CS/CNS/EE/IDS 165: Foundations of Machine Learning

Introduction to Probability

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Outline

Concepts

- Probability space
- Conditional probability and statistical independence.
- Random variables, distributions and densities.
- Expectations and conditional expectations.
- Real and complex Gaussian variables and vectors.
- Inequalities
- Convergence, LLN and CLT.

References

1. T. L. Fine, Probability and Probabilistic Reasoning, Prentice Hall, 2006.
2. D. T. Bersekas and J.N. Tsitsiklis, Introduction to Probability, Athena Scientific, 2002.
3. A. Papoulis, Probability, Random Variables and Stochastic Processes, McGraw-Hill, 4th edition, Dec. 2001.
4. Background Notes.

The Probability Space: Definition

Definition:

A probability space is defined by $(\Omega, \mathcal{F}, \Pr)$

1. Ω is the sample space that contains the set of outcomes.
2. $\mathcal{F}$ is a σ-field of subsets of Ω (events):

   (i) $\Omega \in \mathcal{F}$. (ii) If $\mathcal{E} \in \mathcal{F}$, then $\mathcal{E}^c \in \mathcal{F}$.

   (iii) If $\mathcal{E}_i \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} \mathcal{E}_i \in \mathcal{F}$.
3. $\Pr$ is a function on $\mathcal{F}$ satisfying

   (i) $0 \leq \Pr(\mathcal{E}) \leq 1$. (ii) $P(\Omega) = 1$.

   (iii) If $\mathcal{E}_1, \mathcal{E}_2, \cdots$ are disjoint, then

   $\Pr(\bigcup_{i=1}^{\infty} \mathcal{E}_i) = \sum \Pr(\mathcal{E}_i)$

Why Do We Need Restrictions on Events?

Let $\Omega \triangleq \{(x, y) | x^2 + y^2 = 1\}$. There exists[†] a set $\mathcal{E} \in \Omega$ such that

1. for any rational $\phi, \theta \in [0, 2\pi)$ and $\phi \neq \theta$, the rotation of $\mathcal{E}$ by θ and ϕ are disjoint, *i.e.*, $\mathcal{E}(\theta) \bigcap \mathcal{E}(\phi) = \emptyset$.
2. The union of all $\mathcal{E}$ rotated by rational θ is Ω.

If $\Pr(\mathcal{E}) = x$, then

$$1 = \Pr(\Omega) = \Pr(\bigcup \mathcal{E}(\theta)) = \sum \Pr(\mathcal{E}(\theta)) = \sum x$$

[†] M. Capiński and P. Knopp, *Measure, Integral and Probability*, Springer, 1999.

The Probability Space: Examples

Sample Space Ω

- Picking the "lucky" person out of a class of 30 to receive an A: $\Omega_1 = \{1, 2, \cdots, 29, 30\}$.
- Taking the qualify exam until pass: $\Omega_2 = \{P, FP, FFP, FFFP, \cdots, \}$.
- The time you wake up: $\Omega_3 = \{(00:00, 24:00]\}$
- Throwing a dart to a unit disk: $\Omega_4 = \{(x, y) | x^2 + y^2 \leq 1\}$.

Events

Consider Ω_1

$\mathcal{E}_0$: Someone is lucky: $\mathcal{E}_0 = \Omega_1$.

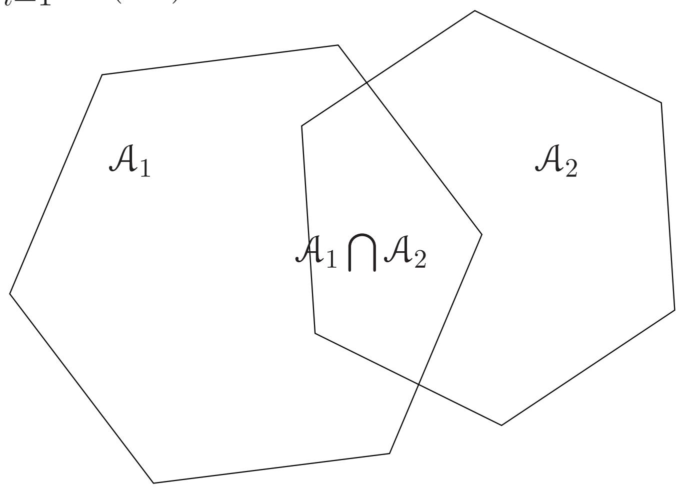
$\mathcal{E}_1$: the "lucky" person has an even ID: $\mathcal{E}_1 = \{2, 4, 6, \cdots, 30\}$.

$\mathcal{E}_2$ The "lucky" person has an even number or a number between 10 and 20. $\mathcal{E}_2 = \mathcal{E}_1 \bigcup \{11, 13, \cdots, 19\}$.

$\mathcal{E}_4$ The "lucky" person has an odd number less than 10. $\mathcal{E}_4 = \mathcal{E}_1^c \bigcap \{1, \cdots, 10\}$.

Elementary Properties

- $\Pr(\mathcal{A}^c) = 1 - \Pr(\mathcal{A}), \quad \Pr(\emptyset) = 0$.
- If $\mathcal{A} \subset \mathcal{B}$, then $\Pr(\mathcal{B}) = \Pr(\mathcal{A}) + \Pr(\mathcal{B} - \mathcal{A}) \geq \Pr(\mathcal{A})$.
- Union bound (Boole's inequality):
  $\Pr(\bigcup_{i=1}^{\infty} \mathcal{A}_i) \leq \sum_{i=1}^{\infty} \Pr(\mathcal{A}_i)$



- Inclusion-exclusion:

$$
\begin{aligned}
\Pr(\mathcal{A}_1 \bigcup \mathcal{A}_2) &= \Pr(\mathcal{A}_1) + \Pr(\mathcal{A}_2) - \Pr(\mathcal{A}_1 \bigcap \mathcal{A}_2) \\
\Pr(\bigcup_{i=1}^{n} \mathcal{A}_i) &= \sum_{i=1}^{n} \Pr(\mathcal{A}_i) - \sum_{i<j} \Pr(\mathcal{A}_i \bigcap \mathcal{A}_j) \\
&\quad + \sum_{i<j<k} \Pr(\mathcal{A}_i \bigcap \mathcal{A}_j \bigcap \mathcal{A}_k) - \cdots \\
&\quad + (-1)^{k+1} \sum_{i_1<i_2<\cdots<i_k} \Pr(\bigcap_{r=1}^{k} \mathcal{A}_{i_r}) + \cdots
\end{aligned}
$$

- Bonferroni's inequality: $\Pr(\bigcap_{i=1}^{n} \mathcal{A}_i) \geq 1 - \sum_{i=1}^{n} \Pr(\mathcal{A}_i^c)$

Sequence of Events

Monotone Convergence

If $\mathcal{E}_i$ increases, *i.e.*, $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$, and let $\mathcal{E} \triangleq \bigcup_{i=1}^{\infty} \mathcal{E}_i$. Then

$$\Pr(\mathcal{E}) = \lim_{i\to\infty} \Pr(\mathcal{E}_i)$$

If $\mathcal{E}_i$ decreases, *i.e.*, $\mathcal{E}_1 \supseteq \mathcal{E}_2 \supseteq \cdots$, and let $\mathcal{E} = \bigcap_{i=1}^{\infty} \mathcal{E}_i$. Then

$$\Pr(\mathcal{E}) = \lim_{i\to\infty} \Pr(\mathcal{E}_i)$$

Limits of Sequences

Let $\{\mathcal{E}_n\}$ be an arbitrary sequence of events. Define limits

$$\mathcal{E}^* = \limsup_{i\to\infty} \mathcal{E}_i \triangleq \bigcap_{i=1}^{\infty}\bigcup_{n=i}^{\infty} \mathcal{E}_n, \quad \mathcal{E}_* = \liminf_{i\to\infty} \mathcal{E}_i \triangleq \bigcup_{i=1}^{\infty}\bigcap_{n=i}^{\infty} \mathcal{E}_n$$

