CS/CNS/EE/IDS 165: Foundations in Machine Learning and Statistical Inference

Sufficient Statistics

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Outline

Concepts

- Parametric statistical model.
- Statistics, sufficient statistics, and minimal sufficient statistics.
- Exponential families.

References

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2. L. L. Scharf, Statistical Signal Processing: Detection, Estimation and Time Series Analysis, Addison-Wesley, Publishing Company, Inc., 1991, Chapter 3.
3. P.J. Bickel and K.A. Doksum, Mathematical Statistics: Basic Ideas and Selected Topics, Prentice Hall, 1977, Chapter 2.
4. T. S. Ferguson, Mathematical Statistics: A Decision Theoretic Approach, Academic Press, 1967, Chapter 3.3.
5. J. Shao, Mathematical Statistics, Springer-Verlag, 1999, Chap. 2.

Motivating Examples

Coin Flip

The experiment of flipping a coin with probability of showing head θ can be modeled by pmfs indexed by θ

$$f(y|\theta) \triangleq \begin{cases} \theta & y = 1 \\ 1-\theta & y = 0 \end{cases}, \quad \theta \in \Theta \triangleq [0,1]$$

Binary signaling in Gaussian noise

The transmission of $\theta \in \{1, -1\}$ over an AWGN channel

$$Y = \theta + N, \quad N \sim \mathcal{N}(0, \sigma^2)$$

with known σ^2 can be modeled by pdfs indexed by $\theta \in \{\pm 1\}$

$$f(y|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y-\theta)^2}{2\sigma^2}\}, \quad \theta \in \Theta \triangleq \{\pm 1\}$$

Channel Estimation

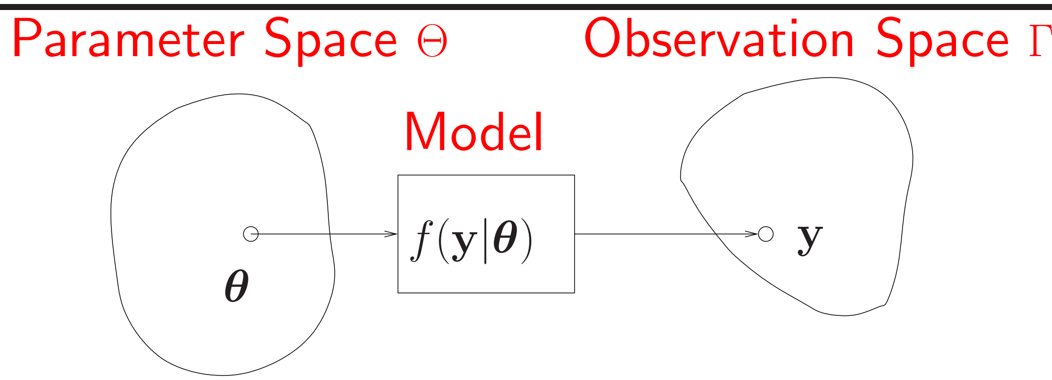
An unknown linear fading channel in Gaussian noise

$$Y_1 = \theta s_1 + N_1, \;\; Y_2 = \theta s_2 + N_2, \quad N_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$$

with known input s_1, s_2 and σ^2 can be modeled by

$$f(y_1, y_2|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y_1 - s_1\theta)^2 + (y_2 - s_2\theta)^2}{2\sigma^2}\}, \quad \theta \in \Theta \triangleq \mathcal{R}$$

Parametric Model



Frequentist Model

The statistic model is defined by the probability density (or pmf) function $f(\mathbf{y}|\boldsymbol{\theta})$ on the observation space Γ indexed by deterministic parameter $\boldsymbol{\theta} \in \Theta$. Note that $f(\mathbf{y}|\boldsymbol{\theta})$ is not the conditional PDF (θ is deterministic); it is merely for notational convenience.

Bayesian Model

If the parameter can be modeled as random with prior pdf $\pi(\theta)$, we then have a Bayesian model

$$f(\mathbf{y}, \boldsymbol{\theta}) = \pi(\boldsymbol{\theta}) f(\mathbf{y}|\boldsymbol{\theta}).$$

The posterior distribution of Θ given observation $\mathbf{y}$ is

$$f(\boldsymbol{\theta}|\mathbf{y}) = \frac{\pi(\boldsymbol{\theta}) f(\mathbf{y}|\boldsymbol{\theta})}{\int \pi(\mathbf{t}) f(\mathbf{y}|\mathbf{t}) d\mathbf{t}}$$

Statistics vs. Probability

In statistics, we are interested in inferring $\boldsymbol{\theta}$ after observing $\mathbf{Y} = \mathbf{y}$. In probability, we are interested in deducing the chance of various outcomes without experiments.

