DQC1-hardness of estimating log-determinants

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In this note, we prove the following result.

Theorem 1. *Given a local Hamiltonian $H = \sum_{j=1}^{r} H_j$ on n qubits, where $r = \mathrm{poly}(n)$, each H_j acts on at most* $\mathrm{poly}(n)$ *qubits and $\|H_j\| \leq \mathrm{poly}(n)$. Suppose further that the condition number $\kappa(H)$ is at most* $\mathrm{poly}(n)$. *Estimating the average log-determinant of H*

$$\frac{1}{2^n} \log\det(H)$$

to accuracy $1/\mathrm{poly}(n)$ *is* DQC1*-hard.*

Notations. We use $\mathbb{I}_n$ to denote the identity matrix of dimension 2^n. We use $|i\rangle\langle i|_1$ for $i \in \{0, 1\}$ to denote the projection onto the space that the first qubit is in $|i\rangle$.

Perturbation theory. An Hermitian matrix H admits a spectral decomposition

$$H = \sum_{k=1}^{K} \lambda_k \Pi_k,$$

where λ_k is an eigenvalue and Π_k is the orthogonal projection onto the corresponding eigenspace. Suppose that an eigenvalue λ_k is separated from the other eigenvalues with a gap of δ. For any Hermitian matrix perturbation E added to H, with $\|E\| \leq \delta/2$, the eigenvalues of H in the λ_k-group are perturbed to new eigenvalues which are $O(\|E\|^2/\delta^2)$-close to the eigenvalues of

$$\lambda_k \Pi_k + \Pi_k E \Pi_k.$$

This statement can be made more rigorous using matrix analysis techniques (see Theorem 2.8 in Chapter V of Stewart and Sun [3]).

Definition 1 (DQC1)**.** *A language L is in* DQC1 *if and only if, for any r-bit string x, there exists a polynomial-time generated quantum circuit U_x acting on $q(r) = \mathrm{poly}(r)$ qubits such that for some specified polynomial $p(r)$,*

$$\begin{aligned}
\mu_{\mathrm{YES}} &:= \mathrm{tr}\left[U_x(|0\rangle\langle 0| \otimes \frac{\mathbb{I}_{q(r)-1}}{2^{q(r)-1}})U_x^\dagger(|1\rangle\langle 1| \otimes \mathbb{I}_{q(r)-1})\right] \geq \frac{1}{2} + \frac{1}{p(r)}, \quad \text{if } x \in L, \\
\mu_{\mathrm{YES}} &:= \mathrm{tr}\left[U_x(|0\rangle\langle 0| \otimes \frac{\mathbb{I}_{q(r)-1}}{2^{q(r)-1}})U_x^\dagger(|1\rangle\langle 1| \otimes \mathbb{I}_{q(r)-1})\right] \leq \frac{1}{2} - \frac{1}{p(r)}, \quad \text{if } x \notin L.
\end{aligned} \tag{1}$$

We show that approximating the normalized log-determinant is DQC1-hard, following the Hamiltonian construction in Brandão [1]. The average of the lowest 2^{N-1} eigenvalues of H_{DQC1} matches the rejection probability of the DQC1 circuit output. Brandão [1], Edenhofer et al. [2] used this construction to prove the DQC1-hardness of estimating other spectral sums $\mathrm{tr}[f(H)]$. Their strategy involved engineering the

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Hamiltonian such that its remaining eigenvalues λ_j are located where $f(\lambda_j)$ is negligible. This ensures the sum over the 2^{N-1} eigenvalues of interest dominates the spectral sum. This strategy does not work for $f(x) = \log(x)$ by direct application.

Instead of eliminating the contribution of the remaining eigenvalues, we use a more detailed perturbation analysis to approximate the contribution of these unwanted eigenvalues. We then subtract this contribution from the total spectral sum, thereby isolating the contribution of the 2^{N-1} eigenvalues of interest.

