Advanced Algorithms

南京大学

尹一通

Constraint Satisfaction Problem (CSP)

- variables: $X = \{x_1, x_2, \dots, x_n\}$
  - each variable ranges over a finite domain Ω
  - an assignment $\sigma \in \Omega^X$ assigns each variable a value in Ω
- constraints: $C_1, C_2, \dots, C_m$
  - each constraint C_i is a Boolean function

    $$C_i : \Omega^{S_i} \to \{\texttt{true}, \texttt{false}\}$$

    defined on a subset of variables $S_i \subseteq X$
  - a constraint C_i is satisfied by an assignment $\sigma \in \Omega^X$ if

    $$C_i(\sigma_{S_i}) = \texttt{true}$$

Constraint Satisfaction Problem (CSP)

- variables: $x_1, x_2, \ldots, x_n \in \Omega$
- constraints: $C_1, C_2, \ldots, C_m$

$$C_i : \Omega^{S_i} \to \{\texttt{true}, \texttt{false}\}$$

Examples: satisfiability, optimization, counting, …

- graph cut: $\Omega = \{0,1\}$, constraints: $x_u \neq x_v$ for each edge uv
- *k*-coloring: $\Omega = [k]$, constraints: $x_u \neq x_v$ for each edge uv
- matching/cover: $\Omega = \{0,1\}$, constraints:

$$\sum_{j \in S_i} x_j \leq 1 \text{ (matching)} \qquad \text{or} \qquad \sum_{j \in S_i} x_j \geq 1 \text{ (cover)}$$

- SAT: $\Omega = \{\mathrm{true}, \mathrm{false}\}$, constraints are *clauses*

Algorithmic Problems for CSP

CSP	Satisfiability	Optimization	Counting
2SAT	**P**	**NP**-hard	**#P**-complete
3SAT	**NP**-complete	**NP**-hard	**#P**-complete
matching	perfect matching **P**	max matching **P**	**#P**-complete
cut (2-coloring)	bipartite test **P**	max-cut **NP**-hard	**FP** (poly-time)
3-coloring	**NP**-complete	max-3-cut **NP**-hard	**#P**-complete

Algorithmic Problems for CSP

Given a CSP instance:

- satisfiability: determine whether ∃ an assignment satisfying all constraints
  - search: return a satisfying assignment

- optimization: find an assignment satisfying as many constraints as possible
  - refutation (dual): find a "proof" of "no assignment can satisfy $>m^*$ constraints" for m^* as small as possible

- counting: estimate the number of satisfying assignments
  - sampling: random sample a satisfying assignments
  - inference: calculate the possibility of a variable being assigned certain value

k-SAT

Instance: a k-CNF formula ϕ.

Determine whether ϕ is satisfiable.

($\exists$ a satisfying assignment σ s.t. $\phi(\sigma) = \text{true}$)

CNF (Conjunctive Normal Form):

$$(x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_2 \vee x_3 \vee x_5) \wedge (x_2 \vee x_4 \vee x_5)$$

k-SAT

Instance: a k-CNF formula ϕ.

Determine whether ϕ is satisfiable.

($\exists$ a satisfying assignment σ s.t. $\phi(\sigma) = \text{true}$)

CNF (Conjunctive Normal Form):

- n Boolean variables: $x_1, x_2, \ldots, x_n \in \{\text{true}, \text{false}\}$
- m clauses: $C_1 \wedge C_2 \wedge \cdots \wedge C_m$
- each clause is in the form $C_i = \ell_{i_1} \vee \ell_{i_2} \vee \cdots \vee \ell_{i_{k_i}}$
- each literal $\ell_{i_j} \in \{x_s, \neg x_s\}$ for some $s \in \{1,2,\ldots,n\}$

k-CNF: (exact-k-CNF)

- each clause contains exactly k variables

k-SAT

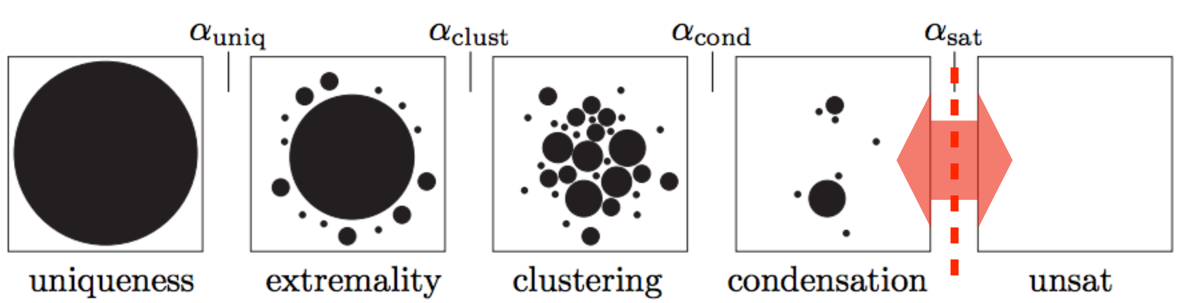
Instance: a k-CNF formula ϕ.

Determine whether ϕ is satisfiable.

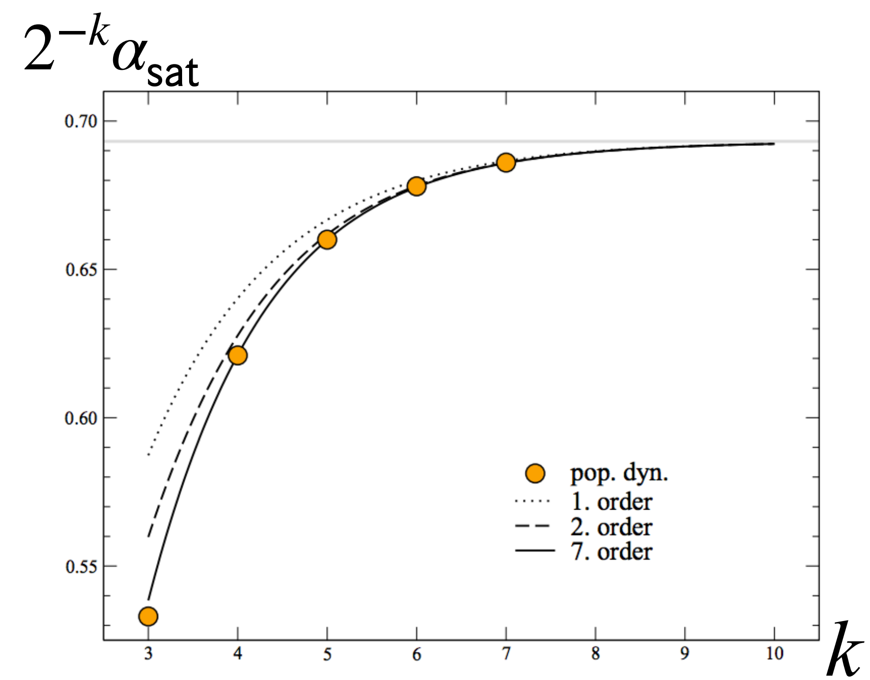
$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

random k-CNF formula with $m = \alpha n$ clauses

phase transition of satisfiability for random CSP:



