Chapter 7
LTI Discrete-Time Systems in the Transform Domain

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Types of Transfer Functions

The analysis methods of discrete LTI systems in the transform domain (ZT and DTFT) .

$$y(n) = x[n] \circledast h[n] \Leftrightarrow Y(z) = X(z)H(z)$$

$$[Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})]$$

$H(z)$ **: transfer function**

$H(e^{j\omega})$ **: frequency response**

Types of Transfer Functions

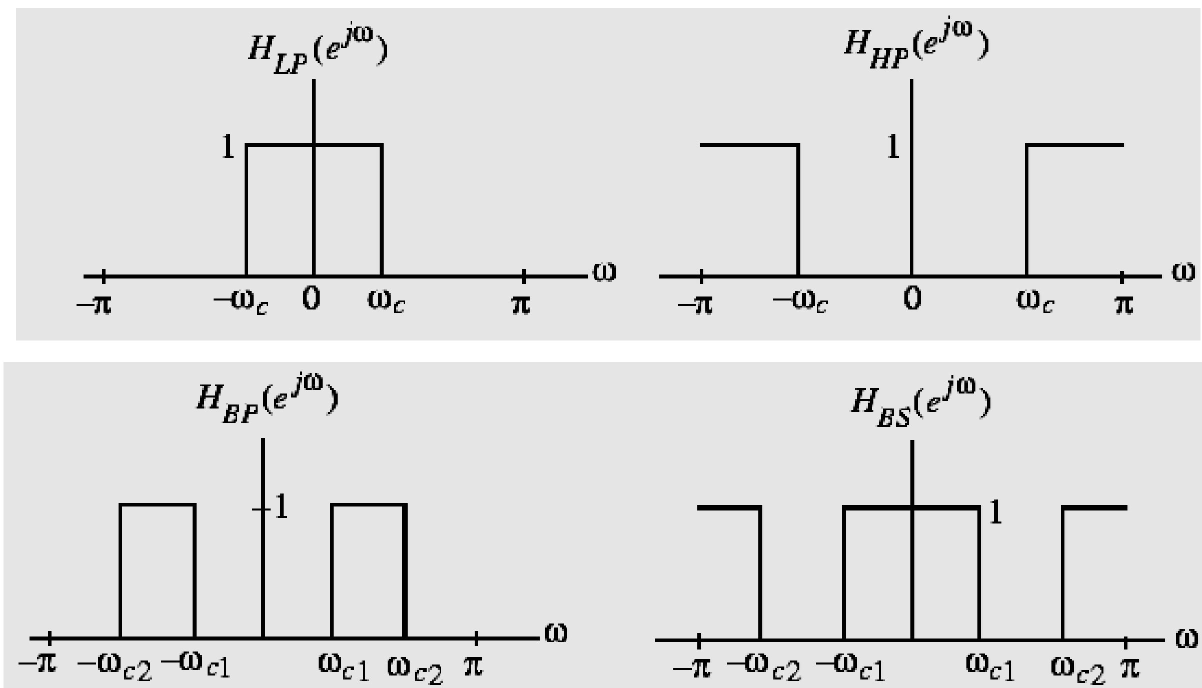
- **The time-domain classification of an LTI digital transfer function sequence is based on the *length of its impulse response* (Criterion #01)**
  - **- Finite impulse response (FIR) transfer function**
  - **- Infinite impulse response (IIR) transfer function**

Types of Transfer Functions

- **In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications**
  - **(1) Classification based on the shape of the magnitude function $|H(e^{j\omega})|$. (Criterion #02)**
  - **(2) Classification based on the form of the phase function $\theta(\omega)$. (Criterion #03)**

Ideal Filters

- **Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:**



Ideal Filters

- **Lowpass filter:** ***Passband***: $0 \le \omega \le \omega_c$

  Stopband: $\omega_c \le \omega \le \pi$
- **Highpass filter:** ***Passband***: $\omega_c \le \omega \le \pi$

  Stopband: $0 \le \omega \le \omega_c$
- **Bandpass filter:**

  Passband: $\omega_{c1} \le \omega \le \omega_{c2}$

  Stopband: $0 \le \omega < \omega_{c1}$ **and** $\omega_{c2} < \omega < \pi$
- **Bandstop filter:**

  Stopband: $\omega_{c1} < \omega < \omega_{c2}$

  Passband: $0 \le \omega \le \omega_{c1}$ **and** $\omega_{c2} \le \omega \le \pi$

Ideal Filters

- **The frequencies ω_c, ω_{c1}, and ω_{c2} are called the *cut-off frequencies***
- **An ideal filter has a magnitude response equal to 1 in the passband and 0 in the stopband, and *has a 0 phase everywhere***

Ideal Filters

- **Earlier in the course we derived the inverse DTFT of the frequency response $H_{LP}(e^{j\omega})$ of the ideal lowpass filter:**

$$h_{LP}[n] = \sin \omega_c n / n\pi, \quad -\infty < n < \infty$$

- **We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable**

Ideal Filters

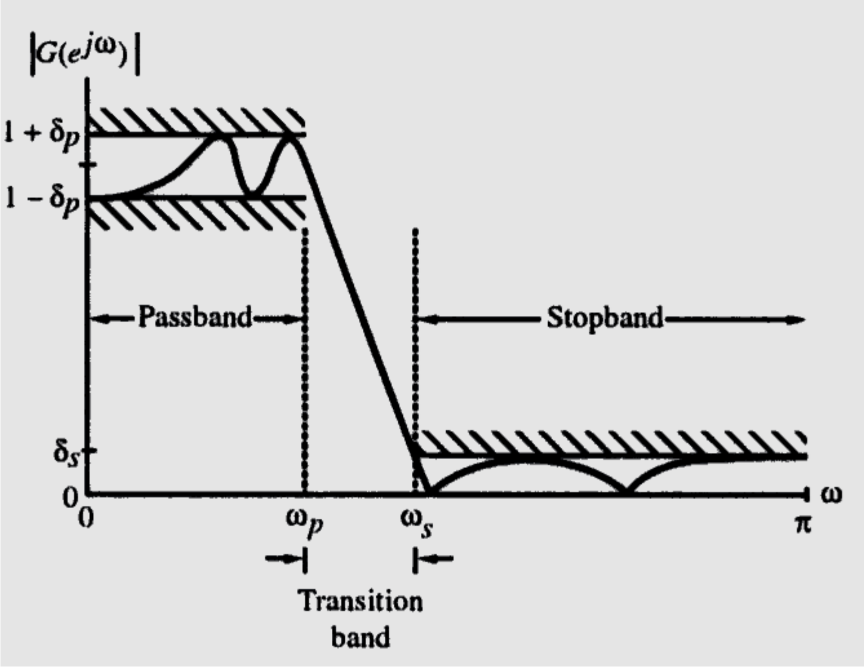
- **Also, $h_{LP}[n]$ is not causal and is of infinite length.**
- **The remaining three ideal filters are also characterized by infinite, noncausal impulse responses and are not absolutely summable**
- **Thus, the ideal filters with the ideal "*brick wall*" frequency responses cannot be realized with finite dimensional LTI filter**

Ideal Filters

- **To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a *transition band* between the passband and the stopband**
- **This permits the magnitude response to decay slowly from its maximum value in the passband to the 0-value in the stopband**

Ideal Filters

- **Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband**



Typical magnitude response specifications of a lowpass filter

Magnitude Characteristics

- **One common classification is based on an ideal magnitude response**
- **A digital filter designed to pass signal components of certain frequencies without distortion should have a frequency response equal to 1 at these frequencies, and should have a frequency response equal to 0 at all other frequencies**

Ideal Filters

- **The range of frequencies where the frequency response takes the value of 1 is called the passband**
- **The range of frequencies where the frequency response takes the value of 0 is called the stopband**

Bounded Real Transfer Functions

- **A causal stable real-coefficient transfer function $H(z)$ is defined as a bounded real (BR) transfer function if**

$$|H(e^{j\omega})| \le 1 \quad \text{for all values of } \omega$$

- **Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs**

Bounded Real Transfer Functions

- **Then the condition** $|H(e^{j\omega})| \le 1$ **implies that**

$$\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2$$

Integrating the above from $-\pi$ to π, and applying Parseval's relation we get

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \le \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

Bounded Real Transfer Functions

- **Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a *passive structure***
- **If $|H(e^{j\omega})|=1$, then the output energy is equal to the input energy, and such a digital filter is therefore a *lossless system***

Bounded Real Transfer Functions

- **A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})|=1$ is thus called a *lossless bounded real (LBR) transfer function***

- **The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity**

Bounded Real Transfer Functions

- **Example: Consider the causal stable IIR transfer function**

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \ \mathrm{ROC}: |z| > |\alpha|, \ 0 < |\alpha| < 1$$

where *K* is a real constant

Its square-magnitude function is given by

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = H(e^{j\omega})H^*(e^{j\omega}) = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

Bounded Real Transfer Functions

- Thus, for $\alpha > 0$, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$

Bounded Real Transfer Functions

- On the other hand, for $\alpha < 0$, the maximum value of $2\alpha\cos\omega$ is equal to -2α at $\omega=\pi$ and the minimum value is equal to 2α at $\omega = 0$
- Here, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1+\alpha)^2$ at $\omega=\pi$ and the minimum value is equal to $K^2/(1-\alpha)^2$ at $\omega=0$