Frequentist vs. Bayesian

Frequentist Viewpoint

- Probability is objective; it is connected to the physical world through the relative frequency of event occurrence.
- Parameters are deterministic and unknown; it does not make sense to calculate $\Pr(\boldsymbol{\theta} \in \mathcal{X} | \mathbf{Y} = \mathbf{y})$.
- Statistical procedures should have well-behaved long-run properties.

Bayesian Viewpoint

- Probability is subjective; it merely describes the degree of a belief. (“tomorrow, 30% chance of snow).
- Even if θ is deterministic, we can assign certain distribution of prior belief.
- The inference of a parameter is made based on the posterior distribution $f(\theta|\mathbf{y})$.

Likelihood Function

Given the observation data $\mathbf{Y} = \mathbf{y}$, then the likelihood function of $\boldsymbol{\theta}$ is a function of the form

$$l(\boldsymbol{\theta}; \mathbf{y}) \triangleq \gamma(\mathbf{y}) f(\mathbf{y}|\boldsymbol{\theta})$$

where $\gamma(\mathbf{y})$ does not depend on θ. A standard choice is when $\gamma(\mathbf{y}) = 1$.

- A likelihood function should be viewed as a function of parameter $\boldsymbol{\theta}$, and it is not uniquely defined.
- Sometimes, it is more convenient to work with log-likelihood function

$$L(\boldsymbol{\theta}; \mathbf{y}) = \log f(\mathbf{y}|\boldsymbol{\theta}).$$

- The average log-likelihood function happens to be the entropy:

$$H_{\boldsymbol{\theta}}(\mathbf{Y}) \triangleq \mathbb{E}_{\boldsymbol{\theta}}(-L(\theta; \mathbf{Y})) = -\int f(\mathbf{y}|\boldsymbol{\theta}) \log f(\mathbf{y}|\boldsymbol{\theta}) d\mathbf{y}$$

  Note that the connection between entropy and likelihood function is only valid when the expectation is taken using the same probability model that the observations are generated.

Example: Uniform Distribution

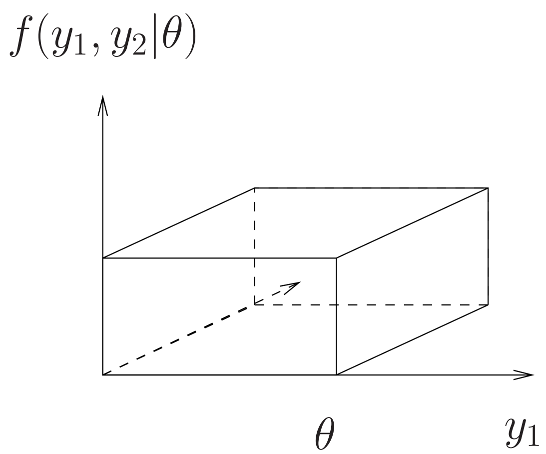
Consider N independent random samples $Y_i \stackrel{i.i.d.}{\sim} \mathcal{U}(0, \theta)$. The parametric model is then given by the PDF

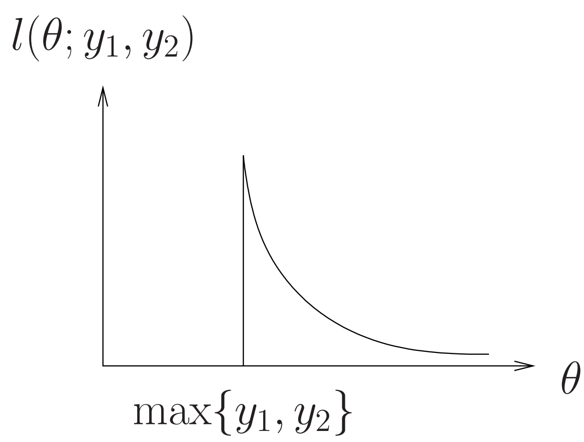
$$f(\mathbf{y}|\theta) = \begin{cases} \frac{1}{\theta^n} & \theta \geq \max\{y_i\} \\ 0 & \text{otherwise} \end{cases}$$

