9.64. For the circuit, we know that the differential equation relating the input $x(t)$ and output $y(t)$ is

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t).$$

Taking the Laplace transform of both sides and simplifying, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}.$$

(a) Note that the poles of $H(s)$ are at

$$\frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2}.$$

If R, L, and C are always positive, then the poles are always in the left half of the s-plane (because the real part of the numerator of the above equation is always negative). Since the system is causal, the ROC is to the right of the right-most pole. Therefore the ROC includes the $j\omega$-axis and the system is stable.

(b) From $H(s)$ we obtain

$$\begin{aligned} H(s)H(-s) &= \frac{1}{L^2C^2s^4 + (RLC^2 - RLC^2)s^3 + (2LC - R^2C^2)s^2 + (RC - RC)s + 1} \\ &= \frac{1}{L^2C^2s^4 + (2LC - R^2C^2)s^2 + 1}. \end{aligned}$$

For this to represent a second order Butterworth filter, we require

$$2LC - R^2C^2 = 0 \quad \Rightarrow \quad R = 2\sqrt{\frac{L}{C}}.$$

9.65. (a) The differential equation relating $v_i(t)$ and $v_o(t)$ may be obtained by putting $x(t) = v_i(t)$ and $y(t) = v_o(t)$ in the differential equation given in the previous problem. Therefore,

$$LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

or

$$\frac{d^2v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$s^2\mathcal{V}_o(s) - s\mathcal{V}_o(0^-) - \mathcal{V}_o'(0^-) + \frac{R}{L}s\mathcal{V}_o(s) - \mathcal{V}_o(0^-) + \frac{1}{LC}\mathcal{V}_o(s) = \frac{1}{LC}\mathcal{V}_i(s) \qquad \text{(S9.65-1)}$$

Now, since $v_i(t) = e^{-3t}u(t)$,

$$\mathcal{V}_i(s) = \frac{1}{s+3}, \qquad \mathcal{R}e\{s\} > -3.$$

Substituting this along with the values of R, L, and C in eq. (S9.65-1), we get

$$\mathcal{V}_o(S) = \frac{2(s^2 + 5s + 7)}{(s+1)(s+2)(s+3)}.$$

The partial fraction expansion of $\mathcal{V}_o(s)$ is

$$\mathcal{V}_o(s) = \frac{3}{s+1} - \frac{2}{s+2} + \frac{1}{s+3}.$$

Taking the inverse Laplace transform, we get

$$v_o(t) = 3e^{-t}u(t) - 2e^{-2t}u(t) + e^{-3t}u(t).$$

9.66. (a) The differential equation relating $i(t)$ and v_2 is

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{v_2}{L}u(t).$$

Also, $i(0^-) = v_1/R$.

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$sI(s) - i(0^-) + \frac{R}{L}I(s) = \frac{v_2}{Ls}.$$

(i) This corresponds to the zero state response of the circuit. Here,

$$I(s) = \frac{v_2}{s(s+1)} = v_2\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

Therefore,

$$i(t) = 2u(t) - 2e^{-t}u(t).$$

(ii) This corresponds to the zero state response of the circuit. Here, $i(0^-) = 4$ and

$$I(s) = \frac{4}{s+1}.$$

Therefore,

$$i(t) = 4e^{-t}u(t).$$

(iii) This corresponds to the total response of the system. It will be the sum of the results of the previous two parts.

$$i(t) = 2u(t) + 2e^{-t}u(t).$$

Chapter 10 Answers

10.1. (a) The given summation may be written as

$$\sum_{n=-1}^{\infty} \frac{1}{2}\left(\frac{1}{2}r^{-1}\right)^n e^{-j\omega n},$$

by replacing z with $re^{j\omega}$. If $r < \frac{1}{2}$, then $\frac{1}{2}r^{-1} > 1$ and the function within the summation grows towards infinity with increasing n. Also, the summation does not converge. But if $r > \frac{1}{2}$, then the summation converges.

(b) The given summation may be written as

$$\sum_{n=1}^{\infty} \frac{1}{2}(2r)^n e^{j\omega n},$$

by replacing z with $re^{j\omega}$. If $r > (1/2)$, then $2r > 1$ and the function within the summation grows towards infinity with increasing n. Also, the summation does not converge. But if $r < \frac{1}{2}$, then the summation converges.

(c) The summation may be written as

$$\sum_{n=0}^{\infty} \frac{r^{-n} + (-r)^{-n}}{2} e^{-j\omega n}$$

by replacing z with $re^{j\omega}$. If $r > 1$, then the function inside the summation grows towards infinity with increasing n. Also, the summation does not converge. But if $r < 1$, then the summation converges.

(d) The summation may be written as

$$\sum_{n=0}^{\infty} (\frac{1}{2}r^{-1})^n \cos(\pi n/4) e^{-j\omega n} + \sum_{n=-\infty}^{0} (\frac{1}{2}r)^{-n} \cos(\pi n/4) e^{-j\omega n}$$

by replacing z with $re^{j\omega}$. The first summation converges for $r > \frac{1}{2}$. The second summation converges for $r < 2$. Therefore, the sum of these two summations converges for $\frac{1}{2} < r < 2$.

10.2. Using eq. (10.3),

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n} \\ &= \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \\ &= \left[\frac{z^{-3}}{125}\right] \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \\ &= \left[\frac{z^{-3}}{125}\right] \frac{1}{1 - \frac{1}{5}z^{-1}} \qquad |z| > \frac{1}{5} \end{aligned}$$

10.3. By using eq. (9.3), we can easily show that

$$\alpha^n u[-n - n_0] \xleftrightarrow{z} \frac{-z^{-n_0}}{1 - \alpha z^{-1}}, \qquad |z| < |\alpha|.$$

We then obtain

$$X(z) = \frac{1}{1 + z^{-1}} + \frac{-z^{-n_0 - 1}}{1 - \alpha z^{-1}}, \qquad 1 < |z| < |\alpha|.$$

Therefore, $|\alpha|$ has to be 2. n_0 can take on any value.

10.4. Using eq. (9.3), we have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{0} (\frac{1}{3})^n \cos(\frac{\pi}{4}n) z^{-n} \\ &= (1/2)\sum_{n=-\infty}^{0} (\frac{1}{3})^n e^{j\pi n/4} z^{-n} + (1/2)\sum_{n=-\infty}^{0} (\frac{1}{3})^n e^{-j\pi n/4} z^{-n} \\ &= (1/2)\sum_{n=0}^{\infty} (\frac{1}{3})^{-n} e^{-j\pi n/4} z^{n} + (1/2)\sum_{n=0}^{\infty} (\frac{1}{3})^{-n} e^{j\pi n/4} z^{n} \\ &= (1/2)\frac{1}{1 - 3e^{-j\pi/4}z} + (1/2)\frac{1}{1 - 3e^{j\pi/4}z}, \qquad |z| < \frac{1}{3} \end{aligned}$$

The poles are at $z = \frac{1}{3}e^{j\pi/4}$ and $z = \frac{1}{3}e^{-j\pi/4}$.

10.5. (a) The given z-transform may be written as

$$X(z) = \frac{z - \frac{1}{2}}{(z - \frac{1}{3})(z - \frac{1}{4})}.$$

Clearly, $X(z)$ has a zero at $z = \frac{1}{2}$. Since in $X(z)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 1, $X(z)$ has a zero at ∞. Therefore, $X(z)$ has one zero in the finite z-plane and one zero at infinity.

(b) The given z-transform may be written as

$$X(z) = \frac{(z-1)(z-2)}{(z-3)(z-4)}.$$

Clearly, $X(z)$ has zeros at $z = 1$ and $z = 2$. Since in $X(z)$, the orders of the numerator and denominator polynomials are identical, $X(z)$ has no zeros at infinity. Therefore, $X(z)$ has two zeros in the finite z-plane and no zeros at infinity.

(c) The given z-transform may be written as

$$X(z) = \frac{(z-1)}{z(z - \frac{1}{4})(z + \frac{1}{4})}.$$

Clearly, $X(z)$ has a zero at $z = 1$. Since in $X(z)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 2, $X(z)$ has two zeros at ∞. Therefore, $X(z)$ has one zero in the finite z-plane and two zeros at infinity.

10.6. (a) No. From property 3 in Section 10.2, we know that for a finite-length signal, the ROC is the entire z-plane. Therefore, there can be no poles in the finite z-plane for a finite-length signal. Clearly, in this problem this is not the case.

(b) No. Since the signal is absolutely summable, the ROC must include the unit circle. Also, since the signal has a pole at $z = 1/2$, the ROC can never be of the form $0 < |z| < r_0$. From property 5 in Section 10.2, we know that the signal cannot be left sided.

(c) Yes. Since the signal is absolutely summable, the ROC must include the unit circle. Since it is given that the signal has a pole at $z = 1/2$, a valid ROC for this signal would be $|z| > 1/2$. From property 4 in Section 10.2 we know that this would correspond to a right-sided signal.

(d) Yes. Since the signal is absolutely summable, the ROC must include the unit circle. Clearly, we can define an ROC which is a ring in the z-plane and includes the unit circle. From property 6 in Section 10.2, we know we ma conclude that the signal could be two sided.

10.7. We may find different signals with the given z-transform by choosing different regions of convergence. The poles of the z-transform are

$$z_0 = \frac{1}{2}j, \qquad z_1 = -\frac{1}{2}j, \qquad z_2 = -\frac{1}{2}, \qquad z_4 = \frac{3}{4}.$$

Based on these pole locations, we may choose from the following regions of convergence:
(i) $0 < |z| < \frac{1}{2}$
(ii) $\frac{1}{2} < |z| < \frac{3}{4}$
(iii) $|z| > \frac{3}{4}$
Therefore, we may have 3 different signals with the given z-transform.

10.8. If

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z), \qquad R,$$

then from Table 10.1 we have

$$\left(\frac{1}{4}\right)^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(4z), \qquad \frac{1}{4}R$$

and

$$\left(\frac{1}{8}\right)^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(8z), \qquad \frac{1}{8}R.$$

Since $\frac{1}{4}R$ includes the unit circle, and $X(z)$ has a pole at $z = 1/2$, we may conclude that R is definitely outside the circle with radius 1/2. The only question we now have to answer is whether R extends to infinity outside this circle of radius 1/2. Since $\frac{1}{8}R$ does not include the unit circle, it is clear that this is not the case. Therefore, R is a ring in the z-plane From property 6 in Section 10.2 we know that $x[n]$ must be a two-sided signal.