Then $\mathcal{E}^*$ is the event that infinitely many of $\{\mathcal{E}_n\}$ occur and $\mathcal{E}^*$ is the event that all except a finite number of $\mathcal{E}_i$ occur, *i.e.*,

$$\begin{aligned}\mathcal{E}^* &= \{\omega \in \Omega : \omega \in \mathcal{E}_i, \text{for infinitely many values of } i\},\\ \mathcal{E}_* &= \{\omega \in \Omega : \omega \in \mathcal{E}_i, \text{for all but finite many of } i\},\end{aligned}$$

Now if we know $\Pr(\mathcal{E}_n)$, what can we say about $\Pr(\mathcal{E}^*)$?

Borel-Cantelli Lemmas

1. If $\sum \Pr(\mathcal{E}_i) < \infty$, then $\Pr(\mathcal{E}^*) = 0$.
2. If $\sum \Pr(\mathcal{E}_i)$ diverges, and $\{\mathcal{E}_n\}$ are independent, then $\Pr(\mathcal{E}^*) = 1$.

Example: Passing the Qualify

Consider the random experiment: taking the Qualify exam. The probability model is given by $(\Omega, \mathcal{F}, P)$ where

- the sample space $\Omega_2 = \{P, FP, FFP, FFFP, \cdots, \}$;
- the σ-field $\mathcal{F}$ includes all subsets of Ω_2, *i.e.*, $\mathcal{F} = 2^\Omega$.
- If the probability of passing is p, and assume that you learned nothing from the last time, then

$$\Pr(\underbrace{FF\cdots F}_{k}P) = (1-p)^k p$$

Q: What is the probability that you will pass in no more than three tries?

$$\mathcal{E} = \{P, FP, FFP\}, \quad \Pr(\mathcal{E}) = p + (1-p)p + (1-p)p^2$$

Q: What is the probability that you pass eventually?

Let $\mathcal{E}_i$ be the event that you pass in no more than i tries. Then $\mathcal{E}_i^c$ is the event that you have not succeeded after i tries.

$$\Pr(\mathcal{E}_i) = 1 - \Pr(\mathcal{E}_i^c) = 1 - (1-p)^i$$

The event of pass eventually is given by

$$\mathcal{E} = \bigcup_{i=1}^{\infty} \mathcal{E}_i, \quad \Pr(\mathcal{E}) = \lim_{i\to\infty} \Pr(\mathcal{E}_i) = 1$$

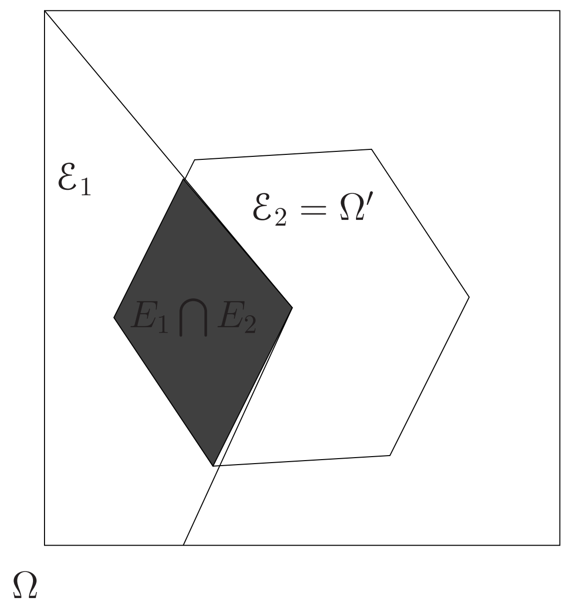
Q: What if your chance of passing increases with the number of tries, you would expect to do better, and $\Pr(\mathcal{E}) = 1$. How about your chance actually decreases with the number of tries?

Conditional Probability

Definition

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two events. Assuming that $\Pr(\mathcal{E}_2) \neq 0$, the conditional probability of the event $\mathcal{E}_1$ given that $\mathcal{E}_2$ has already occurred is given by

$$\Pr(\mathcal{E}_1|\mathcal{E}_2) = \frac{\Pr(\mathcal{E}_1 \bigcap \mathcal{E}_2)}{\Pr(\mathcal{E}_2)}$$



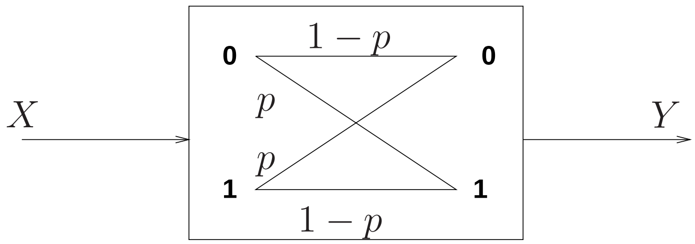
We can think "conditioning" as generating a new probability model (based on the observation of event $\mathcal{E}_2$) from the old by treating $\mathcal{E}_2$ as the new sample space Ω'

Example: Binary Symmetrical Channel

The Channel

The binary symmetric channel (BSC) is defined by the conditional probability

$$\begin{aligned}\Pr(Y=0|X=0) &= \Pr(Y=1|X=1) = 1-p,\\ \Pr(Y=1|X=0) &= \Pr(Y=0|X=1) = p\end{aligned}$$



The Sample Space

$$\Omega = \{(X=x, Y=y), x, y, \in \{0,1\}\} = \{(0,0),(0,1),(1,0),(1,1)\}.$$

The σ-field

$$\mathcal{F} = \{\emptyset, \Omega, \{(0,0)\}, \cdots, \{(1,1)\}, \{(0,0)\} \bigcup \{(0,1)\} \cdots\}$$

The Probability Measure

Suppose that $\{X=0\}$ and $\{X=1\}$ are equally likely.

$$\begin{aligned}\Pr[\{(0,0)\}] &= \Pr(X=0)\Pr(Y=0|X=0) = \frac{1-p}{2},\\ \Pr[\{(1,1)\}] &= \Pr(X=1)\Pr(Y=1|X=1) = \frac{1-p}{2}\\ \Pr[\{(1,0)\}] &= \Pr(X=0)\Pr(Y=1|X=0) = \frac{p}{2},\\ \Pr[\{(0,1)\}] &= \Pr(X=1)\Pr(Y=0|X=1) = \frac{p}{2}\end{aligned}$$

