Evolutionary Computation for Multi-objective Optimization

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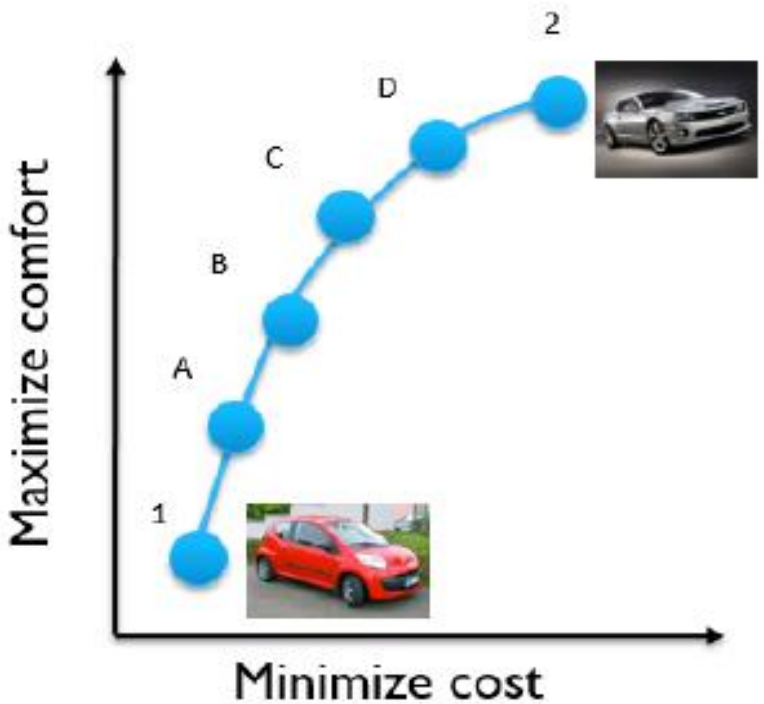
Outline of the Lecture

- Multi-objective optimization: Basic concepts
- Performance assessment and test problems
- Evolutionary computation approaches for multi-objective optimization
- Summary

Introduction to Multi-objective Optimization

- Decision-making: Buying a car



- Features
  - Two or more conflicting objectives
  - There is no single optimal solution
  - Multiple trade-off optimal solutions exist and all such optimal solutions are important

- Any more examples?

Multi-objective Optimization Problems (MOPs)

- MOPs can be modeled as follows:

$$Min/Max \quad F(x) = (f_1(x), f_2(x), \dots, f_m(x))$$

$$\text{Subject to } \; g_j(x) \geq 0,\; h_k(x) = 0,\; x^L \leq x \leq x^U$$

- MOPs involve
  - Finding vectors of decision variables

$$x = (x_1, x_2, \dots, x_n) \in S,$$

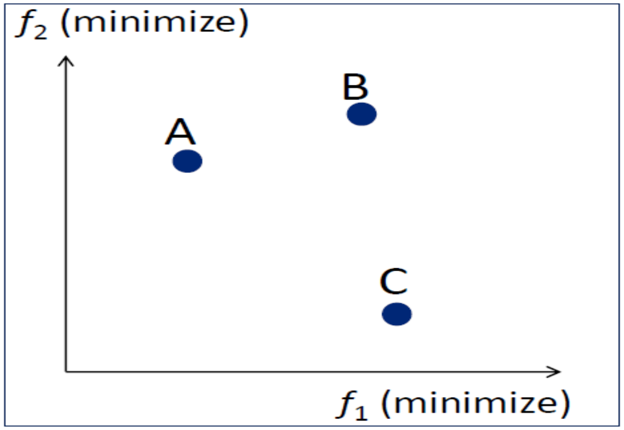
    where S is the decision space or solution space
  - Subject to certain constraints

$$x \in F,\, F \subseteq S,\, F \text{ contains feasible solutions}$$

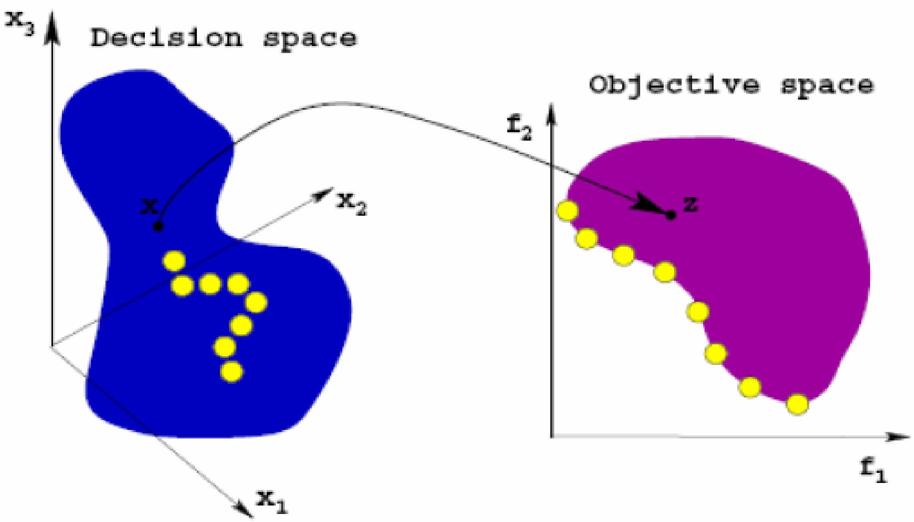
  - Simultaneously optimizing *m* $(\mathrm{m} \geq 2)$ performance criteria expressed as a vector of (often conflicting) objective functions (objective space)

Dominance, Pareto Optimal Set and Front

- Solution x dominates y iff
  - x is no worse than y in all objectives
  - x is better than y in at least one objective
- Example:
  - A dominates B, A doesn't dominate C

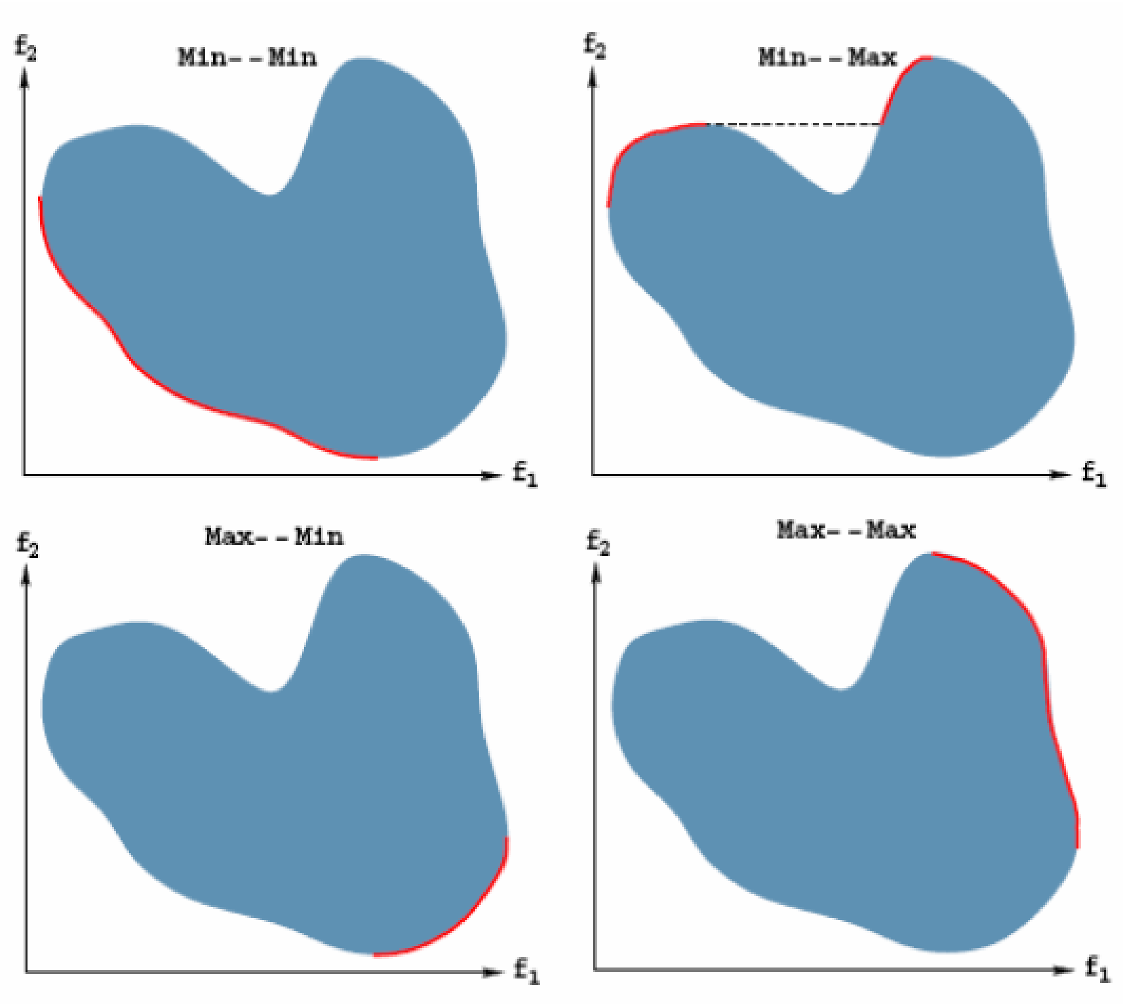


- Pareto optimal set (POS) or Pareto set (PS): the set of all non-dominated solutions in the decision space
- Pareto optimal front (POF) or Pareto front (PF): The image of the POS in the objective space



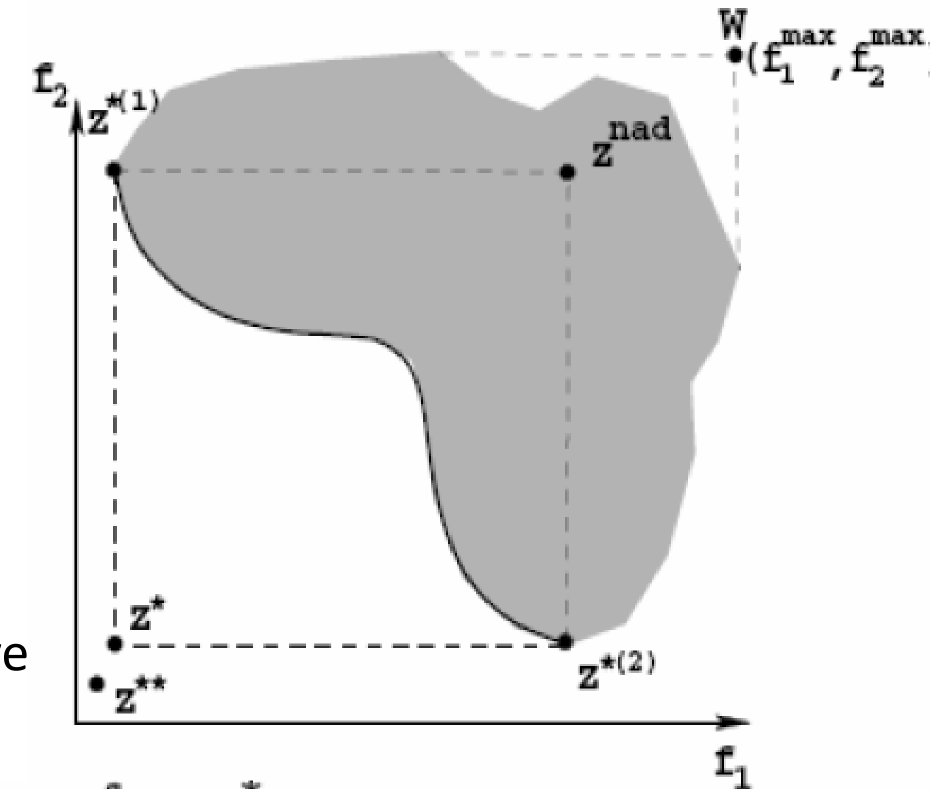
Pareto Optimal Front (POF)

- POF: Depend on the type of objectives
- Always on the boundary of feasible regions



Some Terminologies

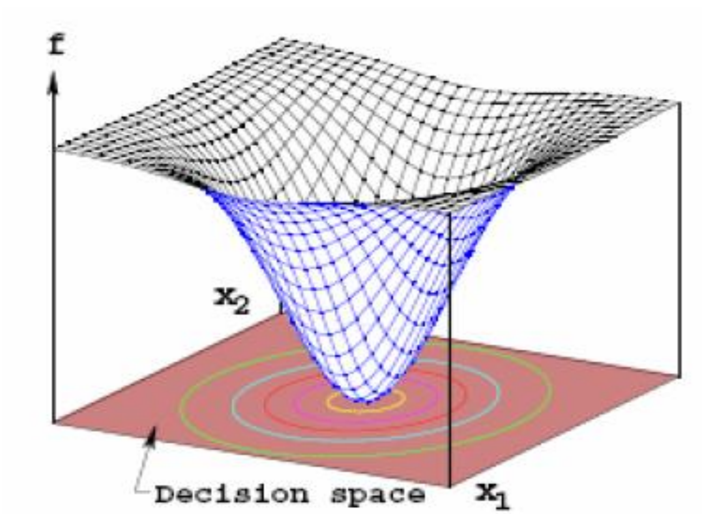
- Ideal point (z*):
  - Lower bound on the POS
- Utopian point (z**):
  - Non-existent point
- Nadir point (z^{nad}):
  - Upper bound on the POS
- Worst point (W):
  - Upper bound on the objective space

