(c) We have

$$x[n] * [h[n] * g[n]] = \left(\frac{1}{2}\right)^n * \delta[n] = \frac{1}{2}^n,$$

$$(x[n] * g[n]) * h[n] = 0 * h[n] = 0,$$

and

$$(x[n] * h[n]) * g[n] = \{(\frac{1}{2})^n \sum_{k=0}^{\infty} 1\} * g[n] = \infty.$$

(d) Let $h(t) = u_1(t)$. Then if the input is $x_1(t) = 0$, the output will be $y_1(t) = 0$. Now if $x_2(t) =$ constant, then $y_2(t) = 0$. Therefore, the system is not invertible.

Now note that

$$|\int_{-\infty}^{t} x_2(\tau)d\tau| = \begin{cases} 0 & \text{if } x_2(t) = 0 \forall t \\ \infty & \text{if } x_2(t) \neq 0 \end{cases}.$$

Therefore, if $\left|\int_{-\infty}^{t} cdt\right|_{t\to\infty} \neq \infty$, then only $x_2(t) = 0$ will yield $y_2(t) = 0$. Therefore the system is invertible.

2.72. We have

$$\delta_\Delta(t) = \frac{1}{\Delta}u(t) * [\delta(t) - \delta(t-T)].$$

Differentiating both sides we get

$$\begin{aligned} \frac{d}{dt}\delta_\Delta t &= \frac{1}{\Delta}u'(t) * [\delta(t) - \delta(t-T)] \\ &= \frac{1}{\Delta}\delta(t) * [\delta(t) - \delta(t-T)] \\ &= \frac{1}{\Delta}[\delta(t) - \delta(t-T)] \end{aligned}$$

2.73. For $k = 1$, $u_{-1}(t) = u(t)$. Therefore, the given statement is true for $k = 1$. Now assume that it is true for some $k > 1$. Then,

$$\begin{aligned} u_{-(k+1)}(t) &= u(t) * u_{-k}(t) \\ &= \int_{-\infty}^{t} u_{-k}(t) = \int_0^t u_{-k}(\tau)d\tau \\ &= \int_0^t \frac{\tau^{k-1}}{(k-1)!}, \quad t \geq 0 \\ &= \left.\frac{\tau^k}{k(k-1)!}\right|_{\tau=t\geq 0} = \frac{t^k}{k!}u(t). \end{aligned}$$

Chapter 3 Answers

3.1. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_3 e^{j3(2\pi/T)t} + a_{-3}e^{-j3(2\pi/T)t} \\ &= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4je^{j3(2\pi/8)t} - 4je^{-j3(2\pi/8)t} \\ &= 4\cos(\frac{\pi}{4}t) - 8\sin(\frac{6\pi}{8}t) \\ &= 4\cos(\frac{\pi}{4}t) + 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2}) \end{aligned}$$

3.2. Using the Fourier series synthesis eq. (3.95).

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\ &= 1 + e^{j(\pi/4)}e^{j2(2\pi/5)n} + e^{-j(\pi/4)}e^{-2j(2\pi/5)n} \\ &\quad +2e^{j(\pi/3)}e^{j4(2\pi/N)n} + 2e^{-j(\pi/3)}a_{-4}e^{-j4(2\pi/N)n} \\ &= 1 + 2\cos(\frac{4\pi}{5}n + \frac{\pi}{4}) + 4\cos(\frac{8\pi}{5}n + \frac{\pi}{3}) \\ &= 1 + 2\sin(\frac{4\pi}{5}n + \frac{3\pi}{4}) + 4\sin(\frac{8\pi}{5}n + \frac{5\pi}{6}) \end{aligned}$$

3.3. The given signal is

$$\begin{aligned} x(t) &= 2 + \frac{1}{2}e^{j(2\pi/3)t} + \frac{1}{2}e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2}e^{j2(2\pi/6)t} + \frac{1}{2}e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coeffcients of $x(t)$ are:

$$a_0 = 2, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_5 = a_{-5}^* = -2j$$

3.4. Since $\omega_0 = \pi$, $T = 2\pi/\omega_0 = 2$. Therefore,

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt$$

Now,

$$a_0 = \frac{1}{2}\int_0^1 1.5dt - \frac{1}{2}\int_1^2 1.5dt = 0$$

and for $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 1.5e^{-jk\pi t}dt - \frac{1}{2}\int_1^2 1.5e^{-jk\pi t}dt \\ &= \frac{3}{2k\pi j}[1 - e{-}jk\pi] \\ &= \frac{3}{k\pi}e^{-jk(\pi/2)}\sin(\frac{k\pi}{2}) \end{aligned}$$

3.5. Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundemental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundemental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \stackrel{FS}{\longleftrightarrow} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \stackrel{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \stackrel{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \stackrel{FS}{\longleftrightarrow} a_{-k}e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \stackrel{FS}{\longleftrightarrow} a_k e^{jk(2\pi/T_1)} + a_{-k}e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k}(a_k + a_{-k})$$

3.6. (a) Comparing $x_1(t)$ with the Fourier series synthesis eq. (3.38), we obtain the Fourier series coefficients of $x_1(t)$ to be

$$a_k = \begin{cases} (\frac{1}{2})^k, & 0 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_1(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is not true for $x_1(t)$, the signal is **not real valued**.

Similarly, the Fourier series coefficients of $x_2(t)$ are

$$a_k = \begin{cases} \cos(k\pi), & 100 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_2(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is true for $x_2(t)$, the signal is **real valued**.

Similarly, the Fourier series coefficients of $x_3(t)$ are

$$a_k = \begin{cases} j\sin(k\pi/2), & 100 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_3(t)$ is real, then a_k has to be conjugate-symmetric, i.e, $a_k = a_{-k}^*$. Since this is true for $x_3(t)$, the signal is **real valued**.

(b) For a signal to be even, its Fourier series coefficients must be even. This is true only for $x_2(t)$.

3.7. Given that

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k$$

we have

$$g(t) = \frac{dx(t)}{dt} \stackrel{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T}a_k.$$

Therefore,

$$a_k = \frac{b_k}{j(2\pi/T)k}, \quad k \neq 0$$

When $k = 0$,

$$a_k = \frac{1}{T}\int_{<T>} x(t)dt = \frac{2}{T} \quad \text{using given information}$$

Therefore,

$$a_k = \begin{cases} \frac{2}{T}, & k = 0 \\ \frac{b_k}{j(2\pi/T)k}, & k \neq 0 \end{cases}.$$

3.8. Since $x(t)$ is real and odd (clue 1), its Fourier series coefficients a_k are purely imaginary and odd (See Table 3.1). Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. Also, since it is given that $a_k = 0$ for $|k| > 1$, the only unknown Fourier series coefficients are a_1 and a_{-1}. Using Parseval's relation,

$$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

for the given signal we have

$$\frac{1}{2}\int_0^2 |x(t)|^2 dt = \sum_{k=-1}^{1} |a_k|^2.$$

Using the information given in clue (4) along with the above equation,

$$|a_1|^2 + |a_{-1}|^2 = 1 \quad \Rightarrow \quad 2|a_1|^2 = 1$$

Therefore,

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}j} \quad \text{or} \quad a_1 = -a_{-1} = -\frac{1}{\sqrt{2}j}$$

The two possible signals which satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} - \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = -\sqrt{2}\sin(\pi t)$$

and

$$x_2(t) = -\frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} + \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = \sqrt{2}\sin(\pi t)$$

3.9. The period of the given signal is 4. Therefore,

$$\begin{aligned} a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}[4 + 8e^{-j\frac{\pi}{2}k}] \end{aligned}$$

This gives

$$a_0 = 3, \quad a_1 = 1 - 2j, \quad a_2 = -1, \quad a_3 = 1 + 2j$$

3.10. Since the Fourier series coefficients repeat every N, we have

$$a_1 = a_{15}, \quad a_2 = a_{16} \quad \text{, and} \quad a_3 = a_{17}$$

Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd. Therefore, $a_0 = 0$ and

$$a_1 = -a_{-1}, \quad a_2 = -a_{-2} \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j, \quad a_{-2} = -2j, \quad a_{-3} = -3j$$

3.11. Since the Fourier series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$. Furthermore, since $x[n]$ is real and even, a_k is also real and even. Therefore, $a_1 = a_{-1} = 5$. We are also given that

$$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50.$$

Using Parseval's relation,

$$\begin{aligned}
\sum_{k=<N>} |a_k|^2 &= 50 \\
\sum_{k=-1}^{8} |a_k|^2 &= 50 \\
|a_{-1}|^2 + |a_1|^2 + a_0^2 + \sum_{k=2}^{8}|a_k|^2 &= 50 \\
a_0^2 + \sum_{k=2}^{8}|a_k|^2 &= 0
\end{aligned}$$

Therefore, $a_k = 0$ for $k = 2, \cdots, 8$. Now using the synthesis eq.(3.94), we have

$$\begin{aligned}
x[n] &= \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^{8} a_k e^{j\frac{2\pi}{10}kn} \\
&= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n} \\
&= 10\cos(\frac{\pi}{5}n)
\end{aligned}$$

3.12. Using the multiplication property (see Table 3.2), we have

$$\begin{aligned}
x_1[n]x_2[n] &\stackrel{FS}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l} = \sum_{k=0}^{3} a_l b_{k-l} \\
&\stackrel{FS}{\longleftrightarrow} a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} \\
&\stackrel{FS}{\longleftrightarrow} b_k + 2b_{k-1} + 2b_{k-2} + 2b_{k-3}
\end{aligned}$$

Since b_k is 1 for all values of k, it is clear that $b_k + 2b_{k-1} + 2b_{k-3} + 2b_{k-3}$ will be 6 for all values of k. Therefore,

$$x_1[n]x_2[n] \stackrel{FS}{\longleftrightarrow} 6, \qquad \text{for all } k.$$

3.13. Let us first evaluate the Fourier series coefficients of $x(t)$. Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd. Therefore, $a_0 = 0$. Now,

$$\begin{aligned}
a_k &= \frac{1}{8}\int_0^8 x(t)e^{-j(2\pi/8)kt}dt \\
&= \frac{1}{8}\int_0^4 e^{-j(2\pi/8)kt}dt - \frac{1}{8}\int_4^8 e^{-j(2\pi/8)kt}dt \\
&= \frac{1}{j\pi k}[1 - e^{-j\pi k}]
\end{aligned}$$

Clearly, the above expression evaluates to zero for all even values of k. Therefore,

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \cdots \\ \frac{2}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by (see Section 3.8)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Since a_k is non zero only for odd values of k, we need to evaluate the above summation only for odd k. Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

is always zero for odd values of k. Therefore,

$$y(t) = 0.$$

3.14. The signal $x[n]$ is periodic with period $N = 4$. Its Fourier series coefficients are

$$\begin{aligned}
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}, \qquad \text{for all } k
\end{aligned}$$

From the results presented in Section 3.8, we know that the output $y[n]$ is given by

$$\begin{aligned}
y[n] &= \sum_{k=0}^{3} a_k H(e^{j(2\pi/4)k})e^{jk(2\pi/4)n} \\
&= \tfrac{1}{4}H(e^{j0})e^{j0} + \tfrac{1}{4}H(e^{j(\pi/2)})e^{j(\pi/2)} \\
&\quad + \tfrac{1}{4}H(e^{j(3\pi/2)})e^{j(3\pi/2)} + \tfrac{1}{4}H(e^{j(\pi)})e^{j(\pi)}
\end{aligned} \qquad \text{(S3.14-1)}$$

From the given information, we know that $y[n]$ is

$$\begin{aligned}
y[n] &= \cos(\frac{5\pi}{2}n + \frac{\pi}{4}) \\
&= \cos(\frac{\pi}{2}n + \frac{\pi}{4}) \\
&= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{-j(\frac{\pi}{2}n + \frac{\pi}{4})} \\
&= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{j(3\frac{\pi}{2}n - \frac{\pi}{4})}
\end{aligned}$$

Comparing this with eq. (S3.14-1), we have

$$H(e^{j0}) = H(e^{j\pi}) = 0$$

and

$$H(e^{j\frac{\pi}{2}}) = 2e^{j\frac{\pi}{4}}, \quad \text{and} \quad H(e^{3j\frac{\pi}{2}}) = 2e^{-j\frac{\pi}{4}}$$

3.15. From the results of Section 3.8,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = 12$. Since $H(j\omega)$ is zero for $|\omega| > 100$, the largest value of $|k|$ for which a_k is nonzero should be such that

$$|k|\omega_0 \leq 100$$

This implies that $|k| \leq 8$. Therefore, for $|k| > 8$, a_k is guaranteed to be zero.

