Digital Signal Processing

IIR Filter Design

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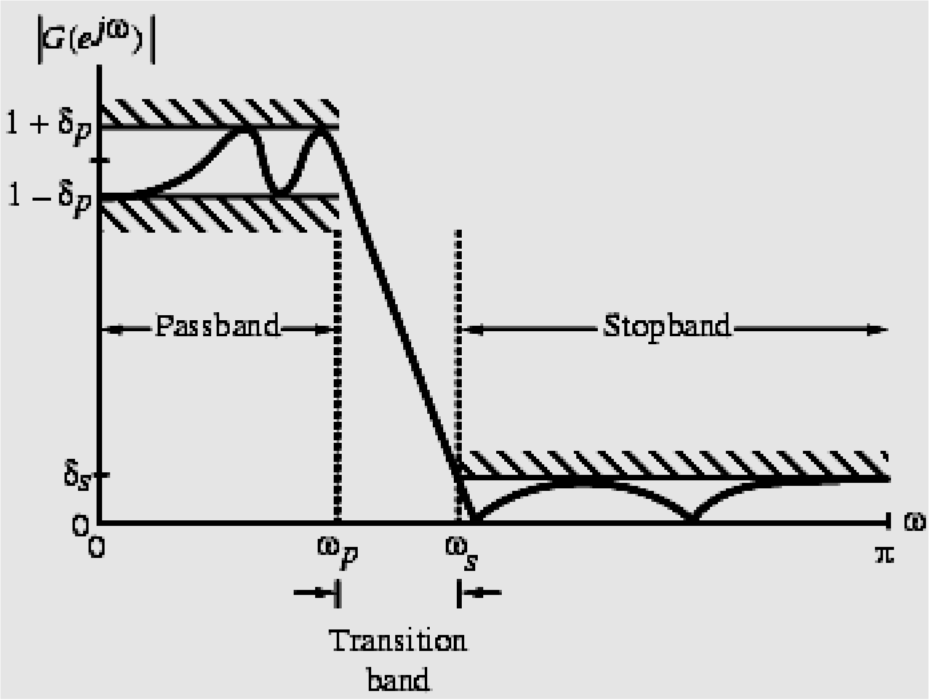
Filter Specification

Example: LPF

$$|G(e^{j\omega})| \le \delta_s \qquad \text{for } \omega_s \le |\omega| \le \pi$$

$$1-\delta_P \le |G(e^{j\omega})| \le 1+\delta_P \qquad \text{for } |\omega| \le \omega_P$$



$$\textit{Passband}: \omega_P$$
$$\textit{Stopband}: \omega_S$$

Filter Selection

The *objective* of filter design is to design casual $H(z)$ that meets the specifications

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

IIR or FIR ?

IIR:

low order to reduce computational complexity, but may not stable.

FIR:

with linear phase ,but high order

$$H[n] = \pm h[N-n]$$

$$H[z] = \sum_{k=0}^{N} h[k] z^{-k}$$

IIR Design

IIR design

The most common practice is to transform $H_a(s)$ into the desired digital transfer function $G(z)$.

convert the digital filter specifications to into analog lowpass prototype filter specifications

↓

determine the analog lowpass filter transfer function meeting these specifications

↓

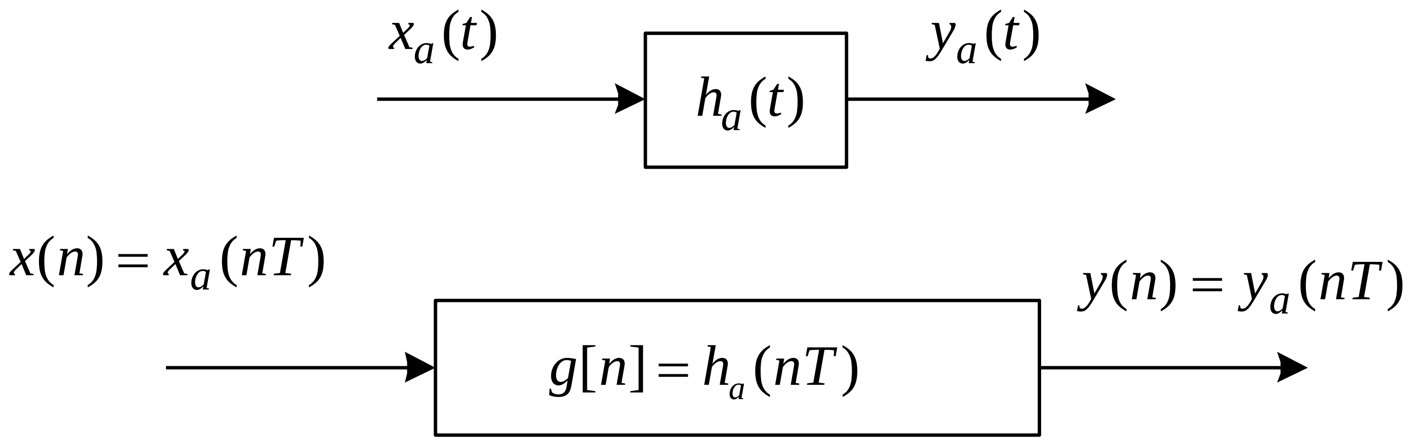
transform the analog filter into the desired digital transfer function

