Discrete-Time Signals In the Transform Domain

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§ 3. Discrete-Time Signals In the Transform Domain

§ 3.1 CTFT (Continuous Time Fourier Translation)

$$x_a(t) \xleftrightarrow{\text{CTFT}} X_a(j\Omega)$$

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

§ 3.2 DTFT(Discrete Time Fourier Translation)

3.2.1 Definition

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \qquad \textbf{... (3.12)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \textbf{... (3.16)}$$

Proof:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

- **The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists.**
- **Then**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin \pi(n-\ell)}{\pi(n-\ell)}$$

- **Now**

L'Hôpital's rule

$$\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases} = \delta[n-\ell]$$

Unit sample sequence

Hence

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] = x[n]$$

- ***Example*** **- The DTFT of the unit sample sequence $\delta[n]$ is given by**

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = e^{-j\omega 0} = 1$$

- ***Example*** **- Consider the causal sequence**

$$x[n] = \alpha^{n}\mu[n], \qquad |\alpha| < 1$$

- **Its DTFT is given by**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^{n} \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^{n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^{n} = \frac{1}{1-\alpha e^{-j\omega}}$$

Geometric Series

as $\left|\alpha e^{-j\omega}\right| = \left|\alpha\right| \times \left|e^{-j\omega}\right| = \left|\alpha\right| < 1$

3.2.2 Basic Properties

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω.
- It is also a periodic function of ω with a period 2π:

$$X(e^{j(\omega_o+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega_o+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n}e^{-j2\pi kn} \qquad = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n} = X(e^{j\omega_o})$$

In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$$

- $X_{\text{re}}(e^{j\omega})$ **and** $X_{\text{im}}(e^{j\omega})$ **are, respectively, the real and imaginary parts of** $X(e^{j\omega})$ **, and are real functions of** ω
- $X(e^{j\omega})$ **can alternately be expressed as**

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

- **The relations between the rectangular and polar forms of $X(e^{j\omega})$ are given by:**

$$X_{re}(e^{j\omega}) = |X(e^{j\omega})|\cos\theta(\omega)$$

$$X_{im}(e^{j\omega}) = |X(e^{j\omega})|\sin\theta(\omega)$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega})$$

$$= X^2_{re}(e^{j\omega})+X^2_{im}(e^{j\omega})$$

$$\tan\theta(\omega) = X_{im}(e^{j\omega})/X_{re}(e^{j\omega})$$

Again

$$X(e^{j\omega}) = |X(e^{j\omega+2\pi k})|e^{j\theta(\omega+2\pi k)}$$

$$= |X(e^{j\omega})|e^{j\theta(\omega)}$$

- $|X(e^{j\omega})|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{\mathrm{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\mathrm{im}}(e^{j\omega})$ are odd functions of ω.

Note: $X(e^{j\omega}) = |X(e^{j\omega})| e^{j[\theta(\omega)+2\pi k]}$

$$= | X(e^{j\omega}) | e^{j\theta(\omega)}$$

for any integer k.

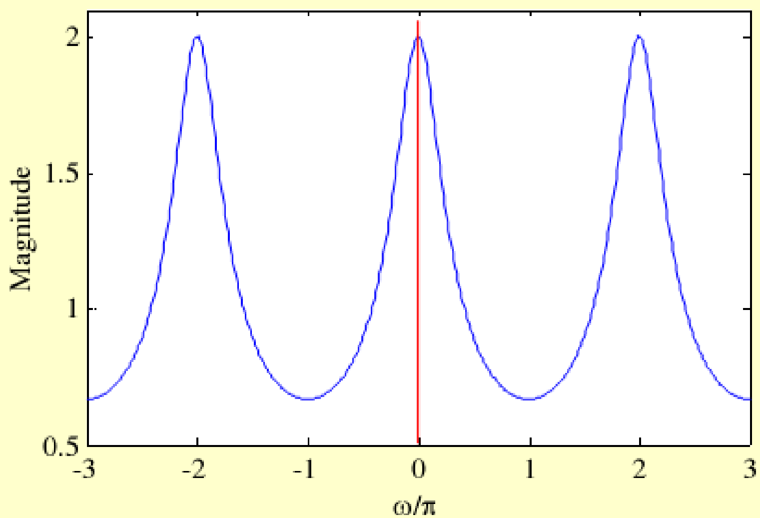
- The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT.

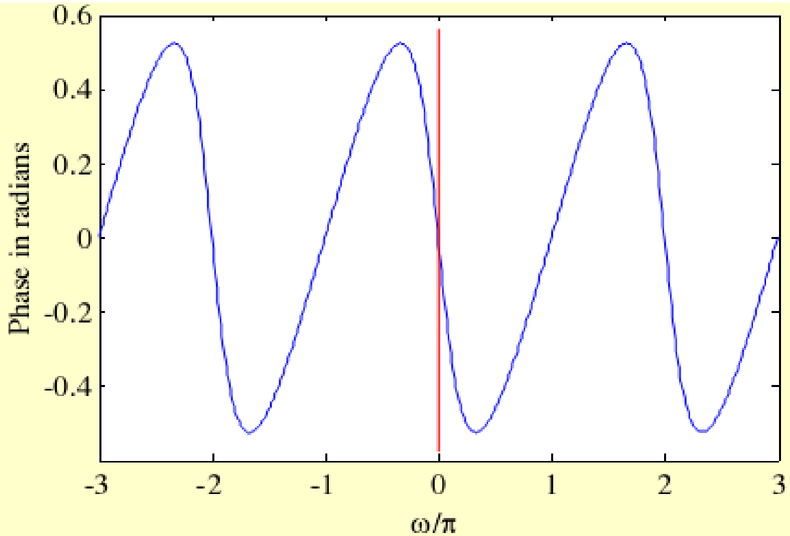
Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) \leq \pi$$

called the principal value

Example **The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$ are shown below**





$$|X(e^{j\omega})| = |X(e^{-j\omega})| \qquad \theta(\omega) = -\theta(-\omega)$$

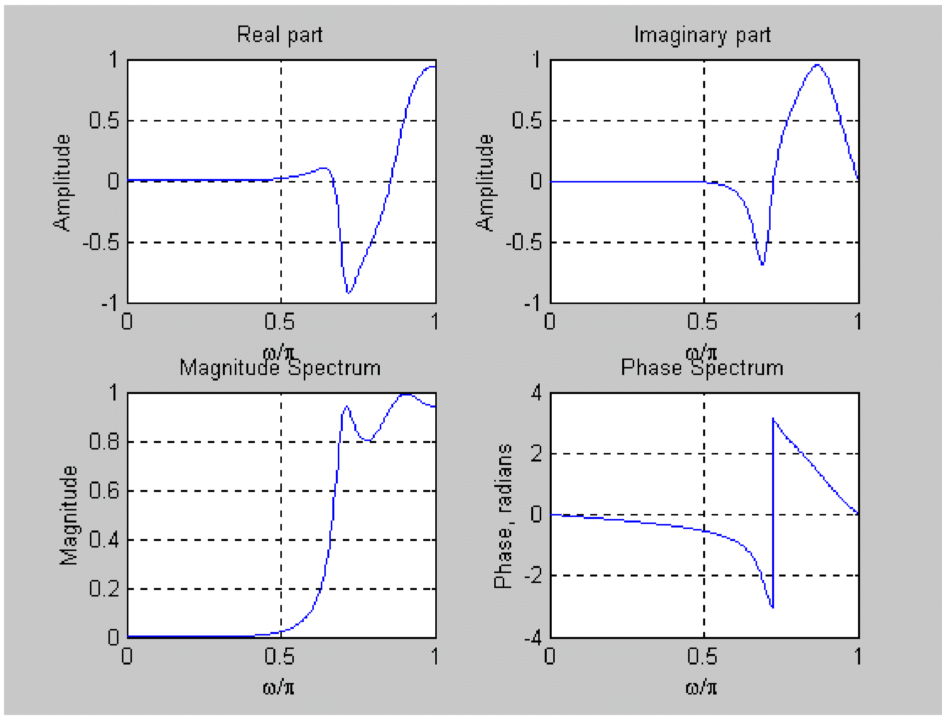
unwrapping

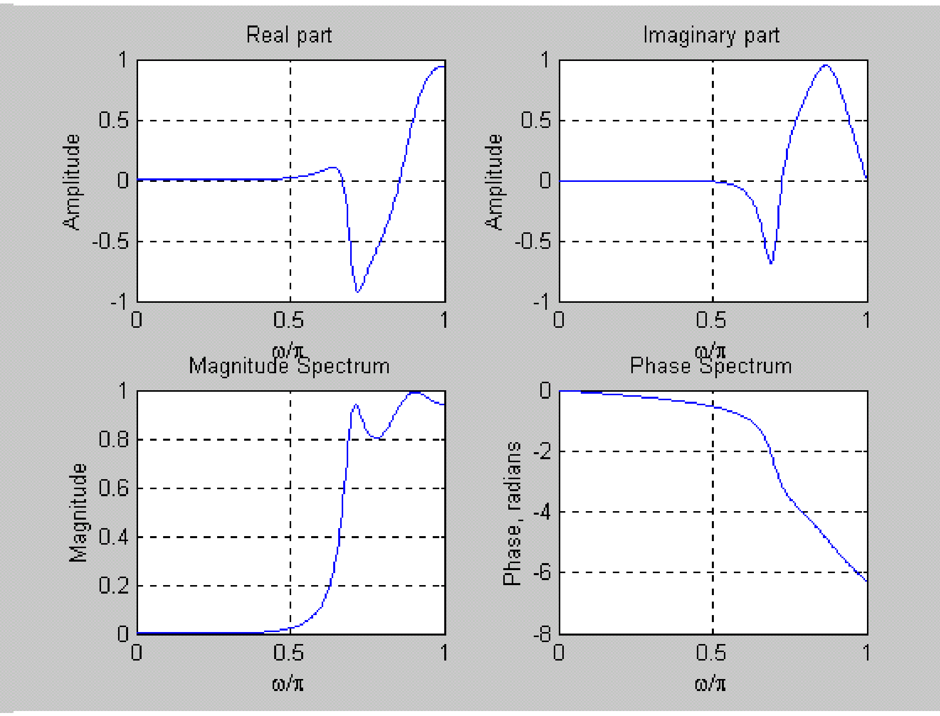
- **The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses.**
- **An alternate type of phase function that is a continuous function of ω is often used.**
- **It is derived from the original phase function by removing the discontinuities of 2π.**

unwrapping

- The process of removing the discontinuities is called "unwrapping"
- The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$
- In some cases, discontinuities of π may be present after unwrapping.

