Linear Algebra

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Vector, vector space

- A vector space V is a set of vectors that is closed under addition and scalar multiplication.
  - $\forall x, y \in V, x + y \in V$
  - $\forall x \in V, c \in \mathbb{R}, cx \in V$
- Example: $\mathbb{R}^n$
- Orthonormal basis of a vector space is a set of vectors $\{v_1, \dots, v_n\}$ that span the whole space and $||v_i|| = 1, v_i^\top v_j = 0, \forall i \neq j$

Norm of vector

A norm is any function $f: \mathbb{R}^n \to \mathbb{R}$ such that

- $f(x) \geq 0, \forall\, x \in \mathbb{R}^n$.
- $f(x) = 0 \;\leftrightarrow x = 0$.
- $f(tx) = |t| f(x), \forall x \in \mathbb{R}^n, t \in \mathbb{R}$
- $f(x + y) \leq f(x) + f(y), \forall\, x, y \in \mathbb{R}^n$

P-norm:

$$||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, p \geq 1$$

Matrix

- A: m x n matrix
- A: $\mathbb{R}^n \rightarrow \mathbb{R}^m$, linear operator, linear transform.

Norms of matrix, trace

- Induced matrix norm:

$$||A||_p = \max_{x \neq 0} \frac{||Ax||_p}{||x||_p}$$

If $p = 2$, $||A||_2 = \max_i \sigma_i$, largest singular value.

- Trace of the matrix: $tr(A) = \sum_i^n A_{ii}$
- Nuclear norm, trace norm: $||A||_* = tr\left(\sqrt{A^\top A}\right) = \sum_i \sigma_i$

Square matrix

- Positive definite:

$$x^\top A x > 0, \forall x \in \mathbb{R}^n$$

- Positive semidefinite if $\exists x, x^\top A x = 0$.
- Indefinite.

Eigenvalue, eigenvectors

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an eigenvalue of A, $\mathrm{v} \in \mathbb{R}^n$ is the eigenvector if
  - $Av = \lambda v, v \neq 0$
- $Av = \lambda v \leftrightarrow (A - \lambda I)v = 0 \leftrightarrow \det(A - \lambda I) = 0$
- $\det(A - \lambda I)$ is a n-degree polynomial in terms of $\lambda \rightarrow$ solution exists.
- The set of eigenvalues is called the *spectrum* of A

Eigenvalue decomposition

- If A is a real symmetric matrix, then
  - All eigenvalues are real.
  - There exists an orthonormal matrix U such that $A = U^\top \Sigma U$,
  - $\Sigma = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

- Any symmetric matrix A can be represented as

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^\top$$

Singular value decomposition

- For a general matrix $A \in \mathbb{R}^{m \times n}$, there always exists orthonormal matrices U, V such that $A = U \Sigma V^\top$, where Σ is a diagonal matrix.
  - $U \in \mathbb{R}^{m \times m}, \Sigma \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n}$
- A can be decomposed as

$$A = \sum_{i=1}^{m} \sigma_i \, u_i v_i^\top$$