Digital Signal Processing

Discrete Time System

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Discrete Time System

System: a operator maps seq. to another seq.

Delay

$$y[n] = x[n - n_d]$$

Moving Average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

Outline

- Classifications of Discrete Time Systems
- Time Domain Characterization of LTI Systems
- Finite-Dimensional LTI System
- Frequency Domain Characterization
- Phase and Group Delay

Classification

Memoryless y[n] depends only on x[n] with same n

Time Invariant Shift in input cause the same shift in output

$$x[n] \Rightarrow y[n] \quad \Rightarrow \quad x[n-n_d] \Rightarrow y[n-n_d]$$

Causality The output of $y[n_0]$ depends only on x[n] with $n \leq n_0$

Stability Bounded input Bounded Output (BIBO)

$$|x[n]| \leq B_x < \infty \quad \Rightarrow \quad |y[n]| \leq B_y < \infty$$

Classification

Linearity

For any input x[n], and scalar

$$x_1[n] \Rightarrow y_1[n] \qquad x_2[n] \Rightarrow y_2[n]$$

$$\alpha x_1[n] \Rightarrow \alpha y_1[n] \qquad \beta x_2[n] \Rightarrow \beta y_2[n]$$

$$\alpha x_1[n] + \beta x_2[n] \Rightarrow \alpha y_1[n] + \beta y_2[n]$$

Question : linear? causal? Time invariant?

$$y[n] = \begin{cases} x[n/L], n = 0, \pm 1L, \pm 2L, \ldots \\ 0, otherwise \end{cases}$$

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LTI System

Let $h_k[n]$ be the response of the system to $\delta[n-k]$

We can write any input x[n] as $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

The output of x[n] is

$h_k[n]$

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} T\{x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h_k[n-k] = \boxed{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

Convolution sum

Time Domain Characterization of LTI

Convolution Sum

$$x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Impulse Response $h[n]$: response of $\delta[n]$

Computation

- Flip $h[k] \rightarrow h[-k]$
- Move: $h[-k] \rightarrow h[n-k]$
- Add: $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$

Time Domain Characterization of LTI

Properties of LTI (Convolution sum)

Commutative

$$x[n] \circledast y[n] = y[n] \circledast x[n]$$

Associative

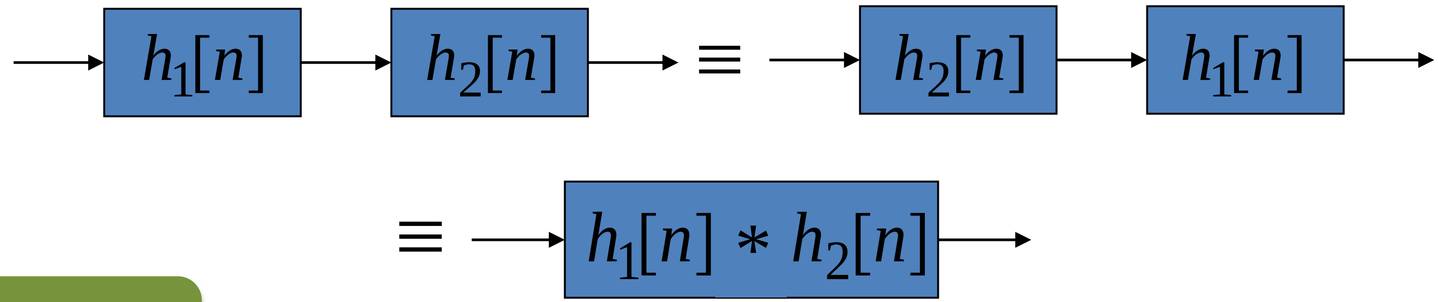
$$(x[n] \circledast h_1[n]) \circledast h_2[n]$$
$$= x[n] \circledast (h_1[n] \circledast h_2[n])$$

Distributive

$$x[n] \circledast (h_1[n] + h_2[n])$$
$$= x[n] \circledast h_1[n] + x[n] \circledast h_2[n]$$