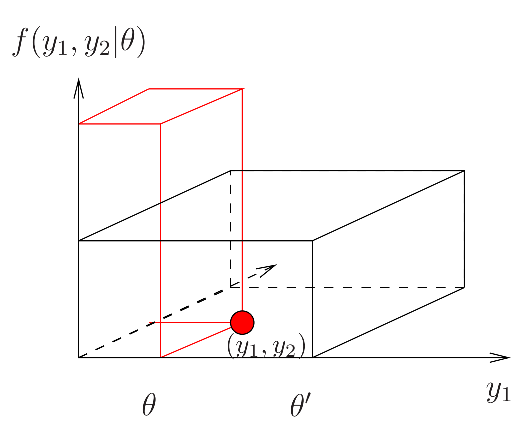
The likelihood function $l(\theta; \mathbf{y})$ defined

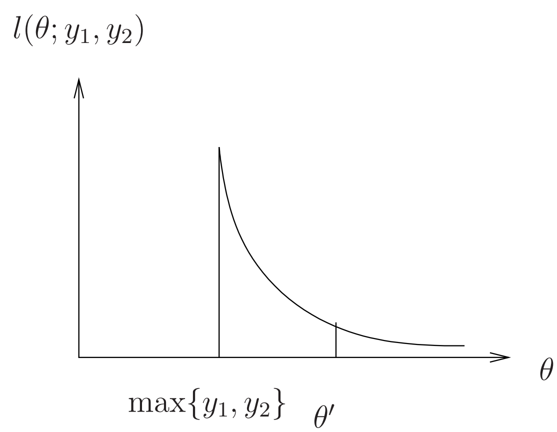
$$l(\theta; \mathbf{y}) \stackrel{\Delta}{=} f(\mathbf{y}|\theta)$$

has a very different look from the PDF.









Examples: The Gaussian Popoulation

Independent Sampling

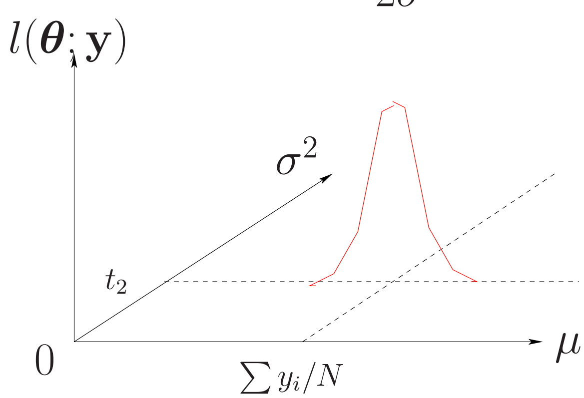
Consider N independent random samples $Y_i \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$. With $\boldsymbol{\theta} = (\mu, \sigma^2) \in \mathcal{R} \times \mathcal{R}^+$, the parametric model is then given by

$$f(\mathbf{y}|\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}^N} \exp\{-\frac{\sum_{i=1}^N y_i^2 - 2\mu \sum_{i=1}^N y_i + N\mu^2}{2\sigma^2}\}$$

The likelihood function can be defined as

$$l(\boldsymbol{\theta}; \mathbf{y}) = \exp\{-N \frac{\frac{1}{N}\sum_{i=1}^N y_i^2 - 2\mu \frac{1}{N}\sum_{i=1}^N y_i + \mu^2 + 2\sigma^2 \ln \sigma}{2\sigma^2}\}$$

$$L(\boldsymbol{\theta}; \mathbf{y}) = -N \frac{\frac{1}{N}\sum_{i=1}^N y_i^2 - 2\mu \frac{1}{N}\sum_{i=1}^N y_i + \mu^2 + 2\sigma^2 \ln \sigma}{2\sigma^2}$$



Remark

- The likelihood function depends only on data summary $(\sum_i y_i, \sum_i y_i^2)$.
- What happens when $N \to \infty$? By the law of large numbers, we have roughly

$$\frac{1}{N} L(\boldsymbol{\theta}; \mathbf{y}) \to -\frac{1 + 2\ln \sigma^2}{2}$$

