The Atiyah-Singer index theorem for Dirac Operators

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Chapter 1

Review of Riemannian manifolds

1.1 Connections and curvature

1.1.1 Connection on $\mathbb{R}^n$

Definition 1. *A* **linear connection** ∇^{LC} *on* $\mathbb{R}^n$ *is a map in*

$$C^\infty(\mathbb{R}^n, T\mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \wedge^1 \otimes T\mathbb{R}^n)$$

such that the image is linear. We also require the following are satisfied:

$$\forall a, b \in \mathbb{R} : \nabla^{LC}(av + bw) = a\nabla^{LC}v + b\nabla^{LC}w \tag{1.1}$$

$$\forall f \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n) : \nabla^{LC}(fv) = df \otimes v + f\nabla^{LC}v \tag{1.2}$$

Remark 1. *Here* $\wedge$ *stands the alternating form bundle. Namely the quotient bundle of* $T^*\mathbb{R}^n$ *with* $x_i \wedge x_j = -x_j \wedge x_i$ *for* $i \neq j$.

Remark 2. *In other standard texts (for example John M Lee's Riemannian Manifolds, page 49), they give equivalent definition by defining it as a map in*

$$T\mathbb{R}^n \times T\mathbb{R}^n \to T\mathbb{R}^n$$

the reader can freely translate between the two language by noticing

$$\nabla^{LC}(v)(u) = \nabla_u(v)$$

Theorem 1. ∇^{LC} *is a linear connection if and only if there exist*

$$\omega \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n) \to C^\infty(\mathbb{R}^n, End(T\mathbb{R}^n) \otimes T^*\mathbb{R}^n)$$

such that $\nabla^{LC} = d + \omega$. *Here* $d(v_1, \cdots v_n) = (dv_1, \cdots dv_n)$.

Proof. This non-intuitive proof is mainly serve to introduce the concept of Christoffel symbols and enable the reader to see how the connection works in coordinate charts. We shall supplement a second proof.

Assume $T\mathbb{R}^n$ has a basis E_j, we use the defining rules for a connection and compute(here we use the summation convention):

$$\begin{aligned}\nabla^{LC}(X)(Y) &= \nabla^{LC}(X_iE_i)(Y_jE_j)\\ &= Y_jE_jX_iE_i + X_i\nabla^{LC}(E_i)(Y_jE_j)\\ &= Y_jE_jX_iE_i + X_iY_j\nabla^{LC}(E_i)(E_j)\\ &= Y_jE_jX_iE_i + X_iY_j\Gamma_{ij}^kE_k\end{aligned}$$

where in the first step we used the fact $E_j = \partial_j$ in de Rham cohomology:

$$\begin{aligned}dX_i \otimes \partial_i[Y] &= [\partial_jX_idx_jY]\partial_i\\ &= [Y_j\partial_jX_i]\partial_i\\ &= Y_jE_jX_iE_i\end{aligned}$$

Here Γ_{ij}^k (the Christoffel symbols) is defined to be the value of $\nabla^{LC}(E_i)(E_j)$. Rearranging the index, we have

$$\nabla^{LC}(X)(Y) = [Y_j\partial_jX_k + X_iY_j\Gamma_{ij}^k]\partial_k \tag{1.3}$$

Now we have:

$$d(X)(Y) = [dX_iE_i]Y = [\partial_jX_id_jY_j\partial_j]\partial_i = [\partial_jX_iY_j]\partial_i = Y_jE_jX_iE_i \tag{1.4}$$

which is precisely the first part of the 1.3. Note for fixed $\Gamma_{ij}^k \in C^\infty(\mathbb{R}^n)$, we can define $\omega(X)(Y)$ as a 1-form by

$$\omega(X)(Y)_k = X_iY_j\Gamma_{ij}^k \tag{1.5}$$

This finishs the proof. □

REMARK 3. *Here we introduce a conceptually more elegant and indeed better proof.*

Proof. We want to show if there exists two connections ∇, ∇', then the difference $\omega = \nabla - \nabla'$ is in $End(T\mathbb{R}^n)$. On the other hand, given a connection ∇, for all $\omega \in End(T\mathbb{R}^n)$ we claim $\nabla' = \nabla + \omega$ is a connection on $\mathbb{R}^n$.

To see how this proved the theorem, we observed that $\mathbb{R}^n$ is equipped with a canonical connection d, such that

$$\nabla_Y(X) = \nabla(X)(Y) = d(X)(Y) = YX_iE_i = YX$$

as we had verified earlier. We verify the two properties:

- $d(aX_1 + bX_2) = ad(X_1) + bd(X_2)$
- $d(fX) = \frac{\partial f}{\partial x_i} dx_i \otimes X + f\partial_i X dx_i = df \otimes X + fd(X)$

So it suffice to verify that for any $Y \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$ and given ∇, $(\nabla - d) \in C^\infty(\mathbb{R}^n, End(T\mathbb{R}^n) \otimes T^*\mathbb{R}^n)$. In other words, the space of connections is an affine space over the space of sections: $C^\infty(\mathbb{R}^n, End(T\mathbb{R}^n) \otimes T^*\mathbb{R}^n)$.

We know both ∇ and d satisfies the first condition, so $\nabla - d$ must satisfy it as well. To check it does belong to the space of sections we verify:

$$\begin{aligned}(d - \nabla)(fX)(Y) &= df(Y) \otimes X + f\nabla(X)(X) - df(Y) \otimes X - fd(X)\\ &= f(\nabla - d)(X)(Y)\end{aligned}$$

Therefore $\nabla - d$ is indeed in $C^\infty(\mathbb{R}^n, End(T\mathbb{R}^n) \otimes T^*\mathbb{R}^n)$, where it evaluates X by $End(T\mathbb{R}^n)$ factor and produce a tangent vector at x, and evaluates Y produce a number in $\mathbb{R}$. This finished the proof. □

REMARK 4. *Here we do not define it formally, but in general the property of ω such that $\omega(fX) = f\omega(X)$ means ω is* **tensoral** *as a 1-form.*

We also wish to point out that in general when M is a manifold, ω is not just a section of the endomorphism bundle but will involve choice of coordinate charts as well.

DEFINITION 2. *We extend the concept of connection to that of an* **exterior covariant derivative***, which is a map:*

$$d^\nabla : \wedge^p(\mathbb{R}^n, T\mathbb{R}^n) \to \wedge^{p+1}(\mathbb{R}^n, T\mathbb{R}^n)$$

that satisfies Leibniz rule:

$$d^\nabla(u \wedge v) = d^\nabla u \wedge v + (-1)^{\deg u} u \wedge dv$$

THEOREM 2. *Let ∇ be a connection. There is a unique exterior covariant derivative that extends ∇.*

Proof. Here extend means $d^\nabla(\sigma)X = \nabla_X \sigma$ for every vector field X and every $\sigma \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$. We may proceed inductively assuming we already defined it on degree $p - 1$ forms, and extend to degree p forms by the Leibniz rule. The resulting exterior derivative is clearly unique.

DEFINITION 3. *The* **curvature** *of ∇^{LC} is defined to be*

$$R_\nabla = \nabla \cdot \nabla \in C^\infty(\mathbb{R}^n, \wedge^2 \otimes End[T\mathbb{R}^n])$$

Lemma 1. *For any $v \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$ we have:*

$$\begin{aligned}R_\nabla(v) &= \nabla(\nabla v)\\ &= (d+\omega)(d+\omega)(v)\\ &= (d+\omega)(dv+\omega v)\\ &= ddv + d(\omega v) + \omega(dv) + (\omega \wedge \omega)v\\ &= 0 + d\omega v - \omega dv + \omega dv + (\omega \wedge \omega)v\\ &= d\omega(v) + \omega \wedge \omega(v)\end{aligned}$$

Remark 5. *We used the formula*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$$

where α is a 1-form.