Example





Wrapped Phase

Unwrapped Phase

Discussion: The Relation Between CTFT and DTFT

If

$$x_a(t) \Leftrightarrow X_a(j\Omega)$$

Sampling in time domain

$$\hat{x}_a(t) = x_a(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT)$$

Its FT

$$\hat{X}_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x_a(nT)\int_{-\infty}^{\infty} \delta(t-nT)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)e^{-j\Omega nT}$$

The discrete signal $x[n] = x_a(nT)$ has DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \quad \text{.......} \ \textbf{(B)}$$

comparing with (A) and (B), when $\omega=\Omega T$,

$$X(e^{j\omega}) = \hat{X}_a(e^{j\omega}) \cdot$$

When sampling frequency is $f_s=1/T$ (Hz), the analog frequency $f=\Omega/2\pi$ (Hz) is

corresponding to its digital frequency $\omega=\Omega T$ (rid) .

In another way, the FT of sampled signal

$$\hat{x}_a(t) = x_a(t)\delta_T(t) \Leftrightarrow \hat{X}_a(j\Omega)$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty} X_a\left[j(\Omega - m\frac{2\pi}{T})\right] \qquad \text{........(C)}$$

When the Nyquist theoriem is satisfied,

$$X(e^{j\omega}) = \frac{1}{T} X_a(j\omega), \text{ where } \omega : (-\pi, \pi)$$

DTFT can be derived from FT by using this relation .

Example You can get DTFT of following sequences by using the relation between FT and DTFT.

$$x_1[n] = e^{j\omega_0 n};\ x_2[n] = \cos(\omega_0 n);\ x_3[n] = 1$$

Solution

Discreting in time $[t \rightarrow nT, T = 1]$,we get:

$$x_1(t) = e^{j\omega_0 t} \Leftrightarrow X_1(j\Omega) = 2\pi\delta(\Omega - \omega_0)$$

$$x_1[n] = e^{j\omega_0 n} \Leftrightarrow X_1(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2m\pi)$$

Discretization in time

Periodization in frequency

In similar way,

$$\cos \omega_0 t \Leftrightarrow \pi[\delta(\Omega + \omega_0) + \delta(\Omega - \omega_0)]$$

$$x_2[n] = \cos \omega_0 n \Leftrightarrow \pi \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2m\pi) + \delta(\omega - \omega_0 - 2m\pi)$$

$$x_3(t) = 1 \Leftrightarrow X_3(j\Omega) = 2\pi\delta(\Omega)$$

$$x_3[n] \Leftrightarrow X_3(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2m\pi)$$

Example Caculate the DTFT of

$$f[n]=\frac{\sin\omega_0 n}{\pi n}$$

Solution **From** $\frac{\sin\omega_0 t}{\pi t}\Leftrightarrow P_{2\omega_0}(\Omega)$

Discreting in time [t→nT, T=1], we get:

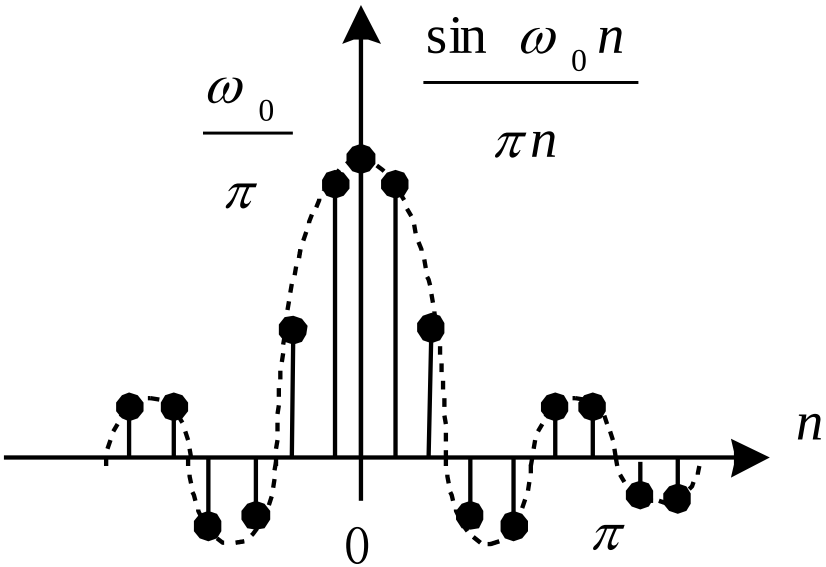
$$f[n]=\frac{\sin\omega_0 n}{\pi n}\Leftrightarrow F(e^{j\omega})=\frac{1}{T}\sum_{m=-\infty}^{\infty}F\ \ j(\omega-2m\pi)$$

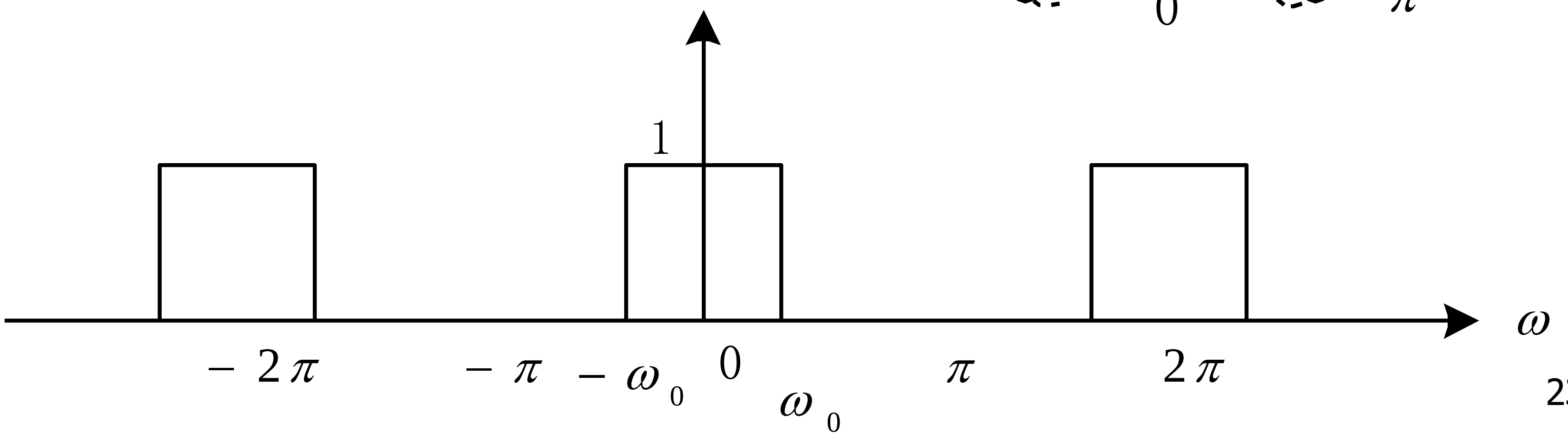
or $$F(e^{j\omega})=\sum_{m=-\infty}^{\infty}P_{2\omega_0}(\omega-2m\pi)$$

Prove by the Inverse-DTFT (IDTFT):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})e^{jn\omega}d\omega$$

$$=\frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{jn\omega}d\omega=\frac{\sin\omega_0 n}{\pi n}$$





3.2.3 Symmetry Relation

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Symmetry relations of the DTFT of a real sequence

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$ $X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{\text{ev}}[n]$ and $x_{\text{od}}[n]$ denote the even and odd parts of $x[n]$, respectively.

3.2.4 Convergence Condition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$$

It is an infinite series sum which may or may not converge.

- **Let**

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

- **Then for** **uniform convergence** **of** $X(e^{j\omega})$

$$\lim_{K\to\infty} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right| = 0$$

- **Now, if $x[n]$ is an absolutely summable sequence, i.e., if**

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right| < \infty$$

- **Then**

$|a+b|<=|a|+|b|$

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x[n] e^{-j\omega n} \right| = \sum_{n=-\infty}^{\infty} \left| x[n] \right| \left| e^{-j\omega n} \right| = \sum_{n=-\infty}^{\infty} \left| x[n] \right| < \infty$$

for all values of ω

- **Thus, the absolute summability of $x[n]$ is a** **sufficient condition** **for the existence of the DTFT $X(e^{j\omega})$**

- **Since**

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 \leq \left(\sum_{n=-\infty}^{\infty} \left|x[n]\right| \right)^2 ,$$

an absolutely summable sequence has always a finite energy.

- **However, a finite-energy sequence is not necessarily absolutely summable.**

- ***Example*** **- The sequence**

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$E_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$ Basel problem

- **But, $x[n]$ is not absolutely summable**

$$\sum_{n=1}^{\infty} \frac{1}{n} = \ln n + C$$

- **To represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a mean-square convergence of $X(e^{j\omega})$:**

$$\lim_{k \to \infty} \int_{-\pi}^{\pi} | X(e^{j\omega}) - X_K(e^{j\omega}) |^2 \, d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}$$

- **Here, the total energy of the error**

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

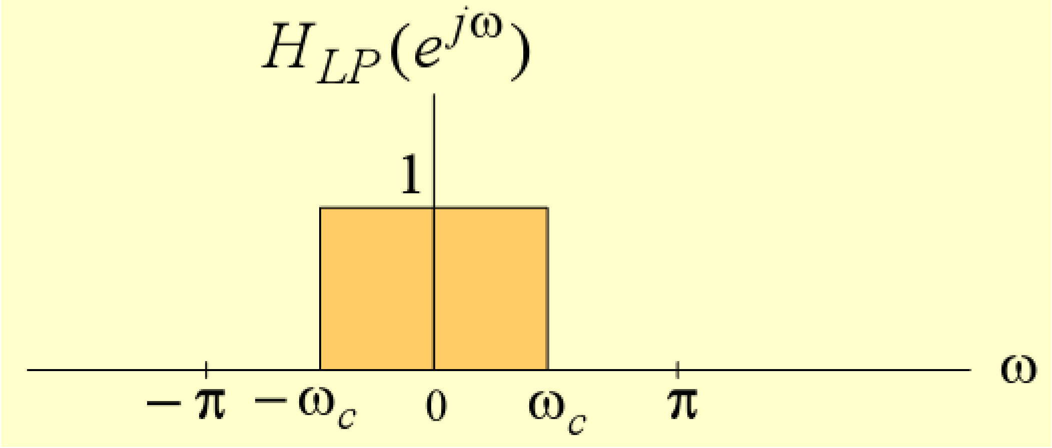
must approach zero at each value of ω **as** ***K*** **goes to** ∞

- **In such a case, the absolute value of the error** $|X(e^{j\omega}) - X_k(e^{j\omega})|$ **may not go to zero as** ***K*** **goes to** ∞ **and the DTFT is no longer bounded**

Example: Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

Shown below



- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty \le n \le \infty$$

n≠0

n=0

- The energy of $h_{LP}[n]$ is given by ω_c/π.
- $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable.