IIR Design

- Apply a mapping from *s*-domain to the *z*-domain so that the essential properties of the analog frequency response are preserved.
- The mapping function should:

(1) The imaginary axis in the *s*-plane be mapped onto the unit circle of the *z*-plane.
(2) A stable analog transfer function be transferred into a stable digital transfer function.

Impulse Invariance Method

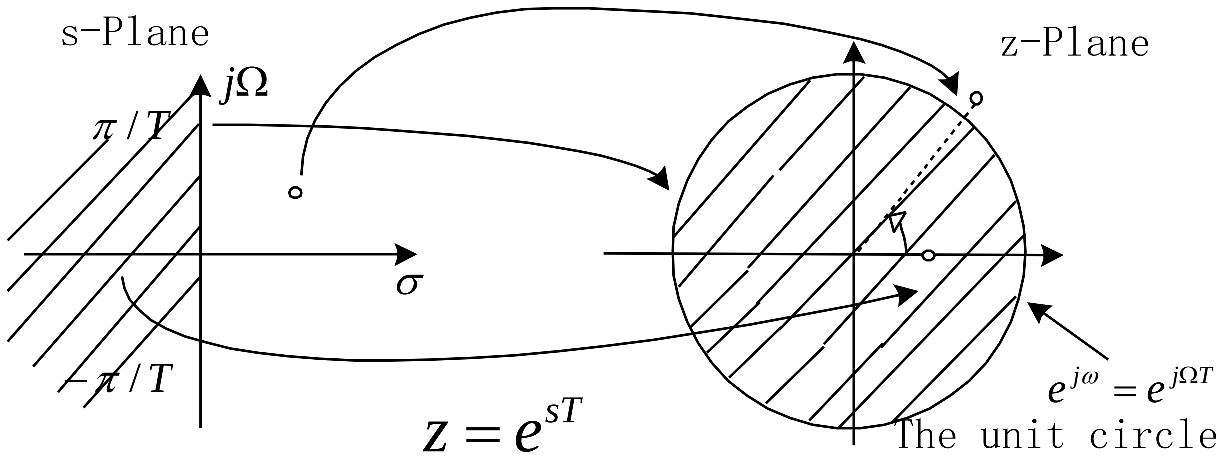


$$G(z)\big|_{z=e^{sT}} = \frac{1}{T}\sum_{m=-\infty}^{\infty} H_a\left(s - jm\frac{2\pi}{T}\right)$$

The mapping function $z = e^{sT}$

IIR Design

Impulse Invariance Method



$$z = e^{sT}$$

- jΩ-axis mapped onto the unit circle, stable region in s-plane mapped into the stable region in z-plane).
- jΩ-axis in the s-plane and the unit circle in $\omega = \Omega T$
- Nyquist

$$G(e^{j\omega}) = \frac{1}{T} H_a(j\Omega) \quad \text{for } |\omega| < \pi$$

Design Steps

$H_a(s)$ is partial-fractional expressed as: ($h_a(t)$ is derived)

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_i} \qquad \mathrm{Re}(s_i) < 0$$

(2) From $g[n] = h_a(nT)$

$$G_1(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{s_i T} z^{-1}} \qquad G(z) = \sum_{i=1}^{N} \frac{T A_i}{1 - e^{s_i T} z^{-1}}$$

