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Date

1. 1) $Y \sim N(\theta, \sigma^2)$, PDF $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)$; $\theta \sim N(\mu, \gamma^2)$, PDF $f(\theta) = \frac{1}{\sqrt{2\pi\gamma^2}} \exp\left(-\frac{(\theta-\mu)^2}{2\gamma^2}\right)$.

$$p(\theta|y) = \frac{\pi(\theta) p(y|\theta)}{\int \pi(t) f(y|t) dt} \propto \pi(\theta) f(y|\theta)$$

$$= \frac{1}{\sqrt{2\pi\gamma^2}} \exp\left(-\frac{(\theta-\mu)^2}{2\gamma^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)$$

$$= \frac{1}{2\pi\gamma\sigma} \exp\left(-\frac{\sigma^2(\theta^2 - 2\theta\mu + \mu^2) + \gamma^2(y^2 - 2\theta y + \theta^2)}{2\gamma^2\sigma^2}\right)$$

$$= \frac{1}{2\pi\gamma\sigma} \exp\left(-\frac{(\sigma^2+\gamma^2)\theta^2 - (2\mu\sigma^2 + 2y\gamma^2)\theta + (\sigma^2\mu^2 + \gamma^2 y^2)}{2\gamma^2\sigma^2}\right)$$

$$= \frac{1}{2\pi\gamma\sigma} \exp\left(-\frac{\theta^2 - 2\cdot\frac{\mu\sigma^2 + y\gamma^2}{\sigma^2+\gamma^2}\theta + \frac{\sigma^2\mu^2 + \gamma^2 y^2}{\sigma^2+\gamma^2}}{\frac{2\gamma^2\sigma^2}{\sigma^2+\gamma^2}}\right).$$

$$\propto \exp\left(-\frac{\left(\theta - \frac{\mu\sigma^2 + y\gamma^2}{\sigma^2+\gamma^2}\right)^2}{2\cdot\frac{\gamma^2\sigma^2}{\sigma^2+\gamma^2}}\right) = N\left(\frac{\mu\sigma^2 + y\gamma^2}{\sigma^2+\gamma^2}, \frac{\gamma^2\sigma^2}{\gamma^2+\sigma^2}\right).$$

Therefore, $p(\theta|y)$ is Gaussian, with mean $\frac{\mu\sigma^2 + y\gamma^2}{\sigma^2+\gamma^2}$, and variance $\frac{\gamma^2\sigma^2}{\gamma^2+\sigma^2}$.

2) Bayesian detector $\delta(y) = \begin{cases} 1, & \text{if } \frac{f(y|\theta_1)}{f(y|\theta_0)} \geq \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})} = \frac{\pi_0}{\pi_1} = \frac{\pi(\theta \leq 0)}{\pi(\theta > 0)} \quad (*) . \\ 0, & \text{otherwise}. \end{cases}$

$$(*): \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)\Big|_{\theta>0}}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\theta)^2}{2\sigma^2}\right)\Big|_{\theta\leq 0}} \geq \frac{1-\pi(\theta>0)}{\pi(\theta>0)} \Leftrightarrow \exp\left(2(\theta_1-\theta_0)y + (\theta_0^2 - \theta_1^2)\right) \geq \frac{1-\pi(\theta>0)}{\pi(\theta>0)}.$$

where $\theta_0 \leq 0$, $\theta_1 > 0$.

Since $\begin{cases} \pi(\theta>0) = \pi\left(\frac{\theta-\mu}{\gamma} > \frac{0-\mu}{\gamma}\right) = Q\left(-\frac{\mu}{\gamma}\right). \\ \pi(\theta \leq 0) = 1 - \pi(\theta > 0) = 1 - Q\left(-\frac{\mu}{\gamma}\right). \end{cases}$

When $\begin{cases} H_0: \theta_0 \leq 0. \\ H_1: \theta_1 > 0, \end{cases}$ with one observation, $(*)$ becomes $1 \geq \frac{1-Q(-\frac{\mu}{\gamma})}{Q(-\frac{\mu}{\gamma})}$.

Therefore, $\delta(y) = \begin{cases} 1, & \text{if } Q\left(-\frac{\mu}{\gamma}\right) \geq \frac{1}{2} \quad \text{i.e. } \frac{\mu}{\gamma} \geq 0. \\ 0, & \text{otherwise}. \end{cases}$

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2. (1) $\mu_0 = \mathbb{E}(H_0) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^{\infty} y \cdot \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy = \sqrt{\frac{2}{\pi}}\sigma.$

$\mu_1 = \mathbb{E}(H_1) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^{\infty} y \cdot \frac{1}{\sigma} e^{-\frac{y}{\sigma}} dy = \sigma.$

Therefore, $\mu_1 > \mu_0$.