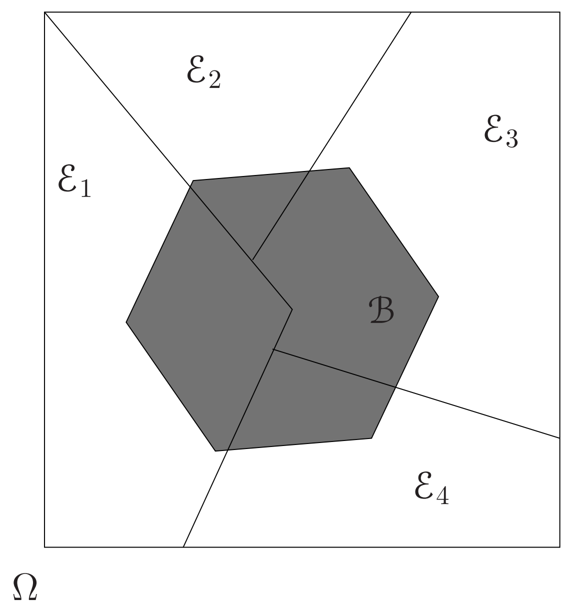
Total Probability Theorem

If $\{\mathcal{E}_i\}$ partition Ω, *i.e.,*

$$\bigcup \mathcal{E}_i = \Omega, \qquad \mathcal{E}_i \bigcap \mathcal{E}_j = \emptyset,$$

then

$$\Pr(\mathcal{B}) = \sum \Pr(\mathcal{E}_i) \Pr(\mathcal{B}|\mathcal{E}_i)$$



The Bayes Formula

$$\Pr(\mathcal{E}_i|\mathcal{B}) = \frac{\Pr(\mathcal{B}|\mathcal{E}_i) \Pr(\mathcal{E}_i)}{\sum \Pr(\mathcal{E}_i) \Pr(\mathcal{B}|\mathcal{E}_i)}$$

Statistical Independence

Definition

Two events $\mathcal{E}_1$ and $\mathcal{E}_2$ are statistically independent if

$$\Pr(\mathcal{E}_1 \bigcap \mathcal{E}_2) = \Pr(\mathcal{E}_1)\Pr(\mathcal{E}_2),$$

which implies that

$$\Pr(\mathcal{E}_1|\mathcal{E}_2) = \Pr(\mathcal{E}_1), \qquad \Pr(\mathcal{E}_2|\mathcal{E}_1) = \Pr(\mathcal{E}_2)$$

Events $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ are statistically independent if

$$\begin{aligned}
\Pr(\mathcal{E}_1 \bigcap \mathcal{E}_2) &= \Pr(\mathcal{E}_1)\Pr(\mathcal{E}_2) \\
\Pr(\mathcal{E}_1 \bigcap \mathcal{E}_3) &= \Pr(\mathcal{E}_1)\Pr(\mathcal{E}_3) \\
\Pr(\mathcal{E}_2 \bigcap \mathcal{E}_3) &= \Pr(\mathcal{E}_2)\Pr(\mathcal{E}_3) \\
\Pr(\mathcal{E}_1 \bigcap \mathcal{E}_2 \bigcap \mathcal{E}_3) &= \Pr(\mathcal{E}_1)\Pr(\mathcal{E}_2)\Pr(\mathcal{E}_3)
\end{aligned}$$

In general, events $\{\mathcal{E}_1, \cdots, \mathcal{E}_n\}$ are statistically independent if

$$\Pr(\mathcal{E}_{i_1} \bigcap \mathcal{E}_{i_2} \bigcap \cdots \bigcap \mathcal{E}_{i_k}) = \Pr(\mathcal{E}_{i_1})\Pr(\mathcal{E}_{i_2})\cdots\Pr(\mathcal{E}_{i_k})$$

for all $\{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$.

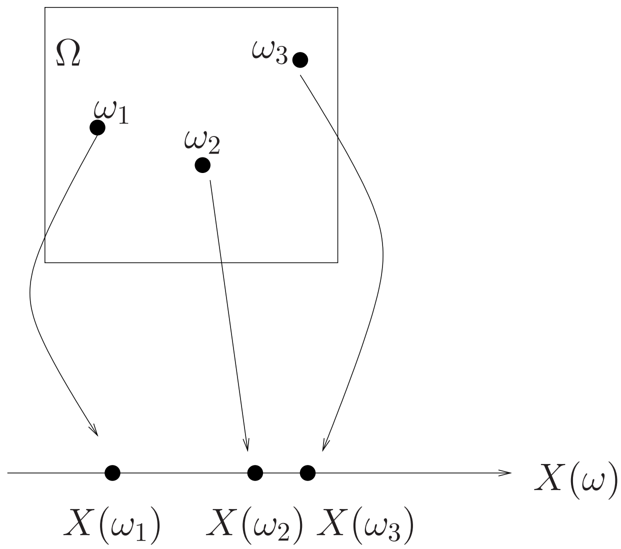
Random Variables

Definition

Given any probability space $(\Omega, \mathcal{F}, \Pr)$, a random variable is a function

$$X : \Omega \to R$$

such that, for all x, $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$.



Notations

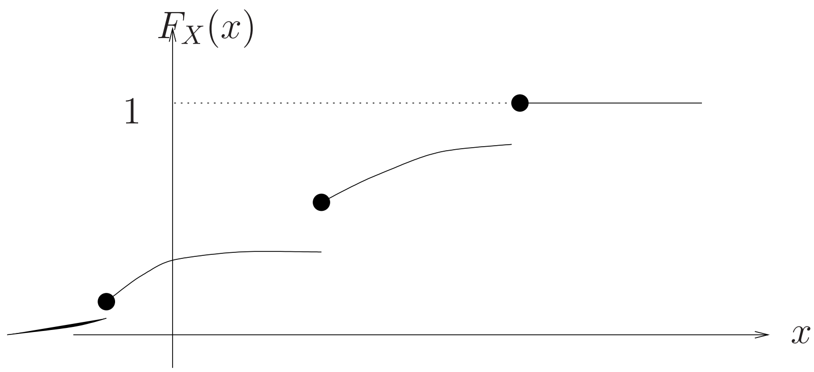
We use capital letters to indicate random variables and their corresponding small letters to indicate their "realizations" [‡]. For example, in $X = x$, X is the random variable (a function) and x is the value that X takes (with some probability).

[‡] We may use small letters to denote random variables when there is no confusion

Cumulative Distribution Function

The cumulative distribution function (CDF) of a random variable X is

$$F_X(x) \triangleq \Pr(X \leq x)$$



Properties

1. $F_X(-\infty) = 0, F_X(\infty) = 1$.
2. If $x < y$, then $F_X(x) \leq F_X(y)$.
3. $F(\cdot)$ is right continuous, *i.e.*, $\lim_{\Delta \to 0^+} F_X(x + \Delta) = F_X(x)$
4. $\Pr(x < X \leq y) = F_X(y) - F_X(x)$.
5. A useful interpretation is

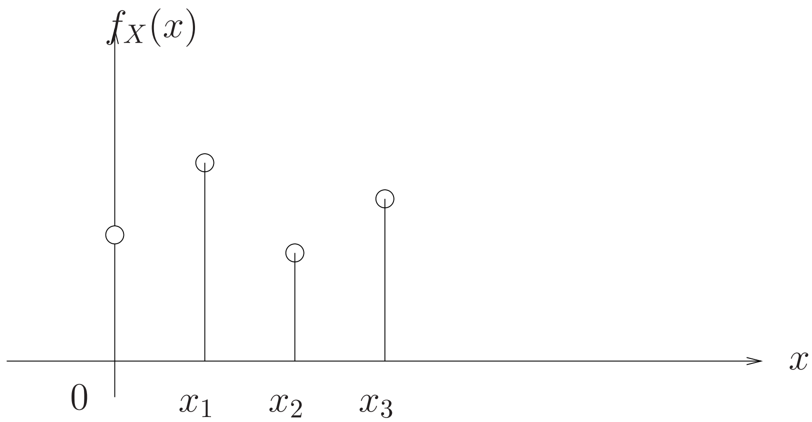
$$\begin{aligned} \Pr(X \in (x, x + dx)) &= F_X(x + dx) - F_X(x) \triangleq dF_X(x) \\ \Pr(X \in \mathcal{A}) &= \int_{\mathcal{A}} dF_X(x) \end{aligned}$$

6. $\Pr(X = x_0) = F_X(x_0) - \lim_{y \uparrow x_0} F_X(y)$.