IIR Design

Proof

$$h_a(t) = \mathcal{L}^{-1}\{H_a(s)\} = \sum_{i=1}^{N} A_i e^{s_i t} u(t)$$

$$g[n] = h_a(nT) = \sum_{i=1}^{N} A_i e^{s_i nT} u[n]$$

$$G_1(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{s_i T} z^{-1}}$$

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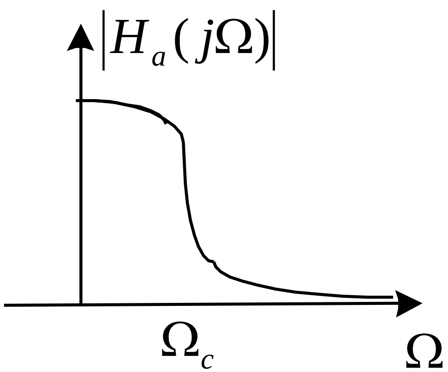
Example

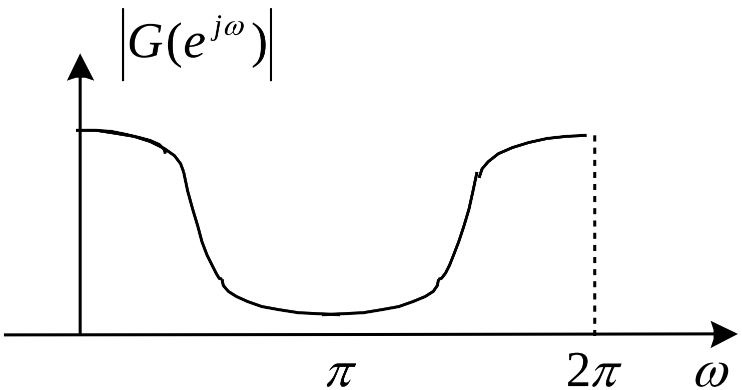
$$H_a(s)=\frac{2}{s^2+3s+2} \Rightarrow H_a(s)=\frac{2}{s+1}-\frac{2}{s+2}$$

$$G(z)=T\frac{2}{1-e^{-T}z^{-1}}-T\frac{2}{1-e^{-2T}z^{-1}}=\frac{2T(e^{-T}-e^{-2T})z^{-1}}{1-(e^{-2T}+e^{-T})z^{-1}+e^{-3T}z^{-2}}$$