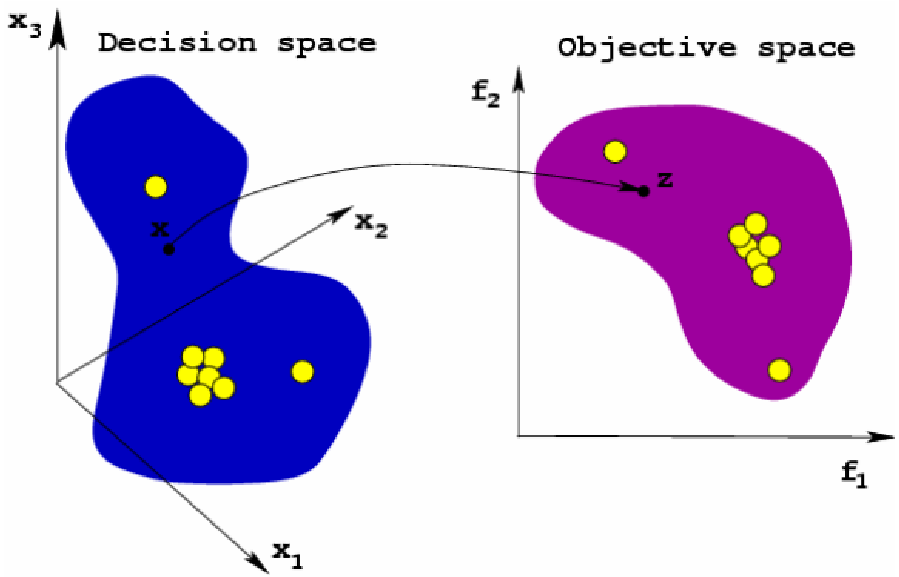


- Normalization: $f_i^{\mathrm{norm}} = \dfrac{f_i - z_i^*}{z_i^{\mathrm{nad}} - z_i^*}$

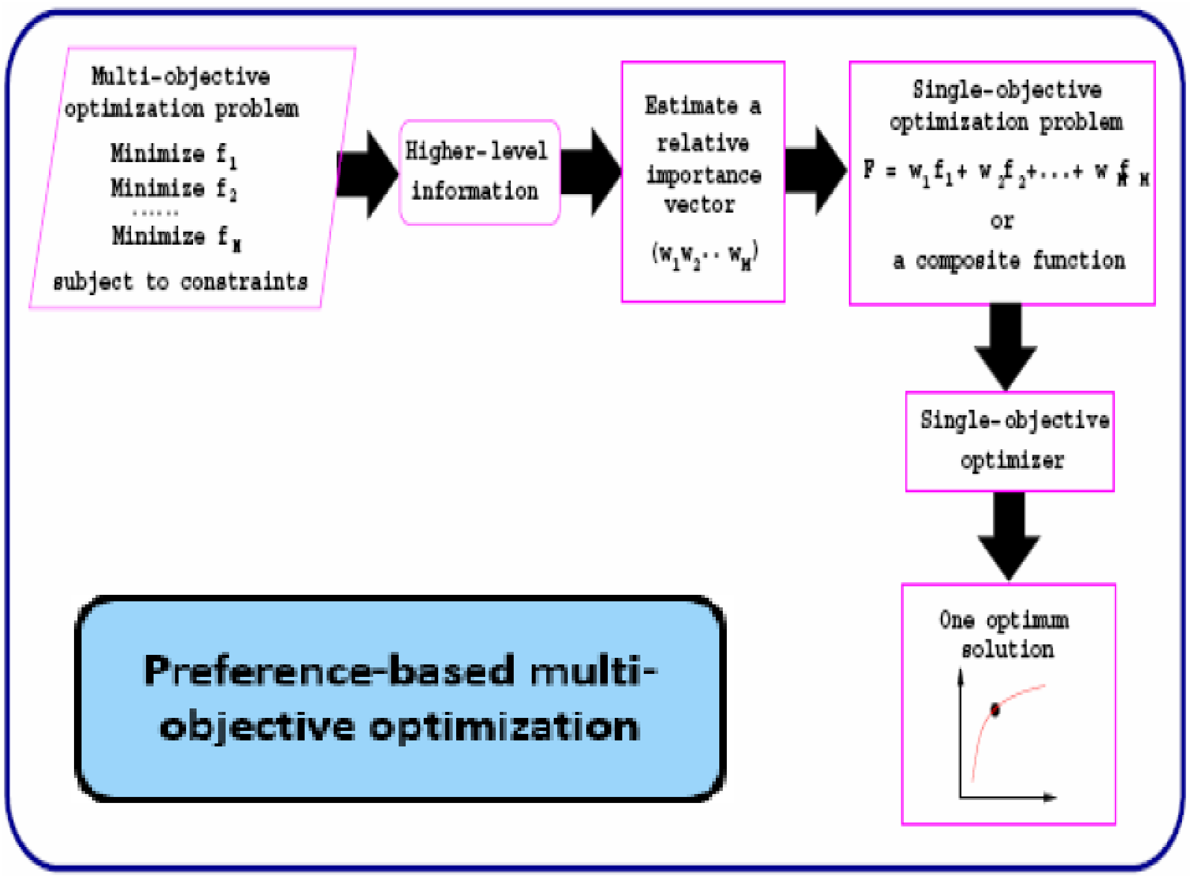
Single Objective vs Multi-objective Optimization

- Single objective:
  - Single optimum or multiple optima with the same objective value
- Multi-objective:
  - A set of non-dominated optima (Pareto optimal set)
  - Two spaces (decision and objective spaces) instead of one



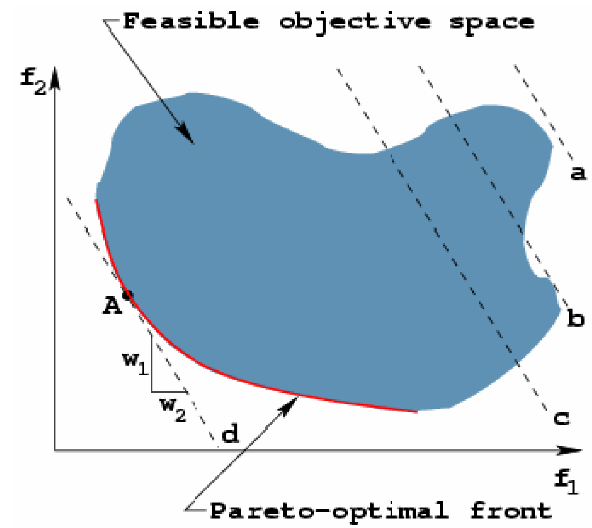


Classical Multi-objective Optimization Methods



- Construct a weighted-sum of objectives and optimize:

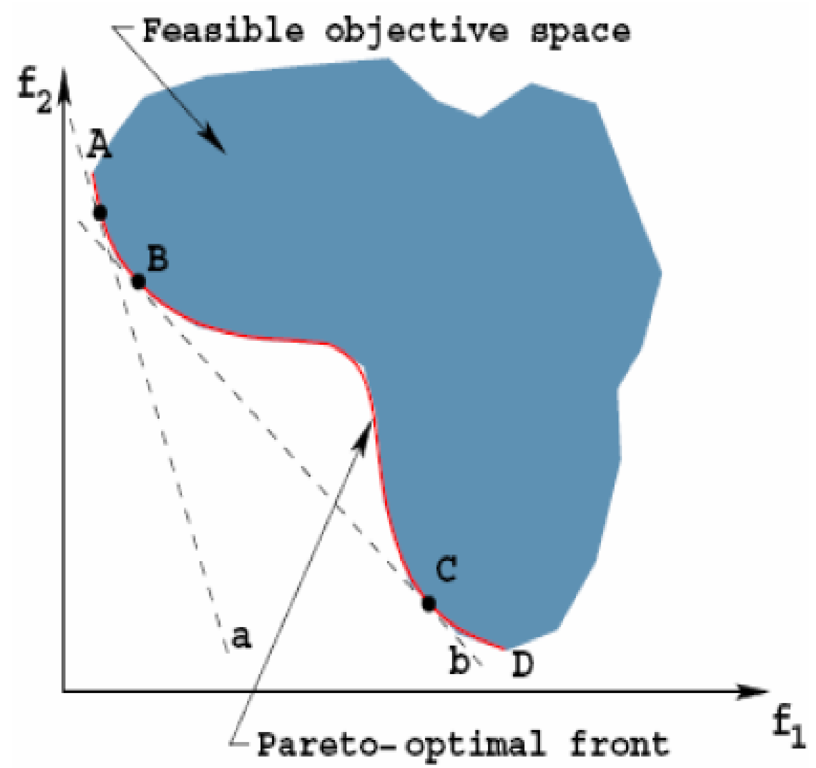
$$F(x) = \sum_{i=1}^{M} w_i f_i(x)$$



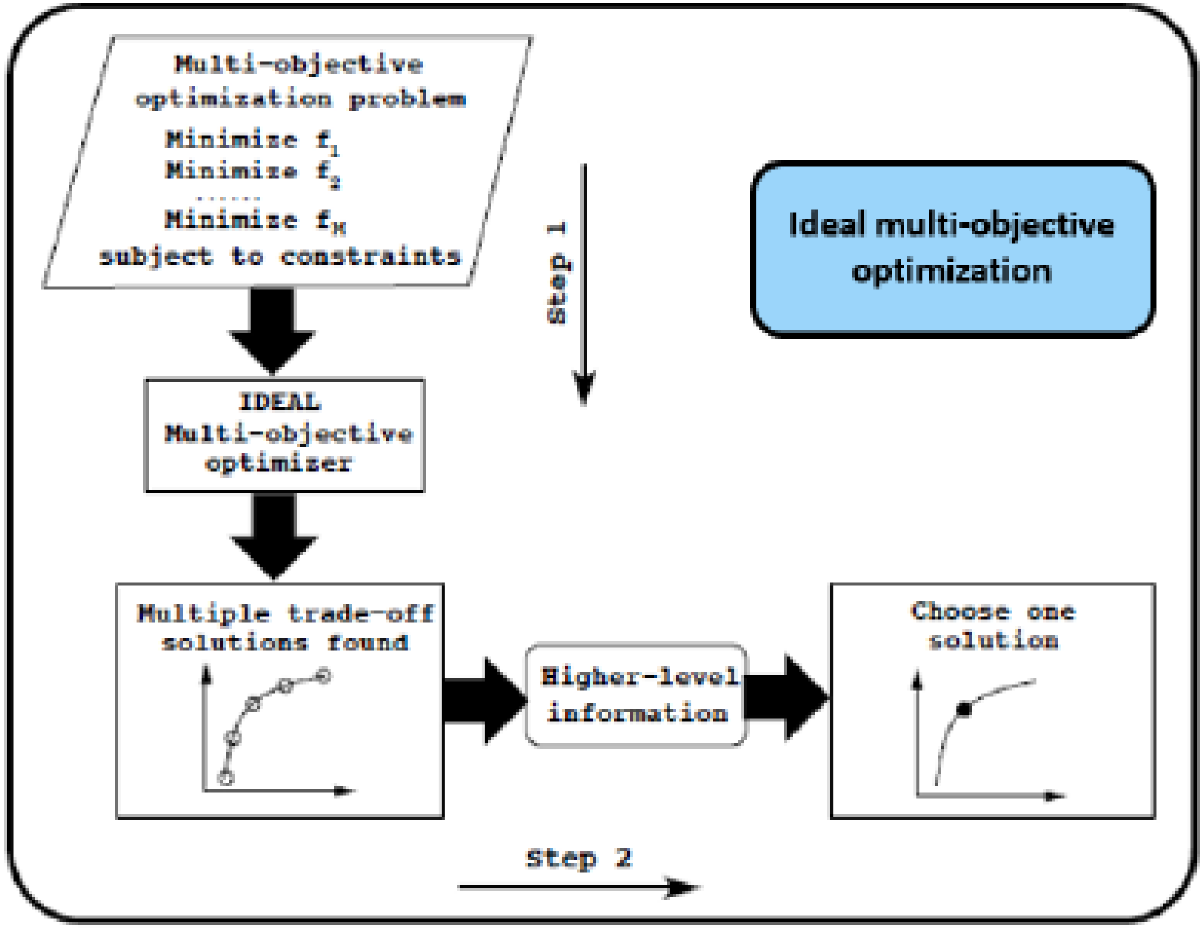
- Advantages: Simple and adequate when a reliable relative preference vector **w** is known

Classical Multi-objective Optimization Methods

- Limitations:
  - Need to know a reliable relative preference vector **w**: Difficult to estimate it without a-priori knowledge
  - The trade-off solution is sensitive to **w**
  - Able to obtain only one solution at a time. Need multiple runs to obtain multiple solutions
  - Unable to find some Pareto-optimal solutions (those in non-convex regions)