Example: Independent Bernoulli Trials

The Model

Suppose that we conduct N independent Bernoulli trials with probability of success $\Pr(Y_i = 1) = \theta$, $\Pr(Y_i = 0) = 1 - \theta$, and $\theta \in \{\theta_1, \theta_2\}$, and $\theta_1 \neq \theta_2$. The parametric model is then given by

$$f(\mathbf{y}|\theta) = \theta^{\sum y_i}(1-\theta)^{N-\sum y_i}$$

Remarks

- Again, the model depends not on the entire $\mathbf{y}$ but only on a single number $t(\mathbf{y}) = \sum_i y_i$—the total number of successes in N trials, *i.e.,* the model can be written as

$$f(\mathbf{y}|\theta) = g(t(\mathbf{y});\theta)$$

- A less obvious but more fundamental fact is that the model depends only on the likelihood ratio

$$r(\mathbf{y}) = \frac{f(\mathbf{y}|\theta_1)}{f(\mathbf{y}|\theta_2)} = \left(\frac{\theta_1}{\theta_2}\right)^{\sum_i y_i}\left(\frac{1-\theta_1}{1-\theta_2}\right)^{n-\sum_i y_i}$$

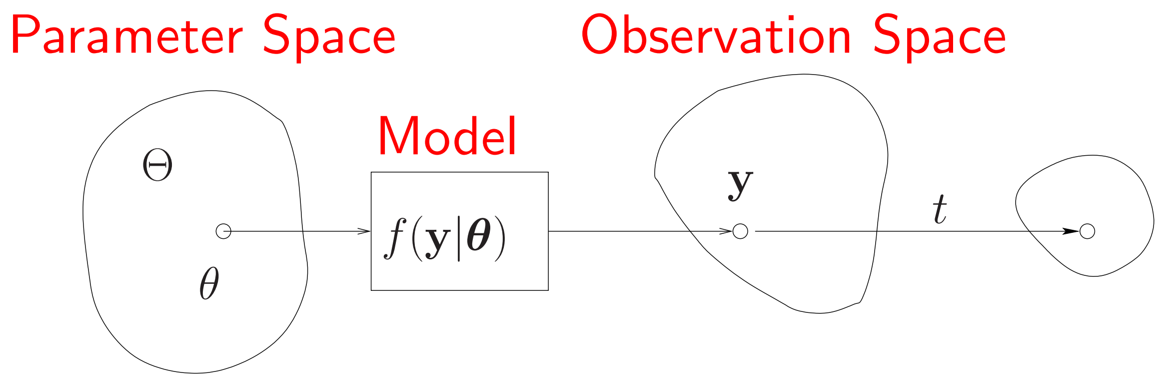
  This follows from

$$r(\mathbf{y}) \to t(\mathbf{y}) \to f(\mathbf{y}|\theta) = q(r(\mathbf{y});\theta)$$

- If we can write $f(\mathbf{y}|\theta) = g(t(\mathbf{y});\theta)$, can we discard $\mathbf{y}$ using only $t(\mathbf{y})$?

Statistics

Given a parametric model $f(\mathbf{y}|\boldsymbol{\theta})$, a (measurable) function $\mathbf{t}(\mathbf{Y})$ of the random observation $\mathbf{Y} \sim f(\mathbf{y}|\boldsymbol{\theta})$ is called a statistic.



- A statistic is a random vector that conveys information about the original parametric model. It often has lower dimension than $\mathbf{y}$ and less complex; it represents a (possibly lossy) data reduction.
- There are many statistics. The original observation $\mathbf{Y}$ is a trivial statistic.
- Statistics are used for inference. It is therefore desirable that (i) they do not loose information about the model—sufficiency and (ii) their dimension is as low as possible—parsimony.

Sufficiency

A statistic $t(\mathbf{Y})$ is a sufficient statistic for model $f(\mathbf{y}|\boldsymbol{\theta})$ if the conditional density of r.v. $\mathbf{Y}$ given $t(\mathbf{Y}) = \mathbf{u}$ is not a function of $\boldsymbol{\theta}$ for all $\mathbf{u}$. A sufficient statistic $t(\mathbf{Y})$ is a minimal sufficient statistic if, for any other sufficient statistic $\tilde{t}$, there is a (measurable) function $h(\cdot)$ such that $t(\mathbf{y}) = h(\tilde{t}(\mathbf{y}))$.