Remark 6. *Writing out in coordinates we have:*

$$R_\nabla(v) = d\omega_{ij}(v) + \omega_{ik} \wedge \omega_{kj}(v)$$

where $d\omega_{ij}$ and $\omega_{ik} \wedge \omega_{kj}$ are $n \times n$ matrices.

Lemma 2. *For all $e, u, v \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$ we have:*

$$R_\nabla(e)(u \wedge v) = \nabla_u \nabla_v(e) - \nabla_v \nabla_u(e) - \nabla_{[u,v]}(e) \tag{1.6}$$

Remark 7. *We switch back to traditional form for the sake of brevity. In our form we have the equation to be:*

$$\nabla(\nabla(e)(v))(u) - \nabla(\nabla(e)(u))(v) + \nabla(e)[u,v]$$

after we proved Lemma 2 we shall use 1.6 as the definition of R_∇.

Proof. Similar to what we did in Lemma 1 we have for any 1-form σ:

$$d^\nabla \sigma(u \wedge v) = d^\nabla(\sigma \cdot v)u - d^\nabla(\sigma \cdot u)v - \sigma[u,v] \tag{1.7}$$

We apply it twice to e:

$$R_\nabla(e)(u \wedge v) = d^\nabla(d^\nabla e)(u \wedge v) = d(de)(u \wedge v) \tag{1.8}$$

By 1.7 with $de = \sigma$ we thus have:

$$R_\nabla(e)(u \wedge v) = d(de \cdot v)u - d(de \cdot u)v - (de)[u,v]$$

Using the definition of the exterior covariant derivative we have:

$$de \cdot u = \nabla_u(e), d(de \cdot v)u = \nabla_u \nabla_v(e)$$

Similarly the other terms correspond to $\nabla_v \nabla_u(e)$ and $\nabla_{[u,v]}(e)$ as well. This finished the proof. □

1.1.2 Connection on Riemannian manifold

DEFINITION 4. *Let $E \to M$ be a vector bundle. A* **connection** *is*

$$\nabla : C^{\infty}(M, E) \to C^{\infty}(M, \wedge^1 \otimes E) \tag{1.9}$$

such that

$$\nabla(fe) = df \otimes e + f \otimes e \tag{1.10}$$

REMARK 8. *If $U \subset M$ can we just restrict ∇_M to be ∇_U by considering*

$$\nabla_U = \chi_U \nabla_M?$$

LEMMA 3. *Let U be a coordinate patch on which E is trivial. Let $e_i : U \to E$ be the local trivilizations. Let $\tau_i : U \to E^*$ be the dual basis. We claim: If $e \in C_c^{\infty}(U, E)$, then*

$$\nabla e \in C_c^{\infty}(u, \wedge^1 \otimes E)$$

if we let

$$\nabla = \tau^{-1}(d + \omega)\tau, \tau \in C^{\infty}(M, E_U^*)$$

Proof. Let $V \subset U$ such that $\overline{V} \subset U$ be compact. Let $\phi \in C_c^{\infty}(U)$ with $\phi \equiv 1$ on V. then we have

$$\phi e_i \in C^{\infty}(M, E)$$

By the definition of connection we have:

$$\nabla(e) = d\phi \otimes e + \omega_{ij}\phi_j \otimes e_i = (d +^{\phi} \omega)e, \forall e \in C_c^{\infty}(V, E)$$

Here of course $d\phi = 0$, so the connection form is controlled by ω. We claim that actually ${}^{\phi}\omega|_V$ does not depend on the 'cut-off' function ϕ at all. This is intuitively clear since the ϕ_j are all locally 1, so any two ϕ, ϕ' must produce the same ${}^{\phi}\omega =^{\phi'} \omega$ as desired on their intersection.

Therefore we may consider a compact exhaustion of U, which always exists for locally compact connected Hausdauff spaces. Then we want to define $\nabla^U(e)_p \in \nabla_p^1 \otimes E_p$. Motivated by the above computation, we defined it by:

$$\nabla^U e_j = \omega_{ij} \otimes e_j \tag{1.11}$$

and let it be extend to the sections by Leibniz rule:

$$\nabla^U(fe) = df_i \otimes e_i + \omega_{ij} f_j \otimes e_i \tag{1.12}$$

Then ∇ satisfies the requirements we put up earlier. Note in this way we defined ∇^U by

$$\nabla = \tau^{-1}(d + \omega)\tau, \tau \in C^{\infty}(M, E_U^*)$$

as the reader can readily check by plug in basis elements e_i. ∎

Definition 5. *Let $s \in C^\infty(M, \wedge^1 \otimes E)$, U a coordinate chart as before. If we have*

$$s|_U = s_i \otimes e_i, s_i \in C^\infty(U, \wedge^1)$$

Then we may define

$$\nabla : C^\infty(M, \wedge^1 \otimes E) \to C^\infty(M, \wedge^2 \otimes E)$$

by

$$\nabla(s) = ds_i \otimes e_i - s_i \wedge \nabla e_i$$

Note here $\nabla e_i = \omega_{ki} e_i$.

Lemma 4. *We claim that the ∇ defined in the previous lemma*

$$\nabla = \tau^{-1}(d + \omega)\tau, \tau \in C^\infty(M, E_U^*)$$

satisfies

$$R_\nabla = \nabla^2 = \tau^{-1}(d\omega + \omega \wedge \omega)\tau$$

such that

$$R_\nabla \in C^\infty(M, \wedge^2 \otimes End(E))$$

Proof. The proof is essentially similar to our proof of lemma 1. So we skip at here. □

Definition 6. *Let $E = TM$ where M is a Riemannian Manifold, then we claim there exist ∇ such that $\forall u \in C^\infty(M, TM)$ we have:*

$$u\langle v, w\rangle = \langle \nabla(v)(u), w\rangle + \langle v, \nabla(w)(u)\rangle$$

In other words we have:

$$\nabla(w)(v) - \nabla(v)(w) = [v, w]$$

This is the well known **Levi-Civita** *connection.*

Remark 9. *We view $\nabla_u(v) = \iota_u(\nabla v)$. Here*

$$\iota_u : C^\infty(M, \wedge^1 \otimes TM) \to C^\infty(M, TM)$$

Our goal is from $\omega = [\omega_{ij}]$, such that $\nabla = \tau^{-1}(d + \omega)\tau$, there is a unique ω satisfies:

- $\omega^T = -\omega$.
- $d\tau = -\omega \wedge \tau$.

Note that by lemma 1 we have:

$$d\tau(v, w) = v\tau(w) - w\tau(v) - \tau[v, w]$$

1.2 Introducing Riemannian Curvature

Discussion. *Let* $\nabla : C^\infty(M, E) \to C^\infty(M, \wedge^1 \otimes E)$ *be a connection on* M.

Lemma 5. *If* $U \subset M$ *and* $e \in C_c^\infty(M, E)$*, then*

$$\nabla(e) \in C_c^\infty(M, \wedge^1 \otimes E) \tag{1.13}$$

Lemma 6. *Given any open set* $\mathcal{U} \subset M$*, there exist a connection*

$$\nabla_U : C^\infty(U, E) \to C^\infty(U, \wedge^1 \otimes E) \tag{1.14}$$

Theorem 3. *Let* $\mathcal{U}$ *be an open set over which* E *is trivial. Let* $e_1, e_2 \cdots e_N$ *be the set of chosen trivilizations. Let* $\tau_1, \tau_2 \cdots \tau_N$ *be the dual basis. Then there exist a matrix of one forms* $\omega = [\omega_{ij}]$ *such that*

$$\nabla^{\mathcal{U}} = \tau^{-1}(d + \omega)\tau \tag{1.15}$$

Definition 7. *We define* $\nabla : C^\infty(M, \wedge^1 \otimes E) \to C^\infty(M, \wedge^2 \otimes E)$.

Lemma 7. *we have:*

$$R_\nabla = \nabla^2 \tag{1.16}$$

and

$$R_\nabla = \tau^{-1}(d\omega + \omega \wedge \omega)\tau \tag{1.17}$$

Here $\mathbb{R}_\nabla \in C^\infty(M, \wedge^2 \otimes Hom(E))$.

