Changhao Xu 2103530.

Date

1. $\delta(y)=1$ if $\frac{f(y|\theta_1)}{f(y|\theta_0)} \geq \frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$

Given $\pi_0 = Pr(H_0)$, $\pi_1 = 1-\pi_0$, $C_{10}=C_{01}=1$, $C_{00}=C_{11}=0$,

$\Rightarrow$ Bayesian detector $\delta(y\in(0,1)) = \begin{cases} 1, & \text{if } \frac{Pr(y\in(0,1)|\theta_1)}{Pr(y\in(0,1)|\theta_0)} = \frac{2y}{1} = 2y \geq \frac{\pi_0}{1-\pi_0} \\ 0, & \text{otherwise.} \end{cases}$

$\delta(y\in(-\infty,0]\cup[1,+\infty)) = \begin{cases} 1, & \text{if } \frac{Pr(y \text{ otherwise}|\theta_1)}{Pr(y \text{ otherwise}|\theta_0)} = \frac{0}{0} = 0 \geq \frac{\pi_0}{1-\pi_0} \\ 0, & \text{otherwise} \end{cases}$

Since $\pi_0\in[0,1]$, $\frac{\pi_0}{1-\pi_0}\geq 0$, when $\pi_0=0$, $\delta(y\in(-\infty,0]\cup[0,+\infty))=0$ as well.

$\Rightarrow \delta(y\in(-\infty,0]\cup[1,+\infty))=0$.

That is, $\begin{cases} \delta(y\in(0,1)) = \begin{cases} 1, & \text{if } y\geq \frac{\pi_0}{2(1-\pi_0)}, \text{ i.e. } \pi_0 \leq \frac{2y}{1+2y}. \\ 0, & \text{otherwise.} \end{cases} \\ \delta(y\in(-\infty,0]\cup[1,+\infty))=0. \end{cases}$

Bayesian risk $V(\pi) = \pi R_S(\theta_0) + (1-\pi)R_S(\theta_1) = \pi_0 Pr(\delta(y)=1; H_0) + (1-\pi_0)Pr(\delta(y)=0; H_1)$.

Case 1: $\frac{2}{3}<\pi_0\leq 1$. $\delta(y\in(0,1))=0$, $\delta(y\in(-\infty,0]\cup[1,+\infty))=0$, i.e. $\delta(y)=0$.

$V(\pi) = \pi_0\times 0 + (1-\pi_0)\times 1 = 1-\pi_0$.

Case 2: $0\leq\pi_0\leq\frac{2}{3}$. $\begin{cases} \delta(y\in(0,\frac{\pi_0}{2(1-\pi_0)}))=0, \ \delta(y\in[\frac{\pi_0}{2(1-\pi_0)},1])=1. \\ \delta(y\in(-\infty,0]\cup[1,+\infty))=0. \end{cases}$

$\Rightarrow \begin{cases} \delta(y\in(-\infty,\frac{\pi_0}{2(1-\pi_0)})\cup[1,+\infty))=0. \\ \delta(y\in[\frac{\pi_0}{2(1-\pi_0)},1))=1. \end{cases}$

$$V(\pi) = \pi_0\int_{\frac{\pi_0}{2(1-\pi_0)}}^{1} 1\,dy + (1-\pi_0)\left[\int_{-\infty}^{\frac{\pi_0}{2(1-\pi_0)}} + \int_1^{+\infty}\right]1\,dy$$

$$= (1-\pi_0) + \int_{\frac{\pi_0}{2(1-\pi_0)}}^{1}[\pi_0\times 1-(1-\pi_0)\times 1]\,dy$$

$$= (1-\pi_0) + (2\pi_0-1)\Big|_{\frac{\pi_0}{2(1-\pi_0)}}^{1}$$

$$= (1-\pi_0) + (2\pi_0-1)\left(1-\frac{\pi_0}{2(1-\pi_0)}\right)$$

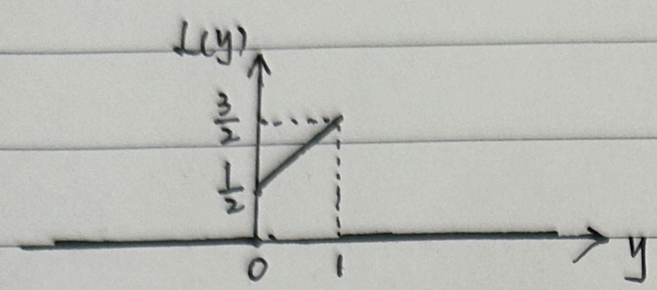
$$= \frac{\pi_0(3-4\pi_0)}{2(1-\pi_0)}$$