Proof of Theorem 1. We follow Kitaev's construction to encode the quantum circuit U_x into a local Hamiltonian. By the definition of $\mathsf{DQC1}$, for any r-bit string x, there exists a quantum circuit $U_x = U_T \dots U_1$ composed of $T = \mathrm{poly}(r)$ gates acting on $N = \mathrm{poly}(r)$ qubits such that U_x satisfies Eq. (1). We encode U_x into the following Hamiltonian acting on $n = N + \log(T+1)$ qubits

$$H_{\mathrm{DQC1}} = H_{\mathrm{out}} + J_{\mathrm{in}} H_{\mathrm{in}} + J_{\mathrm{prop}} H_{\mathrm{prop}}$$

where

$$H_{\mathrm{out}} = (T+1)|0\rangle\langle 0|_1 \otimes |T\rangle\langle T|, \quad H_{\mathrm{in}} = |1\rangle\langle 1|_1 \otimes |0\rangle\langle 0|,$$

and

$$H_{\mathrm{prop}} = \sum_{t=1}^{T} \mathbb{I}_N \otimes |t-1\rangle\langle t-1| - U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t| + \mathbb{I}_N \otimes |t\rangle\langle t|.$$

H_{DQC1} is $O(\log(n))$-local and satisfies the conditions in the theorem statement [1]. Operator H_{prop} can be similarity transformed to

$$H_{\mathrm{prop}} = W\Big(\mathbb{I}_N \otimes R\Big)W^\dagger,$$

where

$$W = \sum_{t=0}^{T} U_t \dots U_1 \otimes |t\rangle\langle t|\,, \quad R = \sum_{t=1}^{T} (|t-1\rangle\langle t-1| - |t\rangle\langle t-1| - |t-1\rangle\langle t| + |t\rangle\langle t|).$$

The operator R admits spectral decomposition $R = \sum_{k=0}^{T} \lambda_k v_k v_k^\dagger$ with eigenvalues

$$\lambda_k = 4\sin^2\left(\frac{\pi k}{2(T+1)}\right), \quad k = 0, \dots, T,$$

which are also eigenvalues of H_{prop}, each with degeneracy 2^N. Let $\mathcal{S}_k$ be the eigenspace of H_{prop} corresponding to the eigenvalue λ_k and Π_k be the projection onto $\mathcal{S}_k$. The eigenvectors of R is

$$v_{k,j} = \sqrt{\frac{2-\delta_{k0}}{T+1}} \sum_{j=0}^{T} \cos\left[\frac{\pi k}{T+1}\Big(j + \frac{1}{2}\Big)\right], \quad j = 0, \dots, T.$$

The k-th eigenspace of H_{prop}, $\mathcal{S}_k$, is spanned by

$$|v_{k,i}\rangle = \sqrt{\frac{2-\delta_{k0}}{T+1}} \sum_{t=0}^{T} \cos\left[\frac{\pi k}{T+1}\Big(t + \frac{1}{2}\Big)\right] U_t \dots U_1 |i\rangle \otimes |t\rangle \tag{2}$$

for $i = 0, \dots, 2^N - 1$. The minimum gap between two different eigenvalues of $J_{\mathrm{prop}} H_{\mathrm{prop}}$ is $\Delta_1 = \Omega(J_{\mathrm{prop}} T^{-2})$. Suppose that $J_{\mathrm{in}} = \Omega(T)$ and $J_{\mathrm{prop}} = \Omega(J_{\mathrm{in}} T^2)$. Regarding $H_{\mathrm{out}} + J_{\mathrm{in}} H_{\mathrm{in}}$ as a perturbation to $J_{\mathrm{prop}} H_{\mathrm{prop}}$, by the eigenvalue perturbation theory, the eigenvalues of $H_{\mathrm{DQC1}} = H_{\mathrm{out}} + J_{\mathrm{in}} H_{\mathrm{in}} + J_{\mathrm{prop}} H_{\mathrm{prop}}$ are

$$\|H_{\mathrm{out}} + J_{\mathrm{in}} H_{\mathrm{in}}\|^2/\Delta_1 = O(J_{\mathrm{in}}^2 T^2 / J_{\mathrm{prop}})$$

close to the eigenvalues of

$$J_{\mathrm{prop}} \lambda_k + \Pi_k (H_{\mathrm{out}} + J_{\mathrm{in}} H_{\mathrm{in}}) \Pi_k, \quad k = 0, \dots, T.$$

