Program Analysis

5. Abstract Interpretation (1): Concrete Semantics

Kihong Heo

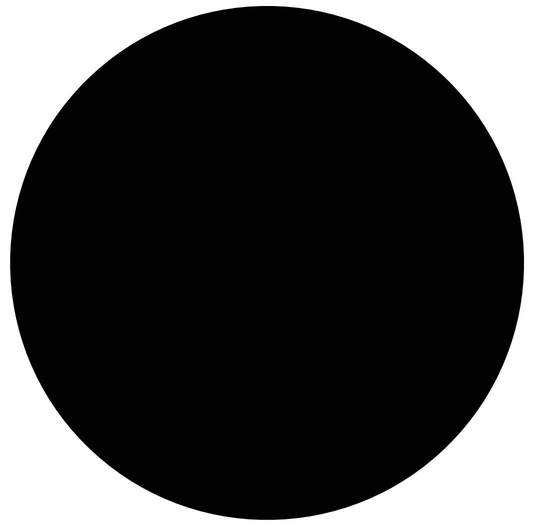
KAIST

Abstract Interpretation

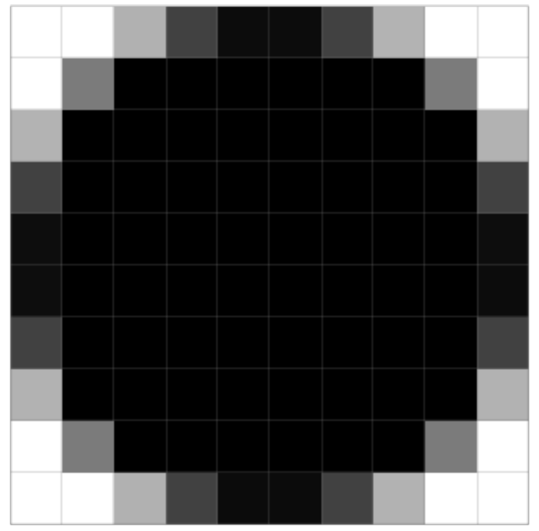
- A **powerful framework** for designing **correct** static analysis
  - Framework: given some inputs, a static analysis comes out
  - Powerful: all static analyses are understood in this framework (e.g., type systems, data-flow analysis, etc)
  - Correct: mathematically proven
- Estcabilished by Patrick and Radhia Cousot
  - *Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints*, 1977
  - *Systematic Design of Program Analysis Frameworks*, 1979

Abstract?

- Concrete (execution, dynamic) vs Abstract (analysis, static)
- Without abstraction, it is undecidable to subsume all possible behavior of SW
  - Recall the Rice's theorem and the Halting problem



Concrete



Abstract

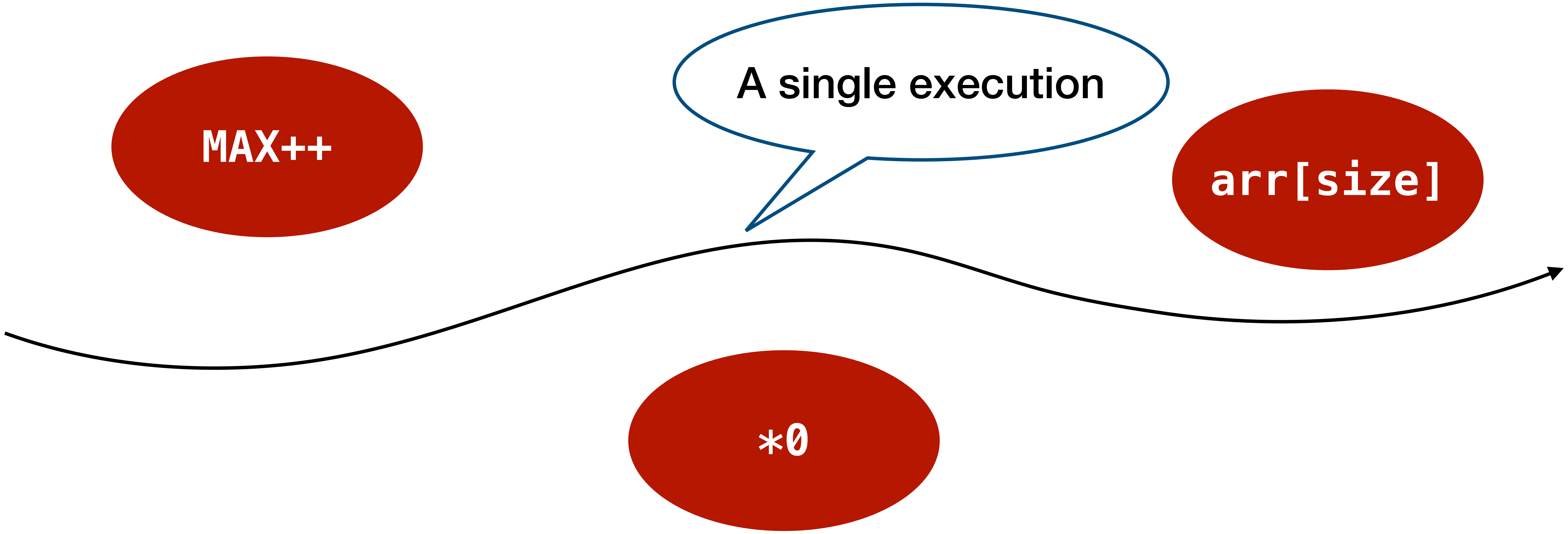
Example

```
x = 3;
while (*) {
  x += 2;
}
x -= 1;
print(x);
```