Changhao Xu 2103530

Date

2. 1) $H_0: f_0(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$ $H_1: f_1(y) = \begin{cases} f_0(y), & Pr = 0.5 \\ f(y) = 2y, & Pr = 0.5, \ y \in [0,1] \end{cases}$

Likelihood ratio $\mathcal{L}(y) = \dfrac{f(y|\theta_1)}{f(y|\theta_0)} = \dfrac{\frac{1}{2} \times f_0(y) + \frac{1}{2} \times 2y}{f_0(y)} = \begin{cases} y + \frac{1}{2}, & 0 \le y \le 1. \\ 0, & \text{otherwise.} \end{cases}$

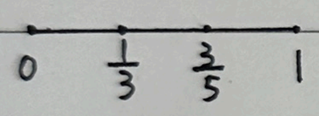


2) $\delta(y) = 1$ if $\mathcal{L}(y) \ge \dfrac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})} = \dfrac{\pi}{1-\pi}$.

$\delta(y \in [0,1]) = \begin{cases} 1, & \text{if } y \ge \frac{\pi}{1-\pi} - \frac{1}{2}. \\ 0, & \text{otherwise} \end{cases}$ $\delta(y \in (-\infty, 0) \cup (1, +\infty)) = 0$.

3) $V(\pi) = \pi R_s(\theta_0) + (1-\pi) R_s(\theta_1) = \pi Pr(\delta(y) = 1; H_0) + (1-\pi) Pr(\delta(y) = 0; H_1)$

$\delta(y) = 1$ if $y \ge \frac{\pi}{1-\pi} - \frac{1}{2}$, i.e. $\pi \le \dfrac{y + 1/2}{y + 3/2}$.



Case 1: $0 \le \pi \le \frac{1}{3}$. $\delta(y \in [0,1]) = 1$. $\delta(y \in (-\infty, 0) \cup (1, +\infty)) = 0$

$V(\pi) = \pi \times 1 + (1-\pi) \times 0 = \pi$.

Case 2: $\frac{3}{5} \le \pi \le 1$. $\delta(y \in [0,1]) = \delta(y \in (-\infty, 0) \cup (1, +\infty)) = 0$; i.e. $\delta(y) = 0$

$V(\pi) = 1 - \pi$.

Case 3: $\frac{1}{3} < \pi < \frac{3}{5}$. $\begin{cases} \delta(y \in [\frac{\pi}{1-\pi} - \frac{1}{2}, 1]) = 1, \ \delta(y \in (0, \frac{\pi}{1-\pi} - \frac{1}{2})) = 0 \\ \delta(y \in (-\infty, 0) \cup (1, +\infty)) = 0. \end{cases}$

$$V(\pi) = \pi \int_{\frac{\pi}{1-\pi} - \frac{1}{2}}^{1} 1 \, dy + (1-\pi) \left[\int_{-\infty}^{\frac{\pi}{1-\pi} - \frac{1}{2}} + \int_{1}^{+\infty} \right] (\tfrac{1}{2} + \tfrac{1}{2} \times 2y) \, dy.$$

$$= (1-\pi) + \int_{\frac{\pi}{1-\pi} - \frac{1}{2}}^{1} \left[\pi - (1-\pi)(\tfrac{1}{2} + y) \right] dy.$$

$$= (1-\pi) + \left[(\tfrac{3}{2}\pi - \tfrac{1}{2}) y + (\pi - 1) \times \tfrac{y^2}{2} \right] \Big|_{\frac{\pi}{1-\pi} - \frac{1}{2}}^{1}$$

$$= (1-\pi) + \frac{(5\pi - 3)^2}{8(\pi - 1)}.$$

$$= \frac{17\pi^2 - 14\pi + 1}{8(\pi - 1)}.$$

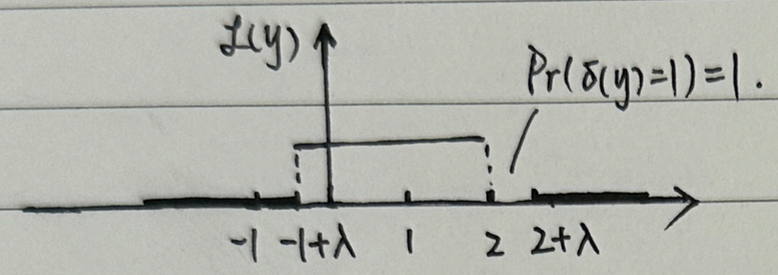
Changhao Xu 2103530

Date

3. 1) $H_0: \theta=0,\ Y\sim U(-1,2)$. $H_1: \theta=1,\ Y\sim U(-1,2)+\lambda$.