Lemma 8. *Let* $E = TM$ *with a Riemannian metric. If we have* ∇ *such that*

$$u\langle v, \omega\rangle = \langle \nabla_u v, \omega\rangle + \langle v, \nabla_u \omega\rangle \tag{1.18}$$

Then we have

$$\nabla_v \omega - \nabla_\omega v = [v, \omega] \tag{1.19}$$

Execrise 1. *A connection* ∇ *on* TM *is* **compatible** *with the metric* g *over* $\mathcal{U} \subset M$ *if and only if*

$$\omega^T = -\omega \tag{1.20}$$

Remark 10. *We have* $d\tau = \omega \wedge \tau$ *in this case.*

Remark 11. *The remark at here needs a fix.*

THEOREM 4. *There exists a unique connection on TM that is compatible with g and is torsion free. This is the* **Levi-Civita** *connection.*

Proof. We define on a coordinate patch

$$\nabla = \tau^{-1}(d + \omega)\tau \tag{1.21}$$

REMARK 12. *The proof at here needs a fix.*

Example 1. *Let M be a 2-dimensional surface. We fix an orthonormal frame e_1, e_2 with τ_1, τ_2 its dual. We set up ω as*

$$\begin{Bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{Bmatrix}$$

where $\omega_{ij} = -\omega_{ji}$. This force ω to be:

$$\begin{Bmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{Bmatrix}$$

Recall we have

$$d\tau_1 = -\omega_{12} \wedge \tau_2 \tag{1.22}$$

$$d\tau_2 = \omega_{12} \wedge \tau_1 \tag{1.23}$$

So in order to find the Levi-Civita connection, we just need to solve ω_{12}! Calculating it we have

$$\tau R \tau^{-1} = d\omega + \omega \wedge \omega$$

But since ω consists of ω_{12} only, $\omega \wedge \omega = 0$. We thus have:

$$d\omega = \begin{Bmatrix} 0 & d\omega_{12} \\ -d\omega_{12} & 0 \end{Bmatrix} = K \begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix} \tau_1 \wedge \tau_2; d\omega_{12} = K\tau_1 \wedge \tau_2 \tag{1.24}$$

Then we have

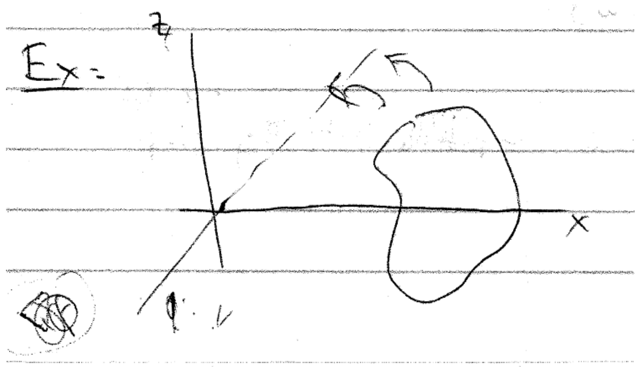
$$R = K \begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix} \tau_1 \wedge \tau_2; \tag{1.25}$$

Execrise 2. *Show that K is well defined (does not depend on the orthonormal frame we chosen earlier).*

Example 2. *Let M be a two dimensional 'torus' formed by rotating the curve*

$$x = f(\phi) > 0, y = g(\phi) \tag{1.26}$$

around the z-axis. Here is a graph:



We thus lead to the following parametrization:

$$x = f(\phi)\cos(\theta), y = f(\phi)\sin(\theta), z = g(\phi) \tag{1.27}$$

This space is a Riemannian manifold. Recall we have the metric inherited from $\mathbb{R}^3$*:*

$$g = dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{1.28}$$

While by our parametrization we have:

$$dx = f'(\phi)\cos(\theta)d\phi - f(\phi)\sin(\theta)d\theta \tag{1.29}$$

$$dy = f'(\phi)\sin(\theta)d\phi + f(\phi)\cos(\theta)d\theta \tag{1.30}$$

$$dz = g'(\phi)d\phi \tag{1.31}$$

Combing together we have:

$$g_M = (f'(\phi)^2 + g'(\phi)^2)d\phi \otimes d\phi + f(\phi)^2 d\theta \otimes d\theta \tag{1.32}$$

Rearranging by letting

$$\tau_1 = \sqrt{f'(\phi) + g'(\phi)}d\phi, \tau_2 = f(\phi)d\theta$$

we claim the above is equal to

$$\tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 \tag{1.33}$$

To find the Levi-Civita connection we have to solve (1.22) and (1.23):

$$d\tau_1 = -\omega_{12} \wedge \tau_2, d\tau_2 = \omega_{12} \wedge \tau_1$$

for ω_{12}.

Since M is a 2-dimensional manifold, we know $d\tau_1 = 0$. If we write

$$\omega_{12} = ad\phi + bd\theta$$

This implies:

$$d\tau_1 = -\omega_{12} \wedge \tau_2 = (ad\phi + bd\theta) \wedge (f'(\phi)d\theta) = 0 \tag{1.34}$$

When we expand it we find $\omega_{12} = bd\theta$ only. Plug into $d\tau_2 = \omega_{12} \wedge \tau_1$ we have:

$$d\tau_2 = F'(\phi)d\phi \wedge d\theta = bd\theta \wedge \sqrt{f'(\phi)^2 + g'(\phi)}d\phi \tag{1.35}$$

Resolving b, we conclude

$$\omega_{12} = -\frac{f'(\phi)}{\sqrt{f'(\phi)^2 + g'(\phi)^2}} \tag{1.36}$$

Remark 13. *The negative sign here is questionable.*
Recall $d\omega_{12} = K\tau_1 \wedge \tau_2$ as we derived in (1.24). We thus conclude:

$$K = \frac{g'(\phi)^2 + f''(\phi) + f'(\phi)g'(\phi)}{f(\phi)(f'(\phi)^2 + g'(\phi)^2)^2} \tag{1.37}$$

Remark 14. *The computation at here looks fishy, I do not think it coincide with my computation.*

1.3 Neat application of Levi-Civita connection

Theorem 5.

$$R(u,v)\omega + R(v,\omega)u + R(\omega,u)v = 0$$

Discussion. *Recall that $\nabla^{LC} = d + \omega$, where $\omega^T = -\omega$, $d\tau = -\omega \wedge \tau$, and $\tau = (\tau_1 \cdots \tau_n)$, τ_js are the duals of an orthogonal trivialization of TM.*

So we have

$$0 = -d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)\omega \wedge d\tau$$

Rearranging and using $d\tau = -\omega \wedge \tau$ we have

$$0 = -d\omega \wedge \tau - \omega \wedge \omega \wedge tau = -(d\omega + \omega \wedge \omega) \wedge \tau = -R \wedge \tau$$

Therefore we conclude that

$$R \wedge \tau = 0$$

Theorem 6. *We have*

$$R(u,v,\omega,z) = \langle R(u,v)\omega, z\rangle$$

Example 3. *Let* $\dim M = 2$*, then we have*

$$\tau R \tau^{-1} = K \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau_1 \wedge \tau_2$$

Here we have

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \end{bmatrix}$$

which are dual to e_1, e_2 *that are orthogonal trivializations.*

Theorem 7. *We have* $K \equiv 0$ *near a point* $p \in M$ *equivalent to exist coordinate system* (y_1, y_2) *near* p *such that* $g = dy_1^2 + dy_2^2 = dy_1 \otimes dy_1 + dy_2 \otimes dy_2$.

