Digital Signal Processing

Z-Transform

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Outline

- Z-Transform
- Rational Z-Transform
- Inverse Z-Transform
- Z-Transform Theorems

Z-Transform

Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

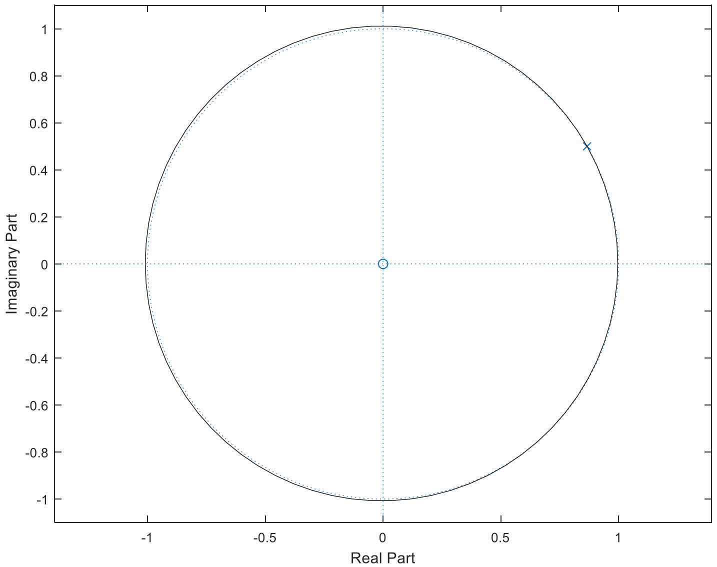
Let $z = re^{j\omega}$

$$X(z)\big|_{Z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} \boxed{x[n]r^{-n}} e^{-j\omega n} = X(re^{j\omega})$$

Converge if $\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$

Region of Convergence (ROC)

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

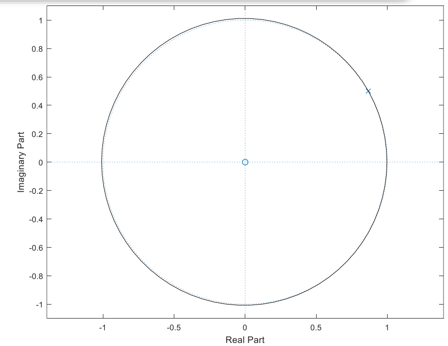


Z-Transform

Example: right sided sequence $x[n]=a^n u[n]$

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}=\sum_{n=0}^{\infty} a^n z^{-n}=\sum_{n=0}^{\infty}(az^{-1})^n$$

$$=\frac{1}{1-az^{-1}} \qquad |z|>|a|$$



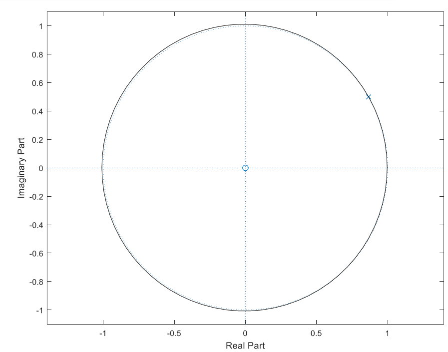
Stability of the sequence?

Stable if $|a|<1$

Example: left sided sequence $x[n]=-a^n u[-n-1]$

$$X(z)=-\sum_{n=-\infty}^{-1} a^n z^{-n}=-\sum_{n=1}^{\infty}(a^{-1}z)^n$$

$$=\frac{1}{1-az^{-1}} \qquad |z|<|a|$$



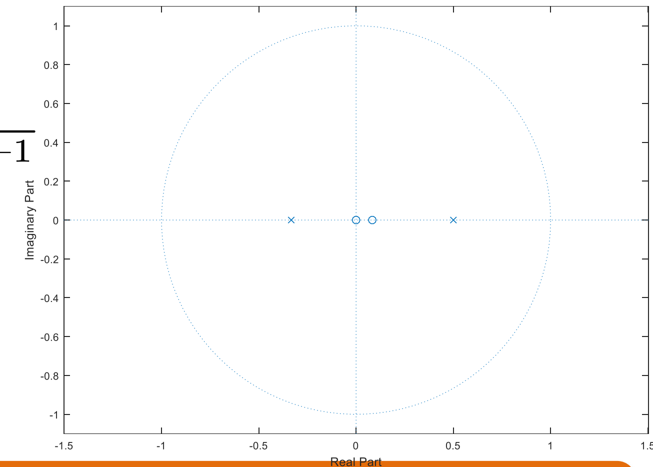
Stability of the sequence?