10.9. Using partial-fraction expansion,

$$X(z) = \frac{2/9}{1 - z^{-1}} + \frac{7/9}{1 + 2z^{-1}}, \qquad |z| > 2.$$

Taking the inverse z-transform,

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}(-2)^n u[n].$$

10.10. We use the approach developed in Example 10.11 to solve this problem.

(a) Since $|z| > \frac{1}{3}$, we may use long division to obtain the power-series expansion of $X($ as shown below.

$$1+\tfrac{1}{3}z^{-1}\,\big)\,1 + z^{-1}\,\big(\,1 + \tfrac{2}{3}z^{-1} + \tfrac{2}{9}z^{-2} + \ldots\ldots$$
$$1 + \tfrac{1}{3}z^{-1}$$
$$\tfrac{2}{3}z^{-1} + \tfrac{2}{9}z^{-2}$$
$$\tfrac{2}{9}z^{-2} + \tfrac{2}{27}z^{-3}$$
$$\vdots$$

Comparing $X(z)$ with the definition of the z-transform in eq. (10.3), we see that

$$x[0] = 1, \qquad x[1] = \frac{2}{3}, \qquad x[2] = -\frac{2}{9}.$$

(b) Since $|z| < \frac{1}{3}$, we may use long division to obtain the power-series expansion of $X(z$ as shown below.

$$\tfrac{1}{3}z^{-1}+1\,\big)\; z^{-1}+1\;\big(\,3 - 6z + 18z^2 \ldots\ldots$$
$$z^{-1} + 3$$
$$-2$$
$$-2 - 6z^{-1}$$
$$6z^{-1}$$
$$\vdots$$

Comparing $X(z)$ with the definition of the z-transform in eq. (10.3), we see that

$$x[0] = 3, \qquad x[-1] = -6, \qquad x[-2] = 18.$$

10.11. Since the ROC includes the entire z-plane, we know that the signal must be finite length. From the finite-sum formula, we have

$$\frac{1}{1024}\left[\frac{1024 - z^{10}}{1 - \frac{1}{2}z^{-1}}\right] = \sum_{n=0}^{9}\left(\frac{1}{2}\right)^n z^{-n}.$$

Comparing this with the definition of the z-transform in eq. (10.3), we obtain

$$x[n] = \begin{cases} (\frac{1}{2})^n, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

10.12. The pole-zero plots for each of the three z-transforms is as shown in Figure S10.12.

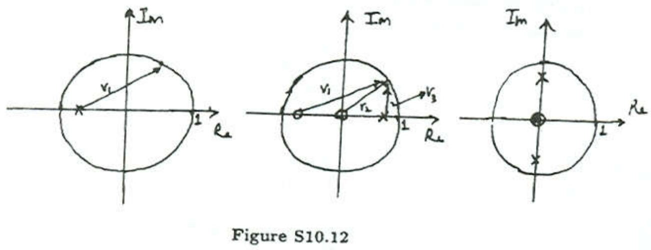

Figure S10.12

(a) From Section 10.4, we know that the magnitude of the Fourier transform may be expressed as

$$|H_1(e^{j\omega})| = \frac{1}{\text{Length of } \vec{v_1}},$$

where $\vec{v_1}$ is as shown in the figure above. Clearly, for small values of ω (ω near zero), the right-hand side of the above equation is small. But as ω approaches π, the right-hand side of the above equation becomes large. Therefore, $H_1(e^{j\omega})$ is approximately highpass.

(b) From Section 10.4, we know that the magnitude of the Fourier transform may be expressed as

$$|H_2(e^{j\omega})| = \frac{(\text{Length of } \vec{v_1})(\text{Length of } \vec{v_2})}{(\text{Length of } \vec{v_3})^2},$$

where $\vec{v_1}$, $\vec{v_1}$, and $\vec{v_3}$ are as shown in the figure above. Clearly, for small values of ω (ω near zero), the numerator of the right-hand side of the above equation is much larger than the denominator. Therefore, $H_1(e^{j\omega})$ is large near $\omega = 0$. But as ω approaches π, the denominator of the right-hand side of the above equation is much larger than the numerator. Therefore, $H_2(e^{j\omega})$ is small near $\omega = \pi$. Therefore, $H_2(e^{j\omega})$ is approximately lowpass.

(c) From Section 10.4, we know that the magnitude of the Fourier transform may be expressed as

$$|H_3 e^{j\omega})| = \frac{(\text{Length of } \vec{v_1})^2}{(\text{Length of } \vec{v_2})(\text{Length of } \vec{v_3})},$$

where $\vec{v_1}$, $\vec{v_1}$, and $\vec{v_3}$ are as shown in the figure above. Clearly, for small values of ω (ω near zero), and for values of ω near π the numerator of the right-hand side of the above equation is almost the same as the denominator. But when $|\omega|$ is near $\pi/2$, the numerator of the right-hand side of the above equation is much larger than the denominator. Therefore, $H_1(e^{j\omega})$ is large near $\omega = \pi/2$. Therefore, $H_1(e^{j\omega})$ is approximately bandpass.

10.13. (a) The signal $g[n]$ is

$$g[n] = \delta[n] - \delta[n-6].$$

Using the definition of the z-transform in eq. (10.3), we obtain

$$G(z) = 1 - z^{-6}, \qquad |z| > 0.$$

(b) From Table 10.1, we have

$$x[n] = \sum_{k=-\infty}^{n} g[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1 - z^{-1}}G(z), \qquad \text{At least} |z| > 1.$$

Therefore,

$$X(z) = \frac{1 - z^{-6}}{1 - z^{-1}}, \qquad |z| > 0.$$

The ROC is $|z| > 0$ because $x[n]$ is a finite-length signal.

10.14. (a) We know that $x[n] * x[n]$ will be triangular signal whose first non-zero value occurs at $n = 0$. Furthermore, we also know that $x[n] * x[n - n_0]$ has its first nonzero value at $n = n_0$. Therefore, $n_0 = 2$.

(b) From Problem 10.13 we have

$$X(z) = \frac{1 - z^{-6}}{1 - z^{-1}}, \qquad |z| > 0.$$

Using the shift property,

$$x[n-2] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-2}\frac{1 - z^{-6}}{1 - z^{-1}}, \qquad |z| > 0.$$

Using the convolution property,

$$g[n] = x[n] * x[n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-2}\left(\frac{1 - z^{-6}}{1 - z^{-1}}\right)^2, \qquad |z| > 0.$$

Since

$$\lim_{z \to \infty} G(z) = 0 = g[0],$$

$G(z)$ does satisfy the initial value theorem.

10.15. Taking the z-transform of $y[n]$, we have

$$Y(z) = \frac{1}{1-\frac{1}{9}z^{-1}}, \qquad |z| > \frac{1}{9}.$$

Now from Table 10.1, we have

$$y_1[n] = y_{(2)}[n] = \begin{cases} y[r], & ,n = 2r \\ 0, & ,n \neq 2r \end{cases} \quad \xleftrightarrow{\mathcal{Z}} Y_1(z) = Y(z^2), \qquad |z| > \frac{1}{3}.$$

Therefore,

$$y_1[0] = 1,\ y_1[1] = 0,\ y_1[2] = \frac{1}{9},\ y_1[3] = 0,\ y_1[4] = \frac{1}{81}, \cdots$$

This may be written as

$$y_1[n] = \frac{1}{2}\left[\left(\frac{1}{3}\right)^n u[n] + (-1)^n \left(\frac{1}{3}\right)^n u[n]\right].$$

If we now choose $x[n]$ to be $\frac{1}{2}\left[\left(\frac{1}{3}\right)^n u[n]\right]$, then

$$Y_1(z) = Y(z^2) = (1/2)[X(z) + X(-z)], \qquad |z| > \frac{1}{3}.$$

Furthermore, since $X(z)$ has only one pole and one zero, this choice of $x[n]$ satisfies both the given conditions.

We may also choose $x[n]$ to be $\frac{1}{2}\left[(-1)^n\left(\frac{1}{3}\right)^n u[n]\right]$. This would still satisfy both given conditions.

10.16. For a system to be both causal and stable, the corresponding z-transform must not have any poles outside the unit circle.

(a) The given z-transform has a pole at infinity. Therefore, it is **not causal**.

(b) The poles of this z-transform are at $z = \frac{1}{4}$ and $z = -\frac{3}{4}$. Therefore, it is **causal**.

(c) This z-transform has a pole at $-\frac{4}{3}$. Therefore, it is **not causal**.

10.17. (a) Since $\lim_{z\to\infty} = 1$, $H(z)$ has no poles at infinity. Furthermore, since $h[n]$ is given to be right-sided, $h[n]$ has to be causal.

(b) Since $h[n]$ is causal, the numerator and denominator polynomials of $H(z)$ have the same order. Since $H(z)$ is given to have two zeros, we may conclude that it also has two poles.

Since $h[n]$ is real, the poles must occur in conjugate pairs. Also, it is given that one of the poles lies on the circle defined by $|z| = \frac{3}{4}$. Therefore, the other pole also lies on the same circle.

Clearly, the ROC for $H(z)$ will be of the form $|z| > \frac{3}{4}$. and will include the unit circle. Therefore, we may conclude that the system is stable.

10.18. (a) Using the analysis of Example 10.28, we may show that

$$H(z) = \frac{1 - 6z^{-1} + 8z^{-2}}{1 - \frac{2}{3}z^{-2} + \frac{1}{9}z^{-2}}.$$

Since $H(z) = Y(z)/X(z)$, we may write

$$Y(z)[1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}] = X(z)[1 - 6z^{-1} + 8z^{-2}].$$

Taking the inverse z-transform we obtain

$$y[n] - \frac{2}{3}y[n-1] + \frac{1}{9}y[n-2] = x[n] - 6x[n-1] + 8x[n-2].$$

(b) $H(z)$ has only two poles. These are both at $z = \frac{1}{3}$. Since the system is causal, the ROC of $H(z)$ will be of the form $|z| > \frac{1}{3}$. Since the ROC includes the unit circle, the system is stable.

10.19. (a) The unilateral z-transform is

$$\begin{aligned} \mathcal{X}(z) &= \sum_{n=0}^{\infty}(\frac{1}{4})^n u[n+5] z^{-n} \\ &= \sum_{n=0}^{\infty}(\frac{1}{4})^n z^{-n} \\ &= \frac{1}{1-(1/4)z^{-1}}, \qquad |z| > \frac{1}{4} \end{aligned}$$

(b) The unilateral z-transform is

$$\begin{aligned} \mathcal{X}(z) &= \sum_{n=0}^{\infty}(\delta[n+3] + \delta[n] + 2^n[-n])z^{-n} \\ &= \sum_{n=0}^{\infty}(0 + \delta[n] + \delta[n])z^{-n} \\ &= 2, \qquad \text{All } z \end{aligned}$$

The unilateral z-transform is

$$\begin{aligned} \mathcal{X}(z) &= \sum_{n=0}^{\infty}(\frac{1}{2})^{|n|} z^{-n} \\ &= \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} \\ &= \frac{1}{1-(1/2)z^{-1}}, \qquad |z| > \frac{1}{2} \end{aligned}$$

10.20. Applying the unilateral z-transform to the given difference equation, we have

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = \mathcal{X}(z).$$

(a) For the zero-input response, assume that $x[n] = 0$. Since we are given that $y[-1] = 2$,

$$z^{-1}\mathcal{Y}(z) + y[-1] + 2\mathcal{Y}(z) = 0 \Rightarrow \mathcal{Y}(z) = \frac{-1}{1 + (1/2)z^{-1}}.$$

Taking the inverse unilateral z-transform,

$$y[n] = -\left(-\frac{1}{2}\right)^n u[n].$$

(b) For the zero-state response, set $y[-1] = 0$. Also, we have

$$\mathcal{X}(z) = \mathcal{UZ}\{(1/2)^n u[n]\} = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}.$$

Therefore,

$$\mathcal{Y}(z) = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(\frac{2}{2+z^{-1}}\right).$$

We use partial fraction expansion followed by the inverse unilateral z-transform to obtain

$$y[n] = \frac{1}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{6}\left(\frac{1}{4}\right)^n u[n].$$

(c) The total response is the sum of the zero-state and zero-input responses. This is

$$y[n] = -\frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{6}\left(\frac{1}{4}\right)^n u[n].$$

10.21. The pole-zero plots are all shown in Figure S10.21.

(a) For $x[n] = \delta[n+5]$,

$$X(z) = z^5, \qquad \text{All } z.$$

The Fourier transform exists because the ROC includes the unit circle.

(b) For $x[n] = \delta[n-5]$,

$$X(z) = z^{-5}, \qquad \text{All } z \text{ except } 0.$$

The Fourier transform exists because the ROC includes the unit circle.

(c) For $x[n] = (-1)^n u[n]$,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-n} \\ &= 1/(1+z^{-1}), \qquad |z| > 1 \end{aligned}$$

The Fourier transform does not exist because the ROC does not include the unit circle.

(d) For $x[n] = (1/2)^{n+1}u[n+3]$,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-3}^{\infty}(1/2)^{n+1}z^{-n} \\ &= \sum_{n=0}^{\infty}(1/2)^{n-2}z^{-n+3} \\ &= 4z^3/(1-(1/2)z^{-1}), \qquad |z| > 1/2 \end{aligned}$$

The Fourier transform exists because the ROC includes the unit circle.