3.16. (a) The given signal $x_1[n]$ is

$$x_1[n] = (-1)^n = e^{j\pi n} = e^{j(2\pi/2)n}$$

Therefore, $x_1[n]$ is periodic with period $N = 2$ and it's Fourier series coefficients in the range $0 \leq k \leq 1$ are

$$a_0 = 0, \quad \text{and} \quad a_1 = 1$$

Using the results derived in Section 3.8, the output $y_1[n]$ is given by

$$\begin{aligned}
y_1[n] &= \sum_{k=0}^{1} a_k H(e^{j2\pi k/2})e^{k(2\pi/2)} \\
&= 0 + a_1 H(e^{j\pi})e^{j\pi} \\
&= 0
\end{aligned}$$

(b) The signal $x_2[n]$ is periodic with period $N = 16$. The signal $x_2[n]$ may be written as

$$\begin{aligned}
x_2[n] &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{-j(2\pi/16)(3)n} \\
&= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n}
\end{aligned}$$

Therefore, the non-zero Fourier series coefficients of $x_2[n]$ in the range $0 \leq k \leq 15$ are

$$a_0 = 1, \quad a_3 = -(j/2)e^{j(\pi/4)}, \quad a_{13} = (j/2)e^{-j(\pi/4)}$$

Using the results derived in Section 3.8, the output $y_2[n]$ is given by

$$\begin{aligned}
y_2[n] &= \sum_{k=0}^{15} a_k H(e^{j2\pi k/16})e^{k(2\pi/16)} \\
&= 0 - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n} \\
&= \sin(\frac{3\pi}{8}n + \frac{\pi}{4})
\end{aligned}$$

(c) The signal $x_3[n]$ may be written as

$$x_3[n] = \left[\left(\frac{1}{2}\right)^n u[n]\right] * \sum_{k=-\infty}^{\infty}\delta[n-4k] = g[n] * r[n]$$

where $g[n] = (\frac{1}{2})^n u[n]$ and $r[n] = \sum_{k=-\infty}^{\infty}\delta[n-4k]$. Therefore, $y_3[n]$ may be obtained by passing the signal $r[n]$ through the filter with frequency response $H(e^{j\omega})$, and then convolving the result with $g[n]$.

The signal $r[n]$ is periodic with period 4 and its Fourier series coeffients are

$$a_k = \frac{1}{4}, \qquad \text{for all } k \text{ (See Problem 3.14)}$$

The output $q[n]$ obtained by passing $r[n]$ through the filter with frequency response $H(e^{j\omega})$ is

$$\begin{aligned}
q[n] &= \sum_{k=0}^{3} a_k H(e^{j2\pi k/4})e^{k(2\pi/4)} \\
&= (1/4)(H(e^{j0})e^{j0} + H(e^{j(\pi/2)})e^{j(\pi/2)} + H(e^{j\pi})e^{j\pi} + H(e^{j3(\pi/2)})e^{j3(\pi/2)}) \\
&= 0
\end{aligned}$$

Therefore, the final output $y_3[n] = q[n] * g[n] = 0$.

3.17. (a) Since complex exponentials are Eigen functions of LTI systems, the input $x_1(t) = e^{j5t}$ has to produce an output of the form Ae^{j5t}, where A is a complex constant. But clearly, in this case the output is not of this form. Therefore, system S_1 is definitely **not LTI**.

(b) This system may be LTI because it satisifies the Eigen function property of LTI systems.

(c) In this case, the output is of the form $y_3(t) = (1/2)e^{j5t} + (1/2)e^{-j5t}$. Clearly, the output contains a complex exponential with frequency -5 which was not present in the input $x_3(t)$. We know that an LTI system can never produce a complex exponential of frequency -5 unless there was complex exponential of the same frequency at its input. Since this is not the case in this problem, S_3 is definitely **not LTI**.

3.18. (a) By using an argument similar to the one used in part (a) of the previous problem, we conclude that S_1 is defintely not LTI.

(b) The output in this case is $y_2[n] = e^{j(3\pi/2)n} = e^{-j(\pi/2)n}$. Clearly this violates the eigen function property of LTI systems. Therefore, S_2 is definitely not LTI.

(c) The output in this case is $y_3[n] = 2e^{j(5\pi/2)n} = 2e^{j(\pi/2)n}$. This does not violate the eigen function property of LTI systems. Therefore, S_3 could possibly be an LTI system.

3.19. (a) Voltage across inductor $= L\frac{dy(t)}{dt}$.
Current through resistor $= \frac{L}{R}\frac{dy(t)}{dt}$.
Input current $x(t)$ = current through resistor + current through inductor
Therefore,

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t).$$

Substituting for R and L we obtain

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

(b) Using the approach outlined in Section 3.10.1, we know that the output of this system will be $H(j\omega)e^{j\omega t}$ when the input is $e^{j\omega t}$. Substituting in the differential equation of part (a),

$$j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

Therefore,

$$H(j\omega) = \frac{1}{1+j\omega}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2}e^{j(2\pi/2\pi)t} + \frac{1}{2}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1} = \frac{1}{2}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} + a_{-1}H(-j)e^{-jt} \\ &= (1/2)(\frac{1}{1+j}ejt + \frac{1}{1-j}e^{-jt}) \\ &= (1/2\sqrt{2})(e^{-j\pi/4}e^{jt} + e^{j\pi/4}e^{-jt}) \\ &= (1/\sqrt{2})\cos(t - \frac{\pi}{4}) \end{aligned}$$

3.20. (a) Current through the capacitor $= C\frac{dy(t)}{dt}$.
Voltage across resistor $= RC\frac{dy(t)}{dt}$.
Voltage across inductor $= LC\frac{d^2y(t)}{dt^2}$.
Input voltage = Voltage across resistor + Voltage across inductor + Voltage across capacitor.
Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t)$$

Substituting for R, L and C, we have

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b) We will now use an approach similar to the one used in part (b) of the previous problem. If we assume that the input is of the form $e^{j\omega t}$, then the output will be of the form $H(j\omega)e^{j\omega t}$. Substituting in the above differential equation and simplifying, we obtain

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

(c) The signal $x(t)$ is periodic with period 2π. Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2j}e^{j(2\pi/2\pi)t} - \frac{1}{2j}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1}^* = \frac{1}{2j}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j)e^{jt} - a_{-1}H(-j)e^{-jt} \\ &= (1/2j)(\frac{1}{j}e^{jt} - \frac{1}{-j}e^{-jt}) \\ &= (-1/2)(e^{jt} + e^{-jt}) \\ &= -\cos(t) \end{aligned}$$

3.21. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_5 e^{j5(2\pi/T)t} + a_{-5}e^{-j5(2\pi/T)t} \\ &= je^{j(2\pi/8)t} - je^{-j(2\pi/8)t} + 2e^{j5(2\pi/8)t} + 2e^{-j5(2\pi/8)t} \\ &= -2\sin(\frac{\pi}{4}t) + 4\cos(\frac{5\pi}{4}t) \\ &= -2\cos(\frac{\pi}{4}t - \pi/2) + 4\cos(\frac{5\pi}{4}t). \end{aligned}$$

3.22. (a) (i) $T = 1$, $a_0 = 0$, $a_k = \frac{j(-1)^k}{k\pi}$, $k \neq 0$.
(ii) Here,

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

$T = 6$, $a_0 = 1/2$, and

$$a_k = \begin{cases} 0, & k \text{ even} \\ \frac{6}{\pi^2 k^2}\sin(\frac{\pi k}{2})\sin(\frac{\pi k}{6}), & k \text{ odd} \end{cases}.$$

(iii) $T = 3$, $a_0 = 1$, and

$$a_k = \frac{3j}{2\pi^2 k^2}[e^{jk2\pi/3}\sin(k2\pi/3) + 2e^{jk\pi/3}\sin(k\pi/3)], \qquad k \neq 0.$$

(iv) $T = 2$, $a_0 = -1/2$, $a_k = \frac{1}{2} - (-1)^k$, $k \neq 0$.
(v) $T = 6$, $\omega_0 = \pi/3$, and

$$a_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi/3}.$$

Note that $a_0 = 0$ and $a_{k\ even} = 0$.
(vi) $T = 4$, $\omega_0 = \pi/2$, $a_0 = 3/4$ and

$$a_k = \frac{e^{-jk\pi/2}\sin(k\pi/2) + e^{-jk\pi/4}\sin(k\pi/4)}{k\pi}, \qquad \forall k.$$

(b) $T = 2$, $a_k = \frac{-1^k}{2(1+jk\pi)}[e - e^{-1}]$ for all k.
(c) $T = 3$, $\omega_0 = 2\pi/3$, $a_0 = 1$ and

$$a_k = \frac{2e^{-j\pi k/3}}{\pi k}\sin(2\pi k/3) + \frac{e^{-j\pi k}}{\pi k}\sin(\pi k).$$

3.23. (a) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/4)}{k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & 1/2 < |t| < 2 \end{cases}.$$

Now, note that $b_0 = 1/4$. Let us define another signal $z(t) = -1/4$ whose only nonzero FS coefficient is $c_0 = -1/4$. The signal $p(t) = y(t) + z(t)$ will have FS coefficients

$$d_k = a_k + c_k = \begin{cases} 0, & k = 0 \\ \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise.} \end{cases}$$

Now note that $a_k = d_k e^{j(\pi/2)k}$. Therefore, the signal $x(t) = p(t+1)$ which is as shown in Figure S2.23(a).

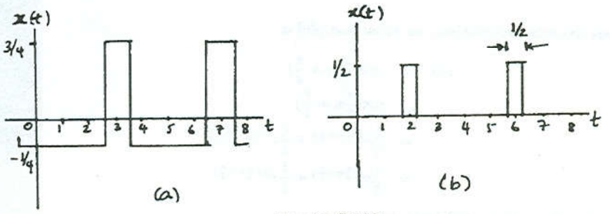

Figure S3.23

(b) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/8)}{2k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1/2, & |t| < 1/4 \\ 0, & 1/4 < |t| < 2 \end{cases}.$$

Now note that $a_k = b_k e^{j\pi k}$. Therefore, the signal $x(t) = y(t+2)$ which is as shown in Figure S2.23(b).