As a result

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

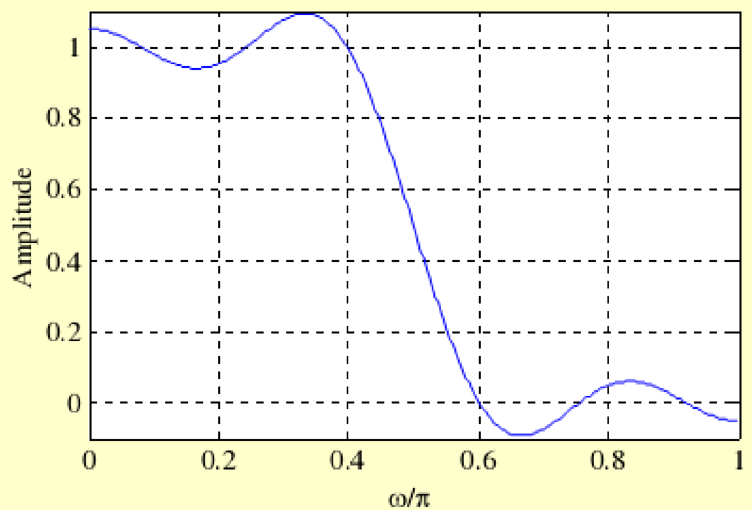
Does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense

- **The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function**

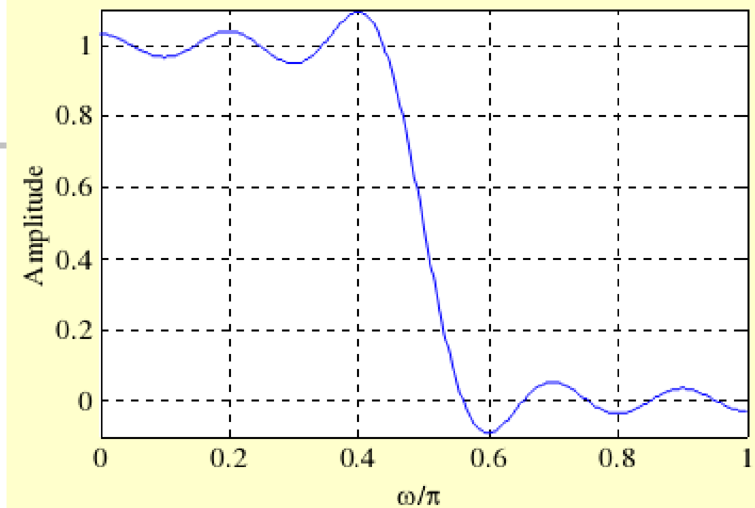
$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

For various values of K as shown in next slide

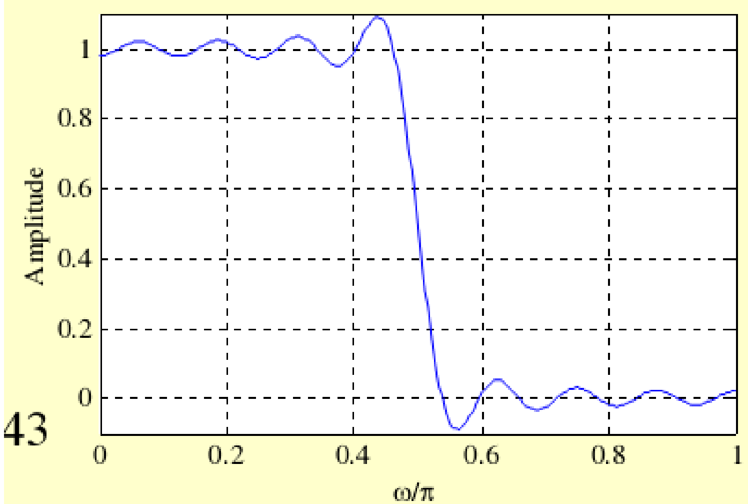
K = 10



K = 20

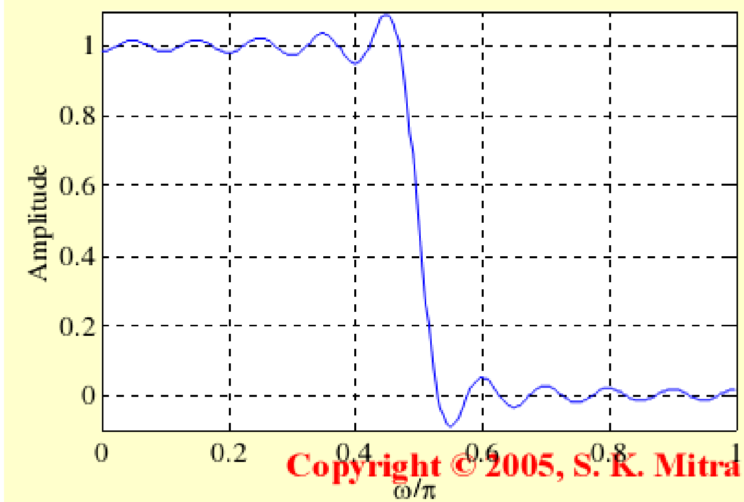


K = 30



43

K = 40



- **As can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$**
- **The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K.**

- As K goes to infinity, the condition

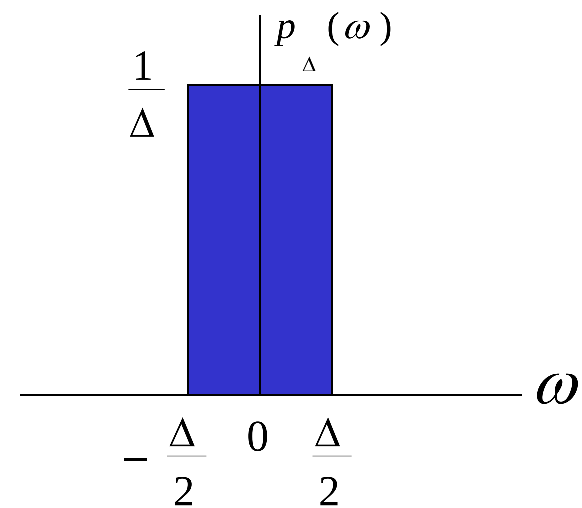
$$\lim_{K \to \infty} \int_{-\pi}^{\pi} | H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega}) |^2 \, d\omega = 0$$

holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ approximation $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the Gibbs phenomenon.

- **The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable.**
- **Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n+\varphi)$ and the exponential sequence $A\alpha^n$.**
- **For this type of sequences, a DTFT representation is possible using the Dirac Delta function $\delta(\omega)$.**

- **A Dirac Delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area**

- **It is the limitation form of a unit area pulse function $p_\Delta(\omega)$ as D goes to zero satisfying**



$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_{\Delta}(\omega)\, d\omega = \int_{-\infty}^{\infty} \delta(\omega)\, d\omega$$

- ***Example*** **- Consider the complex exponential sequence**

$$x[n] = e^{j\omega_o n}$$

Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \le \omega_o \le \pi$$

- **The function**

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

is a periodic function of ω with a period 2π and is called a periodic impulse train or impulse train.

- **To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$ we compute the inverse DTFT of $X(e^{j\omega})$.**

Thus

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega - \omega_o + 2\pi k\right)e^{j\omega n}d\omega$$

$$= \int_{-\pi}^{\pi}\delta(\omega - \omega_o)e^{j\omega n}d\omega = e^{j\omega_o n}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Sequence **DTFT**

$$\delta[n] \quad \leftrightarrow \quad 1$$

$$1 \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

$$e^{j\omega_o n} \quad \leftrightarrow \quad \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

$$\mu[n] \quad \leftrightarrow \quad \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega + 2\pi k)$$

$$\alpha^n \mu[n],\ (|\alpha| < 1) \quad \leftrightarrow \quad \frac{1}{1 - \alpha\, e^{-j\omega}}$$

3.3 DTFT Theorems

- There are a number of important properties of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table 3.4

Type of Property	Sequence	DTFT
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$ag[n]+bh[n]$	$aG(e^{j\omega})+bH(e^{j\omega})$
Time-shifting	$g[n-n_0]$	$e^{-j\omega n_0}G(e^{j\omega})$
Frequency-shifting	$e^{-j\omega_0 n}g[n]$	$G(e^{j(\omega+\omega_0)})$
Differentiation	$ng[n]$	$j dG(e^{j\omega})/d\omega$
Convolution	$g[n]*h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$	

proof of differentiation in frequency domain.

$$g[n] \longleftrightarrow G(e^{j\omega})$$

- Use the definition of $G(e^{j\omega})$ and differentiate both sides, we obtain

$$\frac{dG(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jng[n]e^{-j\omega n}$$

The right-hand side of this equation is the Fourier transform of $-jng[n]$. Therefore, multiplying both sides by j, we see

$$ng[n] \longleftrightarrow j\mathrm{d}G(e^{j\omega})/\mathrm{d}\omega$$

- ***Example 2: Explicit expression with DTFT theorems***
- **Determine the DTFT** $Y(e^{j\omega})$ **of** $y[n] = (n+1)\alpha^n \mu[n], |\alpha|<1$

- ***Example 2: Explicit expression with DTFT theorems***
- **Determine the DTFT $Y(e^{j\omega})$ of $y[n] = (n+1)\alpha^n\mu[n], |\alpha|<1$**

Let $x[n] = \alpha^n\mu[n], |\alpha|<1$

- **We can therefore write**

$$y[n] = nx[n] + x[n]$$

- **The DTFT of $x[n]$ is given by**

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

- **Using the differentiation property of the DTFT, we observe that the DTFT of $nx[n]$ is given by**

differentiation by parts

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- **Next using the linearity property of the DTFT we arrive at**

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

- ***Example 3: Implicit expression with DTFT theorems***
- **Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by**

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- ***Example 3: Implicit expression with DTFT theorems***
- **Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by**

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- **The DTFT of $\delta[n]$ is 1.**
- **Using the time-shifting property of the DTFT we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$.**

- **Using the linearity property we then obtain the frequency-domain representation of**

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as $d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$

- **Solving the above equation we get**

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Example 4: Convolution with DTFT

Convolution (in frequency domain)

$$x[n] \otimes y[n] \Leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

Convolution (in frequency domain)

$$x[n]y[n] \Leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

where the convolution in frequency is calculated *in a periodic interval*, it is called as the periodical convolution.

The DTFT $X(e^{j\omega})$ is a periodic function of ω with a period 2π.