(e) For $x[n] = (-1/3)^n u[-n-2]$,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{-2}(-1/3)^n z^{-n} \\ &= \sum_{n=2}^{\infty}(-1/3)^{-n}z^n \\ &= \sum_{n=0}^{\infty}(-1/3)^{-n-2}z^{n+2} \\ &= 9z^2/(1+3z), \qquad |z| < 1/3 \\ &= 3z/(1+(1/3)z^{-1}), \qquad |z| < 1/3 \end{aligned}$$

The Fourier transform does not exist because the ROC does not include the unit circle.

(f) For $x[n] = (1/4)^n u[-n+3]$,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{3}(1/4)^n z^{-n} \\ &= \sum_{n=-3}^{\infty}(1/4)^{-n}z^n \\ &= \sum_{n=0}^{\infty}(1/4)^{-n+3}z^{n-3} \\ &= (1/64)z^{-3}/(1-4z), \qquad |z| < 1/4 \\ &= (1/16)z^{-4}/(1-(1/4)z^{-1}), \qquad |z| < 1/4 \end{aligned}$$

The Fourier transform does not exist because the ROC does not include the unit circle.

(g) Consider $x_1[n] = 2^n u[-n]$.

$$
\begin{aligned}
X_1(z) &= \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} \\
&= \sum_{n=-\infty}^{0} (2)^n z^{-n} \\
&= \sum_{n=0}^{\infty} (2)^{-n} z^{n} \\
&= 1/(1-(1/2)z), \qquad |z| < 2 \\
&= -2z^{-1}/(1-2z^{-1}), \qquad |z| < 2
\end{aligned}
$$

Consider $x_2[n] = (1/4)^n u[n-1]$.

$$
\begin{aligned}
X_2(z) &= \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} \\
&= \sum_{n=1}^{\infty} (1/4)^n z^{-n} \\
&= \sum_{n=0}^{\infty} (1/4)^{n+1} z^{-n-1} \\
&= (z^{-1}/4)[1/(1-(1/4)z^{-1})], \qquad |z| > 1/4
\end{aligned}
$$

The z-transform of the overall sequence $x[n] = x_1[n] + x_2[n]$ is

$$
X(z) = -\frac{2z^{-1}}{(1-2z^{-1})} + \frac{z^{-1}/4}{1-(1/4)z^{-1}}, \qquad (1/4) < |z| < 2.
$$

The Fourier transform exists because the ROC includes the unit circle.

(h) Consider $x[n] = (1/3)^{n-2} u[n-2]$.

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=2}^{\infty} (1/3)^{n-2} z^{-n} \\
&= \sum_{n=0}^{\infty} (1/3)^{n} z^{-n-2} \\
&= z^{-2}[1/(1-(1/3)z^{-1})], \qquad |z| > 1/3
\end{aligned}
$$

The Fourier transform exists because the ROC includes the unit circle.

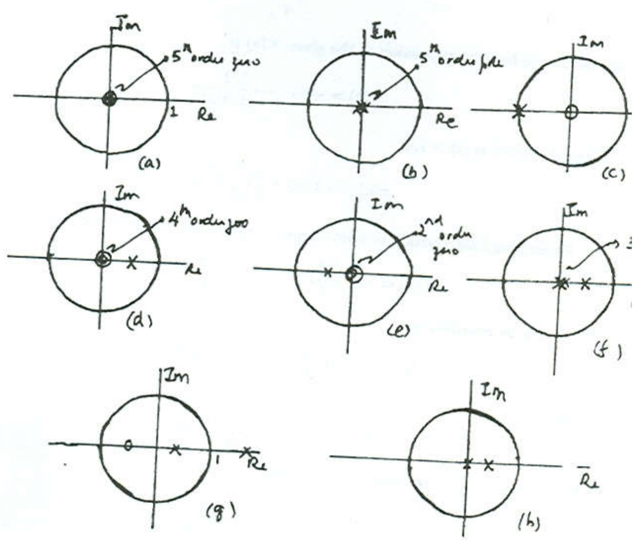

Figure S10.21

10.22. (a) Using the z-transform analysis equation,

$$
\begin{aligned}
X(z) &= (1/2)^{-4} z^4 + (1/2)^{-3} z^3 + (1/2)^{-2} z^2 + (1/2)^{-1} z^1 + (1/2)^0 z^0 \\
&+ (1/2)^1 z^{-1} + (1/2)^2 z^{-2} + (1/2)^3 z^{-3} + (1/2)^4 z^{-4}
\end{aligned}
$$

This may be expressed as

$$
X(z) = (1/2)^{-4} z^4 \left[\frac{1-(1/2)^9 z^{-9}}{1-(1/2)z^{-1}}\right].
$$

This has four zeros at $z = 0$ and 8 more zeros distributed on a circle of radius 1/2. T ROC is the entire z plane. (Although from an inspection of the expression for X(it seems like there is a pole at 1/2, note that there is also a zero at 1/2 which canc with this pole.) Since the ROC includes the unit circle, the Fourier transform exists

(b) Consider the sequence $x_1[n] = (1/2)^{|n|}$. This may be written as

$$
x_1[n] = (1/2)^n u[n] + 2^n u[-n-1].
$$

Now,

$$
(1/2)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-(1/2)z^{-1}}, \qquad |z| > (1/2)
$$

and

$$
(2)^n u[-n-1] \stackrel{z}{\longleftrightarrow} -\frac{1}{1-2z^{-1}}, \qquad |z| < 2.
$$

Therefore,

$$
X_1(z) = \frac{1}{1-(1/2)z^{-1}} - \frac{1}{1-2z^{-1}}, \qquad (1/2) < |z| < 2.
$$

Note that $x[n] = n x_1[n]$. Therefore,

$$
X(z) = -z\frac{d}{dz}X_1(z) = -\frac{(1/2)z^{-1}}{(1-(1/2)z^{-1})^2} + \frac{2z^{-1}}{(1-2z^{-1})^2}.
$$

The ROC is $(1/2) < |z| < 2$. Therefore, the Fourier transform exists.

(c) Write $x[n]$ as

$$
x[n] = n(1/2)^n u[n] - n2^n u[-n-1] = n x_1[n] - n x_2[n]
$$

where

$$
x_1[n] = (1/2)^n u[n] \stackrel{z}{\longleftrightarrow} X_1(z) = \frac{1}{1-(1/2)z^{-1}}, \qquad |z| > (1/2)
$$

and

$$
x_2[n] = (2)^n u[-n-1] \stackrel{z}{\longleftrightarrow} X_2(z) = -\frac{1}{1-2z^{-1}}, \qquad |z| < 2.
$$

Using the differentiation property, we get

$$
X(z) = -z\frac{d}{dz}X_1(z) + z\frac{d}{dz}X_2(z) = -\frac{(1/2)z^{-1}}{(1-(1/2)z^{-1})^2} - \frac{2z^{-1}}{(1-2z^{-1})^2}
$$

The ROC is $(1/2) < |z| < 2$. Therefore, the Fourier transform exists.

(d) The sequence may be written as

$$
x[n] = 4^n \left\{\frac{e^{j[(2\pi n/6)+(\pi/4)]} + e^{-j[(2\pi n/6)+(\pi/4)]}}{2}\right\} u[-n-1].
$$

Now,

$$
4^n e^{j[(2\pi n/6)+(\pi/4)]} u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{e^{j\pi/4}}{2}\frac{1}{1-4e^{j2\pi/6}z^{-1}}, \qquad |z| < 4
$$

and

$$
4^n e^{-j[(2\pi n/6)+(\pi/4)]} u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{e^{-j\pi/4}}{2}\frac{1}{1-4e^{-j2\pi/6}z^{-1}}, \qquad |z| < 4
$$

Therefore,

$$
X(z) = \frac{e^{j\pi/4}}{2}\frac{1}{1-4e^{j2\pi/6}z^{-1}} + \frac{e^{-j\pi/4}}{2}\frac{1}{1-4e^{-j2\pi/6}z^{-1}}, \qquad |z| < 4.
$$

The ROC is $|z| < 4$. Therefore, the Fourier transform exists.

10.23. (i) The partial fraction expansion of the given $X(z)$ is

$$
X(z) = \frac{-1/2}{1-\frac{1}{2}z^{-1}} + \frac{3/2}{1+\frac{1}{2}z^{-1}}.
$$

Since the ROC is $|z| > 1/2$,

$$
x[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n].
$$

Performing long-division in order to get a right-sided sequence, we obtain

$$
X(z) = 1 - z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{16}z^{-4} - \frac{1}{16}z^{-5} + - \cdots.
$$

This may be rewritten as

$$
\begin{aligned}
X(z) &= \frac{3}{2}[1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} + - \cdots] \\
&- \frac{1}{2}[1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \cdots].
\end{aligned}
$$

Therefore,

$$
x[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n].
$$

(ii) The partial fraction expansion of the given $X(z)$ is

$$
X(z) = \frac{-1/2}{1-\frac{1}{2}z^{-1}} + \frac{3/2}{1+\frac{1}{2}z^{-1}}.
$$

Since the ROC is $|z| < 1/2$,

$$
x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{3}{2}\left(-\frac{1}{2}\right)^n u[-n-1].
$$

Performing long-division in order to get a left-sided sequence, we obtain

$$
X(z) = 4z - 4z^2 + 16z^3 - 16z^4 + 64z^5 - 64z^6 + - \cdots.
$$

This may be rewritten as

$$
\begin{aligned}
X(z) &= \frac{3}{2}[2z - 4z^2 + 8z^3 - 16z^4 + - \cdots] \\
&+ \frac{1}{2}[2z + 4z^2 + 8z^3 + 16z^4 + \cdots].
\end{aligned}
$$

Therefore,

$$
x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{3}{2}\left(-\frac{1}{2}\right)^n u[-n-1].
$$

(iii) The partial fraction expansion of the given $X(z)$ is

$$X(z) = -2 + \frac{3/2}{1-(1/2)z^{-1}}.$$

Since the ROC is $|z| > 1/2$,

$$x[n] = -2\delta[n] + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n].$$

Performing long-division in order to get a right-sided sequence, we obtain

$$X(z) = -\frac{1}{2} + \frac{3}{4}z^{-1} + \frac{3}{8}z^{-2} + \frac{3}{16}z^{-3} + \cdots.$$

This may be rewritten as

$$X(z) = -2 + \frac{3}{2}[1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \cdots].$$

Therefore,

$$x[n] = -2\delta[n] + \frac{3}{2}\left(\frac{1}{2}\right)^n u[n].$$

(iv) The partial fraction expansion of the given $X(z)$ is

$$X(z) = -2 + \frac{3/2}{1-(1/2)z^{-1}}.$$

Since the ROC is $|z| < 1/2$,

$$x[n] = -2\delta[n] - \frac{3}{2}\left(\frac{1}{2}\right)^n u[-n-1].$$

Performing long-division in order to get a left-sided sequence, we obtain

$$X(z) = -2 - 3z - 6z^2 - 12z^3 - 24z^4 - \cdots.$$

This may be rewritten as

$$X(z) = -2 - \frac{3}{2}[2z + 4z^2 + 8z^3 + 16z^4 + \cdots].$$

Therefore,

$$x[n] = -2\delta[n] - \frac{3}{2}\left(\frac{1}{2}\right)^n u[-n-1].$$

(v) We may similarly show that in this case,

$$x[n] = 2n(1/2)^n u[n] - n(1/2)^{n+1}u[n+1].$$

(vi) We may similarly show that in this case,

$$x[n] = -2n(1/2)^n u[-n-1] + n(1/2)^{n+1}u[-n-2].$$

10.24. (a) We may write $X(z)$ as

$$X(z) = \frac{1-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}.$$

Therefore,

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}.$$

If $x[n]$ is absolutely summable, then the ROC of $X(z)$ has to include the unit circle. Therefore, the ROC is $|z| > 1/2$. It follow that

$$x[n] = \left(\frac{1}{2}\right)^2 u[n].$$

(b) Carrying out long division on $X(z)$, we get

$$X(z) = 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-2} + - \cdots.$$

Using the analysis equation (10.3), we get

$$x[n] = \delta[n] - \left(-\frac{1}{2}\right)^{n-1} u[n-1].$$

(c) We may write $X(z)$ as

$$X(z) = \frac{3z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{3z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}.$$

