdots M$

- Inverse system is causal if and only if the ROC

$$|z| > \max_k (|c_k|)$$

- If with stable constrain

$$\max_k (|c_k|) < 1$$

Causal and Stable: $\max_k (|c_k|) < 1 \quad |z| > \max_k (|c_k|)$

Relationship between Magnitude & Phase

Magnitude of the System

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)\big|_{z=e^{j\omega}}$$

Define $C(z) = H(z)H^*(1/z^*)$

$$= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=0}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=0}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

From C(z)we can infer H(z)

- Poles, Zeros show up in conjugate reciprocal pair
- one inside unit circle, one outside unit circle

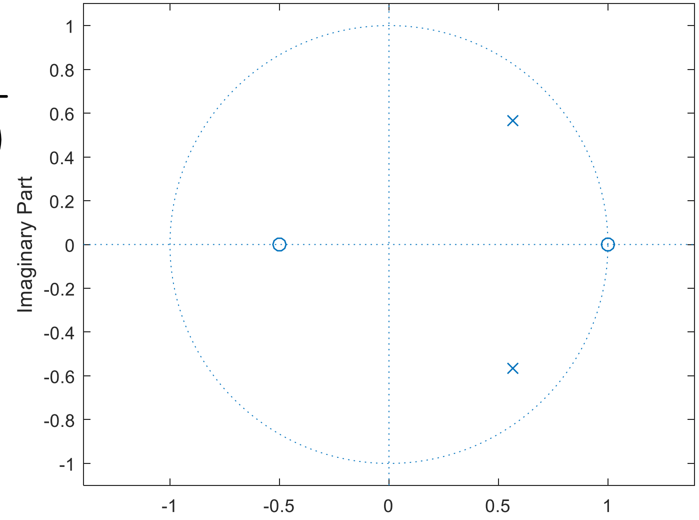
Relationship between Magnitude & Phase

Example

$$H_1(z) = \frac{2(1-z^{-1})(1+0.5z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

Zeros at $z=1, z=-0.5$

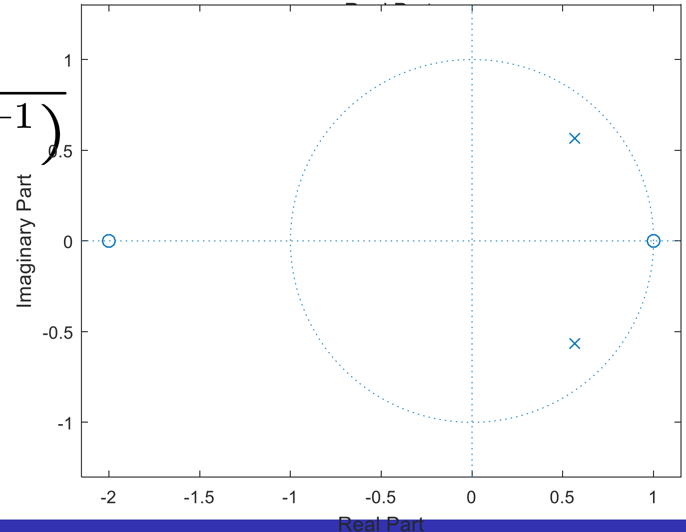
poles at $z=0.8e^{j\pi/4}, z=0.8e^{-j\pi/4}$



$$H_2(z) = \frac{(1-z^{-1})(1+2z^{-1})}{(1-0.8e^{j\pi/4}z^{-1})(1-0.8e^{-j\pi/4}z^{-1})}$$