Example 3.14

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EXAMPLE 3.14 Convolution Sum Computation Using Fourier Transform

Consider the sequences $x[n] = \alpha^n \mu[n], |\alpha| < 1$ and $h[n] = \beta^n \mu[n], |\beta| < 1$. We consider here the implementation of the convolution $y[n] = x[n] \circledast h[n]$ via the DTFT-based approach. We observe from Table 3.3 that the DTFTs of the sequence $x[n]$ and the sequence $h[n]$ are, respectively, $X(e^{j\omega}) = 1/(1 - \alpha e^{-j\omega})$ and $H(e^{j\omega}) = 1/(1 - \beta e^{-j\omega})$. Hence, using the convolution theorem, we note that the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

3.4 Energy Density Spectrum of a Discrete-Time Sequence

- **The total energy of a finite-energy sequence $g[n]$ is given by**

$$E_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- **From Parseval's relation we observe that**

$$E_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

- **The quantity**

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$$

is called the energy density spectrum

- **The area under this curve in the range: $-\pi \le \omega \le \pi$ divided by 2π is the energy of the sequence.**

3.5 Band-Limited Discrete-Time Signals

- **Since the spectrum of a discrete-time signal is a periodic function of ω with a period 2π, a full-band signal has a spectrum occupying the frequency range of $-\pi \le \omega \le \pi$**
- **A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range $-\pi \le \omega \le \pi$**

- **An ideal band-limited signal has a spectrum that is zero outside a frequency range $0<\omega_a\leq|\omega|\leq\omega_b<\pi$, that is**

$$X(e^{j\omega}) = \begin{cases} 0, & 0 \leq |\omega| \leq \omega_a \\ 0, & \omega_b \leq |\omega| \leq \pi \end{cases}$$

An ideal band-limited discrete-time signal cannot be generated in practice.

- **A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal's energy is concentrated.**
- **A lowpass discrete-time real signal has a spectrum occupying the frequency range $0<|\omega|\le\omega_p<\pi$ and has a bandwidth of ω_p.**

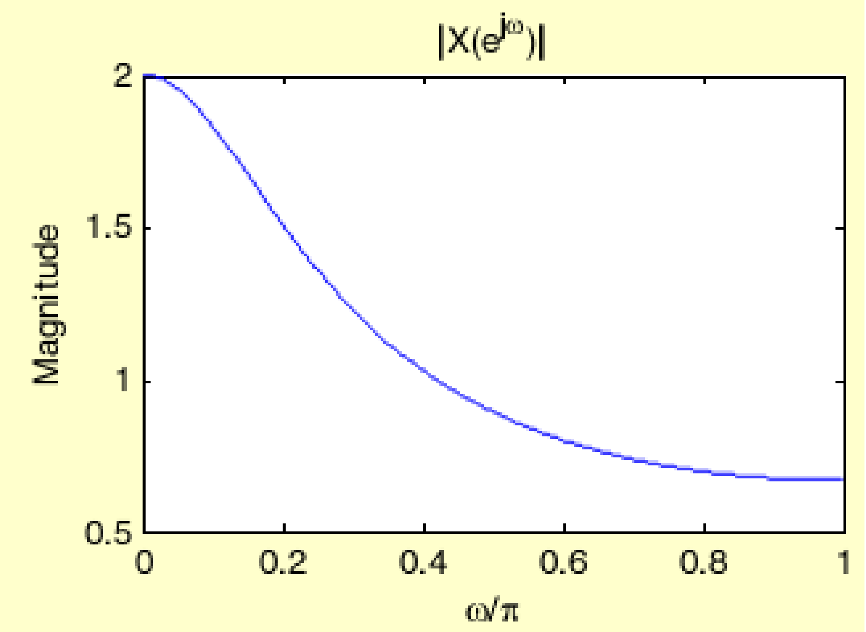
- **A highpass discrete-time real signal has a spectrum occupying the frequency range $0<\omega_p\leq|\omega|<\pi$ and has a bandwidth of $\pi-\omega_p$.**
- **A bandpass discrete-time real signal has a spectrum occupying the frequency range $0<\omega_L\leq|\omega|\leq\omega_H<\pi$ and has a bandwidth of $\omega_H-\omega_L$.**

- ***Example***: **Consider the sequence**

$$x[n] = (0.5)^n \mu[n]$$

Its DTFT is given below on the left along with its magnitude spectrum shown below on the right

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



- **It can be shown that 80% of the energy of this lowpass signal is contained in the frequency range** $0 \leq |\omega| \leq 0.5081\pi$
- **Hence, we can define the 80% bandwidth to be** 0.5081π

- ***Example:*** **Compute the energy of the sequence $h_{LP}[n] = \sin\omega_c n/\pi n, -\infty < n < \infty$**
- **Here**

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{LP}(e^{j\omega})|^2 \, d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

- **Therefore**

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Hence, $h_{LP}[n]$ is a finite-energy lowpass sequence.

3.6 DTFT Computation Using MATLAB

- **The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of**

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \dots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \dots + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_l$

- **For example, the statement**

$$H = \text{freqz}(\text{num, den, w})$$

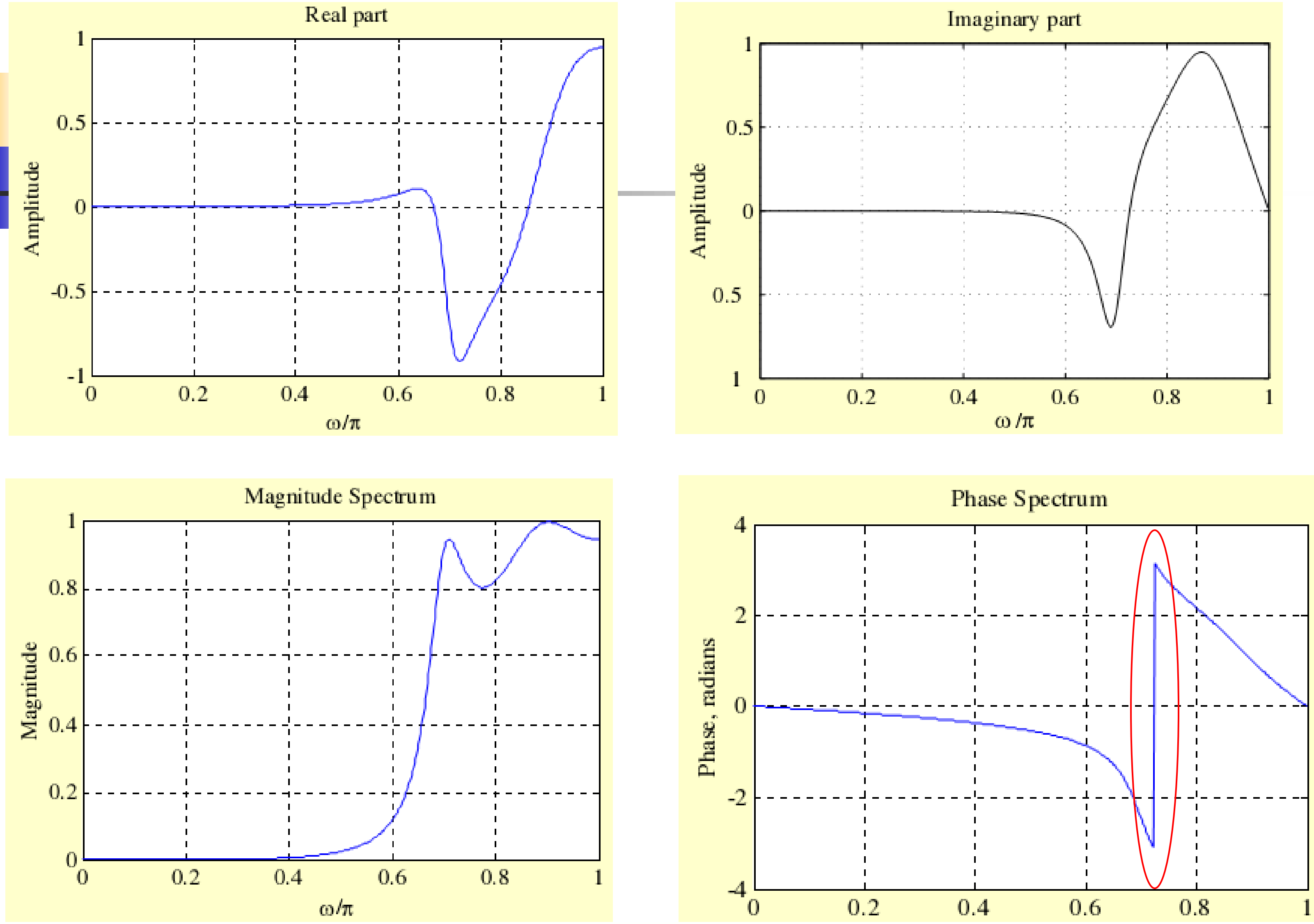
returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector w.

- There are several other forms of the function `freqz`
- The **Program 3_1.m** in the text can be used to compute the values of the DTFT of a real sequence
- It computes the real and imaginary parts, and the magnitude and phase of the DTFT

- ***Example*** **- Plots of the real and imaginary parts, and the magnitude and phase of the DTFT**

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

are shown on the next slide.



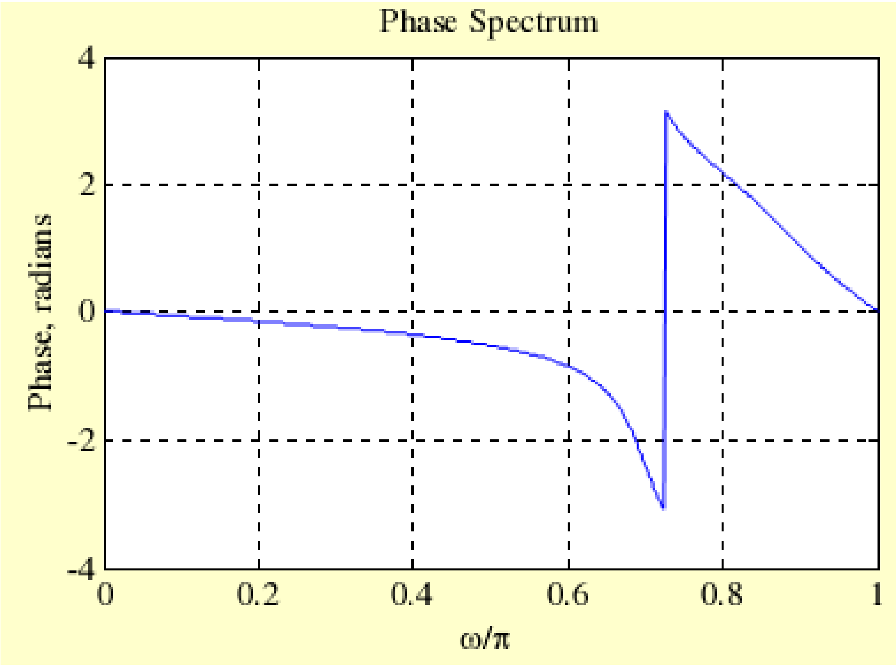
wrapped phase

3.7 The unwrapped phase function

- **In numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range (principle value).**
- **Thus, the phase functions of some sequences exhibit discontinuities of 2π radians in the plot.**