The partial fraction expansion of $X(z)$ is

$$X(z) = \frac{4}{1-\frac{1}{2}z^{-1}} - \frac{4}{1+\frac{1}{4}z^{-1}}.$$

Since $x[n]$ is absolutely summable, the ROC must be $|z| > 1/2$ in order to include the unit circle. It follows that

$$x[n] = 4\left(\frac{1}{2}\right)^2 u[n] - 4\left(-\frac{1}{4}\right)^2 u[n].$$

10.25. (a) The partial fraction expansion of $X(z)$ is

$$X(z) = -\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-z^{-1}}.$$

Since $x[n]$ is right-sided, the ROC has to be $|z| > 1$. Therefore, it follows that

$$x[n] = -\left(\frac{1}{2}\right)^2 u[n] + 2u[n].$$

(b) $X(z)$ may be rewritten as

$$X(z) = \frac{z^2}{(z-\frac{1}{2})(z-1)}.$$

Using partial fraction expansion, we may rewrite this as

$$\begin{aligned} X(z) &= 2z^2\left[-\frac{1}{z-\frac{1}{2}} + \frac{1}{z-1}\right] \\ &= 2z\left[-\frac{z}{z-\frac{1}{2}} + \frac{z}{z-1}\right] \end{aligned}$$

If $x[n]$ is right-sided, then the ROC for this signal is $|z| > 1$. Using this fact we may find the inverse z-transform of the term within square brackets above to be $y[n] = -(1/2)^n u[n] + u[n]$. Note that $X(z) = 2zX(z)$. Therefore, $x[n] = 2y[n+1]$. This gives

$$x[n] = -2\left(\frac{1}{2}\right)^{n+1} u[n+1] + 2u[n+1].$$

Noting that $x[-1] = 0$, we may rewrite this as

$$x[n] = -\left(\frac{1}{2}\right)^n u[n] + 2u[n].$$

This is the answer that we obtained in part (a).

10.26. (a) From part (b) of the previous problem,

$$X(z) = \frac{z^2}{(z-\frac{1}{2})(z-1)}.$$

(b) From part (b) of the previous problem,

$$X(z) = 2z\left[-\frac{z}{z-\frac{1}{2}} + \frac{z}{z-1}\right].$$

(c) If $x[n]$ is left-sided, then the ROC for this signal is $|z| < 1/2$. Using this fact, we may find the inverse z-transform of the term within square brackets above to be $y[n] = (1/2)^n u[-n-1] - u[-n-1]$. Note that $X(z) = 2zX(z)$. Therefore, $x[n] = 2y[n+1]$. This gives

$$x[n] = 2\left(\frac{1}{2}\right)^{n+1} u[-n-2] - 2u[-n-2].$$

10.27. We perform long-division on $X(z)$ so as to obtain a right-sided sequence. This gives us

$$X(z) = z^3 + 4z^2 + 5z + \cdots.$$

Therefore, comparing this with eq. (10.3) we get

$$x[-3] = 1, \quad x[-2] = 4, \quad x[-1] = 5,$$

and $x[n] = 0$ for $n < -3$.

10.28. (a) Using eq. (10.3), we get

$$X(z) = 1 - 0.95z^{-6} = \frac{z^6 - 0.95}{z^6}.$$

(b) Therefore, $X(z)$ has six zeros lying on a circle of radius 0.95 (as shown in Figure S10.28) and 6 poles at $z = 0$.

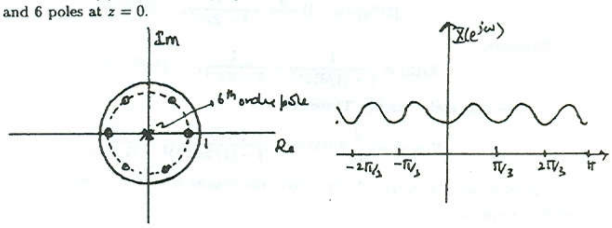

Figure S10.28

(c) The magnitude of the Fourier transform is as shown in Figure S10.28.

10.29. The plots are as shown in Figure S10.29.

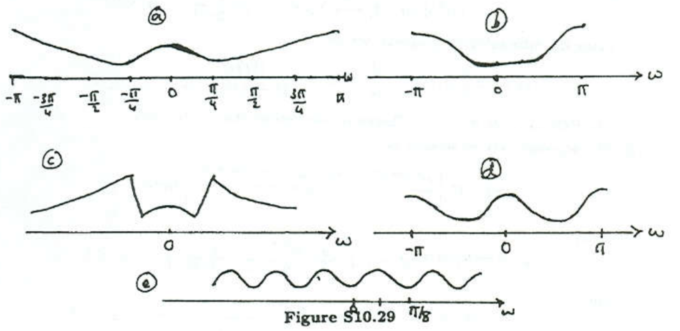

Figure S10.29

10.30. From the given information, we have

$$x_1[n] \xleftrightarrow{Z} X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

and

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}.$$

Using the time shifting property, we get

$$x_1[n+3] \overset{z}{\longleftrightarrow} z^3 X_1(z), \qquad |z| > \frac{1}{2}.$$

Using the time reversal and shift properties, we get

$$x_2[-n+1] \overset{z}{\longleftrightarrow} z^{-1} X_2(z^{-1}), \qquad |z| < 3.$$

Now, using the convolution property, we get

$$y[n] = x_1[n+3] * x_2[-n+1] \overset{z}{\longleftrightarrow} Y(z) = z^2 X_1(z) X_2(z^{-1}), \qquad \frac{1}{2} < |z| < 3$$

Therefore,

$$Y(z) = \frac{z^2}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z)}.$$

10.31. From Clue 1, we know that $x[n]$ is real. Therefore, the poles and zeros of $X(z)$ have to occur in conjugate pairs. Since Clue 4 tells us that $X(z)$ has a pole at $z = (1/2)e^{j\pi/3}$, we can conclude that $X(z)$ must assume another pole at $z = (1/2)e^{-j\pi/3}$. Now, since $X(z)$ has no more poles, we have to assume that $X(z)$ has 2 or less zeros. If $X(z)$ has more than 2 zeros, then $X(z)$ must have poles at infinity. Since Clue 3 tells us that $X(z)$ has 2 zeros at the origin, we know that $X(z)$ must be of the form

$$X(z) = \frac{Az^2}{(z-\frac{1}{2}e^{j\pi/3})(z-\frac{1}{2}e^{-j\pi/3})}.$$

Since Clue 5 tells us that $X(1) = 8/3$, we may conclude that $A = 2$. Therefore,

$$X(z) = \frac{2z^2}{(z-\frac{1}{2}e^{j\pi/3})(z-\frac{1}{2}e^{-j\pi/3})}.$$

Since $x[n]$ is right-sided, the ROC must be $|z| > 1/3$.

10.32. (a) We are given that $h[n] = a^n u[n]$ and $x[n] = u[n] - u[n-N]$. Therefore,

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[n-k]x[k] \\ &= \sum_{k=0}^{N-1} a^{n-k}u[n-k] \end{aligned}$$

Now, $y[n]$ may be evaluated to be

$$y[n] = \begin{cases} 0, & n<0 \\ \sum_{k=0}^{n} a^n a^{-k}, & 0 \le n \le N-1 \\ \sum_{k=0}^{N-1} a^n a^{-k}, & n > N-1 \end{cases}.$$

Simplifying,

$$y[n] = \begin{cases} 0, & n<0 \\ (a^n - a^{-1})/(1-a^{-1}), & 0 \le n \le N-1 \\ a^n(1-a^{-N})/(1-a^{-1}), & n > N-1 \end{cases}.$$

(b) Using Table 10.2, we get

$$H(z) = \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

and

$$X(z) = \frac{1-z^{-N}}{1-z^{-1}}, \qquad \text{All}z.$$

Therefore,

$$Y(z) = X(z)H(z) = \frac{1}{(1-z^{-1})(1-az^{-1})} - \frac{z^{-N}}{(1-z^{-1})(1-az^{-1})}$$

The ROC is $|z| > |a|$. Consider

$$P(z) = \frac{1}{(1-z^{-1})(1-az^{-1})}$$

with ROC $|z| > |a|$. The partial fraction expansion of $P(z)$ is

$$P(z) = \frac{1/(1-a)}{1-z^{-1}} + \frac{1/(1-a^{-1})}{1-az^{-1}}.$$

Therefore,

$$p[n] = \frac{1}{1-a}u[n] + \frac{1}{1-a^{-1}}a^n u[n].$$

Now, note that

$$Y(z) = P(z)[1-z^{-N}].$$

Therefore,

$$y[n] = p[n] - p[n-N] = \frac{1}{1-a}\{u[n]-u[n-N]\} + \frac{1}{1-a^{-1}}\{a^n u[n] - a^{n-N}u[n-N]\}.$$

This may be written as

$$y[n] = \begin{cases} 0, & n<0 \\ (a^n - a^{-1})/(1-a^{-1}), & 0 \le n \le N-1 \\ a^n(1-a^{-N})/(1-a^{-1}), & n > N-1 \end{cases}.$$

This is the same as the result of part (a).

10.33. (a) Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}.$$

The poles of $H(z)$ are at $(1/4) \pm j(\sqrt{3}/4)$. Since $h[n]$ is causal, the ROC has to be $|z| > |(1/4)+j(\sqrt{3}/4)| = (1/2)$.

(b) We have

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}.$$

Therefore,

$$Y(z) = H(z)X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}.$$

The ROC of $Y(z)$ will be the intersection of the ROCs of $X(z)$ and $H(z)$. This implies that the ROC of $Y(z)$ is $|z| > 1/2$. The partial fraction expansion of $Y(z)$ is

$$Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}/2}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}.$$

Using Table 10.2 we get

$$y[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{2}{\sqrt{3}}\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi n}{3}\right)u[n].$$

10.34. (a) Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-z^{-1}-z^{-2}}.$$

The poles of $H(z)$ are at $z = (1/2) \pm (\sqrt{5}/2)$. $H(z)$ has a zero at $z=0$. The pole-zero plot for $H(z)$ is as shown in Figure S10.34. Since $h[n]$ is causal, the ROC for $H(z)$ has to be $|z| > (1/2) + (\sqrt{5}/2)$.

(b) The partial fraction expansion of $H(z)$ is

$$H(z) = -\frac{1/\sqrt{5}}{1-(\frac{1+\sqrt{5}}{2})z^{-1}} + \frac{1/\sqrt{5}}{1-(\frac{1-\sqrt{5}}{2})z^{-1}}.$$

Therefore,

$$h[n] = -\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

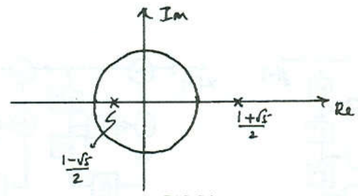

Figure S10.34

(c) Now assuming that the ROC is $(\sqrt{5}/2)-(1/2) < |z| < (1/2)+(\sqrt{5}/2)$, we get

$$h[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[-n-1] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

10.35. Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z-\frac{5}{2}+z^{-1}} = \frac{z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}.$$

The partial fraction expansion of $H(z)$ is

$$H(z) = \frac{-2/3}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{1-2z^{-1}}.$$

If the ROC is $|z| > 2$, then

$$h_1[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(2)^n u[n].$$

If the ROC is $1/2 < |z| < 2$, then

$$h_2[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{3}(2)^n u[-n-1].$$

If the ROC is $|z| < 1/2$, then

$$h_3[n] = \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{2}{3}(2)^n u[-n-1].$$

For each $h_i[n]$, we now need to show that if $y[n] = h_i[n]$ in the difference equation, then $x[n] = \delta[n]$. Consider substituting $h_1[n]$ into the difference equation. This yields

$$\begin{aligned} \tfrac{2}{3} \quad & \left(\tfrac{1}{2}\right)^{n-1}u[n-1] - \tfrac{2}{3}(2)^{n-1}u[n-1] - \tfrac{5}{3}\left(\tfrac{1}{2}\right)^n u[n] \\ & + \tfrac{5}{3}(2)^n u[n] + \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{n+1}u[n+1] - \tfrac{2}{3}(2)^{n+1}u[n+1] = x[n] \end{aligned}$$

Then,

$$x[n] = 0, \quad \text{for } n < -1,$$
$$x[-1] = 2/3 - 2/3 = 0,$$
$$x[n] = 0, \quad \text{for } n > 0.$$

It follows that $x[n] = \delta[n]$. It can similarly be shown that $h_2[n]$ and $h_3[n]$ satisfy the difference equation.