Example

Consider n Bernoulli trials $Y_i \overset{\text{IID}}{\sim} \mathcal{B}(\theta)$. Denote $\mathbf{Y} = (Y_1, \cdots, Y_n)$. We claim that $t(\mathbf{Y}) = \sum Y_i$ is a sufficient statistic.

$$\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y}|t(\mathbf{Y}) = j) &= \frac{\Pr(\mathbf{Y} = \mathbf{y}, t(\mathbf{Y}) = j)}{\Pr(t(\mathbf{Y}) = j)} \\
&= \begin{cases} \frac{\theta^j (1-\theta)^{n-j}}{\binom{n}{j} \theta^j (1-\theta)^{n-j}} & \text{if } t(\mathbf{y}) = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Remarks:

- If we know $t(\mathbf{y})$, then we can discard $\mathbf{y}$ since, given $t(\mathbf{Y}) = t(\mathbf{y})$, the probability of $\mathbf{Y}$ no longer depends on θ; the outcome of $\mathbf{Y} = \mathbf{y}$ is no longer informative.
- How to find sufficient statistics?

The Neyman-Fisher Factorization Theorem

Theorem: A statistic $t(\mathbf{Y})$ is sufficient if and only if the pdf $f(\mathbf{y}|\boldsymbol{\theta})$ has the factorization

$$f(\mathbf{y}|\theta) = g(t(\mathbf{y}); \boldsymbol{\theta})h(\mathbf{y})$$

where g and h are non-negative functions.

Proof for the discrete case: If $f(\mathbf{y}|\boldsymbol{\theta}) = g(t(\mathbf{y}); \boldsymbol{\theta})h(\mathbf{y})$, then

$$\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y}|t(\mathbf{Y}) = \mathbf{u}; \boldsymbol{\theta}) &= \frac{\Pr(\mathbf{Y} = \mathbf{y}, t(\mathbf{Y}) = \mathbf{u}; \boldsymbol{\theta})}{\Pr(t(\mathbf{Y}) = \mathbf{u}; \boldsymbol{\theta})} \\
&= \begin{cases} \frac{g(\mathbf{u}, \boldsymbol{\theta})h(\mathbf{y})}{\Pr(t(\mathbf{Y}) = \mathbf{u}; \boldsymbol{\theta})} & \text{if } t(\mathbf{y}) = \mathbf{u} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

But

$$\Pr(t(\mathbf{Y})) = \mathbf{u}; \boldsymbol{\theta}) = \sum_{\mathbf{y}, t(\mathbf{Y}) = \mathbf{u}} f(\mathbf{y}|\boldsymbol{\theta}) = g(\mathbf{u}; \boldsymbol{\theta}) \sum_{\mathbf{y}, t(\mathbf{y}) = \mathbf{u}} h(\mathbf{y})$$

Hence

$$\Pr(\mathbf{Y} = \mathbf{y}|t(\mathbf{Y}) = \mathbf{u}; \boldsymbol{\theta}) = \begin{cases} \frac{h(\mathbf{y})}{\sum_{\mathbf{y}, t(\mathbf{y}) = \mathbf{u}} h(\mathbf{y})} & \text{if } t(\mathbf{y}) = \mathbf{u} \\ 0 & \text{otherwise} \end{cases}$$

If $t(\mathbf{Y})$ is sufficient, let

$$\begin{aligned}
g(t(\mathbf{y}); \boldsymbol{\theta}) &\triangleq \Pr(t(\mathbf{Y}) = t(\mathbf{y}); \boldsymbol{\theta}), \\
h(\mathbf{y}) &= \Pr(\mathbf{Y} = \mathbf{y}|t(\mathbf{Y}) = \mathbf{t}(\mathbf{y}))
\end{aligned}$$

Then

$$\begin{aligned}
f(\mathbf{y}|\boldsymbol{\theta}) &= \Pr(\mathbf{Y} = \mathbf{y}; \boldsymbol{\theta}) = \Pr(\mathbf{Y} = \mathbf{y}, t(\mathbf{Y}) = t(\mathbf{y}); \theta) \\
&= g(t(\mathbf{y}); \boldsymbol{\theta})h(\mathbf{y})
\end{aligned}$$

Sufficiency of Likelihood

Corollary Consider a binary hypothesis model given by $\mathbf{y} \sim p(\mathbf{y};\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\}$. Define the statistic by the likelihood ratio

$$r(\mathbf{Y}) \triangleq \frac{f(\mathbf{Y}|\theta_1)}{f(\mathbf{Y}|\theta_0)}.$$

We then have $p(\mathbf{y};\boldsymbol{\theta}) = f(\mathbf{y}|\boldsymbol{\theta}_0)g(r(\mathbf{y});\boldsymbol{\theta})$, where

$$g(r(\mathbf{y});\theta) = \begin{cases} 1 & \theta = \theta_0 \\ r(\mathbf{y}) & \theta = \theta_1 \end{cases}$$

By the Neyman-Fisher factorization, $r(\mathbf{Y})$ is a sufficient statistic.