When $T = 1$

$$G(z)=\frac{0.4651z^{-1}}{1-0.5032z^{-1}+0.04979z^{-2}}$$





Bilinear Translation Method

map from s-plane to z-plane

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

For $s = \sigma_0 + j\Omega_0$

$$z = \frac{1+s}{1-s} = \frac{1+\sigma_0 + j\Omega_0}{1-\sigma_0 + j\Omega_0}$$

$$|z|^2 = \frac{(1+\sigma_0)^2 + (\Omega_0)^2}{(1-\sigma_0)^2 + (\Omega_0)^2}$$

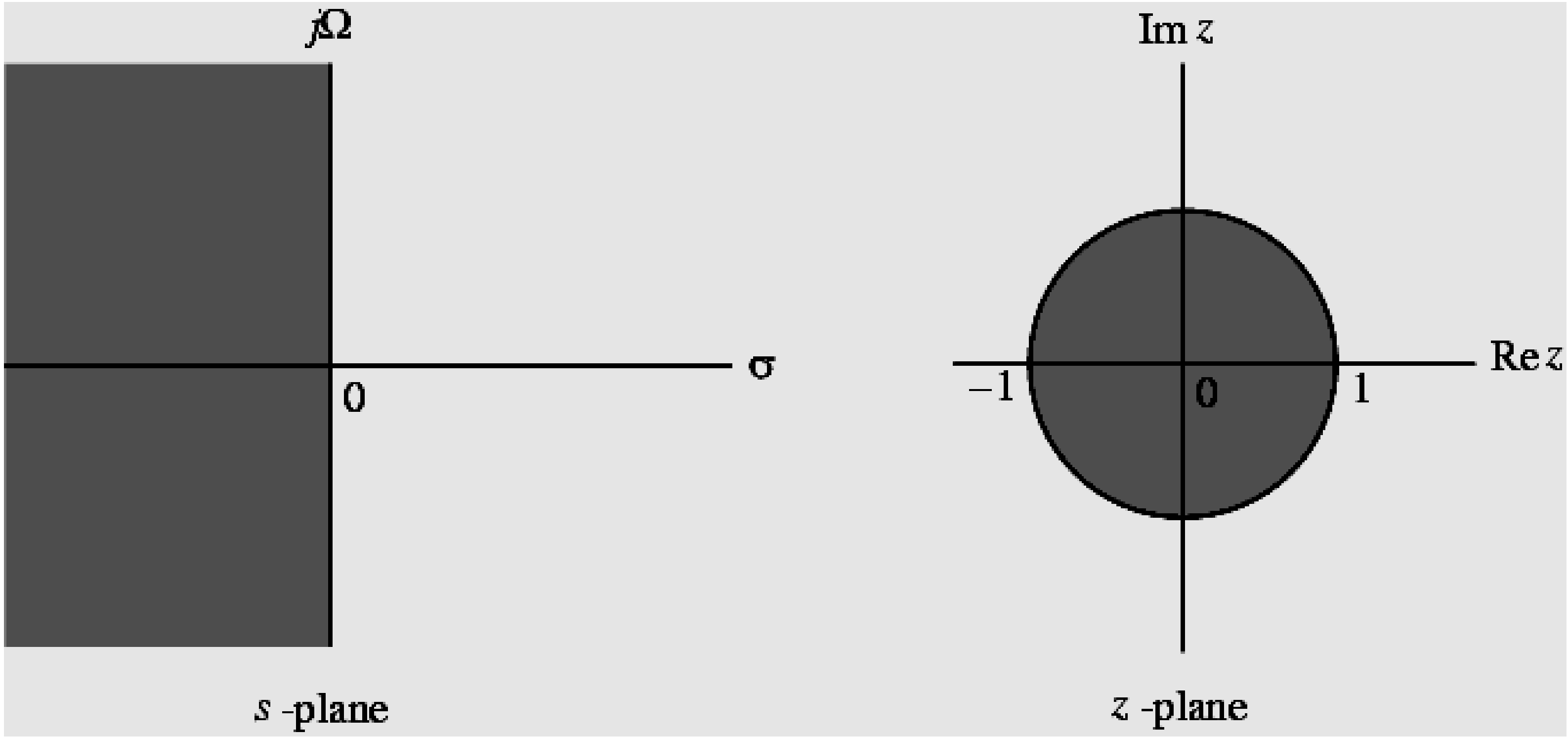
$\sigma_0 = 0$ $\quad |z| = 1$

$\sigma_0 < 0$ $\quad |z| < 1$

$\sigma_0 > 0$ $\quad |z| > 1$

IIR Design

Bilinear Translation Method

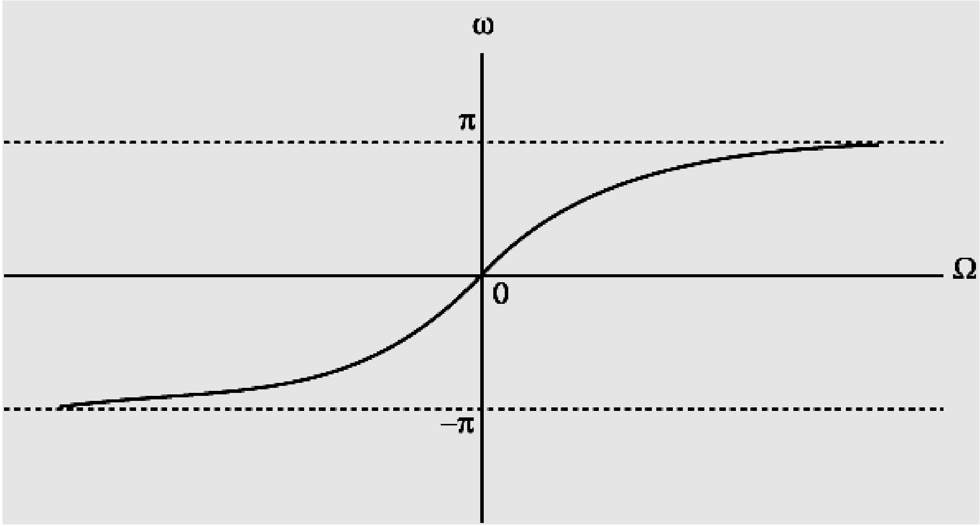


Bilinear Translation Method

- For $z = e^{j\omega}$ with T = 2 we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)} = j\tan(\omega/2)$$