10.36. Taking the z-transform of both sides of the given difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}}.$$

The partial fraction expansion of $H(z)$ is

$$H(z) = -\frac{3/8}{1 - \frac{1}{3}z^{-1}} + \frac{3/8}{1 - 3z^{-1}}.$$

Since $H(z)$ corresponds to a stable system, the ROC has to be $(1/3) < |z| < 3$. Therefore,

$$h[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}(3)^n u[-n-1].$$

10.37. (a) The block-diagram may be redrawn as shown in part (a) of the figure below. This may be treated as a cascade of the two systems shown within the dotted lines in Figure S10.37. These two systems may be interchanged as shown in part (b) of the figure Figure S10.37 without changing the system function of the overall system. From the figure below, it is clear that

$$y[n] = x[n] + \frac{9}{8}x[n-1] - \frac{1}{3}y[n-1] + \frac{2}{9}y[n-2].$$

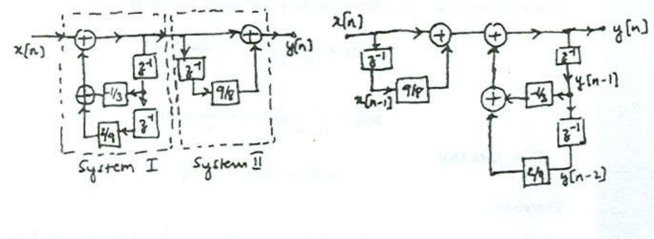

Figure S10.37

(b) Taking the z-transform of the above difference equation and simplifying, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{9}{8}z^{-1}}{1 + \frac{1}{3}z^{-1} - \frac{2}{9}z^{-2}} = \frac{1 + \frac{9}{8}z^{-1}}{(1 + \frac{2}{3}z^{-1})(1 - \frac{1}{3}z^{-1})}.$$

$H(z)$ has poles at $z = 1/3$ and $z = -2/3$. Since the system is causal, the ROC has to be $|z| > 2/3$. The ROC includes the unit circle and hence the system is stable.

10.38. (a) $e_1[n] = f_1[n]$.

(b) $e_2[n] = f_2[n]$.

(c) Using the results of parts (a) and (b), we may redraw the block-diagram as shown in Figure S10.38.

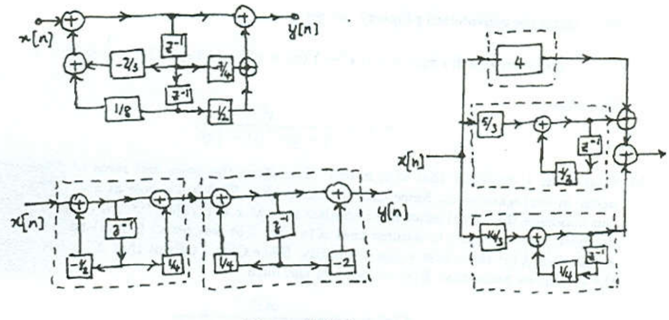

Figure S10.38

(d) Using the approach shown in the examples in the textbook we may draw the block-diagram of $H_1(z) = [1 + (1/4)z^{-1}]/[1 + (1/2)z^{-1}]$ and $H_2(z) = [1 - 2z^{-1}]/[1 - (1/4)z^{-1}]$ as shown in the dotted boxes in the figure below. $H(z)$ is the cascade of these two systems.

(e) Using the approach shown in the examples shown in the textbook, we may draw the block-diagram of $H_1(z) = 4$, $H_2(z) = [5/3]/[1 + (1/2)z^{-1}]$ and $H_3(z) = [-14/3]/[1 - (1/4)z^{-1}]$ as shown in the dotted boxes in the figure below. $H(z)$ is the parallel combination of $H_1(z)$, $H_z(z)$, and $H_3(z)$.

10.39. (a) The direct form block diagram may be drawn as shown in part (a-i) of Figure S10.39 by noting that

$$H_1(z) = \frac{1}{1 - \frac{5}{3}z^{-1} - \frac{11}{36}z^{-2} - \frac{5}{18}z^{-3} + \frac{1}{36}z^{-4}}.$$

The cascade block-diagram is as shown in part (a-ii) of Figure S10.39.

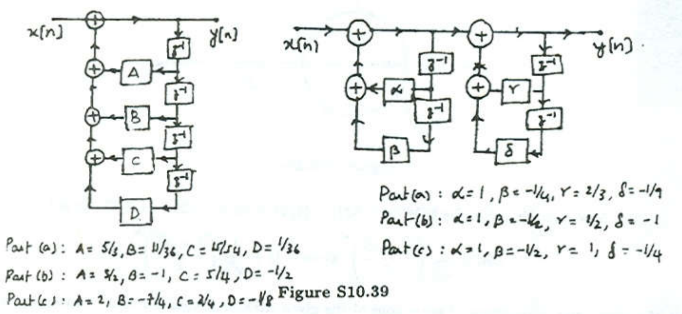

Figure S10.39

Note that

$$H_1(z) = \left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{2}{3}z^{-1}}\right]\left[\frac{1}{1 - \frac{2}{3}z^{-1}}\right].$$

Therefore, $H_1(z)$ may be drawn as a cascade of four systems for which the coefficient multipliers are all real.

(b) The direct form block diagram may be drawn as shown in part (b-i) of Figure S10.39 by noting that

$$H_2(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + 2z^{-2} - \frac{5}{4}z^{-3} + \frac{1}{2}z^{-4}}.$$

The cascade block-diagram is as shown in part (b-ii) of Figure S10.39.

Note that

$$H_2(z) = \left[\frac{1}{1 - \frac{(1+j)}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1-j}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1+j\sqrt{15}}{4}z^{-1}}\right]\left[\frac{1}{1 - \frac{1-j\sqrt{15}}{4}z^{-1}}\right].$$

Therefore, $H_1(z)$ cannot be drawn as a cascade of four systems for which the coefficient multipliers are all real.

(c) The direct form block diagram may be drawn as shown in part (c-i) of the Figure S10.39 by noting that

$$H_3(z) = \frac{1}{1 - 2z^{-1} + \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} + \frac{1}{8}z^{-4}}.$$

The cascade block-diagram is as shown in part (c-ii) of the Figure S10.39.

Note that

$$H_3(z) = \left[\frac{1}{1 - \frac{(1+j)}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1-j}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right].$$

Therefore, $H_1(z)$ cannot be drawn as a cascade of four systems for which the coefficient multipliers are all real.

10.40. The definition of the unilateral z-transform is

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}.$$

(a) Since $x[n] = \delta[n+5]$ is zero in the range $0 \le n \le \infty$, $\mathcal{X}(z) = 0$.

(b) The unilateral Laplace transform of $x[n] = \delta[n-5]$ is

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} \delta[n-5]z^{-n} = e^{-5\omega}.$$

(c) The unilateral Laplace transform of $x[n] = (-1)^n u[n]$ is

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} (-1)^n u[n]z^{-n} = \frac{1}{1 + z^{-1}}, \qquad |z| > 1.$$

(d) The unilateral Laplace transform of $x[n] = (1/2)^n u[n+3]$ is

$$\begin{aligned}
\mathcal{X}(z) &= \sum_{n=0}^{\infty} (1/2)^n u[n+3]z^{-n} \\
&= \sum_{n=0}^{\infty} (1/2)^n z^{-n} \\
&= \frac{1}{1 - (1/2)z^{-1}}, \qquad |z| > 1/2.
\end{aligned}$$

(e) Since $x[n] = (-1/3)^n u[-n-2]$ is zero in the range $0 \le n \le \infty$, $\mathcal{X}(z) = 0$.

(f) The unilateral Laplace transform of $x[n] = (1/4)^n u[-n+3]$ is

$$\begin{aligned}
\mathcal{X}(z) &= \sum_{n=0}^{\infty} (1/4)^n u[-n+3]z^{-n} \\
&= \sum_{n=0}^{3} (1/4)^n z^{-n} \\
&= 1 + \frac{1}{4}z^{-1} + \frac{1}{16}z^{-2} + \frac{1}{64}z^{-3}, \qquad \text{All } z.
\end{aligned}$$

(g) The unilateral Laplace transform of $x[n] = 2^n u[-n] + (1/4)^n u[n-1]$ is

$$\begin{aligned}
\mathcal{X}(z) &= \sum_{n=0}^{\infty} 2^n u[-n] + (1/4)^n u[n-1] z^{-n} \\
&= \sum_{n=0}^{\infty} (1/4)^n z^{-n} \\
&= \frac{1}{1-\frac{1}{4}z^{-1}}, \quad \text{All } z.
\end{aligned}$$

(h) The unilateral Laplace transform of $x[n] = (1/3)^{n-2} u[n-2]$ is

$$\begin{aligned}
\mathcal{X}(z) &= \sum_{n=0}^{\infty} (1/3)^{n-2} u[n-2] z^{-n} \\
&= z^{-2} \sum_{n=0}^{\infty} (1/3)^n z^{-n} \\
&= \frac{z^{-2}}{1-(1/2)z^{-1}}, \quad |z| > 1/2.
\end{aligned}$$

10.41. From the given information,

$$\begin{aligned}
\mathcal{X}_1(z) &= \sum_{n=0}^{\infty} (1/2)^{n+1} u[n+1] z^{-n} \\
&= (1/2) \sum_{n=0}^{\infty} (1/2)^n z^{-n} \\
&= \frac{1/2}{1-(1/2)z^{-1}}, \quad |z| > 1/2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{X}_2(z) &= \sum_{n=0}^{\infty} (1/4)^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} (1/4)^n z^{-n} \\
&= \frac{1}{1-(1/4)z^{-1}}, \quad |z| > 1/4.
\end{aligned}$$

Using Table 10.2 and the time shift property we get

$$X_1(z) = \frac{z}{1-\frac{1}{2}z^{-1}}, \quad |z| > 1/2.$$

and

$$X_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > 1/4.$$

(a) We have

$$G(z) = X_1(z) X_2(z) = \frac{z}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}.$$

The ROC is $|z| > (1/2)$. The partial fraction expansion of $G(z)$ is

$$G(z) = z\left[\frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right].$$

Using Table 10.2 and the time shift property, we get

$$g[n] = 2\left(\frac{1}{2}\right)^{n+1} u[n+1] - \left(\frac{1}{4}\right)^{n+1} u[n+1].$$

(b) We have

$$\mathcal{Q}(z) = \mathcal{X}_1(z)\mathcal{X}_2(z) = \frac{1/2}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}.$$

The ROC of $\mathcal{Q}(z)$ is $|z| > (1/2)$. The partial fraction expansion of $\mathcal{Y}(z)$ is

$$\mathcal{Q}(z) = \frac{1}{2}\left[\frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right].$$

Therefore,

$$q[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{4}\right)^n u[n].$$

Clearly, $q[n] \neq g[n]$ for $n > 0$.