Chapter 2

The road to Gauss-Bonnet

2.1 Definition of Differential Operators

Definition 8. *We define Diff$^m(U)$ as the collection of differential operators*

$$P : C^\infty(U) \to C^\infty(U)$$

such that

$$P = \sum_{|I|<m} a_I \partial_{X_I}, a_I \in C^\infty(U)$$

We denote

$$\textit{Diff}^0(U) = C^\infty(U)$$

and so on. Therefore

$$\triangle \in \textit{Diff}^2(U)$$

Theorem 8. *Let $p, Q \in$ Diff$^m(U)$, then we have:*

- $P + Q \in$ *Diff*(U).
- $a \in C^\infty(U)$ *implies* $aP \in$ *Diff*(U).
- *If $m \leq m'$, then we have*

$$\textit{Diff}^m(U) \subset \textit{Diff}^{m'}(U)$$

Lemma 9.

$$\forall a \in C^\infty(U), \partial_{x_i} \circ a = a\partial_{x_i} + (\partial_{x_i} a)$$

This follows from Leibniz rule.

THEOREM 9. *We have*

$$\mathit{Diff}^m \circ \mathit{Diff}^n \subset \mathit{Diff}^{m+n}$$

DEFINITION 9. *We define the principal symbol. Let $P \in \mathit{Diff}^m(\mu)$, this implies*

$$P = \sum_{|I| \leq m} a_i \partial_{X_I}, a_I, a_I : U \to \mathbb{C}$$

We define the principal symbol by:

$$\sigma_m(P) : T^*(U) \to \mathbb{C} : \xi : \sum_{|I|=m} a_I(q) i\xi(\partial_{x_{i1}}) \cdots i\xi_{\partial_{x_{im}}}$$

where ξ is basis vector for the $T_q U^*$:

$$\xi : T_q(U) \to \mathbb{R}$$

We thus have

$$\sigma_m(P) \in C^\infty(T^*U)$$

Here we use the multi-symbol notation:

$$\partial_{X_I} = \partial_{X_{i1}} \circ \partial_{X_{i2}} \cdots \partial_{X_{in}}$$

Example 4. *The Laplace operator is:*

$$\triangle = -\sum_{i=j}^{n} \partial_{x_i} \circ \partial_{x_j}$$

Example 5. *If $P \in \mathit{Diff}^m(M), Q \in \mathit{Diff}^n(M)$, then we have*

$$\sigma_{m+n}(P+Q) = \sigma_m(P) \circ \sigma_{m'}(Q)$$

If we define $k = \max(m,n)$, then we have

$$\sigma_k(P+Q) = \sigma(P) + \sigma(Q)$$

THEOREM 10. *Let $(y_1 \cdots y_n)$ be another set of coordinate system on U, then we have*

$$P \in DIFF(U) \leftrightarrow P = \sum_{|I| \leq m} b_I \partial_{Y_I}$$

where $b_I \in C^\infty(U)$. In other words, we know

Diff(U) *is coordinate independent.*

Proof. Induction on m, for $m = 0$ this is trivial. For $m = 1$ we have

$$P \in \text{Diff}(U) \leftrightarrow P = \sum_{|I|\leq 1} a_I \partial X_I = a_0 + \sum_{j=1}^{n} a_j \partial_{x_j}$$

and the rest follows from chain rule. Then we induct.

THEOREM 11. *The principal symbol σ is coordinate independent.*

Proof. This is left as an excise.

Now we have seen that

- Diff(U) is coordinate independent.
- for $m \leq n$, we have
$$\text{Diff}^m \subset \text{Diff}^{m-1}$$
Let $\sigma_m(P) = T^*(U) \to \mathbb{C}$. Then we have
$$\sigma_m(P)(\xi) = i^m \sum_{|i|=m} a_I \xi(\partial_i) \cdots \xi(\partial_k)$$
Here we have
$$\sigma_m(P) \in C^\infty(T^* U)$$

THEOREM 12. *We have the following exact sequence:*

$$0 \to \textit{Diff}^{m-1} \to \textit{Diff}^m \to^{\sigma_m} C^\infty(T^*(U)) \to 0$$

Also we know that $\sigma_m(P)$is coordinate free. We have:

$$\sigma_{m+n}(P \circ Q) = \sigma_m(P) \circ \sigma_n(Q)$$

2.2 Differential Operator on Manifolds

Let M be a C^∞ manifold. Given $m \in \{0, 1, 2 \cdots\}$, a m-th order differential operator is a linear map

$$P : C^\infty(M) \to C^\infty(M)$$

such that

$$\forall u \in C^\infty(M), \forall \textit{coordinate patch} U \subset M$$

We have

$$P_u|_U = P_u(u|U), P_u \in \textit{Diff}^m(M)$$

such that we have

$$(Pu)_U = \sum_{|I|\le m} a_I \partial_{X_I}(u|U), \textit{for some} a_I \in C^\infty(U)$$

It satisfies the following properties:

- $$\textit{Diff}^m(M) \circ \textit{Diff}^n(M) \subset \textit{Diff}^{m+n}(M)$$
- *Diff*$^m(M)$ *is a module over* $C^\infty(M)$.

Discussion. *We now re-visit the principal symbol. Let* $P \in$ *Diff*(M)*, we define*

$$\sigma_m(P) : T^*(M) \to \mathbb{C}$$

as follows: Take any coordinate patch U *containing* q *and let*

$$P_u : \textit{local representation of} P \textit{over} U$$

Then we have

$$\sigma_m(P)(\xi) = \sigma_m(P_u)(\xi) \in \mathbb{C}$$

Lemma 10. *Properties of the principal symbol:*

- $\sigma_m(aP_1 + bP_2) = a\sigma_m(P_1) + b\sigma_m(P_2)$
- $$\sigma_{m+n}(P \circ Q) = \sigma_m(P) \times \sigma_n(Q)$$

Lemma 11. *We have the adjointness over Hermitian product spaces: Let* $T : V \to W$ *be a linear map, then its adjoint (if it exists) is a linear map* $A : W \to V$ *such that*

$$\langle TV, W\rangle = \langle V, AW\rangle, \forall v \in V, w \in W$$

Definition 10. *The adjoint is unique and denoted by* T^*.

Lemma 12. *We define an inner product by assuming* M *is an oriented Riemannian compact manifold. Let* dg *be the volume form. Then we have*

$$\int : C^\infty(M) \to \mathbb{C} : f \to \int f dg$$

So we have the inner product defined by the L^2 *fashion:*

$$\langle f, g\rangle : C^\infty(M) \times C^\infty(M) : \int_M f\overline{h} dg$$

THEOREM 13. *Any $P \subset \textit{Diff}^m(M)$ have an adjoint. In other words, there exists*

$$P^* : C^\infty(M) \to C^\infty(M)$$

such that

$$\langle Pf, h \rangle = \langle f, P^* h \rangle$$

Moreover we know that

$$P^* \in \textit{Diff}(M)$$

and we have

$$\sigma_m(P) = \overline{\sigma_m(P)}$$

To prove it we use integration by parts:

$$\int (\partial_{x_j} f)\overline{h} = -\int f \overline{\partial_{x_j} h}$$

We now discuss the case on vector bundles.

Example 6. *We consider the trivial case on the exterior form bundle. We have*

$$d : C^\infty(M, \wedge^k) \to C^\infty(M, \wedge^{k+1})$$

such that

$$(d\alpha)|_\mu = \sum \frac{\partial \alpha_J}{x_j} dx_j \wedge dx_J$$

where

$$\alpha|_\mu = \sum a_J dx_J \in C^\infty(U)$$

Example 7. *We consider the case of the covariant derivative. We have*

$$\nabla : C^\infty(M, E) \to C^\infty(M, \wedge^1 \otimes E)$$

such that

$$(\nabla e)|_U = \sum \frac{\partial \alpha_j}{\partial x_k} dx_k \otimes e_j + \sum b_{ij}^k a_j dx_k \otimes e_i$$

Here we have

$$\sum b_{ij} dx_k = w_{ij} \in C^\infty(M, \wedge^1), e|_U = \sum a_j e_j$$

We can also re-write this as

$$(\nabla e)_U = \sum_{k,i} (\sum_j \delta_{ij} \frac{\partial}{\partial x_j} + b_{ij}^k) a_j dx_k \otimes e_j$$

That's how we get

$$\nabla = d + w$$

If we regard

$$P_{k,i} = \sum_j \delta_{ij} \frac{\partial}{\partial x_j} + b_{ij}^k$$

as a differential operator indexed by k, i, we have each term being a first order differential operator:

$$P_{k,i}(j) = \delta_{ij} \frac{\partial}{\partial x_j} + b_{ij}^k \in \textit{Diff}^1(\mu)$$

Example 8. *We have*

$$(d\alpha)|_U = \sum_{j,J} \frac{\partial \alpha_J}{\partial x_j} dx_j \wedge dx_J$$

where we assume

$$\alpha|_U = \sum \alpha_J dx_J$$

we note that the basis over $\wedge^k$ is dx_J. We finally note:

$$(d\alpha)_U = \sum_{i,J,I} \frac{\alpha_J}{\partial x_j} sgn((i,J),(j,J)) dx_I$$

where

$$sgn((i,J), I)$$

will be defined later.