(c) The only nonzero FS coefficients are $a_1 = a_{-1}^* = j$ and $a_2 = a_{-2}^* = 2j$. Using the FS synthesis equation, we get

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1}e^{-j(2\pi/T)t} + a_2 e^{j2(2\pi/T)t} + a_{-2}e^{-j2(2\pi/T)t} \\ &= je^{j(2\pi/4)t} - je^{-j(2\pi/4)t} + 2je^{j2(2\pi/4)t} - 2je^{-j2(2\pi/4)t} \\ &= -2\sin(\frac{\pi}{2}t) - 4\sin(\pi t) \end{aligned}$$

(d) The FS coefficients a_k may be written as the sum of two sets of FS coefficients b_k and c_k, where

$$b_k = 1, \qquad \text{for all } k$$

and

$$c_k = \begin{cases} 1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

The FS coefficients b_k correspond to the signal

$$y(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

and the FS coefficients c_k correspond to the signal

$$z(t) = \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t}\delta(t - 2k).$$

Therefore,

$$x(t) = y(t) + p(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k) + \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t}\delta(t-2k).$$

3.24. (a) We have

$$a_0 = \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 (2-t)\,dt = 1/2.$$

(b) The signal $g(t) = dx(t)/dt$ is as shown in Figure S3.24.

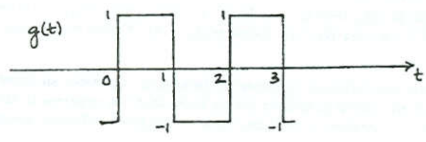

Figure S3.24

The FS coefficients b_k of $g(t)$ may be found as follows:

$$b_0 = \frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt = 0$$

and

$$\begin{aligned} b_k &= \frac{1}{2}\int_0^1 e^{-j\pi kt}dt - \frac{1}{2}\int_1^2 e^{-j\pi kt}dt \\ &= \frac{1}{j\pi k}[1 - e^{-j\pi k}]. \end{aligned}$$

(c) Note that

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\pi a_k.$$

Therefore,

$$a_k = \frac{1}{jk\pi}b_k = -\frac{1}{\pi^2k^2}\{1 - e^{-j\pi k}\}.$$

3.25. (a) The nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1}^* = 1/2j$.

(c) Using the multiplication property, we know that

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Therefore,

$$c_k = a_k * b_k = \frac{1}{4j}\delta[k-2] - \frac{1}{4j}\delta[k+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = (1/4j)$

(d) We have

$$z(t) = \sin(4t)\cos(4t) = \frac{1}{2}\sin(8t).$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2} = (1/4j)$.

3.26. (a) If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a_{-k}^*$. Since this is not true in this case problem, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\frac{2\pi}{T_0}a_k.$$

Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T_0), & \text{otherwise} \end{cases}.$$

Since b_k is not even, $g(t)$ is not even.

3.27. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2}e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4}e^{-j4(2\pi/N)n} \\ &= 2 + 2e^{j\pi/6}e^{j(4\pi/5)n} + 2e^{-j\pi/6}e^{-j(4\pi/5)n} + e^{j\pi/3}e^{j(8\pi/5)n} + e^{-j\pi/3}e^{-j(8\pi/5)n} \\ &= 2 + 4\cos[(4\pi n/5) + \pi/6] + 2\cos[(8\pi n/5) + \pi/3] \\ &= 2 + 4\sin[(4\pi n/5) + 2\pi/3] + 2\sin[(8\pi n/5) + 5\pi/6] \end{aligned}$$

3.28. (a) $N = 7$,

$$a_k = \frac{1}{7}\frac{e^{-j4\pi k/7}\sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period $(0 \le k \le 5)$ may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6}e^{-j\pi k/2}\frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \qquad 1 \le k \le 5.$$

(c) $N = 6$.

$$a_k = 1 + 4\cos(\pi k/3) - 2\cos(2\pi k/3).$$

(d) $N = 12$, a_k over one period $(0 \le k \le 11)$ may be specified as: $a_1 = \frac{1}{4j} = a_{11}^*$, $a_5 = -\frac{1}{4j} = a_7^*$, $a_k = 0$ otherwise.

(e) $N = 4$.

$$a_k = 1 + 2(-1)^k(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}).$$

(f) $N = 12$,

$$\begin{aligned} a_k &= 1 + (1 - \frac{1}{\sqrt{2}})2\cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}})\cos(\frac{\pi k}{2}) \\ &+ 2(1 + \frac{1}{\sqrt{2}})\cos(\frac{5\pi k}{6}) + 2(-1)^k + 2\cos(\frac{2\pi k}{3}). \end{aligned}$$

3.29. (a) $N = 8$. Over one period $(0 \le n \le 7)$,

$$x[n] = 4\delta[n-1] + 4\delta[n-7] + 4j\delta[n-3] - 4j\delta[n-5].$$

(b) $N = 8$. Over one period $(0 \le n \le 7)$,

$$x[n] = \frac{1}{2j}\left[\frac{-e^{j\frac{3\pi n}{4}}\sin\{\frac{7}{2}(\frac{\pi n}{4} + \frac{\pi}{3})\}}{\sin\{\frac{1}{2}(\frac{\pi n}{4} + \frac{\pi}{3})\}} + \frac{e^{j\frac{3\pi n}{4}}\sin\{\frac{7}{2}(\frac{\pi n}{4} - \frac{\pi}{3})\}}{\sin\{\frac{1}{2}(\frac{\pi n}{4} - \frac{\pi}{3})\}}\right].$$

(c) $N = 8$. Over one period $(0 \le n \le 7)$,

$$x[n] = 1 + (-1)^n + 2\cos(\frac{\pi n}{4}) + 2\cos(\frac{3\pi n}{4}).$$

(d) $N = 8$. Over one period $(0 \le n \le 7)$,

$$x[n] = 2 + 2\cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right).$$

3.30. (a) The nonzero FS coefficients of $x(t)$ are $a_0 = 1$, $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1}^* = e^{-j\pi/4}/2$.

(c) Using the multiplication property, we know that

$$z[n] = x[n]y[n] \overset{FS}{\longleftrightarrow} c_k = \sum_{l=-2}^{2} a_l b_{k-l}.$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

(d) We have

$$\begin{aligned} z[n] &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)\cos\left(\frac{2\pi}{6}n\right) \\ &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \frac{1}{2}\left[\sin(\frac{4\pi}{6}n + \frac{\pi}{4}) + \sin(\frac{\pi}{4})\right] \end{aligned}$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

3.31. (a) $g[n]$ is as shown in Figure S3.31. Clearly, $g[n]$ has a fundamental period of 10.

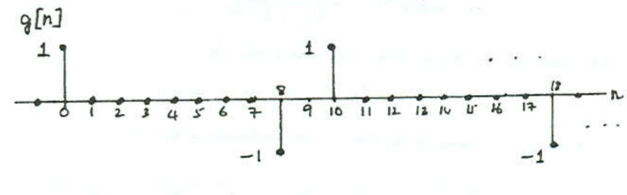

Figure S3.31

(b) The Fourier series coefficiennts of $g[n]$ are $b_k = (1/10)[1 - e^{-j(2\pi/10)8k}]$.

(c) Since $g[n] = x[n] - x[n-1]$, the FS coeffcients a_k and b_k must be related as

$$b_k = a_k - e^{-j(2\pi/10)k}a_k.$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-j(2\pi/10)k}} = \frac{(1/10)[1 - e^{-j(2\pi/10)8k}]}{1 - e^{-j(2\pi/10)k}}.$$

3.32. (a) The four equations are

$$a_0 + a_1 + a_2 + a_3 = 1, \qquad a_0 + ja_1 - a_2 - ja_3 = 0$$

$$a_0 - a_1 + a_2 - a_3 = 2, \qquad a_0 - ja_1 - a_2 + ja_3 = -1.$$

Solving, we get $a_0 = 1/2$, $a_1 = -\frac{1+j}{4}$, $a_2 = -1$, $a_3 = -\frac{1-j}{4}$.

(b) By direct calculation,

$$a_k = \frac{1}{4}[1 + 2e^{-jk\pi} - e^{-jk3\pi/2}].$$

This is the same as the answer we obtained in part (a) for $0 \le k \le 3$.

3.33. We will first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. From the discussion in Section 3.9.2 we know that the response to this input will be $y(t) = H(j\omega)e^{j\omega t}$. Therefore, substituing these in the given differential equation, we get

$$H(j\omega)j\omega e^{j\omega t} + 4e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

From eq. (3.124), we know that

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

when the input is $x(t)$. $x(t)$ has the Fourier series coefficients a_k and fundamental frequency ω_0. Therefore, the Fourier series coefficients of $y(t)$ are $a_k H(jk\omega_0)$.

(a) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(4+j2\pi)}, \qquad b_{-1} = a_{-1}H(-j2\pi) = \frac{1}{2(4-j2\pi)}.$$

(b) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_2 = a_{-2}^* = 1/2j$ and $a_3 = a_{-3}^* = e^{j\pi/4}/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_2 = a_2 H(j4\pi) = \frac{1}{2j(4+j4\pi)}, \qquad b_{-2} = a_{-2}H(-j4\pi) = -\frac{1}{2j(4-j4\pi)},$$

$$b_3 = a_3 H(j6\pi) = \frac{e^{j\pi/4}}{2(4+j6\pi)}, \qquad b_{-3} = a_{-3}H(-j6\pi) = -\frac{e^{-j\pi/4}}{2(4-j6\pi)}.$$

3.34. The frequency response of the system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|t|}e^{-j\omega t}dt = \frac{1}{4+j\omega} + \frac{1}{4-j\omega}.$$

(a) Here, $T=1$ and $\omega_0 = 2\pi$ and $a_k = 1$ for all k. The FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}.$$

(b) Here, $T=2$ and $\omega_0 = \pi$ and

$$a_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 0, & k \text{ even} \\ \frac{1}{4+j\pi k} + \frac{1}{4-j\pi k}, & k \text{ odd} \end{cases}.$$

(c) Here, $T=1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k=0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k=0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}\left[\frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}\right], & k \text{ odd} \end{cases}.$$

3.35. We know that the Fourier series coefficient of $y(t)$ are $b_k = H(jk\omega_0)a_k$, where ω_0 is the fundamental frequency of $x(t)$ and a_k are the FS coefficients of $x(t)$.

If $y(t)$ is identical to $x(t)$, then $b_k = a_k$ for all k. Noting that $H(j\omega) = 0$ for $|\omega| \geq 250$, we know that $H(jk\omega_0) = 0$ for $|k| \geq 18$ (because $\omega_0 = 14$). Therefore, a_k must be zero for $|k| \geq 18$.

3.36. We will first evaluate the frequency response of the system. Consider an input $x[n]$ of the form $e^{j\omega n}$. From the discussion in Section 3.9 we know that the response to this input will be $y[n] = H(e^{j\omega})e^{j\omega n}$. Therefore, substituing these in the given difference equation, we get

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}e^{-j\omega}e^{j\omega n}H(e^{j\omega}) = e^{j\omega n}.$$

Therefore,

$$H(j\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}.$$

From eq. (3.131), we know that

$$y[n] = \sum_{k=<N>} a_k H(e^{j2\pi k/N})e^{jk(2\pi/N)n}$$

when the input is $x[n]$. $x[n]$ has the Fourier series coefficients a_k and fundamental frequency $2\pi/N$. Therefore, the Fourier series coefficients of $y[n]$ are $a_k H(e^{j2\pi k/N})$.