Probability Mass Function

For discrete random variables, *i.e.*, X takes values in a countable set $\{x_i\}$. The probability mass function (PMF) of is given by

$$f_X(x) \triangleq \Pr(X = x)$$



The PMF is related to CDF by

$$F_X(x) = \sum_{u:u \leq x} f_X(u)$$

For any event $\mathcal{E}$, we have

$$\Pr(\mathcal{E}) = \sum_{u \in \mathcal{E}} f_X(u)$$

To unify notations, we also write the above as

$$\Pr(\mathcal{E}) = \int_{\mathcal{E}} f_X(x) dx = \int_{\mathcal{E}} dF_X(x)$$

Probability Density Function

A random variable is continuous if its distribution function can be expressed as

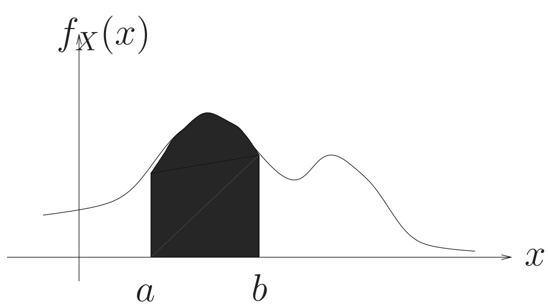
$$F_X(x) = \int_{-\infty}^{x} f_X(u)du \tag{1}$$

for some integrable function $f_X : \mathcal{R} \to [0, \infty)$. Function $f_X(x)$ is the probability density function (pdf) of X:

$$f_X(x) = \frac{d}{dx}F_X(x).$$

Properties:

- $f_X(u) \geq 0$.
- $\int_{-\infty}^{\infty} f_X(u)du = 1$.
- $\int_a^b f_X(u)du = \Pr(a < X \leq b)$.
- $\Pr(\mathcal{E}) = \int_{\mathcal{E}} f_X(u)du$.



Random Vectors

Given a random vector $\mathbf{X} = [X_1, \cdots, X_n]$ defined on the probability space $(\Omega, \mathcal{F}, P)$,

- the joint density distribution function is given by

$$F_{\mathbf{X}}(\mathbf{x}) = \Pr(\mathbf{X} \leq \mathbf{x}) \triangleq \Pr(X_1 \leq x_1, \cdots, X_n \leq x_n).$$

- The joint density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{X}}(\mathbf{x})$$

- The marginal distribution of X_i is given by

$$F_{X_i}(x) \triangleq \Pr(X_i < x) = F_{\mathbf{X}}(\infty, \cdots, \infty, \underbrace{x}_{ith}, \infty, \cdots, \infty)$$

- The marginal density is given by

$$f_{X_i}(x) = \frac{d}{dx} F_{X_i}(x) = \int f_{\mathbf{X}}(\mathbf{x}) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n$$

Independent Random Variables

Recall Independent Events

- $\mathcal{A}$ and $\mathcal{B}$ are statistically independent if

$$\Pr(\mathcal{A} \bigcap \mathcal{B}) = \Pr(A) \Pr(B)$$

- Events $\{A, B, C\}$ are statistically independent if

$$\begin{aligned} \Pr(\mathcal{A} \bigcap \mathcal{B}) &= \Pr(\mathcal{A}) P(\mathcal{B}) \\ \Pr(\mathcal{A} \bigcap \mathcal{C}) &= \Pr(\mathcal{A}) \Pr(\mathcal{C}) \\ \Pr(\mathcal{C} \bigcap \mathcal{B}) &= \Pr(\mathcal{C}) \Pr(\mathcal{B}) \\ \Pr(\mathcal{A} \bigcap \mathcal{B} \bigcap \mathcal{C}) &= \Pr(\mathcal{A}) \Pr(\mathcal{B}) \Pr(\mathcal{C}) \end{aligned}$$

Independent Random Variables

We call n random variables $\mathbf{X} = (X_1, \cdots, X_n)$ statistically independent if

$$F_{\mathbf{X}}(\mathbf{x}) = F_{X_1}(x_1) \cdots F_{X_n}(x_n)$$

or equivalently

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

Conditioning on Random Variables

Conditional Distribution

Consider random variables X and Y with joint distribution (or density) function $F_{X,Y}(x,y)$ ($f_{X,Y}(x,y)$). The conditional distribution of X given $Y = y$ is defined as

$$F_{X|Y}(x|y) \triangleq \Pr(X \leq x | Y = y) = \lim_{\epsilon \downarrow 0} \frac{\Pr(X \leq x, y < Y \leq y + \epsilon)}{\Pr(y < Y \leq y + \epsilon)}$$

The conditional density function of $F_{X|Y}$, written as $f_{X|Y}$, is given by

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)} & f_Y(y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $f_Y(y) = \int f_{X,Y}(u,y)du$ is the marginal pdf of Y.
Further,

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(u|y)du$$

If X and Y are independent, $f_{X|Y}(x|y) = f_X(x)$.

Example: Consider independent random variables X and N such that

$$Y = X + N,$$

where X is discrete with PMF $f_X(x)$ and N is continuous with PDF $f_N(n)$. Then

$$\begin{aligned} F_{Y|X}(y|x) &= \Pr(Y \leq y | X = x) = \frac{\Pr(N \leq y - x, X = x)}{f_X(x)} = F_N(y - x) \\ F_{X|y}(x|y) &= \Pr(X = x | Y = y) = \lim_{\epsilon \downarrow 0} \frac{\Pr(X = x, y < Y \leq y + \epsilon)}{\Pr(y < Y \leq y + \epsilon)} = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} \\ f_{Y|X}(y|x) &= f_N(y - x) \end{aligned}$$

Expectation of Random Variables

Definition

For a random variable X

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x dF_X(x), \quad \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) dF_X(x)$$

Properties

1. The indicator function of an event $\mathcal{E}$ is defined as

$$1_{\mathcal{E}}(x) = \begin{cases} 1 & x \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

   We then have

$$\Pr(\mathcal{E}) = \int_{\mathcal{E}} dF_X(x) = \mathbb{E}(1_{\mathcal{E}}(X))$$

2. If X is nonnegative random variable with CDF F,

$$\mathbb{E}(X) = \int_0^{\infty} (1 - F_X(x)) dx$$

3. Linearity: $\mathbb{E}(\alpha X + \beta Y) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y)$.

4. If X and Y are independent, then $\mathbb{E}(h(X)g(Y)) = \mathbb{E}(h(X))\mathbb{E}(g(Y))$.

5. Variance and Covariance

$$\begin{aligned} \text{Var}(X) &\triangleq \mathbb{E}(X - \mathbb{E}(X))^2, \\ \text{Cov}(X, Y) &\triangleq \mathbb{E}(\mathbb{E}(X - \mathbb{E}(X))\mathbb{E}(Y - \mathbb{E}(Y))). \end{aligned}$$

   The standard deviation of X is $\sqrt{\text{Var}(X)}$.

