# A Summary of Error Propagation

Suppose you measure some quantities $a, b, c, \ldots$ with uncertainties $\delta a, \delta b, \delta c, \ldots$. Now you want to calculate some other quantity $Q$ which depends on $a$ and $b$ and so forth. What is the uncertainty in $Q$? The answer can get a little complicated, but it should be no surprise that the uncertainties $\delta a, \delta b$, etc. "propagate" to the uncertainty of $Q$. Here are some rules which you will occasionally need; all of them assume that the quantities $a, b$, etc. have errors which are *uncorrelated* and *random*. (These rules can all be derived from the Gaussian equation for normally-distributed errors, but you are not expected to be able to derive them, merely to be able to use them.)

## 1 Addition or Subtraction

If $Q$ is some combination of sums and differences, i.e.

$$Q = a + b + \cdots + c - (x + y + \cdots + z), \tag{1}$$

then

$$\boxed{\delta Q = \sqrt{(\delta a)^2 + (\delta b)^2 + \cdots + (\delta c)^2 + (\delta x)^2 + (\delta y)^2 + \cdots + (\delta z)^2}.} \tag{2}$$

In words, this means that the uncertainties add *in quadrature* (that's the fancy math word for the square root of the sum of squares). In particular, if $Q = a + b$ or $a - b$, then

$$\boxed{\delta Q = \sqrt{(\delta a)^2 + (\delta b)^2}.} \tag{3}$$

Example: suppose you measure the height $H$ of a door and get $2.00 \pm 0.03$ m. This means that $H = 2.00$ m and $\delta H = 0.03$ m. The door has a knob which is a height $h = 0.88 \pm 0.04$ m from the bottom of the door. Then the distance from the doorknob to the top of the door is $Q = H - h = 1.12$ m. What is the uncertainty in $Q$? Using equation (3),

$$\begin{aligned}
\delta Q &= \sqrt{(\delta H)^2 + (\delta h)^2} && (4)\\
&= \sqrt{(0.03 \text{ m})^2 + (0.04 \text{ m})^2} && (5)\\
&= \sqrt{0.0009 \text{ m}^2 + 0.0016 \text{ m}^2} && (6)\\
&= \sqrt{0.0025 \text{ m}^2} = 0.05 \text{ m}. && (7)
\end{aligned}$$

So $Q = 1.12 \pm 0.05$ m.

You might reasonably wonder, "Why isn't $\delta Q$ just equal to $\delta a + \delta b$? After all, if I add $a \pm \delta a$ to $b \pm \delta b$, the answer is definitely at least $a + b - (\delta a + \delta b)$ and at most $a + b + (\delta a + \delta b)$, right?" The answer has to do with the probabalistic nature of the uncertainties $\delta a$ and $\delta b$; remember, they represent 68% confidence intervals. So 32% of the time, the true value of $a$ is outside of the range bounded by $a \pm \delta a$; likewise for $b$. But how often is the sum outside of the range bounded by $a + b \pm (\delta a + \delta b)$? A little bit less often than that. In order for it to be higher than $a + b + (\delta a + \delta b)$, for instance, you'd need either both $a$ and $b$ to be on the high end of the expected range (or one of them to be very high), which is less likely than one being high and the other being low. (Remember that this formula assumes that the uncertainties in $a$ and $b$ are *uncorrelated* with each other.) So $\delta a + \delta b$ is actually a slight *overestimate* of the uncertainty in $a + b$. If you were to go through the math in detail, you'd arrive at the conclusion that the expected uncertainty is given by equation (3), rather than by the simpler expression $\delta a + \delta b$.

There is good news, though. The more complicated expression in equation (3) has a very nice feature: it puts more weight on the larger uncertainty. In particular, when one of the uncertainties is significantly greater than the other, the more certain quantity contributes essentially nothing to the uncertainty of the sum. For instance, if $\delta a = 5$ cm and $\delta b = 1$ cm, then equation (3) gives

$$\begin{aligned} \delta(a+b) &= \sqrt{(\delta a)^2 + (\delta b)^2} && (8) \\ &= \sqrt{(5\text{ cm})^2 + (1\text{ cm})^2} && (9) \\ &= \sqrt{25\text{ cm}^2 + 1\text{ cm}^2} && (10) \\ &= 5.1\text{ cm}. && (11) \end{aligned}$$

Since we generally round uncertainties to one significant figure anyway, 5.1 isn't noticeably different from 5. So the 1 cm uncertainty in $b$ didn't end up mattering in our final answer. As a general rule of thumb, when you are adding two uncertain quantities and one uncertainty is more than twice as big as the other, you can just use the larger uncertainty as the uncertainty of the sum, and neglect the smaller uncertainty entirely. (However, if you are adding more than two quantities together, you probably shouldn't neglect the smaller uncertainties unless they are at most 1/3 as big as the largest uncertainty.)