Ideal Multi-objective Optimization Methods

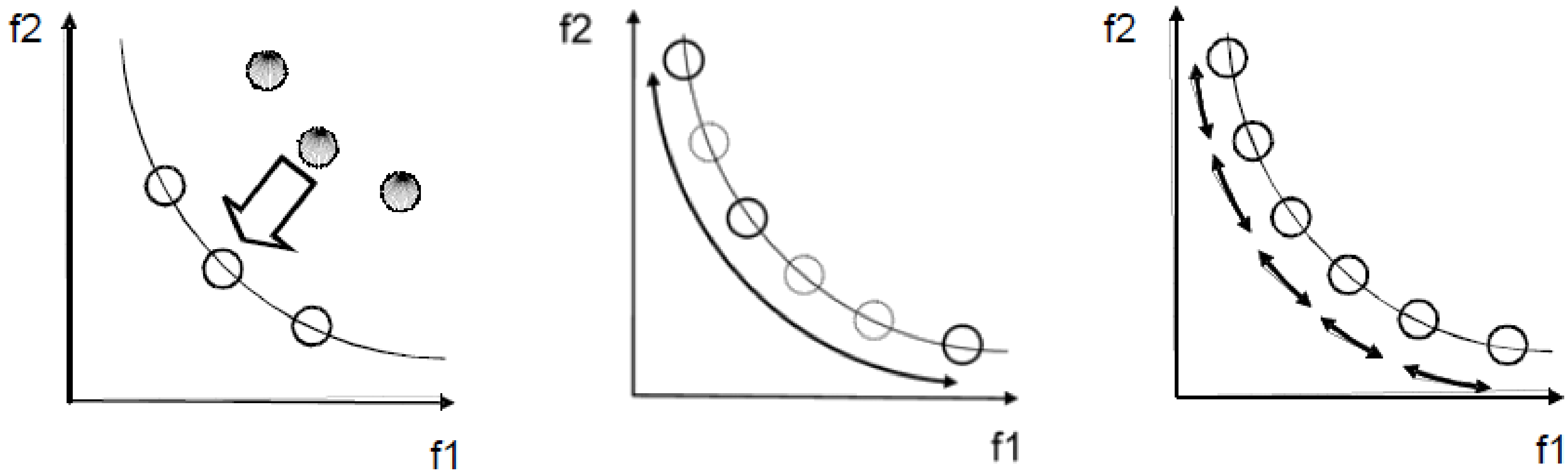


- Advantages:
  - More methodical practical and less subjective
  - Can obtain a set of trade-off solutions in one run
  - High-level information can be used to compare trade-off solutions to choose one for usage
- Challenge:
  - Difficult to obtain all the trade-off optimal solutions

Evolutionary Multi-objective Optimization (EMO)

- Advantages of evolutionary algorithms (EAs) for MOPs:
  - Low requirements on the problem characteristics
  - Easy to integrate heuristic knowledge if available
  - Easy to use (usually inexpensive to develop)
  - User-interaction possible
  - Capable to obtain a set of optimal solutions

Three Goals in EMO



Close to the Pareto front (Convergence)

Widely spread (Extensity)

Equally distributed (Uniformity)

- Usually, extensity and uniformity are combined and called diversity

Performance Assessment

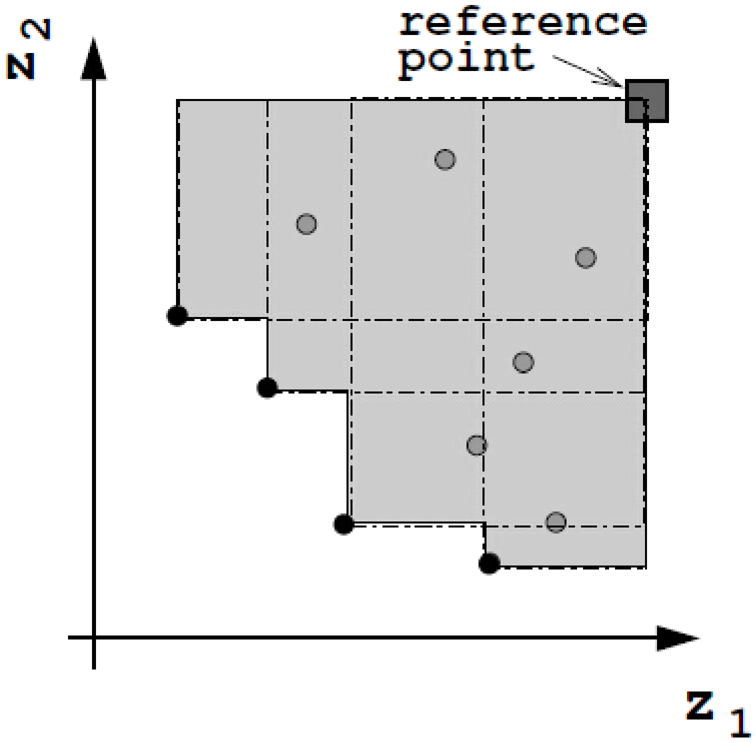
Assess Pareto front approximations obtained by stochastic search algorithms

- **Convergence**: GD [Velduizen and Lamont, 1998], Convergence Measure [Deb and Jain, 2002], Coverage [Zitzler and Thiele, 1999], Dominance Ranking [Knowles et al, 2006]
- **Diversity (both extensity and uniformity):** Δ Metric [Deb et al, 2002], Sigma Diversity Metric [Mostaghim and Teich, 2005], Diversity Measure [Deb and Jain, 2002], DCI [Li et al, 2014]
- **Extensity**: Maximum Spread [Zitzler et al, 2000; Goh and Tan, 2007], Overall Pareto Spread [Wu and Azarm, 2001], Spread Assessment [Li and Zheng, 2009]
- **Uniformity**: Spacing [Schott, 1995], Uniform Distribution [Tan et al, 2002], Entropy Measure [Farhang-Mehr and Azarm, 2003]
- **Comprehensive Assessment (both convergence and diversity):** Hypervolume [Zitzler and Thiele, 1999], ε-indicator [Zitzler et al, 2003], IGD [Bosman and Thierens, 2003], G-Metric [Lizarraga et al, 2008], Average Hausdorff Distance [Schutze, 2012], IGD^+ [Ishibuchi et al, 2014], PCI [Li et al, 2015]

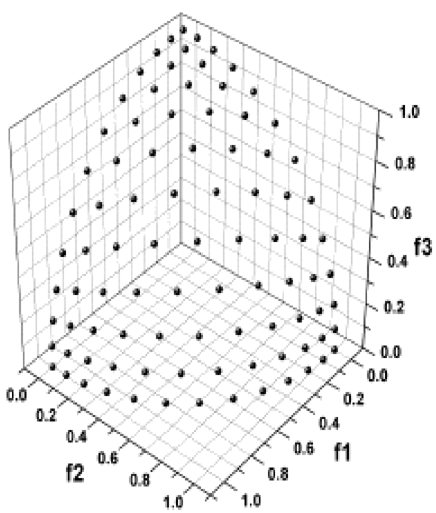
Hypervolume (HV) [Zitzler & Thiele, 1999]

HV calculates the volume of the objective space dominated by an approximation set and bounded by a reference point

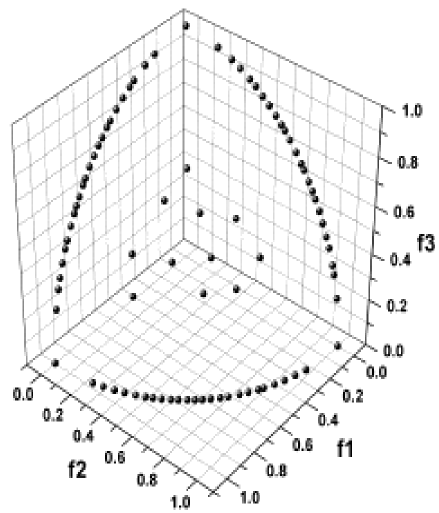
- Good theoretical properties, Pareto dominance compliant, no need of a reference set
- Heavy computational cost, need of a reference point, preferring the knee and boundary solutions to well-distributed solutions



HV Prefers Boundary Solutions: Example



(a) MOEA/D



(b) IBEA

Reference point	MOEA/D	IBEA
(1.0, 1.0, 1.0)	4.1413E–1	4.1525E–1
(1.1, 1.1, 1.1)	7.4484E–1	7.4596E–1
(1.2, 1.2, 1.2)	1.1418E+0	1.1430E+0
(1.4, 1.4, 1.4)	2.1578E+0	2.1590E+0
(1.7, 1.7, 1.7)	4.3268E+0	4.3280E+0
(2.0, 2.0, 2.0)	7.4138E+0	7.4150E+0
(2.5, 2.5, 2.5)	1.5039E+1	1.5040E+1
(3.0, 3.0, 3.0)	2.6414E+1	2.6415E+1

Two Pareto optimal sets on DTLZ2 are obtained by MOEA/D and IBEA.

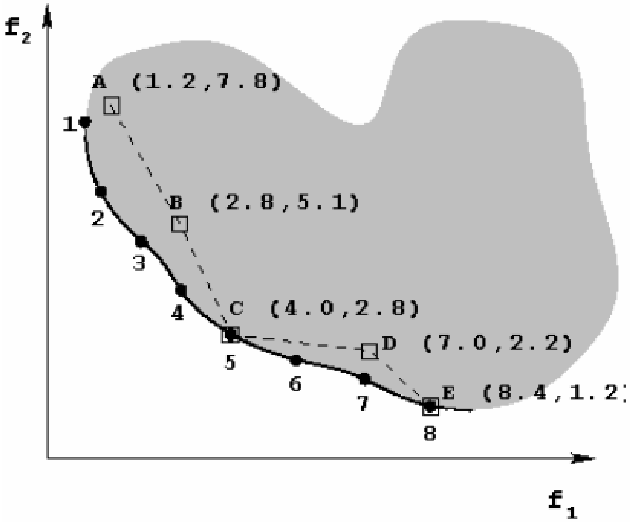
HV results of the two sets under different reference points.

Inverted Generational Distance

IGD calculates the average Euclidean distance from each point in a reference set to its closest solution in an approximation set

$$IGD(\mathrm{PF}^*, \mathrm{PF}) = \frac{\sum_{p \in \mathrm{PF}} d(p, \mathrm{PF}^*)}{|\mathrm{PF}|}$$

- PF is a reference set from the true POF, PF* is an approximate POF
- Quadratic time complexity, no need of parameter settings
- Need a reference set (as a substitution of the Pareto front)



How to specify a proper reference set?

A practical method of constructing a reference set is to use the non-dominated solutions of all the tested approximation sets

Multi-objective Test Problems

- **Continuous functions**: SCH [Veldhuizen, 1999], ZDT [Zitzler et al, 2000], DTLZ [Deb et al, 2005], WFG [Huband et al, 2006], UF [Zhang et al, 2009], Distance Minimization Problem [Köppen and Yoshida, 2007; Ishibuchi et al, 2010], Rectangle [Li et al, 2014]
- **Combinatorial functions**: Knapsack [Zitzler and Thiele, 1999], TSP [Corne and Knowles, 2007], MNK-Landscapes [Aguirre and Tanaka, 2004]

Problem	m	d	Properties	Problem	m	d	Properties	Problem	m	d	Properties
SCH1	2	1	Convex	WFG7	2	22	Concave, Biased	UF2	2	30	Convex, Complex PS
SCH2	2	1	Discontinuous	WFG8	2	22	Concave, Nonseparable, Biased	UF3	2	30	Convex, Complex PS
KUR	2	3	Discontinuous	WFG9	2	22	Concave, Nonseparable, Deceptive, Biased	UF4	2	30	Concave, Complex PS
ZDT1	2	30	Convex	VNT1	3	2	Convex	UF5	2	30	Linear, Discrete, Complex PS
ZDT2	2	30	Concave	VNT2	3	2	Mixed	UF6	2	30	Linear, Discontinuous, Complex PS
ZDT3	2	30	Discontinuous	VNT3	3	2	Mixed, Degenerate	UF7	2	30	Linear, Complex PS
ZDT4	2	10	Convex, Multimodal	DTLZ1	3	7	Linear, Multimodal	UF8	3	30	Concave, Complex PS
ZDT6	2	10	Concave, Multimodal, Biased	DTLZ2	3	12	Concave	UF9	3	30	Linear, Discontinuous, Complex PS
WFG1	2	22	Mixed, Biased	DTLZ3	3	12	Concave, Multimodal	UF10	3	30	Concave, Complex PS
WFG2	2	22	Convex, Discontinuous, Nonseparable	DTLZ4	3	12	Concave, Biased	DTLZ2(4)	4	13	Concave
WFG3	2	22	Linear, Degenerate, Nonseparable	DTLZ5	3	12	Concave, Degenerate	DTLZ2(6)	6	15	Concave
WFG4	2	22	Concave, Multimodal	DTLZ6	3	12	Concave, Degenerate, Biased	DTLZ2(10)	10	19	Concave
WFG5	2	22	Concave, Deceptive	DTLZ7	3	22	Mixed, Discontinuous, Multimodal	DTLZ5(2,10)	10	19	Concave, Degenerate
WFG6	2	22	Concave, Nonseparable	UF1	2	30	Convex, Complex PS	DTLZ5(3,10)	10	19	Concave, Degenerate