Bounded Real Transfer Functions

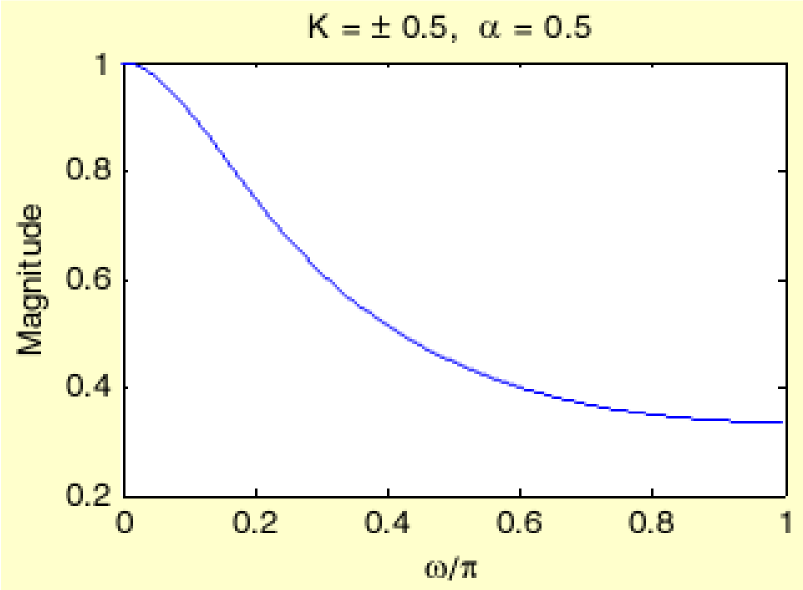
- **Hence,**

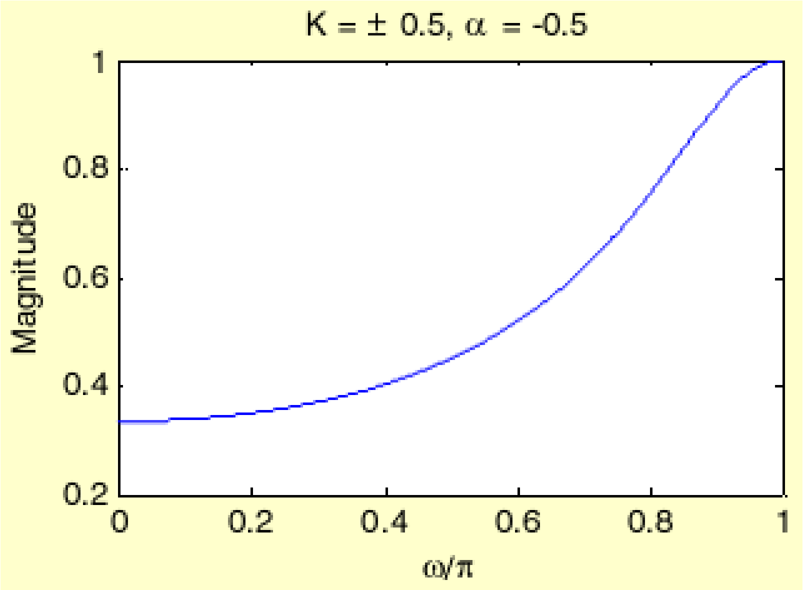
$$H(z) = \frac{K}{1-\alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

is a BR function for $K = \pm(1-|\alpha|)$,

Plots of the magnitude function for $\alpha = \pm 0.5$ with values of K chosen to make $H(z)$ a BR function are shown on the next slide

Bounded Real Transfer Functions





Lowpass filter **Highpass filter**

Allpass Transfer Function

Definition

- **An IIR transfer function *A*(*z*) with unity magnitude response for all frequencies, i.e.,**

$$\left|A(e^{j\omega})\right|^2 = 1, \text{ for all } \omega$$

is called an allpass transfer function

- **An *M*-th order causal real-coefficient allpass transfer function is of the form**

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

Allpass Transfer Function

- **If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:**

$$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

then it follows that $A_M(z)$ can be written as:

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- **Note from the above that if $z=re^{j\varphi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = (1/r)e^{-j\varphi}$**

Allpass Transfer Function

- **The numerator of a real-coefficient allpass transfer function is said to be the mirror-image polynomial of the denominator, and vice versa**

- **We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-*M* polynomial $D_M(z)$, i.e.,**

$$\tilde{D}_M(z) = z^{-M} D_M(z^{-1})$$

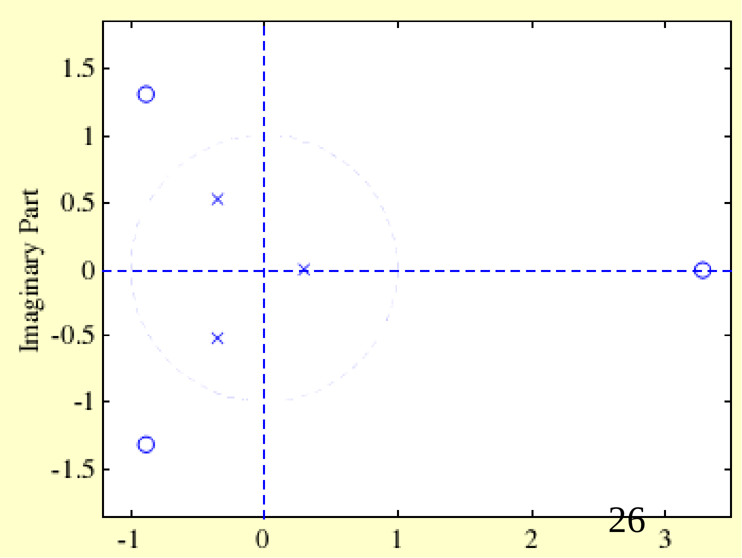
Allpass Transfer Function

- **The expression**

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** **in the *z*-plane**

$$A_3(z) = \frac{-0.2 + 0.18 z^{-1} + 0.4 z^{-2} + z^{-3}}{1 + 0.4 z^{-1} + 0.18 z^{-2} - 0.2 z^{-3}}$$



Allpass Transfer Function

- To show that $|A_M(e^{j\omega})|=1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

Hence

$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

Allpass Transfer Function

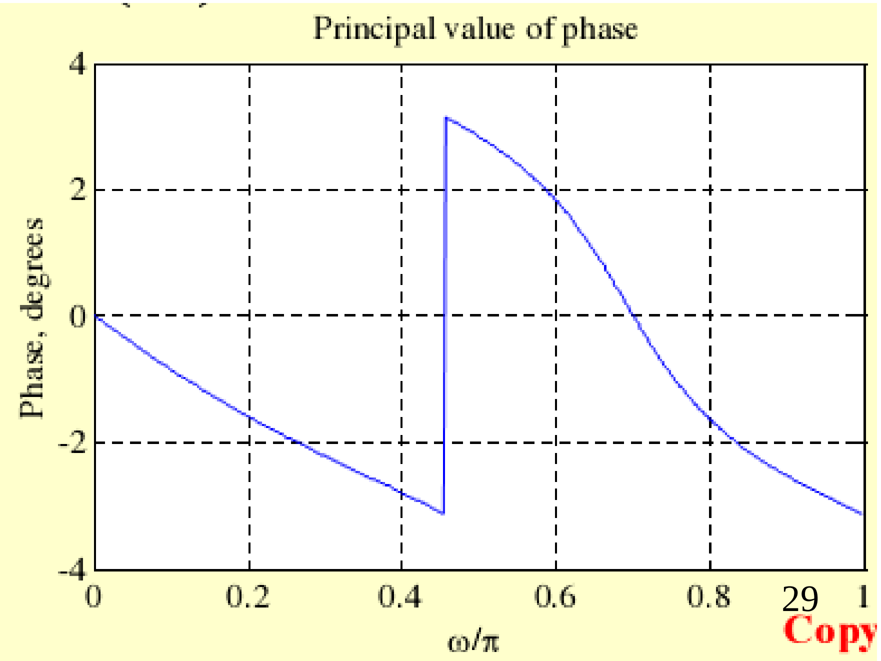
- **Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane**
- **Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle**

Allpass Transfer Function

- **Figure below shows the principal value of the phase of the 3rd-order allpass function**