[Ding, Sly, Sun, **STOC**'15]
[Krząkała, Montanari, Ricci-Tersenghi, Semerjian, Zdeborová, **PNAS**'07]
[Achlioptas, Naor, Peres, **Nature**'05]



k-SAT

Instance: a k-CNF formula ϕ.

Determine whether ϕ is satisfiable.

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

k-CNF: (exact-k-CNF)

- each clause contains exactly k variables

degree d:

(shares variables)

- each clause intersects with $\leq d$ other clauses

Theorem: $d \leq 2^{k-2}$ ⇒ ϕ is always satisfiable

The *Lovász Local Lemma*

ϕ: a k-CNF formula of degree d

The *Lovász Local Lemma* (*LLL*) for k-SAT:

Theorem: $d \leq 2^{k-2}$ ⇨ ϕ is always satisfiable

Algorithmic *LLL* for k-SAT:

Theorem (Moser 2009): $\exists$ constant $c > 0$

$d \leq 2^{k-c}$ ⇨ satisfying assignment can be found in time $O(n + km)$ *w.h.p.*

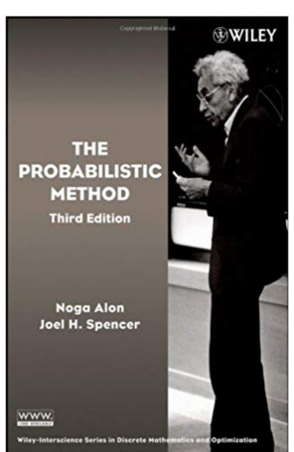
The *Probabilistic* Method

ϕ: a k-CNF formula of degree d

Theorem: $d \le 2^{k-2}$ $\Rightarrow$ ϕ is always satisfiable

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

NaïveRandomGuess(ϕ)
sample a uniform random assignment $X_1, X_2, \ldots, X_n \in \{\text{true}, \text{false}\}$;



The *Probabilistic Method*: $\Pr[\ \phi(\boldsymbol{X})=\text{true}\ ] > 0$ $\Rightarrow$ $\exists$ a satisfying assignment (ϕ is satisfiable)

The *Probabilistic Method*

ϕ: a k-CNF formula of degree d

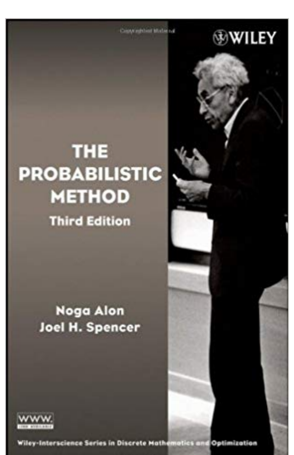
Theorem: $d \le 2^{k-2}$ $\Longrightarrow$ ϕ is always satisfiable

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

sample a uniform random assignment

$$X_1, X_2, \ldots, X_n \in \{\text{true}, \text{false}\};$$

bad event A_i: clause C_i is unsatisfied



The *Probabilistic Method*:

$$\Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0 \Longrightarrow \exists \text{ a satisfying assignment}$$

(ϕ is satisfiable)

The *Lovász* Sieve

m bad event: $A_1, A_2, \ldots, A_m$

Goal: $$\Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0 \qquad (\bigstar)$$

- union bound: $\sum_{i=1}^{m} \Pr[A_i] < 1 \Rightarrow (\bigstar)$
- principle of inclusion exclusion (PIE):

容斥定理

$$\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S \neq \emptyset}} (-1)^{|S|-1} \Pr\left[\bigwedge_{i \in S} A_i\right] < 1 \Rightarrow (\bigstar)$$

- *LLL*: every A_i is independent of all but $\leq d$ other bad events (degree $\leq d$)

$$\forall i: \quad \Pr[A_i] \leq \frac{1}{4d} \Rightarrow (\bigstar)$$

?

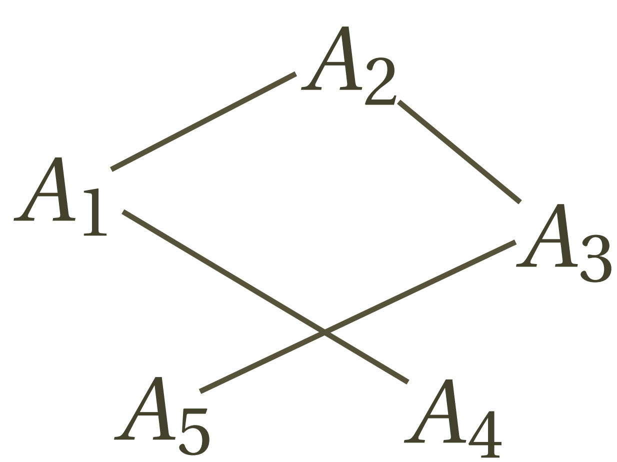
m bad event: $A_1, A_2, \ldots, A_m$

every A_i is independent of all but $\le d$ other bad events

Lovász Local Lemma (Erdos-Lovász 1975):

$$\forall i: \quad \Pr[A_i] \le \frac{1}{4d} \quad \Longrightarrow \quad \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

Example:



dependency graph

(max degree d)

$A_1(X_1, X_4)$

$A_2(X_1, X_2)$

$A_3(X_2, X_3)$

$A_4(X_4)$

$A_5(X_3)$

X_1, X_2, X_3, X_4

are mutually independent

m bad event: $A_1, A_2, \ldots, A_m$

every A_i is independent of all but $\leq d$ other bad events

Lovász Local Lemma (Lovász 1977):

$$\forall i: \quad \Pr[A_i] \leq \frac{1}{\mathrm{e}(d+1)} \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

$$\Uparrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = \frac{1}{d+1}$$

Lovász Local Lemma (asymmetric version):

$$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$$
$$\forall i: \quad \Pr[A_i] \leq \alpha_i \prod_{j \sim i} (1 - \alpha_j) \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > \prod_{i=1}^{m} (1 - \alpha_i)$$