2.3 Differential Operator On Manifolds, II

Diff(M) *consists of linear maps*

$$P : C^\infty(M) \to C^\infty(M)$$

such that for any coordinate patch $\mathcal{M} \subset M$, there exist $P_U \in \textit{Diff}^m(M)$, such that

$$\forall \mu \in C^\infty(M), (P\mu)|_U = P_\mu(U|_U$$

Assuming that M is a Riemannian compact oriented manifold. Recall that, given $P \in \textit{Diff}^m(M)$, a coordinate cover $\{U_j\}$ of M, and a coordinate partition subordinate to $\{U_j\}$, the operator

$$Q = \sum_j \phi_j Q_j$$

where

$$Q_j = (P^*_{U_j}$$

satisfies

$$\langle Pu, v\rangle = \rangle u, Qv\rangle, \forall u, v \in C^\infty(M)$$

Therefore, given $\mu \in C^\infty(M)$, we have

$$Q(\mu) = \sum_j \phi_j Q_j(\mu|_{U_j}) \in C^\infty(M)$$

To discuss other topics we need the following theorem:

THEOREM 14. *Let U, V be overlapping coordinate patches, let $\phi \in C^\infty_c(U)$.*

- *If $v \in C^\infty(U \cap V)$, then $\phi(v) \in C^\infty(V)$.*
- *If $P \in$ Diff$^m(U)$, then*

$$\phi P \in \textit{Diff}^m(V)$$

Proof. Exercise. We note that $\phi(v)$ could be extended by 0 trivially in V.

Using this theorem, we can prove that any coordinate patch $V \subset M, \forall j,\ v \in C^\infty(V)$, we have

$$\phi_j Q_j(v) \in C^\infty(V)$$

Therefore

$$Q \text{ satisfies the definition of } \mathrm{Diff}^m(M) \text{ and } Q \in \mathrm{Diff}^m(M).$$

2.4 Vector Bundles

Let V, W be vector spaces. Let $T : V \to W$ be linear map such that

$$Te_j = T_{ij} f_i$$

THEOREM 15. *We have*

$$T : V \to W \in Hom(V, W) \leftrightarrow \exists \alpha_i \in V^*, \exists w_i \in W^*$$

and constants τ_{ij}, such that we have

$$\forall e \in V, T(e) = \sum_{i,j} \tau_{ij} \alpha_j(e) w_i$$

Proof. This is clear.

Let E, F be complex vector bundles over M. We define

$$\text{Diff}^m(M, E, F) = \{P : C^\infty(M, E) \to C^\infty(M, F)\}$$

such that for any coordinate patch, exists a trivialization

$$e_1 \cdots e_{N_1} \in E_U, f_1 \cdots f_{N_2} \in F_U$$

and

$$P_{ij} \in \text{Diff}^m(U)$$

such that

$$(Pe)|_U = \sum (P_{ij}\alpha_j) f_i, e|_U = \sum a_j e_j$$

Example 9. *We consider the trivial example. We have*

$$d : C^\infty(M, \wedge^k) \to C^\infty(M, \wedge^{k+1})$$

Let U be a coordinate patch and $\alpha \in C^\infty(M, \wedge^n)$. We have

$$\alpha|_U = \sum_J \alpha_J dx_J, i \leq j_1 \leq j_2 \cdots j_n \leq n$$

We thus have

$$(d\alpha)|_U = \sum_{j,J} \frac{\partial \alpha_j}{\partial x_J} dx_j \wedge dx_J$$

But it does not satisfy the description earlier, because $dx_j \wedge dx_J$ is not a basis itself.

To solve this problem we define

$$(d\alpha)_U = \sum_{i,j,I} sgn((i, J), J) \frac{\partial \alpha_J}{\partial x_j} dx_J, 1 \leq i_1 < i_2 \cdots < i_n < n$$

Remark 15. *I think here is a typo in the notes, the x_J in the notes is x_I, which does not make sense here. We continue with the example:*

Proof. So we have

$$(d\alpha)|_U = \sum_I \sum_J (P_{IJ}\alpha_J) dx_I$$

where we have

$$P_{IJ} = \sum_{j=1}^n syn((i, J), I)\partial x_j$$

Now we have

$$d \in \text{Diff}^1(M, \wedge^k, \wedge^{k+1})$$

Example 10. *We now revisit the covariant derivative. We have*

$$\nabla(e)|_U = \sum da_j \otimes e_j|_U + \sum a_j(\nabla e_j)|_U$$

and we have

$$(\nabla e_j)_U = \sum_{k,i} b_{ij}^k dx_k \otimes e_i, b_{ij}^k \in C^\infty(U)$$

Thus

$$(\nabla e)|_U = \sum \frac{\partial a_j}{\partial x_k} dx_k \otimes e_i + \sum_{i,j,k} \sum_{i,j}^{k} a_j dx_k \otimes e_i$$

Remark 16. *I think Maurcio lost a_j in the formula. I add it for completeness.*

Proof. which equals

$$\sum_{k,i}(\sum_j \delta_{ij}\frac{\partial}{\partial x_j} + b_{ij}^k)a_j dx_k \otimes e_j = \sum_{k,i,j}(P_{k,i,j}a_j)dx_k \otimes e_i$$

where

$$P_{k,i,j} = \delta_{jk}\frac{\partial}{\partial x_k} + b_{ij}^k \in D^1(M)$$

therefore

$$\nabla \in \text{Diff}^1(M, E, \wedge^1 \otimes E)$$

2.5 Adjoints

Assume that E, F are Hermitian vector bundles. Given $e_1, e_2 \in C^\infty(M, E)$, we have

$$(e_1, e_2) \in C^\infty(M)$$

defined pointwise. We assume M is a Riemannian, compact, oriented manifold such that

$$\langle e_1, e_2 \rangle = \int_M (e_1, e_2) dg$$

Let $P \in \text{Diff}^m(M, E, F)$ such that

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

recall the definition of the adjoint for vector spaces. We have the similar definition for vector bundles. Using it we have the following theorem:

THEOREM 16. *Given $P \in \mathit{Diff}^m(M,E,F)$, it has an adjoint*

$$P^* : C^\infty(M,F) \to C^\infty(M,E)$$

Moreover, we have

$$P^* \in \mathbf{\mathit{Diff}}^m(M,F,E)$$

2.6 Principal Symbol

We now review the principal symbol. We have

$$\mathbf{\mathit{Diff}}^{m-1}(M,E,F) \to \mathbf{\mathit{Diff}}^m(M,E,F)$$

What is the quotient

$$\mathbf{\mathit{Diff}}^m(M,E,F)/\mathbf{\mathit{Diff}}^m(M,E,F)?$$

Given

$$P \in \mathbf{\mathit{Diff}}^m(M)$$

recall that for all $\xi \in T^*_q M$ we have $\sigma_m(P)(\xi) \in \mathbb{C}$. We now have

$$\sigma_m(P)(\xi) = \sigma_m(P_U)(\xi)$$

after restricting to a coordinate patch. Now let

$$P \in \mathbf{\mathit{Diff}}^m(M,E,F)$$

and

$$\xi \in T^*_q(M)$$

We choose a coordinate patch U with $f \in U$ and choose coordinate functions e_i, f_i so that we have

$$P = \sum P_{ij} e^*_j \otimes f_i$$

We have

$$\sigma_m(P)(\xi) = \sum_{i,j} \sigma_m(P_{ij})(\xi) e^*_j \otimes f_i(q), \in Hom(E_q, F_q)$$

where of course

$$e^*_j \otimes f_i(q) \in Hom(E,F)$$

and

$$\sigma_m(P_{ij})$$

is a polynomial of ξ.We can thus conclude that

$$\sigma_m(P) : T_q^*(M) \to Hom(E_q, F_q)$$

but actually we have something stronger:

$$\sigma_m(P) : T^*(M) \to Hom(E, F)$$

between bundles. Note that if $e \in E_q$*, then we can write*

$$e = \sum a_j e_j(q), a_j \in \mathbb{C}$$

Then we have

$$\sigma_m(P)(\xi) \in Hom(E_p, E_q)$$

and

$$\sigma_m(P)(\xi)(e) = \sum \sigma_m(P_{ij})(\xi) a_j f_j(q)$$

Example 11. *We consider the familiar example* d *and* ∇*. We have*

$$\sigma_1(\xi)(e) = \sum_{k,i,j} \sigma_1(P_{k,i,j})(\xi)(e) dx_k \otimes e_i$$

which equals

$$i \sum \delta_{j,k} \xi(\partial x_k) dx_k \otimes e_i$$

After equating $\partial_{x_k}(dx_k) = 1$ *we have*

$$i\xi \otimes e$$

This is clear, for example from the definition of $\nabla = d + \omega$*, and we know the principal symbol of* d *is*

$$\sigma_1(d)(\xi) = i\xi$$

2.7 Reivew of previous section

Let $P \in$ *Diff*$^m(M, E, F)$*, this implies*

$$P : C^\infty(M, E) \to C^\infty(M, F)$$

The map is linear over $\mathbb{C}$ *and valid over any coordinate patch* U *and trivialization* $e_1 \cdots e_N ; f_1 \cdots f_N$ *of* E_U, F_U *respectively. We now have*

$$P|_U = \beta^{-1}[P_{ij}]\alpha, \alpha = [e_1^* \cdots e_N^*]^T, \beta = [f_1^* \cdots f_N^*]^T$$

such that for all $e \in C^\infty(M,E)$, we have

$$Pe|_U = \sum (P_{ij}a_j) f_i, e|_U = \sum a_j e_j$$

Note that we have

$$P|_U = \sum P_{ij} e_j^* \otimes f_i$$

We also have $P \in$ Diff$^m(M,E,F)$ implies $P|_U = \sum P_{ij}\alpha_j \otimes g_i$ for some $\alpha_j \in C^\infty(U,E^*)$, $g_* \in C^\infty(U,F)$. Then we have

$$P_{ij} \in \textit{Diff}^m(U)$$

Example 12. *We consider the known example. We have*

$$\nabla : C^\infty(E) \to C^\infty(M, \mathbb{C} \wedge^1 \otimes E)$$

and we know

$$\nabla(e) = \sum_{k,i} \sum_j (\delta_{i,j}\partial_{x_k} + b^n_{ij}) a_j dx_k \otimes e_i$$

If we assume

$$e|_U = \sum a_j e_j$$

Then we would have

$$\nabla e_j = \sum b^n_{ij} dx_k \otimes e_i$$

Remark 17. *I think Maurcio has a typo here, e_j in his equation should be e_i. We continue.*

Proof. which equals

$$\sum_{k,i,j} (P_{k,i,j} a_j) dx_k \otimes e_i$$

where

$$P_{k,i,j} = \delta_{i,j}\partial_{x_k} + b^k_{ij} \in C^\infty(U)$$

as before.

Remark 18. *Why should $\delta_{i,j}\partial_{x_k}$ in $C^\infty(U)$?*

We now have

$$(\nabla e)|_U = \nabla(\sum a_j e_j) = \sum da_j \otimes e_j + a_j \nabla e_j = \sum \frac{\partial a_j}{\partial x_k} dx_k \otimes e_j + \sum_j^n a_j \sum_{k,i}^n b^k_{ij} dx_k \otimes e_i$$

Rearranging we have this equal to

$$(\sum_{i,j,k} \partial_k e_j^* \otimes (dx_k \otimes e_j) + \sum b^k_{ij} e_j^* \otimes dx_k \otimes e_i)$$

Example 13. *We now revisit another familiar example. We have*

$$d : C^{\infty}(M, \wedge^{k}) \to C^{\infty}(M, \wedge^{k+1})$$

Remark 19. *I think there is a typo in Maucrio's notes. We continue.*

Proof. We thus have

$$(d\alpha)|_{U} = \sum sgn((j, J), I)\frac{\partial \alpha_{J}}{\partial x_{j}} dx_{I}$$

Here we assume

$$\alpha|_{U} = \sum \alpha_{J} dx_{J} = \sum_{I,J} (P_{IJ}\alpha_{J}) dx_{I}, P_{IJ} = \sum sgn(j, J)\partial_{j}$$

Remark 20. *I think there is another typo here, we cannot have*

$$\partial_{j} a_{J}$$

appeared in the summation. We continue.

Proof. Therefore we have

$$\begin{aligned} d\alpha &= \sum \partial_{j}\alpha_{J} dx_{j} \wedge dx_{J} && (2.1)\\ &= \sum \partial_{x_{j}}(dx_{J}) \otimes (dx_{j} \wedge dx_{J}) && (2.2)\\ &= \sum (\text{Diff}^{1})\alpha_{J} \otimes g_{i,J} && (2.3) \end{aligned}$$

Remark 21. *I think (2.2) and (2.3) does not make much sense. The equation need some heavy editing. We continue.*

Proof. We thus conclude that d is a differential operator.

Definition 11. *We define a differential operator on U as*

$$P|_{U} = \beta^{-1}[P_{ij}]\alpha$$

and

$$\sigma_{m}(P)(\xi) = \beta_{q}^{-1}(\sigma_{m}(P_{ij})(\xi))\alpha_{q} : E_{q} \to F_{q}$$

Also, if

$$P|_{U} = \sum Q_{ij}\alpha_{j} \otimes g_{i} \in C^{\infty}(M, F), Q_{ij} \in C^{\infty}(M, E)$$

We then have

$$\sigma_{m}(P)(\xi) = \sum \sigma_{m}(Q_{ij})(\xi)\alpha_{j}(q) \otimes g_{i}(q) : E_{q} \to E_{q}$$

We note that

$$\sigma_{m}(P) \in C^{\infty}(T^{*}M, Hom(E, F))$$

and it is fiber preserving.

Example 14. *We now revisit our example earlier: We have*

$$\sigma_1(\nabla)(\xi) = i\xi$$

because we have

$$\nabla = \sum \partial_k e_j^* \otimes (dx_k \otimes e_j) + \triangle$$

where the $\triangle$ denotes lower order terms from ω we ignore. We now have

$$\sigma_1(\nabla)(\xi) = \sum i\xi(\partial_k) e_j^* \otimes dx_k \otimes e_j$$

But know that

$$\xi = \sum \xi(\partial_k) dx_k$$

and

$$id_E = \sum e_j^* \otimes e_j$$

Therefore we have

$$\sigma_1(\nabla)(\xi) = i\xi \otimes id : E \to \mathbb{C} \wedge^1 \otimes E$$