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

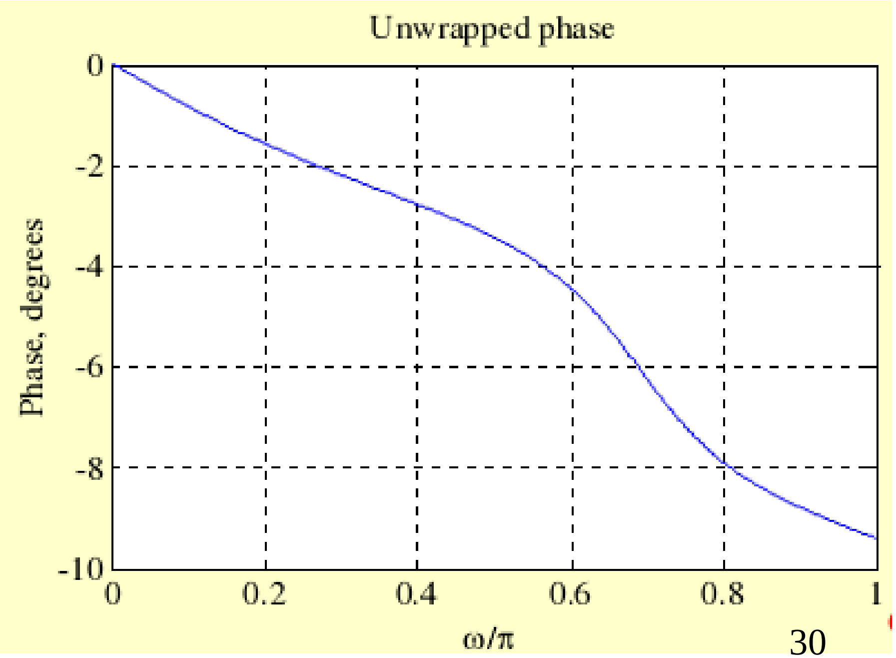
•Note the discontinuity by the amount of 2π in the phase $\theta(\omega)$



Allpass Transfer Function

- **If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below**

Note: The unwrapped phase function is a continuous function of ω



Allpass Transfer Function

- **The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω**

Properties

- **(1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a *lossless structure***

Allpass Transfer Function

(2) The magnitude function of a stable allpass function *A*(*z*) satisfies:

$$|A(z)| \begin{cases} < 1, & \text{for } |z| > 1 \\ = 1, & \text{for } |z| = 1 \\ > 1, & \text{for } |z| < 1 \end{cases}$$

(Problem 7.2)

(3) Let $\tau(\omega)$ denote the group delay function of an allpass filter *A*(*z*), i.e.,

$$\tau_g(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

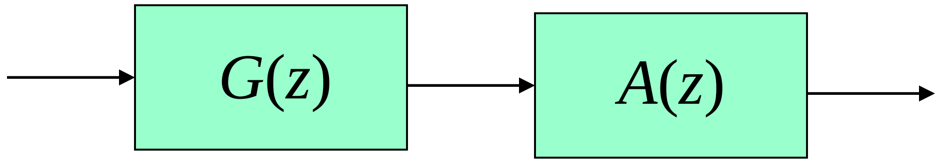
(Problem 7.3)

Allpass Transfer Function

A Simple Application

- **A simple but often used application of an allpass filter is as a delay equalizer**
  - **Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response**
  - **The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a *constant group delay in the band of interest***

Allpass Transfer Function



- **Since** $|A(e^{j\omega})|=1$**, we have**

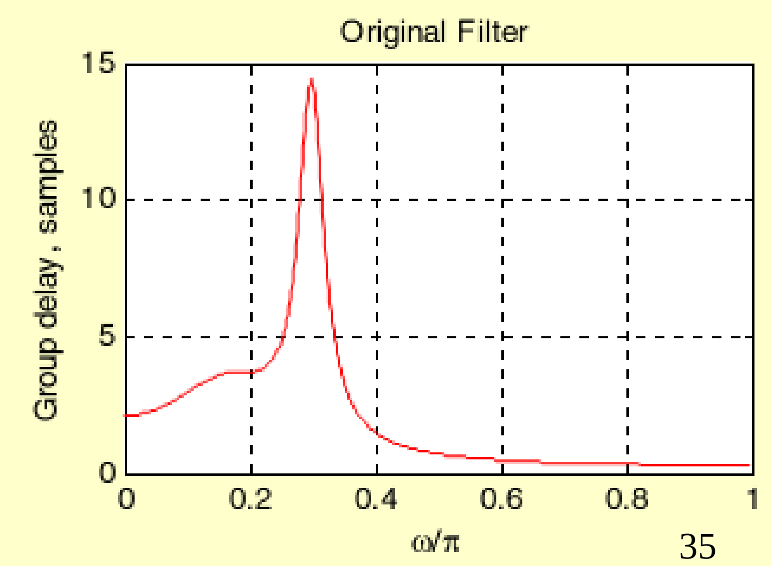
$$|G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})|$$

Overall group delay is the given by the sum of the group delays of $G(z)$ **and** $A(z)$

Allpass Transfer Function

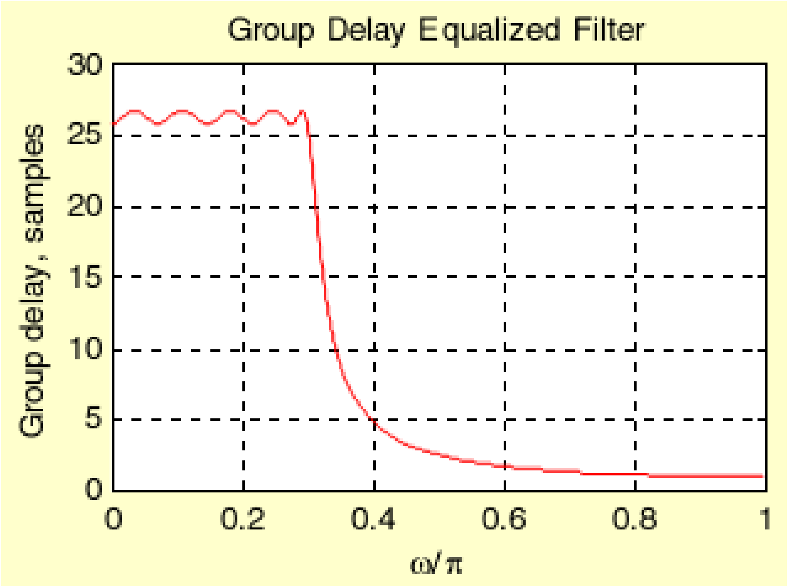
- ***Example*: Figure below shows the group delay of a 4th order elliptic filter with the following specifications: ω_p=0.3π, δ_p=1dB, δ_s=35dB**

The nonlinear phase response



Allpass Transfer Function

- **Figure below shows the group delay of the original elliptic filter cascaded with an 8th order allpass section designed to equalize the group delay in the passband**



Classification Based on Phase Characteristics

- **A second classification of a transfer function is with respect to its phase characteristics**
- **In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband**

Zero-Phase Transfer Functions

- **One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a zero phase characteristic**
- **However, it is impossible to design a causal digital filter with a zero phase**

Zero-Phase Transfer Functions

- **For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement**
- **One zero-phase filtering scheme is sketched below**

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow v[n] \qquad u[n] \longrightarrow \boxed{H(z)} \longrightarrow w[n]$$

$$u[n]=v[-n], \qquad y[n]=w[-n]$$

Zero-Phase Transfer Functions

- **It is easy to verify the above scheme in the frequency domain**
- **Let $X(e^{j\omega})$,$V(e^{j\omega})$,$U(e^{j\omega})$,$W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively**
- **From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide**

Zero-Phase Transfer Functions

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow v[n] \qquad u[n] \longrightarrow \boxed{H(z)} \longrightarrow w[n]$$

$$u[n]=v[-n], \qquad y[n]=w[-n]$$

$V(e^{j\omega})= H(e^{j\omega})X(e^{j\omega})$, $W(e^{j\omega})=H(e^{j\omega})U(e^{j\omega})$

$U(e^{j\omega})= V^*(e^{j\omega})$, $\qquad Y(e^{j\omega})= W^*(e^{j\omega})$

- **Combining the above equations we get**

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega})$$
$$= H^*(e^{j\omega})V(e^{j\omega}) = H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega})$$
$$= |H(e^{j\omega})|^2X(e^{j\omega})$$

- **This is a zero-phase filter with a frequency response $|H(e^{j\omega})|^2$**

Zero-Phase Transfer Functions

- **The function fftfilt implements the above zero-phase filtering scheme**
- **In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a linear-phase characteristic in the frequency band of interest**

Linear-Phase Transfer Functions

- **The most general type of a filter with a linear phase has a frequency response given by**

$$H(e^{j\omega}) = e^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$

- **Note also** $|H(e^{j\omega})| = 1$

$$\tau_g(\omega) = D$$

Linear-Phase Transfer Functions

- **The output $y[n]$ of this filter to an input $x[n]=Ae^{j\omega n}$ (eigenfunction) is then given by**

$$y[n]=e^{-j\omega D}Ae^{j\omega n}=Ae^{j\omega(n-D)}$$

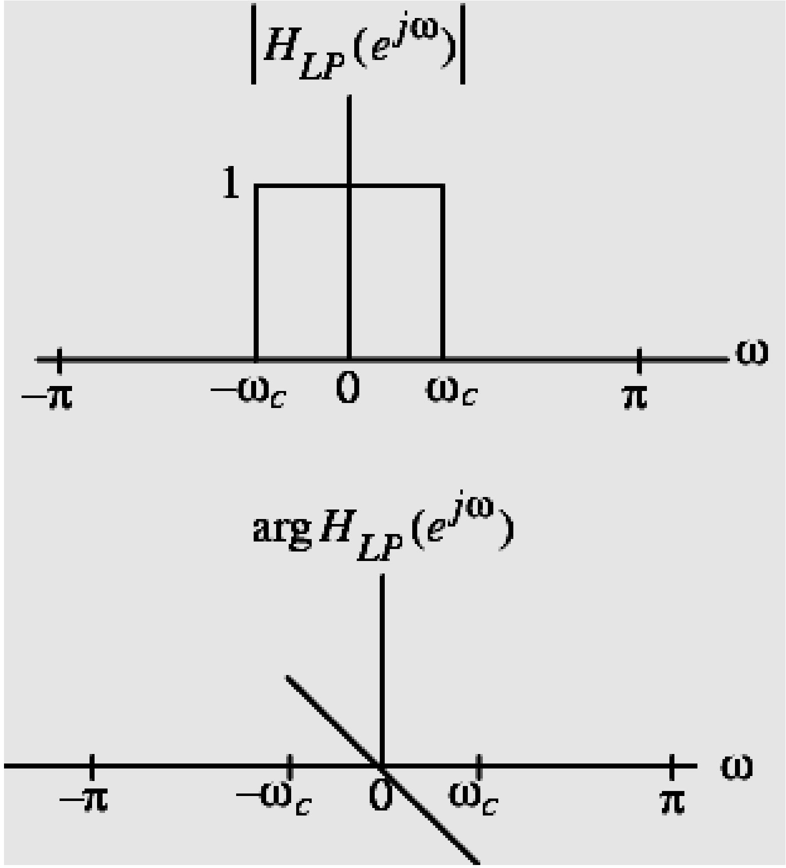
- **If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t=nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D**