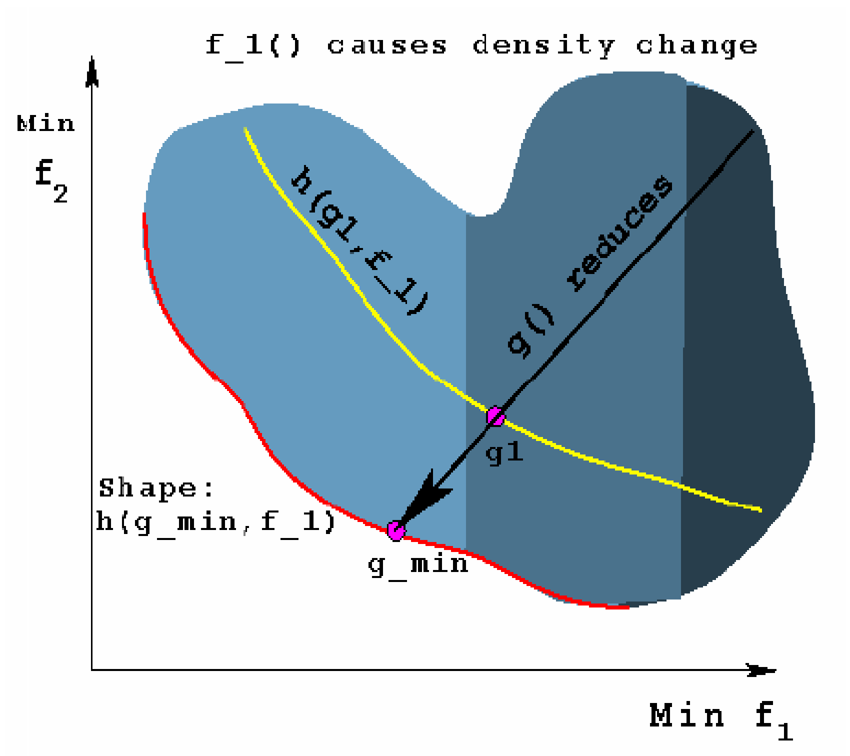
Properties of some continuous problems, where *m* and *d* denote the number of objective and decision variables, respectively

Zitzler-Deb-Thiele (ZDT) Test Problems

- Pareto-optimal front is controllable and known
- Two-objective problems:

$$\begin{aligned} \text{Min.} \quad & f_1(\mathbf{x}) = f_1(\mathbf{x}_I), \\ \text{Min.} \quad & f_2(\mathbf{x}) = g(\mathbf{x}_{II})h(f_1, g). \end{aligned}$$

- Choose $f_1()$, g() and $h()$ functions to introduce different features and difficulties



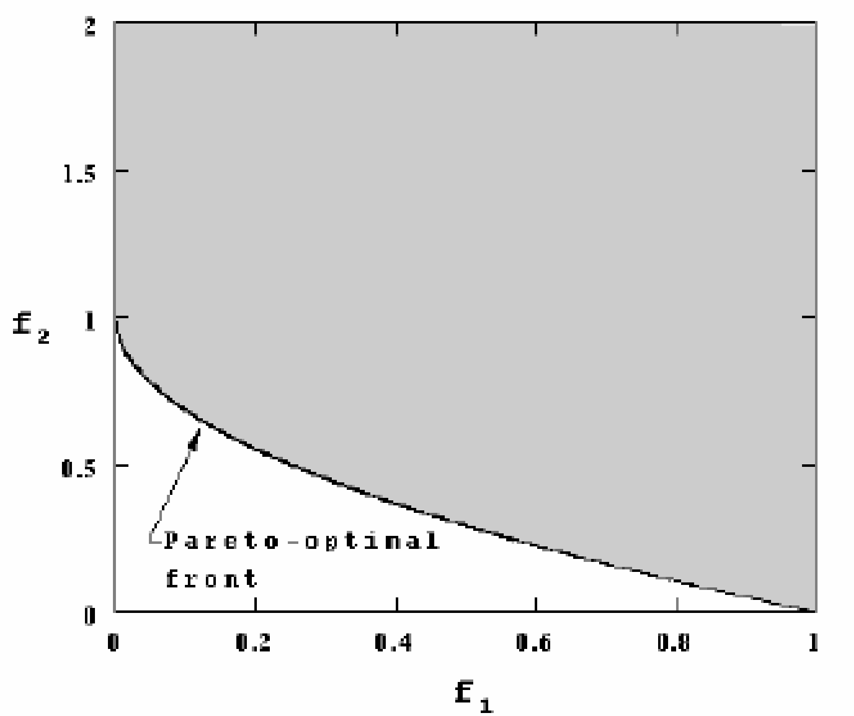
ZDT Test Problems (ZDT1 and ZDT2)

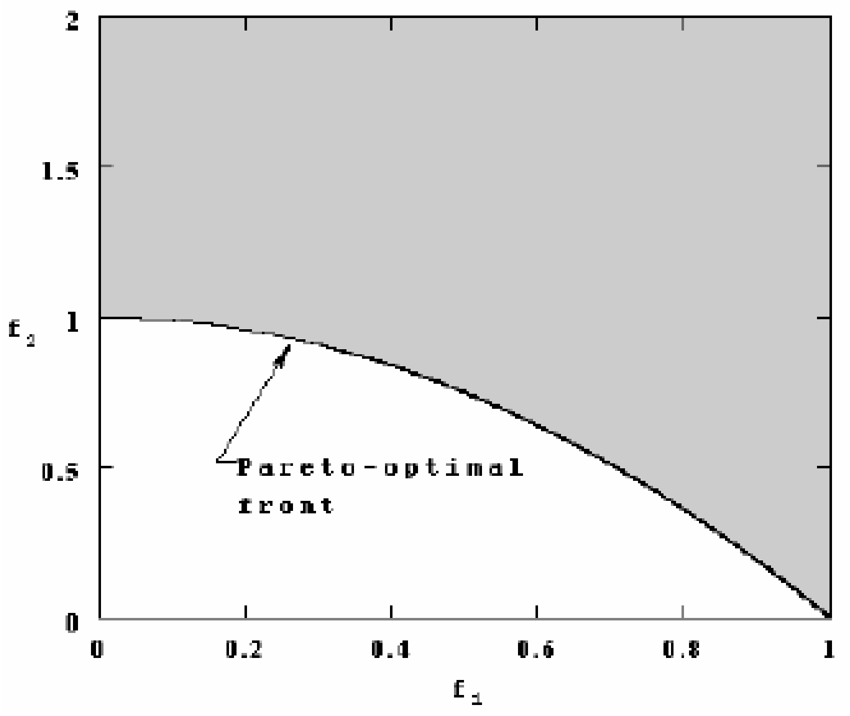
ZDT1

$$\begin{aligned} f_1(\mathbf{x}) &= x_1, \\ g(\mathbf{x}) &= 1 + \tfrac{9}{n-1}\sum\nolimits_{i=2}^{n} x_i, \\ h(f_1, g) &= 1 - \sqrt{f_1/g}. \end{aligned}$$

ZDT2

$$\begin{aligned} f_1(\mathbf{x}) &= x_1, \\ g(\mathbf{x}) &= 1 + \tfrac{9}{n-1}\sum\nolimits_{i=2}^{n} x_i, \\ h(f_1, g) &= 1 - (f_1/g)^2. \end{aligned}$$





ZDT Test Problems (ZDT3 and ZDT4)

ZDT3

$$f_1 = x_1,$$

$$g = 1 + \tfrac{9}{n-1}\sum_{i=2}^{n} x_i,$$

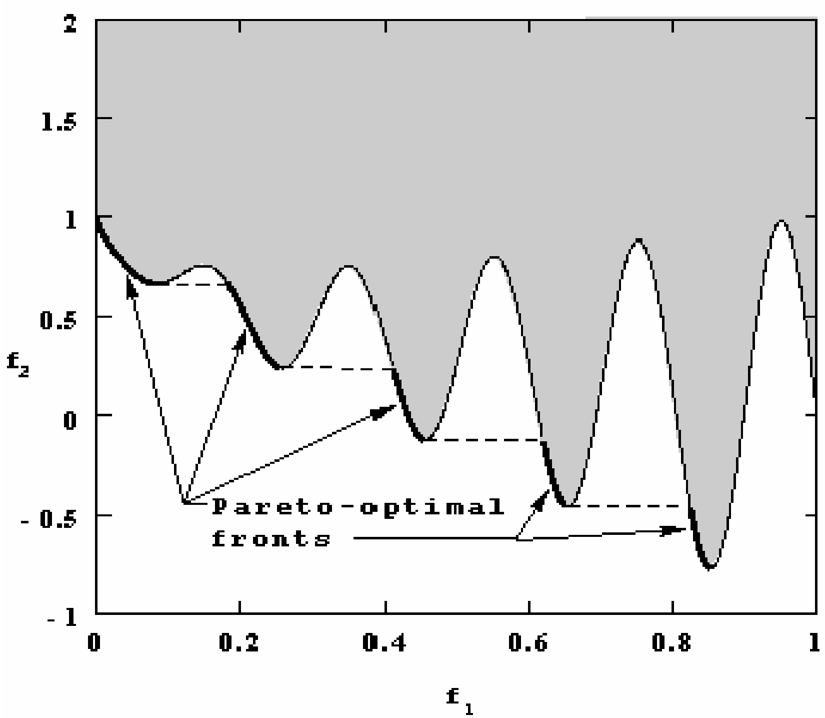
$$h = 1 - \sqrt{f_1/g} - (f_1/g)\sin(10\pi f_1).$$

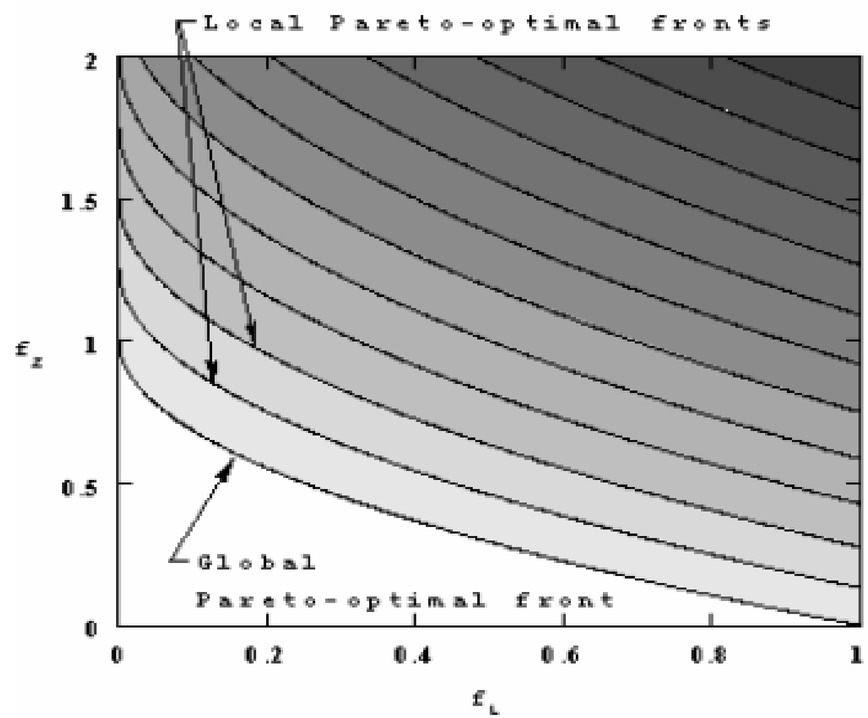
ZDT4

$$f_1 = x_1,$$

$$g = 10n - 9 + \sum_{i=2}^{n}(x_i^2 - 10\cos(4\pi x_i)),$$

$$h = 1 - \sqrt{f_1/g}.$$





ZDT Test Problems (ZDT5 and ZDT6)

ZDT5

$$f_1 = 1 + u(x_1),\ g = \sum_{i=2}^{11} v(u(x_i))$$

$$v = \begin{cases} 2 + u(x_i) & \text{if } u(x_i) < 5, \\ 1 & \text{if } u(x_i) = 5, \end{cases}$$