6. X and Y are uncorrelated if Cov$(X, Y) = 0$.

7. For a real random vector $\mathbf{X} = [X_1, \cdots, X_n]^T$,

$$\text{Mean:} \quad \mathbb{E}(\mathbf{X}) = [\mathbb{E}(X_1), \cdots, \mathbb{E}(X_n)]^T$$

$$\text{Covariance:} \quad \text{Cov}(\mathbf{X}, \mathbf{X}) = \mathbb{E}(\mathbf{X} - \mathbb{E}(\mathbf{X}))(\mathbf{X} - \mathbb{E}(\mathbf{X}))^T$$

- Cov$(\mathbf{X}, \mathbf{X})$ is always positive (semi) definite.
- If $\mathbf{X}$ is a vector of uncorrelated random variables, then Cov$(\mathbf{X}, \mathbf{X})$ is diagonal with variances as diagonal entries.

Conditional Expectation

The conditional expectation of $g(\mathbf{X})$ given $\mathbf{Y} = \mathbf{y}$ is given by

$$\mathbb{E}(g(\mathbf{X})|\mathbf{Y} = \mathbf{y}) = \int g(\mathbf{x}) f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) d\mathbf{x}$$

Note that $\mathbb{E}(g(\mathbf{X})|\mathbf{Y} = \mathbf{y})$ is a function of $\mathbf{y}$.

Conditional Mean as a Random Variable

- We denote $\mathbb{E}(g(\mathbf{X})|\mathbf{Y})$ as the random variable that takes the value $\mathbb{E}(g(\mathbf{X})|\mathbf{Y} = \mathbf{y})$ when $\mathbf{Y} = \mathbf{y}$.
- Successive conditioning:

$$\mathbb{E}(g(\mathbf{X})) = \mathbb{E}(\mathbb{E}(g(\mathbf{X})|\mathbf{Y}))$$

As an example, suppose that $Y \sim \mathcal{U}(0,1)$ and $X \sim \mathcal{U}(0,Y)$.

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(\frac{Y}{2}) = \frac{1}{4} \\ \mathbb{E}(X^2) &= \mathbb{E}(\mathbb{E}(X^2|Y)) = \mathbb{E}(\frac{Y^2}{3}) = \frac{1}{9} \end{aligned}$$

Product Expectation Theorem

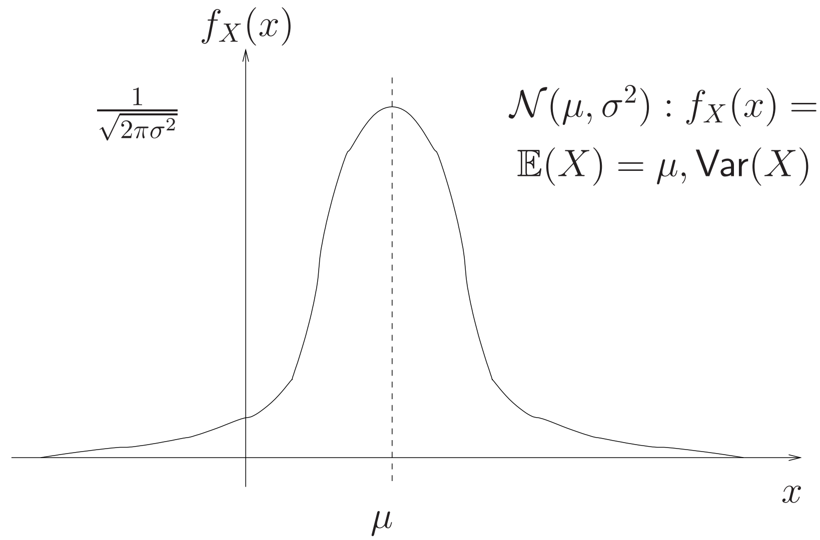
If $g(Y)$ is bounded and $\mathbb{E}(h(X)) \leq \infty$, then

$$\mathbb{E}(h(X)g(Y)) = \mathbb{E}(g(Y)\mathbb{E}(h(X)|Y))$$

A special case is when $g(y) = 1$ and $h(x) = x$

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

The Gaussian Random Variable

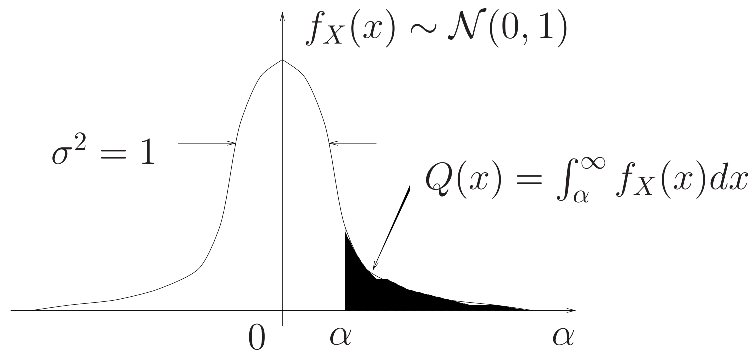


$\mathcal{N}(\mu,\sigma^2) : f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$

$\mathbb{E}(X) = \mu, \mathsf{Var}(X) = \sigma^2.$

The $Q(\cdot)$ function

$$Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{u^2}{2}} du$$



Properties

1. Probability: If $X \sim \mathcal{N}(\mu,\sigma^2)$, then

$$\Pr[X > \alpha] = Q(\frac{\alpha-\mu}{\sigma}), \quad \Pr(X < \alpha) = Q(\frac{\mu-\alpha}{\sigma})$$

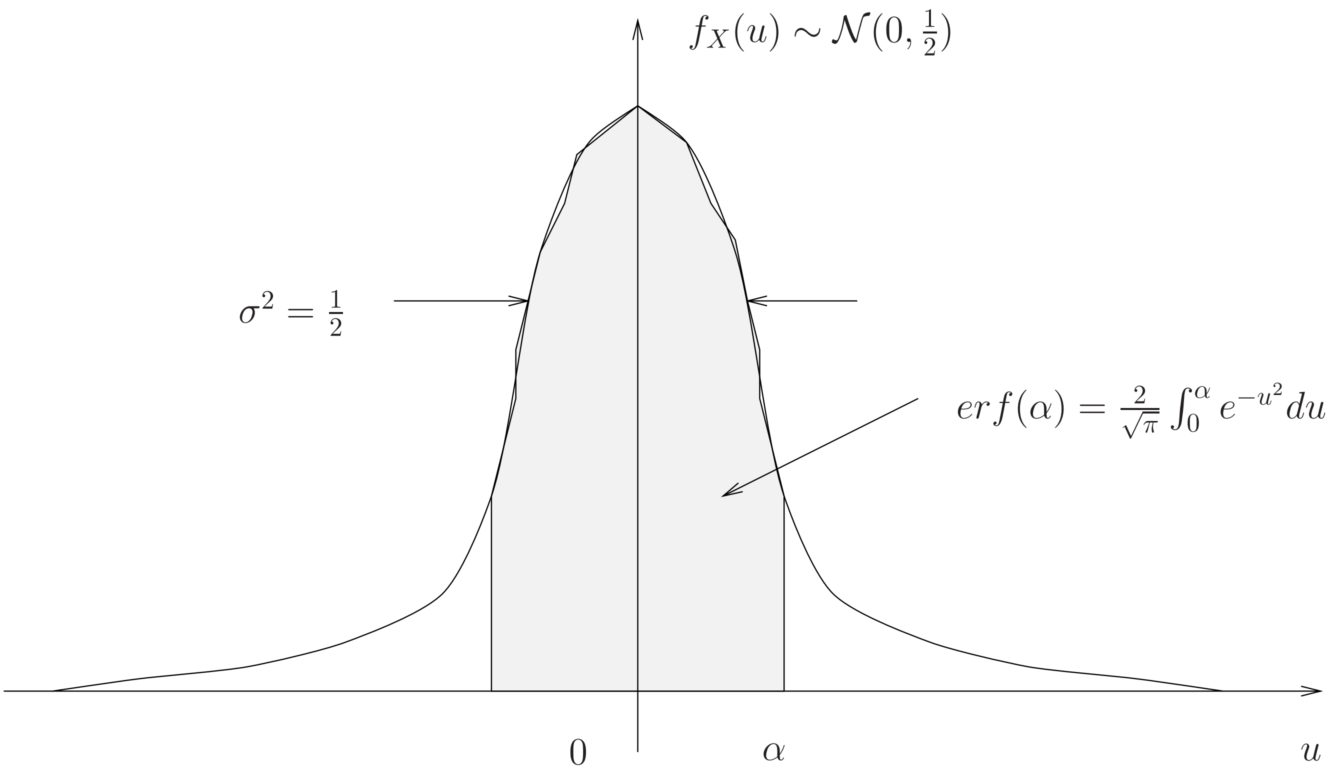
2. Bounds:

$$(1 - \frac{1}{x^2})\frac{e^{-x^2/2}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{1}{2}e^{-x^2/2}$$

$Q(\cdot)$, erf$(\cdot)$, and erfc$(\cdot)$

Definitions:

$$
\begin{aligned}
erf(\alpha) &\stackrel{\Delta}{=} \frac{2}{\sqrt{\pi}}\int_0^{\alpha} e^{-u^2}du \\
erfc(\alpha) &\stackrel{\Delta}{=} \frac{2}{\sqrt{\pi}}\int_{\alpha}^{\infty} e^{-u^2}du = 1 - erf(\alpha)
\end{aligned}
$$



Relations

$$
\begin{aligned}
Q(\alpha) &= \frac{1}{2}erfc(\frac{\alpha}{\sqrt{2}}) = \frac{1}{2}(1 - erf(\frac{\alpha}{\sqrt{2}})) \\
erfc(\alpha) &= 2Q(\sqrt{2}\alpha)
\end{aligned}
$$

Gaussian Random Vectors

A random vector $\mathbf{X} = [X_1, \cdots, X_n]^T$ is Gaussian if

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \text{det}(\boldsymbol{\Sigma})}} exp\{-\frac{1}{2}(\mathbf{X} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})\}$$

where

$$\begin{aligned}
\boldsymbol{\mu} &= \mathbb{E}(\mathbf{X}) = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{pmatrix} \\
\boldsymbol{\Sigma} &= \mathbb{E}\{(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\} \\
&= \begin{pmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) & \cdots & \text{Cov}(X_1, X_n) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) & \cdots & \text{Cov}(X_2, X_n) \\ \vdots & & & \vdots \\ \text{Cov}(X_n, X_1) & \text{Cov}(X_n, X_2) & \cdots & \text{Cov}(X_n, X_n) \end{pmatrix}
\end{aligned}$$

- Random variables $X_1, \cdots, X_n$ are called jointly Gaussian.
- The Gaussian distribution is completely specified by the mean and the covariance.

Properties of Gaussian Random Vectors

Suppose that $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

- Jointly Gaussian implies marginally Gaussian. In particular,

  $$X_i \sim \mathcal{N}(\mathbb{E}(X_i), \mathsf{Cov}(X_i, X_i)).$$

  Any sub-vector of $\mathbf{X}$ is Gaussian. (The converse is not true in general!)

- For any matrix $\mathbf{A}$ and vector $\mathbf{b}$, $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is Gaussian and

  $$\mathbf{Y} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^t).$$

  Proof:

  $$\begin{aligned} \mathbb{E}(\mathbf{Y}) &= \mathbf{A}\mathbb{E}(\mathbf{X}) + \mathbf{b} \\ \mathsf{Cov}(\mathbf{Y}, \mathbf{Y}) &= \mathbb{E}(\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^t \mathbf{A}^t) = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^t \end{aligned}$$

- Uncorrelated Gaussian random variables are independent.

- If

  $$\begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_y \\ \boldsymbol{\mu}_z \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{yy} & \boldsymbol{\Sigma}_{yz} \\ \boldsymbol{\Sigma}_{zy} & \boldsymbol{\Sigma}_{zz} \end{bmatrix} \right), \qquad (2)$$

  $f_{\mathbf{Y}|\mathbf{Z}}(\mathbf{y}|\mathbf{z})$ is the complex Gaussian density with

  $$\begin{aligned} \mathbb{E}(\mathbf{y}|\mathbf{z}) &= \boldsymbol{\mu}_y + \boldsymbol{\Sigma}_{yz}\boldsymbol{\Sigma}_{zz}^{-1}(\mathbf{z} - \boldsymbol{\mu}_z) \\ \mathsf{Cov}(\mathbf{y}, \mathbf{y}^H|\mathbf{z}) &= \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yz}\Sigma_{zz}^{-1}\boldsymbol{\Sigma}_{zy} \end{aligned}$$

Complex Random Vectors

Definition

The probability space of a complex random vector $\mathbf{X} = \mathbf{X}_R + j\mathbf{X}_I$ is defined by the joint distribution of $\mathbf{X}_R$ and $\mathbf{X}_I$. A complex random vector $\mathbf{X}$ is proper (or symmetrical) if

$$\text{Cov}(\mathbf{X}\mathbf{X}^T) = \mathbf{0} \;\Rightarrow \left\{ \begin{array}{l} \text{Cov}(\mathbf{X}_R, \mathbf{X}_R^t) = \text{Cov}(\mathbf{X}_I, \mathbf{X}_I^t) \\ \text{Cov}(\mathbf{X}_R, \mathbf{X}_I^t) = -\text{Cov}(\mathbf{X}_I, \mathbf{X}_R^t) \end{array} \right.$$

Remarks

- If $\mathbf{X}$ is symmetrical, then all second-order statistics of $\mathbf{X}$ is contained in $\text{Cov}(\mathbf{X}, \mathbf{X}^H)$.

$$\begin{aligned} \text{Cov}(\mathbf{X}, \mathbf{X}^H) &= \text{Cov}(\mathbf{X}_R, \mathbf{X}_R^T) + \text{Cov}(\mathbf{X}_I, \mathbf{X}_I^T) \\ &\quad -j(\text{Cov}(\mathbf{X}_R, \mathbf{X}_I^T) - \text{Cov}(\mathbf{X}_I, \mathbf{X}_R^T)) \\ &= 2\text{Cov}(\mathbf{X}_R, \mathbf{X}_R^T) + 2j\text{Cov}(\mathbf{x}_I, \mathbf{x}_R^t) \end{aligned}$$

- If $\mathbf{X}$ is proper, then $\mathbf{A}\mathbf{X} + \mathbf{b}$ is also proper (invariant under affine transforms).
- For proper complex random vectors, we can use complex arithmetics at a lower dimension by changing transpose to Hermitian.