Linear-Phase Transfer Functions

- **If D is an integer, then $y[n]$ is identical to $x[n]$, but delayed by D samples ($y[n] = x[n-D]$)**
- **If D is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$**
- **In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed D units of time**

Linear-Phase Transfer Functions

- Figure right shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband



Linear-Phase Transfer Functions

- **Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape**
- **<u>Example:</u> Determine the impulse response of an ideal lowpass filter with a linear phase response:**

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_0}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

Linear-Phase Transfer Functions

- **Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at**

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_0)}{\pi (n - n_0)}, \quad -\infty < n < +\infty$$

- **As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable**

Linear-Phase Transfer Functions

- **By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed**
- **The truncated approximation may or may not exhibit linear phase, depending on the value of n_0 chosen**

Linear-Phase Transfer Functions

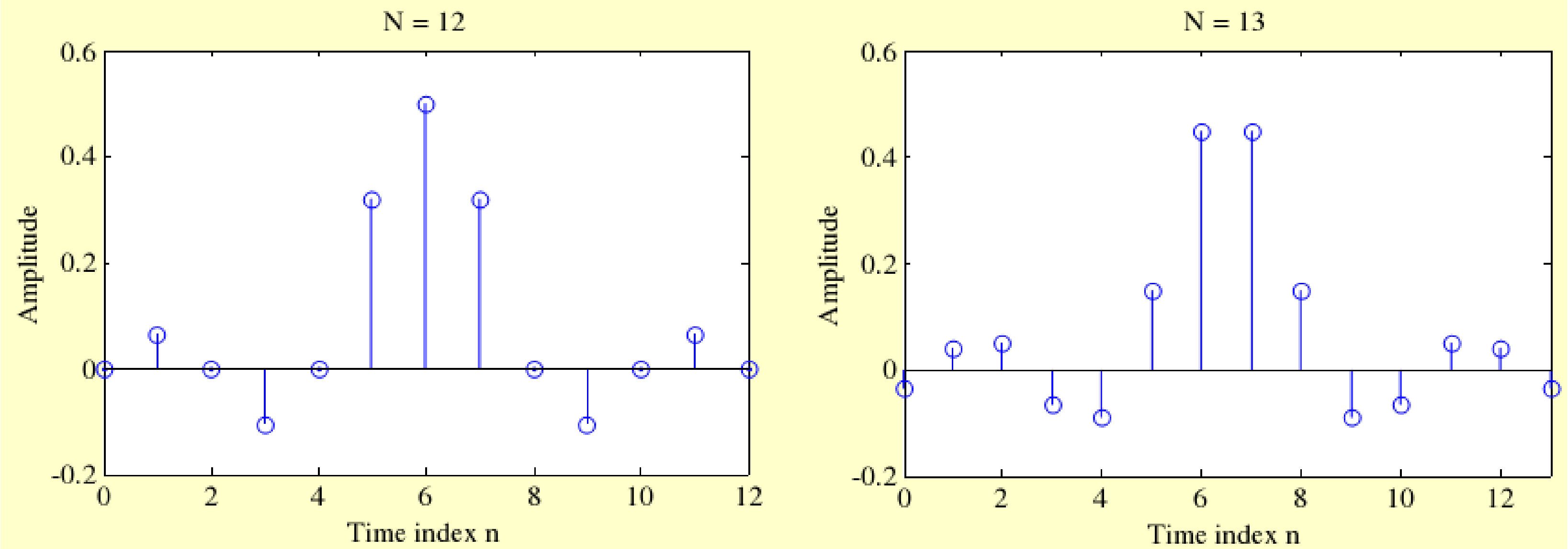
- **If we choose $n_0 = N/2$ with N a positive integer, the truncated and shifted approximation**

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi (n - N/2)}, \; 0 \le n \le N$$

will be a length N+1 causal linear-phase FIR filter

Linear-Phase Transfer Functions

- **Figure below shows the filter coefficients obtained using the function sinc for two different values of *N***



Linear-Phase Transfer Functions

- **Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:**

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} \hat{h}_{LP}[n] e^{-j\omega n} = e^{-j\omega N/2} \tilde{H}_{LP}(\omega)$$

where $\tilde{H}_{LP}(\omega)$, called the zero-phase response or amplitude response, is a real function of ω

Types of Linear-Phase FIR Transfer Functions

- **It is nearly impossible to design a linear-phase IIR transfer function**
- **It is always possible to design an FIR transfer function with an exact linear-phase response**
- **We now develop the forms of the linear-phase FIR transfer function $H(z)$ with real impulse response $h[n]$**
- **Consider a causal FIR transfer function $H(z)$ of length N+1, i.e., of order N:**

$$H(z) = \sum_{n=0}^{N} h[n]\, z^{-n}$$

Types of Linear-Phase FIR Transfer Functions

- **If $H(z)$ is to have a linear-phase, its frequency response must be of the form**

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$$

where c and β are constants, and $\breve{H}(\omega)$, called the **amplitude response**, **also called the** **zero-phase response**, **is a real function of ω**

Types of Linear-Phase FIR Transfer Functions

- **For a real impulse response, the magnitude response $|H(e^{j\omega})|$ is an even function, i.e., $|H(e^{j\omega})| = |H(e^{-j\omega})|$**

- **Since $|H(e^{j\omega})| = |\breve{H}(\omega)|$, the amplitude response is then either an even function or an odd function of ω, i.e.**

$$\breve{H}(-\omega) = \pm \breve{H}(\omega)$$

Types of Linear-Phase FIR Transfer Functions

- **The frequency response satisfies the relation $H(e^{j\omega})=H^*(e^{-j\omega})$, or equivalently, the relation**

$$e^{j(c\omega+\beta)}H(\omega) = e^{-j(-c\omega+\beta)}H(-\omega)$$

If $H(\omega)$ is an even function, then the above relation leads to $e^{j\beta}=e^{-j\beta}$ implying that either $\beta=0$ or $\beta=\pi$

Types of Linear-Phase FIR Transfer Functions

From

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$$

We have

$$\breve{H}(\omega) = e^{-j(c\omega+\beta)}H(e^{j\omega})$$

Substituting the value of *β* in the above we get

$$\breve{H}(\omega) = \pm e^{-jc\omega}H(e^{j\omega}) = \pm\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

Types of Linear-Phase FIR Transfer Functions

- **Replacing ω with $-\omega$ in the previous equation we get**

$$\breve{H}(-\omega) = \pm \sum_{l=0}^{N} h[l] e^{j\omega(c+l)}$$

Making a change of variable $l=N-n$, we rewrite the above equation as

$$\breve{H}(-\omega) = \pm \sum_{l=0}^{N} h[N-n] e^{j\omega(c+N-n)}$$

Types of Linear-Phase FIR Transfer Functions

As $\breve{H}(\omega) = \breve{H}(-\omega)$ **, we have:**

$$h[n]e^{-j\omega(c+n)} = h[N-n]e^{j\omega(c+N-n)}$$

The above leads to the condition

$$h[n] = h[N-n],\ 0 \le n \le N$$

with $c = -N/2$

Thus, the FIR filter with an even amplitude response will have a linear phase if it has a symmetric impulse response

Types of Linear-Phase FIR Transfer Functions

If $\mathrm{H}(\omega)$ **is an odd function of** ω**, then from**

$$e^{j(c\omega+\beta)}H(\omega) = e^{-j(-c\omega+\beta)}H(-\omega)$$

We get $e^{j\beta} = -e^{-j\beta}$ **as** $H(-\omega) = -H(\omega)$

The above is satisfied if $\beta = \pi/2$ **or** $\beta = -\pi/2$

Then

$$H(e^{j\omega}) = e^{j(c\omega+\beta)}H(\omega)$$

Reduces to

$$H(e^{j\omega}) = je^{jc\omega}H(\omega)$$

Types of Linear-Phase FIR Transfer Functions

- **The last equation can be rewritten as**

$$\breve{H}(\omega) = -je^{-jc\omega}H(e^{j\omega}) = -j\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