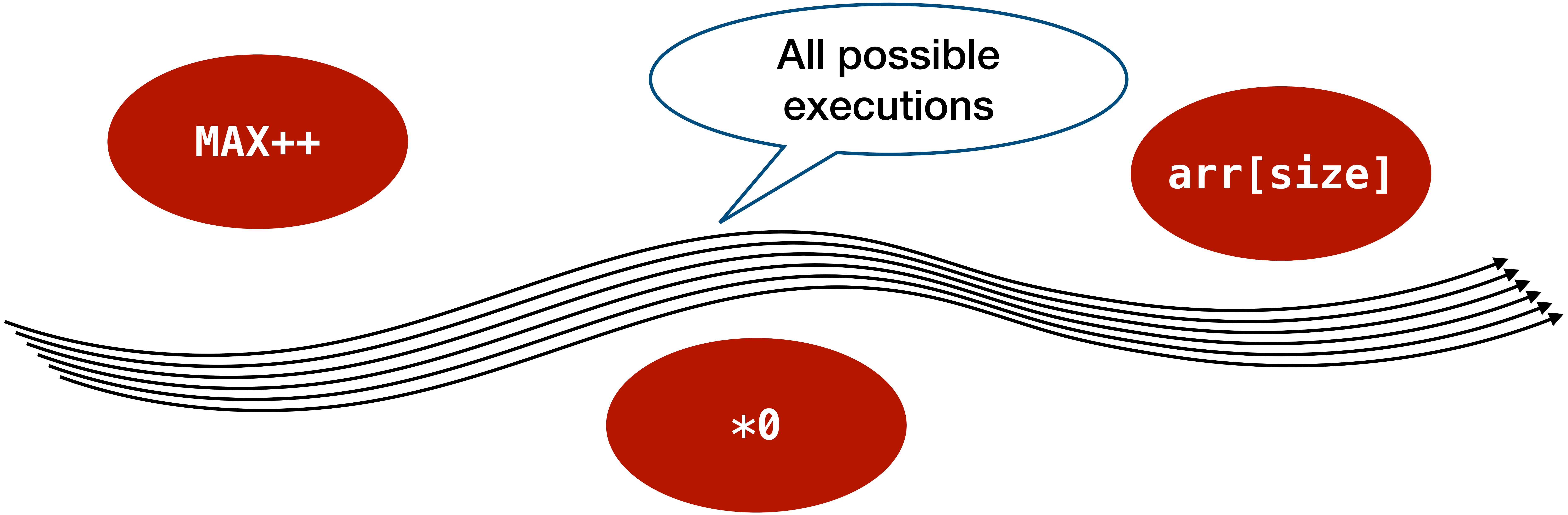
Q: What are the possible output values?

- Concrete interpretation : 2, 4, …, uncomputable (infinitely many possibilities)
- Abstract interpretation 1 : “integers” (good)
- Abstract interpretation 2 : “positive integers” (better)
- Abstract interpretation 3 : “positive even integers” (best)