Remarks:

- For the general discrete model $\Theta = \{\theta_1, \cdots, \theta_M\}$, the M-dimensional vector of likelihood functions $l(\mathbf{y}) = [p(\mathbf{y};\theta_1), \cdots, p(\mathbf{y};\theta_M)]$ or the $M-1$ dimensional vectors of likelihood ratios

$$r(\mathbf{Y}) = [\frac{f(\mathbf{Y}|\theta_2)}{f(\mathbf{Y}|\theta_1)}, \cdots, \frac{f(\mathbf{Y}|\theta_M)}{f(\mathbf{Y}|\theta_1)}]$$

  are also sufficient statistics.

- If we broaden the notion of statistic whose values are functions of θ, the the likelihood function $r(\boldsymbol{\theta};\mathbf{Y})$ is minimal sufficient (Dynkin,1951)[†].

[†] E.B. Dynkin, "Necessary and sufficient statistics for families of distributions," *Sel. Transl. Math., Stat., and Prob.*, vol. 1, pp. 23–41, 1951.

Examples

I.I.D. Gaussian Model: Consider $Y_i \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$.

$$pf(\mathbf{y}|\mu, \sigma^2) = \frac{1}{(\sqrt{2\pi\sigma^2})^n} e^{\{-\frac{\sum y_i^2}{2\sigma^2} + \frac{\mu \sum y_i}{\sigma^2} - \frac{n\mu^2}{2\sigma^2}\}} \to \mathbf{t}(\mathbf{y}) = \begin{pmatrix} \sum_i y_i \\ \sum_i y_i^2 \end{pmatrix}.$$

I.I.D. Poisson Model: Consider $Y_i \overset{i.i.d.}{\sim} \mathcal{P}(\lambda)$.

$$f(\mathbf{y}|\lambda) = \frac{\lambda^{\sum y_i} e^{-n\lambda}}{\prod y_i!} \to t(\mathbf{y}) = \sum_i y_i.$$

Extreme Statistic. Suppose $Y_i \overset{i.i.d.}{\sim} \mathcal{U}(0, \theta)$.

$$\begin{aligned} f(\mathbf{y}|\theta) &= \begin{cases} \frac{1}{\theta^{-n}} & 0 < y_i < \theta, \quad \forall i \\ 0 & \text{otherwise} \end{cases} = h(\mathbf{y}) g(\theta, \max_i y_i) \\ h(\mathbf{y}) &= \begin{cases} 1 & y_i > 0, \quad \forall i \\ 0 & \text{otherwise} \end{cases} \qquad g(\theta, t) = \begin{cases} \frac{1}{\theta^{-n}} & t < \theta, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Channel Estimation in AWGN. Given

$$y_n = x_0 s_n + x_1 s_{n-1} + w_n, \quad w_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2), \quad n = 0, 1, \cdots, N-1,$$

To estimate $\boldsymbol{\theta} = [x_0 \; x_1]^T$ with known s_n, let

$$\mathbf{y} = \begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} s_0 & s_{-1} \\ s_1 & s_0 \\ \vdots & \vdots \\ s_{N-1} & s_{N-2} \end{pmatrix}.$$

Then

$$\begin{aligned} f(\mathbf{y}|\boldsymbol{\theta}) &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} exp\{-\frac{||\mathbf{y} - \mathbf{S}\boldsymbol{\theta}||^2}{2\sigma^2}\} \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} exp\{\frac{2\mathbf{y}'\mathbf{S}\boldsymbol{\theta} - ||\mathbf{S}\boldsymbol{\theta}||^2}{2\sigma^2}\} exp\{-\frac{||\mathbf{y}||^2}{2\sigma^2}\} \end{aligned}$$

Hence

$$t(\mathbf{y}) = \mathbf{y}'\mathbf{S} = \begin{pmatrix} \sum_i s_i y_i \\ \sum_i s_{i-1} y_i \end{pmatrix}$$

