Quantum and Classical Query Complexities of Functions of Matrices

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  - Quantum recommendation systems: computing $A_{\geq\delta}|i\rangle$, where $A_{\geq\delta}$ is the truncation by keeping singular values larger than δ [Kerenidis-Prakash, '16].
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- All these problems can be described as functions of matrices: computing $f(A)|\mathbf{b}\rangle$.
- Can be solved by a similar idea to HHL, but more efficiently by quantum singular value transform [Gilyén-Su-Low-Wiebe, '18].

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Still works if f is not a polynomial, just consider its polynomial approximation.

This talk

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Problem (Approximate an entry of $f(A)$)

Let $f(x) : [-1,1] \to [-1,1]$ be a function, let A be sparse and Hermitian with $\|A\| \le 1$. Given two indices i, j and accuracy ε, compute $\langle i|f(A)|j\rangle \pm \varepsilon$.

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For a sparse matrix $A = (A_{i,j})$, we are given 2 oracles:

$$(i,j) \longrightarrow \boxed{\mathcal{O}_1} \longrightarrow p_{i,j}$$

$$(i,j) \longrightarrow \boxed{\mathcal{O}_2} \longrightarrow A_{i,j}$$

where p_{ij} is the index of the j-th nonzero entry in the i-th row. The query complexity is the minimal number of calls to the oracles to solve the problem.

An example: matrix powers A^d

Classical algorithms:

- Assume A is s-sparse, then by definition

$$(A^d)_{i,j} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{d-1}} A_{i,k_1} A_{k_1,k_2} \cdots A_{k_{d-1},j}$$

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Quantum algorithms:

- Upper bound by QSVT $O(s\sqrt{d}/\varepsilon)$
- Our result (lower bound): $\Omega(\sqrt{d})$

General case (our results)

Assume f is continuous, A is sparse and Hermitian, then computing $f(A)_{i,j} \pm \varepsilon$ costs

	Quantum algorithm	Classical algorithm
Upper bound	$O(sd/\varepsilon)$	$O(s^{d-1})$
Lower bound	$\Omega(d)$	$\Omega((s/2)^{(d-1)/6})$

where $d = \widetilde{\deg}_\varepsilon(f)$ is the approximate degree:

$$\widetilde{\deg}_\varepsilon(f) = \min\{d : |f(x) - g(x)| \leq \varepsilon, \forall x \in [-1,1], \; g(x) \text{ is a polynomial of degree } d\}.$$

The quantum lower bound is similar to the famous polynomial method for Boolean functions [Beals, Buhrman, Cleve, Mosca, de Wolf, FOCS '98].

Key theorem in the proofs

Theorem (Key theorem)

Let $f : [-1,1] \to [-1,1]$ be continuous with odd and even parts f_{odd}, f_{even}, then

- *there is a symmetric tridiagonal matrix*

$$A = \begin{pmatrix} 0 & b_1 & & & \\ b_1 & 0 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & 0 \end{pmatrix}_{n\times n}$$

  satisfying $b_i \neq 0$, $\|A\| \leq 1$ and $f(A)_{1,n} = \varepsilon$, where $n = \widetilde{\deg}_\varepsilon(f_{\mathrm{odd}}) + O(1)$.
- *A similar result for f_{even}.*

Proof. linear semi-infinite programming + dual polynomial method + properties of tridiagonal matrices. ■

Lower bound's proof of quantum algorithms

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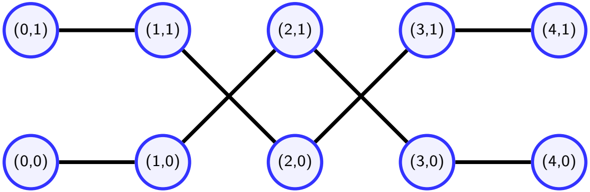
We construct a weighted graph G:

- **vertices:** (i, t), where $i \in \{0, 1, \ldots, n\}, t \in \{0, 1\}$
- **edges:** an edge between $(i-1, t)$ and $(i, t \oplus x_i)$
- **weights:** to be determined

For example, $(x_1, x_2, x_3, x_4) = (0, 1, 1, 0)$, then G is



Lower bound's proof of quantum algorithms

Essentially, G consists of two paths

$$(0,0) - (1, x_1) - (2, x_1 \oplus x_2) - \cdots - (n, x_1 \oplus \cdots \oplus x_n)$$
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Let A be the adjacency matrix of G (essentially two symmetric tridiagonal matrices).

- **Case 1:** if $x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0$, then $\langle 0,0|f(A)|n,1\rangle = 0$

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As a result, if we can estimate $\langle 0,0|f(A)|n,1\rangle$, we then can solve the parity problem, which is known hard. It is very important to ensure that $n = \widetilde{\deg}(f) + O(1)$ in the key theorem. This idea is inspired by the no fast-forwarding theorem. [Berry, Ahokas, Cleve, Sanders, Comm. Math. Phys. '07]

Lower bound's proof of classical algorithms

Forrelation problem (Aaronson & Ambainis, 2015):

Given $g_1, g_2 : \{0,1\}^n \to \{\pm 1\}$, let $D_i = \text{diag}(g_i(x) : x \in \{0,1\}^n)$, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, define

$$\begin{aligned} \Phi(g_1, g_2) &:= \langle 0^n | H^{\otimes n} D_1 H^{\otimes n} D_2 H^{\otimes n} | 0^n \rangle \\ &= \frac{1}{2^{3n/2}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} g_1(x) g_2(y). \end{aligned}$$

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For this problem, the classical query complexity is lower bounded by $\Omega(\sqrt{2^n}/n)$, while the quantum query complexity is $O(1)$.

Feynman's clock construction

Let $U = U_{N-1} \cdots U_2 U_1$ be a unitary operator, define

$$A = \begin{pmatrix} 0 & b_1 U_1^\dagger & & & \\ b_1 U_1 & 0 & b_2 U_2^\dagger & & \\ & b_2 U_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}$$

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Let $|\psi_t\rangle := |t\rangle \otimes U_t \cdots U_1 |0\rangle$, then

$$A|\psi_t\rangle = b_{t-1}|\psi_{t-1}\rangle + b_{t+1}|\psi_{t+1}\rangle$$

In subspace $\{|\psi_t\rangle : t = 0, 1, \ldots, N-1\}$, A is a symmetric tridiagonal matrix.

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In the Forrelation problem, we have $U = H^{\otimes n} D_1 H^{\otimes n} D_2 H^{\otimes n}$. To ensure A is sparse in the clock construction, we decompose

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Now $N = 3n + 2$,

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Lower bound	$\Omega(d)$	$\Omega((s/2)^{(d-1)/6})$

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