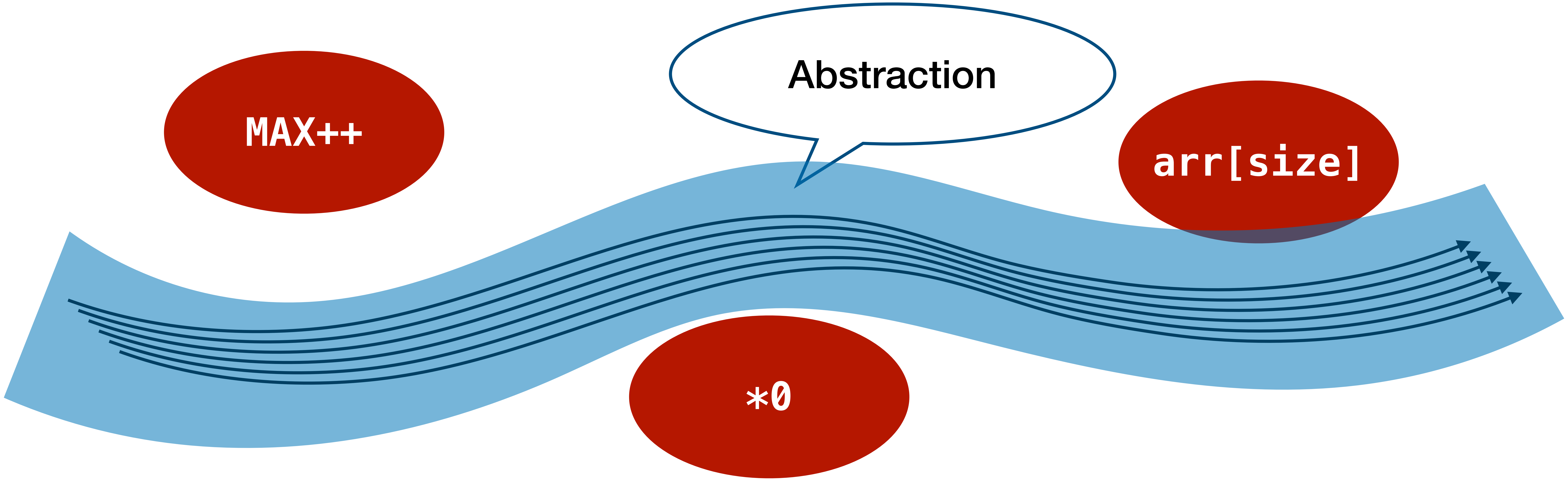
Abstraction of Executions



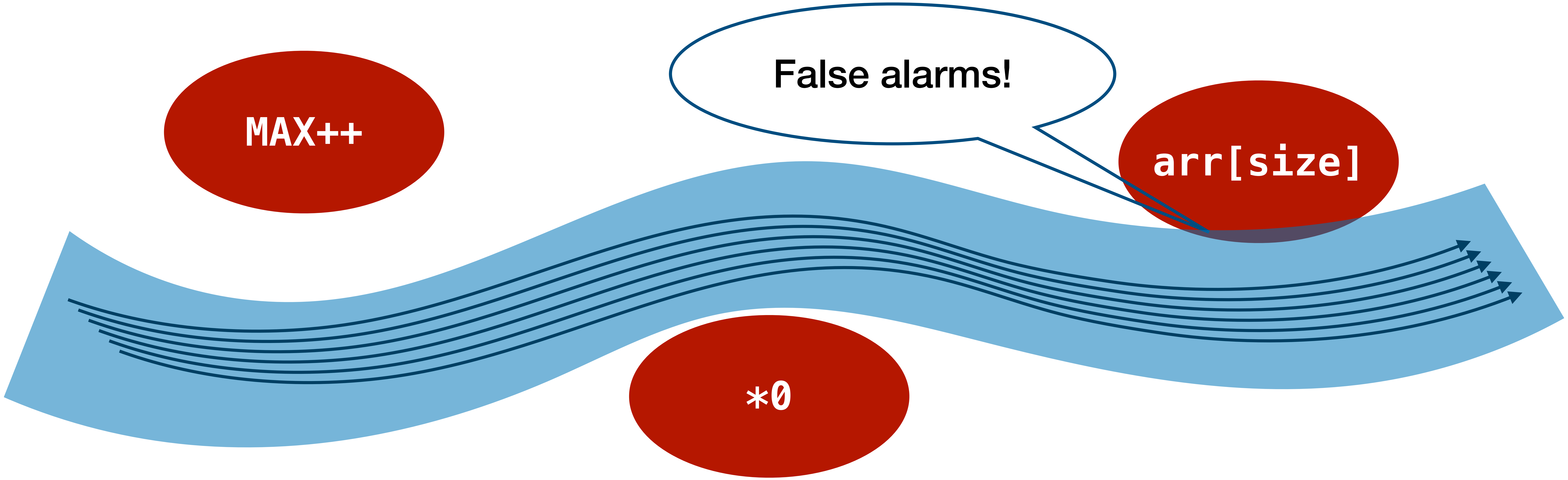
Abstraction of Executions


All possible executions

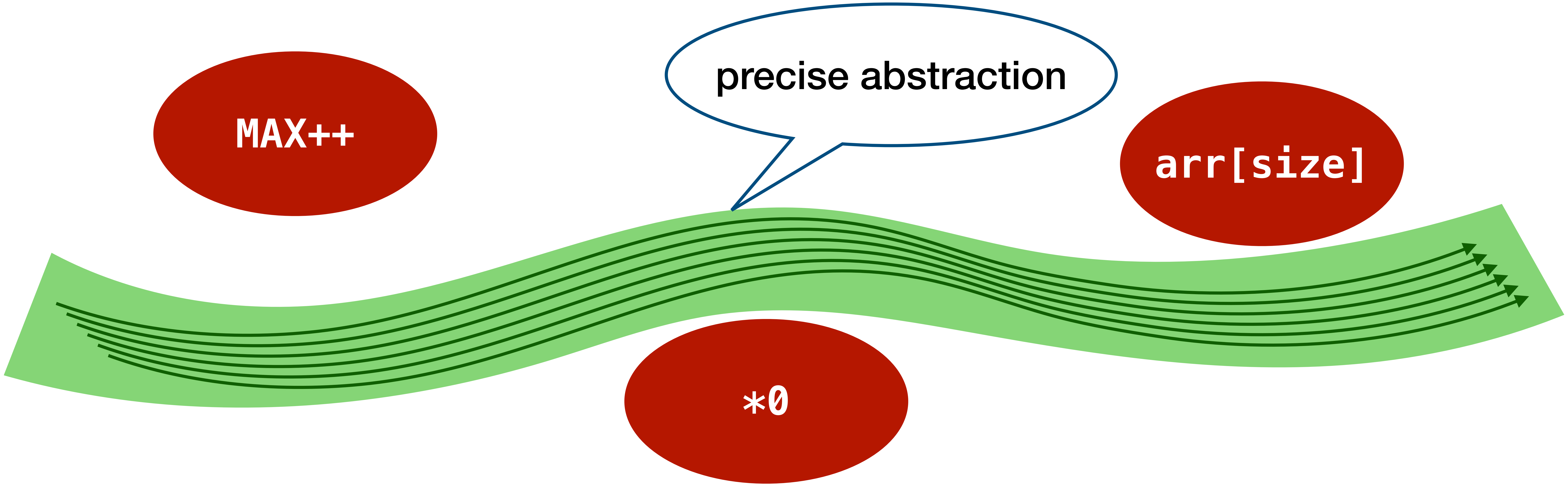