(a) Here, $N=4$ and the nonzero FS coefficients of $x[n]$ are $a_3 = a_{-3}^* = 1/2j$. Therefore, the nonzero FS coefficients of $y[n]$ are

$$b_3 = a_1 H(e^{3j\pi/4}) = \frac{1}{2j(1-(1/4)e^{-j3\pi/4})}, \qquad b_{-3} = a_{-1}H(e^{-3j\pi/4}) = \frac{-1}{2j(1-(1/4)e^{j3\pi/4})}.$$

(b) Here, $N=8$ and the nonzero FS coefficients of $x[n]$ are $a_1 = a_{-1} = 1/2$ and $a_2 = a_{-2} = 1$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(e^{j\pi/4}) = \frac{1}{2(1-(1/4)e^{-j\pi/4})}, \qquad b_{-1} = a_{-1}H(e^{-j\pi/4}) = \frac{1}{2(1-(1/4)e^{j\pi/4})},$$

$$b_2 = a_2 H(e^{j\pi/2}) = \frac{1}{(1-(1/4)e^{-j\pi/2})}, \qquad b_{-2} = a_{-2}H(e^{-j\pi/2}) = \frac{1}{(1-(1/4)e^{j\pi/2})}.$$

3.37. The frequency response of the system may be easily shown to be

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{1}{1-2e^{-j\omega}}.$$

(a) The Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{4}, \qquad \text{for all } k.$$

Also, $N=4$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2k\pi/N}) = \frac{1}{4}\left[\frac{1}{1-\frac{1}{2}e^{-j\pi k/2}} - \frac{1}{1-2e^{-j\pi k/2}}\right].$$

(b) In this case, the Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{6}[1 + 2\cos(k\pi/3)], \qquad \text{for all} k.$$

Also, $N=6$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2k\pi/N}) = \frac{1}{6}[1+2\cos(k\pi/3)]\left[\frac{1}{1-\frac{1}{2}e^{-j\pi k/3}} - \frac{1}{1-2e^{-j\pi k/3}}\right].$$

3.38. The frequency response of the system may be evaluated as

$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}.$$

For $x[n]$, $N=4$ and $\omega_0 = \pi/2$. The FS coefficients of the input $x[n]$ are

$$a_k = \frac{1}{4}, \qquad \text{for all } n.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4}[1 - e^{jk\pi/2} + e^{-jk\pi/2}].$$

3.39. Let the FS coefficients of the input be a_k. The FS coeffients of the output are of the form

$$b_k = a_k H(e^{jk\omega_0}),$$

where $\omega_0 = 2\pi/3$. Note that in the range $0 \leq k \leq 2$, $H(e^{jk\omega_0}) = 0$ for $k = 1,2$. Therefore, only b_0 has a nonzero value among b_k in the range $0 \leq k \leq 2$.

3.40. Let the Fourier series coefficients of $x(t)$ be a_k.

(a) $x(t-t_0)$ is also periodic with period T. The Fourier series coefficients b_k of $x(t-t_0)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0)e^{-jk(2\pi/T)t}dt \\ &= \frac{e^{-jk(2\pi/T)t_0}}{T}\int_T x(\tau)e^{-jk(2\pi/T)\tau}d\tau \\ &= e^{-jk(2\pi/T)t_0}a_k \end{aligned}$$

Similarly, the Fourier series coefficients of $x(t+t_0)$ are

$$c_k = e^{jk(2\pi/T)t_0}a_k.$$

Finally, the Fourier series coefficients of $x(t-t_0) + x(t+t_0)$ are

$$d_k = b_k + c_k = e^{-jk(2\pi/T)t_0}a_k + e^{jk(2\pi/T)t_0}a_k = 2\cos(k2\pi t_0/T)a_k.$$

(b) Note that $\mathcal{E}v\{x(t)\} = [x(t) + x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(-t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_T x(\tau)e^{jk(2\pi/T)\tau}d\tau \\ &= a_{-k} \end{aligned}$$

Therefore, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}}{2}.$$

(c) Note that $\mathcal{R}e\{x(t)\} = [x(t) + x^*(t)]/2$. The FS coefficients of $x^*(t)$ are

$$b_k = \frac{1}{T}\int_T x^*(t)e^{-jk(2\pi/T)t}dt.$$

Conjugating both sides, we get

$$b_k^* = \frac{1}{T}\int_T x(t)e^{jk(2\pi/T)t}dt = a_{-k}.$$

Therefore, the FS coefficients of $\mathcal{R}e\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}^*}{2}.$$

(d) The Fourier series synthesis equation gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}.$$

Differentiating both sides wrt t twice, we get

$$\frac{d^2x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} -k^2\frac{4\pi^2}{T^2}a_k e^{j(2\pi/T)kt}.$$

By inspection, we know that the Fourier series coefficients of $d^2x(t)/dt^2$ are $-k\frac{4\pi^2}{T^2}a_k$.

(e) The period of $x(3t)$ is a third of the period of $x(t)$. Therefore, the signal $x(3t-1)$ is periodic with period $T/3$. The Fourier series coefficents of $x(3t)$ are still a_k. Using the analysis of part (a), we know that the Fourier series coefficients of $x(3t-1)$ is $e^{-jk(6\pi/T)}a_k$.

3.41. Since $a_k = a_{-k}$, we require that $x(t) = x(-t)$. Also, note that since $a_k = a_{k+2}$, we require that

$$x(t) = x(t)e^{-j(4\pi/3)t}.$$

This in turn implies that $x(t)$ may have nonzero values only for $t = 0, \pm 1.5, \pm 3, \pm 4.5, \ldots$

Since $\int_{-0.5}^{0.5} x(t) = 1$, we may conclude that $x(t) = \delta(t)$ for $-0.5 \le t \le 0.5$. Also, since $\int_{0.5}^{1.5} x(t)dt = 2$, we may conclude that $x(t) = 2\delta(t - 3/2)$ in the range $0.5 \le t \le 3/2$. Therefore, $x(t)$ may be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - k3) + 2\sum_{k=-\infty}^{\infty} \delta(t - 3k - 3/2).$$

3.42. (a) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x^*(t)$ are a^*_{-k}. Now, we know that is $x(t)$ is real, then $x(t) = x^*(t)$. Therefore, $a_k = a^*_{-k}$. Note that this implies $a_0 = a_0^*$. Therefore, a_0 must be real.

(b) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k}. If $x(t)$ is even, then $x(t) = x(-t)$. This implies that

$$a_k = a_{-k}. \qquad \text{(S3.42-1)}$$

This implies that the FS coefficients are even. From the previous part, we know that if $x(t)$ is real, then

$$a_k = a^*_{-k}. \qquad \text{(S3.42-2)}$$

Using eqs. (S3.42-1) and (S3.42-2), we know that $a_k = a_k^*$. Therefore, a_k is real for all k. Hence, we may conclude that a_k is real and even.

(c) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k}. If $x(t)$ is odd, then $x(t) = -x(-t)$. This implies that

$$a_k = -a_{-k}. \qquad \text{(S3 42-3)}$$

This implies that the FS coefficients are odd. From the previous part, we know that if $x(t)$ is real, then

$$a_k = a^*_{-k}. \qquad \text{(S3.42-4)}$$

Using eqs. (S3.42-3) and (S3.42-4), we know that $a_k = -a_k^*$. Therefore, a_k is imaginary for all k. Hence, we may conclude that a_k is real and even. Noting that eq. (S3.42-3) requires that $a_0 = -a_0$, we may also conclude that $a_0 = 0$.

(d) Note that $\mathcal{E}v\{x(t)\} = [x(t) + x(-t)]/2$. From the previous parts, we know that the FS coefficients of $\mathcal{E}v\{x(t)\}$ will be $[a_k + a_{-k}]/2$. Using eq. (S3.43-2), we may write the FS coefficients of $\mathcal{E}v\{x(t)\}$ as $[a_k + a_k^*]/2 = \mathcal{R}e\{a_k\}$.

(e) Note that $\mathcal{O}d\{x(t)\} = [x(t) - x(-t)]/2$. From the previous parts, we know that the FS coefficients of $\mathcal{O}d\{x(t)\}$ will be $[a_k - a_{-k}]/2$. Using eq. (S3.43-2), we may write the FS coefficients of $\mathcal{O}d\{x(t)\}$ as $[a_k - a_k^*]/2 = j\mathcal{I}m\{a_k\}$.

3.43. (a) (i) We have

$$x(t) = \sum_{\text{odd } k} a_k e^{jk\frac{2\pi}{T}t}.$$

Therefore,

$$x(t + T/2) = \sum_{\text{odd } k} a_k e^{jk\frac{2\pi}{T}t} e^{jk\pi}.$$

Since $e^{jk\pi} = -1$ for k odd,

$$x(t + T/2) = -x(t).$$

(ii) The Fourier series coefficients of $x(t)$ are

$$\begin{aligned} a_k &= \frac{1}{T}\int_0^{T/2} x(t)e^{-jk\omega_0 t}dt + \frac{1}{T}\int_{T/2}^{T} x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_0^{T/2} [x(t) + x(t + T/2)e^{-jk\pi}]e^{-jk\omega_0 t}dt \end{aligned}$$

Note that the right-hand side of the above equation evaluates to zero for even values of k if $x(t) = -x(t + T/2)$.

(b) The function is as shown in Figure S3.43.
Note that $T = 2$ and $\omega_0 = \pi$. Therefore,

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{1}{jk\pi} + \frac{2}{k^2\pi^2} & k \text{ odd} \end{cases}$$

(c) No. For an even harmonic signal we may follow the reasoning of part (a-i) to show that $x(t) = x(t + T/2)$. In this case, the fundamental period is $T/2$.

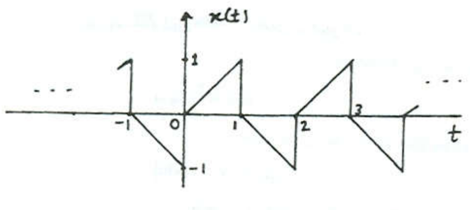

Figure S3.43

(d) (1) If a_1 or a_{-1} is nonzero, then

$$x(t) = a_{\pm 1}e^{\pm j2\pi t/T} + \cdots.$$

and

$$x(t + t_0) = a_{\pm 1}e^{\pm j\frac{2\pi}{T}(t+t_0)} + \cdots$$

The smallest value of $|t_0|$ (other than $|t_0| = 0$ for which $e^{\pm j\frac{2\pi}{T}t_0} = 1$ is the fundamental period. Only then is

$$x(t + t_0) = a_{\pm 1}e^{\pm j2\pi t/T} + \cdots = x(t).$$

Therefore, t_0 has to be the fundamental period.

(2) The period of $x(t)$ is the least common multiple of the periods of $e^{jk(2\pi/T)t}$ and $e^{jl(2\pi/T)t}$. The period of $e^{jk(2\pi/T)t}$ is T/k and the period of $e^{jl(2\pi/T)t}$ and T/l. Since k and l have no common factors, the least common multiple of T/k and T/l is T.

4. The only unknown FS coefficients are a_1, a_{-1}, a_2, and a_{-2}. Since $x(t)$ is real, $a_1 = a^*_{-1}$, and $a_2 = a^*_{-2}$. Since a_1 is real, $a_1 = a_{-1}$. Now, $x(t)$ is of the form

$$x(t) = A_1\cos(\omega_0 t) + A_2\cos(2\omega_0 t + \theta),$$

where $\omega_0 = 2\pi/6$. From this we get

$$x(t - 3) = A_1\cos(\omega_0 t - 3\omega_0) + A_2\cos(2\omega_0 t + \theta - 6\omega_0).$$

Now if we need $x(t) = -x(t - 3)$, then $3\omega_0$ and $6\omega_0$ should both be odd multiples of π. Clearly, this is impossible. Therefore, $a_2 = a_{-2} = 0$ and

$$x(t) = A_1\cos(\omega_0 t).$$

Now, using Parseval's relation on Clue 5, we get

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = \frac{1}{2}.$$

Therefore, $|a_1| = 1/2$. Since a_1 is positive, we have $a_1 = a_{-1} = 1/2$. Therefore, $x(t) = \cos(\pi t/3)$.