$j \sim i$: A_i and A_j are adjacent in the dependency graph

m bad event: $A_1, A_2, \ldots, A_m$

every A_i is independent of all but $\leq d$ other bad events

Lovász Local Lemma (Erdos-Lovász 1975):

$$\forall i: \quad \Pr[A_i] \leq \frac{1}{4d} \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

$$\Uparrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = \frac{1}{2d}$$

Lovász Local Lemma (asymmetric version):

$$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$$

$$\forall i: \quad \Pr[A_i] \leq \alpha_i \prod_{j \sim i} (1 - \alpha_j) \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > \prod_{i=1}^{m} (1 - \alpha_i)$$

$j \sim i$: A_i and A_j are adjacent in the dependency graph

The *Lovász Local Lemma*

ϕ: a k-CNF formula of degree d

Theorem: $d \le 2^{k-2}$ ⇒ ϕ is always satisfiable

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

sample a uniform random assignment

$$X_1, X_2, \ldots, X_n \in \{\text{true}, \text{false}\};$$

bad event A_i: clause C_i is unsatisfied

$$\forall i: \quad \Pr[A_i] = 2^{-k} \le \frac{1}{4d} \quad \xRightarrow{LLL} \quad \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

(k-CNF)

(ϕ is satisfiable)

Algorithmic LLL

ϕ: a k-CNF formula of degree d

The *Lovász Local Lemma* (*LLL*) for k-SAT:

Theorem: $d \leq 2^{k-2}$ ⇒ ϕ is always satisfiable

Algorithmic *LLL* for k-SAT:

Theorem (Moser 2009): $\exists$ constant $c > 0$

$d \leq 2^{k-c}$ ⇒ satisfying assignment can be found in time $O(n + km)$ *w.h.p.*

Moser's Algorithm

ϕ: a k-CNF formula of degree d

Solve(ϕ)

sample a uniform random
  assignment $X_1, X_2, \ldots, X_n$;
while $\exists$ unsatisfied clause C
  Fix(C);

Fix(C)

resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
  Fix(D); (including C itself)

ϕ: a k-CNF formula of degree d with m clauses on n variables

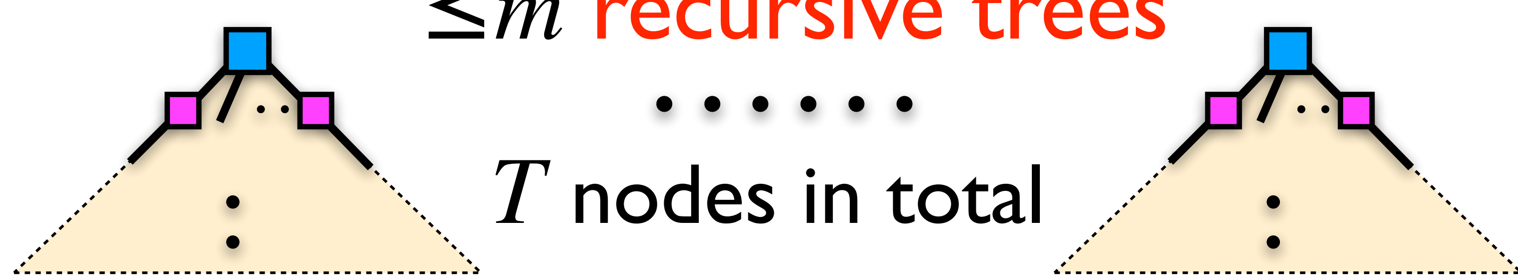
Solve(ϕ)

sample a uniform random
    assignment $x_1, x_2, \ldots, x_n$;
while $\exists$ unsatisfied clause C
    Fix(C);

Fix(C)

resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
    Fix(D);

- terminate $\Rightarrow$ successfully return a satisfying solution
- top-level: **Fix**(C) returned $\Rightarrow$ C remains satisfied



- T: total # of calls to **Fix**(C)
  (including both top-level and recursive calls)
- total cost: $n + kT$ (total # of random bits)

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)

sample a uniform random assignment $x_1, x_2, \ldots, x_n$;

while $\exists$ unsatisfied clause C

  Fix(C);

Fix(C)

resample variables in C uniformly at random;

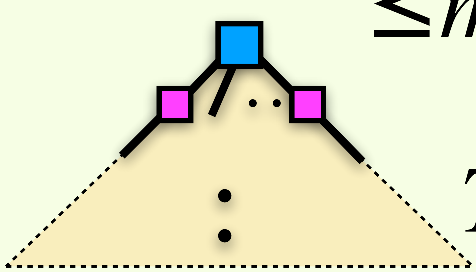
while $\exists$ unsatisfied clause D intersecting C

  Fix(D);

$n + kT$ random bits

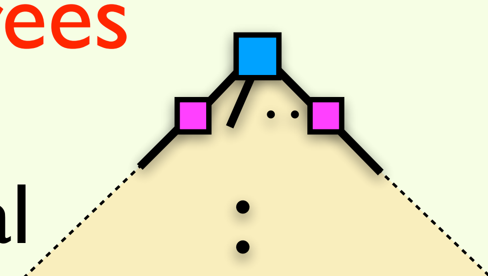
recover

$\leq m$ recursive trees



······

T nodes in total



+

final assignment $x_1, x_2, \ldots, x_n$

Observation:

Fix(C) is called $\Rightarrow$ assignment of C is uniquely determined

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)

sample a uniform random
    assignment $x_1, x_2, \ldots, x_n$;
while $\exists$ unsatisfied clause C
    Fix(C);

Fix(C)

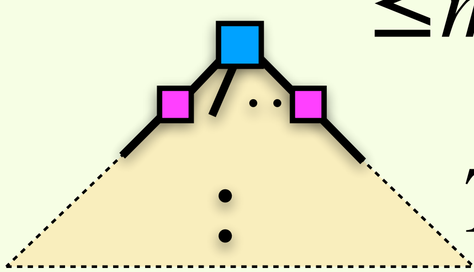
resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
    Fix(D);