As a special case of this, if you add a quantity with an uncertainty to an *exact* number, the uncertainty in the sum is just equal to the uncertainty in the original uncertain quantity.

## 2 Multiplication or Division

If

$$Q = \frac{ab\cdots c}{xy\cdots z}, \tag{12}$$

then

$$\boxed{\frac{\delta Q}{|Q|} = \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2 + \cdots + \left(\frac{\delta c}{c}\right)^2 + \left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2 + \cdots + \left(\frac{\delta z}{z}\right)^2}.} \tag{13}$$

What this means is that the *fractional uncertainties* add in quadrature. In practice, it is usually simplest to convert all of the uncertainties into *percentages* before applying the formula.

Example: a bird flies a distance $d = 120 \pm 3$ m during a time $t = 20.0 \pm 1.2$ s. The average speed of the bird is $v = d/t = 6$ m/s. What is the uncertainty of $v$?

$$\begin{aligned} \frac{\delta v}{v} &= \sqrt{\left(\frac{\delta d}{d}\right)^2 + \left(\frac{\delta t}{t}\right)^2} && (14) \\ &= \sqrt{\left(\frac{3\text{ m}}{120\text{ m}}\right)^2 + \left(\frac{1.2\text{ s}}{20.0\text{ s}}\right)^2} && (15) \\ &= \sqrt{(2.5\%)^2 + (6\%)^2} && (16) \\ &= \sqrt{0.000625 + 0.0036} = 6.5\%. && (17) \end{aligned}$$

So

$$\begin{aligned} \delta v &= v(6.5\%) & (18) \\ &= (6 \text{ m/s})(6.5\%) & (19) \\ &= 0.39 \text{ m/s}. & (20) \end{aligned}$$

So the speed of the bird is $v = 6.0 \pm 0.4$ m/s. Note that as we saw with addition, the formula becomes much simpler if one of the fractional uncertainties is significantly larger than the other. At the point when we noticed that $t$ was 6% uncertain and $d$ was only 2.5% uncertain, we could have just used 6% for the final uncertainty and gotten the same final result (0.36 m/s, which also rounds to 0.4).

The special case of multiplication or division by an exact number is easy to handle: since the exact number has 0% uncertainty, the final product or quotient has the same percent uncertainty as the original number. For example, if you measure the diameter of a sphere to be $d = 1.00 \pm 0.08$ cm, then the fractional uncertainty in $d$ is 8%. Now suppose you want to know the uncertainty in the radius. The radius is just $r = d/2 = 0.50$ cm. Then the fractional uncertainty in $r$ is also 8%. 8% of 0.50 is 0.04, so $r = 0.50 \pm 0.04$ cm.

## 3 Raising to a Power

If $n$ is an exact number and $Q = x^n$, then

$$\delta Q = |n| x^{n-1} \delta x, \tag{21}$$

or equivalently,

$$\boxed{\frac{\delta Q}{|Q|} = |n| \frac{\delta x}{|x|}} \tag{22}$$

The second form is probably easier to remember: the fractional (or percent) uncertainty gets multiplied by $|n|$ when you raise $x$ to the $n$th power.

There is a very important special case here, namely $n = -1$. In this case the rule says that the percent uncertainty is unchanged if you take the reciprocal of a quantity. (This, incidentally, is why multiplication and division are treated exactly the same way in section 2 above.)

Example: the period of an oscillation is measured to be $T = 0.20 \pm 0.01$ s. Thus the frequency is $f = 1/T = 5$ Hz. What is the uncertainty in $f$? Answer: the percent uncertainty in $T$ was $0.01/0.20 = 5\%$. Thus the percent uncertainty in $f$ is also 5%, which means that $\delta f = 0.25$ Hz. So $f = 5.0 \pm 0.3$ Hz (after rounding).

## 4 More Complicated Formulas

Occasionally you may run into more complicated formulas and need to propagate uncertainties through them. Generally, the above rules, when used in combination, will be sufficient to solve most error propagation problems.

Example: a ball is tossed straight up into the air with initial speed $v_0 = 4.0 \pm 0.2$ m/s. After a time $t = 0.60 \pm 0.06$ s, the height of the ball is $y = v_0 t - \frac{1}{2} g t^2 = 0.636$ m. What is the uncertainty of $y$? Assume $g = 9.80$ m/s$^2$ (no uncertainty in $g$).