Complex Gaussian Random Vectors

Random vector $\mathbf{x}$ is complex Gaussian if

1. $\mathbf{X}$ is symetrical
2. $\begin{pmatrix} \mathbf{X}_R \\ \mathbf{X}_I \end{pmatrix}$ is Gaussian.

Properties

- Distribution: $\mathbf{X} \sim \mathcal{N}_c(\mu, \Sigma)$ implies

$$
\begin{aligned}
E(\mathbf{X}) &= \mu, cov(\mathbf{x}, \mathbf{x}^H) = \Sigma, \\
f_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{\pi^n |\Sigma|} exp\{-(\mathbf{x} - \mu)^H \Sigma^{-1} (\mathbf{x} - \mu)\}.
\end{aligned}
$$

- When $\mathbf{X}_R, \mathbf{X}_I \sim \mathcal{N}(0, \frac{N_0}{2}\mathbf{I})$, $\mathbf{X} \sim \mathcal{N}_c(0, N_0\mathbf{I})$,

$$p(\mathbf{x}) = \frac{1}{\pi^n N_0^n} exp\{-\frac{||\mathbf{x}||^2}{N_0}\}.$$

- A userful case: If $\mathbf{X} = \mathbf{S} + \mathbf{N}$ where $\mathbf{S}$ and $\mathbf{N}$ are independent, $\mathbf{N} \sim \mathcal{N}(0, N_0\mathbf{I})$,

$$f_{\mathbf{X}|\mathbf{S}}(\mathbf{x}|\mathbf{s}) = \frac{1}{\pi^n N_0^n} exp\{-\frac{||\mathbf{x} - \mathbf{s}||^2}{N_0}\}$$

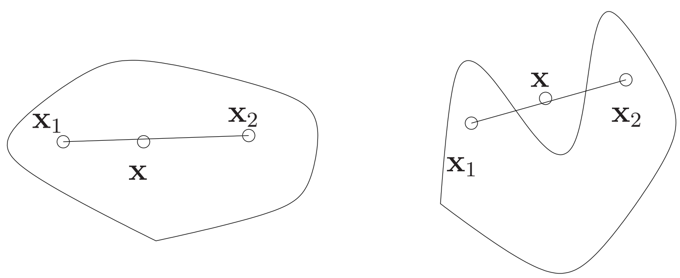
Convexity and Jensen's Inequality

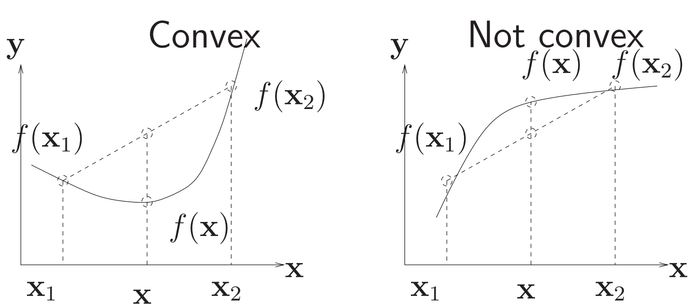
Convex Set and Convex Function

A set $\mathcal{X}$ in $\mathcal{R}^n$ or $\mathcal{C}^n$ is convex if, for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\theta \in [0,1]$, $\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in \mathcal{X}$. A real valued function $f(\cdot)$ on a convex set $\mathcal{X}$ is convex (convex $\cup$) if, for every $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\theta \in [0,1]$,

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \leq \theta f(\mathbf{x}_1) + (1-\theta)f(\mathbf{x}_2)$$

A function is strictly convex if the strict inequality holds. A function $f$is concave (convex $\cap$) if $-f$ is convex.





Convex function Concave function

Jensen's Inequality

Let f be a real valued convex function. Then

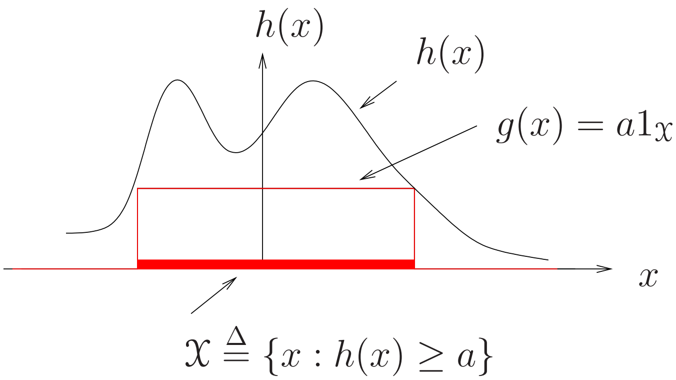
$$f(\mathbb{E}(\mathbf{x})) \leq \mathbb{E}(f(\mathbf{x}))$$

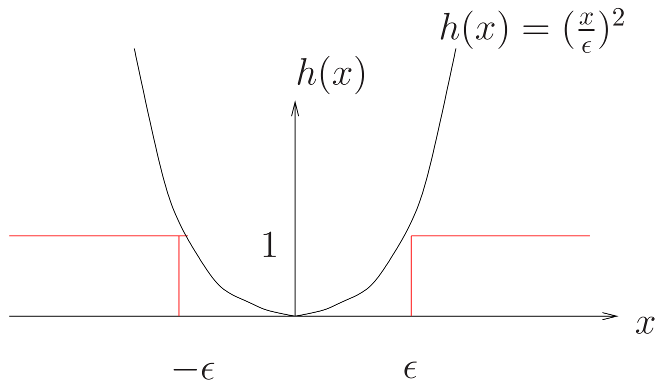
For concave f, the inequality is reversed.

Markov and Chebyshev Inequalities

The Markov Inequality: For any non-negative function $h(\cdot)$,

$$\Pr[h(X) \geq a] \leq \frac{\mathbb{E}(h(X))}{a} \quad \forall a > 0.$$





Chebyshev Inequality: Setting $h(x) = |x - \mathbb{E}(X)|^2$,

$$\Pr[\frac{|X - \mathbb{E}(X)|}{\epsilon} \geq 1] \leq \frac{\mathsf{Var}(X)}{\epsilon^2}$$

As an application, for i.i.d. X_i and $\mathbb{E}(X_i) = p$,

$$Y_N = \frac{1}{N}\sum_{i=1}^{N} X_i \rightarrow \Pr(|Y_N - p| > \epsilon) \leq \frac{\mathsf{Var}(X)}{N\epsilon^2}$$

The probability of Y_N deviates from its mean decreases with $O(\frac{1}{N})$.