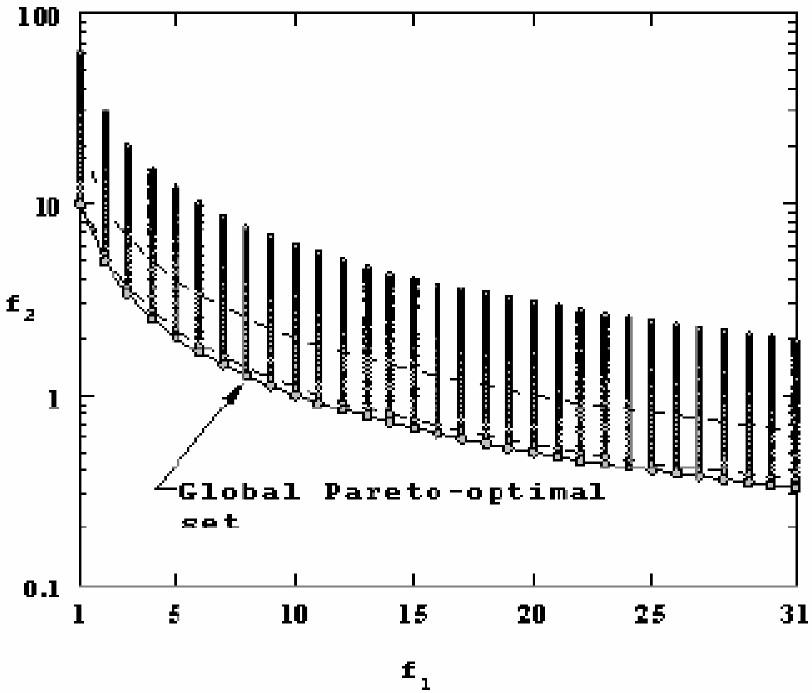
$$h = 1/f_1(\mathbf{x})$$

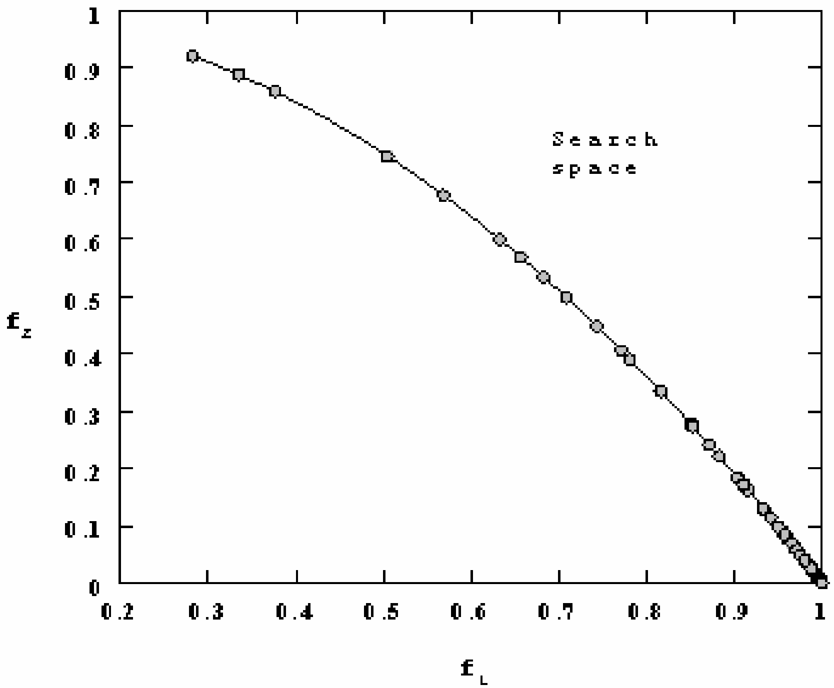
ZDT6

$$f_1 = 1 - \exp(-4x_1)\sin^6(6\pi x_1),$$

$$g = 1 + 9\left[\left(\sum_{i=2}^{10} x_i\right)/9\right]^{0.25},$$

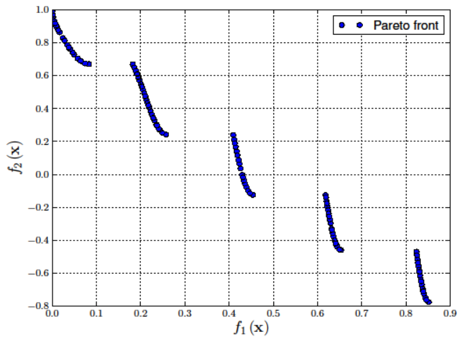
$$h = 1 - (f_1/g)^2.$$



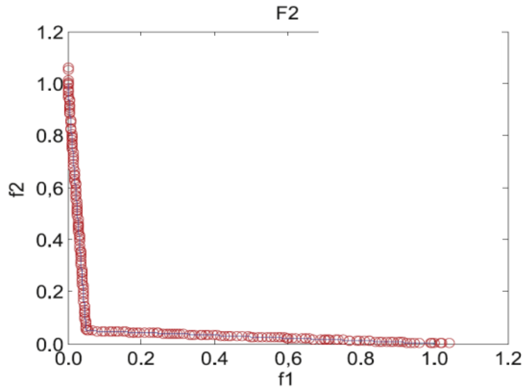


MOPs with Complex PFs

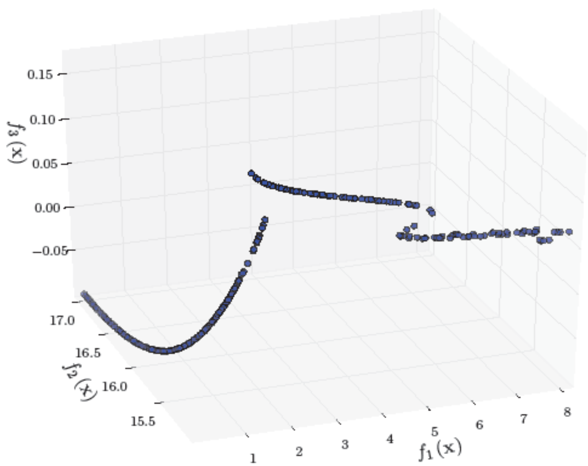
In real-world applications & scientific research, many MOPs have complex PFs



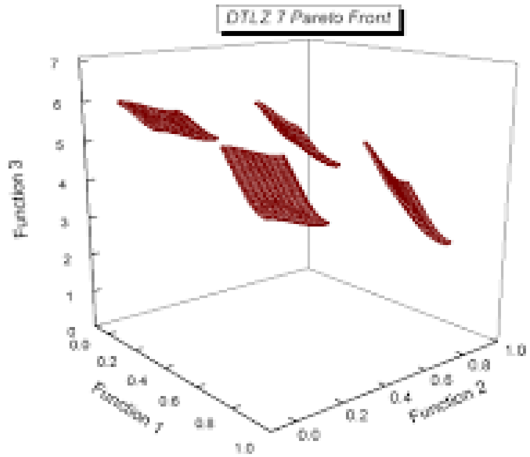
Disconnectivity



Extreme convexity



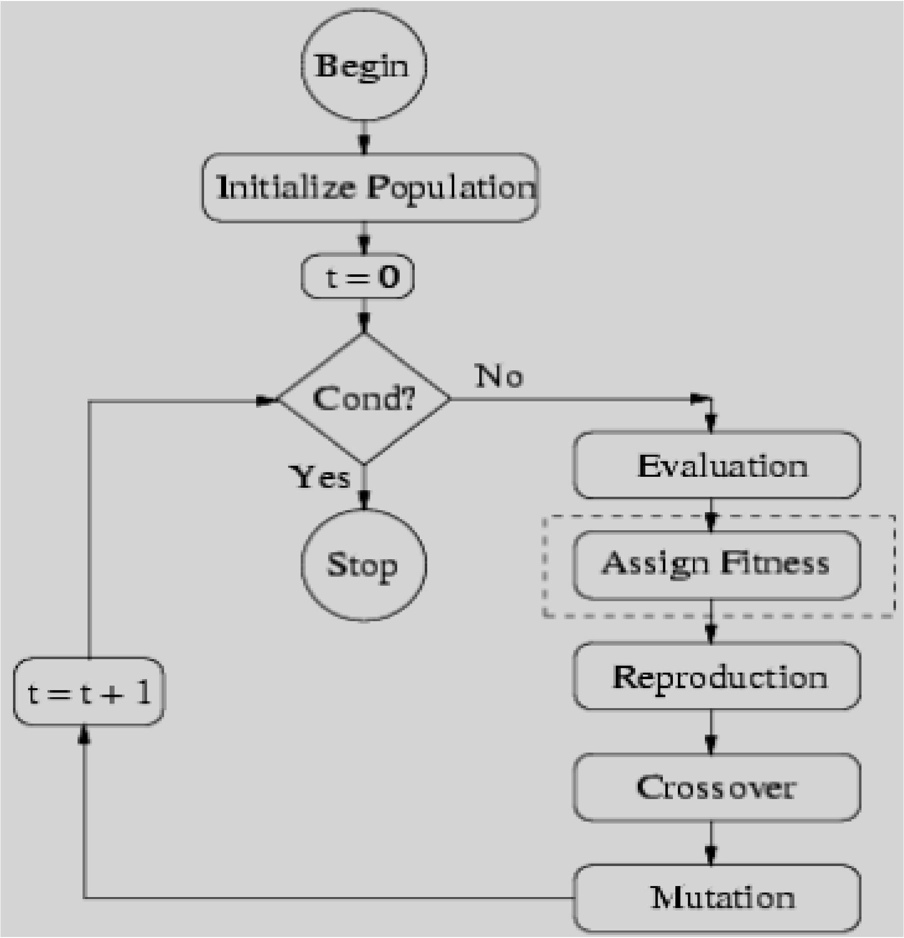
degeneration



Disconnectivity

EMO Framework vs Single Objective EAs

- EMO framework includes an extra fitness assignment step
- Fitness assignment: Modify the fitness calculation
  - Emphasize non-dominated solutions for convergence
  - Emphasize less-crowded solutions for diversity



EMO Algorithm Frameworks

- Pareto-based (domination-based) algorithms
  - NSGA-II [Deb et al, 2002], SPEA2 [Zitzler et al, 2001], PAES [Knowles and Corne, 2000]
- Indicator-based algorithms
  - IBEA [Zitzler and Kunzli, 2004], SMS-EMOA [Beume et al, 2007]
- Decomposition-based algorithms
  - MOEA/D [Zhang and Li, 2007], MSOPS [Hughes, 2003]

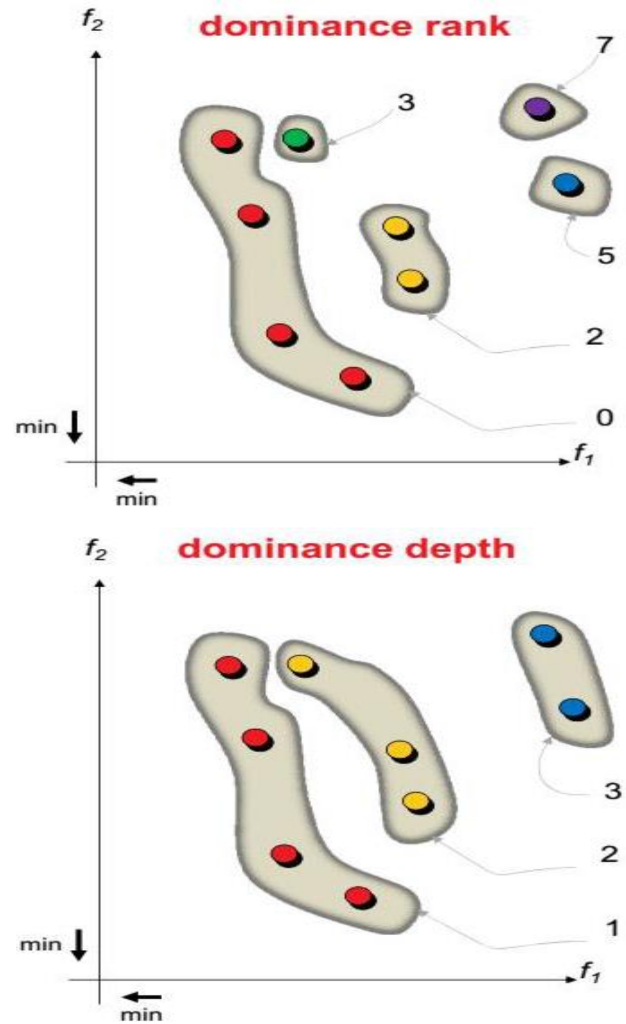
Pareto-based Algorithms

Pareto-based algorithm = Pareto-based selection + diversity maintenance

- An MOP is optimized by simultaneously optimizing all objectives
- Fitness assignment is based on Pareto dominance principle
- An explicit diversity preservation scheme is necessary
- Examples: NSGA-II, SPEA2, PAES

Pareto-based Selection Criterion

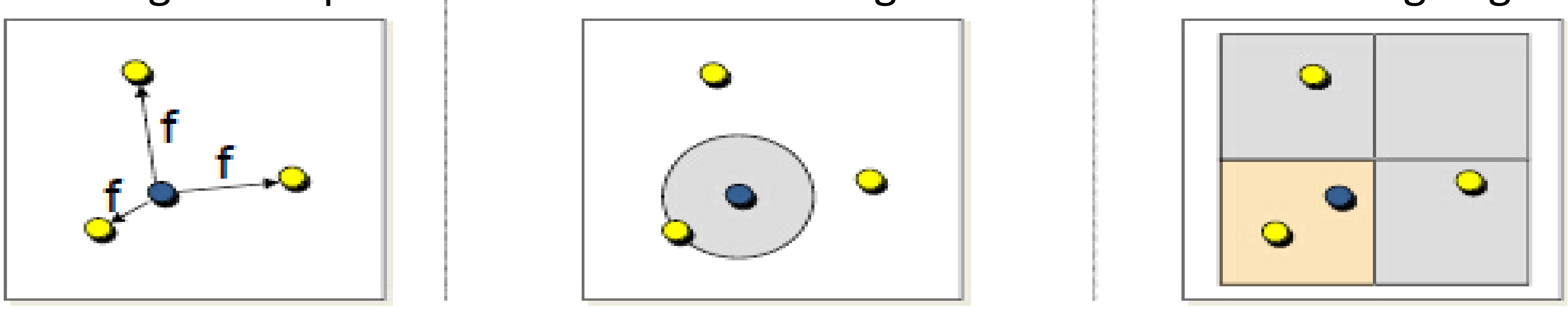
- **Dominance rank**: By how many individuals is an individual dominated?
  - MOGA [Goldberg, 1989], NPGA [Horn et al, 1994]
- **Dominance count**: How many individuals does an individual dominate?
  - SPEA [Zitzler and Thiele, 1999], SPEA2 [Zitzler et al, 2001]
- **Dominance depth**: At which non-dominated front is an individual located?
  - NSGA [Srinivas and Deb, 1994], NSGA-II [Deb et al, 2002]



