Lab06_Report

申长硕 大数据学院 PB22020518

问题引入

背景

对于提高图像的分辨率或降低图像的噪声，FFT(Fast Fourier Transformation)是一种常见的方法。通过FFT将灰度像素模式的图像信息转换成频域信息，做进一步处理之后再通过逆变换将频域信息转换回图像信息。

实验目的

通过FFT于FFT_inv实现对给定函数的Fourier分析以及重建。

数学分析

FFT实现图像处理过程

FFT实现提高分辨率或降噪的过程一般是：

1. 实现FFT，将灰度图像转换为频域
2. 零域部分的可视化与集中
3. 应用低/高同滤波器过滤频率
4. 离散
5. 实现FFT逆变换生成图像数据

DFT与FFT

DFT是离散傅里叶变换，其计算可以直接表示为：

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{n-1} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} \rho^0 & \rho^0 & \rho^0 & \cdots & \rho^0 \\ \rho^0 & \rho^1 & \rho^2 & \cdots & \rho^{n-1} \\ \rho^0 & \rho^2 & \rho^4 & \cdots & \rho^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^0 & \rho^{n-1} & \rho^{2(n-1)} & \cdots & \rho^{(n-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

其中$\rho = e^{-j\frac{2\pi}{n}}$

其逆变换矩阵可表示为：

$$F_n^{-1}=\begin{bmatrix}\rho^0 & \rho^0 & \rho^0 & \cdots & \rho^0\\ \rho^0 & \rho^{-1} & \rho^{-2} & \cdots & \rho^{-(n-1)}\\ \rho^0 & \rho^{-2} & \rho^{-4} & \cdots & \rho^{-2(n-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \rho^0 & rho^{-(n-1)} & \rho^{-2(n-1)} & \cdots & \rho^{-(n-1)^2}\end{bmatrix}$$

快速傅立叶变换采用分而治之(Divide and conquer)策略。

逐次分半算法：若n=2^m为偶数，记 $\omega_n=e^{-i2\pi/n}$, 多项式 $p(z)=\frac{1}{n}\sum_{k=0}^{n-1}f_kz^k$,则

$$g_l=p(\omega_n^l),\quad l=0,1,\cdots,n-1,$$

即计算向量f的离散傅立叶变换等价于求多项式p(z)在n个点 $\{1,\omega_n,\omega_n^2,\cdots,\omega_n^{n-1}\}$ 处的值。

将$p(x)$的系数按偶次项和奇数项分开，构造多项式

$$p_0(z)=\frac{1}{m}\sum_{k=0}^{m-1}f_{2k}z^k,\quad p_1(z)=\frac{1}{m}\sum_{k=0}^{m-1}f_{2k+1}z^k,$$

则

$$p(z)=\frac{p_0(z^2)+zp_1(z^2)}{2}.$$

问题转化为求$p_0(z)$和$p_1(z)$在$\{1,\omega_n^2,\omega_n^4,\cdots,\omega_n^{2(n-1)}\}$的值。

利用单位根的性质有

$$\omega_n^{2k}=e^{-i\frac{2\pi}{n}2k}=e^{-i\frac{2\pi}{m}k}=\omega_m^k,\quad k=0,1,\cdots,n-1,$$

故前面的集合仅有m个不同的值，即$\{1,\omega_m,\omega_m^2,\cdots,\omega_m^{m-1}\}$.

对于$k=0,1,\cdots,m-1$，有

$$g_k=p(\omega_n^k)=\frac{p_0(\omega_n^{2k})+\omega_n^kp_1(\omega_n^{2k})}{2}=\frac{p_0(\omega_m^k)+\omega_n^kp_1(\omega_m^k)}{2},$$

$$g_{k+m}=p(\omega_n^{k+m})=\frac{p_0(\omega_n^{2k+n})+\omega_n^{k+m}p_1(\omega_n^{2k+n})}{2}=\frac{p_0(\omega_m^k)-\omega_n^kp_1(\omega_m^k)}{2}$$