Note that

$$
\begin{aligned}
&\Pi_k(H_{\text{out}} + J_{\text{in}}H_{\text{in}})\Pi_k \\
&= \Big(\sum_{i=0}^{2^N-1} |v_{k,i}\rangle\langle v_{k,i}|\Big)\Big((T+1)\,|0\rangle\langle 0|_1 \otimes |T\rangle\langle T| + J_{\text{in}}\,|1\rangle\langle 1|_1 \otimes |0\rangle\langle 0|\Big)\Big(\sum_{i=0}^{2^N-1} |v_{k,i}\rangle\langle v_{k,i}|\Big) \\
&= \frac{2-\delta_{k0}}{T+1}\sum_{i,j=0}^{2^N-1} |v_{k,i}\rangle\langle v_{k,j}| \cdot \Big((T+1)\langle i|\cos\Big[\frac{(T+\frac{1}{2})\pi k}{T+1}\Big]U_x^\dagger\,|0\rangle\langle 0|_1\,U_x|j\rangle + \cos\Big[\frac{\pi k}{2(T+1)}\Big]J_{\text{in}}\langle i|\,|1\rangle\langle 1|_1\,|j\rangle\Big) \\
&= (2-\delta_{k0})\sum_{i,j=0}^{2^N-1} |v_{k,i}\rangle\langle v_{k,j}| \cdot \Big((-1)^k\cos\Big[\frac{\frac{1}{2}\pi k}{T+1}\Big]\langle i|U_x^\dagger\,|0\rangle\langle 0|_1\,U_x|j\rangle + \cos\Big[\frac{\frac{1}{2}\pi k}{T+1}\Big]\frac{J_{\text{in}}}{T+1}\langle i|\,|1\rangle\langle 1|_1\,|j\rangle\Big).
\end{aligned}
$$

For $k \geq 1$, the eigenvalues of $J_{\text{prop}}H_{\text{prop}}$ in $\mathcal{S}_k$ is perturbed to

$$J_{\text{prop}}\lambda_k + O\Big(\frac{J_{\text{in}}}{T}\Big) + O\Big(\frac{J_{\text{in}}^2T^2}{J_{\text{prop}}}\Big).$$

For $k = 0$, operator $\Pi_0 J_{\text{in}}H_{\text{in}}\Pi_0$ has two dimension-2^{N-1} degenerate eigenspace with eigenvalue 0 and $J_{\text{in}}/(T+1)$. Regarding $\Pi_0 H_{\text{out}}\Pi_0$ as a perturbation to $\Pi_0 J_{\text{in}}H_{\text{in}}\Pi_0$, by the eigenvalue perturbation theory, the 2^{N-1} 0-eigenvalues are perturbed to the eigenvalues of

$$|0\rangle\langle 0|_1\,U_x^\dagger\,|0\rangle\langle 0|_1\,U_x\,|0\rangle\langle 0|_1 + O\Big(\frac{T}{J_{\text{in}}}\Big).$$

and the 2^{N-1} $J_{\text{in}}/(T+1)$-eigenvalues are perturbed to

$$\frac{J_{\text{in}}}{T+1} + O(1).$$

Suppose $J_{\text{prop}} = \Omega(J_{\text{in}}^3T)$. Taking the first perturbation into account, the eigenvalues of H_{DQC1} consist of

1. 2^{N-1} eigenvalues of

$$|0\rangle\langle 0|_1\,U_x^\dagger\,|0\rangle\langle 0|_1\,U_x\,|0\rangle\langle 0|_1 + O\Big(\frac{T}{J_{\text{in}}}\Big) + O\Big(\frac{J_{\text{in}}^2T^2}{J_{\text{prop}}}\Big) = |0\rangle\langle 0|_1\,U_x^\dagger\,|0\rangle\langle 0|_1\,U_x\,|0\rangle\langle 0|_1 + O\Big(\frac{T}{J_{\text{in}}}\Big).$$

2. 2^{N-1} eigenvalues

$$\frac{J_{\text{in}}}{T+1} + O(1) + O\Big(\frac{J_{\text{in}}^2T^2}{J_{\text{prop}}}\Big) = \frac{J_{\text{in}}}{T+1} + O(1).$$

3. $T2^N$ eigenvalues

$$J_{\text{prop}}\lambda_k + O\Big(\frac{J_{\text{in}}}{T}\Big) + O\Big(\frac{J_{\text{in}}^2T^2}{J_{\text{prop}}}\Big) = J_{\text{prop}}\lambda_k + O\Big(\frac{J_{\text{in}}}{T}\Big).$$

   for $k = 1, \ldots, T$.