3.45. By inspection, we may conclude that the FS coefficients of $x(t)$ are

$$\gamma_k = \begin{cases} a_0, & k = 0 \\ B_k + jC_k, & k > 0 \\ B_k - jC_k, & k < 0 \end{cases}.$$

(a) We know from Problem 3.42 that if $x(t)$ is real, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are $\mathcal{R}e\{\gamma_k\}$. Therefore,

$$\alpha_0 = a_0, \qquad \alpha_k = B_{|k|}$$

We know from Problem 3.42 that if $x(t)$ is real, the FS coefficients of $\mathcal{O}d\{x(t)\}$ are $j\mathcal{I}m\{\gamma_k\}$. Therefore,

$$\beta_0 = 0, \qquad \beta_k = \begin{cases} jC_k, & k > 0 \\ -jC_k, & k < 0 \end{cases}$$

(b) $\alpha_k = \alpha_{-k}$ and $\beta_k = -\beta_{-k}$

(c) The signal is

$$y(t) = 1 + \mathcal{E}v\{x(t)\} + \frac{1}{2}\mathcal{E}v\{z(t)\} - \mathcal{O}d\{z(t)\}.$$

This is as shown in Figure S3.45.

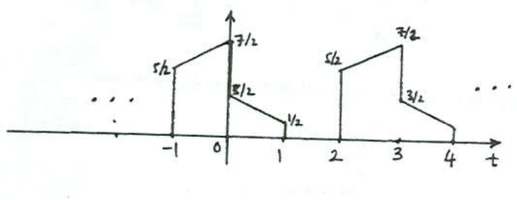

Figure S3.45

3.46. (a) The Fourier series coefficients of $x(t)$ are

$$\begin{aligned} c_k &= \frac{1}{T}\int_T \sum_n \sum_l a_n b_l e^{j(n+l)\omega_0 t}e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\sum_n \sum_l a_n b_l \delta(k - (n + l)) \\ &= \sum_n a_n b_{k-n} \end{aligned}$$

(b) (i) Here, $T_0 = 3$ and $\omega_0 = 2\pi/3$. Therefore,

$$c_k = [\frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)] * \frac{2\sin(k2\pi/3)}{3k2\pi/3}.$$

Simplifying,

$$c_k = \frac{\sin\{(k-30)2\pi/3\}}{3(k-30)2\pi/3} + \frac{\sin\{(k+30)2\pi/3\}}{3(k+30)2\pi/3}$$

and $c_{\pm 30} = 1/3$.

(ii) We may express $x_2(t)$ as

$$x_2(t) = \text{sum of two shifted square waves } \times \cos(20\pi t).$$

Here, $T_0 = 3$, $\omega_0 = 2\pi/3$. Therefore,

$$\begin{aligned} c_k &= \frac{1}{3}e^{-j(k-30)(2\pi/3)}\frac{\sin\{(k-30)2\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3}e^{-j(k+30)(2\pi/3)}\frac{\sin\{(k+30)2\pi/3\}}{(k+30)2\pi/3} \\ &+ \frac{1}{3}e^{-j(k-30)(\pi/3)}\frac{\sin\{(k-30)\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3}e^{-j(k+30)(\pi/3)}\frac{\sin\{(k+30)\pi/3\}}{(k+30)2\pi/3}\end{aligned}$$

(iii) Here, $T_0 = 4$, $\omega_0 = \pi/2$. Therefore,

$$c_k = \left[\frac{1}{2}\delta(k-40) + \frac{1}{2}\delta(k+40)\right] * \frac{j[k\omega_0 + e^{-1}\{\sin k\omega_0 - \cos k\omega_0\}]}{2[1+(k\omega_0)^2]}.$$

Simplifying,

$$\begin{aligned} c_k &= \frac{j[(k-40)\omega_0 + e^{-1}\{\sin(k-40)\omega_0 - \cos(k-40)\omega_0\}]}{4[1+\{(k-40)\omega_0\}^2]} \\ &+ \frac{j[(k+40)\omega_0 + e^{-1}\{\sin(k+40)\omega_0 - \cos(k+40)\omega_0\}]}{4[1+\{(k+40)\omega_0\}^2]}\end{aligned}$$

(c) From Problem 3.42, we know that $b_k = a^*_{-k}$. From part (a), we know that the FS coefficients of $z(t) = x(t)y(t) = x(t)x^*(t) = |x(t)|^2$ will be

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{n-k} = \sum_{n=-\infty}^{\infty} a_n a_{n+k}.$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 e^{-j(2\pi/T_0)kt}dt = \sum_{n=-\infty}^{\infty} a_n a^*_{n+k}.$$

Putting $k = 0$ in this equation, we get

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

3.48. (a) The FS coefficients of $x[n - n_0]$ are

$$\begin{aligned} \hat{a}_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]e^{-j2\pi nk/N} \\ &= \frac{1}{N}e^{-j\frac{2\pi n_0 k}{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} \\ &= e^{-j2\pi k n_0/N}a_k\end{aligned}$$

(b) Using the results of part (a), the FS coefficients of $x[n] - x[n-1]$ are given by

$$\hat{a}_k = a_k - e^{-j2\pi k/n}a_k = [1 - e^{-j2\pi k/n}]a_k.$$

(c) Using the results of part (a), the FS coefficients of $x[n] - x[n - N/2]$ are given by

$$\hat{a}_k = a_k[1 - e^{-jk\pi}] = \begin{cases} 0, & k \text{ even} \\ 2a_k, & k \text{ odd}\end{cases}.$$

(d) Note that $x[n]+x[n+N/2]$ has a period of $N/2$. The FS coefficients of $x[n]+x[n-N/2]$ are given by

$$\hat{a}_k = \frac{2}{N}\sum_{n=0}^{\frac{N}{2}-1}\left[x[n] + x[n+\frac{N}{2}]\right]e^{-j4\pi nk/N} = 2a_{2k}$$

for $0 \le k \le (N/2 - 1)$.

(e) The FS coefficients of $x^*[-n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x^*[-n]e^{-j2\pi nk/N} = a^*_k.$$

(f) With N even the FS coefficients of $(-1)^n x[n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi n/N)(k-\frac{N}{2})} = a_{k-N/2}$$

(g) With N odd, the period of $(-1)^n x[n]$ is $2N$. Therefore, the FS coefficients are

$$\hat{a}_k = \frac{1}{2N}\left[\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})} + \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})}e^{-j\pi(k-N)}\right].$$

Note that for k odd $\frac{k-N}{2}$ is an integer and $k - N$ is an even integer. Also, for k even, $k - N$ is an odd integer and $e^{-j\pi(k-N)} = -1$. Therefore,

$$\hat{a}_k = \begin{cases} a_{\frac{k-N}{2}}, & k \text{ odd} \\ 0, & k \text{ even}\end{cases}.$$

(h) Here.

$$y[n] = \frac{1}{2}[x[n] + (-1)^n x[n]].$$

For N even,

$$\hat{a}_k = \frac{1}{2}[a_k + a_{k-\frac{N}{2}}].$$

For N odd,

$$\hat{a}_(k) = \begin{cases} \frac{1}{2}[a_k + a_{\frac{k-N}{2}}], & k \text{ even} \\ \frac{1}{2}a_k, & k \text{ odd}\end{cases}.$$

3.49. (a) The FS coefficients are given by

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}} \\ &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} + \frac{1}{N}\sum_{n=N/2}^{N-1} x[n]e^{-j\frac{2\pi nk}{N}} \\ &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} + \frac{e^{-j\pi k}}{N}\sum_{n=0}^{(N/2)-1} x[n+N/2]e^{-j\frac{2\pi nk}{N}} = 0 \\ &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} - \frac{e^{-j\pi k}}{N}\sum_{n=0}^{(N/2)-1} x[n]e^{-j\frac{2\pi nk}{N}} \\ &= 0, \quad \text{for } k \text{ even.}\end{aligned}$$

(b) By adopting an approach similar to part (a), we may show that

$$\begin{aligned} a_k &= \frac{1}{N}\left[\sum_{n=0}^{\frac{N}{4}-1}\{1 - e^{-jk\pi/2} + e^{-j\pi k} - e^{-j\frac{3\pi k}{2}}\}x[n]e^{-j\frac{2\pi nk}{N}}\right] \\ &= 0, \quad \text{for } k = 4r, r \in \mathcal{I}\end{aligned}$$

(c) If N/M is an integer, we may generalize the approach of part (a) to show that

$$a_k = \frac{1}{N}\left[\sum_{k=0}^{B-1}\{1 - e^{-j2\pi r} + e^{-j4\pi r} - \cdots + e^{-j2\pi(M-1)r}\}x[n]e^{-j\frac{2\pi nk}{N}}\right].$$

where $B = N/M$ and $r = k/m$. From the above equation, it is clear that

$$a_k = 0, \quad \text{if } k = rM, r \in \mathcal{I}.$$

3.50. From Table 3.2, we know that if

$$x[n] \overset{FS}{\longleftrightarrow} a_k,$$

then

$$(-1)^n x[n] = e^{j(2\pi/N)(N/2)n}x[n] \overset{FS}{\longleftrightarrow} a_{k-N/2}.$$

In this case, $N = 8$. Therefore,

$$(-1)^n x[n] \overset{FS}{\longleftrightarrow} a_{k-4}.$$

Since it is given that $a_k = -a_{k-4}$, we have

$$x[n] = -(-1)^n x[n].$$

This implies that $x[0] = x[\pm 2] = x[\pm 4] = \cdots = 0$.

We are also given that $x[1] = x[5] = \cdots = 1$ and $x[3] = x[7] = -1$. Therefore, one period of $x[n]$ is as shown in Figure S3.50.

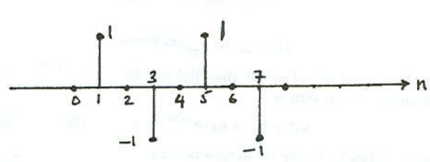

Figure S3.50

3.51. We have

$$e^{j4(2\pi/8)n}x[n] = e^{j\pi n}x[n] = (-1)^n x[n] \overset{FS}{\longleftrightarrow} a_{k-4}$$

and therefore,

$$(-1)^{n+1}x[n] \overset{FS}{\longleftrightarrow} -a_{k-4}.$$

If $a_k = -a_{k-4}$, then $x[0] = x[\pm 2] = x[\pm 4] = \cdots = 0$. Now, note that in the signal $p[n] = x[n-1]$, $p[\pm 1] = p[\pm 3] = \cdots = 0$. Now let us plot the signal $z[n] = (1 + (-1)^n)/2$. This is as shown in Figure S3.51.

Clearly, the signal $y[n] = z[n]p[n] = p[n]$ because $p[n]$ is zero whenever $z[n]$ is zero. Therefore, $y[n] = x[n-1]$. The FS coefficients of $y[n]$ are $a_k e^{-j(2\pi/8)}$.