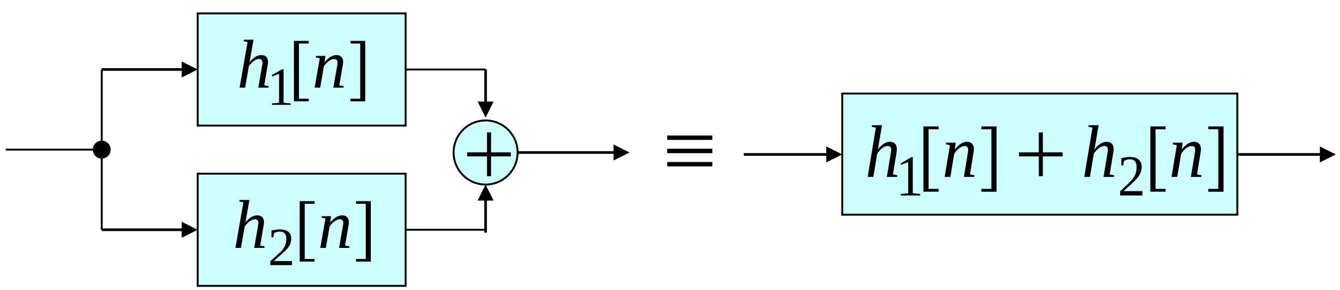
Time Domain Characterization of LTI

Cascade



$$h[n] = h_1[n] \circledast h_2[n]$$

Parallel



$$h[n] = h_1[n] + h_2[n]$$

Time Domain Characterization of LTI

Stability of LTI:

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \le \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]|$$

$$< B_x \sum_{k=-\infty}^{\infty} |h[k]| < B_x S < \infty$$

Bounded Input Bounded Output (BIBO)

Time Domain Characterization of LTI

Causality

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Causality: $y[n]=0$ for $x[n]=0 \quad n<0$

$$h[n]=0 \quad n<0$$

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Finite Dimensional LTI

Linear Constant Coefficient Different Equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

Accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$$

$$y[n] - y[n-1] = x[n]$$

Finite Dimensional LTI

Linear Constant Coefficient Different Equation

Moving Average

$$y[n] = \frac{1}{M}\sum_{k=0}^{M} x[n-k]$$

$$h[n] = \frac{1}{M+1}(u[n] - u[n-M])$$
$$= \frac{1}{M+1}(\delta[n] - \delta[n-M]) \circledast u[n]$$

Accumulator

difference

2 ways of represent the given system

Linear Constant Coefficient Different Equation

$$y[n] = y_p[n] + y_h[n]$$

$y_p[n]$: particular solution, depends on input x[n]

$y_c[n]$ ($y_h[n]$): homogeneous solution when x[n]=0

$$\sum_{k=0}^{N} d_k y_h[n-k] = 0$$

$$y_h[n] = d_0 \lambda_0^{\ n} + d_1 \lambda_1^{\ n-1} + d_2 \lambda_2^{n-2} + \ldots + d_N \lambda_N^{n-N}$$

λ : the root of $\sum_{k=0}^{N} d_k \lambda^{N-k} = 0$

Finite Dimensional LTI

Linear Constant Coefficient Different Equation

$$y[n] = y_p[n] + y_c[n]$$

$y_p[n]$: particular solution, depends on input x[n]

Input Signal	Particular Solution
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \cdots + K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \cdots + K_M)$
$A\sin(\omega_0 n)$	$K_1 \cos(\omega_0 n) + K_2 \sin(\omega_0 n)$

Finite Dimensional LTI

Example $y[n] + y[n-1] - 6y[n-2] = x[n]$

$x[n] = 8u[n]$ and initial condition $y[-1] = 1, y[-2] = -1$

Let $x[n] = 0$, and $y[n] = \lambda^n$ $\quad \sum_{k=0}^{N} d_k \lambda^{N-k} = 0$

$$\lambda^n + \lambda^{n-1} - 6\lambda^{n-2} = 0 \Longrightarrow \lambda^{n-2}(\lambda+3)(\lambda-2) = 0$$