$$\Omega = \tan(\omega/2)$$

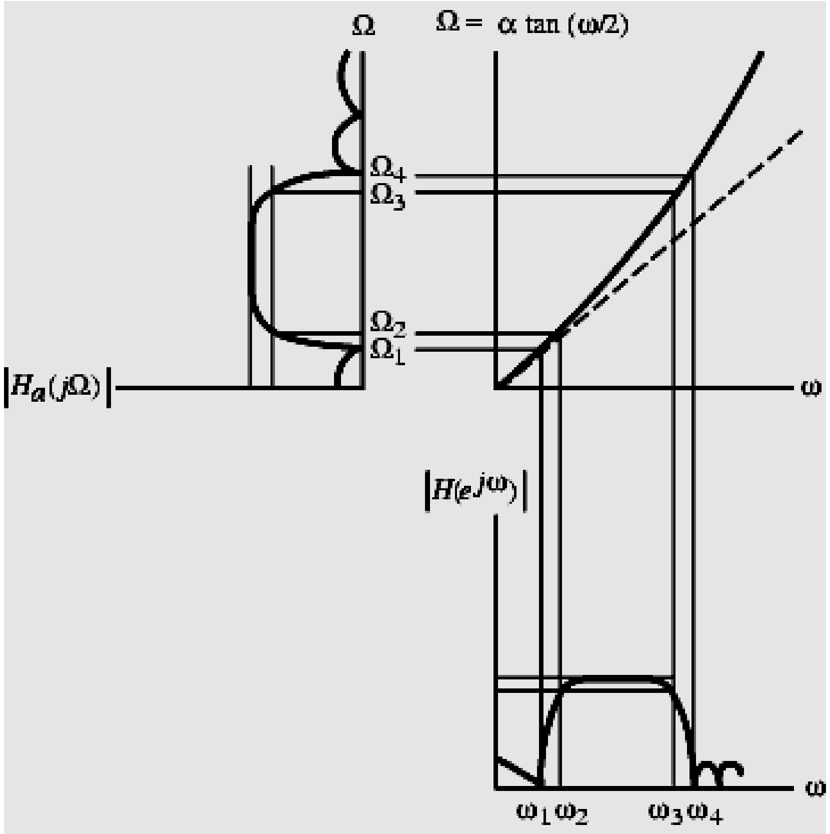


frequency warping

IIR Design

Bilinear Translation Method

Thus, to develop a digital filter meeting a specified magnitude, we must ***first prewarp*** the critical band edge frequencies(ω_P and ω_S) to find their analog Equivalents (Ω_S and Ω_P) using Eq.(9.18).



IIR Design

Bilinear Translation Method

- Pre-warp (ω_p, ω_s) to find their analog equivalents (Ω_p, Ω_s)
- Design the analog filter $H_a(s)$
- Design the digital filter $G(z)$ by applying bilinear transformation to $H_a(s)$
- Transformation does not preserve phase response of analog filter

IIR Design

Example

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

$$G(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)} = \frac{\Omega_c (1 - z^{-1})}{(1 - z^{-1}) + \Omega_c (1 + z^{-1})}$$

$$= \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \qquad \alpha = \frac{1 - \Omega_c}{1 + \Omega_c} = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}$$

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Example

$$H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + Bs + \Omega_0^2}$$

$|H_a(j\Omega_0)| = 0$ $|H_a(j0)| = |H_a(j\infty)| = 1$

- Ω_0 is called the notch frequency
- If $|H_a(j\Omega_2)| = |H_a(j\Omega_1)| = 1/\sqrt{2}$ $B = \Omega_2 - \Omega_1$ is the 3-dB notch bandwidth