3.52. (a) If $x[n]$ is real, $x[n] = x^*[n]$. Therefore,

$$a_{-k} = \sum_n x[n]e^{j2\pi nk/N} = a^*_k.$$

From this result, we get $b_{-k} = b_k$ and $c_{-k} = -c_k$.

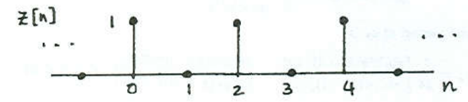

Figure S3.51

(b) If N is even, then

$$a_{N/2} = \frac{1}{N}\sum_n x[n]e^{-j\pi n} = \frac{1}{N}\sum_n (-1)^n x[n] = \text{real}.$$

(c) If N is odd, then

$$\begin{aligned}
x[n] &= \sum_{k=-(N-1)/2}^{(N-1)/2} a_k e^{j(2\pi/N)kn} \\
&= \sum_{k=0}^{(N-1)/2} a_k e^{j(2\pi/N)kn} + \sum_{k=1}^{(N-1)/2} a_k^* e^{-j(2\pi/N)kn} \quad \text{(From (a))} \\
&= a_0 + \sum_{k=1}^{(N-1)/2} (b_k + jc_k)e^{j(2\pi/N)kn} \sum_{k=1}^{(N-1)/2} (b_k - jc_k)e^{-j(2\pi/N)kn} \\
&= a_0 + 2\sum_{k=1}^{(N-1)/2} b_k \cos(2\pi kn/N) - c_k \sin(2\pi kn/N).
\end{aligned}$$

If N is even, then

$$\begin{aligned}
x[n] &= \sum_{k=0}^{N-1} a_k e^{j(2\pi/N)kn} \\
&= a_0 + (-1)^n a_{N/2} + 2\sum_{k=1}^{(N-2)/2} a_k e^{j(2\pi/N)kn} + a_{N-k}e^{j(2\pi/N)(N-k)n} \\
&= a_0 + (-1)^n a_{N/2} + 2\sum_{k=1}^{(N-2)/2} a_k e^{j(2\pi/N)kn} - a_k^* e^{-j(2\pi/N)kn} \quad \text{(From (a))} \\
&= a_0 + (-1)^n a_{N/2} + 2\sum_{k=1}^{(N-1)/2} b_k \cos(2\pi kn/N) - c_k \sin(2\pi kn/N).
\end{aligned}$$

(d) If $a_k = A_k e^{j\theta_k}$, then $b_k = A\cos(\theta_k)$ and $c_k = A\sin(\theta_k)$. Substituting in the result of the previous part, we get for N odd:

$$\begin{aligned}
x[n] &= a_0 + 2\sum_{k=1}^{(N-1)/2} A\cos(\theta_k)\cos(2\pi kn/N) - c_k \sin(\theta_k)\sin(2\pi kn/N) \\
&= a_0 + 2\sum_{k=1}^{(N-1)/2} A_k \cos\{\frac{2\pi nk}{N} + \theta_k\}.
\end{aligned}$$

Similarly, for N even,

$$\begin{aligned}
x[n] &= a_0 + (-1)^n a_{N/2} + 2\sum_{k=1}^{(N-1)/2} A\cos(\theta k)\cos(2\pi kn/N) - c_k \sin(\theta k)\sin(2\pi kn/N) \\
&= a_0 + (-1)^n a_{N/2} + 2\sum_{1}^{(N-2)/2} A_k \cos\{\frac{2\pi nk}{N} + \theta_k\}.
\end{aligned}$$

(e) The signal is:

$$y[n] = d.c\{x[n]\} - d.c.\{z[n]\} + \mathcal{E}v\{z\} + \mathcal{O}d\{x\} - 2\mathcal{O}d\{z\}.$$

This is as shown Figure S3.52.

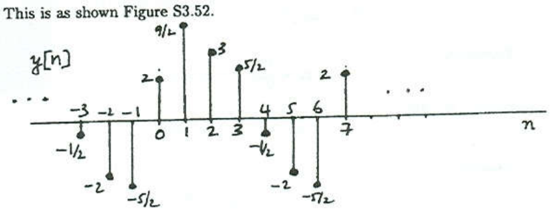

Figure S3.52

3.53. We have

$$a_k = \frac{1}{N}\sum_{<N>} x[n]e^{-j(2\pi/N)kn}.$$

Note that

$$a_0 = \frac{1}{N}\sum_{<N>} x[n]$$

which is real if $x[n]$ is real.

(a) If N is even, then

$$a_{N/2} = \frac{1}{N}\sum_{<N>} x[n]e^{-j\pi n} = \frac{1}{N}\sum_{<N>} x[n](-1)^n.$$

Clearly, $a_{N/2}$ is also real if $x[n]$ is real.

(b) If N is odd, only a_0 is guaranteed to be real.

3.54. (a) Let $k = pN$, $p \in \mathcal{I}$. Then,

$$a[pN] = \sum_{n=0}^{N-1} e^{j(2\pi/N)pNn} = \sum_{n=0}^{N-1} e^{j2\pi pn} = \sum_{n=0}^{N-1} 1 = N.$$

(b) Using the finite sum formula, we have

$$a[k] = \frac{1 - e^{j2\pi k}}{1 - e^{j(2\pi/N)k}} = 0, \quad \text{if } k \neq pN, p \in \mathcal{I}.$$

(c) Let

$$a[k] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)kn},$$

where q is some arbitrary integer. By putting $k = pN$, we may again easily show that

$$a[pN] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)pNn} = \sum_{n=q}^{q+N-1} e^{j2\pi pn} = \sum_{n=q}^{q+N-1} 1 = N.$$

Now,

$$a[k] = e^{j(2\pi/N)kq}\sum_{n=0}^{N-1} e^{j(2\pi/N)kn}.$$

Using part (b), we may argue that $a[k] = 0$ for $k \neq pN$, $p \in \mathcal{I}$.

3.55. (a) Note that

$$x_m[n+mN] = \begin{cases} x[\frac{n}{m} + N], & n = 0, \pm m, \cdots \\ 0, & \text{otherwise} \end{cases} = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \cdots \\ 0, & \text{otherwise} \end{cases} = x_m[n].$$

Therefore, $x_{(m)}[n]$ is periodic with period mN.

(b) The time-scaling operation discussed in this problem is a linear operation. Therefore, if $x[n] = v[n] + w[n]$, then, $x_m[n] = v_m[n] + w_m[n]$.

(c) Let us consider

$$y[n] = \frac{1}{m}\sum_{l=0}^{m-1} e^{j(2\pi/mN)(k_0+lN)n} = \frac{1}{m}e^{j(2\pi/mN)k_0 n}\sum_{l=0}^{m-1} e^{j(2\pi/m)ln}.$$

This may be written as [From Problem 3.54]

$$y[n] = \begin{cases} e^{j(2\pi/mN)k_0 n}, & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(S3.55-1)}$$

Now, also note that by applying time-scaling on $x[n]$, we get

$$x_{(m)}[n] = \begin{cases} e^{j(2\pi/mN)k_0 n}, & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(S3.55-2)}$$

Comparing eqs. (S3.55-1) and (S3.55-2), we see that $y[n] = x_{(m)}[n]$.

(d) We have

$$b_k = \frac{1}{mN}\sum_{n=0}^{mN-1} x_{(m)}[n]e^{-j(2\pi/mN)kn}.$$

We know that only every mth value in the above summation is nonzero. Therefore,

$$\begin{aligned}
b_k &= \frac{1}{mN}\sum_{n=0}^{N-1} x_{(m)}[nm]e^{-j(2\pi/mN)kmn} \\
&= \frac{1}{mN}\sum_{n=0}^{N-1} x_{(m)}[nm]e^{-j(2\pi/N)kn}
\end{aligned}$$

Note that $x_{(m)}[nM] = x[n]$. Therefore,

$$b_k = \frac{1}{mN}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn} = \frac{a_k}{m}.$$

3.56. (a) We have

$$x[n] \overset{FS}{\longleftrightarrow} a_k \quad \text{and} \quad x^*[n] \overset{FS}{\longleftrightarrow} a^*_{-k}.$$

Using the multiplication property,

$$x[n]x^*[n] = |x[n]|^2 \overset{FS}{\longleftrightarrow} \sum_{l=<N>} a_l a^*_{l+k}.$$

(b) From above, it is clear that the answer is yes.

3.57. (a) We have

$$x[n]y[n] = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_k b_l e^{j(2\pi/N)(k+l)n}.$$

Putting $l' = k + l$, we get

$$x[n]y[n] = \sum_{k=0}^{(N-1)}\sum_{l'=k}^{(k+N-1)} a_k b_{l'-k} e^{j(2\pi/N)l'n}.$$

But since both $b_{l'-k}$ and $e^{j(2\pi/N)l'n}$ are periodic with period N, we may rewrite this as

$$x[n]y[n] = \sum_{k=0}^{N-1}\sum_{l'=0}^{N-1} a_k b_{l'-k} e^{j(2\pi/N)l'n} = \sum_{l=0}^{N-1}\left[\sum_{k=0}^{N-1} a_k b_{l-k}\right] e^{j(2\pi/N)ln}.$$

Therefore,

$$c_k = \sum_{k=0}^{N-1} a_k b_{l-k}.$$

By interchanging a_k and b_k, we may show that

$$c_k = \sum_{k=0}^{N-1} b_k a_{l-k}.$$

(b) Note that since both a_k and b_k are peroidic with period N, we may rewrite the above summation as

$$c_k = \sum_{<N>} a_k b_{l-k} = \sum_{<N>} b_k a_{l-k}.$$

(c) (i) Here,

$$c_k = \sum_{l=0}^{N-1} \frac{1}{2}[\delta[l-3] + \delta[l-N+3]] a_{k-l}.$$

Therefore,

$$c_k = \frac{1}{2}a_{k-3} + \frac{1}{2}a_{k+3-N}.$$

(ii) Period=N. Also,

$$b_k = \frac{1}{N}, \quad \text{for all } k.$$

Therefore,

$$c_k = \frac{1}{N}\sum_{l=0}^{N-1} a_l.$$

(iii) Here,

$$b_k = \frac{1}{N}[1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}].$$

Therefore,

$$c_k = \frac{1}{N}\sum_{l=0}^{N-1}[1 + e^{-j2\pi l/3} + e^{-j4\pi l/3}] a_{k-l}.$$

(d) Period=12. Also,

$$x[n] \overset{FS}{\longleftrightarrow} a_2 = a_{10} = 1/2, \text{ All other } a_k = 0, \qquad 0 \le k \le 11$$

and

$$y[n] \overset{FS}{\longleftrightarrow} b_k = (\frac{1}{12})\frac{\sin 7\pi k/12}{\sin \pi k/12}, \qquad 0 \le k \le 11.$$

Therefore one period of c_k is,

$$c_k = \frac{1}{24}\left[\frac{\sin\{7\pi(k-2)/12\}}{\sin\{\pi(k-2)/12\}} + \frac{\sin\{7\pi(k-10)/12\}}{\sin\{\pi(k-10)/12\}}\right], \ 0 \le k \le 11$$

(e) Using the FS analysis equation, we have

$$N \sum_{l=<N>} a_l b_{k-l} = \sum_{<N>} x[n]y[n]e^{-j(2\pi/N)kn}.$$

Putting $k = 0$ in this, we get

$$N \sum_{l=<N>} a_l b_{-l} = \sum_{<N>} x[n]y[n].$$

Now let $y[n] = x^*[n]$. Then $b_l = a^*_{-l}$. Therefore,

$$N \sum_{l=<N>} a_l a_l^* = \sum_{<N>} x[n]x^*[n].$$

Therefore,

$$N \sum_{l=<N>} |a_l|^2 = \sum_{<N>} |x[n]|^2.$$

3.58. (a) We have

$$z[n+N] = \sum_{<L>} x[r]y[n+N-r].$$

Since $y[n]$ is periodic with period N, $y[n+N-r] = y[n-r]$. Therefore,

$$z[n+N] = \sum_{<L>} x[r]y[n-r] = z[n].$$

Therefore, $z[n]$ is also periodic with period N.