The K-Parameter Exponential Family

Definition: A family of distributions is said to be a K-parameter exponential family if there exist $c_1(\boldsymbol{\theta}), \cdots, c_K(\boldsymbol{\theta}), d(\boldsymbol{\theta})$, $t_1(\mathbf{y}), \cdots, t_K(\mathbf{y}), s(\mathbf{y})$ and a set $\mathcal{A}$ such that

$$f(\mathbf{y}|\boldsymbol{\theta}) = \exp\{\sum_{i=1}^{K} c_i(\boldsymbol{\theta}) t_i(\mathbf{y}) + d(\boldsymbol{\theta}) + s(\mathbf{y})\} 1_{\mathcal{A}}(\mathbf{y})$$

where $I_{\mathcal{A}}(\mathbf{y})$ is the indicator function not related to $\boldsymbol{\theta}$. It is often more convenient to use the canonical form (or the natural representation) of the exponential distribution

$$f(\mathbf{y}|\boldsymbol{\eta}) = \exp\{\sum_{i=1}^{K} \eta_i t_i(\mathbf{y}) + d(\boldsymbol{\eta}) + s(\mathbf{y})\} 1_{\mathcal{A}}(\mathbf{y}).$$

Theorem: Let $\{f(\mathbf{y}|\boldsymbol{\theta}), \boldsymbol{\theta} \in \boldsymbol{\Lambda}\}$ be a K-parameter exponential family, *i.e.*,

$$f(\mathbf{y}|\boldsymbol{\theta}) = \exp\{\sum_{i=1}^{K} c_i(\boldsymbol{\theta}) t_i(\mathbf{y}) + d(\boldsymbol{\theta}) + s(\mathbf{y})\} I_{\mathcal{A}}(\mathbf{y})$$

If $\{\mathbf{c}(\boldsymbol{\theta}) = [c_1(\boldsymbol{\theta}), \cdots, c_K(\boldsymbol{\theta})], \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ has an interior point, then $t(\mathbf{y}) = [t_1(\mathbf{y}), \cdots, t_K(\mathbf{y})]^T$ is minimal sufficient.

Proof: The sufficiency of $\mathbf{t}(\mathbf{y})$ follows the Neyman-Fisher factorization. The minimality is implied by the completeness of $\mathbf{t}(\mathbf{y})$, which will be discussed later. The reason for the existence of "interior point" is to prevent the trivial cases such as by splitting $c_1(\boldsymbol{\theta}) = c_{11}(\boldsymbol{\theta}) + c_{12}(\boldsymbol{\theta})$ thus increasing the dimension of the statistic.

Examples of Exponential Family

These belong to the exponential family

1. Gaussian. $Y_i \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$.

2. Binomial: $Y \sim \mathcal{B}(\theta, n)$

$$f(k|\theta) = \binom{n}{k} \theta^k (1-\theta)^{(n-k)} = \binom{n}{k} e^{k \ln \frac{\theta}{1-\theta} + n \ln(1-\theta)}$$

3. Multinomial: In n independent trials with s outcomes in each trial. Let p_i be the probability for the ith outcome. Let y_i be the number of trials that have the ith outcome.

$$\begin{aligned} f(y_1, \cdots, y_s | p_1, \cdots, p_s) &= \frac{n!}{y_1! \cdots y_s!} p_1^{y_1} \cdots p_s^{y_s} \\ &= exp(k_1 \ln p_1 + \cdots + k_s \ln p_s) h(\mathbf{y}) I_{\mathcal{A}}(\mathbf{y}) \end{aligned}$$

4. Poisson. $Y_i \overset{i.i.d.}{\sim} \mathcal{P}(\theta)$

$$f(y_1, \cdots, y_n | \theta) = \frac{\theta^{\sum y_i}}{\prod y_i!} e^{-n\theta} = \exp\{\sum y_i \ln \theta - n\theta\} h(\mathbf{y})$$

These do not belong to the exponential family

1. Uniform. $Y \sim \mathcal{U}(0, \theta)$.

$$f(y|\theta) = \frac{1}{\theta} I_{(0,\theta)}(y)$$

2.

$$f(y|\theta) = 2\frac{y+\theta}{1+2\theta} = exp\{\ln 2(y+\theta) - \ln(1+2\theta)\}, \; 0 < y < 1, \;\; \theta > 0$$