FFT的实现过程

直接摆对应算法：

```
Algorithm FFT

n <- length[f]
if n == 1 then
return f
end if

ω_n <- e^(-i2π/n)
ω <- 1
```

```
f^0 <- (f_0, f_2, ..., f_(n-2))
f^1 <- (f_1, f_3, ..., f_(n-1))

g^0 <- FFT(f^0)
g^1 <- FFT(f^1)

for k <- 0 to n/2 - 1 do
g_k <- (g^0_k + ωg^1_k) / 2
g_(k+n/2) <- (g^0_k - ωg^1_k) / 2
ω <- ωω_n
end for

return g
```

```
Algorithm IFFT

n <- length[f]
if n == 1 then
return f
end if

ω_n <- e^(i2π/n)
ω <- 1

f^0 <- (f_0, f_2, ..., f_(n-2))
f^1 <- (f_1, f_3, ..., f_(n-1))

g^0 <- IFFT(f^0)
g^1 <- IFFT(f^1)

for k <- 0 to n/2 - 1 do
g_k <- g^0_k + ωg^1_k
g_(k+n/2) <- g^0_k - ωg^1_k
ω <- ωω_n
end for

return g
```

算法设计及代码实现

定义FFT_Visualization类

实现fft和ifft方法

```
def calc_fft(self):
    def fft(x):
        N = len(x)
        if N <= 1:
            return x
        # 分别计算偶数项和奇数项
        even = fft(x[::2])
```

```
            odd = fft(x[1::2])
            # 计算g_k
            T = [np.exp(-2j * np.pi * k / N) * odd[k] for k in range(N // 2)]
            return [(even[k] + T[k])/2 for k in range(N // 2)] + [(even[k] - T[k])/2
for k in range(N // 2)]

        self.F = fft(self.func_values)

def calc_ifft(self):
    def ifft(X):
        N = len(X)
        if N <= 1:
            return X
        # 分别计算偶数项和奇数项
        even = ifft(X[::2])
        odd = ifft(X[1::2])
        T = [np.exp(2j * np.pi * k / N) * odd[k] for k in range(N // 2)]
        return [(even[k] + T[k]) for k in range(N // 2)] + [(even[k] - T[k]) for
k in range(N // 2)]

    if not self.F:
        self.calc_fft()

    self.iF = ifft(self.F)
```

对应结果的可视化

fft

```
def fft_plot(self):
    if self.F is None:
        self.calc_fft()

    print("G's sequence:")
    print([(np.real(num), np.imag(num)) for num in self.F])

    # 创建一个图像，只包含三维图像
    fig1 = plt.figure(figsize=(12, 12))
    ax1 = fig1.add_subplot(111, projection='3d')
    ax1.plot(range(len(self.F)), np.real(self.F), np.imag(self.F), label='Complex
Sequence', color='green')
    ax1.plot(range(len(self.F)), np.real(self.F), np.zeros_like(self.F),
label='Real Projection', color='blue')
    ax1.plot(range(len(self.F)), np.zeros_like(self.F), np.imag(self.F),
label='Imaginary Projection', color='red')
    ax1.set_title('FFT of $f_1(t)$ (n = {}) - 3D Plot'.format(self.size))
    ax1.set_xlabel('Index')
    ax1.set_ylabel('Real Part')
    ax1.set_zlabel('Imaginary Part')
    ax1.legend()
    plt.show()

    # 再创建一个图像，只包含基本图像
    fig2 = plt.figure(figsize=(12, 9))

    plt.subplot(3, 1, 1)
    plt.plot(np.abs(self.F), label='Absolute', color='blue')
```

```
    plt.title('FFT of $f_1(t)$ (n = {}) - Real Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Real Part')
    plt.legend()

    plt.subplot(3, 1, 2)
    plt.plot(np.real(self.F), label='Real', color='blue')
    plt.title('FFT of $f_1(t)$ (n = {}) - Real Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Real Part')
    plt.legend()

    plt.subplot(3, 1, 3)
    plt.plot(np.imag(self.F), label='Imaginary', color='red')
    plt.title('FFT of $f_1(t)$ (n = {}) - Imaginary Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Imaginary Part')
    plt.legend()

    plt.tight_layout()
    plt.show()
```

ifft

```
def ifft_plot(self):
    if not self.iF:
        self.calc_ifft()

    # 绘制逆 FFT 结果的实部和虚部
    plt.figure(figsize=(12, 6))

    plt.subplot(2, 1, 1)
    plt.plot(np.real(self.iF), label='Real', color='blue')
    plt.title('IFFT of FFT of $f_1(t)$ (n = {}) - Real Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Real Part')
    plt.legend()

    plt.subplot(2, 1, 2)
    plt.plot(np.imag(self.iF), label='Imaginary', color='red')
    plt.title('IFFT of FFT of $f_1(t)$ (n = {}) - Imaginary
Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Imaginary Part')
    plt.legend()

    plt.tight_layout()
    plt.show()
```