$$\lambda_1 = -3,\ \lambda_1 = 2$$

$$y_h[n] = \alpha_1(-3)^n + \alpha(2)^n$$

The coefficients can be found by the initial conditions

Finite Dimensional LTI

Example $y[n]+y[n-1]-6y[n-2]=x[n]$

$x[n]=8u[n]$ and initial condition $y[-1]=1, y[-2]=-1$

Particular solution $y_p=k$

$k+k-6k=8u[n] \Rightarrow k=-2$

$$y[n]=y_p[n]+y_h[n]$$

$$=\alpha_1(-3)^n+\alpha(2)^n-2 \quad \text{for } n\geq 0$$

$y[-1]=1, y[-2]=-1 \Rightarrow \alpha_1=-1.8, \alpha_2=4.8$

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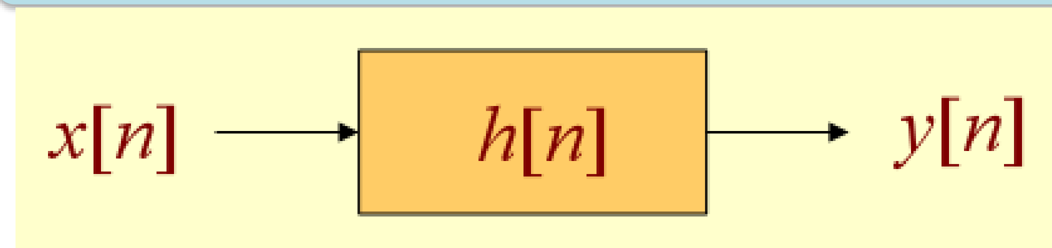
Frequency Domain Characterization

Frequency Response



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$x[n] = e^{j\omega n} \quad \text{for} \quad -\infty < n < \infty$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]e^{j\omega k} = \sum_{k=-\infty}^{\infty} h[n-k]e^{j\omega(n-k)} e^{j\omega n}$$

$$= H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega}$$

- For input signal $e^{j\omega n}$, the output is the complex exponential of the same frequency multiplied by a complex constant $H(e^{j\omega})$

- $e^{j\omega n}$: eigen-function of the system.

Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega k}$$

- $H(e^{j\omega})$: the frequency response of the LTI discrete-time system
- $H(e^{j\omega})$ provides a frequency-domain description of the system
- $H(e^{j\omega})$ is the DTFT of the impulse response $\{h[n]\}$

Frequency Domain Characterization

Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega k}$$

$H(e^{j\omega})$ complex function of ω with a period 2π

$$H(e^{j\omega}) = H_{\mathrm{re}}(e^{j\omega}) + jH_{\mathrm{im}}(e^{j\omega})$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$$

Frequency Domain Characterization

Frequency Response

- Magnitude and phase functions are real functions of ω
- frequency response is a complex function of ω.
- If $h[n]$ is real then the magnitude function is an even function of ω:

$$|H(e^{j\omega})| = |H(e^{-j\omega})|$$

and the phase function is an odd function of ω:

$$\theta(\omega) = -\theta(-\omega)$$

$H_{\mathrm{re}}(e^{j\omega})$ even

$H_{\mathrm{im}}(e^{j\omega})$ Odd

Frequency Domain Characterization

Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega k}$$

Example: Freq. Response of Ideal Delay

$$y(n) = x[n - n_d] \qquad h(n) = \delta[n - n_d]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega k} = e^{-j\omega n_d}$$

$$|H(e^{j\omega})| = 1$$

$$\theta(\omega) = arg(H(e^{j\omega})) = -\omega n_d$$

Frequency Domain Characterization

Example: Freq. Response of moving average

$$y(n) = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \qquad h(n) = \frac{1}{M} \quad \text{for} \quad 0 \le n \le M-1$$

$$H(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} e^{j\omega k} = \frac{1}{M}\frac{1-e^{j\omega M}}{1-e^{j\omega}} = \frac{1}{M}\frac{\sin(M\omega/2)}{\sin(w/2)} e^{-j(M-1)\omega/2}$$

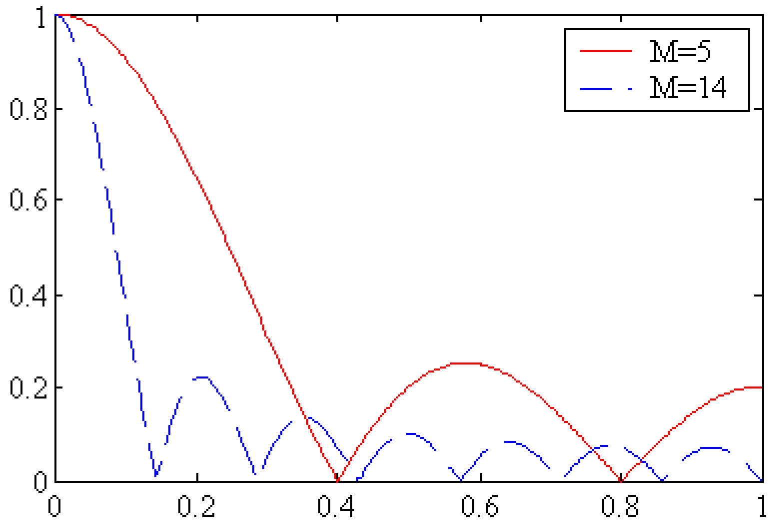
$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