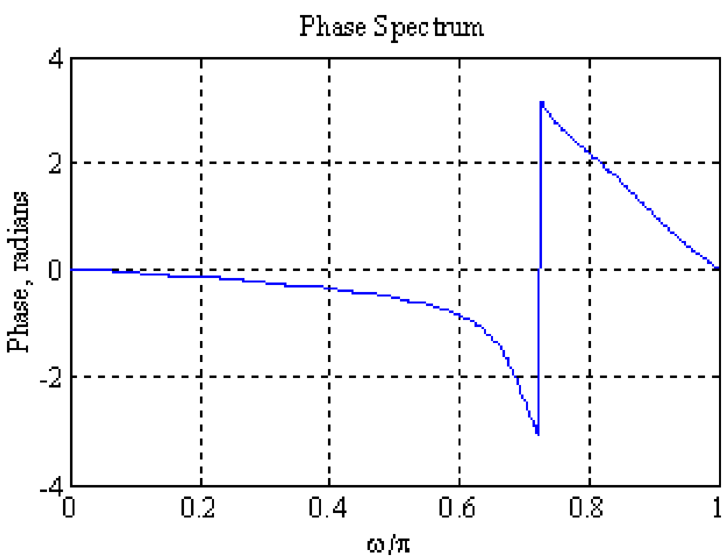
- **For example, there is a discontinuity of 2π at ω = 0.72 in the phase response below**

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

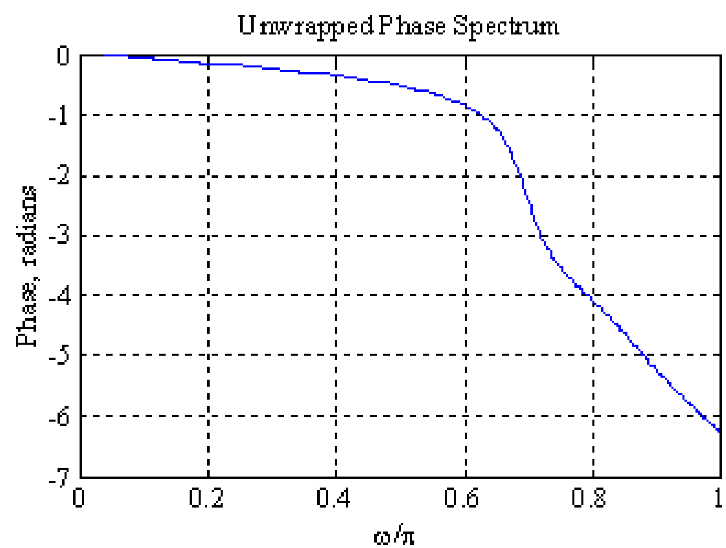


- **In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π.**
- **Process of discontinuity removal is called unwrapping the phase.**
- **The unwrapped phase function will be denoted as $\theta_c(\omega)$.**

- **Note: The phase spectrum displays a discontinuity of 2π at $\omega = 0.72$**
- **This discontinuity can be removed using the function *unwrap* as indicated below.**



Wrapped Phase



Unwrapped Phase

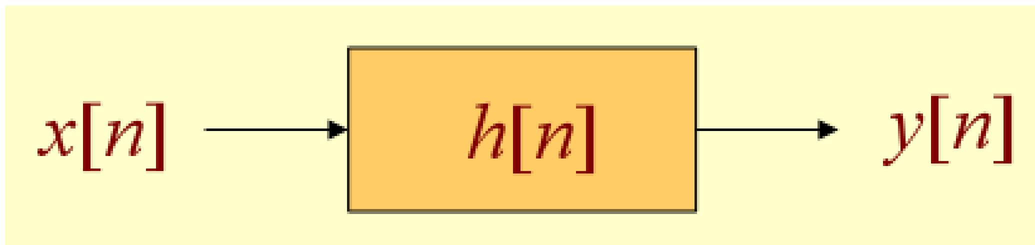
3.8 Frequency Response of An LTI Discrete-Time system

3.8.1 Definition (3.8.1—3.8.3)

- **Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies.**

- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the **superposition** property.
- An important property of an LTI system is that for certain types of input signals, called **eigenfunctions**, the output signal is the input signal multiplied by a complex constant.
- We consider here one such eigenfunction as the input.

- **Consider the LTI discrete-time system with an impulse response $\{h[n]\}$ shown below**



Its input-output relationship in the time-domain is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- **If the input is of the form $x[n] = e^{j\omega n}$,**

$$-\infty < n < \infty$$

- **Then it follows that the output is given by**

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

- **Then we can write $y[n] = H(e^{j\omega})e^{j\omega n}$**
- **Thus for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$.**
- **Thus $e^{j\omega n}$ is an eigenfunction of the system.**

- **The quantity $H(e^{j\omega})$ is called the frequency response of the LTI discrete-time system**

- **$H(e^{j\omega})$ provides a frequency-domain description of the system**

- **$H(e^{j\omega})$ is precisely the DTFT of the impulse response $\{h[n]\}$ of the system**

- $H(e^{j\omega})$, in general, is a complex function of ω with a period 2π.
- It can be expressed in terms of its real and imaginary parts

$$H(e^{j\omega}) = H_{\mathrm{re}}(e^{j\omega}) + jH_{\mathrm{im}}(e^{j\omega})$$

or, in terms of its magnitude and phase,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{H(e^{j\omega})\}$$

- **The function $|H(e^{j\omega})|$ is called the magnitude response and the function $\theta(\omega)$ is called the phase response of the LTI discrete-time system.**
- **Design specifications for the LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both.**

- **In some cases, the magnitude function is specified in decibels as**

$$G(\omega) = 20\log_{10}|H(e^{j\omega})| \quad \text{dB}$$

where $G(\omega)$ is called the gain function

- **The negative of the gain function**

$$A(\omega) = -G(\omega)$$

is called the attenuation or loss function

- **Note: Magnitude and phase functions are real functions of ω, whereas the frequency response is a complex function of ω.**
- **If the impulse response $h[n]$ is real then the magnitude function is an even function of ω:**

$$|H(e^{j\omega})| = |H(e^{-j\omega})|$$

and the phase function is an odd function of ω:

$$\theta(\omega) = -\theta(-\omega)$$

- **Likewise, for a real impulse response $h[n]$, $H_{re}(e^{j\omega})$ is even and $H_{im}(e^{j\omega})$ is odd**
- **Example: Consider the M-point moving average filter with an impulse response given by**

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

Its frequency response is then given by

$$H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n}$$

$$H(e^{j\omega}) = \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=M}^{\infty} e^{-j\omega n}\right)$$

$$= \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n}\right)(1 - e^{-jM\omega}) = \frac{1}{M} \times \frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$

$$= \frac{1}{M} \times \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-j(M-1)\omega/2}$$

- **Thus, the magnitude response of the M-point moving average filter is given by**

$$\left|H(e^{j\omega})\right| = \left|\frac{1}{M}\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

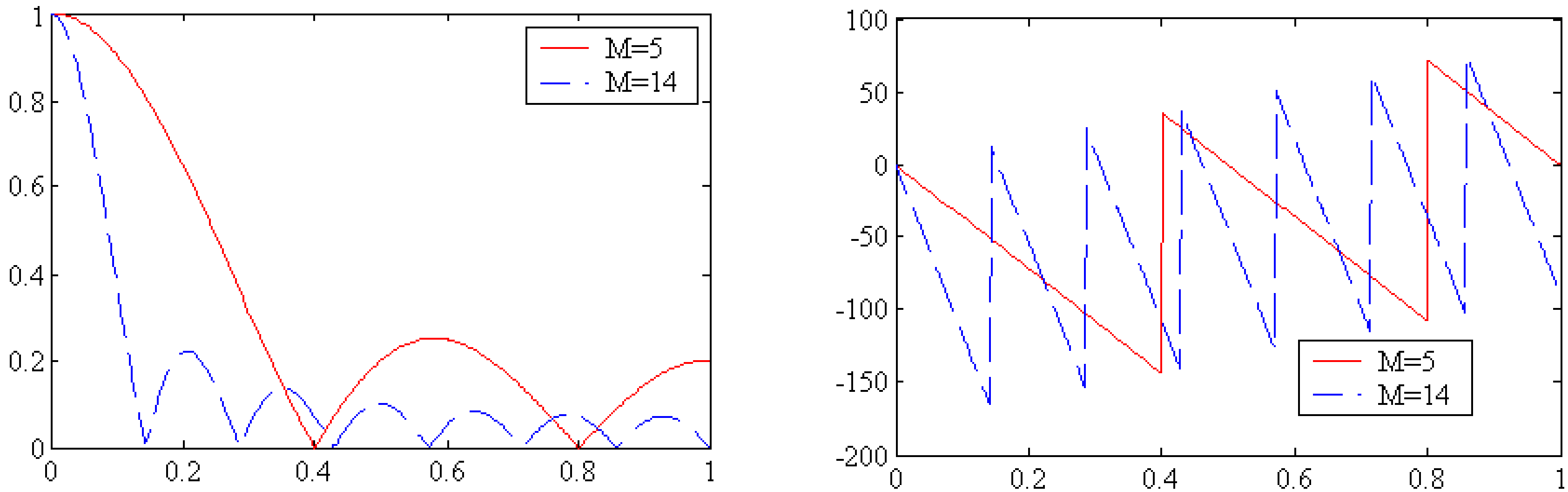
And the phase response is given by

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi\sum_{k=0}^{M/2}\mu(\omega - \frac{2\pi k}{M})$$

3.8.4 Frequency Response Computation Using MATLAB

- The function freqz(h,1,w) can be used to determine the values of the frequency response vector h at a set of given frequency points w
- From h, the real and imaginary parts can be computed using the functions real and imag, and the magnitude and phase functions using the functions abs and angle

- ***Example*** - **Program 3_2.m can be used to generate the magnitude and phase responses of an M-point moving average filter as shown below**



3.8.5 Steady-State and Transient Responses

- **Note that the frequency response also determines the steady-state response of an LTI discrete-time system to a sinusoidal input.**
- ***Example*** **Determine the steady-state output $y[n]$ of a real coefficient LTI discrete-time system with a frequency response $H(e^{j\omega})$ for an input**

$$x[n] = A\cos(\omega_0 n + \varphi), \; -\infty < n < \infty$$

- **We can express the input $x[n]$ as**

$$x[n] = g[n] + g^*[n]$$

where

$$g[n] = 0.5Ae^{j\varphi}e^{j\omega_0 n}$$

Now the output of the system for an input $e^{j\omega_0 n}$ (Eigen function) is simply

$$H(e^{j\omega_0})e^{j\omega_0 n}$$

- **Because of linearity, the response $v[n]$ to an input $g[n]$ is given by**

$$v[n] = 0.5Ae^{j\varphi}H(e^{j\omega_0})e^{j\omega_0 n}$$

Likewise, the output $v^*[n]$ to the input $g^*[n]$ is

$$v^*[n] = 0.5Ae^{-j\varphi}H(e^{-j\omega_0})e^{-j\omega_0 n}$$

- **Combining the last two equations we get**

$$y[n] = v[n] + v^*[n]$$

$$= 0.5Ae^{j\varphi}H(e^{j\omega_0})e^{j\omega_0 n} + 0.5Ae^{-j\varphi}H(e^{-j\omega_0})e^{-j\omega_0 n}$$

$$= 0.5A|H(e^{j\omega_0})|\{e^{j\theta(\omega_0)}e^{j\varphi}e^{j\omega_0 n} + e^{-j\theta(\omega_0)}e^{-j\varphi}e^{-j\omega_0 n}\}$$

$$= A|H(e^{j\omega_0})|\cos[\omega_0 n+\theta(\omega_0)+\varphi]$$

The red terms are from the system:

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$$

The other terms are from the signal:

$$x[n] = A\cos(\omega_0 n + \varphi), -\infty < n < \infty$$

- **Thus, the output $y[n]$ has the same sinusoidal waveform (frequency is the same) as the input with two differences:**

(1) The amplitude is multiplied by $|H(e^{j\omega_0})|$, the value of the magnitude function at $\omega = \omega_0$

(2) The output has a phase lag relative to the input by an amount $\theta(\omega_0)$, the value of the phase function at $\omega = \omega_0$

- **The expression for the steady-state response developed earlier assumes that the system is *initially relaxed* before the application of the input $x[n]$**
- **In practice, excitation $x[n]$ to a discrete-time system is usually a right-sided sequence applied at some sample index $n = n_0$**
- **We develop the expression for the output for such an input**

- **Without any loss of generality, assume $x[n] = 0$ for $n < 0$, and $h[n] = 0$ for $n < 0$**
- **From the input-output relation**

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{0} h[k]x[n-k] + \sum_{k=0}^{n} h[k]x[n-k] + \sum_{k=n+1}^{\infty} h[k]x[n-k]$$

$$= 0 + \sum_{k=0}^{n} h[k]x[n-k] + 0 = \sum_{k=0}^{n} h[k]x[n-k]$$

We observe that for an input $x[n] = e^{j\omega n}\mu[n]$

$$y[n] = \sum_{k=0}^{n} h[k]e^{j\omega(n-k)}\underbrace{\mu[n-k]}_{=1} = \sum_{k=0}^{n} h[k]e^{j\omega(n-k)}$$

- The output for $n < 0$ is $y[n] = 0$
- The output for $n \geq 0$ is given by

$$y[n] = \left(\sum_{k=0}^{n} h[k] e^{-j\omega k}\right) e^{j\omega n}$$

$$= \left(\sum_{k=0}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n}$$

$$y[n] = \underbrace{H(e^{j\omega}) e^{j\omega n}}_{steady-state\ response} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k}\right) e^{j\omega n}}_{Transient\ response}$$

The first term on the RHS is the same as that obtained when the input is applied at $n = 0$ to an *initially relaxed* system and is the steady-state response:

$$y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$$

- The second term on the RHS is called the transient response:

$$y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

To determine the effect of the above term on the total output response, we observe

$$\left|y_{tr}[n]\right| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega(k-n)}\right| \le \sum_{k=n+1}^{\infty} \left|h[k]e^{-j\omega(k-n)}\right| = \sum_{k=n+1}^{\infty} \left|h[k]\right| \le \sum_{k=0}^{n} \left|h[k]\right| + \sum_{k=n+1}^{\infty} \left|h[k]\right| = \sum_{k=0}^{\infty} \left|h[k]\right|$$

For a causal, stable LTI IIR discrete-time system, $h[n]$ is absolutely summable, as a result, the transient response $y_{rt}[n]$ is a bounded sequence

- **Moreover, as $n \to \infty$,**

$$\sum_{k=n+1}^{\infty} |h[k]| \to 0$$

And hence, the transient response **decays to zero** **as n gets very large**

3.8.7 The Concept of Filtering

- **One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components**
- **Such systems are called digital filters and one of the main subjects of discussion in this course**

- **The key to the filtering process is**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- **It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences**

- **Thus, by appropriately choosing the values of the magnitude function $|H(e^{j\omega})|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others**

- **To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function**

$$\left|H(e^{j\omega})\right| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- **We apply an input**

$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n,$$

$$0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system

- **Because of linearity, the output of this system is of the form**

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right) + B\left|H(e^{j\omega_2})\right|\cos\left(\omega_2 n + \theta(\omega_2)\right)$$

- **As** $\left|H(e^{j\omega_1})\right| = 1, \quad \left|H(e^{j\omega_2})\right| = 0$

the output reduces to

$$y[n] = A\left|H(e^{j\omega_1})\right| \cos\left(\omega_1 n + \theta(\omega_1)\right)$$

• Thus, the system acts like a lowpass filter

• In the following example, we consider the design of a very simple digital filter

Example **- The input consists of a sum of two sinusoidal sequences of angular frequencies ω_1=0.1 rad/sample and ω_2=0.4 rad/sample**

We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component

- **For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:** $h[0] = h[2] = \alpha, h[1] = \beta$

- **The convolution sum description of this filter is then given by**

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

- $y[n]$ **and** $x[n]$ **are, respectively, the output and the input sequences**
- **Design Objective: Choose suitable values of** α **and** β **so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample**

- **Now, the frequency response of the FIR filter is given by**

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$

$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$

$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega}$$

$$= (2\alpha\cos\omega + \beta)e^{-j\omega}$$

$$\boxed{\mathrm{DTFT}: H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

- **The magnitude and phase functions are**

$$|H(e^{j\omega})| = 2\alpha\cos\omega + \beta$$

$$\theta(\omega) = -\omega$$

- **In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero.**
- **Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one.**

Thus, the two conditions that must be satisfied are

$$|H(e^{j0.1})| = 2\alpha\cos(0.1)+\beta = 0$$

$$|H(e^{j0.4})| = 2\alpha\cos(0.4)+\beta = 1$$

- **Solving the above two equations we get**

$$\alpha = -6.76$$

$$\beta = 13.46$$

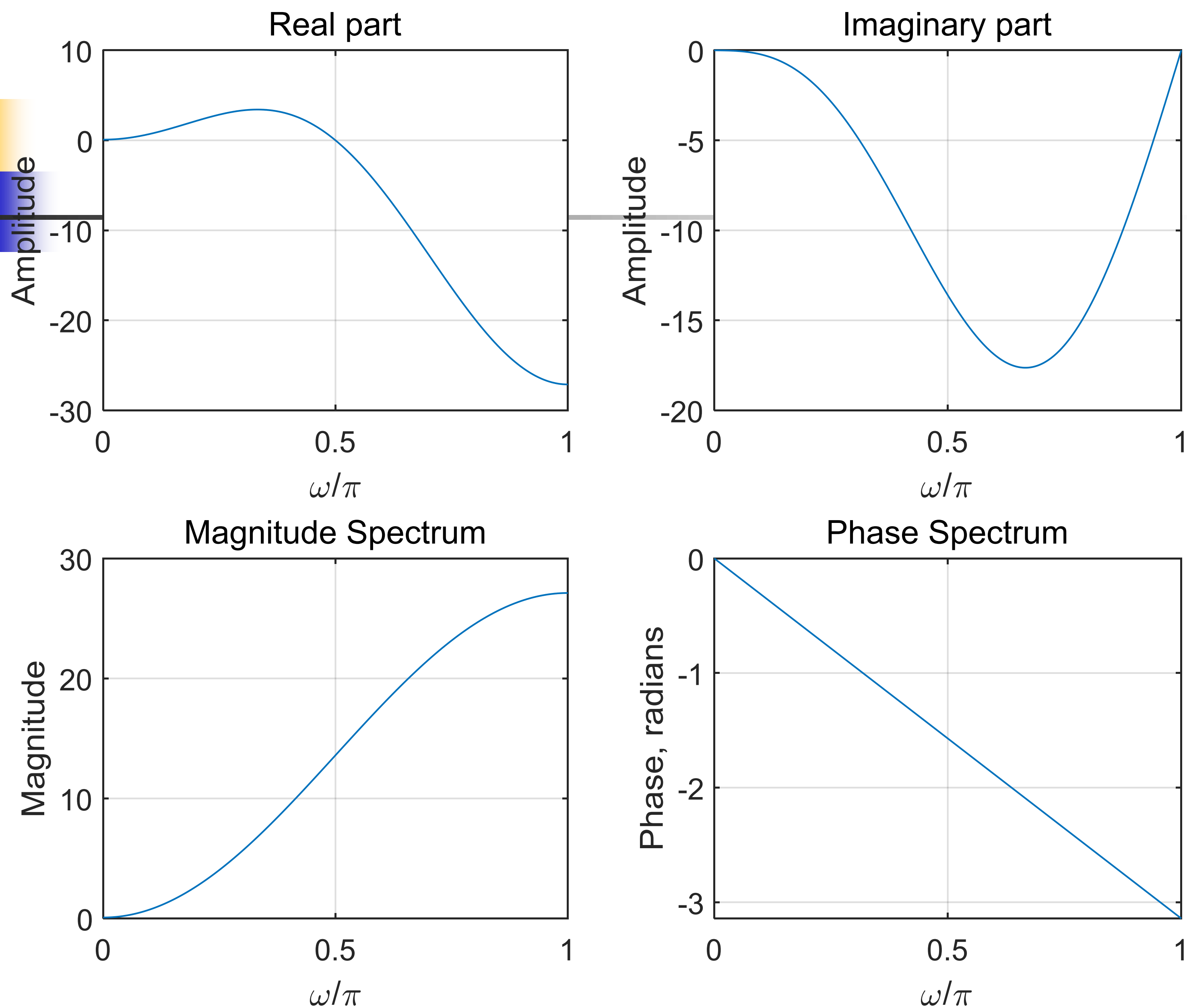
- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76(x[n]+x[n-2])+13.46x[n-1]$$