Abstraction of Executions



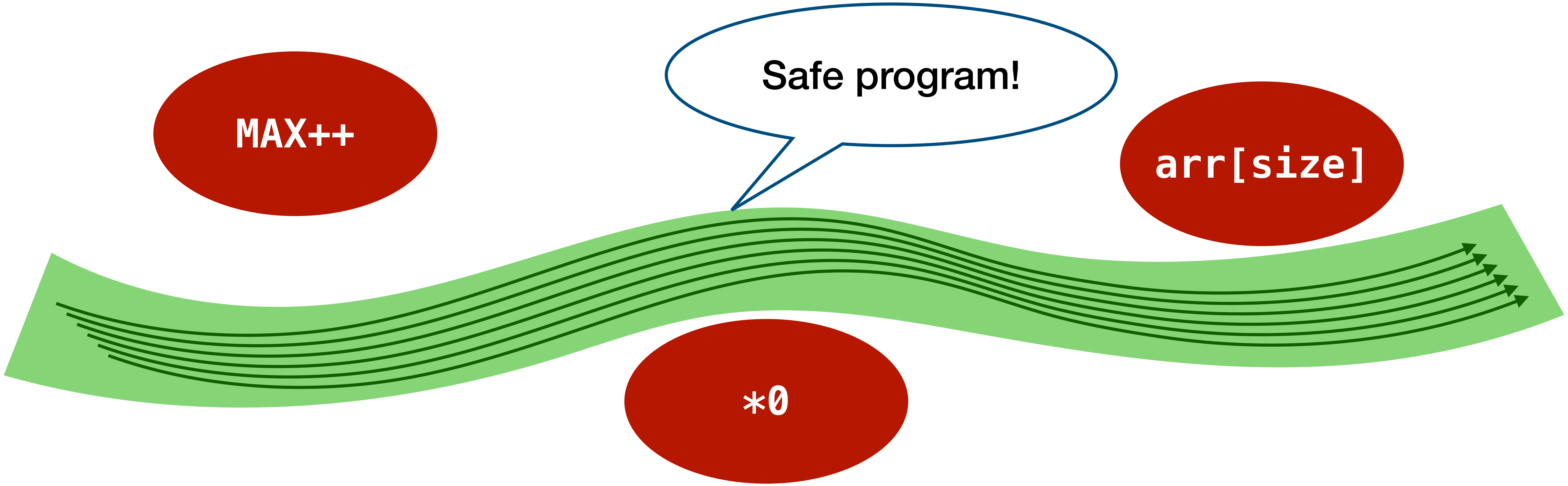
Abstraction of Executions



Abstraction of Executions



Abstraction of Executions

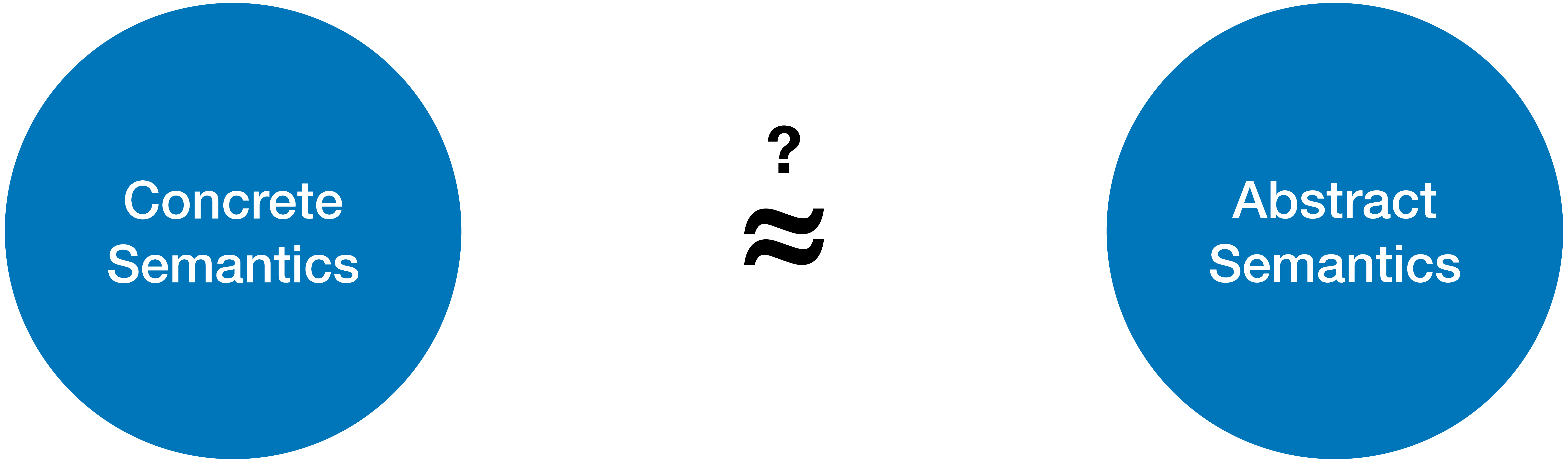


How to analyze?