A Lower Bound

If h is a non-negative uniformly bounded by M, then

$$\Pr(h(X) \geq a) \geq \frac{\mathbb{E}(h(X)) - a}{M - a}, \quad a \in [0, M).$$

Chernoff Bound

If we want to have exponentially decaying probability, we may need the Chernoff bound. Let X be a random variable. For any $\lambda > 0$ and τ,

$$\Pr[X \geq \tau] \leq \exp\{-\lambda\tau + \phi_X(\lambda)\}$$

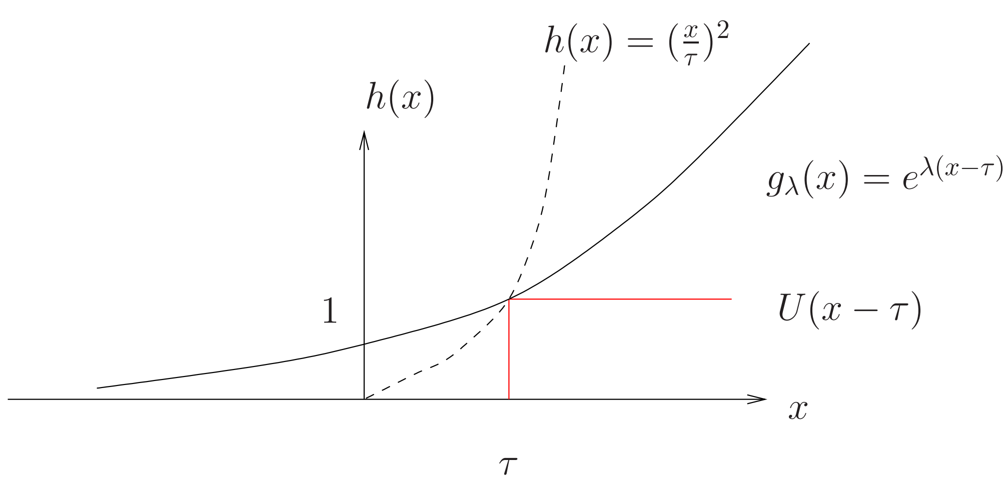
where

$$\phi_X(\lambda) \triangleq \ln \mathbb{E}(e^{\lambda X})$$

is the cumulant generating function. Similarly, we also have

$$\Pr[X \leq \tau] \leq \exp\{\lambda\tau + \phi_X(-\lambda)\}$$

Proof: Use the Markov inequality with $h(X) = e^{\lambda X}$ and $a = e^{\lambda\tau}$



Remark: The Chernoff bound can be tightened by optimizing λ.

An Application of the Chernoff Bound

Consider

$$Y_N \triangleq \frac{1}{N}\sum_{i=1}^{N} X_i, \quad X_i \overset{\text{i.i.d.}}{\sim} \mathcal{B}(p)$$

By the Chernoff bound,

$$\begin{aligned}\Pr[Y_N \geq a] &= \Pr[\sum X_i \geq Na] \leq e^{-N\lambda a}\mathbb{E}(e^{\lambda \sum X_i}) \\ &= e^{-N\lambda a}[\mathbb{E}(e^{\lambda X_i})]^N \\ &= [\mathbb{E}(e^{\lambda(X_i - a)})]^N\end{aligned}$$

The best λ is given by solving

$$\frac{d}{d\lambda}\mathbb{E}(e^{\lambda(X_i-a)})|_{\lambda=\lambda_o} = 0 \to \frac{\mathbb{E}(X_i e^{\lambda_o X_i})}{\mathbb{E}(e^{\lambda_o X_i})} = a$$

For Bernoulli r.v. and $a \in (p, 1]$,

$$\frac{pe^{\lambda_o}}{pe^{\lambda_o} + (1-p)} = a \to \lambda_o = \ln\frac{a(1-p)}{p(1-a)} > 0$$

Thus,

$$\Pr[Y_N \geq a] \leq [(\frac{p}{a})^a(\frac{1-p}{1-a})^{1-a}]^N = \exp\{-ND(\mathcal{B}(a)||\mathcal{B}(p))\}$$

where

$$D(P_1||P_2) \triangleq \mathbb{E}_{P_1}(\log\frac{P_1}{P_2})$$

is the Kullback-Leibler divergence, which is always positive.

Weak Convergence and Weak LLN

Definition

Suppose X and $\{X_n, n = 1, 2, \cdots\}$ are random variables defined on the same probability space. We say that the sequence (X_n) converges in probability, denoted as $X_n \xrightarrow{P} X$ if, for all ϵ,

$$\Pr(|X_n - X| \geq \epsilon) \to 0 \quad \text{as } n \to \infty$$

Example

Let X_n be independent variables with PMF

$$\Pr(X_n = 1) = 1 - \frac{1}{n} \quad \Pr(X_n = n) = \frac{1}{n}$$

For any $\epsilon > 0$,

$$\Pr(|X_n - 1| > \epsilon) = \Pr(X_n = n) = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

Therefore $X_n \xrightarrow{P} 1$.

The Weak Law of Large Numbers

Let X_i be a sequence of i.i.d. random variables with mean μ and variance σ^2. Then,

$$\bar{X}_n \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \mu$$

Proof: Use the Chebyshev Inequality for $X = \frac{1}{N} \sum_{i=1}^{N} X_i$.

Strong Convergence and Strong LLN

Definition

The sequence (X_n) converges almost surely (or strongly), denoted by $X_n \xrightarrow{\text{a.s.}} X$, if

$$\Pr(\omega \in \Omega : X_n(\omega) \to X(\omega)) = \Pr(X_n \to X) = 1 \text{ as } n \to \infty$$

Equivalently, $X_n \xrightarrow{\text{a.s.}} X$ if $\forall \epsilon > 0$ and $\delta \in (0,1)$, there exists n_0 such that, for all $n > n_0$,

$$\Pr(\bigcap_{m>n} \{|X_m - X| \leq \epsilon\}) > 1 - \delta$$

Example Revisited Let X_n be independent variables with PMF

$$\Pr(X_n = 1) = 1 - \frac{1}{n} \quad \Pr(X_n = n) = \frac{1}{n}$$

For every $\epsilon > 0$, $\delta \in (0,1)$, and $N > n$,

$$\begin{aligned}\Pr(\bigcap_{m>n} \{|X_m - 1| \leq \epsilon\}) &\leq \Pr(\bigcap_{m=n+1}^{N} \{|X_m - 1| \leq \epsilon\}) = \prod_{m=n+1}^{N} \Pr(|X_m - 1| \leq \epsilon) \\ &= \prod_{m=n+1}^{N} (1 - \frac{1}{m}) = \frac{n}{N} \leq 1 - \delta\end{aligned}$$

Strong Law of Large Numbers

Suppose (X_n) are i.i.d. random variables with mean μ and $\mathbb{E}(|X|^4) < \infty$. Then

$$\bar{X}_n \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mu$$

We can show that

$$\Pr(|\bar{X}_n - \mu| > \epsilon) \leq \frac{A}{n^2},$$

where A is a constant. By the Borel-Cantellis Lemma, $\{|\bar{X}_n - \mu| > \epsilon\}$ happens only finite number of times.

Convergence in Distribution and CLT

Definition

Suppose X and $\{X_n, n = 1, 2, \cdots\}$ are random variables defined on the same probability space. We say that the sequence (X_n) with CDF $F_{X_n}(x)$ converges in distribution to X with CDF $F_X(x)$, denoted as $X_n \xrightarrow{D} X$, if $F_{X_n}(x) \to F_X(x)$ for all x where $F_X(x)$ is continuous.

Central Limit Theorem

Let $\{X_n\}$ be i.i.d. random variables with mean μ and variance σ^2. Denote $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1)$$

The law of the iterative lograrithm

If $\{X_i\}$ are i.i.d. with mean μ and variance σ^2. Then

$$\Pr(\limsup_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{2n\log\log n}} = 1) = 1$$

This means that the event, with probability 1, the event

$$\{\frac{S_n - n\mu}{\sigma} > \alpha\sqrt{2n\log\log n}\}$$

should happen only finite number of times if $\alpha > 1$ and infinitely many times if $\alpha < 1$.