(b) The FS coefficients of $z[n]$ are

$$\begin{aligned} c_l &= \frac{1}{N}\sum_{n=<N>}\sum_{k=<N>} a_k b_{n-k} e^{-j2\pi nl/N} \\ &= \frac{1}{N}\sum_{k=<N>} a_k e^{-j2\pi kl/N} \sum_{n=<N>} b_{n-k} e^{-j2\pi(n-k)l/N} \\ &= \frac{1}{N} N a_l N b_l \\ &= N a_l b_l. \end{aligned}$$

(c) Here, $n = 8$. The nonzero FS coefficients in the range $0 \le k \le 7$ for $x[n]$ are $a_3 = a_5^* = 1/2j$. Note that for $y[n]$, we need only evaluate b_3 and b_5. We have

$$b_3 = b_5^* = \frac{1}{4(1 - e^{-j3\pi/4})}.$$

Therefore, the only nonzero FS coefficients in the range $0 \le k \le 7$ for the periodic convolution of these signals are $c_3 = 8a_3b_3$ and $c_5 = 8a_5b_5$.

(d) Here,

$$x[n] \overset{FS}{\longleftrightarrow} a_k = \frac{1}{16j}\left[\frac{1 - e^{j(3\pi/7 - \pi k/4)4}}{1 - e^{-j(3\pi/7-\pi k/4)}} - \frac{1 - e^{j(3\pi/7+\pi k/4)4}}{1 - e^{-j(3\pi/7+\pi k/4)}}\right]$$

and

$$y[n] \overset{FS}{\longleftrightarrow} b_k = \frac{1}{8}\left[\frac{1 - (1/2)^8}{1 - (1/2)e^{-jk\pi/4}}\right].$$

Therefore,

$$z[n] = x[n]y[n] \overset{FS}{\longleftrightarrow} 8a_kb_k.$$

3.59. (a) Note that the signal $x(t)$ is periodic with period NT. The FS coefficients of $x(t)$ are

$$a_k = \frac{1}{NT}\int_0^{NT}\left[\sum_{p=-\infty}^{\infty} x[p]\delta(t - pT)\right] e^{-j(2\pi/NT)kt}dt.$$

Note that the limits of the summation may be changed in accordance with the limits of the integration so that we get

$$a_k = \frac{1}{NT}\int_0^{NT}\left[\sum_{p=0}^{N-1} x[p]\delta(t - pT)\right] e^{-j(2\pi/NT)kt}dt.$$

Interchanging the summation and the integration and simplifying

$$\begin{aligned} a_k &= (1/NT)\sum_{p=0}^{N-1} x[p]\int_0^{NT}\delta(t - pT)e^{-j(2\pi/NT)kt}dt \\ &= (1/NT)\sum_{p=0}^{N-1} x[p]e^{-j(2\pi/N)pk} \\ &= (1/T)\left[(1/N)\sum_{p=0}^{N-1} x[p]e^{-j(2\pi/N)pk}\right]. \end{aligned}$$

Note that the term within brackets on the RHS of the above equation constitutes the FS coefficients of the signal $x[n]$. Since, this is periodic with period N, a_k must also be periodic with period N.

(b) If the FS coefficients of $x(t)$ are periodic with period N, then

$$a_k = a_{k-N}.$$

This implies that

$$x(t) = x(t)e^{j(2\pi/T)Nt}.$$

This is possible only if $x(t)$ is zero for all t other than when $(2\pi/T)Nt = 2\pi k$, where $k \in \mathcal{I}$. Therefore, $x(t)$ is of the form

$$x(t) = \sum_{k=-\infty}^{\infty} g[k]\delta(t - kT/N).$$

(c) A simple example would be $x(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT)$.

3.60. (a) The system is not LTI. $(1/2)^n$ is an eigen function of LTI systems. Therefore, the output should have been of the form $K(1/2)^n$, where K is a complex constant.

(b) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = (1-(1/2)e^{-j\omega})/(1-(1/4)e^{-j\omega})$. The system is unique.

(c) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = (1 - (1/2)e^{-j\omega})/(1 - (1/4)e^{j\omega})$. The system is unique.

(d) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j/8}) = 2$.

(e) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = 2$. The system is unique.

(f) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j\pi/2}) = 2(1 - e^{j\pi/2})$.

(g) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j\pi/3}) = 1 - j\sqrt{3}$.

(h) Note that $x[n]$ and $y_1[n]$ are periodic with the same fundamental frequency. Therefore, it is possible to find an LTI system with this input-output relationship without violating the Eigen function property. The system is not unique because $H(e^{j\omega})$ needs to be have specific values only for $H(e^{j(2\pi/12)k})$. The rest of $H(e^{j\omega})$ may be chosen arbitrarily.

(i) Note that $x[n]$ and $y_1[n]$ are not periodic with the same fundamental frequency. Furthermore, note that $y_2[n]$ has 2/3 the period of $x[n]$. Therefore, $y[n]$ will be made up of complex exponentials which are not present in $x[n]$. This violates the eigen function property of LTI systems. Therefore, the system cannot be LTI.

3.61. (a) For this system,

$$x(t) \to \boxed{\delta(t)} \to x(t).$$

Therefore, all functions are eigenfunctions with an eigenvalue of one.

(b) The following is an eigen function with an eigen value of 1:

$$x(t) = \sum_k \delta(t - kT).$$

The following is an eigen function with an eigen value of 1/2:

$$x(t) = \sum_k (\frac{1}{2})^k \delta(t - kT).$$

The following is an eigen function with an eigen value of 2:

$$x(t) = \sum_k (2)^k \delta(t - kT).$$

(c) If $h(t)$ is real and even then $H(\omega)$ is real and even.

$$e^{j\omega t} \rightarrow \boxed{H(j\omega)} \rightarrow H(j\omega)e^{j\omega t}$$

and

$$e^{-j\omega t} \rightarrow \boxed{H(j\omega)} \rightarrow H(-j\omega)e^{-j\omega t} = H(j\omega)e^{-j\omega t}.$$

From these two statements, we may argue that

$$\cos(\omega t) = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}] \rightarrow \boxed{H(j\omega)} \rightarrow H(j\omega)\cos(\omega t).$$

Therefore, $\cos(\omega t)$ is an eigenfunction. We may similarly show hat $\sin(\omega t)$ is an eigenfunction.

(d) We have

$$\phi(t) \rightarrow \boxed{u(t)} \rightarrow \lambda\phi(t).$$

Therefore,

$$\lambda\phi(t) = \int_{-\infty}^{t} \phi(\tau)d\tau.$$

Differentiating both sides wrt t, we get

$$\lambda\phi'(t) = \phi(t).$$

Let $\phi(0) = \phi_0$. Then

$$\phi(t) = \phi_0 e^{t/\lambda}.$$

3.62. (a) The fundamental period of the input is $T = 2\pi$. The fundamental period of the input is $T = \pi$. The signals are as shown in Figure S3.62.

(b) The Fourier series coeffients of the output are

$$b_k = \frac{2(-1)^k}{\pi(1 - 4k^2)}.$$

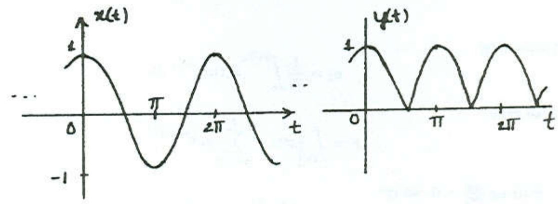

Figure S3.62

(c) The dc component of the input is 0. The dc component of the output is $2/\pi$.

3.63. The average energy per period is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_k |\alpha_k|^2 = \sum_k \alpha^{2|k|} = \frac{1+\alpha^2}{1-\alpha^2}.$$

We want N such that

$$\sum_{-N+1}^{N-1} |\alpha_k|^2 = 0.9\frac{1+\alpha^2}{1-\alpha^2}.$$

This implies that

$$\frac{1 - 2\alpha^{2N} + 2\alpha^2}{1-\alpha^2} = \frac{1+\alpha^2}{1-\alpha^2}.$$

Solving,

$$N = \frac{\log[1.45\alpha^2 + 0.95]}{2\log\alpha},$$

and

$$\frac{\pi N}{4} < W < \frac{(N-1)\pi}{4}.$$

3.64. (a) Due to linearity, we have

$$y(t) = \sum_k c_k \lambda_k \phi_k(t).$$

(b) Let

$$x_1(t) \longrightarrow y_1(t) \quad \text{and} \quad x_2(t) \longrightarrow y_2(t).$$

Also, let

$$x_3(t) = ax_1(t) + bx_2(t) \longrightarrow y_3(t).$$

Then,

$$\begin{aligned} y_3(t) &= t^2[ax_1''(t) + bx_2''(t)] + t[ax_1'(t) + bx_2'(t)] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, the system is linear.

Now consider

$$x_4(t) = x(t - t_0) \rightarrow y_4(t).$$

We have

$$y_4(t) = t^2\frac{d^2x(t-t_0)}{dt^2} + t\frac{dx(t-t_0)}{dt} \neq y(t - t_0)$$

Therefore, the system is not time invariant.

(c) For inputs of the form $\phi_k(t) = t^k$, the output is

$$y(t) = k^2 t^k = k^2\phi_k(t).$$

Therefore, $\phi_k(t)$ are eigenfunctions with eigenvalue $\lambda_k = k^2$.

(d) The output is

$$y(t) = 10^3 t^{-10} + 3t + 8t^4.$$

3.65. (a) Pairs (a) and (b) are orthogonal. Pairs (c) and (d) are not orthogonal.

(b) Orthogonal, but not orthonormal. $A_m = 1/\omega_0$.

(c) Orthonormal.

(d) We have

$$\int_{t_0}^{t_0+T} e^{jm\omega_0\tau}e^{-jn\omega_0\tau}d\tau = e^{j(m-n)\omega_0 t_0}\frac{[e^{j(m-n)2\pi} - 1]}{(m-n)\omega_0}$$

This evaluates to 0 when $m \neq n$ and to jT when $m = n$. Therefore, the functions are orthogonal but not orthonormal.

(e) We have

$$\begin{aligned} \int_{-T}^{T} x_e(t)x_o(t)dt &= \frac{1}{4}\int_{-T}^{T}[x(t) + x(-t)][x(t) - x(-t)]dt \\ &= \frac{1}{4}\int_{-T}^{T} x^2(t)dt - \frac{1}{4}\int_{-T}^{T} x^2(-t)dt \\ &= 0. \end{aligned}$$

(f) Consider

$$\int_a^b \frac{1}{\sqrt{A_k}}\phi_k(t)\frac{1}{\sqrt{A_l}}\phi_l^*(t)dt = \frac{1}{\sqrt{A_k A_l}}\int_a^b\int_a^b \phi_k(t)\phi_l^*(t)dt.$$

This valuates to zero for $k \neq l$. For $k = l$, it evaluates to $A_k/A_k = 1$. Therefore, the functions are orthonormal.