- Interpret the target program
  - with abstract semantics (= analyzer’s concern)
  - not concrete semantics (= interpreter’s and compiler’s concern)
- Example

	Concrete	**Abstract 1**	**Abstract 2**	**Abstract 3**
`x = 3;`	{3}	Int	Pos	PosOdd
`while (∗) {`				
`  x += 2;`				
`}`	{3, 5, 7, …}	Int	Pos	PosOdd
`x -= 1;`	{2, 4, 6, …}	Int	Pos	PosEven
`print(x);`				

Principles



Concrete Semantics

$\overset{?}{\approx}$

Abstract Semantics

- How to guarantee soundness?
- How to guarantee termination?
- How to design more precise abstraction?
- How to compute abstract semantics?

Practice

- Guidance for a lot of design choices in practice such as
  - Soundness vs Scalability vs Precision vs Usability vs …
  - Characteristics of target programs and properties
  - Optimizations of program analyzers

Abstract Interpretation Framework

- Abstract interpretation concerns
  - Concrete semantics: $[\![C]\!] = \mathsf{lfp} F \in \mathbb{D}$
  - Abstract semantics: $[\![C]\!]^\sharp = \bigsqcup_{i \geq 0} F^{\sharp i}(\bot) \in \mathbb{D}^\sharp$
- Requirements:
  - Relationship between $\mathbb{D}$ and $\mathbb{D}^\sharp$
  - Relationship between $F \in \mathbb{D} \to \mathbb{D}$ and $F^\sharp \in \mathbb{D}^\sharp \to \mathbb{D}^\sharp$
- Guarantees:
  - Correctness (soundness): $[\![C]\!] \approx [\![C]\!]^\sharp$
  - Computability: $[\![C]\!]^\sharp$ is computable within finite time

Design of Static Analysis

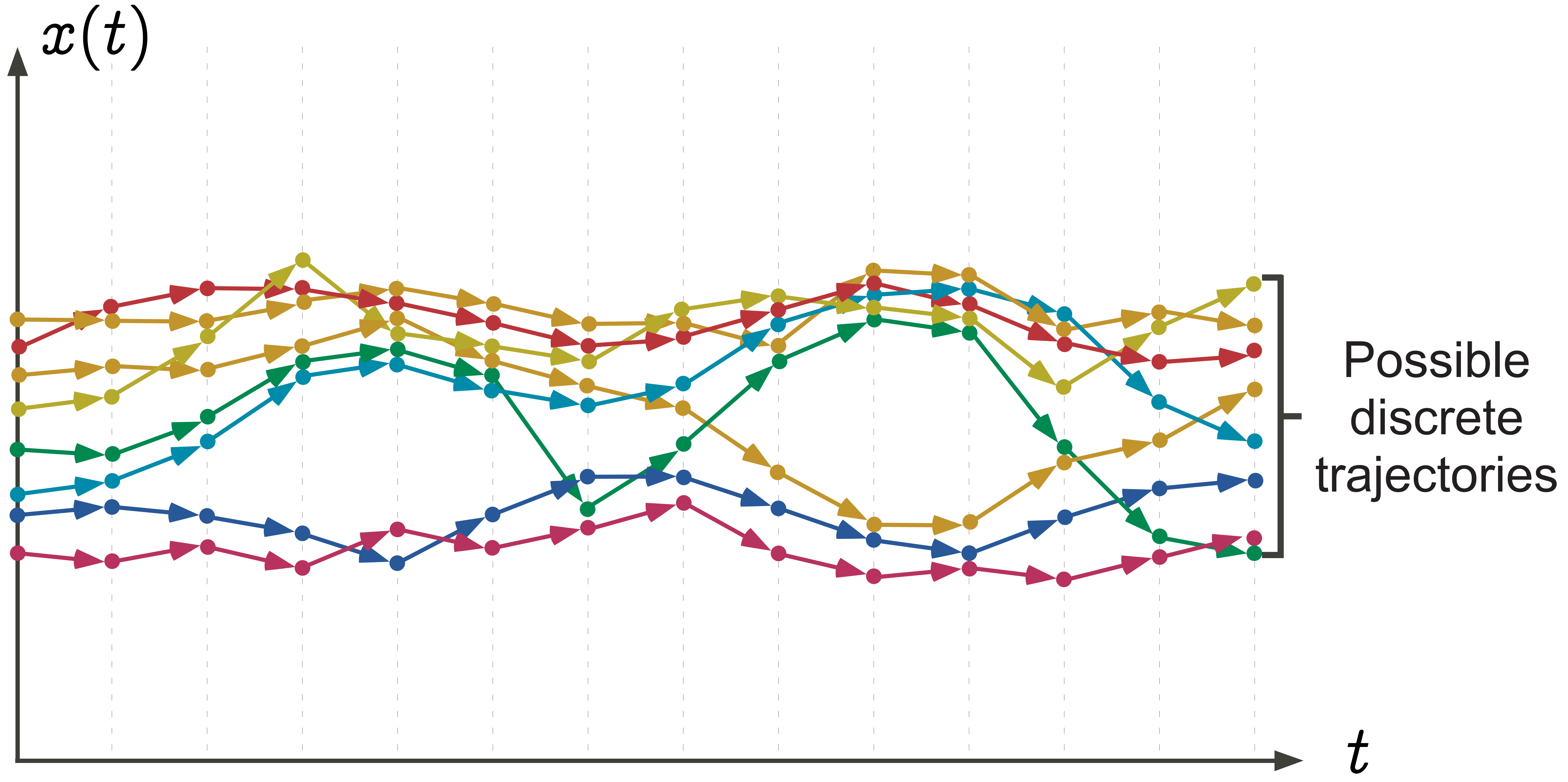
- Goal: conservative and terminating static analysis
- Design principles:
  - Define concrete semantics
  - Define abstract semantics (sound w.r.t the concrete semantics)
- Computation & implementation:
  - Abstract semantics of a program: the least fixed point of the semantic function
  - Static analyzer: compute the least fixed point within finite time