Example 15. *Similarly we have*

$$\sigma_1(d)(\xi) = \sum i\xi(\partial_j)(dx_J)^* \otimes dx_j \wedge dx_J$$

Using the fact that

$$\xi = \sum \xi(\partial_j) dx_j$$

We have

$$\sigma(d)(\xi) = i\xi \wedge : C\wedge^k \to \mathbb{C}\wedge^{k+1}$$

2.8 Adjoints, Review

Recall that if we assume M is compact, oriented Riemannian manifold and $\langle\rangle$ is a Hermitian inner product. Then we have

$$C^\infty(M,E) \times C^\infty(M,E) \to \mathbb{C}$$

such that

$$(e_1, e_2) = \int \langle e_1, e_2 \rangle dg$$

THEOREM 17. *For all $P \in Diff^m(M,E,F)$, there exist $P^* \in Diff^m(M,F,E)$ such that*

$$(Pe,f) = (e,P^*f), \forall e \in C^\infty(M,E), f \in C^\infty(M,F)$$

such that

$$P : C^\infty(M,E) \to C^\infty(M,F)$$

has a formal adjoint

$$P^* : C^\infty(M,F) \to C^\infty(M,E)$$

Moreover we have

$$\sigma_m(P^*)(\xi) = \overline{\sigma_m(P)(\xi)}$$

Recall that

$$\sigma_m(P)(\xi) : E_q \to F_q$$

and

$$\sigma_m(P^*)(\xi) = \textit{adjoint}$$

Example 16. *We have*

$$d : C^\infty(M, \mathbb{C}\wedge^k) \to C^\infty(M, \mathbb{C}\wedge^{k+1})$$

so we have

$$d^* : C^\infty(M, \mathbb{C}\wedge^{k+1}) \to C^\infty(M, \mathbb{C}\wedge^k)$$

Therefore we have

$$\sigma(d^*)(\xi) = (\sigma_1(d)(\xi))^* = (i\xi\wedge)^* = -(i\xi\wedge)^*$$

LEMMA 13. *We claim that the adjoint of*

$$\xi\wedge : \mathbb{C}\wedge^k \to C\wedge^{k+1}$$

is

$$\beta^\# \rfloor$$

So we have

$$\sigma_1(d^*)(\xi) = -i\xi^\# \rfloor$$

Example 17. *For any $\alpha \in \wedge_q^k$, we have*

$$\langle \xi \wedge \alpha, \beta \rangle = \langle \alpha, \xi^\# \rfloor \beta \rangle$$

Example 18. The Hodge Star
Now consider

$$D = d + d^* : C^\infty(M,E) \to C^\infty(M,E)$$

Here E is the exterior form bundle

$$\mathbb{C}\wedge^0 \otimes \cdots \mathbb{C}\wedge^n$$

Then $D \in Diff^1(M,E)$ and

$$\sigma_1(D)(\xi) = i(\xi\wedge^+ - \xi^\# \rfloor) : E \to E$$

When composed twice we have:

$$\sigma_1(D)(\xi) \circ \sigma_1(D)(\xi) : E \to E$$

can be computed directly by

$$\xi^\#(\xi) = \langle \xi, \xi \rangle$$

This example motivate the following definition:

Definition 12. *Let E be any complex hermitian vector bundle and $D \in Diff^1(M,E)$. Then D is a* ***Dirac Operator*** *if D is self adjoint ($D^2 = D$) and*

$$\sigma_1(D)(\xi)^2 = |\xi|^2$$

2.9 Index of a Dirac Operator

Now let D be a Dirac operator:

$$D : C^\infty(M,E) \to C^\infty(M,E)$$

and assume E is $\mathbb{Z}_2$ graded: $E = E^+ \oplus E^-, \dim E^+ = \dim E^-$. Assume that

$$D : C^\infty(M,E^\pm) \to C^\infty(M,E^\mp)$$

In other words, we have:

$$Z : E \to E$$

by

$$Z = \begin{cases} 1 & on E^+ \\ -1 & on E^- \\ \end{cases}$$

Then we have $D \circ Z = -Z \circ D$.

Example 19. *If $e \in E^+$, we have:*

$$D \circ Z(e) = D(e) = -Z \circ (D) = -Z(D(e)) = D(e)$$

Example 20. *Let $E = \mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd}$ and $\dim M$ is even. Then we can define*

$$D^+ = D : C^\infty(M, E^+) \to C^\infty(M, E^-)$$

and

$$D^- : D : C^\infty(M, E^-) \to C^\infty(M, E^+)$$

Here our $D^\pm$ is the adjoint of $D^\mp$. In other words

$$(D^+(e), f) = (e, D^-(f))$$

REMARK 22. *I belive Prof. Loya used D as map between E^+_m and E^-_m fiber wise once we fixed m, not literally a map on sections.*

THEOREM 18. *$D^+ = D : C^\infty(M, E^+) \to C^\infty(M, E^-)$ is Fredholm. This means:*

- $\dim \ker D^+ < \infty$
- $\dim \textit{coker} D^+ < \infty$

DEFINITION 13. *We define the **index** of a differential operator by*

$$Ind D^+ = \dim \ker D^+ - \dim \textit{coker} D^+$$

2.10 Hodge decomposition Theorem

Let M be a compact, oriented Riemannian manifold. Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be an elliptic differential operator. Let E, F be Hermitian vector bundles over M.

THEOREM 19. *$P : C^\infty(M, E) \to C^\infty(M, F)$ is Fredholm. Moreover, we have:*

- $\dim \ker P < \infty$
- $\dim \ker P^* < \infty$
- $C^\infty(M, F) = PC^\infty(M, E) \oplus \ker P^*$.

We will prove the above theorem in the case $P = D$, a generalized Dirac operator. We review the theorem's application when $P = D = d + d^*$:

LEMMA 14. *We have*

$$D : d + d^* : C^\infty(M, \wedge^*) \to C^\infty(M, \wedge^*)$$

Then $\dim \ker D < \infty$ *and*

$$C^\infty(M, \wedge^*) = DC^\infty(M, \wedge^*) \oplus \ker D$$

This is the ***Hodge decomposition****.*

COROLLARY 1. *We have*

$$\ker D \cong H^*_{DR}(M)$$

We note that

$$D^2 = \triangle$$

is the Laplace-Beltrami operator, and $\ker D \cong \ker D^2 \cong H^*_{DR}(M)$*.*

Proof. Let $\alpha \in \ker D$. This implies $\alpha \in C^\infty(M, \wedge^*)$ with

$$d\alpha + d^*(\alpha) = 0 \to d^* d\alpha + {d^*}^2\alpha = 0 \to d^* d\alpha = 0$$

We construct a map

$$\alpha \to [\alpha] \in H^*_{DR}$$

We have to verify the map is well defined. We note that $d\alpha = 0$ if and only if

$$(d\alpha, d\alpha) = (\alpha, d^* d\alpha) = 0$$

but we know $d^* d\alpha = 0$. This implies

$$\alpha \to [\alpha] \in H^*_{DR}(M)$$

is well defined. We now prove it is indeed an isomorphism. Let $\alpha, \beta \in \ker D$ such that $\alpha \neq \beta$. If $\alpha = \beta + d\gamma$, then we have

$$D\alpha = D(\beta + d\gamma) = 0 \leftrightarrow Dd\gamma = 0 \leftrightarrow d^* d\gamma = 0$$

Therefore

$$\langle d\gamma + d^*\gamma, d\gamma + d^*\gamma \rangle = 2\langle \gamma, dd^*\gamma \rangle = 0 \to \gamma \in \ker D$$

By our reasoning earlier we then have $d\gamma = 0$. Therefore the map

$$\alpha \to [\alpha]$$

is unique. We also have to prove it is surjective. Let $a \in H^*_{DR}(M) \neq 0$, we must show that $a \in \ker(D)$. By Hodge decomposition Theorem we may decompose $a = a_1 + a_2$, where $a_1 = D(\beta) \in DC^\infty(M, \wedge^*)$. Then we have

$$da = 0 \leftrightarrow da_1 = 0 \leftrightarrow (dd + dd^*)\beta = 0 \leftrightarrow dd^*(\beta) = 0 \leftrightarrow D\beta = 0 \leftrightarrow a_1 = 0$$

So we concluded the isomorphism.

Discussion. *We now wish to prove Hodge decomposition under assumptions that $\mathcal{D}$ is a Dirac operator, E is $\mathbb{Z}_2$-graded with Z defined earlier. So*

$$\mathcal{D} : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$$

Before we prove it we provide a few examples.