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{(1+\Omega_0^2) - 2(1-\Omega_0^2)z^{-1} + (1+\Omega_0^2)z^{-2}}{(1+\Omega_0^2+B) - 2(1-\Omega_0^2)z^{-1} + (1+\Omega_0^2-B)z^{-2}}$$

$$= \frac{1+\alpha}{2}\,\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

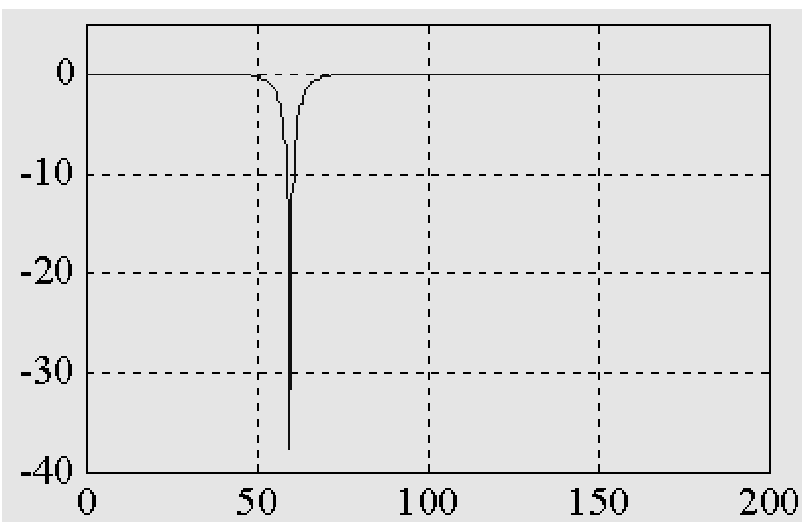
$$\alpha = \frac{1+\Omega_0^2-B}{1+\Omega_0^2+B} = \frac{1-\tan(B_w/2)}{1+\tan(B_w/2)} \qquad \beta = \frac{1-\Omega_0^2}{1+\Omega_0^2} = \cos(\omega_0)$$

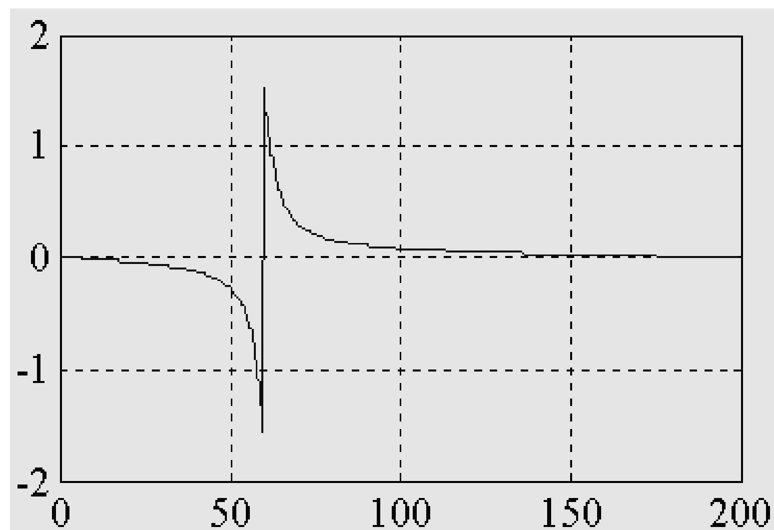
IIR Design

Example

$$H_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + Bs + \Omega_o^2}$$

$$G(z) = \frac{0.954965 - 1.1226287z^{-1} + 0.954965z^{-2}}{1 - 1.1226287z^{-1} + 0.90993z^{-2}}$$





Case Study

Design $G(z)$ with a maximally flat magnitude, and a pass ripple not exceeding 0.5dB, and the minimum stopband attenuation 15dB.

$$20\log\left|G\left(e^{j\omega_p}\right)\right| > -0.5dB \qquad 20\log\left|G\left(e^{j\omega_s}\right)\right| < -15dB$$

$$\omega_P = 0.25\pi, \omega_S = 0.55\pi$$

IIR Design

(1) Prewarpping

$$\Omega_P = \tan\left(\frac{\omega_P}{2}\right) = \tan\left(\frac{0.25\pi}{2}\right) = 0.4142136$$

$$\Omega_S = \tan\left(\frac{\omega_S}{2}\right) = \tan\left(\frac{0.55\pi}{2}\right) = 1.1708496$$

(2) Design the parent analog filter $H_a(s)$

From the specifications we obtain

pass ripple 0.5dB $\Rightarrow$

$$20\log\left(1/\sqrt{1+\varepsilon^2}\right) = -0.5$$

$$\varepsilon = 0.122$$

stopband attenuation 15dB $\Rightarrow$

$$20\log(1/A) = -15$$

$$A^2 = 31.6228$$

IIR Design

$$k = \frac{\Omega_s}{\Omega_p} = \frac{1.1708496}{0.4142136} = 2.8266809$$

$$k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = 0.0631234$$

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(15.841979)}{\log_{10}(2.8266814)} = 2.6586997$$

The least order of Butterworth LPF is $N = 3$.