As $-\breve{H}(-\omega) = \breve{H}(\omega)$ **, from the above we get**

$$-\breve{H}(-\omega) = j\sum_{l=0}^{N} h[l]e^{j\omega(c+l)}$$

Types of Linear-Phase FIR Transfer Functions

- **Making a change of variable *l*=*N*−*n*, we rewrite the last equation as**

$$-\breve{H}(-\omega) = j\sum_{l=0}^{N} h[l]e^{j\omega(c+l)}$$

Equating the above with

$$\breve{H}(\omega) = -j\sum_{n=0}^{N} h[n]e^{-j\omega(c+n)}$$

We arrive at the condition for linear phase as

Types of Linear-Phase FIR Transfer Functions

$$h[n] = -h[N-n],\ \ 0 \le n \le N$$

with $c = -N/2$

Therefore a FIR filter with an odd amplitude response will have linear-phase response if it has an antisymmetric impulse response

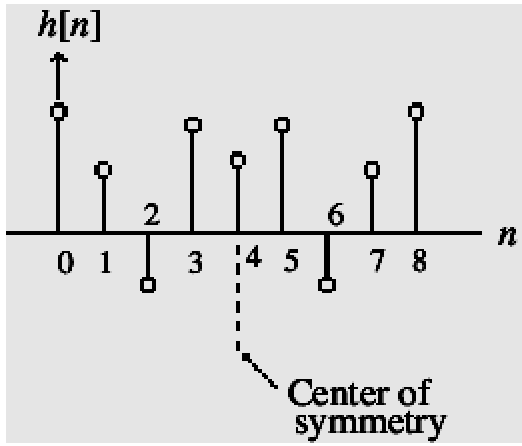
Types of Linear-Phase FIR Transfer Functions

- **Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions**
- **For an antisymmetric FIR filter of odd length, i.e., N even**

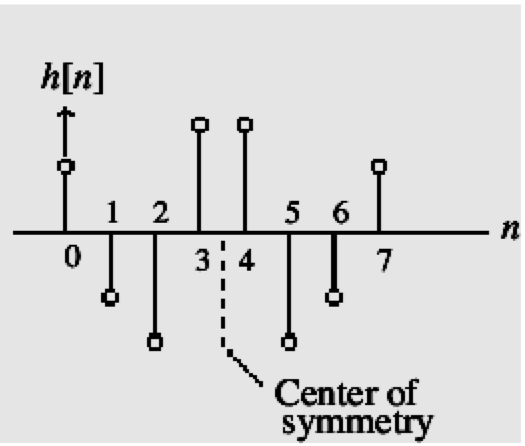
$$h[N/2] = 0$$

- **We examine next the each of the 4 cases**

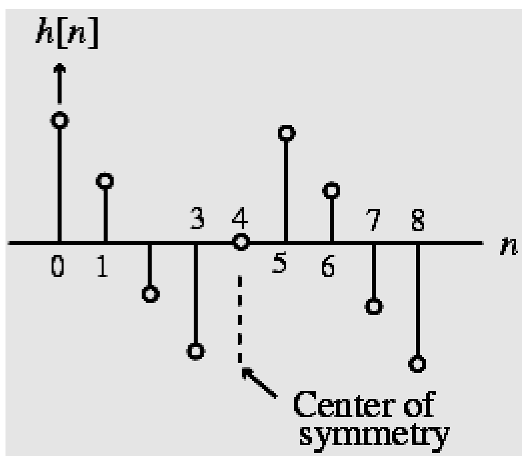
Types of Linear-Phase FIR Transfer Functions



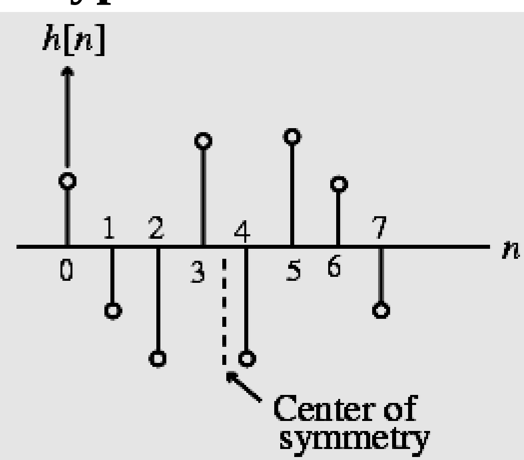
Type 1: $N = 8$



Type 2: $N = 7$



Type 3: $N = 8$



Type 4: $N = 7$

Types of Linear-Phase FIR Transfer Functions

If the impulse response of FIR filter *h*[*n*] is real causal (*N*+1)-point (*n*:[0, *N*]) sequence, and satisfied with :

$$h[n] = h[N-n]$$ **(symmetric) or**

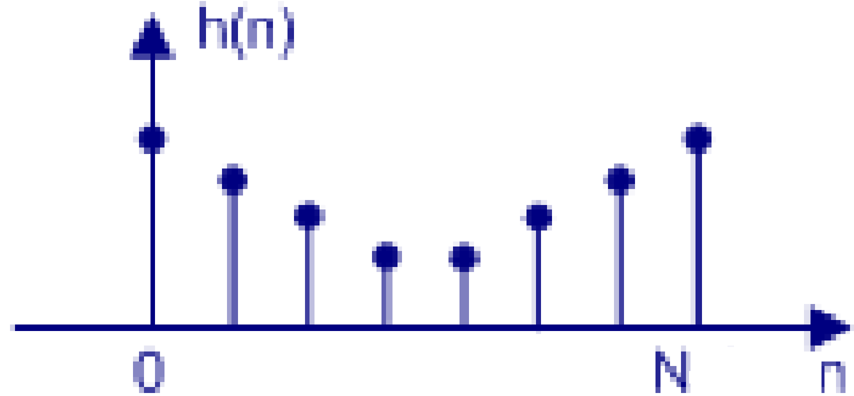
$$h[n] = -h[N-n]$$ **(antisymmetric), then**

$$h[n] \Leftrightarrow H(e^{jw}) = \breve{H}(\omega)e^{-jN\omega/2} = |\breve{H}(\omega)|e^{-jN\omega/2}e^{j\beta}$$

or

$$h[n] \Leftrightarrow H(e^{jw}) = j\breve{H}(\omega)e^{-jN\omega/2} = |\breve{H}(\omega)|e^{-jN\omega/2+\pi/2}e^{j\beta}$$

where the amplitude response $\breve{H}(\omega)$ **can become negative,**



$$\beta = 0, \pi \quad for \ \breve{H}(\omega) \geq 0, or < 0$$

The magnitude and phase of FIR filter are given by:

$$\varphi(\omega) = \begin{cases} -\dfrac{N\omega}{2}, & for \ \breve{H}(\omega) \geq 0 \\ -\dfrac{N\omega}{2} - \pi, & for \ \breve{H}(\omega) < 0 \end{cases}$$

$$|H(e^{j\omega})| = |\breve{H}(\omega)|$$

or $$\varphi(\omega) = \begin{cases} -\dfrac{N\omega}{2} + \dfrac{\pi}{2}, & for \ \breve{H}(\omega) \geq 0 \\ -\dfrac{N\omega}{2} - \dfrac{\pi}{2}, & for \ \breve{H}(\omega) < 0 \end{cases}$$

The group delay is $\tau(\omega) = N/2$ **.**

Proof: when $h[n]$ is symmetric,

$$h[n] = h[N-n]$$

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = \frac{1}{2}\sum_{n=0}^{N} h[n]\left[z^{-n} + z^{-N} z^{n}\right]$$

$$= z^{-\frac{N}{2}} \sum_{n=0}^{N} h[n]\left[\frac{1}{2} Z^{-(n-\frac{N}{2})} + \frac{1}{2} Z^{(n-\frac{N}{2})}\right]$$

So,

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{n=0}^{N} h[n] \cos[(n - N/2)\omega]$$

$$= e^{-jN\omega/2} \breve{H}(\omega) \qquad \text{.....} \ \textbf{(1)}$$

In similar way,we can get:

$$H(e^{j\omega}) = -je^{-jN\omega/2}\sum_{n=0}^{N} h[n]\sin[(n-N/2)\omega]$$

$$= je^{-jN\omega/2}\breve{H}(\omega) \qquad \text{.....} \quad \textbf{(2)}$$

when *h*[*n*] is antisymmetric.