$n + kT$ random bits

encode

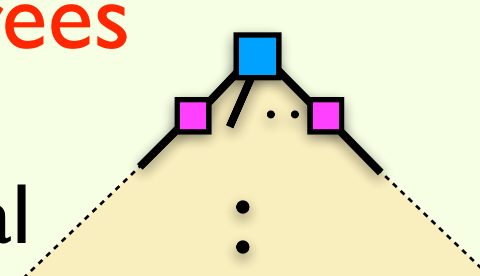
1-1 mapping Enc_ϕ

$\leq m$ recursive trees



$\cdots\cdots$

T nodes in total



+

final assignment
$x_1, x_2, \ldots, x_n$

represented by succinct representation:

$\leq m \log m + T(\log_2 d + O(1))$ bits

n bits

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)

sample a uniform random assignment $x_1, x_2, \ldots, x_n$;
while $\exists$ unsatisfied clause C
Fix(C); lexicographic order

Fix(C)

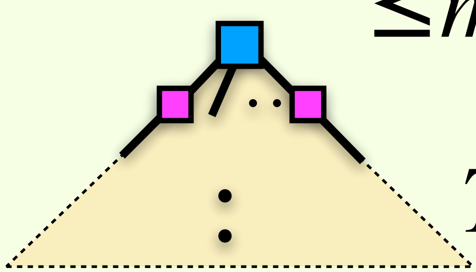
resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
Fix(D);

$n + kT$ random bits

encode

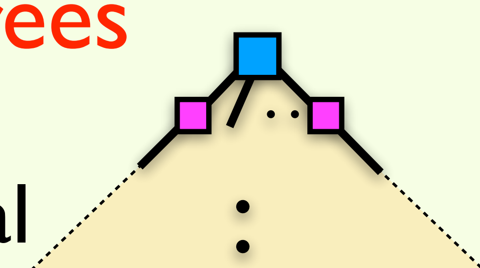
1-1 mapping Enc$_\phi$

$\leq m$ recursive trees



······

T nodes in total



\+

final assignment $x_1, x_2, \ldots, x_n$

represented by succinct representation:

$\leq m + T(\log_2 d + \mathrm{O}(1))$ bits

n bits

- an m-bit vector to indicate the root nodes
- O(1) bits to record the stack operation for each recursive call

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)

sample a uniform random assignment $x_1, x_2, \ldots, x_n$;

while $\exists$ unsatisfied clause C

Fix(C); lexicographic order

Fix(C)

resample variables in C uniformly at random;

while $\exists$ unsatisfied clause D intersecting C

Fix(D);

$n + kT$ random bits

encode

1-1 mapping Enc_ϕ

Incompressibility Theorem (Kolmogorov):

N uniform random bits cannot be encoded to less than N - l bits with probability at least 1-O(2^{-l}).

$\leq m + T(\log_2 d + O(1))$ bits + n bits

w.h.p.: $n + kT - \log_2 n \leq m + T(\log_2 d + O(1)) + n$

$\Leftrightarrow (k - \log_2 d - O(1))T \leq m + \log_2 n$

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)
sample a uniform random assignment $x_1, x_2, \ldots, x_n$; while $\exists$ unsatisfied clause C **Fix**(C); lexicographic order

Fix(C)
resample variables in C uniformly at random; while $\exists$ unsatisfied clause D intersecting C **Fix**(D);

- T: total # of calls to **Fix**(C) (including both top-level and recursive calls)
- total cost: $n + kT$

w.h.p.: $(k - \log_2 d - O(1))T \leq m + \log_2 n$

$d \leq 2^{k-c}$ $\Rightarrow$ $T \leq m + \log_2 n$

for some constant c

satisfying assignment can be found in time $O(n + k(m + \log n))$ *w.h.p.*

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)
sample a uniform random assignment $x_1, x_2, \ldots, x_n$;
while $\exists$ unsatisfied clause C
Fix(C); lexicographic order

Fix(C)
resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
Fix(D);

- T: total # of calls to **Fix**(C) (including both top-level and recursive calls)
- total cost: $n + kT$

w.h.p.: $$(k - \log_2 d - O(1))T \le m + \log_2 n$$

Theorem (Moser 2009): $\exists$ constant $c > 0$

$d \le 2^{k-c}$ $\Rightarrow$ satisfying assignment can be found in time $O(n + km)$ *w.h.p.*

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)
sample a uniform random assignment $x_1, x_2, \ldots, x_n$;
while ∃ unsatisfied clause C
Fix(C); lexicographic order

Fix(C)
resample variables in C uniformly at random;
while ∃ unsatisfied clause D intersecting C
Fix(D);

- T: total # of calls to **Fix**(C) **Why should T be finite?** (including both top-level and recursive calls)

Incompressibility Theorem (Kolmogorov):

Does this hold when *N* is random?

N uniform random bits cannot be encoded to less than N - l bits with probability at least $1\text{-}O(2^{-l})$.

Theorem (Moser 2009): ∃ constant $c > 0$

$d \leq 2^{k-c}$ ⇒ satisfying assignment can be found in time $O(n + km)$ *w.h.p.*

ϕ: a k-CNF formula of degree d with m clauses on n variables

Solve(ϕ)

sample a uniform random
  assignment $x_1, x_2, \ldots, x_n$;
while $\exists$ unsatisfied clause C
  Fix(C); lexicographic order

Fix(C)

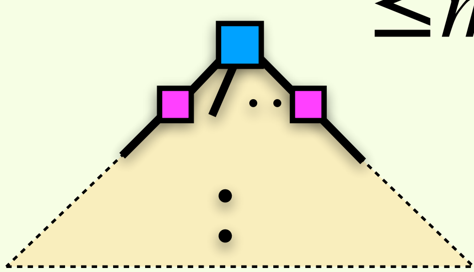
resample variables in C uniformly at random;
while $\exists$ unsatisfied clause D intersecting C
  Fix(D);

- $n + kt$ random bits where $t = 2(m + \log n)$ is fixed

encode

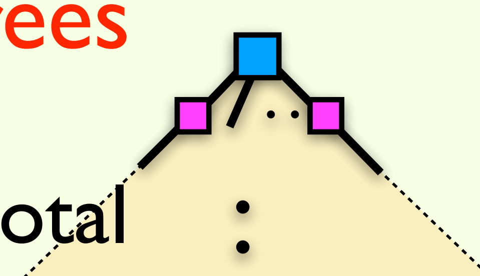
- used as the random bits for the algorithm;
- force to terminate the algorithm if used up;