Diversity Maintenance

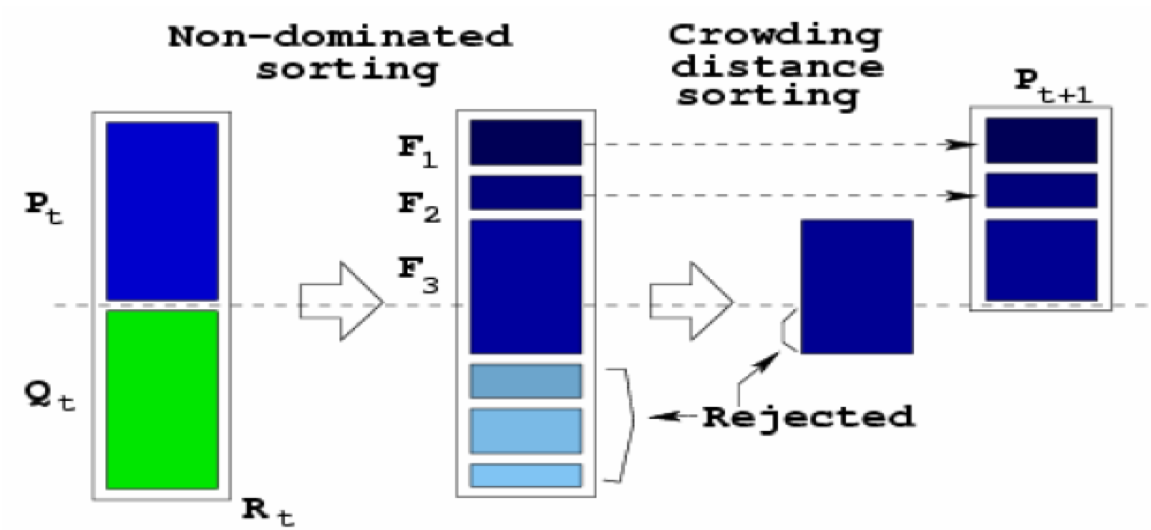
- Niching Technique
  - Calculate the distance function of an individual to others in its niche [Horn et al, 1994; Fonseca and Fleming, 1995]
- K-th Nearest Neighbor
  - Calculate the distance of an individual to its k-th nearest neighbor [Zitzler et al, 2001]
- Grid Crowding Degree
  - Count the number of individuals in a box [Knowles and Corne, 2000, Corne et al, 2001, Knowles and Corne, 2003]

Niching technique | *K*-th nearest neighbor | Grid crowding degree



Elitist Non-dominated Sorting GA (NSGA-II)

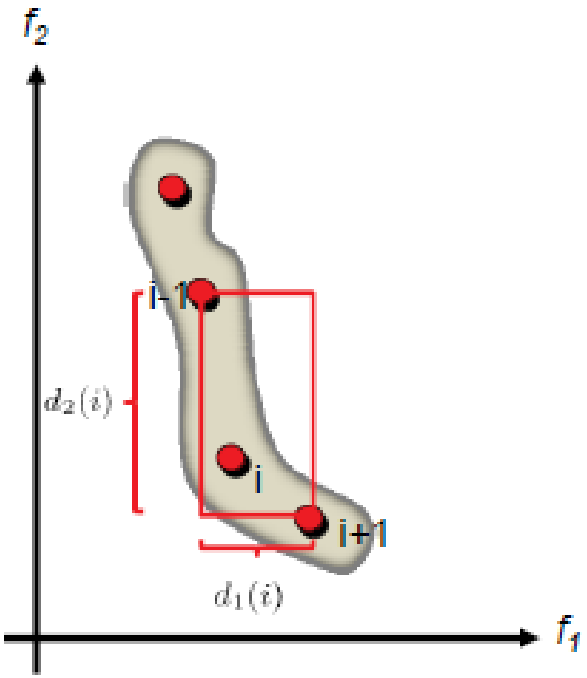
- Non-dominated solutions are emphasized for converging to POF
- Elites are preserved for a fast and reliable convergence to POF
- Solutions of larger crowding distance (CD) are emphasized for maintaining diversity
- Use crowded tournament selection for mating



NSGA-II: Crowding Distance (CD)

- Sort solutions regarding each objective
- Compute CD based on the distance of a solution to its neighbours in each objective

$$\mathrm{CD}(i) = \frac{d_1(i)}{f_{1,\max} - f_{1,\min}} + \cdots + \frac{d_m(i)}{f_{m,\max} - f_{m,\min}}$$



K. Deb, A. Pratap, S. Agarwal, T. Meyarivan. A fast and elitist multiobjective genetic algorithm: NSGA-II. IEEE Trans on Evolutionary Computation, 6(2): 182-197, 2002

Indicator-based Algorithms

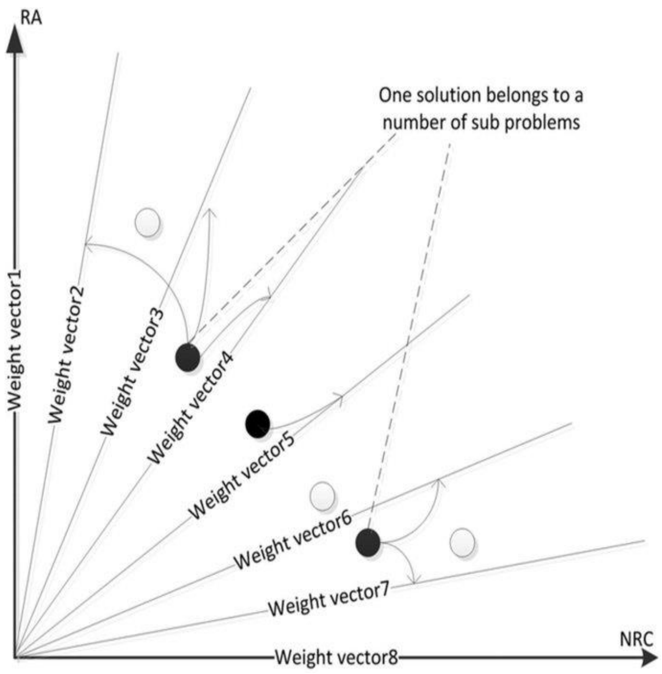
Utilize a performance indicator, e.g., hypervolume (HV), to measure the fitness of solutions

- An MOP is optimized by simultaneously optimizing all objectives
- High selection pressure towards Pareto front
- Examples:
  - IBEA [Zitzler and Kunzli, 2004]; SMS-EMOA [Beume et al, 2007]; HypE [Bader and Zitzler 2011]
- Hard to maintain uniformity of solutions

Decomposition-based Algorithms

Decompose an MOP into a set of scalar subproblems and solve them collaboratively: MOEA/D [Zhang and Li, 2007]

- MOEA/D [Zhang and Li, 2007]
  - Optimize all subproblems simultaneously by an EA
  - Use best solutions of neighbor subproblems for mating
  - Keep the best solution for each subproblem
  - Update information of neighbor subproblems
  - Use external archive for non-dominated solutions
- Many subsequent variants and enhancements



Q. Zhang and H. Li. MOEA/D: A Multiobjective Evolutionary Algorithm Based on Decomposition. IEEE Trans on Evolutionary Computation, 11(6): 712-731, 2007

MOEA/D: Decomposition Methods

Weighted Sum (WS)

$$\begin{aligned} \min \quad & g^{ws}(x|w,z^*) = \textstyle\sum_{i=1}^{m}(w_i|f_i(x) - z_i^*|) \\ s.t. \quad & x \in \Omega_x. \end{aligned}$$

Tchebycheff (TCH)

$$\begin{aligned} \min \quad & g^{te}(x|w,z^*) = \max_{1\le i\le m}\left(\frac{1}{w_i}|f_i(x) - z_i^*|\right) \\ s.t. \quad & x \in \Omega_x, \end{aligned}$$

z* is the ideal reference point

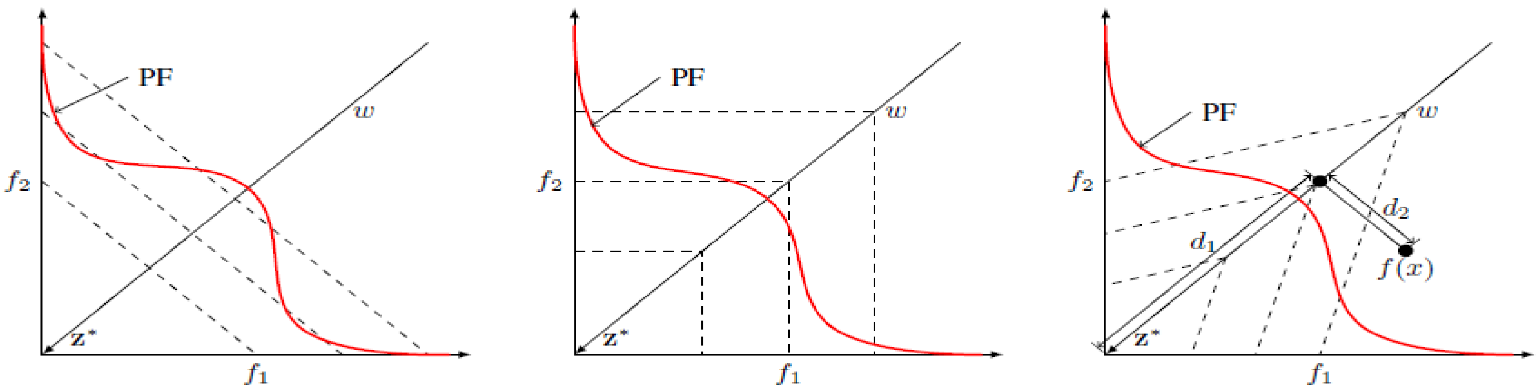
Penalty Boundary Intersection (PBI)

$$\begin{aligned} \min \quad & g^{pbi}(x|w,z^*) = d_1 + \theta d_2 \\ s.t. \quad & x \in \Omega_x, \end{aligned}$$

$$d_1 = \frac{\|(f(x) - z^*)^T w\|}{\|w\|},$$

$$d_2 = \|f(x) - (z^* + d_1\frac{w}{\|w\|})\|.$$

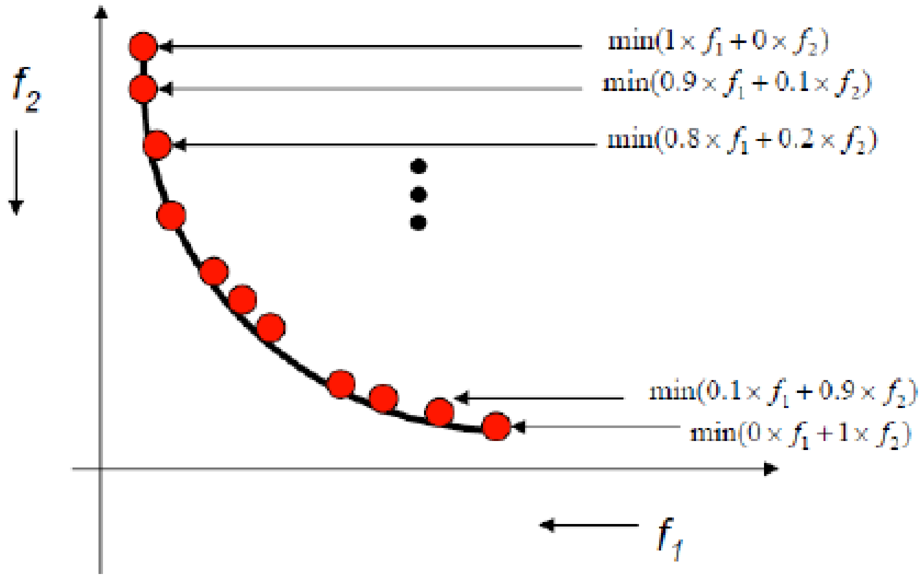
θ is a user-defined penalty factor.