IIR Design

the 3-dB frequency

$$\Omega_c = (\varepsilon)^{-1/N}\Omega_p = 1.419915\Omega_p = 0.588148$$

third-order normalized lowpass Butterworth transfer function

$$H_{an}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

which has 3-dB frequency at $\Omega_c = 1$.

$$H_a(s) = H_{an}\left(\frac{s}{\Omega_c}\right) = \frac{0.203451}{(s+0.588148)(s^2+0.588148s+0.245918)}$$

Digital filter $G(z)$ by applying bilinear transformation

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

IIR Design—The Impulse Invariance Method

The third-order lowpass Butterworth with 3-dB frequency at Ω_c.

$$H_a(s) = H_{an}\left(\frac{s}{\Omega_c}\right) = \frac{1}{1 + 2\frac{s}{\Omega_c} + 2\left(\frac{s}{\Omega_c}\right)^2 + \left(\frac{s}{\Omega_c}\right)^3}$$

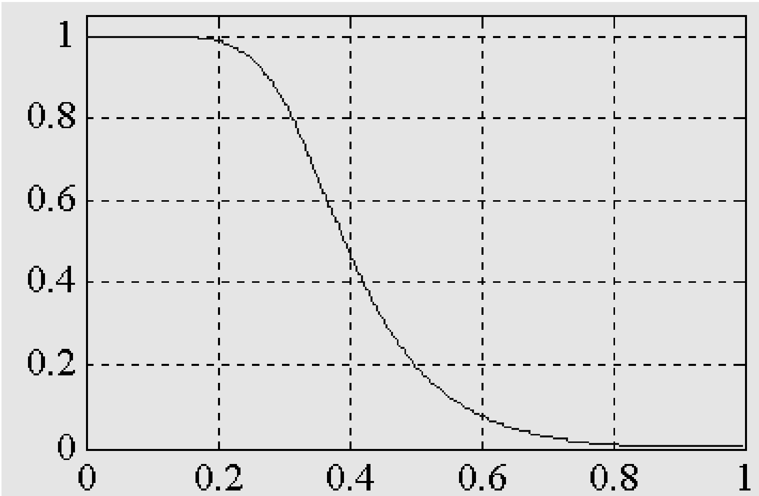
$$H_a(s) = \frac{\Omega_c^3}{(s+\Omega_c)\left(s - \Omega_c e^{j2\pi/3}\right)\left(s - \Omega_c e^{-j2\pi/3}\right)}$$

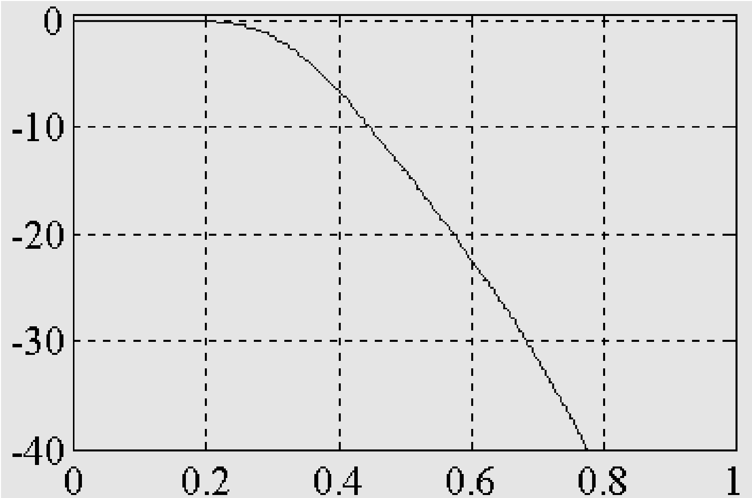
$$= \frac{\Omega_c}{s+\Omega_c} + \frac{-\left(\Omega_c/\sqrt{3}\right)e^{j\pi/6}}{s + \Omega_c\left(1 - j\sqrt{3}\right)/2} + \frac{-\left(\Omega_c/\sqrt{3}\right)e^{-j\pi/6}}{s + \Omega_c\left(1 + j\sqrt{3}\right)/2}$$