Define Standard Semantics

- Formalization of a **single program execution**
  - Recall Lecture 2 and 3 (operational and denotation semantics)
- What to describe: different choices depending on the purpose
  - E.g., denotational, operational, etc
- In this lecture, we will use denotational semantics
  - Recall the denotational semantics the simple imperative language

$$[\![C]\!] : \mathbb{M} \to \mathbb{M}$$

Define Standard Semantics



*from Patrick Cousot's slides

Standard Semantics

- Define a semantic domain $\mathbb{D} = \mathbb{M} \rightarrow \mathbb{M}$ (CPO)
- Define a semantic function $F : \mathbb{D} \rightarrow \mathbb{D}$ (continuous)
- Semantics of a program: the least fixed point of F

$$\mathbf{lfp} F = \bigsqcup_{i \geq 0} F^i(\bot)$$

Standard Semantics of Commands

$$
\begin{array}{rcl}
[\![C]\!] & : & \mathbb{M} \to \mathbb{M} \\
[\![\texttt{skip}]\!] & = & \lambda m.m \\
[\![C_0\,\texttt{;}\,C_1]\!] & = & \lambda m.[\![C_1]\!]([\![C_0]\!](m)) \\
[\![x \,\texttt{:=}\, E]\!] & = & \lambda m.m\{x \mapsto [\![E]\!](m)\} \\
[\![\texttt{input}(x)]\!] & = & \lambda m.m\{x \mapsto n\} \\
[\![\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2]\!] & = & \lambda m. \begin{cases} [\![C_1]\!](m) & \text{if } [\![B]\!](m) = \texttt{true} \\ [\![C_2]\!](m) & \text{if } [\![B]\!](m) = \texttt{false} \end{cases} \\
[\![\texttt{while}\ B\ C]\!] & = & \mathbf{lfp}\lambda X. \left(\lambda m. \begin{cases} X([\![C]\!](m)) & \text{if } [\![B]\!](m) = \texttt{true} \\ m & \text{if } [\![B]\!](m) = \texttt{false} \end{cases} \right)
\end{array}
$$

Standard Semantics of Programs

- What is the semantic function F?
  - Straightforward from the definition of $[\![C]\!]$ (because of compositionally!)

$$F : (\mathbb{M} \to \mathbb{M}) \to (\mathbb{M} \to \mathbb{M})$$
$$F = \lambda X.[\![C]\!]$$

- Then, the semantics of a program C

$$\mathbf{lfp}F : \mathbb{M} \to \mathbb{M}$$
$$\mathbf{lfp}F = [\![C]\!]$$

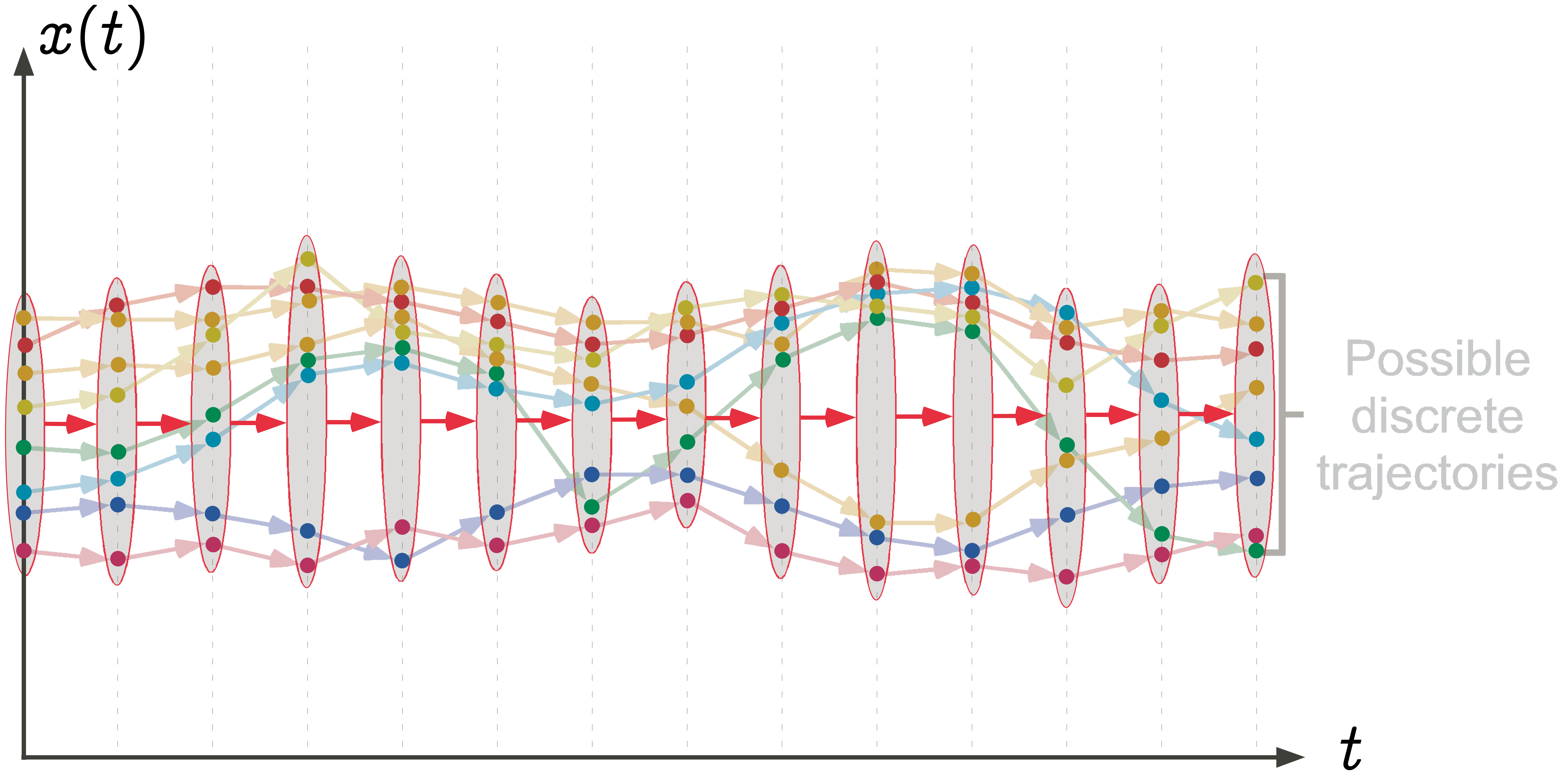
- In this lecture, we will consider only $[\![C]\!]$