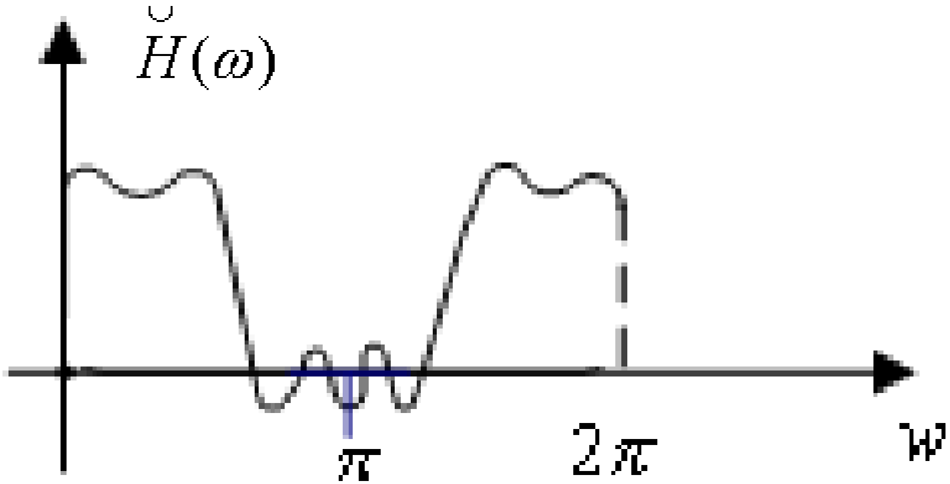
Type 1: *h*[*n*]=*h*[*N*−*n*] with Odd Length

From (1),

$$\breve{H}(\omega) = \sum_{n=0}^{N} h[n]\cos\left[\omega(n-\frac{N}{2})\right]$$

$$\breve{H}(\omega) = h[\frac{N}{2}] + \sum_{n=1}^{N/2} 2h[\frac{N}{2}-n]\cos[\omega n]$$

Because cos$n\omega$ is symmetric at ω=0, π , 2π, therefore, $\breve{H}(\omega)$ is symmetric at ω=0, π , 2π.



Type 2: $h[n]=h[N-n]$ with Even Length

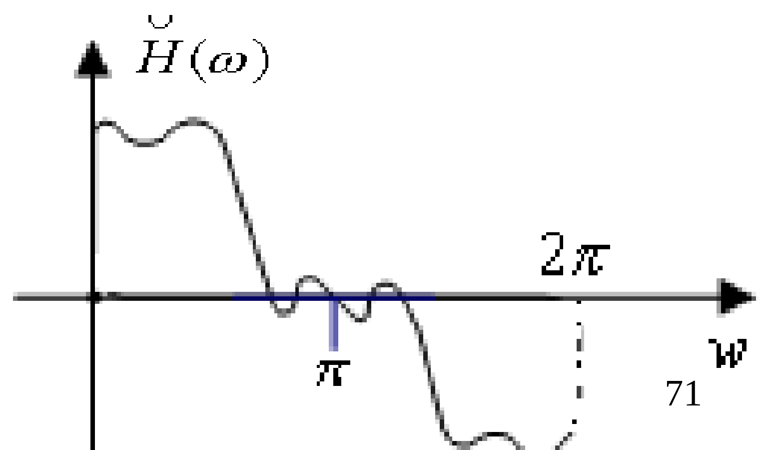
From (1),

$$\breve{H}(\omega) = \sum_{n=0}^{N} h[n]\cos\left[\omega\left(n - \frac{N}{2}\right)\right]$$

$$h\left[\frac{N}{2}\right] = 0$$

$$\breve{H}(\omega) = \sum_{n=1}^{(N+1)/2} 2h\left[\frac{N+1}{2} - n\right]\cos\left[\omega\left(n - \frac{1}{2}\right)\right]$$

Because cos[*ω*(*n*−1/2)] is antisymmetric at *ω*=*π*, therefore, $\breve{H}(\omega)$ **is antisymmetric at *ω*=*π* (** $\breve{H}(\pi) = 0$ **).**

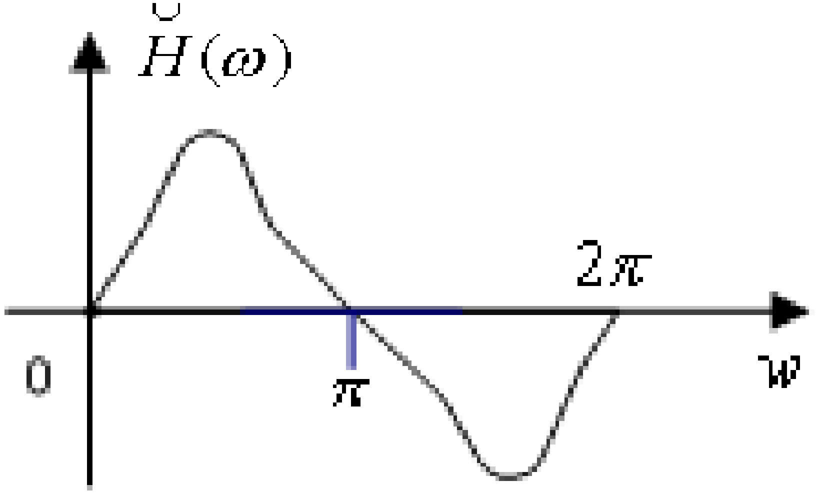


Type 3: $h[n]=-h[N-n]$ with Odd Length

From (2),

$$\breve{H}(\omega) = 2\sum_{n=1}^{N/2} h[\frac{N}{2} - n]\sin(n\omega)$$

Because $\sin n\omega$ is zero and antisymmetric at $\omega=0, \pi, 2\pi$, therefore, $\breve{H}(\omega)$ is antisymmetric at $\omega=0, \pi, 2\pi$. ($\breve{H}(\omega)=0$ at $\omega=0, \pi, 2\pi$)



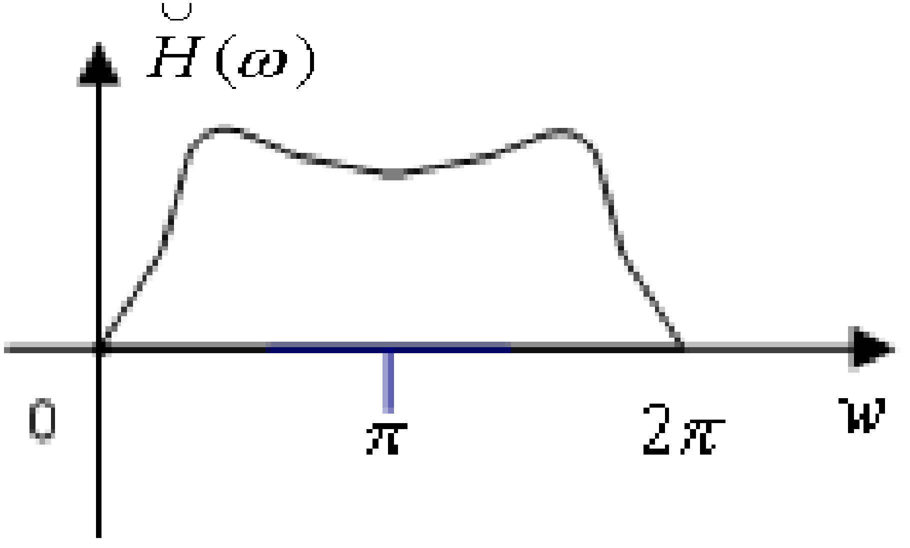
Type 4: $h[n]=-h[N-n]$ with Even Length

From (2),

$$\breve{H}(\omega) = 2\sum_{n=1}^{(N+1)/2} h[\frac{N+1}{2} - n]\sin((n - \frac{1}{2})\omega)$$

Because $\sin[\omega(n-1/2)]$ is zero at $\omega=0, 2\pi$.

Therefore, $\breve{H}(\omega)$ is symmetric at $\omega=0, 2\pi$.



Zero Locations of Linear-Phase FIR Transfer Functions

An Linear-Phase FIR Transfer Functions

satisfies

$$H(z) = z^{-N} H(z^{-1})$$

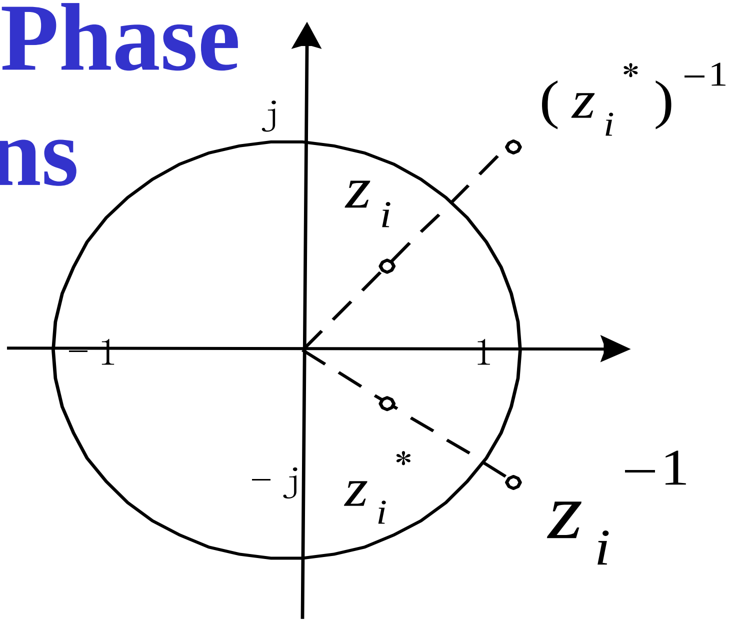
$$H(z)$$

$$H(z) = -z^{-N} H(z^{-1})$$

If $h[n]$ is real, z_i is a zero of $H(z)$, then, z_i^{-1} is also a zero of $H(z)$.

That means the zeros of $H(z)$ are symmetric on the unit circle of z-plane.