(2) $P_F = \Pr\left(y > \frac{\mu_0+\mu_1}{2}; H_0\right) = \int_{\frac{\mu_0+\mu_1}{2}}^{\infty} \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy = 2 \cdot \int_{\frac{\mu_0+\mu_1}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{y}{\sigma})^2}{2}} d\left(\frac{y}{\sigma}\right)$

$= 2Q\left(\frac{\mu_0+\mu_1}{2\sigma}\right) = 2Q\left(\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right).$

$P_D = \Pr\left(y > \frac{\mu_0+\mu_1}{2}; H_1\right) = \int_{\frac{\mu_0+\mu_1}{2}}^{\infty} \frac{1}{\sigma} e^{-\frac{y}{\sigma}} dy = -e^{-\frac{y}{\sigma}}\Big|_{\frac{\mu_0+\mu_1}{2}}^{\infty} = e^{-\frac{\mu_0+\mu_1}{2\sigma}} = \exp\left(-\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right).$

(3) $L(y) = \frac{f(y|\theta_1)}{f(y|\theta_0)} = \frac{\frac{1}{\sigma} e^{-\frac{y}{\sigma}}}{\frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}} = \sqrt{\frac{\pi}{2}} e^{\frac{y^2-2\sigma y}{2\sigma^2}}.$

(4) $L(y) \geq \tau_0$ i.e. $\sqrt{\frac{\pi}{2}} e^{\frac{y^2-2\sigma y}{2\sigma^2}} \geq \tau_0. \Leftrightarrow e^{\frac{1}{2}(\frac{y}{\sigma}-1)^2-\frac{1}{2}} \geq \sqrt{\frac{2}{\pi}}\tau_0.$

$\Leftrightarrow \frac{1}{2}\left(\frac{y}{\sigma}-1\right)^2 \geq \ln\left(\sqrt{\frac{2e}{\pi}}\tau_0\right)$

$\Leftrightarrow \left|\frac{y}{\sigma}-1\right| \geq \underbrace{\sqrt{2\ln\left(\sqrt{\frac{2e}{\pi}}\tau_0\right)}}_{\tau_1}.$

Therefore, $L(y) \geq \tau_0 \Leftrightarrow \left|\frac{y}{\sigma}-1\right| \geq \tau_1$.

(5) $\delta(y) = \begin{cases} 1, & \text{if } \left|\frac{y}{\sigma}-1\right| \geq \tau_1 \Leftrightarrow y \geq \sigma(1+\tau_1) \text{ or } y \leq \sigma(1-\tau_1). \\ 0, & \text{otherwise}. \end{cases}$

When $\tau_1 < 1$, $1-\tau_1 > 0$. $P_F = \Pr\left(\left|\frac{y}{\sigma}-1\right| \geq \tau_1; H_0\right) = \left(\int_0^{\sigma(1-\tau_1)} + \int_{\sigma(1+\tau_1)}^{\infty}\right) \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy$

$= \left(\int_0^{\infty} - \int_{\sigma(1-\tau_1)}^{\infty} + \int_{\sigma(1+\tau_1)}^{\infty}\right) \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy$

$= 1 - 2Q(1-\tau_1) + 2Q(1+\tau_1).$

(6) When $\tau_1 > 1$. $y \leq \sigma(1-\tau_1) < 0$ do not apply. Only $y \geq \sigma(1+\tau_1)$ applies.

$P_F = \Pr\left(\left|\frac{y}{\sigma}-1\right| \geq \tau_1; H_0\right) = \int_{\sigma(1+\tau_1)}^{\infty} \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy = 2Q(1+\tau_1) = \alpha.$

$\tau_1 = Q^{-1}\left(\frac{\alpha}{2}\right) - 1.$ From $\tau_1 > 1 \Rightarrow Q^{-1}\left(\frac{\alpha}{2}\right) - 1 > 1 \Rightarrow Q^{-1}\left(\frac{\alpha}{2}\right) > 2.$

$\delta(y) = \begin{cases} 1, & \text{if } \left|\frac{y}{\sigma}-1\right| \geq Q^{-1}\left(\frac{\alpha}{2}\right) - 1 \text{ i.e. } y \geq \sigma \cdot Q^{-1}\left(\frac{\alpha}{2}\right) \\ 0, & \text{otherwise}. \end{cases}$

(7) If $\tau_1 > 1$, from (2): $P_F = 2Q\left(\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right) = \alpha. \Rightarrow Q^{-1}\left(\frac{\alpha}{2}\right) = Q^{-1}\left(Q\left(\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right)\right)$

$= \frac{\sqrt{\frac{2}{\pi}}+1}{2} < 2.$

Therefore, the detector from (2) is NOT optimal for $\tau_1 > 1$ case.

If $\tau_1 < 1$ case, $1 - 2Q(1-\tau_1) + 2Q(1+\tau_1) = \alpha = 2Q\left(\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right).$

$\tau_1 \approx 0.6541 < 1$. applies.

Therefore, detector from (2) is optimal for $\alpha \doteq 2Q\left(\frac{\sqrt{\frac{2}{\pi}}+1}{2}\right)$.