Define Concrete Semantics

- Formalization of **all possible** program executions
  - So-called collecting semantics
  - Usually a simple extension of the standard semantics
- What to describe: different choices depending on the purposes (recall, property)
  - Some are more expressive than others
  - E.g., traces (sequence of states), reachable states (set of states), etc
- In this lecture, we will use reachable states for concrete semantics

$$\llbracket C \rrbracket : \mathbb{M} \to \mathbb{M} \quad \xrightarrow{\textbf{collecting}} \quad \llbracket C \rrbracket_{\wp} : \wp(\mathbb{M}) \to \wp(\mathbb{M})$$

Transitions of Sets of States

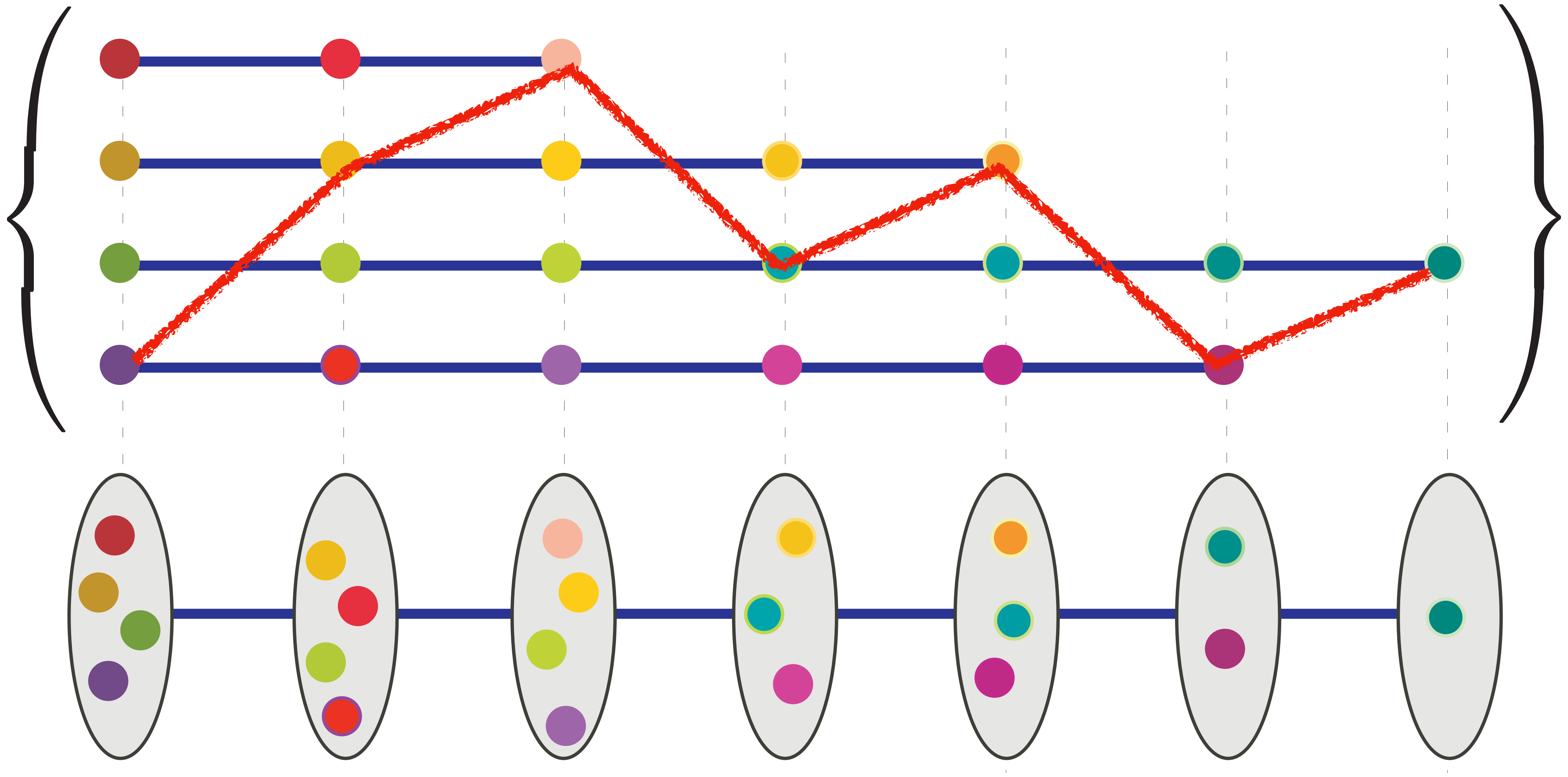


*from Patrick Cousot's slides

Traces vs Reachable States

Traces

Reachable States



Can Answer:

- Can variable `p` be `NULL` at line 10?
- Can buffer index `i` be larger than size `s`?
- ...

Can't Answer:

- Does the red trace exist?
- ...

*from Patrick Cousot's slides

Concrete Semantics

- Define a concrete domain $\mathbb{D}$ (CPO)
- Define a semantic function $F : \mathbb{D} \to \mathbb{D}$ (continuous)
- Then the concrete semantics is defined as the least fixed point of the semantic function F :

$$\mathbf{lfp} F = \bigsqcup_{i \geq 0} F^i(\bot)$$

Example

- Define a concrete semantics of the simple language using denotational semantics
  - Concrete domain $\mathbb{D} = \wp(\mathbb{M}) \rightarrow \wp(\mathbb{M})$
  - Define a semantic function $F : \mathbb{D} \rightarrow \mathbb{D}$
  - Concrete semantics $\mathbf{lfp} F \in \mathbb{D}$
- Q: How to define F?

Concrete Semantics of Expressions

$$\llbracket E \rrbracket_\wp \;:\; \wp(\mathbb{M}) \to \wp(\mathbb{Z})$$

$$\begin{aligned}
\llbracket n \rrbracket_\wp &= \lambda M.\{n\} \\