Zeros at $z=1, z=-2$

poles at $z=0.8e^{j\pi/4}, z=0.8e^{-j\pi/4}$



Relationship between Magnitude & Phase

Example

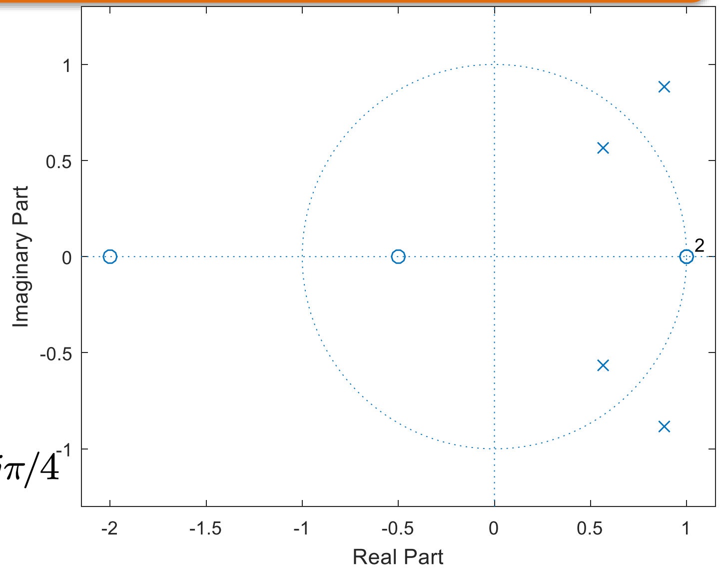
$$C_1(z) = H_1(z) H_1^*(1/z^*)$$

Zeros at $z=1, z=-0.5$

$z=1, z=-2$

poles at $z=0.8e^{j\pi/4}, z=0.8e^{-j\pi/4}$

$z=1/0.8e^{j\pi/4}, z=1/0.8e^{-j\pi/4}$



$$C_2(z) = H_2(z) H_2^*(1/z^*)$$

Zeros at $z=1, z=-2$

$z=1, z=-0.5$

poles at $z=0.8e^{j\pi/4}, z=0.8e^{-j\pi/4}$

$z=1/0.8e^{j\pi/4}, z=1/0.8e^{-j\pi/4}$

$$\boxed{C_1(z) = C_2(z)}$$

Relationship between Magnitude & Phase

All Pass System

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \quad \Rightarrow \quad H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$H_{ap}(e^{j\omega}) = e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \quad \Rightarrow \quad |H_{ap}(e^{j\omega})| = 1$$

$$\boxed{\text{All Frequencies Passes Through}}$$

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k-1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

Relationship between Magnitude & Phase

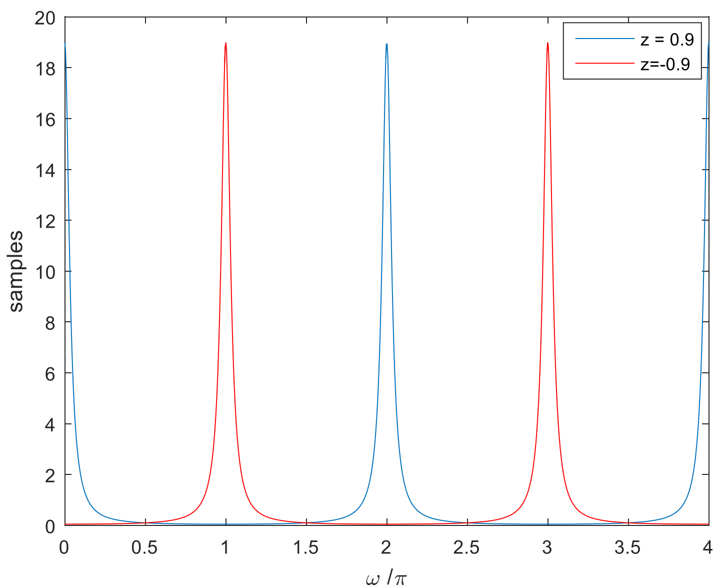
All Pass System

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \qquad a = re^{j\theta}u \qquad z = e^{j\omega}$$

$$\angle H_{ap}(e^{j\omega}) = \angle \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}} = -\omega - 2\,\mathrm{atan}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(w-\theta)}\right]$$

$$\tau_d = -\frac{d\angle H_{ap}(e^{j\omega})}{d\omega} = \frac{1-r^2}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

- Causal System: r<1
- Group delay: positive



Relationship between Magnitude & Phase

Minimum Phase System

System with poles and zeros are inside the unit circle

Any Rational System can be expressed as

$$H(z) = H_{\min}(z) H_{ap}(z)$$

Proof: assume just 1 zeros outside unit circle

$$H(z) = H_1(z)\left(z^{-1} - c^*\right) \qquad |C| < 1$$

Minimum phase

$$= H_1(z)\left(z^{-1} - c^*\right)\frac{1 - cz^{-1}}{1 - cz^{-1}} = H_1(z)\left(1 - cz^{-1}\right)\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

$$H_{\min}(z) = H_1(z)\left(1 - cz^{-1}\right) \qquad H_{ap}(z) = \frac{z^{-1} - c^*}{1 - cz^{-1}}$$

Example: Minimum Phase and All Pass Decomposition

$$H(z) = \frac{(1+3z^{-1})}{(1+1/2z^{-1})} \qquad |z| > 1/2$$

$$H(z) = \frac{3}{1+1/2z^{-1}}(z^{-1}+1/3) = \frac{3(1+1/3z^{-1})}{1+1/2z^{-1}} \frac{(z^{-1}+1/3)}{1+1/3z^{-1}}$$

$$H_{\min}(z) = \frac{3(1+1/3z^{-1})}{1+1/2z^{-1}} \qquad H_{ap}(z) = \frac{(z^{-1}+1/3)}{1+1/3z^{-1}}$$