digital transfer function

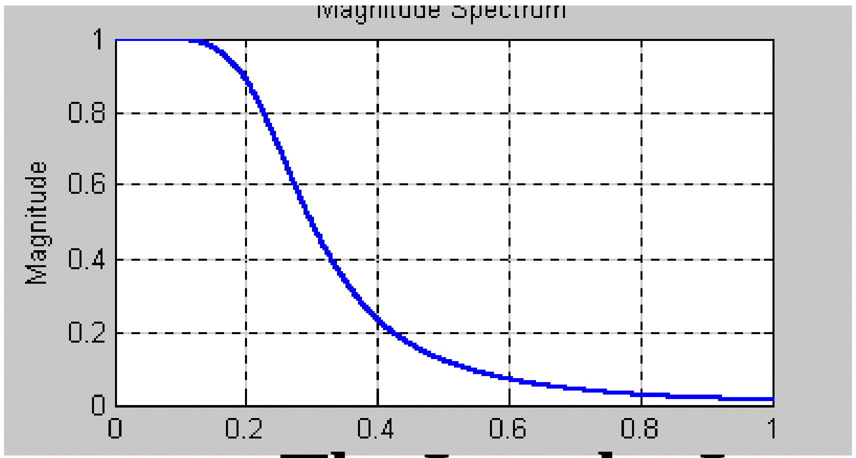
$$G(z) = \frac{\omega_c}{1 - e^{\omega_c} z^{-1}} + \frac{-\left(\omega_c/\sqrt{3}\right)e^{j\pi/6}}{1 + e^{-\omega_c\left(1 - j\sqrt{3}\right)/2} z^{-1}} + \frac{-\left(\omega_c/\sqrt{3}\right)e^{-j\pi/6}}{1 + e^{\omega_c\left(1 + j\sqrt{3}\right)/2} z^{-1}} \quad \omega_c = \pi/4$$

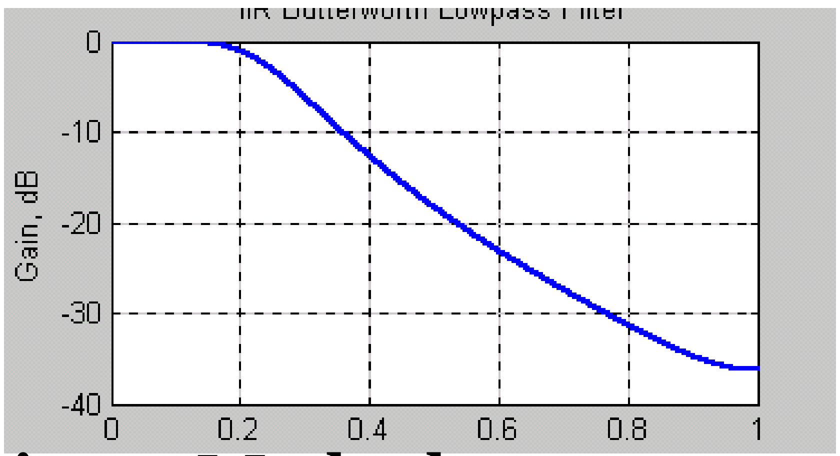
IIR Design





The Bilinear Translation Method





The Impulse Invariance Method

IIR Design with Matlab

Read and exercise by yourself!

<u>Note</u>: LPF design directly in digital form

1. Order Estimation

```
[N,Wn] = buttord(Wp, Ws, Rp, Rs);
```

2. Filter Design

```
[b,a] = butter(N,Wn);
% Getting:G(z)=B(z)/A(z)
```

3. Other type filter

```
[b,a] = cheby1(N,Rp,Wp,'high');
%Wn=[W1, W2]
```

IIR Design with Matlab

Note: Digital LPF design in analog form by Bilinear Transform

1. Order Estimation

```
[N,Wn] = ellipord(Wp,Ws,Rp,Rs,'s');
```

2. Filter Design

```
[bt,at] = ellip(N,Wn);
%Getting: Ha(z) = Bt(z)/At(z)
```

3. Bilinear Transform

```
[num,den]= bilinear(b,a,0.5);
% 0.5 means T=2,Fs=0.5
% Getting:G(z)=B(z)/A(z)
```