Answer: Let's start by naming some things. Let $a = v_0 t = 2.4$ m and $b = \frac{1}{2} g t^2 = 1.764$ m. Then using the multiplication rule, we can get the uncertainty in $a$:

$$\begin{aligned} \delta a &= |a|\sqrt{\left(\frac{\delta v_0}{v_0}\right)^2 + \left(\frac{\delta t}{t}\right)^2} && (23)\\ &= v_0 t\sqrt{(0.05)^2 + (0.10)^2} && (24)\\ &= (2.4\text{ m})(0.112) = 0.27\text{ m}. && (25)\end{aligned}$$

For $\delta b$, we can use the power rule:

$$\begin{aligned} \delta b &= 2b\left(\frac{\delta t}{t}\right) && (26)\\ &= 2(1.764\text{ m})(0.10) && (27)\\ &= 0.35\text{ m}. && (28)\end{aligned}$$

Finally, $y = a + b$, so we can get $\delta y$ from the sum rule:

$$\begin{aligned} \delta y &= \sqrt{(\delta a)^2 + (\delta b)^2} && (29)\\ &= \sqrt{(0.27\text{ m})^2 + 0.35\text{ m})^2} = 0.44\text{ m} && (30)\end{aligned}$$

Thus $y$ would be properly reported as $0.6 \pm 0.4$ m.

One catch is the rule that the errors being propagated must be *uncorrelated.* Practically speaking, this means that you have to write your equation so that the same variable does not appear more than once. This is best illustrated by an example. Suppose you have a variable $x$ with uncertainty $\delta x$. You want to calculate the uncertainty propagated to $Q$, which is given by $Q = x^3$. You might think, "well, $Q$ is just $x$ times $x$ times $x$, so I can use the formula for multiplication of three quantities, equation (13)." Let's see: $\delta Q/Q = \sqrt{3}\delta x/x$, so $\delta Q = \sqrt{3}x^2\delta x$. But this is the wrong answer—what happened?

What happened was that equation (13) does not apply if the three quantities have correlated uncertainties. (Obviously, any variable is correlated with itself.) Instead, you have to use equation (22), and you would get the proper answer ($\delta Q = 3x^2\delta x$).

The same sort of thing can happen in subtler ways with expressions like $Q = \frac{x}{x+y}$. You can't use the usual rule when both the numerator and denominator contain $x$, so you have to first rewrite $Q$ as $\frac{1}{1+y/x}$, and then apply several rules: first the quotient rule to get the uncertainty in $y/x$, then the addition rule to get the uncertainty in $1 + y/x$ (fortunately this one is trivial), and then the power rule to get the uncertainty in $(1 + y/x)^{-1}$.

# 5 Comparing Two Quantities

One of the most important applications of error propagation is comparing two quantities with uncertainty. For example, suppose Ann and Billy both measure the speed of a moving ball. Ann measures $3.6 \pm 0.2$ m/s and Billy gets $3.3 \pm 0.3$ m/s. Do the two measurements agree? If the two values were slightly closer together, or if the two uncertainties were slightly larger, the answer would be a pretty clear yes. If the uncertainties were smaller, or the values further apart, it would be a pretty clear no. But as it is, the answer isn't clear at all.

Here's where error propagation comes to the rescue. Let's call Ann's result $A \pm \delta A$, and Billy's result $B \pm \delta B$. To see if they agree, we compute the *difference* $D = A - B$. Using error propagation, we can figure out the uncertainty $\delta D$. Then the question of whether $A$ agrees with $B$, with uncertainties on both sides, has been simplified to the question of whether $D$ agrees with zero, with uncertainty on only one side.

Let's work out the result of our example. Using the rule for a sum (or difference), we get

$$\delta D = \sqrt{(\delta A)^2 + (\delta B)^2} \tag{31}$$

$$= \sqrt{(0.2\ \mathrm{m/s})^2 + 0.3\ \mathrm{m/s})^2} = 0.36\ \mathrm{m/s}. \tag{32}$$

Since $D = 0.3 \pm 0.4$ m/s, we see that zero is comfortably within the uncertainty range of $D$, so the two measurements agree.

This raises an interesting question: have we now shown that Ann's measurement and Billy's measurement are equal? The answer is no—we've merely shown that they *could* be equal. There is no experiment you can perform to prove that two quantities are equal; the best you can do is to measure them both so precisely that you can put a very tight bound on the amount by which they might differ. (For example, if we had found that $D = 0.003 \pm 0.004$ m/s, you might be more convinced that $A = B$.)

However, it *is* possible to show that two quantities are *not* equal, at least to a high degree of confidence. If you follow the above procedure and find that $D$ is 3 times as big as $\delta D$, that puts serious doubt into the hypothesis that the two quantities are equal, because $D = 0$ is three standard deviations away from your observed result. That could just be due to chance, but the odds are on your side: you can be 99.7% confident that the two quantities were actually different.

This is often how discoveries are made in science: you start by making a prediction of what you'll see *if whatever you're trying to discover isn't actually there.* This prediction is called the *null hypothesis.* Then you compare the null hypothesis to your experimental result. If the two differ by three standard deviations, you can be 99.7% confident that the null hypothesis was wrong. But if the null hypothesis was wrong, then you've made a discovery! (Of course, if it is an important discovery, the scientific community will want to repeat or extend your experiment, to increase the level of confidence in the result beyond a simple $3\sigma$ measurement.)