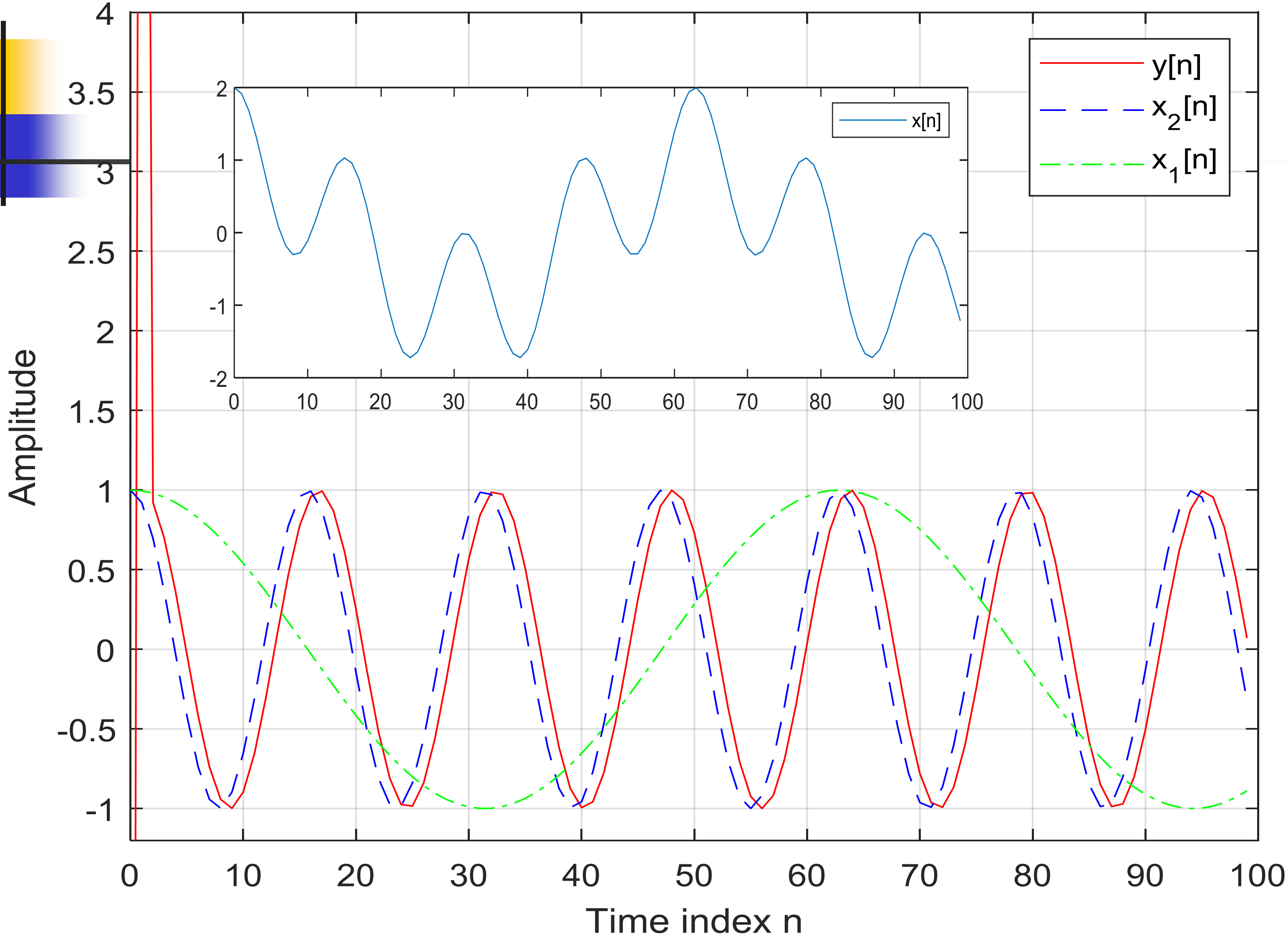
where the input is

$$x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]$$

- Program 3_3.m can be used to verify the filtering action of the above system
- Figure below shows the plots generated by running this program



Real part
Imaginary part
Magnitude Spectrum
Phase Spectrum
Amplitude
Magnitude
Phase, radians
ω/π



y[n]
$x_2[n]$
$x_1[n]$
x[n]
Amplitude
Time index n

- **The first seven samples of the output are shown below**

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
		Transient Part		
0	1.0	1.0	2.0	−13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	−0.0291995	0.8918614	0.3623572
5	0.8775825	−0.4161468	0.4614357	−0.0292002
6	0.8253356	−0.7373937	0.0879419	−0.4161467

Steady-state

- **From this table, it can be seen that, neglecting the least significant digit,**

$$y[n] = \cos(0.4(n-1)) \qquad \text{for } n \geq 2$$

- **Computation of the present value of the output requires the knowledge of the present and two previous input samples**
- **Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$**

- **Therefore, first two output samples constitute the transient part of the output**
- **Since the impulse response is of length 3, the steady-state is reached at $n=N=2$**
- **Note also that the output is delayed version of the high-frequency component $\cos(0.4n)$ of the input, and the delay is one sample period**

3.9 Phase and Group Delays

3.9.1 Definition

- **If the input $x[n]$ to an LTI system $H(e^{j\omega})$ is a sinusoidal signal of frequency ω_0:**

$$x[n] = A\cos(\omega_0 n+\varphi), \quad -\infty < n < \infty$$

then, the output $y[n]$ is also a sinusoidal signal of the same frequency ω_0 but lagging in phase $\theta(\omega_0)$ by radians:

$$y[n] = A|H(e^{j\omega})|\cos[\omega_0 n+\theta(\omega_0)+\varphi],$$

$$-\infty < n < \infty$$

- **We can rewrite the output expression as**

$$y[n] = A|H(e^{j\omega})|\cos\{\omega_0[n-\tau_p(\omega_0)]+\varphi\},$$

Where $\tau_p(\omega_0) = -\theta(\omega_0)/\omega_0$

is called the phase delay.

The minus sign in front indicates phase lag.

Thus, the output $y[n]$ **is a time-delayed version of the input** $x[n]$**.**

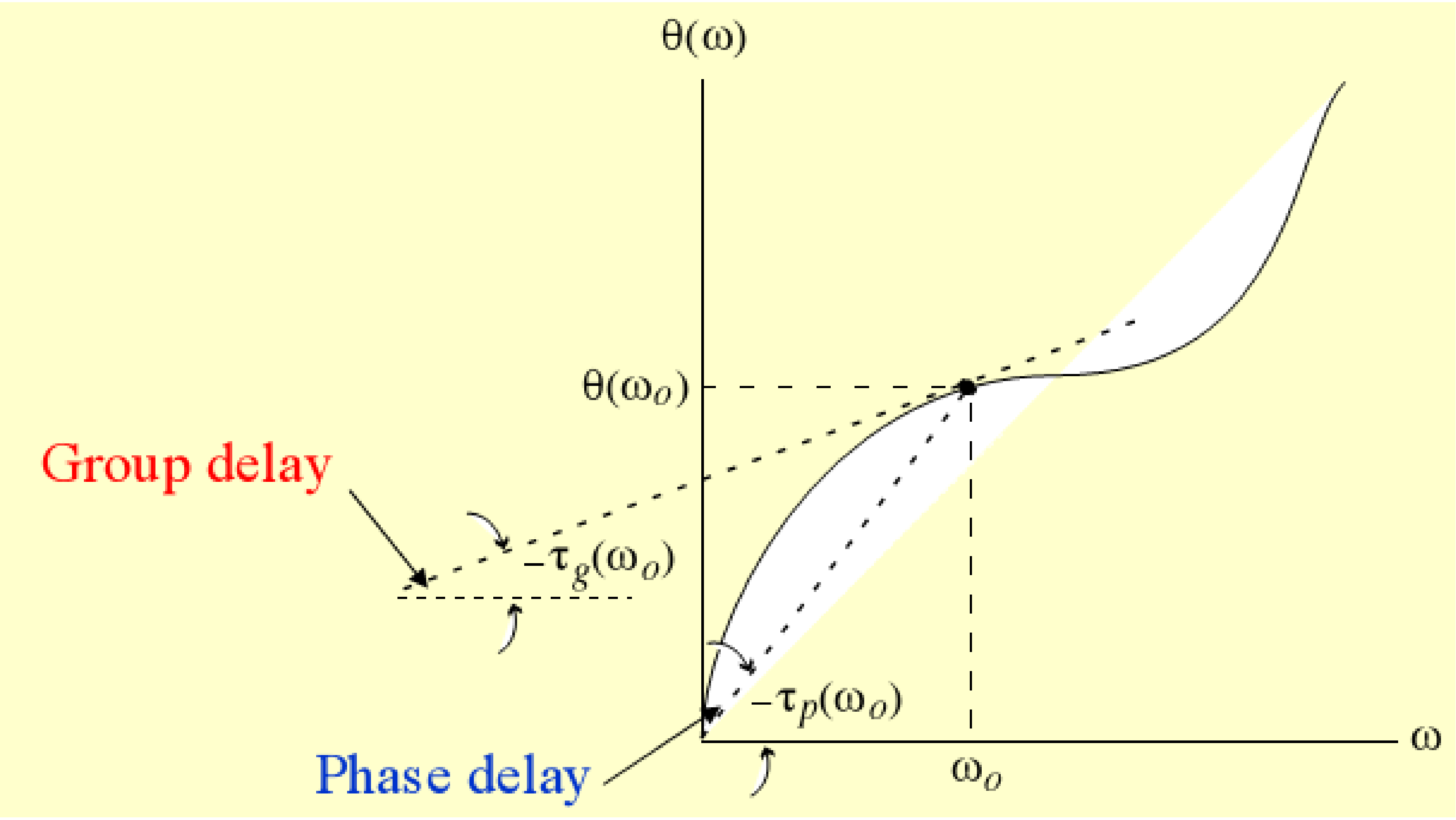
- **In general, $y[n]$ will not be delayed replica of $x[n]$ unless the phase delay $\tau_p(\omega_0)$ is an integer.**
- **When the input is composed of many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays.**

- **In this case, the signal delay is determined using the <u>group delay</u> defined by**

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} \qquad \textbf{(3.117)}$$

- **In defining the group delay, it is assumed that the phase function is unwrapped so that its derivatives exist.**

- **A graphical comparison of the two types of delays are indicated below**



- ***Example***: **The phase function of the FIR filter** $y[n] = \alpha x[n]+\beta x[n-1]+\alpha x[n-2]$ **is** $\theta(\omega) = -\omega$
- **Hence its group delay is given by** $\tau_g(\omega) =$ 1 **verifying the result obtained earlier by simulation.**

$$\boxed{\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}}$$

- ***Example***: **for the M-point moving average filter**

$$h[n] = \begin{cases} \dfrac{1}{M}, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

The phase function is

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=0}^{M/2} \mu(\omega - \frac{2\pi k}{M})$$

Hence its group delay is $\tau_g(\omega) = \dfrac{M-1}{2}$

- **Physical significance of the two delays are better understood by examining the continuous-time case**
- **Consider an LTI continuous-time system with a frequency response**

$$H_a(j\Omega) = \left|H_a(j\Omega)\right| e^{j\theta_a(\Omega)}$$

And excited by a narrow-band amplitude modulated continuous-time signal

$$x_a(t) = a(t)\cos(\Omega_c t)$$

- $a(t)$ **is a lowpass modulating signal with a band-limited continuous-time Fourier transform given by**

$$|A(j\Omega)| = 0, \;\; |\Omega| > \Omega_0$$

and $\cos(\Omega_c t)$ **is the high-frequency carrier signal.**

- **Assume that in the frequency range $\Omega_c - \Omega_0 < |\Omega| < \Omega_c + \Omega_0$ the frequency response of the continuous-time system has a constant magnitude and a linear phase:**

$$|H_a(j\Omega)| = |H_a(j\Omega_c)|$$

$$\theta_a(\Omega) = \theta_a(\Omega_c) - (\Omega - \Omega_c)\frac{d\theta_a(\Omega)}{d\Omega}\Big|_{\Omega=\Omega_c}$$

$$= -\Omega_c \tau_p(\Omega_c) + (\Omega - \Omega_c)\tau_g(\Omega_c)$$

$$f(t)\cos\omega_0 t \longleftrightarrow 1/2\,\{F[j(\omega+\omega_0)]+F[j(\omega-\omega_0)]\}$$