\llbracket x \rrbracket_\wp &= \lambda M.\{m(x) \mid m \in M\} \\
\llbracket E_1 \odot E_2 \rrbracket_\wp &= \lambda M.\{v_1 \odot v_2 \mid v_1 \in \llbracket E_1 \rrbracket_\wp(M), v_2 \in \llbracket E_2 \rrbracket_\wp(M)\}
\end{aligned}$$

$$\llbracket B \rrbracket_\wp \;:\; \wp(\mathbb{M}) \to \wp(\mathbb{M})$$

$$\begin{aligned}
\llbracket \mathsf{true} \rrbracket_\wp &= \lambda M.M \\
\llbracket \mathsf{false} \rrbracket_\wp &= \lambda M.\emptyset \\
\llbracket E_1 \otimes E_2 \rrbracket_\wp &= \lambda M.\{m \in M \mid \llbracket E_1 \rrbracket(m) \otimes \llbracket E_2 \rrbracket(m) = \mathsf{true}\}
\end{aligned}$$

Concrete Semantics of Commands

$$[\![C]\!]_{\wp} \; : \; \wp(\mathbb{M}) \to \wp(\mathbb{M})$$

$$\begin{array}{rcl}
[\![\texttt{skip}]\!]_{\wp} & = & \lambda M.M \\
[\![C_0\,\texttt{;}\,C_1]\!]_{\wp} & = & \lambda M.[\![C_1]\!]_{\wp} \circ [\![C_0]\!]_{\wp}(M) \\
[\![x \,\texttt{:=}\, E]\!]_{\wp} & = & \lambda M.\{m\{x \mapsto [\![E]\!](m)\} \mid m \in M\} \\
[\![\texttt{input}(x)]\!]_{\wp} & = & \lambda M.\{m\{x \mapsto n\} \mid m \in M, n \in \mathbb{Z}\} \\
[\![\texttt{if } B \texttt{ then } C_1 \texttt{ else } C_2]\!]_{\wp} & = & \lambda M.[\![C_1]\!]_{\wp} \circ [\![B]\!]_{\wp}(M) \cup [\![C_2]\!]_{\wp} \circ [\![\neg B]\!]_{\wp}(M) \\
[\![\texttt{while } B\ C]\!]_{\wp} & = & \lambda M.[\![\neg B]\!]_{\wp}\big(\mathbf{lfp}\lambda X.M \cup [\![C]\!]_{\wp} \circ [\![B]\!]_{\wp}(X)\big)
\end{array}$$

Summary

- Abstract interpretation: a **framework** for designing correct static analysis
- Concrete semantics: collection of **all possible behaviors** of a program
    - Usually a simple extension of the standard semantics
    - Defined as a least fixed point of a concrete semantic function
- Plan: define and compute a **sound abstract semantics** of the program