10.42. (a) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) + 3z^{-1}\mathcal{Y}(z) + 3y[-1] = \mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = \frac{-3}{1+3z^{-1}}.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = -3(-3)^n u[n] = (-3)^{n+1} u[n].$$

Now, since it is given that $x[n] = (1/2)^n u[n]$, we have

$$\mathcal{X}(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > 1/2.$$

Setting $y[-1]$ to be zero, we get

$$\mathcal{Y}(z) + 3z^{-1}\mathcal{Y}(z) = \frac{1}{1-\frac{1}{2}z^{-1}}.$$

Therefore,

$$\mathcal{Y}(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}.$$

The partial fraction expansion of $\mathcal{Y}(z)$ is

$$\mathcal{Y}(z) = \frac{1/7}{1-\frac{1}{2}z^{-1}} + \frac{6/7}{1+3z^{-1}}.$$

The inverse unilateral z-transform gives the zero-state response

$$y_{zs}[n] = \frac{1}{7}\left(\frac{1}{2}\right)^n u[n] + \frac{6}{7}(-3)^n u[n].$$

(b) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = 0.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = 0.$$

Now, since it is given that $x[n] = u[n]$, we have

$$\mathcal{X}(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1.$$

Setting $y[-1]$ to be zero, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) = \frac{1}{1-z^{-1}} - \frac{(1/2)z^{-1}}{1-z^{-1}}.$$

Therefore,

$$\mathcal{Y}(z) = \frac{1}{1-z^{-1}}.$$

The inverse unilateral z-transform gives the zero-state response

$$y_{zs}[n] = u[n].$$

(c) Taking the unilateral z-transform of both sides of the given difference equation, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

Setting $\mathcal{X}(z) = 0$, we get

$$\mathcal{Y}(z) = \frac{1/2}{1-\frac{1}{2}z^{-1}}.$$

The inverse unilateral z-transform gives the zero-input response

$$y_{zi}[n] = \left(\frac{1}{2}\right)^{n+1} u[n].$$

Since the input $x[n]$ is the same as the one used in the part (b), the zero-state response is still

$$y_{zs}[n] = u[n].$$

10.43. (a) First let us determine the z-transform $X_1(z)$ of the sequence $x_1[n] = x[-n]$ in terms of $X(z)$:

$$\begin{aligned}
X_1(z) &= \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n] z^{n} \\
&= X(1/z)
\end{aligned}$$

Therefore, if $x[n] = x[-n]$, then $X(z) = X(1/z)$.

(b) If z_0 is a pole, then $1/X(z_0) = 0$. From the result of part (a), we know that $X(z_0) = X(1/z_0)$. Therefore, $1/X(z_0) = 1/X(1/z_0) = 0$. This implies that there is a pole at $1/z_0$.

If z_0 is a zero, then $X(z_0) = 0$. From the result of part (a), we know that $X(z_0) = X(1/z_0) = 0$. This implies that there is a zero at $1/z_0$.

(c) (1) In this case,

$$X(z) = z + z^{-1} = \frac{1+z^2}{z}, \quad |z| > 0.$$

$X(z)$ has zeros $z_1 = j$ and $z_2 = -j$. Also, $X(z)$ has the poles $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = 1/z_1$ and $p_1 = 1/p_2$, which proves that the statement of (b) is true.

(2) In this case,

$$X(z) = z - \frac{5}{2} + z^{-1} = \frac{1-\frac{5}{2}z+z^2}{z}, \quad |z| > 0.$$

$X(z)$ has zeros $z_1 = -1/2$ and $z_2 = -2$. Also, $X(z)$ has the poles $p_1 = 0$ and $p_2 = \infty$. Clearly, $z_2 = 1/z_1$ and $p_1 = 1/p_2$, which proves that the statement of (b) is true.

10.44. (a) Using the shift property, we get

$$\mathcal{Z}\{\Delta x[n]\} = X(z) - z^{-1}X(z) = (1-z^{-1})X(z).$$

(b) The z-transform $X_1(z)$ is given by

$$\begin{aligned}X_1(z) &= \sum_{n=-\infty}^{\infty} x_1[n]z^{-n}\\ &= \sum_{n=-\infty}^{\infty} x[n]z^{-2n}\\ &= X(2z).\end{aligned}$$

(c) Let us define a signal $g[n] = \{x[n]+(-1)^n x[n]\}/2$. Note that $g[2n] = x[2n]$ and $g[n] = 0$ for n odd. Also, using Table 10.1, we get

$$G(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z).$$

The z-transform $X_1(z)$ is given by

$$\begin{aligned}X_1(z) &= \sum_{n=-\infty}^{\infty} x_1[n]z^{-n}\\ &= \sum_{n=-\infty}^{\infty} g[2n]z^{-n}\\ &= \sum_{\substack{n=-\infty\\ \text{even}}}^{\infty} g[n]z^{-n/2}\\ &= \sum_{n=-\infty}^{\infty} g[n]z^{-n/2}\\ &= G(z^{1/2})\\ &= \frac{1}{2}X(z^{1/2}) + \frac{1}{2}X(-z^{1/2}).\end{aligned}$$

10.45. In each part of this problem, we assume that the signal obtained by taking the inverse z-transform is called $x[n]$.

(a) Yes. The order of the numerator is equal to the order of the denominator in the given z-transform. Therefore, we can perform long-division to expand the z-transform such that the highest power of z in the expansion is 0. This would make $x[n] = 0$ for $n < 0$.

(b) No. This z-transform can be obtained by multiplying the z-transform of the previous part by z. Hence, its inverse is the inverse of the previous part shifted by 1 to the left. This implies that the resultant signal is not zero at $n = -1$.

(c) Yes. We can perform long-division to expand the z-transform such that the highest power of z in the expansion is -1. This would make $x[n] = 0$ for $n \le 0$.

(d) No. When long-division is used to expand the z-transform, the highest power of z in the expansion is 1. This would make $x[-1] \ne 0$.

10.46. (a) Taking the z-transform of both sides of the difference equation relating $x[n]$ and $s[n]$ and simplifying, we get

$$H_1(z) = \frac{X(z)}{S(z)} = 1 - z^{-8}e^{-8\alpha} = \frac{z^8 - e^{-8\alpha}}{z^8}.$$

The system has an 8th order pole at $z = 0$ and 8 zeros distributed around a circle of radius $e^{-\alpha}$. This is shown in Figure S10.46. The ROC is everywhere on the z-plane except at $z = 0$.

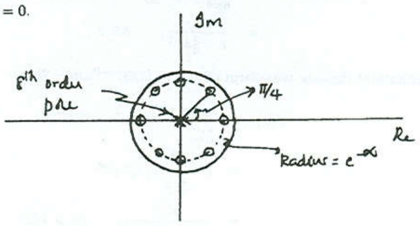

Figure S10.46

(b) We have

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{S(z)}{X(z)} = \frac{1}{H_1(z)}.$$

Therefore,

$$H_2(z) = \frac{1}{1 - z^{-8}e^{-8\alpha}} = \frac{z^8}{z^8 - e^{-8\alpha}}.$$

There are two possible ROCs for $H_2(z)$: $|z| < e^{-\alpha}$ or $|z| > e^{-\alpha}$. If the ROC is $|z| < e^{-\alpha}$, then the ROC does not include the unit circle. This in turn implies that the system would be unstable and anti-causal. If the ROC is $|z| > e^{-\alpha}$, then the ROC includes the unit circle. This in turn implies that the system would be stable and causal.

(c) We have

$$H_2(z) = \frac{1}{1 - z^{-8}e^{-8\alpha}}.$$

We need to choose the ROC to be $|z| > e^{-\alpha}$ in order to get s stable system. Now consider

$$P(z) = \frac{1}{1 - z^{-1}e^{-8\alpha}}$$

with ROC $|z| > e^{-\alpha}$. Taking the inverse z-transform, we get

$$p[n] = e^{-8\alpha n}u[n].$$

Now, note that

$$H_2(z) = P(z^8).$$

From Table 10.1 we know that

$$h_2[n] = \begin{cases} p[n/8] = e^{-\alpha n}, & n = 0, \pm 8, \pm 16, \cdots \\ 0, & \text{otherwise}\end{cases}$$

10.47. (a) From Clue 1, we have $H(-2) = 0$. From Clue 2, we know that when

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

we have

$$Y(z) = 1 + \frac{a}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}.$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + a - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}.$$

Substituting $z = -2$ in the above equation and noting that $H(-2) = 0$, we get

$$a = -\frac{9}{8}.$$

(b) The response to the signal $x[n] = 1 = 1^n$ will be $y[n] = H(1)x[n]$. Therefore,

$$y[n] = H(1) = \frac{1}{4}.$$

10.48. From the pole-zero diagram, we may write

$$H_1(z) = A\frac{(z - \frac{3}{4}e^{j\pi/4})(z - \frac{3}{4}e^{-j\pi/4})}{(z - \frac{3}{4}e^{j3\pi/4})(z - \frac{3}{4}e^{-j3\pi/4})}$$

and

$$H_2(z) = B\frac{(z - \frac{1}{2}e^{j3\pi/4})(z - \frac{1}{2}e^{-j3\pi/4})}{(z - \frac{1}{2}e^{j\pi/4})(z - \frac{1}{2}e^{-j\pi/4})}$$

where A and B are constants. Now note that

$$H_2(z) = \frac{B}{A}H_1\left(\frac{3}{2}ze^{j\pi}\right) = \frac{B}{A}H_1\left(-\frac{3}{2}z\right).$$

Using the property 10.5.3 of the z-transform (see Table 10.1), we get

$$h_2[n] = \frac{B}{A}\left(-\frac{2}{3}\right)^n h_1[n].$$

We may rewrite this as

$$h_2[n] = g[n]h_1[n],$$

where $g[n] = (B/A)(-2/3)^n$. Note that since both $h_1[n]$ and $h_2[n]$ are causal, we may assume that $g[n] = 0$ for $n < 0$. Therefore,

$$g[n] = \frac{B}{A}\left(-\frac{2}{3}\right)^n u[n].$$

Now, clue 3 also states that $\sum_{k=0}^{\infty}|g[k]| = 3$. Therefore,

$$\sum_{k=0}^{\infty}\frac{B}{A}\left(-\frac{2}{3}\right)^k = 3$$

or

$$\frac{B}{A}\frac{1}{1 - 2/3} = 3 \quad\Rightarrow\quad \frac{B}{A} = 1.$$

Therefore,

$$g[n] = \left(-\frac{2}{3}\right)^n u[n].$$

10.49. (a) We may write the left side of eq. (P10.49-1) as

$$\sum_{n=N_1}^{\infty}|x[n]|r_1^{-n} = \sum_{n=N_1}^{\infty}|x[n]|\left(r_0\frac{r_1}{r_0}\right)^{-n} = \sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}\left(\frac{r_1}{r_0}\right)^{-n} \qquad \text{(S10.49-1)}$$

Since $r_1 \ge r_0$, the sequence $(r_1/r_0)^{-n}$ decays with increasing n, i.e., as $n \to \infty$ $(r_1/r_0)^{-n} \to 0$. Therefore, $(r_1/r_0)^{-n} \le (r_1/r_0)^{-N_1}$ for $n \ge N_1$. Substituting this in eq.(S10.49-1), we get

$$\sum_{n=N_1}^{\infty}|x[n]|r_1^{-n} = \sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}\left(\frac{r_1}{r_0}\right)^{-n} \le \left(\frac{r_1}{r_0}\right)^{-N_1}\sum_{n=N_1}^{\infty}|x[n]|r_0^{-n}.$$

Therefore, $A = (r_1/r_0)^{-N_1} = (r_0/r_1)^{N_1}$.

(b) The above inequality shows that if $X(z)$ has the finite bound B for $|z| = r_0$, then $X(z)$ has the finite bound $(r_0/r_1)^{N_1}B$ for $|z| = r_1 \ge r_0$. Thus, $X(z)$ converges for $|z| = r_1 \ge r_0$ and Property 4 of Section 10.2 follows.

(c) Consider a left-sided sequence $x[n]$ such that

$$x[n] = 0, \qquad n > N_2$$

and for which

$$\sum_{n=-\infty}^{\infty}|x[n]|r_0^{-n} = \sum_{n=-\infty}^{N_2}|x[n]|r_0^{-n}.$$

Then we need to show that if $r_1 \le r_0$,

$$\sum_{n=-\infty}^{N_2} |x[n]| r_1^{-n} \le P \sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n}. \tag{S10.49-2}$$

where P is a positive constant.