$\leq m$ recursive trees



$\cdots\cdots$

$T \leq t$ nodes in total



+ final assignment $x_1, x_2, \ldots, x_n$

(maybe unsatisfying)

$\leq m + T(\log_2 d + O(1))$ bits + n bits

+ $(t-T)k$ unused random bits

w.h.p.: $(k - \log_2 d - O(1))T \leq m + \log_2 n$

$d \leq 2^{k-c}$ for some constant c $\Rightarrow$ $T \leq m + \log_2 n$

Algorithmic LLL

ϕ: a k-CNF formula of degree d

The *Lovász Local Lemma* (*LLL*) for k-SAT:

Theorem: $d \le 2^{k-2}$ $\Rightarrow$ ϕ is always satisfiable

Algorithmic *LLL* for k-SAT:

Theorem (Moser 2009): $\exists$ constant $c > 0$

$d \le 2^{k-c}$ $\Rightarrow$ satisfying assignment can be found in time $\mathrm{O}(n + km)$ *w.h.p.*

The *Lovász Local Lemma*

m bad event: $A_1, A_2, \ldots, A_m$

every A_i is independent of all but $\le d$ other bad events

Lovász Local Lemma (Lovász 1977):

$$\forall i: \quad \Pr[A_i] \le \frac{1}{e(d+1)} \quad \Rightarrow \quad \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

Lovász Local Lemma (asymmetric version):

$$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$$
$$\forall i: \quad \Pr[A_i] \le \alpha_i \prod_{j \sim i} (1 - \alpha_j) \quad \Rightarrow \quad \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > \prod_{i=1}^{m} (1 - \alpha_i)$$

$j \sim i$: A_i and A_j are adjacent in the dependency graph

The *Lovász Local Lemma*

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \emptyset\}$

Lovász Local Lemma (asymmetric version):

$$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$$
$$\forall i: \quad \Pr[A_i] \leq \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j) \quad \Longrightarrow \quad \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > \prod_{i=1}^{m} (1 - \alpha_i)$$

The *Lovász Local Lemma*

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \text{vbl}(A_i) \cap \text{vbl}(A_j) \neq \emptyset\}$

Lovász Local Lemma (asymmetric version):

$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$

$$\forall i: \quad \Pr[A_i] \leq \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

$\Rightarrow$

$\exists$ an assignment of $X_1, \ldots, X_n$ avoiding all bad events $A_1, \ldots, A_m$

Moser-Tardos Algorithm

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \text{vbl}(A_i) \cap \text{vbl}(A_j) \neq \emptyset\}$

Assumption: The followings can be done efficiently:

- draw an independent sample of a random variable X_j.
- check whether a bad event A_i occurs on current $X_1, \ldots, X_n$.

Moser-Tardos Algorithm:

sample all $X_1, \ldots, X_n$;
while $\exists$ an occurring bad event A_i:
  resample all $X_j \in \text{vbl}(A_i)$;

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \emptyset\}$

Moser-Tardos Algorithm:
sample all $X_1, \ldots, X_n$;
while $\exists$ an occurring bad event A_i:
    resample all $X_j \in \mathrm{vbl}(A_i)$;

Lovász Local Lemma (Moser-Tardos 2010):

$\exists \alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1)$
$\forall i: \quad \Pr[A_i] \leq \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$

$\Rightarrow$ a satisfying assignment is returned within $\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_i}$ resamples in expectation

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \emptyset\}$

Moser-Tardos Algorithm:
sample all $X_1, \ldots, X_n$;
while $\exists$ an occurring bad event A_i:
    resample all $X_j \in \mathrm{vbl}(A_i)$;

Lovász Local Lemma (Moser-Tardos 2010):

$$\forall i: \quad \Pr[A_i] \leq \frac{1}{\mathrm{e}(d+1)}$$

where $d \triangleq \max_i |\Gamma(A_i)|$

$\Longrightarrow$ a satisfying assignment is returned within m/d resamples in expectation

- n mutually independent random variables: $X_1, \ldots, X_n$
- m bad events: $A_1, A_2, \ldots, A_m$, determined by $X_1, \ldots, X_n$
- vbl(A_i): set of variables on which A_i is defined
- neighborhood: $\Gamma(A_i) \triangleq \{A_j \mid j \neq i \wedge \mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \emptyset\}$

Moser-Tardos Algorithm:
sample all $X_1, \ldots, X_n$;
while $\exists$ an occurring bad event A_i:
resample all $X_j \in \mathrm{vbl}(A_i)$;

Lovász Local Lemma (Moser-Tardos 2010):

$$\forall i: \quad \Pr[A_i] \le \frac{1}{4d}$$

where $d \triangleq \max_i |\Gamma(A_i)|$

a satisfying assignment is returned within $m/(2d\text{-}1)$ resamples in expectation

k-SAT

ϕ: a k-CNF formula of degree d

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

Moser-Tardos Algorithm:

sample a uniform random assignment $x_1, x_2, \ldots, x_n \in \{\text{true}, \text{false}\}$;

while $\exists$ an unsatisfied clause C:

  resample values of variables in C uniformly at random;

bad event A_i: clause C_i is unsatisfied

$$\forall i: \quad \Pr[A_i] = 2^{-k} \quad \leq \frac{1}{4d}$$

(assuming $d \leq 2^{k-2}$)

$\Rightarrow$ a satisfying assignment is returned within $m/(2d\text{-}1)$ resamples in expectation

k-SAT

ϕ: a k-CNF formula of degree d

$$\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

Moser-Tardos Algorithm:

sample a uniform random assignment $x_1, x_2, \ldots, x_n \in \{\text{true}, \text{false}\}$;

while $\exists$ an unsatisfied clause C:

resample values of variables in C uniformly at random;

Theorem (Moser-Tardos 2010):

$d \leq 2^{k-2}$ $\Rightarrow$ satisfying assignment can be found in time $O(n + km/d)$ in expectation

- mutually independent random variables: $\mathcal{X} \triangleq \{X_1, \ldots, X_n\}$
- bad events: $\mathcal{A} \triangleq \{A_1, A_2, \ldots, A_m\}$
- $\forall A \in \mathcal{A}$, $\mathsf{vbl}(A) \subseteq \mathcal{X}$: set of variables determining A
- neighborhood: $\forall A \in \mathcal{A}, \Gamma(A) \triangleq \{B \neq A \mid \mathsf{vbl}(A) \cap \mathsf{vbl}(B) \neq \emptyset\}$