Zero Locations of Linear-Phase FIR Transfer Functions



Discussion:

(a) Type 2 FIR has zero at $z=-1$ ($\omega=\pi$), it can not be used to design a highpass filter.

(b) Type 3 FIR has zeros $z=-1$ ($\omega=\pi$), $z=1$ ($\omega=0$), it can not be used to design a highpass, lowpass or bandstop filter.

Zero Locations of Linear-Phase FIR Transfer Functions

(c) Type 4 FIR has zeros at $z=1$ ($\omega=0, 2\pi$), it can not be used to design a lowpass filter.

(d) Type 1 FIR has no such restrictions and can be used to design almost any type of filter.

Simple Digital Filters

Example 1. Lowpass and Highpass

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) \qquad H_1(z) = \frac{1}{2}(1 - z^{-1})$$

$$H_0(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

$$H_1(e^{j\omega}) = je^{-j\omega/2}\sin(\omega/2)$$

Linear Phase?

Type?

Example 2. The M-order FIR Comb Filter

$$y[n] = \frac{1}{M}\{x[n] + x[n-1] + x[n-2] + \ldots + x[n-M+1]\}$$

The impulse response

Simple Digital Filters

$$h[n] = \frac{1}{M}[u[n] - u[n-M]]$$

$$H(z) = \frac{1}{M}\frac{1 - z^{-M}}{1 - z^{-1}}$$

$$H(e^{j\omega}) = \frac{1}{M}\frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$

$$\left|H(e^{j\omega})\right| = \frac{1}{M}\left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

Linear-phase?

Examples of Type I Linear-Phase FIR Transfer Functions

Type 1: Symmetric Impulse Response with Odd Length

- **In this case, the degree *N* is even**
- **Assume *N*=8 for simplicity**
- **The transfer function *H*(*z*) is given by**

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$$
$$+ h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

Examples of Type I Linear-Phase FIR Transfer Functions

- **Because of symmetry, we have *h*[0]=*h*[8], *h*[1] = *h*[7], *h*[2] = *h*[6], and *h*[3] = *h*[5]**
- **Thus, we can write:**

$$H(z) = h[0]\left(1+z^{-8}\right)+h[1]\left(z^{-1}+z^{-7}\right)+h[2]\left(z^{-2}+z^{-6}\right)+h[3]\left(z^{-3}+z^{-5}\right)+h[4]z^{-4}$$

$$= z^{-4}\left\{h[0]\left(z^{4}+z^{-4}\right)+h[1]\left(z^{3}+z^{-3}\right)+h[2]\left(z^{2}+z^{-2}\right)+h[3]\left(z+z^{-1}\right)+h[4]\right\}$$

Examples of Type I Linear-Phase FIR Transfer Functions

- **The corresponding frequency response is then given by**

$$H(e^{j\omega}) = e^{-j4\omega}\{2h[0]\cos(4\omega) + 2h[1]\cos(3\omega)$$

$$+ 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]\}$$

- **The quantity inside the braces is a real function of ω, and can assume positive or negative values in the range $0 \le |\omega| \le \pi$**

Examples of Type I Linear-Phase FIR Transfer Functions

- **The phase function here is given by**

$$\theta(\omega) = -4\omega + \beta$$

where β is either 0 or π, and hence, it is a linear function of ω in the generalized sense

- **The group delay is given by**

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4$$

indicating a constant group delay of 4 samples

Examples of Type I Linear-Phase FIR Transfer Functions

- **In the general case for Type 1 FIR filters, the frequency response is of the form**

$$H(e^{j\omega}) = e^{-jN\omega/2}\breve{H}(\omega)$$

where the amplitude response $\breve{H}(\omega)$ **, also called the zero-phase response, is of the form**

$$\breve{H}(\omega) = h[\tfrac{N}{2}] + 2\sum_{n=1}^{N/2} h[\tfrac{N}{2} - n]\cos(\omega n)$$

Minimum-Phase and Maximum-Phase Transfer Functions

- **Consider the two 1st-order transfer functions:**

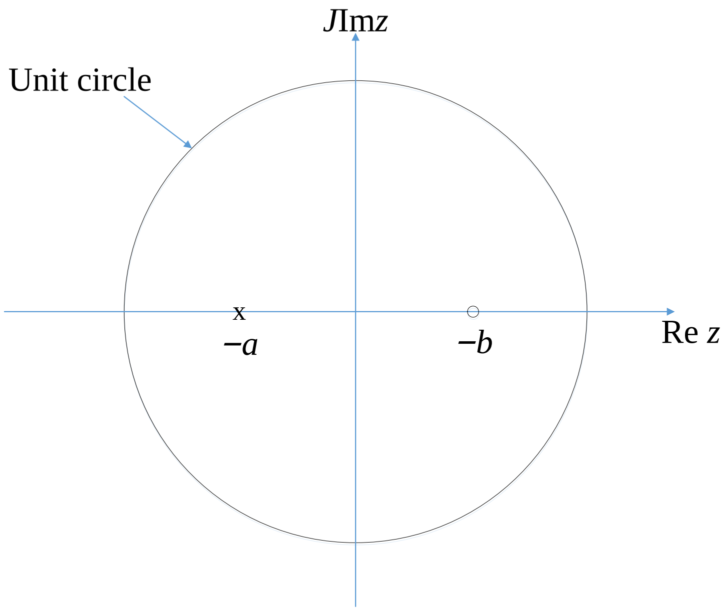
$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

- **Both transfer functions have a pole inside the unit circle at the same location $z=-a$ and are stable**

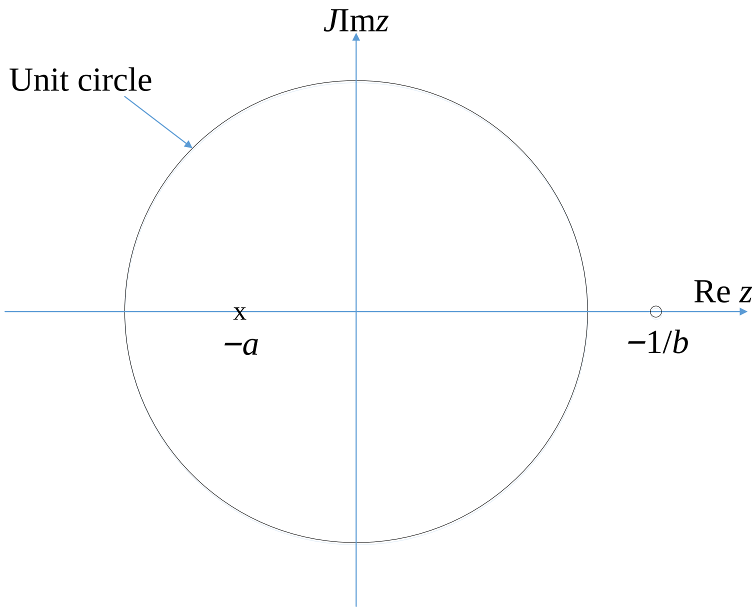
- **But the zero of $H_1(z)$ is inside the unit circle at $z=-b$, whereas, the zero of $H_2(z)$ is at $z=-1/b$ situated in a mirror-image symmetry**

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions



$H_1(z)$



$H_2(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- **However, both transfer functions have an identical magnitude function as**

