CS 165 Winter 2023 *Last updated:* February 6, 2023

Midterm Exam

Due at 12:00 pm, Feb 9.

1 Bayesian detection

Let Y be a Gaussian random variable with unknown mean Θ and known variance σ^2. Suppose the mean Θ is itself a Gaussian random variable $\Theta \sim \mathcal{N}(\mu, \gamma^2)$ where both μ and γ^2 are known.

1. Show that the conditional distribution of θ given $Y = y$ is Gaussian. Confirm that the mean and variance are $\frac{\mu\sigma^2 + y\gamma^2}{\gamma^2+\sigma^2}$, $\frac{\gamma^2\sigma^2}{\gamma^2+\sigma^2}$ respectively.

   (Hint: $p(\theta|y) \propto \pi(\theta)p(y|\theta)$)

2. Suppose that an observation $Y = y$ is made from the above Gaussian shifted mean model. Consider the hypothesis $\mathcal{H}_0 : \theta \leq 0$ versus $\mathcal{H}_1 : \theta > 0$. Find the Bayesian detector for the uniform cost with $C_{ij} = 1$ when $i \neq j$ and $C_{ij} = 0$ otherwise.

2 Gaussian vs exponential

Consider the following simple binary hypotheses:

$$\mathcal{H}_0 : Y \sim \begin{cases} \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad vs. \quad \mathcal{H}_1 : Y \sim \begin{cases} \frac{1}{\sigma} e^{-\frac{y}{\sigma}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

both having the same variance with different means (μ_0 and μ_1, respectively).

1. Find μ_0 and μ_1 in terms of σ. Confirm $\mu_1 > \mu_0$.

2. Consider first the simple threshold detector that compares the observation $Y = y$ with the means of the two distributions, *i.e.*,

   $$\delta(y) = \begin{cases} 1 & y > \frac{\mu_0+\mu_1}{2} \\ 0 & \text{otherwise} \end{cases}$$

   Find the size and power of the detector. You may express the solution in terms of the Q function ($Q(x) = P(X > x)$ where X is a standard Gaussian random variable).

3. Compute the likelihood ratio $L(y)$.

4. Argue that testing $L(y)$ against threshold τ_0 is equivalent to testing $|\frac{y}{\sigma} - 1|$ against another threshold τ_1. (Hint: $e^{x^2-ax} = c \cdot e^{|x-a/2|^2}$ where a and c are constants.)

5. Consider the case where $\tau_1 < 1$. Find the false alarm in terms of τ_1 and the Q function.

6. Consider the case where $\tau_1 > 1$. Derive the Neyman-Pearson detector for a given size α.

7. Is the detector from part (b) optimal for some α?