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
    resample all $X \in \mathsf{vbl}(A)$;

Lovász Local Lemma (Moser-Tardos 2010):

$\exists \alpha : \mathcal{A} \to [0,1)$

$\forall A \in \mathcal{A}: \quad \Pr[A] \leq \alpha_A \prod_{B \in \Gamma(A)} (1 - \alpha_B)$ ⇒ a satisfying assignment is returned within $\sum_{A \in \mathcal{A}} \frac{\alpha_A}{1 - \alpha_A}$ resamples in expectation

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
resample all $X \in \mathsf{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathcal{A}$$

random sequence of resampled events

$$\forall A \in \mathcal{A}, \quad N_A \triangleq |\{i \mid \Lambda_i = A\}|$$

total # of times that A is resampled

Lovász Local Lemma (Moser-Tardos 2010):

$\exists \alpha : \mathcal{A} \to [0,1)$

$$\forall A \in \mathcal{A}: \quad \Pr[A] \le \alpha_A \prod_{B \in \Gamma(A)} (1 - \alpha_B)$$

$\Rightarrow$

$\forall A \in \mathcal{A}$:

$$\mathbb{E}[N_A] \le \frac{\alpha_A}{1 - \alpha_A}$$

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
resample all $X \in \mathsf{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathcal{A}$$

random sequence of resampled events

witness tree: A witness tree τ is a labeled tree in which every vertex v is labeled by an event $A_v \in \mathcal{A}$, such that *siblings* have distinct labels.

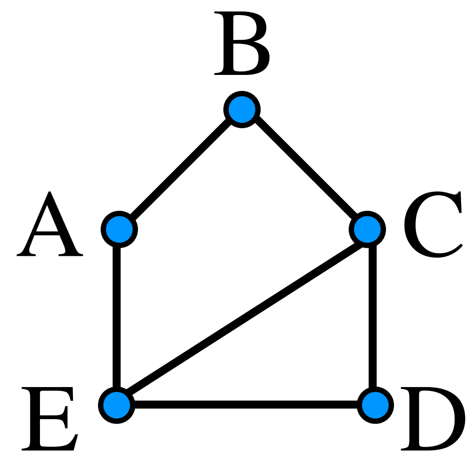
$T(\Lambda, t)$ is a witness tree constructed from exe-log Λ:

- initially, T is a single root with label Λ_t
- for $i = t\text{-}1, t\text{-}2, \ldots, 1$
  - if $\exists$ a vertex v in T with label $A_v \in \Gamma^+(\Lambda_i)$
    - add a new child u to the *deepest* such v and label it with Λ_i
- $T(\Lambda, t)$ is the resulting T

inclusive neighborhood: $\Gamma^+(A) \triangleq \{B \in \mathcal{A} \mid \mathsf{vbl}(A) \cap \mathsf{vbl}(B) \neq \emptyset\}$

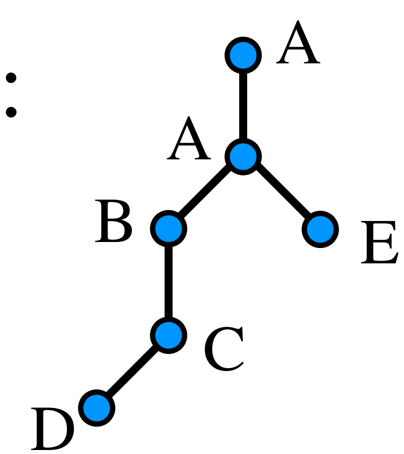
$$= \Gamma(A) \cup \{A\}$$

dependency graph:



exe-log Λ: D, C, E, D, B, A, C, A, D, ...

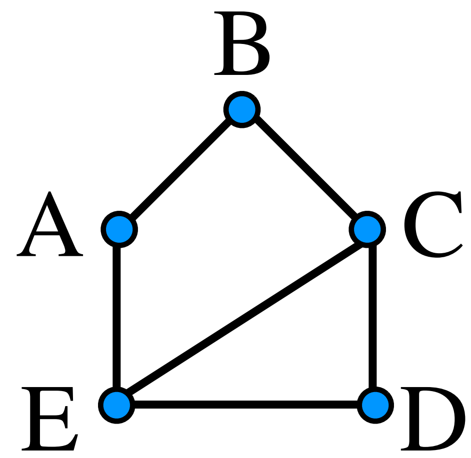
$T(\Lambda, 8)$:



$T(\Lambda, t)$ is a witness tree constructed from exe-log Λ:

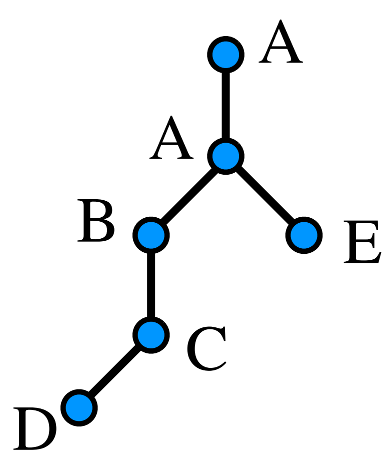
- initially, T is a single root with label Λ_t
- for $i = t\text{-}1, t\text{-}2, \ldots, 1$
  - if $\exists$ a vertex v in T with label $A_v \in \Gamma^+(\Lambda_i)$
    - add a new child u to the *deepest* such v and label it with Λ_i
- $T(\Lambda, t)$ is the resulting T

dependency graph:

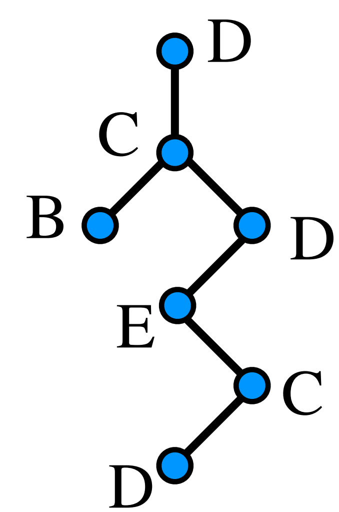


exe-log Λ: D, C, E, D, B, A, C, A, D, ...