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

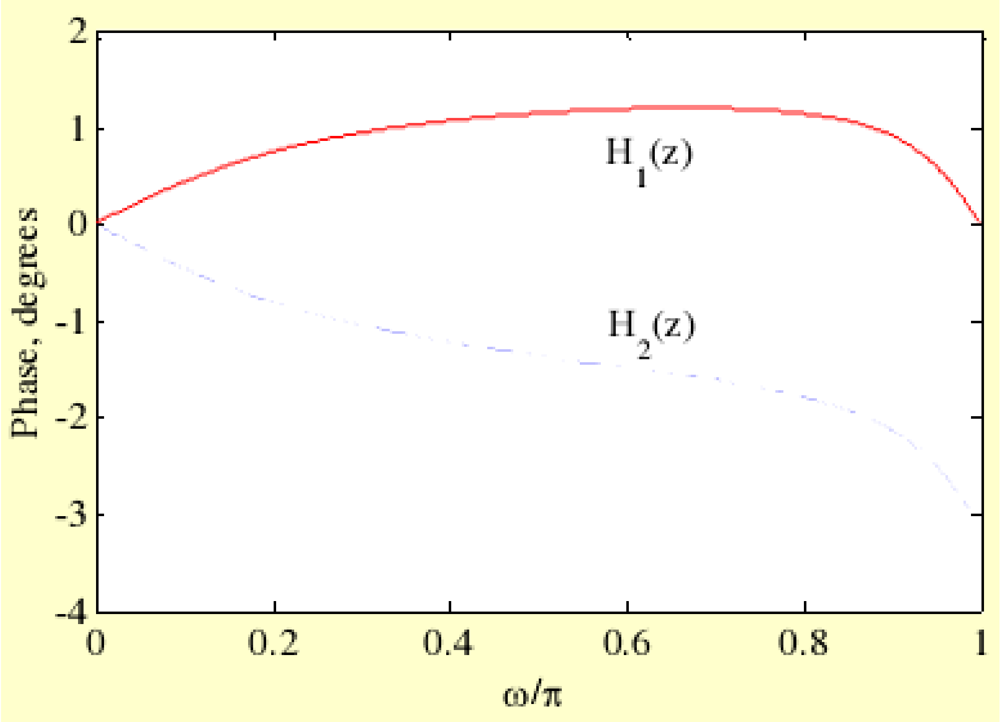
The corresponding phase functions are

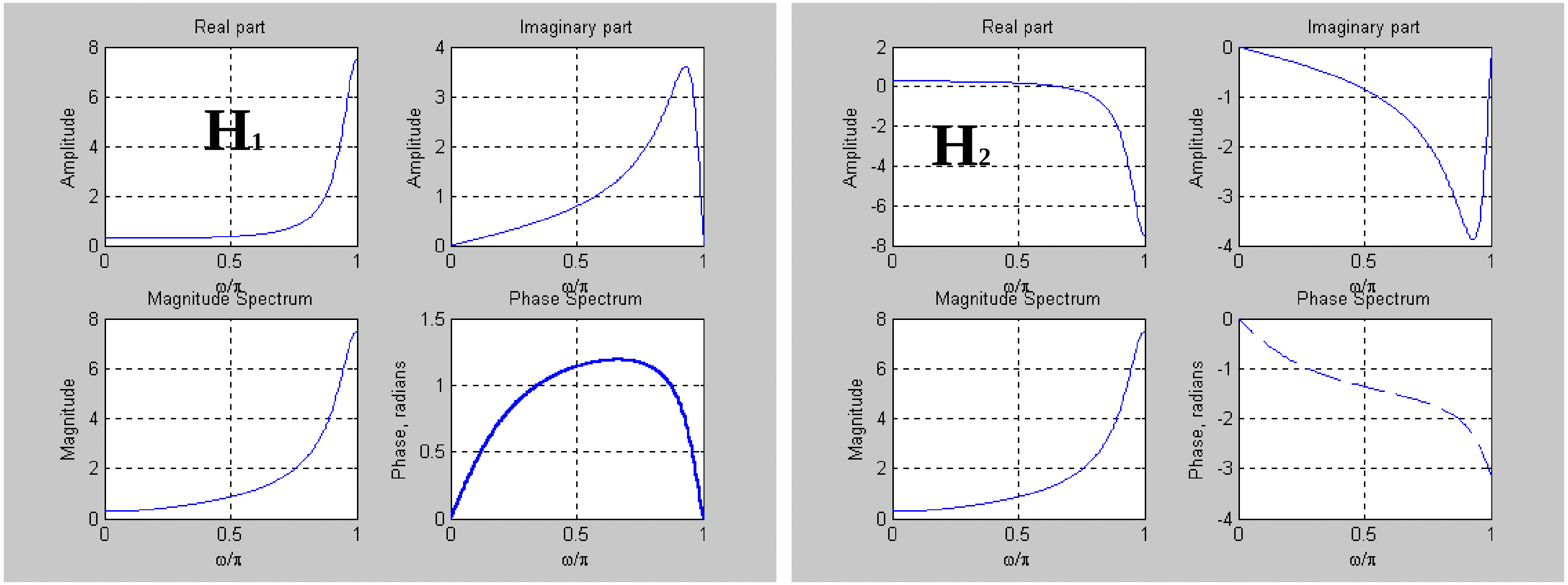
$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

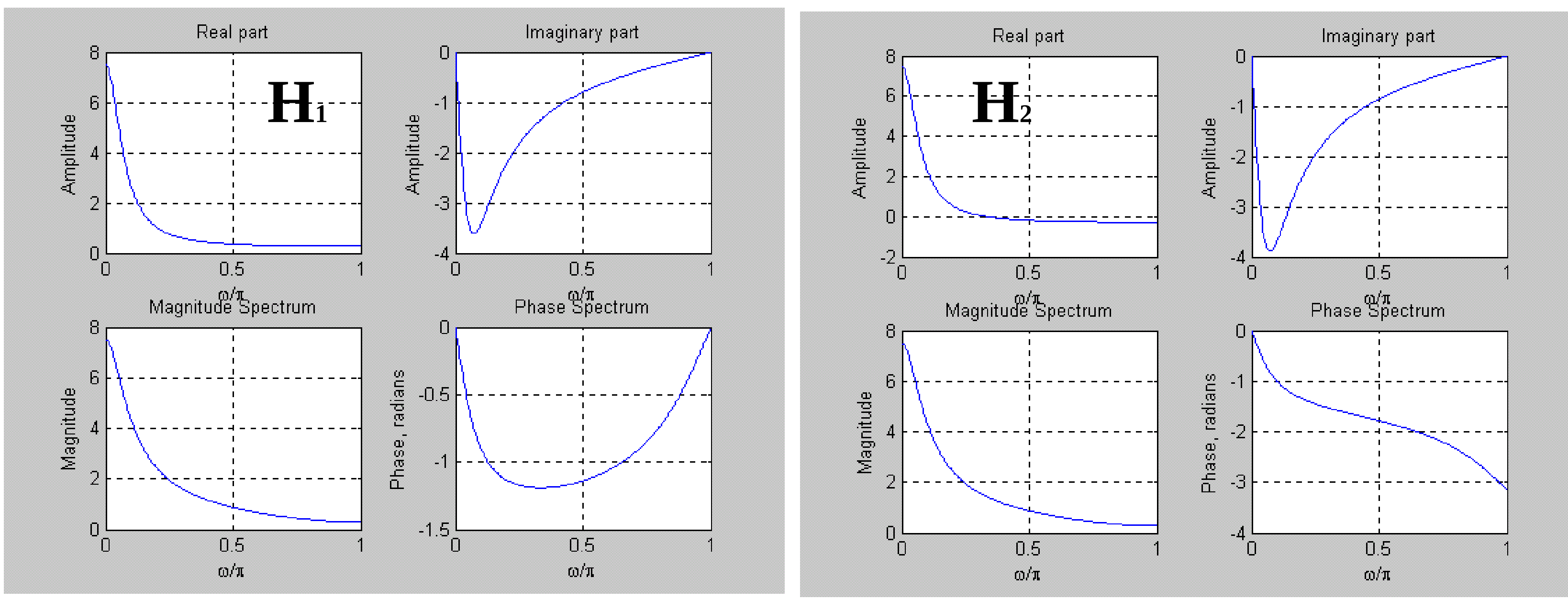
Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for $a = 0.8$ and $b = -0.5$





a = 0.8 and b = −0.5



a = −0.8 and b = 0.5

Minimum-Phase and Maximum-Phase Transfer Functions

- **From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$**
- **The excess phase lag property of $H_2(z)$ with respect to $H_1(z)$ can also be explained by observing that we can write**

$$H_2(z) = \frac{bz+1}{z+a} = \underbrace{\left(\frac{z+b}{z+a}\right)}_{H_1(z)} \underbrace{\left(\frac{bz+1}{z+b}\right)}_{A(z)}$$

Minimum-Phase and Maximum-Phase Transfer Functions

- **Where $A(z) = (bz+1)/(z+b)$ is a stable allpass function**
- **The phase functions of $H_1(z)$ and $H_2(z)$ are thus related through**

$$arg[H_2(e^{j\omega})] = arg[H_1(e^{j\omega})] + arg[A(e^{j\omega})]$$

- **As the *unwrapped phase function of a stable first-order allpass function is a negative function of ω*, it follows from the above that $H_2(z)$ has indeed an excess phase lag with respect to $H_1(z)$**

Minimum-Phase and Maximum-Phase Transfer Functions

- **Generalizing the above result, let $H_m(z)$ be a causal stable transfer function with all zeros inside the unit circle and let $H(z)$ be another causal stable transfer function satisfying $|H(e^{j\omega})| = |H_m(e^{j\omega})|$**
- **These two transfer functions are then related through $H(z) = H_m(z)A(z)$ where $A(z)$ is a causal stable allpass function**

Minimum-Phase and Maximum-Phase Transfer Functions

- **The unwrapped phase functions of $H_m(z)$ and $H(z)$ are thus related through**

$$\arg[H(e^{j\omega})] = \arg[H_m(e^{j\omega})] + \arg[A(e^{j\omega})]$$

- **$H(z)$ has an excess phase lag with respect to $H_m(z)$**
- **A causal stable transfer function with all zeros inside the unit circle is called a *minimum-phase transfer function***

Minimum-Phase and Maximum-Phase Transfer Functions

- **A causal stable transfer function with all zeros outside the unit circle is called a** ***maximum-phase transfer function***
- **A causal stable transfer function with zeros inside and outside the unit circle is called a** ***mixed-phase transfer function***

Minimum-Phase and Maximum-Phase Transfer Functions

- **<u>Example</u>: Find the corresponding minimum-phase transfer function for the following mixed-phase transfer function:**

$$H(z) = \frac{2(1+0.3z^{-1})(0.4-z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}$$

- **We can rewrite *H*(*z*) as**

$$H(z) = \underbrace{\left[\frac{2(1+0.3z^{-1})(1-0.4z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}\right]}_{\text{Minimum-phase transfer function}} \underbrace{\left(\frac{0.4-z^{-1}}{1-0.4z^{-1}}\right)}_{\text{Allpass transfer function}}$$

Thanks!

Any questions?