误差

```
def calc_error(self):
    '''计算逆 FFT 结果和原始函数之间的误差'''
    if not self.iF:
        print("Please perform inverse FFT first.")
        return
```

```python
        func_values_ifft = np.real(self.iF)

        # 计算绝对误差
        self.absolute_errors = [abs((self.func_values[i] - func_values_ifft[i])) for
i in range(self.size)]

    def plot_errors(self, tolerance=1e-6):
        '''
        绘制误差随着索引的变化
        '''
        if not self.absolute_errors:
            self.calc_error()

        max_error = np.max(self.absolute_errors)
        print(f'max absolute error: {max_error}')
        if max_error < tolerance:
            print("Inverse FFT result is close to original function.")
        else:
            print("Inverse FFT result is not close to original function.")

        # 绘制误差图像
        plt.figure(figsize=(10, 6))
        plt.plot(range(self.size), self.absolute_errors, color='blue',
label='Absolute Error')
        plt.axhline(y=tolerance, color='red', linestyle='--', label='Tolerance')
        plt.title('Errors of Inverse FFT Result')
        plt.xlabel('Index')
        plt.ylabel('Absolute Error')
        plt.legend()
        plt.grid(True)
        plt.show()
```

低通滤波

```python
    def low_pass_filter(self, ratio=0.25):
        """
        对频域信号进行低频滤波，保留前 ratio 比例的频率分量，其余置零。
        """
        if not self.F:
            self.calc_fft()

        num_freq_components = len(self.F)
        num_components_to_keep = int(ratio * num_freq_components)

        self.filtered_F = np.zeros_like(self.F)
        self.filtered_F[:num_components_to_keep] = self.F[:num_components_to_keep]

        # return filtered_F

    def low_pass_filter_and_plot(self, ratio=0.25):
        """
        对频域信号进行低频滤波并绘制滤波后的结果。
        """
        self.low_pass_filter(ratio)

        # 绘制 FFT 结果的实部和虚部
        plt.figure(figsize=(12, 6))
```

```
    plt.subplot(2, 1, 1)
    plt.plot(np.real(self.filtered_F), label='Real', color='blue')
    plt.title('Filtered FFT of $f_1(t)$ (n = {}) - Real Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Real Part')
    plt.legend()

    plt.subplot(2, 1, 2)
    plt.plot(np.imag(self.filtered_F), label='Imaginary', color='red')
    plt.title('Filtered FFT of $f_1(t)$ (n = {}) - Imaginary
Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Imaginary Part')
    plt.legend()

    plt.tight_layout()
    plt.show()
```

低通后ifft

```
def filter_ifft(self):
    """
    计算滤波后的逆 FFT。
    """
    if self.filtered_F is None:
        self.low_pass_filter()
        print("Please perform low-pass filtering first.")

    def ifft(X):
        N = len(X)
        if N <= 1:
            return X
        # 分别计算偶数项和奇数项
        even = ifft(X[::2])
        odd = ifft(X[1::2])
        T = [np.exp(2j * np.pi * k / N) * odd[k] for k in range(N // 2)]
        return [(even[k] + T[k]) for k in range(N // 2)] + [(even[k] - T[k]) for
k in range(N // 2)]

    self.filtered_iF = ifft(self.filtered_F)

def plot_filtered_ifft(self):
    """
    绘制滤波后的逆 FFT 结果的实部和虚部。
    """
    if self.filtered_iF is None:
        self.filter_ifft()

    # 绘制逆 FFT 结果的实部和虚部
    plt.figure(figsize=(12, 6))

    plt.subplot(2, 1, 1)
    plt.plot(np.real(self.filtered_iF), label='Real', color='blue')
    plt.title('Filtered IFFT of Filtered FFT of $f_1(t)$ (n = {}) - Real
Part'.format(self.size))
```

```python
    plt.xlabel('Index')
    plt.ylabel('Real Part')
    plt.legend()

    plt.subplot(2, 1, 2)
    plt.plot(np.imag(self.filtered_iF), label='Imaginary', color='red')
    plt.title('Filtered IFFT of Filtered FFT of $f_1(t)$ (n = {}) - Imaginary
Part'.format(self.size))
    plt.xlabel('Index')
    plt.ylabel('Imaginary Part')
    plt.legend()

    plt.tight_layout()
    plt.show()
```