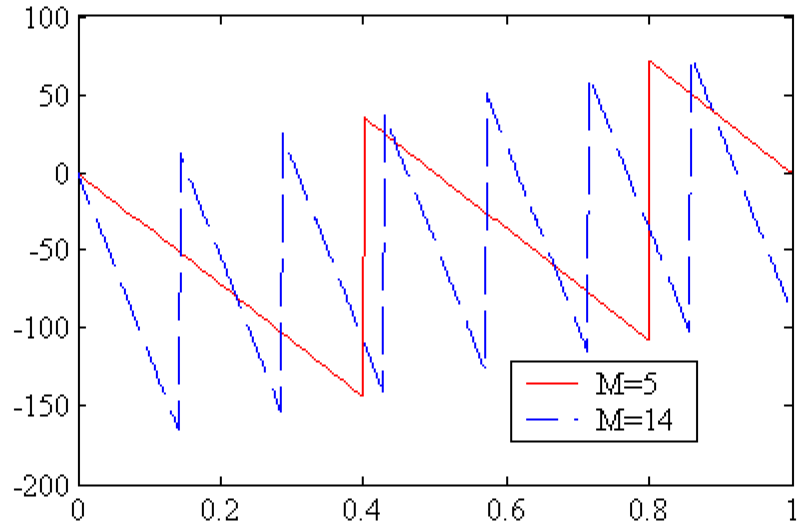
$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi\sum_{k=0}^{M/2}\mu(\omega - \frac{2\pi k}{M})$$

Frequency Domain Characterization

Frequency Response Calculation Matlab

- Use freqz(h,1,w) to the frequency response of h at a set of given frequency points w





Frequency Domain Characterization

Example: Freq. Response of sinusoidal

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}\mathrm{e}^{j\phi}e^{j\omega_0 n} + \frac{A}{2}\mathrm{e}^{-j\phi}e^{-j\omega_0 n}$$

$$y_1[n] = H(e^{j\omega_0})\frac{A}{2}\mathrm{e}^{j\phi}e^{j\omega_0 n} \quad y_2[n] = H(e^{-j\omega_0})\frac{A}{2}\mathrm{e}^{-j\phi}e^{-j\omega_0 n}$$

$$y[n] = \frac{A}{2}\left[H(e^{j\omega_0})\mathrm{e}^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})\mathrm{e}^{-j\phi}e^{-j\omega_0 n}\right]$$

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})|e^{j\theta(\omega_0)}$$

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta(\omega_0))$$

sinusoidal input sinusoidal output

Frequency Domain Characterization

Example: Freq. Response of causal exponential

$$x[n]=e^{j\omega n}u[n]$$

For n>0 $\quad y[n]=h[n]\circledast x[n]$ for $n\geq 0$

$$y[n]=\sum_{k=0}^{n}h[k]x[n-k]=\sum_{k=0}^{n}h[k]e^{-j\omega k}e^{j\omega n}$$

$$=H(e^{j\omega})e^{j\omega n}-\left(\sum_{k=n+1}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n}$$

Steady state response

transit state response

Bounded if $\quad |y_t[n]|=\left|-\left(\sum_{k=n+1}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n}\right|\leq\sum_{k=n+1}^{\infty}|h[k]|$

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Phase and Group Delay

Response of $x[n] = A\cos(\omega_0 n + \phi)$

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta(\omega_0))$$

$\theta(\omega_0)$: phase of the system

$\tau_p(\omega_0) = -\theta(\omega_0)/\omega_0$: phase delay of the system

$$\tau_g(\omega_0) = -\frac{d}{d\omega}\theta(\omega)$$