We may write the left side of eq. (S10.49-2) as

$$\sum_{n=-\infty}^{N_2} |x[n]| r_1^{-n} = \sum_{n=-\infty}^{N_2} |x[n]| \left(r_0 \frac{r_1}{r_0}\right)^{-n} = \sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n} \left(\frac{r_1}{r_0}\right)^{-n}. \tag{S10.49-3}$$

Since $r_1 \le r_0$, the sequence $(r_1/r_0)^{-n}$ decays with decreasing n, i.e., as $n \to -\infty$ $(r_1/r_0)^{-n} \to 0$. Therefore, $(r_1/r_0)^{-n} \le (r_1/r_0)^{-N_2}$ for $n \le N_2$. Substituting this in eq.(S10.49-3), we get

$$\sum_{n=-\infty}^{N_2} |x[n]| r_1^{-n} = \sum_{n=-\infty}^{N_2} |x[n]| r_0^{-n} \left(\frac{r_1}{r_0}\right)^{-n} \le \left(\frac{r_1}{r_0}\right)^{-N_2} \sum_{n=-\infty}^{N_2} |x[n]| r_0^{\,n}$$

Therefore, $P = (r_1/r_0)^{-N_2} = (r_0/r_1)^{N_2}$.

The above inequality shows that if $X(z)$ has the finite bound B for $|z| = r_0$, then $X(z)$ has the finite bound $(r_0/r_1)^{N_2} B$ for $|z| = r_1 \le r_0$. Thus, $X(z)$ converges for $|z| = r_1 \le r_0$ and Property 5 of Section 10.2 follows.

10.50. (a) From the given pole-zero plot, we get

$$H(z) = A \frac{z^{-1} - a}{1 - az^{-1}},$$

where A is some constant. Therefore,

$$H(e^{j\omega}) = A \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}}$$

and

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) H^* e^{j\omega} = |A|^2 \left[\frac{e^{-j\omega} - a}{1 - ae^{-j\omega}}\right] \left[\frac{e^{j\omega} - a}{1 - ae^{j\omega}}\right].$$

Therefore,

$$|H(e^{j\omega})|^2 = |A|^2 \frac{1 - ae^{-j\omega} - ae^{j\omega} + a^2}{1 - ae^{-j\omega} - ae^{j\omega} + a^2} = |A|^2.$$

This implies that $|H(e^{j\omega})| = |A|$ =constant.

(b) We get $|v_1|^2 = 1 + a^2 - 2a\cos(\omega)$.

(c) We get

$$|v_2|^2 = 1 + \frac{1}{a^2} - \frac{2}{a}\cos\omega = \frac{1}{a^2}[a^2 + 1 + 2a\cos\omega] = \frac{1}{a^2}|v_1|^2$$

10.51. (a) We know that for a real sequence $x[n]$, $x[n] = x^*[n]$. Let us first find the z-transform of $y[n] = x^*[n]$ in terms of $X(z)$, the z-transform of $x[n]$. We have

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} \\
&= \left[\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n}\right]^* \\
&= [X(z^*)]^* = X^*(z^*).
\end{aligned}$$

Now, since $x[n] = x^*[n]$, we have $\mathcal{Z}\{x[n]\} = \mathcal{Z}\{x^*[n]\}$ which in turn implies that $X(z) = X^*(z^*)$.

(b) If $X(z)$ has a pole at $z = z_0$, then $1/X(z_0) = 0$. From the result of the previous part, we know that

$$\frac{1}{X^*(z_0^*)} = 0.$$

Conjugating both sides, we get $1/X(z_0^*) = 0$. This implies that $X(z)$ has a pole at z_0^*.

If $X(z)$ has a zero at $z = z_0$, then $X(z_0) = 0$. From the result of the previous part, we know that

$$X^*(z_0^*) = 0.$$

Conjugating both sides, we get $X(z_0^*) = 0$. This implies that $X(z)$ has a zero at z_0^*.

(c) (1) The z-transform of the given sequence is

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}, \qquad |z| > 1/2.$$

Clearly, $X(z)$ has a pole at $z = 1/2$ and a zero at $z = 0$ and the property of part (b) holds.

(2) The z-transform of the given sequence is

$$X(z) = 1 - \frac{1}{2}z^{1} + \frac{1}{4}z^{-2} = \frac{z^2 - (1/2)z + (1/4)}{z^2}, \qquad |z| > 0.$$

$X(z)$ has two zeros at $z = 1/2$ and two poles at $z = 0$. The property of part (b) still holds.

(d) Now, from part (b) of problem 10.43 we know that if $x[n]$ and $X(z)$ has a pole at $z_0 = \rho e^{j\theta}$, then $X(z)$ must have a pole at $(1/z_0) = (1/\rho)e^{-j\theta}$.

If $x[n]$ is real and $X(z)$ has a pole at $z_0 = \rho e^{j\theta}$, then from part (b) we know that $X(z)$ must have a pole at $z_0^* = \rho e^{-j\theta}$. Now, from part (b) of problem 10.43 we know that if $x[n]$ and $X(z)$ has a pole at $z_0^* = \rho e^{-j\theta}$, then $X(z)$ must have a pole at $(1/z_0^*) = (1/\rho)e^{j\theta}$.

A similar argument may be constructed for zeros.

10.52. We have

$$\begin{aligned}
X_2(z) &= \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x_1[-n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x_1[n] z^{n} \\
&= X_1(z^{-1}) = X_1(1/z).
\end{aligned}$$

Using an argument similar to the one used on part (b) of problem 10.43, we may argue that if $X_1(z)$ has a pole (or zero) at $z = z_0$, then $X_2(z)$ must have a pole (or zero) at $z = 1/z_0$.

10.53. Let us assume that $x[n]$ is a sequence with z-transform $X(z)$ which has the ROC $\alpha < |z| < \beta$.

(a) (1) The z-transform of the sequence $y[n] = x[n - n_0]$ is

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}
\end{aligned}$$

Substituting $m = n - n_0$ in the above equation, we get

$$\begin{aligned}
Y(z) &= \sum_{m=-\infty}^{\infty} x[m] z^{-m-n_0} \\
&= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} \\
&= z^{-n_0} X(z).
\end{aligned}$$

Clearly, $Y(z)$ converges where $X(z)$ converges except for the addition or deletion of $z = 0$ because of the z^{-n_0} term. Therefore, the ROC of $Y(z)$ is $\alpha < |z| < \beta$ except for the possible addition or deletion of $z = 0$ in the ROC.

(2) The z-transform of the sequence $y[n] = z_0^n x[n]$ is

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n] (z/z_0)^{-n} \\
&= X(z/z_0)
\end{aligned}$$

Since $X(z)$ converges for $\alpha < |z| < \beta$, $Y(z)$ converges for $\alpha < |z/z_0| < \beta$. Therefore, the ROC of $Y(z)$ is $|z_0|\alpha < |z| < |z_0|\beta$.

(3) The z-transform of the sequence $y[n] = x[-n]$ is

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n] z^{n} \\
&= X(1/z)
\end{aligned}$$

Since $X(z)$ converges for $\alpha < |z| < \beta$, $Y(z)$ converges for $\alpha < |1/z| < \beta$. Therefore, the ROC of $Y(z)$ is $(1/\beta) < |z| < (1/\alpha)$.

(b) (1) From Problem 10.51(a), we know that the z-transform of the sequence $y[n] = x^*[n]$ is $Y(z) = X^*(z^*)$. The ROC of $Y(z)$ is the same as the ROC of $X(z)$.

(2) Suppose that the ROC of $x[n]$ is $\alpha < |z| < \beta$. From subpart (2) of part (a), the z-transform of $y[n] = z_0^n x[n]$ is

$$Y(z) = X(z/z_0)$$

with ROC $|z_0|\alpha < |z| < |z_0|\beta$. Therefore, $R_y = |z_0| R_x$.

10.54. (a) Let $x[n] = 0$ for $n > 0$. Then,

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=-\infty}^{0} x[n] z^{-n} \\
&= x[0] + x[-1]z + x[-2]z^2 + \cdots
\end{aligned}$$

Therefore,

$$\lim_{z \to 0} X(z) = x[0].$$

(b) Let $x[n] = 0$ for $n < 0$. Then,

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=0}^{\infty} x[n] z^{-n} \\
&= x[0] + x[1]z^{-1} + x[-2]z^{-2} + \cdots
\end{aligned}$$

Therefore,

$$\lim_{z\to\infty} z(X(z)-x[0]) = \lim_{z\to\infty} z\{x[1]z^{-1}+x[-2]z^{-2}+\cdots\} = x[1].$$

10.55. (a) From the initial value theorem, we have

$$\lim_{z\to\infty} X(z) = x[0] = \text{non-zero and finite.}$$

Therefore, as $z \to \infty$, $X(z)$ tends to a finite non-zero value. This implies that $X(z)$ has neither poles nor zeros at infinity.

(b) A rational z-transform is made up of factors of the form $1/(z-a)$ and $(z-b)$. Note that the factor $1/(z-a)$ has a pole at $z=a$ and a zero at $z=\infty$. Also note that the factor $(z-b)$ has a zero at $z=b$ and a pole at $z=\infty$. From the results of part (a), we know that a causal sequence has no poles or zero at infinity. Therefore, all zeros at infinity contributed by factors of the form $1/(z-a)$ must be cancelled out by the poles at infinity contributed by factors of the form $(z-b)$ This implies that the number of factors of the form $(z-b)$ equals the number of factors of the form $1/(z-a)$. Consequently, the number of zeros in the finite z-plane must equal the number of poles in the finite z-plane.

10.56. (a) The z-transform of $x_3[n]$ is

$$\begin{aligned}
X_3(z) &= \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right]z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k]\left[\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}\right] \\
&= \sum_{k=-\infty}^{\infty} x_1[k]\mathcal{Z}\{x_2[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x_1[k]\hat{X}_2(z)
\end{aligned}$$

(b) Using the time shifting property (10.5.2), we get

$$\hat{X}_2(z) = \mathcal{Z}\{x_2[n-k]\} = z^{-k}X_2(z),$$

where $X_2(z)$ is the z-transform of $x_2[n]$. Substituting in the result of part (a), we get

$$X_3(z) = X_2(z)\sum_{k=-\infty}^{\infty} x_1[k]z^{-k}.$$

(c) Noting that the z-transform of $x_1[n]$ may be written as

$$X_1(z) = \sum_{k=-\infty}^{\infty} x_1[k]z^{-k},$$

we may rewrite the result of part (b) as

$$X_3(z) = X_1(z)X_2(z).$$

10.57. (a) $X_1(z)$ is a polynomial of order N_1 in z^{-1}. $X_2(z)$ is a polynomial of order N_2 i Therefore, $Y(z) = X_1(z)X_2(z)$ is a polynomial of order N_1+N_2 in z^{-1}. This i that $M = N_1+N_2$.

(b) By noting that $y[0]$ is the coefficient of the z^0 term in $Y(z)$, $y[1]$ is the coeffici the z^{-1} term in $Y(z)$, and $y[2]$ is the coefficient of the z^{-2} term in $Y(z)$, we get

$$\begin{aligned}
y[0] &= x_1[0]x_2[0], \\
y[1] &= x_1[0]x_2[1] + x_1[1]x_2[0], \\
y[2] &= x_1[0]x_2[2] + x_1[1]x_2[1] + x_1[2]x_2[0].
\end{aligned}$$

(c) We note the pattern that emerges from part (b). The k-th point in the sequence is the coefficient of z^{-k} in $Y(z)$. The z^{-k} term of $Y(z)$ is formed by the following (the product of the z^0 term of $X_1(z)$ with the z^{-k} term of $X_2(z)$) + (the produ the z^{-1} term of $X_1(z)$ with the z^{-k+1} term of $X_2(z)$) + (the product of the z^{-2} of $X_1(z)$ with the z^{-k+2} term of $X_2(z)$) + ... + the (product of the z^{-N_1} ter $X_1(z)$ with the z^{-k+N_1} term of $X_2(z)$).
Therefore,

$$y[k] = \sum_{m=0}^{N_1} x_1[m]x_2[k-m].$$

Since $x_1[m]=0$ for $m > N_1$ and $m < 0$, we may rewrite this as

$$y[k] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[k-m].$$

10.58. Consider a causal and stable system with system function $H(z)$. Let its inverse syst have the system function $H_i(z)$. The poles of $H(z)$ are the zeros of $H_i(z)$ and the zeros $H(z)$ are the poles of $H_i(z)$.