Note that the average of the eigenvalues of $|0\rangle\langle 0|_1\,U_x^\dagger\,|0\rangle\langle 0|_1\,U_x\,|0\rangle\langle 0|_1$ is

$$\frac{1}{2^{N-1}}\operatorname{tr}\big[|0\rangle\langle 0|_1\,U_x^\dagger\,|0\rangle\langle 0|_1\,U_x\,|0\rangle\langle 0|_1\big] = 1 - \operatorname{tr}\Big[U_x(|0\rangle\langle 0| \otimes \frac{\mathbb{I}_{N-1}}{2^{N-1}})U_x^\dagger(|1\rangle\langle 1| \otimes \mathbb{I}_{N-1})\Big] = 1 - \mu_{\text{YES}}.$$

For $\eta = O(J_{\text{in}}/T)$, the average log-determant of $\eta\mathbb{I}_n + H_{\text{DQC1}}$ is

$$
\begin{aligned}
&\frac{1}{(T+1)2^N}\Big(\log\det\Big[\eta\mathbb{I}_{N-1} + |0\rangle\!\langle 0|_1\, U_x^\dagger\, |0\rangle\!\langle 0|_1\, U_x\, |0\rangle\!\langle 0|_1 + O\Big(\frac{T}{J_{\text{in}}}\Big)\Big] + 2^{N-1}\log\Big(\eta + \frac{J_{\text{in}}}{T+1} + O(1)\Big)\\
&\quad + \sum_{k=1}^{T} 2^N \log\Big(\eta + J_{\text{prop}}\lambda_k + O\Big(\frac{J_{\text{in}}}{T}\Big)\Big)\Big)\\
&= \frac{1}{(T+1)2^N}\Big(2^{N-1}\Big(\log(\eta) + \frac{1-\mu_{\text{YES}}}{\eta} + O\Big(\frac{1}{\eta^2}\Big) + O\Big(\frac{T}{J_{\text{in}}\eta}\Big)\Big) + 2^{N-1}\Big(\log\Big(\eta + \frac{J_{\text{in}}}{T+1}\Big) + O\Big(\frac{T}{J_{\text{in}}}\Big)\Big)\\
&\quad + \sum_{k=1}^{T} 2^N \Big(\log\Big(\eta + J_{\text{prop}}\lambda_k\Big) + O\Big(\frac{J_{\text{in}}T}{J_{\text{prop}}}\Big)\Big)\Big)\\
&= \frac{1}{2(T+1)}\Big(\log(\eta) + \frac{1-\mu_{\text{YES}}}{\eta} + \log\Big(\eta + \frac{J_{\text{in}}}{T+1}\Big) + 2\sum_{k=1}^{T}\log\Big(\eta + J_{\text{prop}}\lambda_k\Big)\Big) + O\Big(\frac{1}{T\eta^2} + \frac{1}{J_{\text{in}}} + \frac{J_{\text{in}}T}{J_{\text{prop}}}\Big)
\end{aligned}
$$

where the first equation follows from $\log(x+\delta) = \log(x) + \delta/x + O(\delta^2/x^2)$ and $\lambda_k = \Omega(T^{-2})$. For any polynomial $p(n)$, take

$$
\eta = \Theta(p(n)), \quad J_{\text{in}} = \Omega(Tp^2(n)), \quad J_{\text{prop}} = \Omega(T^4p^6(n)),
$$

and then all previous assumptions $J_{\text{in}} = \Omega(T)$, $J_{\text{prop}} = \Omega(J_{\text{in}}T^2)$, $\eta = O(J_{\text{in}}/T)$, $J_{\text{prop}} = \Omega(J_{\text{in}}^3 T)$ are true and the average log-determant of $\eta\mathbb{I}_n + H_{\text{DQC1}}$ is $O(1/(T\eta p(n))$ close to

$$
\frac{1}{2(T+1)}\Big(\log(\eta) + \frac{1-\mu_{\text{YES}}}{\eta} + \log\Big(\eta + \frac{J_{\text{in}}}{T+1}\Big) + 2\sum_{k=1}^{T}\log\Big(\eta + J_{\text{prop}}\lambda_k\Big)\Big)
$$

Except for μ_{YES}, other terms can be computed efficiently. Thus, if we can estimate $\log\det(\eta\mathbb{I}_n + H_{\text{DQC1}})/2^n$ to accuracy $O(T^{-1}p^{-2}(n))$, then we can obtain an $O(1/p(n))$-approximation of μ_{YES}. □

References

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