(g) We have

$$\begin{aligned} \int_a^b |x(t)|^2 dt &= \int_a^b x(t)x^*(t)dt \\ &= \int_a^b \sum_i a_i\phi_i(t)\sum_j a_j\phi_j^*(t)dt \\ &= \sum_i\sum_j a_i a_j^* \int_a^b \phi_i(t)\phi_j^*(t)dt \\ &= \sum_i |a_i|^2. \end{aligned}$$

(h) We have

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty} h_i(T-\tau)\phi_j(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \phi_i(\tau)\phi_j(\tau)d\tau \\ &= \delta_{ij} = 1 \text{ for } i = j \text{ and } 0 \text{ for } i \neq j. \end{aligned}$$

3.66. (a) We have

$$E = \int_a^b \left[x(t) - \sum_{k=-N}^{N} a_k\phi_k(t)\right]\left[x^*(t) - \sum_{k=-N}^{N} a_k^*\phi_k^*(t)\right]dt$$

Now, let $a_i = b_i + jc_i$. Then

$$\frac{\partial E}{\partial b_i} = 0 = -\int_a^b \phi_i^*(t)x(t)dt + 2b_i - \int_a^b \phi_i(t)x^t(t)dt$$

and

$$\frac{\partial E}{\partial c_i} = 0 = j\int_a^b \phi_i(t)x^*(t)dt + 2c_i - j\int_a^b \phi_i^*(t)x(t)dt.$$

Mutliplying the last equation by j and adding to the one before, we get

$$2b_i + 2jc_i = 2\int_a^b x(t)\phi^*(t)dt.$$

This implies that

$$a_i = \int_a^b x(t)\phi^*(t)dt.$$

(b) In this case, a_i would be

$$a_i = \frac{1}{A_i}\int_a^b x(t)\phi_i^*(t)dt.$$

(c) Choosing

$$a_k = \frac{1}{T_0}\int_b^{b+T_0} x(t)e^{-jk\omega_0 t}dt,$$

we have

$$E = \int_{T_0}\left|x(t) - \sum_{k=-N}^{N} a_k e^{j\omega_0 kt}\right|^2 dt.$$

Putting $\frac{\partial E}{\partial a_k} = 0$, we get

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt.$$

(d) $a_0 = 2/\pi$, $a_1 = a_3 = 0$, $a_2 = 2(1-2\sqrt{2})/\pi$, $a_4 = (1/\pi)[2 - 4\cos(\pi/8) + 4\cos(3\pi/8)]$.

(e) We have

$$\begin{aligned}\int_0^1 \sum_i (a_i\phi_i(t))^*[x(t) - \sum_i a_i\phi_i(t)]dt &= \sum_i a_i^* \int_0^1 x(t)\phi_i^*(t)dt \\ &\quad -\sum_i\sum_j a_i^* a_j \int_0^1 \phi_i^*(t)\phi_j(t)dt \\ &= \sum_i a_i^* a_i - \sum_i a_i^* a_i = 0\end{aligned}$$

(f) Not orthogonal. Example: $\int_0^1 \phi_0(t)\phi_1(t) = \int_0^1 tdt = 1 \neq 0$.

(g) Here,

$$a_0 = \int_0^1 e^t\phi_0^*(t)dt = e - 1.$$

(h) Here, $\hat{x}(t) = a_0 + a_1 t$. Therefore,

$$E = \int_0^1 (e^t - a_0 - a_1 t)(e^t - a_0 - a_1 t)dt.$$

Setting $\partial E/\partial a_0 = 0 = \partial E/\partial a_1$, we get $a_0 = 2(2e-5)$ and $a_1 = 6(3-e)$.

3.67. (a) From eq. (P3.67-1) and (P3.67-4), we get

$$\sum_{n=-\infty}^{\infty} j2\pi n b_n(x)e^{j2\pi nt} = \frac{1}{2}k^2 \sum_{n=-\infty}^{\infty} \frac{\partial^2 b_n(x)}{\partial x^2}e^{j2\pi nt}.$$

Equating coefficients of $e^{j2\pi nt}$ on both sides, we get

$$\frac{\partial^2 b_n(x)}{\partial x^2} = \frac{j4\pi n}{k^2}b_n(x).$$

(b) Since $s^2 = 4\pi jn/k^2$,

$$s = \pm\frac{2\sqrt{\pi n}e^{j\pi/4}}{k}.$$

For $n > 0$,

$$s = \frac{\sqrt{2\pi n}(1+j)}{k}$$

is a stable solution. For $n < 0$,

$$s = -\frac{\sqrt{2\pi|n|}(1-j)}{k}$$

is a stable solution. Also, $b_n(0) = a_n$ and

$$b_n(x) = \begin{cases} a_n e^{-\sqrt{2\pi n}(1+j)x/k}, & n > 0 \\ a_n e^{-\sqrt{2\pi|n|}(1-j)x/k}, & n < 0 \end{cases}.$$

(c) $b_0 = 2$. $b_1 = (1/2j)e^{-(1+j)\pi}$, $b_{-1} = -(1/2j)e^{-(1-j)\pi}$.

$$T(k\sqrt{\pi/2}, t) = 2 + e^{-\pi}\sin(2\pi t - \pi).$$

Phase reversed.

3.68. (a) $x(\theta) = r(\theta)\cos(\theta) = \frac{1}{2}r(\theta)e^{j\theta} + \frac{1}{2}r(\theta)e^{-j\theta}$. If

$$x(\theta) = \sum_{k=-\infty}^{\infty} b_k e^{jk\theta},$$

then $b_k = (1/2)a_{k+1} + (1/2)a_{k-1}$.

(b) $x(\theta) \xleftrightarrow{FS} b_k$. Then $x(\theta) = r(\theta + \pi/4)$. The sketch is as shown in Figure S3.68.

(c) $b_0 = a_0$. Rest of b_k is all zero. Therefore, the sketch will be a circle of radius a_0 as shown in Figure S3.68.

(d) (i) $r(\theta) = r(-\theta)$. Even. Sketch as shown in Figure S3.68.

(ii) $r(\theta + k\pi) = r(\theta)$. Sketch as shown in Figure S3.68.

(iii) $r(\theta + k\pi/2) = r(\theta)$. Sketch as shown in Figure S3.68.

3.69. (a) $\sum_{n=-N}^{N} \phi_k[n]\phi_k^*[m] = \sum_{n=-N}^{N} \delta[n-k]\delta[n-m]$. This is 1 for $k = m$ and 0 for $k \neq m$. Therefore, orthogonal.

(b) We have

$$\sum_{n=r}^{r+N-1} \phi_k[n]\phi_m^*[n] = e^{j(2\pi/N)r(k-m)}\left[\frac{1 - e^{j2\pi(k-m)}}{1 - e^{j(2\pi/N)(k-m)}}\right] = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases}.$$

Therefore, orthogonal.

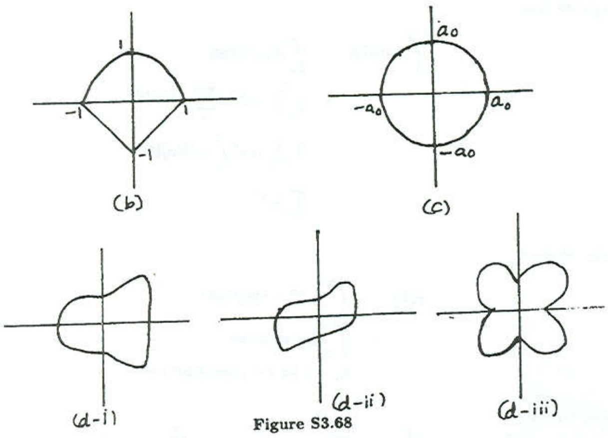

Figure S3.68

(c) We have

$$\begin{aligned}\sum_{n=N_1}^{N_2} |x[n]|^2 &= \sum_{n=N_1}^{N_2}\sum_{i=1}^{M} a_i\phi_i[n]\sum_{k=1}^{M} a_k^*\phi_k^*[n] \\ &= \sum_{k=1}^{M}\sum_{i=1}^{M} a_i a_k^* \sum_{n=N_1}^{N_2} \phi_k^*[n]\phi_i[n] \\ &= \sum_{k=1}^{M}\sum_{i=1}^{M} a_i a_k^* A_i\delta[i-k] = \sum_{i=1}^{M}|a_i|^2 A_i\end{aligned}$$

(d) Let $a_i = b_i + jc_i$. Then

$$\begin{aligned}E &= \sum_{n=N_1}^{N_2} |x[n]|^2 + \sum_{i=1}^{M}(b_i^2 + c_i^2)A_i - \sum_{n=N_1}^{N_2} x[n]\sum_{i=1}^{M}(b_i - jc_i)\phi_i^*[n] \\ &\quad - \sum_{n=N_1}^{N_2} x^*[n]\sum_{i=1}^{M}(b_i + jc_i)\phi_i[n]\end{aligned}$$

Set $\partial E/\partial b_i = 0$. Then

$$b_i = [2A_i]^{-1}\left[\sum_{n=N_1}^{N_2}\{x[n]\phi_i^*[n] + x^*[n]\phi_i[n]\}\right] = \frac{1}{A_i}\mathcal{R}e\left\{\sum_{n=N_1}^{N_2} x[n]\phi_i^*[n]\right\}.$$

Similarly,

$$c_i = \frac{1}{A_i}\mathcal{I}m\left\{\sum_{n=N_1}^{N_2} x[n]\phi_i^*[n]\right\}.$$

Therefore,

$$a_i = b_i + jc_i = \frac{1}{A_i}\sum_{n=N_1}^{N_2} x[n]\phi_i^*[n].$$

(e) $\phi_i[n] = \delta[n-i]$. Then,

$$a_i = \sum_{n=N_1}^{N_2} x[n]\delta[n-i] = x[i].$$

3.70. (a) We get

$$a_{mn} = \frac{1}{T_1 T_2}\int_0^{T_1}\int_0^{T_2} x(t_1, t_2)e^{-jm\omega_1 t_1}e^{-jn\omega_2 t_2}dt_1 dt_2.$$

(b) (i) $T_1 = 1$, $T_2 = \pi$. $a_{11} = 1/2$, $a_{-1,-1} = 1/2$. Rest of the coefficients are all zero.

(ii) Here,

$$a_{mn} = \begin{cases} 1/(\pi^2 mn), & m, n \text{ odd} \\ 0, & \text{otherwise} \end{cases}.$$

3.71. (a) The differential equation $f_s(t)$ and $f(t)$ is

$$\frac{B}{K}\frac{df_s(t)}{dt} + f_s(t) = f(t).$$

The frequency response of this system may be easily shown to be

$$H(j\omega) = \frac{1}{1 + (B/K)j\omega}.$$

Note that for $\omega = 0$, $H(j\omega) = 1$ and for $\omega \to \infty$, $H(j\omega) = 0$. Therefore, the system approximates a lowpass filter.

(b) The differential equation $f_d(t)$ and $f(t)$ is

$$\frac{df_d(t)}{dt} + \frac{K}{B}f_d(t) = \frac{df(t)}{dt}.$$

The frequency response of this system may be easily shown to be

$$H(j\omega) = \frac{j\omega}{j\omega + (K/B)}.$$

Note that for $\omega = 0$, $H(j\omega) = 0$ and for $\omega \to \infty$, $H(j\omega) = 1$. Therefore, the system approoximates a highpass filter.