Relationship between Magnitude & Phase

Minimum Phase System

non-min. phase system $H(z) = H_{\min}(z) H_{ap}(z)$

Its Phase： $\angle H(e^{j\omega}) = \angle H_{\min}(e^{j\omega}) + \angle H_{ap}(e^{j\omega})$

Group delay

$$\tau(H(e^{j\omega})) = \tau(\angle H_{\min}(e^{j\omega})) + \tau(\angle H_{ap}(e^{j\omega}))$$

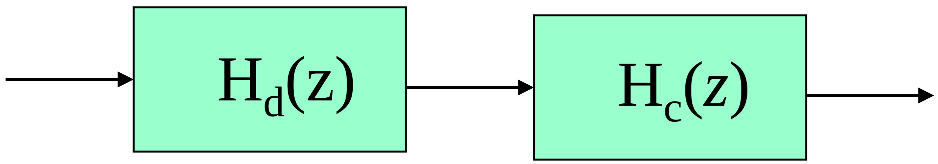
Positive Group delay for stable All-Pass System

$\Downarrow$

Minimum Phase System: Minimum Group delay
(Minimum Phase Lag)

Relationship between Magnitude & Phase

Application: Frequency Response Compensation



$H_d(z)$: distortion system

Hc(z): compensation system

$$H_d(z) = H_{\min}(z) H_{ap}(z)$$

Overall system $$G(z) = H_d(z) H_c(z) = H_{ap}(z)$$

$$H_c(z) = 1/H_{\min}(z)$$

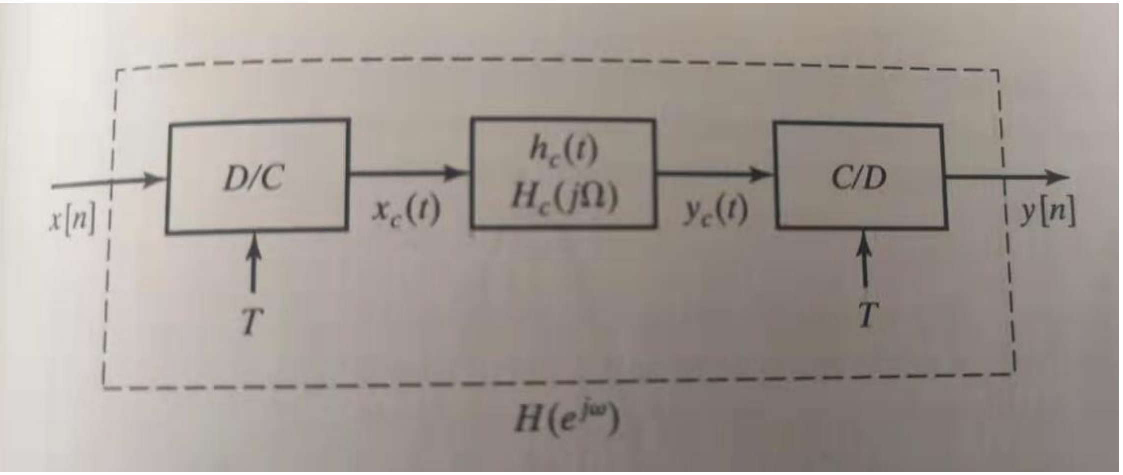
Linear Phase System

Delay in time domain -> phase shift in frequency domain

$$H(e^{j\omega}) = e^{-j\omega\alpha}$$
$$|\omega| < \pi$$

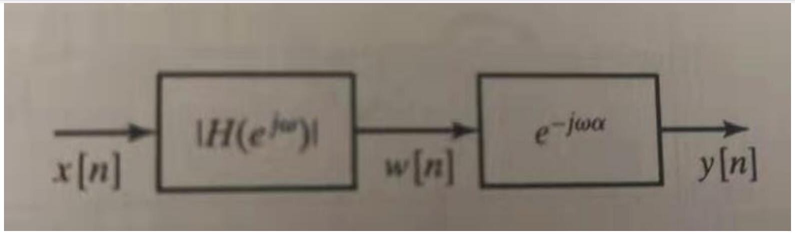
$$|H(e^{j\omega})| = 1$$
$$\angle H(e^{j\omega})| = -\omega\alpha$$

$$h[n] = \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}$$



General Linear Phase System

We can decompose the system



$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

$$= H_1(e^{j\omega}) H_2(e^{j\omega}) \qquad H_1(e^{j\omega}) = |H(e^{j\omega})| = 1$$