(a) WS (b) TCH (c) PBI

Illustration of three scalarizing functions on weight vector w, where dashed lines are contour lines

Weighted Sum Decomposition Illustration



$$\lambda^1 = (1,0), \qquad g(x,\lambda^1) = 1 \times f_1 + 0 \times f_2$$

$$\lambda^2 = (0.9, 0.1) \qquad g(x,\lambda^2) = 0.9 \times f_1 + 0.1 \times f_2$$

$$\vdots$$

$$\lambda^{11} = (0,1) \qquad g(x,\lambda^{11}) = 0 \times f_1 + 1 \times f_2$$

Weight vectors and corresponding weighted sum sub-problems

MOEA/D: Pseudo-code

MOEA/D with PBI Decomposition

Step 1: Initialization

- **Step 1.1 Neighborhood:** Determine the T closest weight vectors to each weight vector
- **Step 1.2** Initial Population - randomly generated
- **Step 1.3** Function Evaluation
- **Step 1.4** Reference point (z) - Initialization

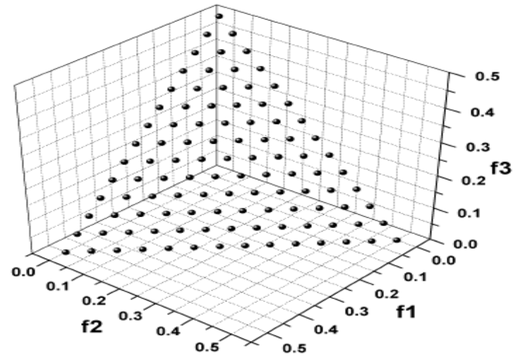
Step 2: Update
For $i = 1, \ldots, N$, **do**

- **Step 2.1 Reproduction:** Offpsring generation via **local mating** using **genetic operators**
- **Step 2.2 Repair:** Offspring repair using heuristics
- **Step 2.3 Function Evaluation:** Offspring
- **Step 2.4 Update of Reference point (z):** Offspring is used to update z
- **Step 2.5 Replacement:** If offspring is a better solution that the existing solutions to neighboring subproblems, then they are replaced by the offspring

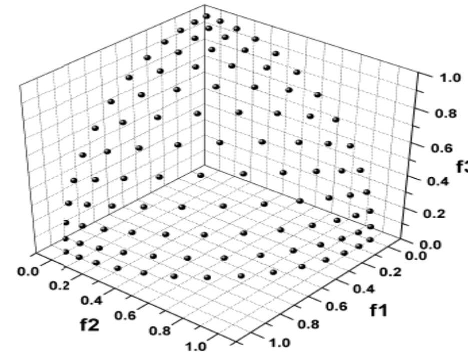
Step 3: Stopping Criteria
If termination criterion is satisfied, then obtain approximation to PO the PF else go to **Step 2**

MOEA/D: Empirical Results

Work perfectly on some problems

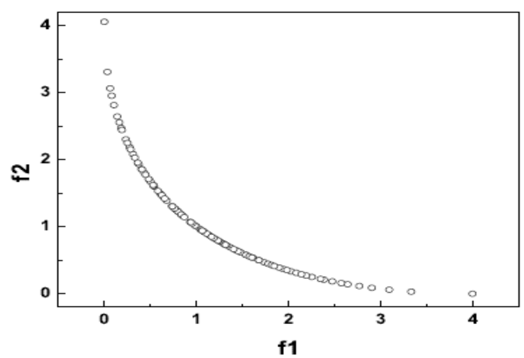


DTLZ1

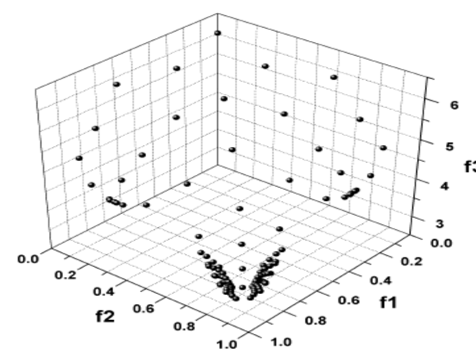


DTLZ2

But struggle on some other problems



SCH1



DTLZ7

Two open issues:

1. MOEA/D loses diversity due to duplicate solutions when optimizing disconnected MOPs
2. MOEA/D cannot obtain uniformly distributed solutions if the PF is extremely shaped

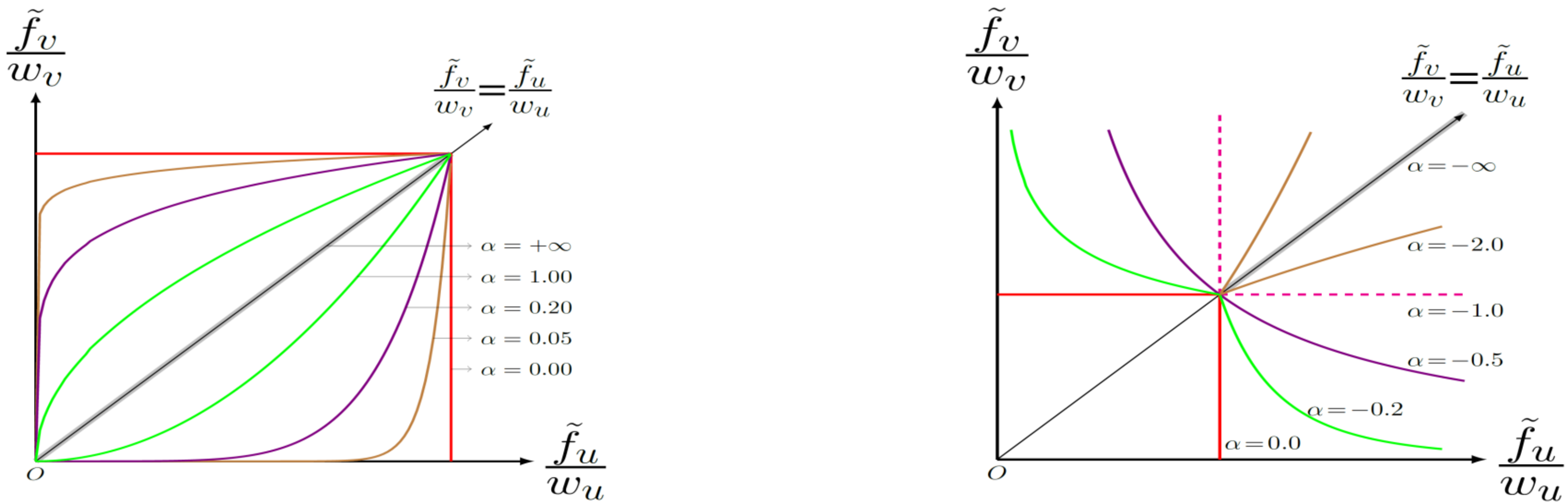
New Scalarizing Functions - MSF

A. Multiplicative Scalarizing Function (MSF)

$$g^{msf}(x|w,z^*) = \frac{\left[\max\limits_{1\le i\le m}\left(\frac{1}{w_i}|f_i(x)-z_i^*|\right)\right]^{1+\alpha}}{\left[\min\limits_{1\le i\le m}\left(\frac{1}{w_i}|f_i(x)-z_i^*|\right)\right]^{\alpha}}$$

Where α is a parameter controlling the geometry of contour lines

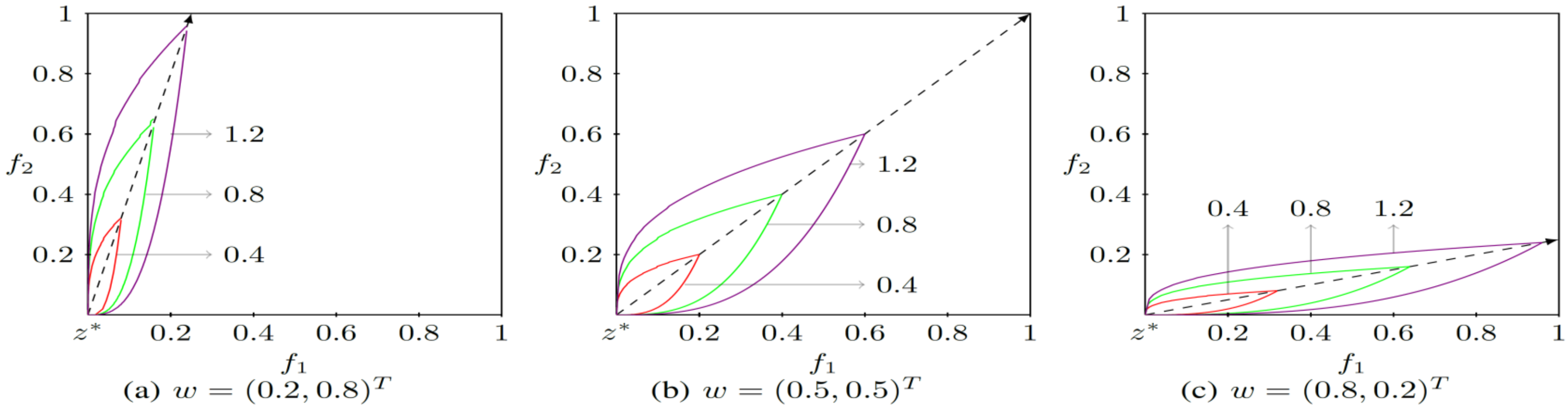
When $\alpha = 0$, MSF degenerates to TCH.



(a) $\alpha \ge 0$ (b) $\alpha \le 0$

Fig. 3: Contour lines of MSF with different α values.

New Scalarizing Functions - MSF

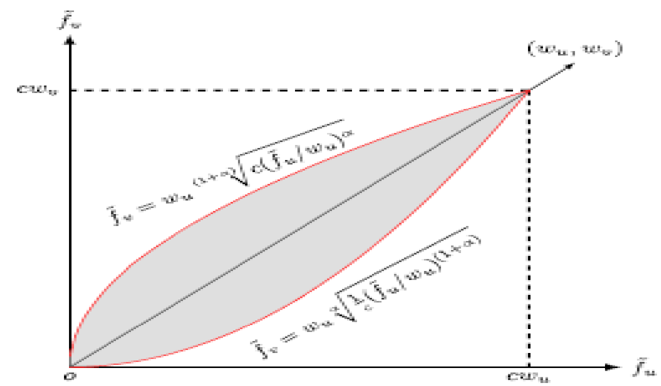


(a) $w = (0.2, 0.8)^T$ (b) $w = (0.5, 0.5)^T$ (c) $w = (0.8, 0.2)^T$

Fig. 5: Contour lines of MSF with $\alpha = 0.5$.

Theorem 1: In bi-objective optimization, the maximum size of improvement region enclosed by MSF at contour value c is equal to

$$\Delta_c = \frac{w_1 w_2 c^2}{2\alpha + 1}$$



Improvement region (shaded area) of MSF.

Two observations:

- ✓ Subproblems with different weight vectors have different size of improvement regions
- ✓ The improvement size for subproblems can be adjusted via α in MSF, whereas this is not possible in existing scalarizing functions.

New Scalarizing Functions - PSF

B. Penalty-based Scalarizing Function (PSF)

Inspired by the idea of PBI that controls diversity by penalizing solutions far from a weight vector, we modify the weighted Chebycheff function in the following way:

$$g^{psf}(x|w, z^*) = \max_{1 \le i \le m} \left(\frac{1}{w_i} |f_i(x) - z_i^*| \right) + \alpha d \qquad (10)$$

$$d = \frac{\sqrt{||f(x) - z^*||^2 ||w||^2 - ||(f(x) - z^*)^T w||^2}}{||w||} \qquad (11)$$

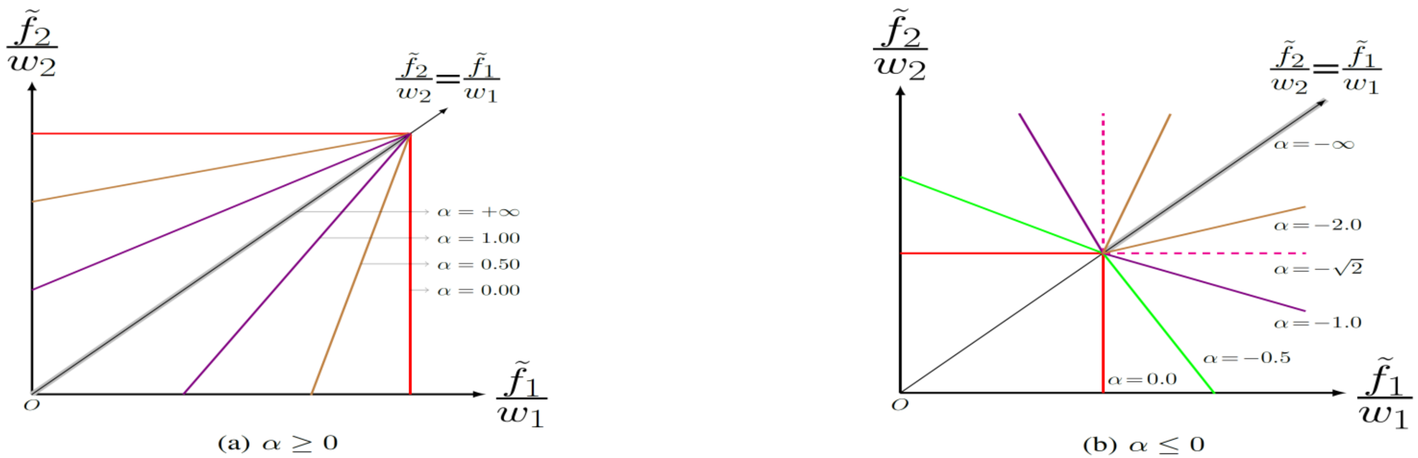
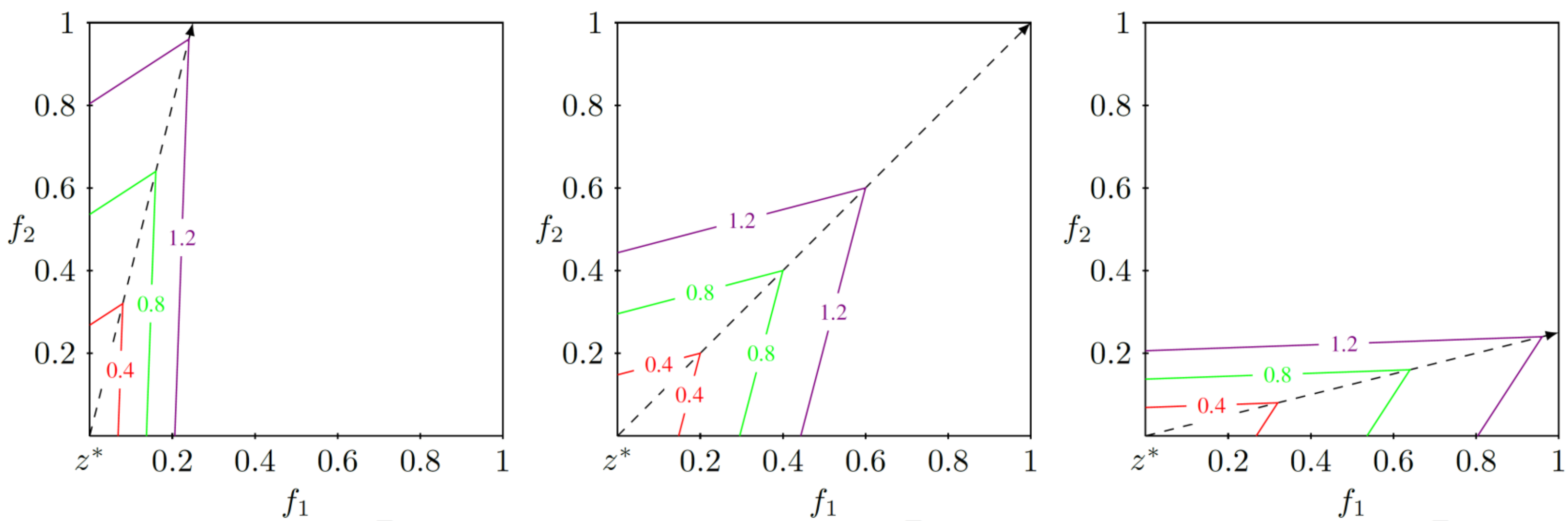


Fig. 6: Contour lines of PSF with different α values

Where α is a parameter controlling the geometry of contour lines.