$H_0: f=\begin{cases}\frac{1}{3}, & -1\le y\le 2.\\ 0, & \text{otherwise}.\end{cases}$ $H_1: f=\begin{cases}\frac{1}{3}, & -1+\lambda\le y\le 2+\lambda.\\ 0, & \text{otherwise}.\end{cases}$

For $\begin{cases} y\in(-\infty,-1+\lambda]\cup[2+\lambda,+\infty),\ \mathcal{L}(y)=0 \text{ due to uniform distribution.}\\ y\in(2,2+\lambda],\ \delta(y)=1.\ \mathcal{L}(y)=\infty.\\ y\in[-1+\lambda,2],\ \mathcal{L}(y)=1.\ \delta(y)=\begin{cases}1, & \text{if } \mathcal{L}(y)\ge\tau\\ 0, & \text{otherwise}.\end{cases}\end{cases}$ $\Rightarrow$ However, since $\mathcal{L}(y)=\begin{cases}0\\1\\\infty,\end{cases}$



It is difficult to give $\mathcal{L}(y)\ge\tau$ directly.

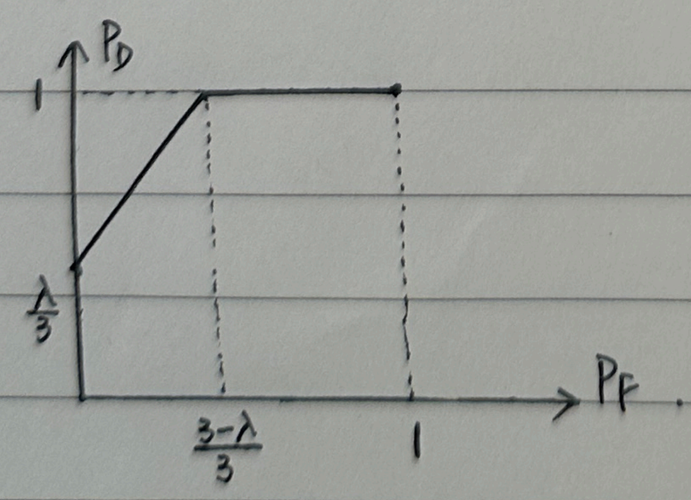
Given that $\delta(y)$ depends on y, we can define $\delta(y)=\begin{cases}1, & \text{if } y\ge\tau^*.\\ 0, & \text{otherwise}.\end{cases}$

$P_F=Pr(D=1\mid H_0)=\int_{\tau^*}^{2}\frac{1}{3}dy=\frac{1}{3}(2-\tau^*)=\alpha.\ \Rightarrow \tau^*=2-3\alpha.$

Therefore, $\delta(y)=\begin{cases}1, & \text{if } y\ge 2-3\alpha.\\ 0, & \text{otherwise}.\end{cases}$

2). If $\tau^*\le -1+\lambda$, i.e. $2-3\alpha\le -1+\lambda$. $\Rightarrow \alpha\ge\frac{3-\lambda}{3}$.

$\begin{cases} P_D=Pr(D=1\mid H_1)=\frac{1}{3}[2-(-1+\lambda)]+\underbrace{Pr(y\in(2,2+\lambda];H_1)}_{\frac{1}{3}\lambda.}=1.\\ P_F=\alpha.\end{cases}$

If $\tau^*>-1+\lambda$, i.e. $\alpha<\frac{3-\lambda}{3}$.

$\begin{cases} P_D=Pr(D=1\mid H_1)=\int_{\tau^*}^{2+\lambda}\frac{1}{3}dy=\frac{1}{3}[2+\lambda-(2-3\alpha)]=\alpha+\frac{\lambda}{3}.\\ P_F=\alpha.\end{cases}$