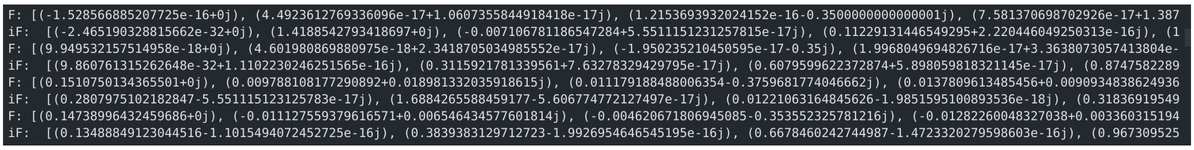
分析与思考

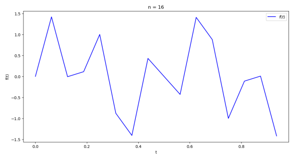
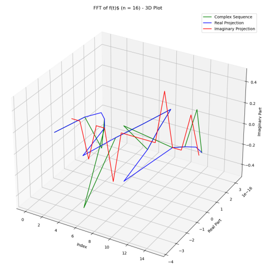
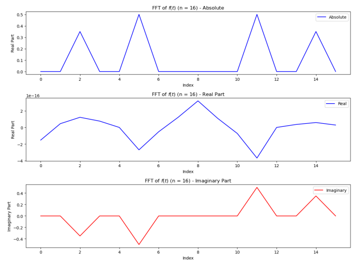
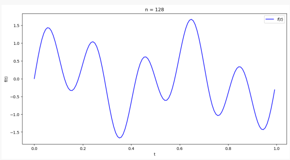
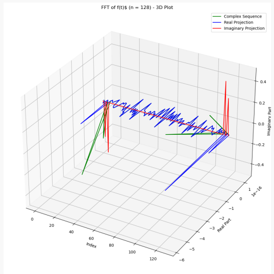
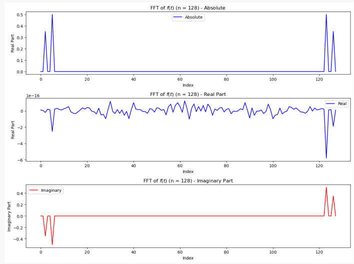
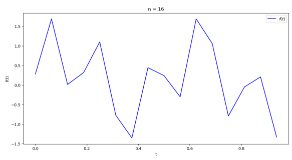
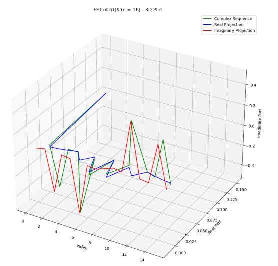
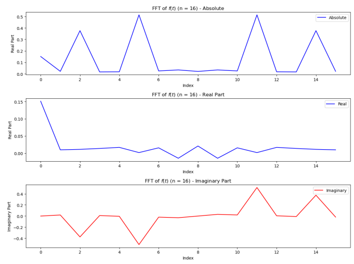
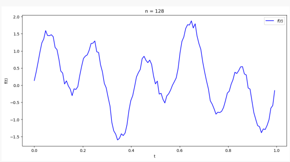
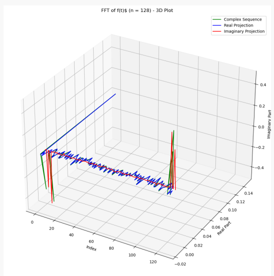
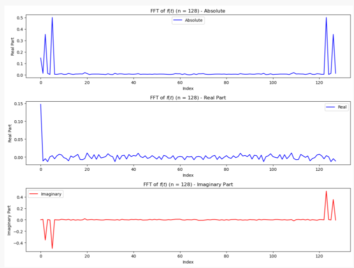
输出结果