For $H(z)$ to correspond to a be causal and stable system, all its poles must be witl the unit circle. Similarly, for $H_i(z)$ to correspond to a be causal and stable system, all poles must be within the unit circle. Since the poles of $H_i(z)$ are the zeros of $H(z)$, t previous statement implies that the zeros of $H(z)$ must be within the unit circle. Therefo all poles and zeros of a minimum-phase system must lie within the unit circle.

0.59. (a) From Figure S10.59, we have

$$W_1(z) = X(z) - \frac{k}{3}z^{-1}W_1(z) \quad\Rightarrow\quad W_1(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}}.$$

Also,

$$W_2(z) = -\frac{k}{4}z^{-1}W_1(z) = -X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Therefore, $Y(z) = W_1(z) + W_2(z)$ will be

$$Y(z) = X(z)\frac{1}{1+\frac{k}{3}z^{-1}} - X(z)\frac{\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Finally,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{k}{4}z^{-1}}{1+\frac{k}{3}z^{-1}}.$$

Since $H(z)$ corresponds to a causal filter, the ROC will be $|z| > |k|/3$.

(b) For the system to be stable, the ROC of $H(z)$ must include the unit circle. This is possible only if $|k|/3 < 1$. This implies that $|k|$ has to be less than 3.

(c) If $k=1$, then

$$H(z) = \frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{3}z^{-1}}.$$

The response to $x[n] = (2/3)^n$ will be of the form

$$y[n] = x[n]H(2/3) = \frac{5}{12}(2/3)^n.$$

60. The unilateral z-transform of $y[n] = x[n+1]$ is

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n]z^{-n} \\
&= y[0] + y[1]z^{-1} + y[2]z^{-2} + \cdots \\
&= x[1] + x[2]z^{-1} + x[3]z^{-2} + \cdots \\
&= z\{x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots\} - zx[0] \\
&= z\mathcal{X}(z) - zx[0].
\end{aligned}$$

10.61. (a) The unilateral z-transform of $y[n] = x[n+3]$ is

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n]z^{-n} \\
&= \sum_{n=0}^{\infty} x[n+3]z^{-n} \\
&= \sum_{n=-3}^{\infty} x[n+3]z^{-n} - x[0]z^3 - x[1]z^2 - x[2]z \\
&= \sum_{n=0}^{\infty} x[n]z^{-n+3} - x[0]z^3 - x[1]z^2 - x[2]z \\
&= z^3\sum_{n=0}^{\infty} x[n]z^{-n} - x[0]z^3 - x[1]z^2 - x[2]z \\
&= z^3\mathcal{X}(z) - x[0]z^3 - x[1]z^2 - x[2]z
\end{aligned}$$

(b) The unilateral z-transform of $y[n] = x[n-3]$ is

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n]z^{-n} \\
&= \sum_{n=0}^{\infty} x[n-3]z^{-n} \\
&= \sum_{n=3}^{\infty} x[n-3]z^{-n} + x[-1]z^{-2} + x[-2]z^{-1} + x[-3] \\
&= \sum_{n=0}^{\infty} x[n]z^{-n-3} + x[-1]z^{-2} + x[-2]z^{-1} + x[-3] \\
&= z^{-3}\sum_{n=0}^{\infty} x[n]z^{-n} + x[-1]z^{-2} + x[-2]z^{-1} + x[-3] \\
&= z^{-3}\mathcal{X}(z) + x[-1]z^{-2} + x[-2]z^{-1} + x[-3]
\end{aligned}$$

(c) We have

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{m=0}^{\infty} x[n-m].$$

Therefore,

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{m=0}^{\infty} z^{-m}\mathcal{X}(z) + \sum_{m=1}^{\infty} z^{-m}\sum_{l=1}^{m} x[-l]z^{l} \\
&= \frac{\mathcal{X}(z)}{1-z^{-1}} + \sum_{m=1}^{\infty} z^{-m}\sum_{l=1}^{m} x[-l]z^{l}
\end{aligned}$$

10.62. Note that

$$\phi_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k] = x[n] * x[-n].$$

Now, applying the convolution property, the z-transform of $\phi_{xx}[n]$ is

$$\Phi_{xx}(z) = X(z)\mathcal{Z}\{x[-n]\}.$$

From the time-reversal property we know that the z-transform of $x[-n]$ is $X(1/z)$. Therefore,

$$\Phi_{xx}(z) = X(z)X(1/z).$$

10.63. (a) Since the ROC is $|z| < 1/2$, the sequence is left-sided. Using the power-series expansion, we get

$$\log(1-2z) = -\sum_{n=1}^{\infty} \frac{2^n z^n}{n} = -\sum_{n=-\infty}^{-1} -\frac{2^{-n}z^{-n}}{n}.$$

Therefore,

$$x[n] = \frac{2^{-n}}{n}u[-n-1].$$

(b) Since the ROC is $|z| > 1/2$, the sequence is right-sided. Using the power-series expansion, we get

$$\log(1-(1/2)z^{-1}) = -\sum_{n=1}^{\infty} \frac{(1/2)^n z^{-n}}{n}.$$

Therefore,

$$x[n] = -\frac{2^{-n}}{n}u[n-1].$$

10.64. Let us define $Y(z)$ to be

$$Y(z) = -z\frac{d}{dz}X(z).$$

Then using the differentiation property of the z-transform, we get

$$y[n] = nx[n].$$

(a) Now,

$$Y(z) = -z\frac{d}{dz}X(z) = z\frac{2}{1-2z} = -\frac{1}{1-\frac{1}{2}z^{-1}}.$$

Noting that the ROC of $Y(z)$ is $|z| < (1/2)$ (the same as the ROC of $X(z)$), we get

$$y[n] = \left(\frac{1}{2}\right)^n u[-n-1].$$

Therefore,

$$x[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1] = \frac{2^{-n}}{n}u[-n-1].$$

This is same as the answer obtained for Problem 10.63(a).

(b) In this part,

$$Y(z) = -z\frac{d}{dz}X(z) = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}.$$

Noting that the ROC of $Y(z)$ is $|z| > (1/2)$ (the same as the ROC of $X(z)$), we get

$$y[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

Therefore,

$$x[n] = -\frac{1}{n}\left(\frac{1}{2}\right)^n u[n-1] = -\frac{2^{-n}}{n}u[n-1].$$

This is same as the answer obtained for Problem 10.63(b).

10.65. (a) From the given $H_c(s)$, we get

$$|H_c(j\omega)| = \frac{|a-j\omega|}{|a+j\omega|} = \frac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}} = 1.$$

(b) Applying the bilinear transformation, we get

$$H_d(z) = \frac{a - \frac{1-z^{-1}}{1+z^{-1}}}{a + \frac{1-z^{-1}}{1+z^{-1}}} = \frac{a-1}{a+1}\left[\frac{1+z^{-1}\frac{a+1}{a-1}}{1+z^{-1}\frac{a-1}{a+1}}\right].$$

Therefore, $H_d(z)$ has a pole at $z = (a-1)/(a+1)$ and a zero at $z = (a+1)/(a-1)$. Since a is real and positive,

$$\left|\frac{a-1}{a+1}\right| \le 1 \quad \text{and} \quad \left|\frac{a+1}{a-1}\right| \ge 1.$$

Therefore, the pole of $H_d(z)$ lies inside the unit circle and the zero of $H_d(z)$ lies outside the unit circle.

(c) $H_d(z)$ may be rewritten as

$$H(z) = \frac{(a-1)+z^{-1}(a+1)}{(a+1)+z^{-1}(a-1)}.$$

Therefore,

$$|H(e^{j\omega})| = \frac{|(a-1)+e^{-j\omega}(a+1)|}{|(a+1)+e^{-j\omega}(a-1)|} = \frac{|(a-1)+(\cos\omega - j\sin\omega)(a+1)|}{|(a+1)+(\cos\omega - j\sin\omega)(a-1)|}.$$

This may be written as

$$\begin{aligned} |H(e^{j\omega})| &= \frac{\sqrt{(a-1)^2+\cos^2\omega(a+1)^2+2(a+1)(a-1)\cos\omega+(a+1)^2\sin^2\omega}}{\sqrt{(a+1)^2+\cos^2\omega(a-1)^2+2(a+1)(a-1)\cos\omega+(a-1)^2\sin^2\omega}} \\ &= \frac{\sqrt{(a-1)^2+(a+1)^2+2(a+1)(a-1)\cos\omega}}{\sqrt{(a+1)^2+(a-1)^2+2(a+1)(a-1)\cos\omega}} = 1 \end{aligned}$$

10.66. (a) We are given that

$$H_d(z) = H_c\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

Therefore,

$$H_d(e^{j\omega}) = H_c\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = H_c\left(\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right) = H_c\left(j\tan\frac{\omega}{2}\right).$$

(b) From the given $H_c(s)$, we get

$$H_c(0) = \frac{1}{(e^{j\pi/4})(e^{-j\pi/4})} = 1$$

and

$$H_c(\infty) = \frac{1}{\lim_{s\to\infty}(s+e^{j\pi/4})(s+e^{-j\pi/4})} = 0.$$

Now,

$$\begin{aligned} |H_c(j\omega)| &= \frac{1}{|(j\omega+e^{j\pi/4})(j\omega+e^{-j\pi/4})|} = \frac{1}{|-\omega^2+2\cos(\pi/4)j\omega+1|} \\ &= \frac{1}{\sqrt{(1-\omega^2)^2+4\omega\cos^2(\pi/4)\omega^2}}. \end{aligned}$$

Clearly, $|H_c(j\omega)|$ decreases monotonically with increasing ω.

(c) (1) We are given that

$$H_d(z) = H_c\left(\frac{1-z^{-1}}{1+z^{-1}}\right).$$

Therefore,

$$H_d(z) = \frac{1}{(\frac{1-z^{-1}}{1+z^{-1}}+e^{j\pi/4})(\frac{1-z^{-1}}{1+z^{-1}}+e^{-j\pi/4})}.$$

This may be rewritten as

$$H_d(z) = \frac{1}{(1+e^{j\pi/4})(1+e^{-j\pi/4})}\frac{(1+z^{-1})^2}{[1-z^{-1}\frac{1+e^{j\pi/4}}{1-e^{j\pi/4}}][1-z^{-1}\frac{1+e^{-j\pi/4}}{1-e^{-j\pi/4}}]}.$$

Therefore, $H_d(z)$ has exactly two poles which lie at $z = -(1+e^{j\pi/4})/(1-e^{j\pi/4})$ and $z = -(1+e^{-j\pi/4})/(1-e^{-j\pi/4})$. It can be easily shown that both these poles lie inside the unit circle.

(2) From the result of part (a), we have

$$H_d(e^{j0}) = H_c(j\tan 0) = H_c(j0) = 1.$$

(3) We have

$$\begin{aligned} |H_d(e^{j\omega})| &= \left|H_c\left(j\tan\tfrac{\omega}{2}\right)\right| = \frac{1}{|1-\tan^2(\omega/2)+\sqrt{2}j\tan(\omega/2)|} \\ &= \frac{1}{\sqrt{(1-\tan^2(\omega/2))^2+2\tan^2(\omega/2)}} = \frac{1}{1+\tan^4(\omega/2)}. \end{aligned}$$

As ω increases from 0 to *pi*, $\tan(\omega/2)$ increases monotonically from 0 to ∞. Therefore, $|H_d(e^{j\omega})|$ decreases monotonically from 1 to 0.

(4) The half-power frequency ω_d satisfies the relationship

$$|H_d(e^{j\omega_d})|^2 = \frac{1}{2} = \left|H_c\left(j\tan\frac{\omega_d}{2}\right)\right|^2.$$

We know that $|H_c(j)|^2 = 1/2$. Therefore,

$$j\tan\frac{\omega_d}{2} = j \quad \Rightarrow \quad \omega_d = \pi/2.$$