Stable if $|a|>1$

Example: Sum of 2 sequences $x[n]=(1/2)^n u[n]+(-1/3)^n u[n]$

$$X(z)=X_1(z)+X_2(z)$$
$$=\frac{1}{1-(1/2)z^{-1}}+\frac{1}{1-(-1/3)z^{-1}}$$
$$=\frac{2z(z-1/12)}{(z-1/2)(z+1/3)}$$

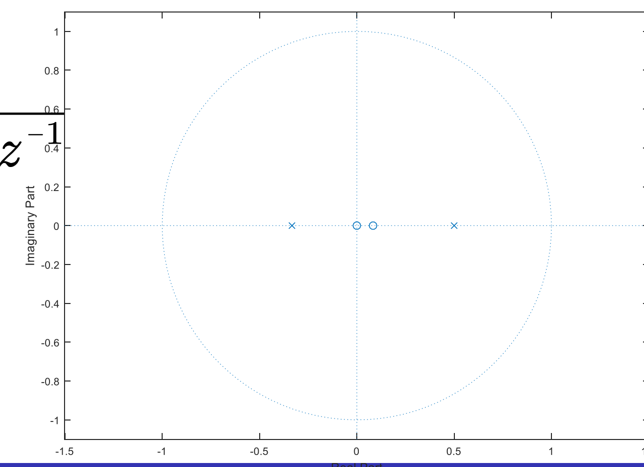
ROC $|z|>1/2$



Example: 2 sided sequence $x[n]=(-1/3)^n u[n]-(1/2)^n u[-n-1]$

$$X(z)=X_1(z)+X_2(z)$$
$$=\frac{1}{1-(1/2)z^{-1}}+\frac{1}{1-(-1/3)z^{-1}}$$
$$=\frac{2z(z-1/12)}{(z-1/2)(z+1/3)}$$

ROC $1/3<|z|<1/2$



Z-Transform

Example: Finite length sequences $x[n]=a^n\,(u[n]-u[n-N])$

$$X(z)=\sum_{n=0}^{N-1}a^n z^{-n}=\frac{1-(az^{-1})^N}{1-az^{-1}}=\frac{1}{z^N-1}\frac{z^N-a^N}{z-a}$$

ROC $$X(z)=\sum_{n=0}^{N-1}|a^n z^{-n}|<\infty$$

$$|z|\neq 0 \qquad |a|<\infty$$

1. ROC is a ring or disk
2. ROC contains no pole
3. Right side sequence: outside the largest pole
4. Left side sequence: inside the smallest pole
5. Two side sequence: ring
6. Finite length sequence: entire z-plane (except z=0 or z= ∞)

Z-Transform pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

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- Computation of Convolution Sum

LTI System in Rational Form

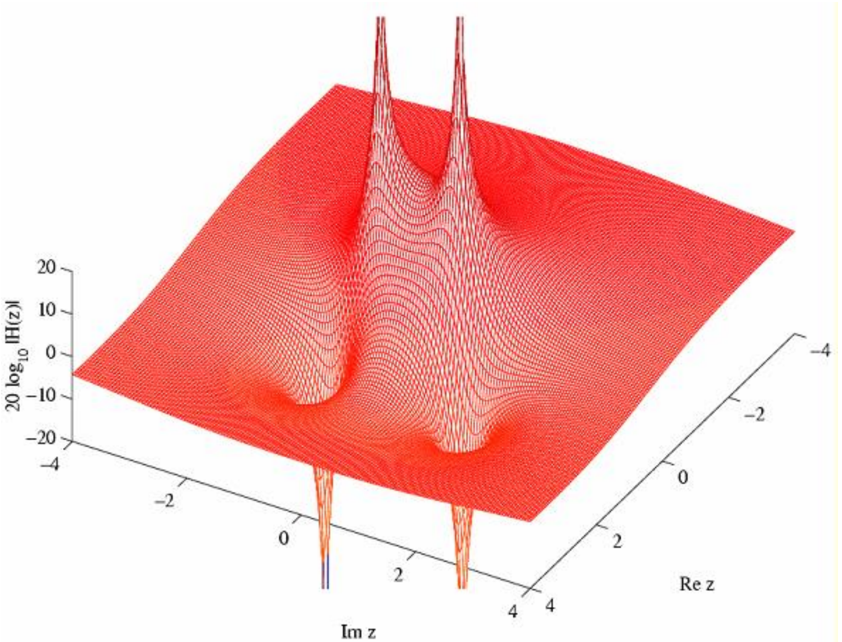
$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_N z^{-N}}$$