$H_1(e^{j\omega})$ Zero phase system $\qquad H_2(e^{j\omega}) = e^{\angle H(e^{j\omega})} = e^{-j\omega\alpha}$

Example $\quad h_{lp}[n] = \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \qquad \omega_c = 0.4\pi \quad \alpha = n_d$

$$h_{lp}[2n_d - n] = \frac{\sin\omega_c(2n_d - n - n_d)}{\pi(2n_d - n - n_d)} = \frac{\sin\omega_c(n_d - n)}{\pi(n_d - n)} = h_{lp}[n]$$

$h[2\alpha - n] = h[n]$ Even Symmetric

Generalized Linear Phase System

Type I: $h[n]=h[M-n] \quad 0\le n\le M$ Even M

$$H(e^{j\omega})=\sum_{n=0}^{M}h[n]e^{-j\omega n}=\sum_{n=0}^{M/2-1}h[n]e^{-j\omega n}+\sum_{n=0}^{M/2-1}h[M-n]e^{-j\omega(M-n)}+h[M/2]e^{-j\omega M/2}$$

$$=e^{-j\omega M/2}\sum_{n=0}^{M/2-1}\left(h[n]e^{j\omega(M/2)}e^{-j\omega n}+h[n]e^{-j\omega(M/2)}e^{j\omega n}\right)+h[M/2]e^{-j\omega M/2}$$

$$=e^{-j\omega M/2}\sum_{k=0}^{M/2}\left(a[k]\cos\omega k\right)$$

$$a[0]=h[M/2]$$

$$a[k]=2h[(M/2)-k]$$

Type II: $h[n]=h[M-n] \quad 0\le n\le M$ M Odd

$$H(e^{j\omega})=e^{-j\omega M/2}\sum_{k=1}^{(M+1)/2}\left(b[k]\cos\omega(k-1/2)\right)$$

$$b[k]=2h[((M+1)/2)-k]$$

Generalized Linear Phase System

Type III: $h[n] = -h[M-n] \quad 0 \le n \le M$ M even

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + \sum_{n=0}^{M/2-1} h[M-n]e^{-j\omega(M-n)} + h[M/2]e^{-j\omega M/2}$$

$$= je^{-j\omega M/2} \sum_{k=1}^{M/2} (c[k]\sin\omega k)$$

$$c[k] = 2h[(M/2)-k]$$

Type IV: $h[n] = -h[M-n] \quad 0 \le n \le M$ M Odd

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = je^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} (d[k]\sin\omega(k-1/2))$$

$$d[k] = 2h[((M+1)/2)-k]$$

Locations of Zero for FIR Linear Phase Sys.

Type I, II: $h[n] = h[M-n] \quad 0 \le n \le M$

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = \sum_{n=0}^{M} h[M-n] z^{-n} = \sum_{k=M}^{0} h[k] z^{k-M}$$

$$= z^{-M} \sum_{k=M}^{0} h[k] z^{k} = z^{-M} H(z^{-1})$$

If z_0 is the zero $H(z_0) = (z_0)^{-M} H(z_0^{-1}) = 0$ $1/z_0$ is also a zero

$$z_0 = re^{j\theta} \Rightarrow z_0^{-1} = r^{-1} e^{-j\theta}$$

Real h[n], zeros are conjugate pairs, so the zeros are

$$z = re^{\pm j\theta}, r^{-1} e^{\pm j\theta}$$

For $z = -1$ $H(-1) = (-1)^{-M} H(-1)$

Odd M $H(-1) = -H(-1) = 0$

Locations of Zero for FIR Linear Phase Sys.

Type III，IV: $h[n] = -h[M-n] \quad 0 \le n \le M$

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = -\sum_{n=0}^{M} h[M-n] z^{-n} = -z^{-M} H(z^{-1})$$

If z_0 is the zero $H(z_0) = -(z_0)^{-M} H(z_0^{-1}) = 0$ $1/z_0$ is also a zero

$$z_0 = re^{j\theta} \quad \Rightarrow \quad z_0^{-1} = r^{-1} e^{-j\theta}$$

Real h[n], zeros are conjugate pairs, so the zeros are

$$z = re^{\pm j\theta}, r^{-1} e^{\pm j\theta}$$

For $z = 1$ $\quad H(1) = -H(1)$

For $z = -1$ $\quad H(-1) = -(-1)^{-M} H(-1)$

even M $\quad H(-1) = -H(-1) = 0$

Eg： Length of HPF must be odd