When $\alpha = 0$, MSF degenerates to TCH.

New Scalarizing Functions - PSF



(a) $w = (0.2, 0.8)^T$ (b) $w = (0.5, 0.5)^T$ (c) $w = (0.8, 0.2)^T$

Fig. 7: Contour lines of PSF with $\alpha = 1.0$.

Efficient MOEA/D with MSF/PSF

Efficient MOEA/D:

- ✓ Adaptive scalarizing strategy

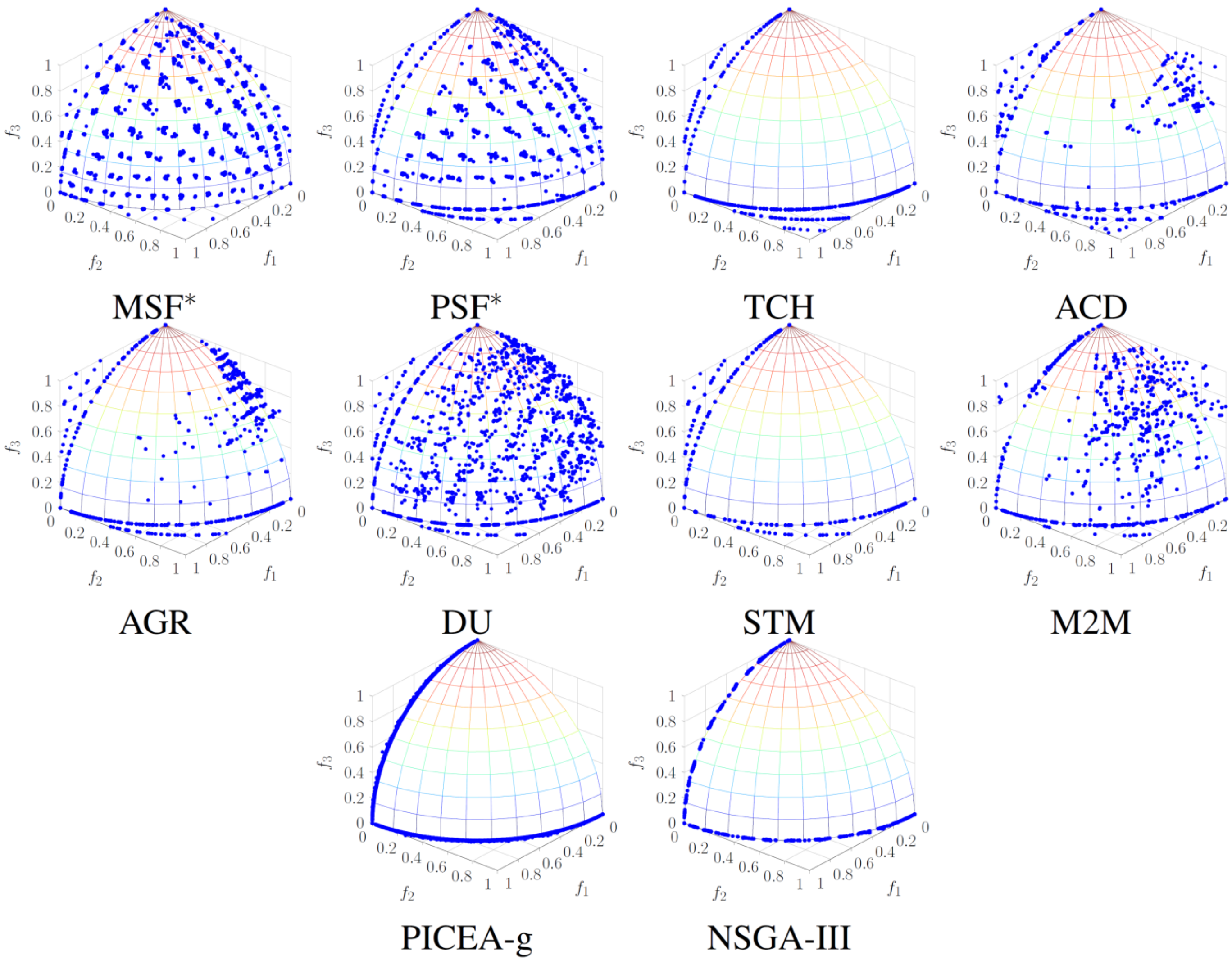
$$\alpha = \beta(1 - \frac{gen}{MaxGen})\left\{m \min_{1 \leq i \leq m}(w_i)\right\}$$

- ✓ Mating selection in neighbourhood.
- ✓ Solution replacement in most suitable subproblems

```
Choose a scalarizing function SF for MOEA/D;
gen ← 1;
while gen ≤ MaxGen do
    Update α for the selected SF according to Eq. (15);
    for i ← 1 to N do
        Randomly select indexes r_1 and r_2 from B(i);
        Apply genetic operators on individuals x^{r_1}, x^{r_2}
          to produce a new solution y;
        Evaluate the objective vector of y, and update z*;
        Find the T most suitable subproblems for y:
          S = {s_1, s_2, ..., s_T};
        c ← 0 ;
        for j ← 0 to T do
            if x ≺_SF x^{s_j} then
                x^j ← y and c ← c + 1;
            end
            if c ≥ n_r then
                break;
            end
        end
    end
    Update z^{nad} using P, gen ← gen + 1;
end
```

S. Jiang and S. Yang. Scalarizing functions in decomposition-based multiobjective evolutionary algorithms. IEEE Trans. on Evolutionary Computation, 22(2): 296-313, 2018

Efficient MOEA/D: Experimental Results



Efficient MOEA/D: Experimental Results

TABLE II: Best, median, and worst HVD values obtained by different algorithms

Prob.	MSF*	PSF*	TCH	ACD	AGR	DU	STM	M2M	PICEA-g	NSGA-III
	1.39E-02	1.43E-02	7.66E-02	1.59E-02	2.73E-02	1.53E-02	**1.32E-02**	2.48E-02	5.81E-01	1.64E-02
MOP1	**1.43E-02**	1.46E-02	5.46E-01‡	1.61E-02‡	2.91E-02‡	1.58E-02‡	6.80E-02‡	2.70E-02‡	5.97E-01‡	1.71E-02‡
	1.50E-02	**1.49E-02**	5.82E-01	1.64E-02	3.56E-02	1.69E-02	5.39E-01	2.88E-02	6.08E-01	3.95E-02
	6.30E-03	6.48E-03	1.48E-01	6.90E-03	1.02E-02	8.39E-03	6.42E-03	1.76E-02	2.85E-01	8.97E-03
MOP2	**6.39E-03**	6.55E-03	2.22E-01‡	8.29E-03‡	1.82E-02‡	8.64E-03‡	6.78E-03	1.91E-02‡	3.12E-01‡	9.79E-03‡
	6.53E-03	8.36E-03	3.33E-01	9.51E-03	2.64E-01	1.20E-02	1.84E-01	1.36E-01	3.33E-01	9.81E-02
	6.08E-03	6.29E-03	2.15E-01	6.13E-03	1.03E-02	8.12E-03	2.15E-01	1.26E-02	2.15E-01	9.38E-03
MOP3	**6.76E-03**	6.80E-03	3.48E-01‡	7.16E-03†	1.24E-02‡	8.69E-03‡	2.15E-01‡	1.53E-02‡	2.15E-01	2.10E-01‡
	7.94E-03	1.45E-02	4.15E-01	1.03E-02	1.55E-01	3.19E-02	2.82E-01	1.88E-02	2.82E-01	2.15E-01
	3.45E-03	6.48E-03	2.78E-01	7.21E-03	4.35E-03	4.26E-03	**3.06E-03**	8.72E-03	3.57E-01	5.60E-03
MOP4	9.45E-03	**9.57E-03**	3.12E-01‡	1.33E-02‡	1.31E-02	1.02E-02	2.29E-01	1.48E-02‡	3.77E-01$^{\diamond}$	2.58E-01‡
	2.10E-02	**1.54E-02**	3.48E-01	2.82E-02	1.25E-01	1.33E-02	2.45E-01	2.47E-02	3.96E-01	2.87E-01
	1.38E-02	1.45E-02	4.72E-01	1.70E-02	2.73E-02	2.33E-02	2.94E-02	2.92E-02	3.13E-01	2.50E-02
MOP5	1.63E-02	**1.60E-02**	4.72E-01‡	1.77E-02	3.12E-02‡	2.46E-02‡	3.13E-01‡	3.55E-02‡	4.72E-01‡	3.14E-02‡
	2.09E-02	**1.87E-02**	4.72E-01	1.88E-02	4.63E-02	3.95E-02	3.13E-01	5.45E-02	4.72E-01	3.34E-01
	5.15E-02	5.25E-02	2.46E-01	1.20E-01	9.28E-02	6.23E-02	5.77E-02	1.53E-01	2.40E-01	6.56E-02
MOP6	**5.25E-02**	5.31E-02	3.47E-01‡	2.17E-01‡	2.00E-01‡	7.66E-02‡	1.83E-01‡	1.91E-01‡	3.38E-01‡	1.74E-01‡
	6.32E-02	6.50E-02	3.47E-01	3.47E-01	2.91E-01	1.20E-01	2.46E-01	2.96E-01	3.39E-01	2.00E-01
	9.58E-02	1.08E-01	2.60E-01	1.70E-01	1.58E-01	9.67E-02	1.58E-01	1.63E-01	2.66E-01	1.77E-01
MOP7	**1.03E-01**	1.11E-01	2.72E-01‡	2.62E-01‡	2.65E-01‡	1.13E-01	2.55E-01‡	2.14E-01‡	2.66E-01‡	1.81E-01‡
	1.60E-01	2.10E-01	2.72E-01	2.98E-01	2.72E-01	1.88E-01	2.72E-01	1.20E+00	2.67E-01	2.87E-01
	4.44E-02	**4.38E-02**	8.65E-02	6.59E-02	6.94E-02	9.25E-02	5.62E-02	2.63E-01	4.75E-01	5.31E-02
MOP8	5.81E-02	**5.46E-02**	2.02E-01‡	1.16E-01‡	1.01E-01‡	1.01E-01‡	6.88E-02‡	4.29E-01‡	6.11E-01‡	5.74E-02
	9.58E-02	1.09E-01	3.81E-01	1.54E-01	1.52E-01	1.71E-01	3.15E-01	7.30E-01	7.28E-01	**6.24E-02**
	9.25E-02	**8.05E-02**	1.59E-01	1.32E-01	1.04E-01	9.71E-02	1.58E-01	2.05E-01	1.70E-01	1.76E-01
MOP9	**1.01E-01**	1.03E-01	6.15E-01‡	3.96E-01‡	1.72E-01‡	1.03E-01‡	1.59E-01‡	2.48E-01‡	6.08E-01‡	1.79E-01‡
	1.71E-01	3.25E-01	6.76E-01	6.20E-01	3.58E-01	2.23E-01	4.78E-01	4.63E-01	6.76E-01	1.81E-01

† and ◇ indicates MSF* and PSF* significantly outperform the corresponding algorithm, respectively.
‡ indicates both MSF* and PSF* significantly outperform the corresponding algorithm.

- Efficient MOEA/D generally works better than all the compared algorithms
- Decomposition-based algorithms obtain better results than dominance-based ones
- Some many-objective optimizers like DU, PICEA-g and NSGA-III are not effective in multiobjective optimization

Summary

- EC for MOPs has received great attention over several decades
  - Many MOEAs developed
- Still active and needs more research
  - More efficient algorithms
  - More real-world applications
  - Theoretical analysis
  - Many-objective optimization (>3 objectives)