FFT的结果

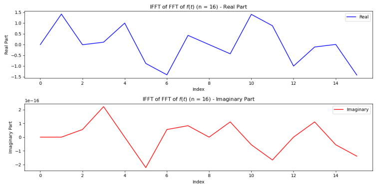
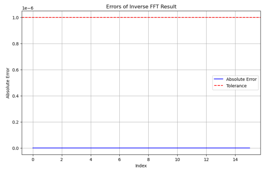
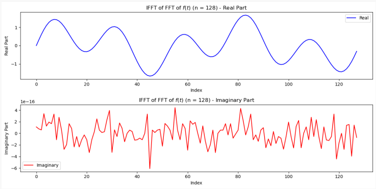
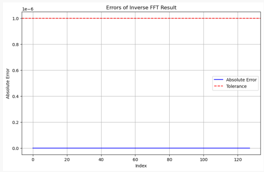
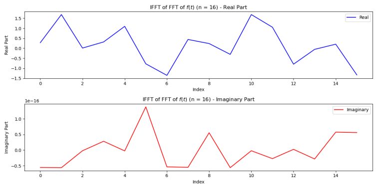
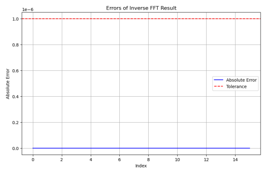
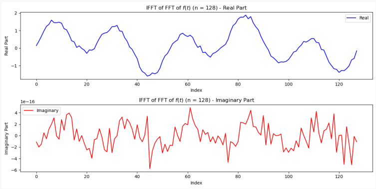
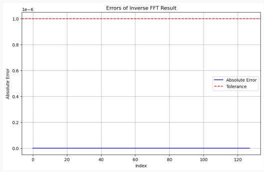
四组计算出的结果：