$T(\Lambda, 8)$:



$T(\Lambda, 9)$:



$T(\Lambda, t)$ is a witness tree constructed from exe-log Λ:

- initially, T is a single root with label Λ_t
- for $i = t\text{-}1, t\text{-}2, \ldots, 1$
  - if $\exists$ a vertex v in T with label $A_v \in \Gamma^{+}(\Lambda_i)$
    - add a new child u to the *deepest* such v and label it with Λ_i
- $T(\Lambda, t)$ is the resulting T

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
resample all $X \in \mathrm{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathcal{A}$$

random sequence of resampled events

witness tree: A witness tree τ is a labeled tree in which every vertex v is labeled by an event $A_v \in \mathcal{A}$, such that *siblings* have distinct labels.

$T(\Lambda, t)$ is a witness tree constructed from exe-log Λ:

- initially, T is a single root with label Λ_t
- for $i = t\text{-}1, t\text{-}2, \ldots, 1$
  - if $\exists$ a vertex v in T with label $A_v \in \Gamma^+(\Lambda_i)$
    - add a new child u to the *deepest* such v and label it with Λ_i
- $T(\Lambda, t)$ is the resulting T

$T(\Lambda, s) \neq T(\Lambda, t)$ if $s \neq t$

$\mathcal{T}_A$: set of all witness trees with root-label A

$$\Longrightarrow \quad \mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

Moser-Tardos Algorithm:
sample all $X \in \mathscr{X}$;
while $\exists$ an occurring event $A \in \mathscr{A}$:
resample all $X \in \text{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A}$$

random sequence of resampled events

$T(\Lambda, t)$ is a witness tree constructed from exe-log Λ:

- initially, T is a single root with label Λ_t
- for $i = t\text{-}1, t\text{-}2, \ldots, 1$
    - if $\exists$ a vertex v in T with label $A_v \in \Gamma^+(\Lambda_i)$
        - add a new child u to the *deepest* such v and label it with Λ_i
- $T(\Lambda, t)$ is the resulting T

Lemma 1 For any particular witness tree τ:

$$\Pr[\exists t, T(\Lambda, t) = \tau] \leq \prod_{v \in \tau} \Pr[A_v]$$

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
resample all $X \in \mathrm{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathcal{A}$$

random sequence of resampled events

Lemma 1 For any particular witness tree τ:

$$\Pr[\exists t, T(\Lambda, t) = \tau] \le \prod_{v \in \tau} \Pr[A_v]$$

$N_A = |\{ i \mid \Lambda_i = A \}|$ total # of times that A is resampled

$$\mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

$\mathcal{T}_{\mathrm{A}}$: set of all witness trees with root-label A

(lemma 1) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \Pr[A_v]$$

Moser-Tardos Algorithm:
sample all $X \in \mathscr{X}$;
while $\exists$ an occurring event $A \in \mathscr{A}$:
resample all $X \in$ vbl(A);

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A}$$

random sequence of resampled events

LLL condition: $\exists \alpha : \mathcal{A} \to [0,1)$

$$\forall A \in \mathcal{A} : \quad \Pr[A] \le \alpha_A \prod_{B \in \Gamma(A)} (1 - \alpha_B)$$

$$\mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

$\mathcal{T}_\mathrm{A}$: set of all witness trees with root-label A

(lemma 1) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \Pr[A_v]$$

(*LLL* cond.) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha(B)) \right]$$

goal: $$\le \frac{\alpha_A}{1 - \alpha_A}$$

Moser-Tardos Algorithm:

sample all $X \in \mathscr{X}$;

while $\exists$ an occurring event $A \in \mathscr{A}$:

  resample all $X \in \mathsf{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A}$$

random sequence of resampled events

Lemma 1 For any particular witness tree τ:

$$\Pr[\exists t, T(\Lambda, t) = \tau] \leq \prod_{v \in \tau} \Pr[A_v]$$

$X_i^{(t)}$: t-th sampling of variable $X_i \in \mathcal{X}$

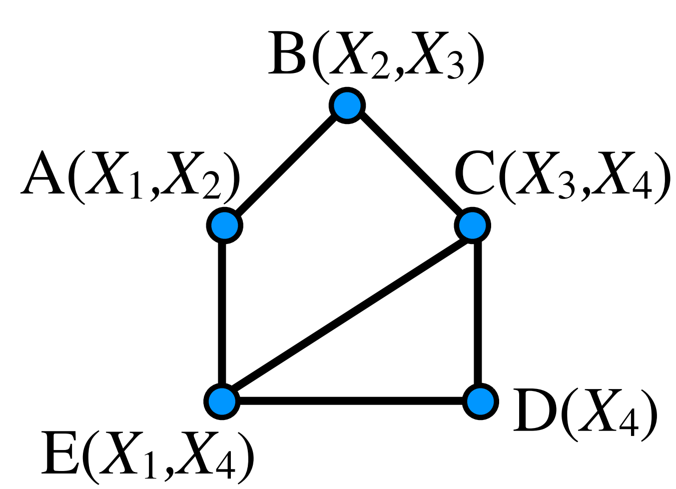
X_1 : $X_1^{(0)}, X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)}, \ldots$

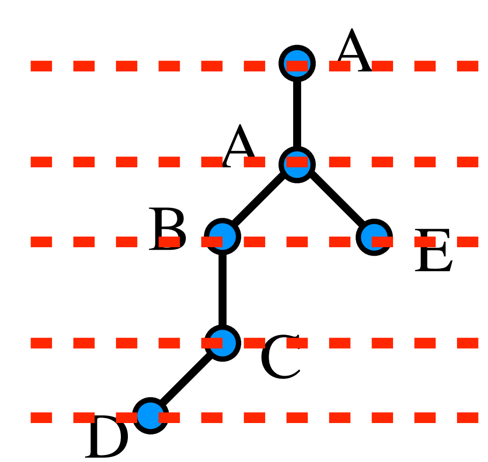
X_2 : $X_2^{(0)}, X_2^{(1)}, X_2^{(2)}, X_2^{(3)}, X_2^{(4)}, \ldots$

X_3 : $X_3^{(0)}, X_3^{(1)}, X_3^{(2)}, X_3^{(3)}, X_3^{(4)}, \ldots$

X_4 : $X_4^{(0)}, X_4^{(1)}, X_4^{(2)}, X_4^{(3)}, X_4^{(4)}, \ldots$

exe-log Λ: D,C,E,D,B,A,C,A,D, ...





Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
resample all $X \in \text{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathcal{A}$$

random sequence of resampled events

Lemma 1 For any particular witness tree τ:

$$\Pr[\exists t, T(\Lambda, t) = \tau] \le \prod_{v \in \tau} \Pr[A_v]$$

$N_A = |\{\, i \mid \Lambda_i = A \,\}|$ total # of times that A is resampled

$$\mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

$\mathcal{T}_{\mathrm{A}}$: set of all witness trees with root-label A

(lemma 1) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \Pr[A_v]$$

Moser-Tardos Algorithm:
sample all $X \in \mathscr{X}$;
while $\exists$ an occurring event $A \in \mathscr{A}$:
resample all $X \in \text{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A}$$

random sequence of resampled events

LLL condition: $\exists \alpha : \mathcal{A} \to [0,1)$

$$\forall A \in \mathcal{A}: \quad \Pr[A] \le \alpha_A \prod_{B \in \Gamma(A)} (1-\alpha_B)$$

$$\mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

$\mathcal{T}_{\text{A}}$: set of all witness trees with root-label A

(lemma 1) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \Pr[A_v]$$

(*LLL* cond.) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha(B)) \right]$$

goal: $$\le \frac{\alpha_A}{1-\alpha_A}$$

grow a random witness tree $T_A \in \mathcal{T}_A$:

- initially, T_A is a single root with label A
- for $i = 1, 2, \ldots$
  - for every vertex v at depth i (root has depth 1) in T_A
  - for every $B \in \Gamma^+(A_v)$:
    - add a new child u to v independently with probability α_B;
    - and label it with B;
- stop if no new child added for an entire level

inclusive neighborhood: $\Gamma^+(A) \triangleq \{B \in \mathscr{A} \mid \mathsf{vbl}(A) \cap \mathsf{vbl}(B) \neq \emptyset\}$

$= \Gamma(A) \cup \{A\}$

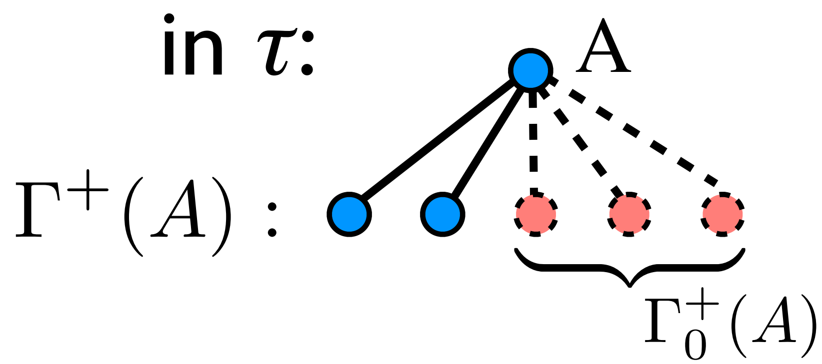
Lemma 2 For any particular witness tree $\tau \in \mathcal{T}_A$:

$$\Pr[T_A = \tau] = \frac{1-\alpha_A}{\alpha_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha_B) \right]$$

Lemma 2 For any particular witness tree $\tau \in \mathcal{T}_A$:

$$\Pr[T_A = \tau] = \frac{1-\alpha_A}{\alpha_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha_B) \right]$$

in τ:



$$\Pr[T_A = \tau] = \frac{1}{\alpha_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma_0^+(A_v)} (1 - \alpha_B) \right]$$

$$= \frac{1-\alpha_A}{\alpha_A} \prod_{v \in \tau} \left[\frac{\alpha(A_v)}{1-\alpha(A_v)} \prod_{B \in \Gamma^+(A_v)} (1 - \alpha_B) \right]$$

$$= \frac{1-\alpha_A}{\alpha_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha_B) \right]$$

Moser-Tardos Algorithm:
sample all $X \in \mathscr{X}$;
while $\exists$ an occurring event $A \in \mathscr{A}$:
resample all $X \in \mathrm{vbl}(A)$;

execution log Λ:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A}$$

random sequence of resampled events

LLL condition: $\exists \alpha : \mathcal{A} \to [0,1)$

$$\forall A \in \mathcal{A} : \quad \Pr[A] \le \alpha_A \prod_{B \in \Gamma(A)} (1 - \alpha_B)$$

$$\mathbf{E}[N_A] = \sum_{\tau \in \mathcal{T}_A} \Pr[\exists t, T(\Lambda, t) = \tau]$$

$\mathcal{T}_A$: set of all witness trees with root-label A

(lemma 1) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \Pr[A_v]$$

(*LLL* cond.) $$\le \sum_{\tau \in \mathcal{T}_A} \prod_{v \in \tau} \left[\alpha(A_v) \prod_{B \in \Gamma(A_v)} (1 - \alpha(B)) \right]$$

(lemma 2) $$\le \frac{\alpha_A}{1 - \alpha_A} \sum_{\tau \in \mathcal{T}_A} \Pr[T_A = \tau] \le \frac{\alpha_A}{1 - \alpha_A}$$

- mutually independent random variables: $\mathcal{X} \triangleq \{X_1, \ldots, X_n\}$
- bad events: $\mathcal{A} \triangleq \{A_1, A_2, \ldots, A_m\}$
- $\forall A \in \mathcal{A}$, $\mathsf{vbl}(A) \subseteq \mathcal{X}$: set of variables determining A
- neighborhood: $\forall A \in \mathcal{A}$, $\Gamma(A) \triangleq \{B \neq A \mid \mathsf{vbl}(A) \cap \mathsf{vbl}(B) \neq \emptyset\}$

Moser-Tardos Algorithm:
sample all $X \in \mathcal{X}$;
while $\exists$ an occurring event $A \in \mathcal{A}$:
    resample all $X \in \mathsf{vbl}(A)$;

Lovász Local Lemma (Moser-Tardos 2010):

$\exists \alpha : \mathcal{A} \to [0,1)$

$\forall A \in \mathcal{A}: \quad \Pr[A] \le \alpha_A \prod_{B \in \Gamma(A)} (1 - \alpha_B)$

$\Rightarrow$ a satisfying assignment is returned within $\sum_{A \in \mathcal{A}} \frac{\alpha_A}{1 - \alpha_A}$ resamples in expectation