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No.

Date . .

1. (a) Normal distribution $N(\theta, 1)$

PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2}\right)$.

, then $L(\theta; y) = \log f(y;\theta) = \log \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i-\theta)^2}{2}\right)$

$= \log\left[\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i-\theta)^2\right)\right]$

$= -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(y_i-\theta)^2$.

Since $L(\theta; y) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\left[\sum_{i=1}^{n} y_i^2 \underbrace{- 2\theta \sum_{i=1}^{n} y_i}_{g(t(y);\theta)} + n\theta^2\right]$.

then $\sum_{i=1}^{n} y_i$ is a sufficient statistic.

(b). Exponential distribution $\mathcal{E}(\theta)$

PDF $f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x \geq 0. \\ 0, & x < 0. \end{cases}$

Then $L(\theta; y) = \log f(y;\theta) = \log\left(\frac{1}{\theta^n} \cdot e^{-\sum_{i=1}^{n} y_i/\theta}\right) = -n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} y_i$, for $y_i \geq 0$.

Similarly, $\sum_{i=1}^{n} y_i$ is a sufficient statistic.

(c) Poisson distribution $P(\theta)$.

PDF $f_X(x) = e^{-\theta} \cdot \theta^x / x!$

$L(\theta; y) = \log f(y;\theta) = \log\left(e^{-n\theta} \cdot \frac{\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}\right) = -n\theta + \sum_{i=1}^{n} y_i \cdot \log\theta - \log\left(\prod_{i=1}^{n} y_i!\right)$

then $\sum_{i=1}^{n} y_i$ is a sufficient statistic.

(d). Bernoulli $B(\theta)$.

PDF $f_X(x) = \begin{cases} \theta, & x=1 \\ (1-\theta), & x=0. \end{cases} = \theta^x (1-\theta)^{1-x}$ for $x = 0, 1$.

$L(\theta; y) = \log f(y;\theta) = \log\left(\theta^{\sum_{i=1}^{n} y_i} (1-\theta)^{n - \sum_{i=1}^{n} y_i}\right) = \sum_{i=1}^{n} y_i \cdot \log\theta + \left(n - \sum_{i=1}^{n} y_i\right) \cdot \log(1-\theta)$.

then $\sum_{i=1}^{n} y_i$ is a sufficient statistic.

(e) Uniform distribution $\mathcal{U}(0, \theta)$.

See next page, thanks!

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1. (e). PDF $f_X(x) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{otherwise.} \end{cases}$

$$l(\theta; y) = \log f(y;\theta) = \begin{cases} \log\left(\frac{1}{\theta^n}\right) = -n\log\theta, & \text{if } \theta \geq \max\{y_i\} \ \& \ \min\{y_i\} \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

Since this PDF doesn't contain y, based on the boundary conditions, we can choose $t(y) = \max\{y_i\}$ as a sufficient statistic.

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2. (a) $X \sim B(\frac{1}{4})$. $Y \sim \begin{cases} N(\theta,1), & X=0 \\ N(\theta,2), & X=1. \end{cases}$ i.e. $f(y_i;\theta) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\theta)^2}{2}}, & X=0 \\ \frac{1}{\sqrt{2\pi \times 2}} e^{-\frac{(y_i-\theta)^2}{2\times 2}}, & X=1. \end{cases}$

$$l(\theta;y) \triangleq f(y;\theta) = \prod_{i=1}^{n} f(y_i;\theta).$$

$$= \begin{cases} (2\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i-\theta)^2\right), & X=0 \\ (4\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{4}\sum_{i=1}^{n}(y_i-\theta)^2\right), & X=1. \end{cases}$$

$$= \begin{cases} (2\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{n}y_i^2\right) \cdot \exp\left(\theta\sum_{i=1}^{n}y_i - \frac{n}{2}\theta^2\right), & X=0. \\ (4\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{4}\sum_{i=1}^{n}y_i^2\right) \cdot \exp\left(\frac{\theta}{2}\sum_{i=1}^{n}y_i - \frac{n}{4}\theta^2\right), & X=1. \end{cases}$$

Since the PDF $f(y;\theta)$ is a function of $\begin{cases} \sim \exp\left(\theta\sum_{i=1}^{n}y_i\right)\cdots, & X=0 \\ \sim \exp\left(\frac{\theta}{2}\sum_{i=1}^{n}y_i\right)\cdots, & X=1. \end{cases}$

From Neyman-Fisher theorem, $\left(\sum_{i=1}^{n}y_i, X\right)$ is a sufficient statistic.

(b). $f(y;\theta) = \begin{cases} (2\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2}\sum_{i=1}^{n}y_i^2\right) \cdot \exp\left(\theta\sum_{i=1}^{n}y_i - \frac{n}{2}\theta^2\right), & X=0 \\ (2\pi)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{4}\sum_{i=1}^{n}y_i^2\right) \cdot \exp\left(\frac{\theta}{2}\sum_{i=1}^{n}y_i - \frac{n}{4}\theta^2\right), & X=1. \end{cases}$ (*).

Given $X \sim B(\frac{1}{4})$, i.e. $f_X(x) = \begin{cases} \frac{1}{4}, & X=1 \\ \frac{3}{4}, & X=0. \end{cases} \Leftrightarrow f_X(x) = \left(\frac{1}{4}\right)^X \left(\frac{3}{4}\right)^{1-X}, X=0 \text{ or } 1.$

Therefore, $f(y;\theta) = (*) = h(y)\, g(t(y);\theta)$, $t(y)$ is a function of both $\sum_{i=1}^{n}y_i$ and X.

Since X cannot be removed, $\sum_{i=1}^{n}y_i$ is NOT a sufficient statistic.