由于数据实在太多，所以仅截屏打印，具体可见`run.ipynb`，主要实现下面的可视化，而且已经使用绝对误差进行验证最终结果是正确的

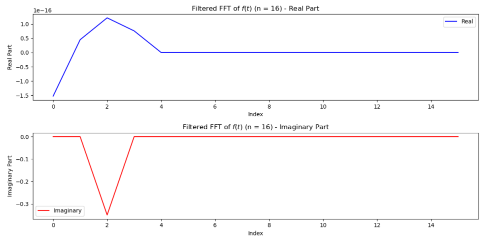
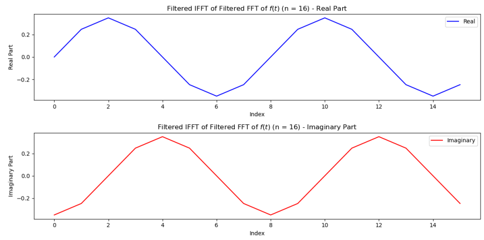
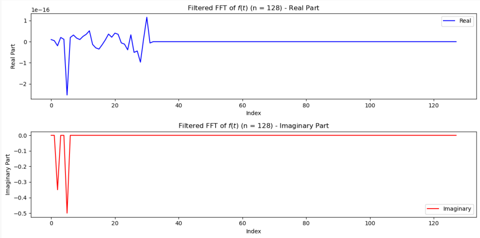
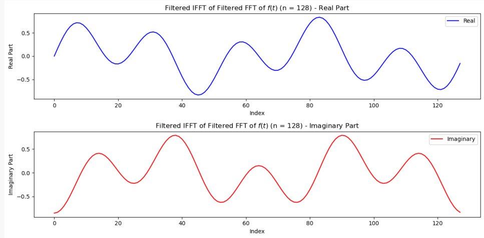
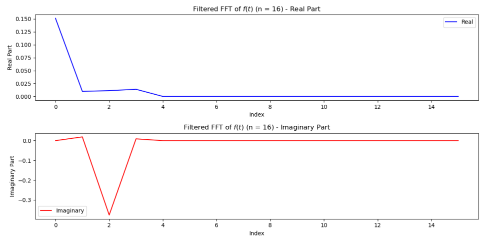
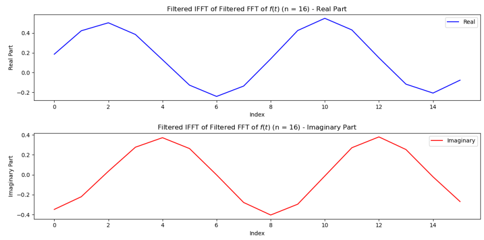
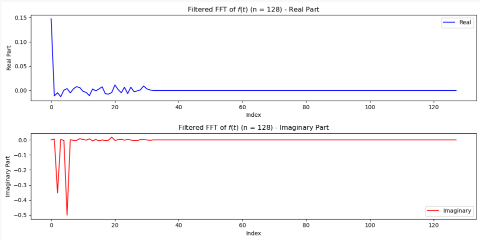
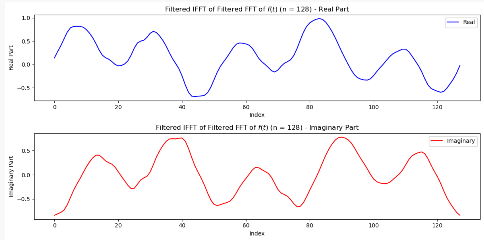

```
F: [(-1.528566885207725e-16+0j), (4.4923612769336096e-17+1.0607355844918418e-17j), (1.2153693932024152e-16-0.35000000000000001j), (7.581370698702926e-17+1.387
iF:  [(-2.465190328815662e-32+0j), (1.4188542793418697+0j), (-0.007106781186547284+5.5511151231257815e-17j), (0.11229131446549295+2.220446049250313e-16j), (1
F: [(9.949532157514958e-18+0j), (4.601980869880975e-18+2.3418705034985552e-17j), (-1.950235210450595e-17-0.35j), (1.9968049694826716e-17+3.3638073057413804e-
iF:  [(9.860761315262648e-32+1.1102230246251565e-16j), (0.3115921781339561+7.63278329429795e-17j), (0.6079599622372874+5.898059818321145e-17j), (0.8747582289
F: [(0.1510750134365501+0j), (0.009788108177290892+0.018981332035918615j), (0.011179188488006354-0.3759681774046662j), (0.0137809613485456+0.0090934838624936
iF:  [(0.2807975102182847-5.551115123125783e-17j), (1.6884265588459177-5.606774772127497e-17j), (0.01221063164845626-1.9851595100893536e-18j), (0.31836919549
F: [(0.14738996432459686+0j), (-0.011127559379616571+0.006546434577601814j), (-0.004620671806945085-0.353552325781216j), (-0.01282260048327038+0.003360315194
iF:  [(0.13488849123044516-1.1015494072452725e-16j), (0.3839383129712723-1.9926954646545195e-16j), (0.6678460242744987-1.4723320279598603e-16j), (0.967309525
```

函数	函数图像	FFT总图像	绝对值及各部分图像
f_{1*n1}			
f_{1*n2}			
f_{2*n1}			
f_{2*n2}			

IFFT的结果

函数	ifft结果	误差	绝对误差最大值
f_{1*n1}			2.220446049250313e-16
f_{1*n2}			4.440892098500626e-16
f_{2*n1}			2.220446049250313e-16
f_{2*n2}			4.996003610813204e-16

低通滤波

函数	低通fft	低通ifft
f_{1*n1}		
f_{1*n2}		
f_{2*n1}		
f_{2*n2}		

总结与分析

分析总结

1. 本实验主要实现：
    1. FFT和IFFT的函数实现
    2. 快速傅里叶变换及反变换过程中的可视化
    3. 低通滤波正反傅里叶变换及其可视化
2. 经过观察：
    1. FFT是将时域上的信号反映到频域上，方便进行后续处理
    2. 反变换则与之相反
    3. 低通滤波就是直接按某比例将高频的幅频置为0
    4. 经过观察，低通之后的反傅里叶得到的结果相较原函数相对更加光滑
    5. 其实也就是留下低频的周期函数分量，去掉小周期的周期函数从而去掉“毛刺”
    6. 经过实际观察，使用低通滤波却是可以实现这样的效果