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

zeros

poles



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Inverse Z-Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$$

Integration around a counterclockwise closed circular contour centered at the origin and with radius r.

Inspection $X(Z) = \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad \Rightarrow \quad x[n] = a^n u[n]$

Factorization $X(Z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad A_k = (1 - d_k z^{-1}) X(Z)|_{z=d_k}$

Example $X(Z) = \frac{1}{(1 - 1/4z^{-1})(1 - 1/2z^{-1})} \quad |z| > \frac{1}{2}$

$$X(Z) = \frac{-1}{(1 - 1/4z^{-1})} + \frac{2}{(1 - 1/2z^{-1})} \quad x[n] = (2(1/2)^n - (1/4)^n) u[n]$$

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Z-Transform Theorems

Property	Sequence	z -Transform	ROC
	$g[n]$	$G(z)$	$\mathcal{R}_g$
	$h[n]$	$H(z)$	$\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g-} < |z| < R_{g+}$ and $\mathcal{R}_h$ denotes the region $R_{h-} < |z| < R_{h+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g+} < |z| < 1/R_{g-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g-}R_{h-} < |z| < R_{g+}R_{h+}$.

Z-Transform Theorems

Differentiation $\quad nx[n] \longleftrightarrow -z\dfrac{d}{dz}X[z] \quad$ ROC: same

Proof

$$X(z) = \sum_{n=-\infty}^{N-1} x[n]z^{-n} \Rightarrow \frac{d}{dz}X(z) = \sum_{n=-\infty}^{N-1} -nx[n]z^{-n-1}$$

$$-z\frac{d}{dz}X(z) = \sum_{n=-\infty}^{N-1} nx[n]z^{-n}z = \sum_{n=-\infty}^{N-1} nx[n]z^{-n} \longleftrightarrow nx[n]$$

Example : $\quad X(z) = \ln(1+az^{-1}) \quad |z|>a$

$$-z\frac{d}{dz}X(z) = -z\frac{-az^{-2}}{1+az^{-1}} = \frac{az^{-1}}{1+az^{-1}}$$

$$\frac{1}{1+az^{-1}} \longleftrightarrow (-a)^n u[n] \qquad az^{-1}\frac{1}{1+az^{-1}} \longleftrightarrow (-a)^{n-1}u[n-1]$$

$$\ln(1+az^{-1}) \longleftrightarrow (-1)^{n-1}\frac{1}{n}a^n u[n-1]$$

Z-Transform Theorems

Time Reversal $x^*[-n] \longleftrightarrow X^*(1/z)$ ROC: $1/\mathrm{ROC}_x$

Proof

$$\sum_{n=-\infty}^{\infty} x^*[-n]z^{-n} = \left(\sum_{n=-\infty}^{\infty} x[-n]z^{n}\right)^* = \left(\sum_{n=-\infty}^{\infty} x[-n](1/z)^{-n}\right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n](1/z)^{n}\right)^* = \left(\sum_{n=-\infty}^{\infty} x[n](1/z^*)^{-n}\right)^*$$

Convolution $x[n] \circledast y[n] \longleftrightarrow X(z)Y(z)$ ROC: $\mathrm{ROC}_x \cap \mathrm{ROC}_y$

Proof

$$x[n] \circledast y[n] \longleftrightarrow \sum_{n=-\infty}^{\infty} x[n] \circledast y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[n-k]z^{n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]z^{n} = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} y[m]z^{-m-k}$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{n=-\infty}^{\infty} y[m]z^{-m} = X(z)Y(z)$$