- **Now, the CTFT of $x_a(t)$ is given by**

$$X_a(j\Omega)=\frac{1}{2}\left\{A\left[j(\Omega+\Omega_c)\right]+A\left[j(\Omega-\Omega_c)\right]\right\}$$

•Also, because of the band-limiting constraint $X_a(j\Omega)=0$ outside the frequency range

$$\Omega_c-\Omega_0<|\Omega|<\Omega_c+\Omega_0$$

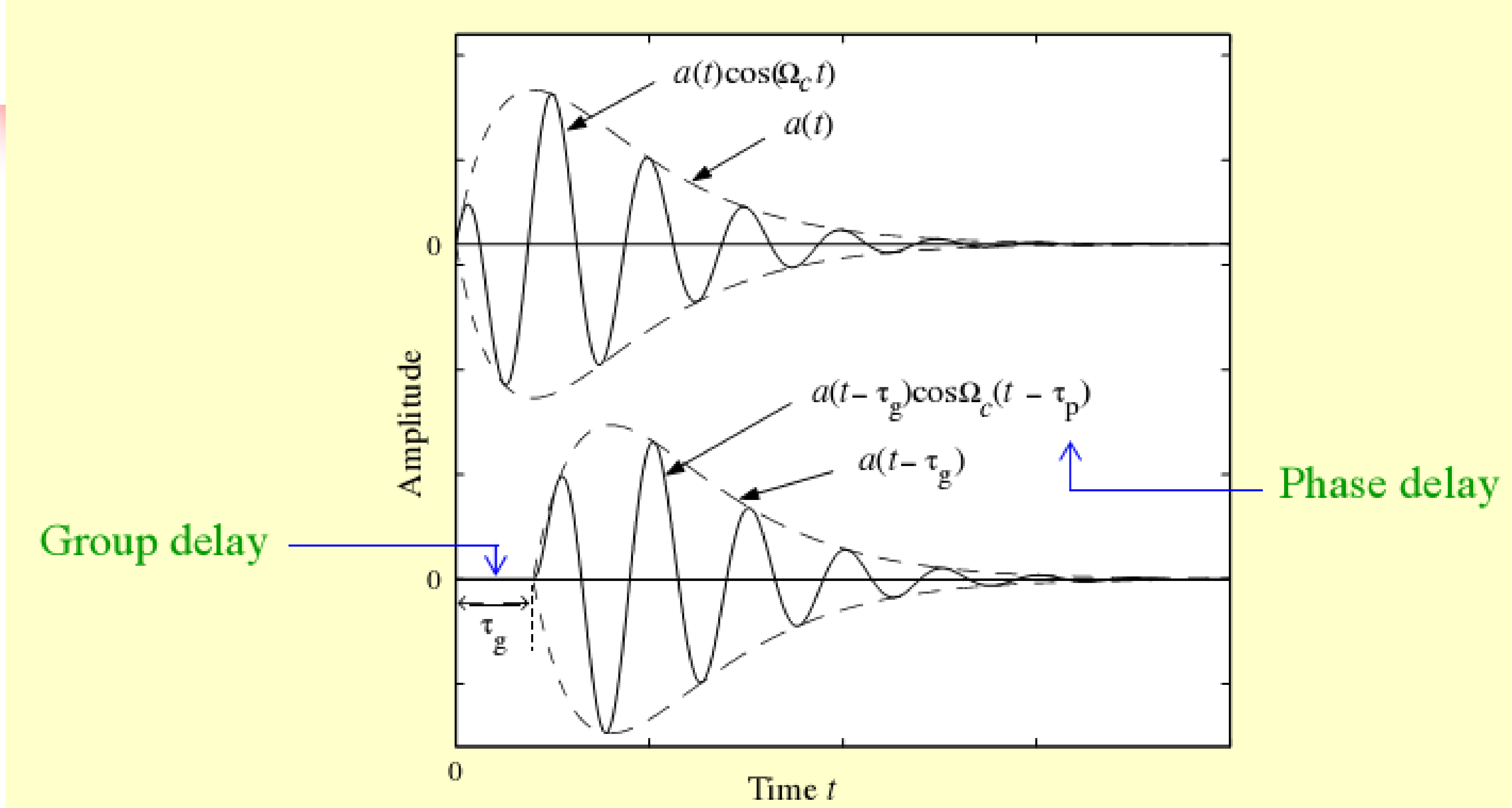
•As a result, the output response $y_a(t)$ of the LTI continuous-time system is given by

$$y_a(t)=a[t-\tau_g(\Omega_c)]\cos\Omega_c[t-\tau_p(\Omega_c)],$$

assuming $|H_a(j\Omega_c)|=1$

- As can be seen from the above equation, **the group delay $\tau_g(\Omega_c)$ is precisely the delay of the envelope $a(t)$ of the input signal $x_a(t)$**, **whereas, the phase delay $\tau_p(\Omega_c)$ is the delay of the carrier**.

- The figure below illustrates the effects of the two delays on an amplitude modulated sinusoidal signal.

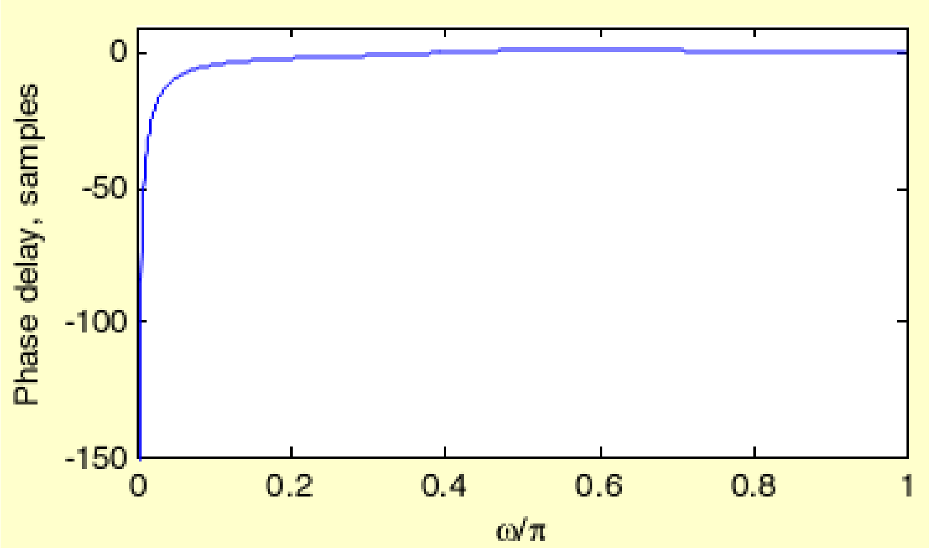

$a(t)\cos(\Omega_c t)$
$a(t)$
$a(t-\tau_g)\cos\Omega_c(t-\tau_p)$
$a(t-\tau_g)$
Phase delay
Group delay
τ_g
0
0
0
Amplitude
Time t

- **The waveform of the underlying continuous-time output shows distortion when the group delay is not constant over the bandwidth of the modulated signal**
- **If the distortion is unacceptable, an all-pass delay equalizer is usually cascaded with the LTI system so that the overall group delay is approximately linear over the frequency range of interest while keeping the magnitude response of the original LTI system unchanged.**

3.9.2 Phase and Group Delay Computation using MATLAB

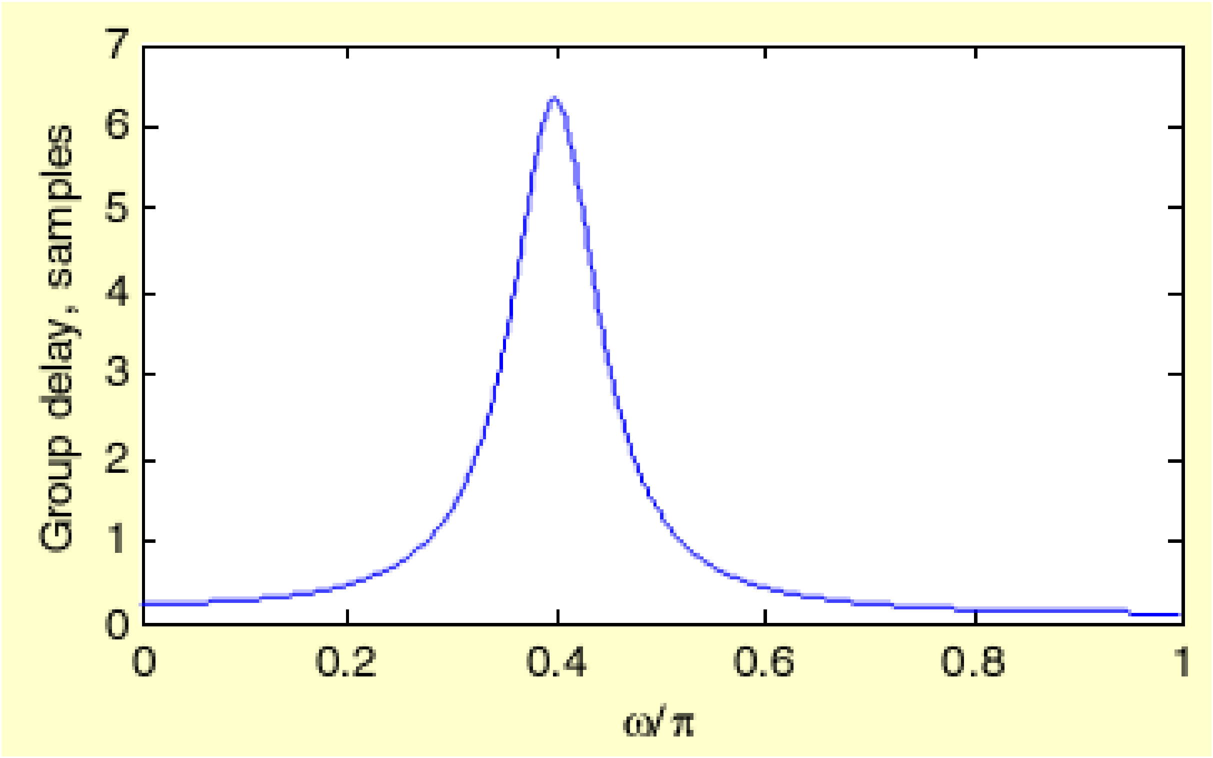
- **Phase delay can be computed using the function [phi,w]= phasedelay(num,den,1024)**
- **Figure below shows the phase delay of the DTFT**

$$H(e^{j\omega}) = \frac{0.1367(1 - e^{-j2\omega})}{1 - 0.5335e^{-j\omega} + 0.7265e^{-j2\omega}}$$



- **Group delay can be computed using the function [gd,w] = grpdelay(num,den,1024)**
- **Figure below shows the group delay of the DTFT**

$$H(e^{j\omega}) = \frac{0.1367(1-e^{-j2\omega})}{1-0.5335e^{-j\omega}+0.7265e^{-j2\omega}}$$



Thanks!

Any questions?