Example 21.

$$\mathcal{D} = d + d^* : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp), \wedge^* = \wedge^{even} \oplus \wedge^{odd}$$

Example 22.

$$M = \mathbb{R}^2, \mathcal{D} = \begin{pmatrix} 0 & \overline{\partial}^* \\ \overline{\partial} & 0 \end{pmatrix} : C^\infty(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2) \to C^\infty(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2)$$

Here

$$\overline{\partial} = \partial_x + i\partial_y, \overline{\partial}^* = \partial_x - i\partial_y$$

and

$$\sigma(D)(\xi)^2 = [i \begin{bmatrix} 0 & -\xi(\partial_x + i\xi(\partial_y) \\ \xi(\partial_x) + i\xi(\partial_y) & 0 \end{bmatrix}]^2 = |\xi|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

REMARK 23. *I think this is the example of a Cauchy-Riemann operator.*

Discussion. *In general, we can always write*

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} : C^\infty(M, E^+ \oplus E^-) \to C^\infty(M, E^+ \oplus E^-)$$

We will now compute the index of $\mathcal{D}^+$. Recall by Theorem 19 for P elliptic, we have

$$indP = \dim \ker P - \dim \ker P^*$$

LEMMA 15. *In our case it translates to*

$$indD^+ = \dim \ker D^+ - \dim \ker D^-$$

because $\mathcal{D}$ is self-adjoint.

Proof. We can prove this using Hodge decomposition:

$$C^\infty(M, E) = \mathcal{D}C^\infty(M, E) \oplus \ker \mathcal{D}$$

which showed coker$\mathcal{D}^+ = \ker D^-$.

REMARK 24. *The index of $\mathcal{D}$ is zero, so we are only interested in the index of $\mathcal{D}^+$.*

LEMMA 16. *Recall $\mathcal{D} \circ Z = -Z \circ \mathcal{D}$, we have:*

$$ind\mathcal{D}^{+} = sgn(Z : \ker \mathcal{D} \to \ker \mathcal{D}) = \#\text{positive eigenvalues - negative eigenvalues of } Z$$

where Z acts on $\ker \mathcal{D} \cong H^*_{DR}(M)$.

Example 23. *We have*

$$d + d^* : C^{\infty}(M, \mathbb{C}\wedge^*) \to C^{\infty}(M, \mathbb{C}\wedge^*), E = \mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd}$$

So we have

$$IndD^{+} = syn(Z : H^*_{DR}(M) \to H^*_{DR}(M)) = \chi(M)$$

where

$$Z = \begin{cases} 1 & \text{on even forms} \\ -1 & \text{on odd forms} \end{cases}$$

REMARK 25. *Many nice objects can be written in terms of the indexes of the Dirac operator.*

2.11 Determinants and Traces

Let V be a finite dimensional vector space. We have

$$\det : Hom(V) \to \mathbb{C}$$

Here if we write

$$f \in Hom(V) = \sum a_{ij} v_i \otimes v_j$$

Then we have

$$\det(f) = \sum_{\sigma} sgn(\sigma) a_{i\sigma(1)} \cdots a_{n\sigma(n)}$$

Now let $\mathcal{A}$ equal to a commutative algebra over $\mathbb{C}$. We have:

$$\det : \mathcal{A} \otimes Hom(V) \to \mathcal{A}; Tr : \mathcal{A} \otimes Hom(V) \to \mathcal{A};$$

If $f \in \mathcal{A} \otimes Hom(V)$, then we have:

$$f = \sum a_{ij} \otimes v_i \otimes v_j^*$$

and we have

$$\det(f) = \sum_{\sigma} sgn(\sigma) a_{i\sigma(1)} \cdots a_{n\sigma(n)}$$

2.12 $\hat{A}$-genus

Let $\mathcal{A}_0$ be a nilpotent algebra over $\mathbb{C}$. This means that there exist $n_0 \in \mathbb{N}$ such that

$$\prod_{i=1}^{K} a_i = 0, \forall a_i \in A, K > n_0$$

Example 24. *Consider*

$$\mathcal{A}_0 = \mathbb{C} \wedge_p^2 \oplus \mathbb{C} \wedge_p^4 \oplus \cdots \mathbb{C} \wedge_p^{2m}, 2m = \dim M$$

Here we let

$$\mathcal{A} = \mathbb{C} \oplus \mathcal{A}_0$$

or

$$\mathcal{A} = \mathbb{C} \wedge_p^{even}$$

Discussion. *Consider*

$$\frac{z}{\sinh(z)} = 1 + \sum_{k=1}^{\infty} b_k z^{2k}$$

Let $T \in \mathcal{A}_0 \otimes Hom(V)$, then we have

$$\frac{T}{\sinh(T)} = 1 + \sum_{k=1}^{\infty} b_k T^{2k} \in \mathcal{A} \otimes Hom(V)$$

because the second term is in $\mathcal{A}_0 \otimes Hom(V)$. This implies

$$\det\left(\frac{T}{\sinh(T)}\right) \in \mathcal{A}$$

by out discussion earlier in 2.11. We now consider

$$(1+z)^{1/2} = 1 + \sum_{k=1}^{n_0} \binom{\frac{1}{2}}{k} z^k$$

Example 25. *Let*

$$\det\left(\frac{T}{\sinh(T)}\right) = 1 + \beta, \beta \in \mathcal{A}_0$$

Then we have

$$\sqrt{\det\left(\frac{T}{\sinh(T)}\right)} = (1+\beta)^{1/2} = 1 + \sum_{i=1}^{n_0} \binom{\frac{1}{2}}{k} \beta^k \in \mathbb{C} \oplus \mathcal{A}_0 = \mathcal{A}$$

This implies that

$$\hat{A}(T) = (\det)^{\frac{1}{2}}\left(\frac{T}{\sinh(T)}\right) \in \mathcal{A}$$

2.13 Relative Chern Character

Let $S \in Hom(V)$ *and* $T \in \mathcal{A}_0 \otimes Hom(V)$*. We have*

$$e^z = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!}$$

Therefore

$$Se^T = S(1 + \sum_{i=1}^{n_0} \frac{1}{k!} T^k) \in \mathcal{A} \otimes Hom(V)$$

Then we have

$$ch_S(T) = Tr(Se^{T/2\pi i}) \in \mathcal{A}$$

Now let $\mathcal{D}$ *be a Dirac operator as before.* M *is a compact oriented Riemannian manifold. Let* $\mathcal{R}$ *be the Riemannian curvature operator:*

$$\mathcal{R} \in C^\infty(M, \wedge^2 \otimes Hom(TM))$$

Example 26. *Let* $p \in M$*,* $\mathcal{R}_p \in \wedge_p^2 \otimes Hom(T_pM) \subset \mathcal{A}_0 \otimes Hom(V)$*.*

Here

$$\mathcal{A}_0 = \mathbb{C} \wedge_p^2 \oplus \cdots \mathbb{C} \wedge_p^{2m}, V = T_pM$$

Hence:

$$\hat{A}(R_p) = (\det)^{1/2}(\frac{\mathcal{R}_p/2\pi i}{\sinh(\mathcal{R}_p/2\pi i)}) \in \mathcal{A}$$

This implies

$$\mathcal{A}(\mathcal{R}) \in C^\infty(M, \mathbb{C}\wedge^{even})$$

2.14 From Principle symbol to Atiyah-Singer

Recall that we defined

$$\sigma(D) : T^*(M) \to Hom(E)$$

Recall that $\mathcal{R}$ *is the Riemnannian tensor inside of* $\mathbb{C}^\infty(M, \wedge^2 M \otimes T^*M \otimes T^*M)$*. Note that:*

$$T^*M \times T^*M \to Hom(E)$$

can given by

$$(\epsilon, \delta) \to \sigma(\mathcal{D})(\epsilon) \cdot \sigma(D)(\delta)$$

Therefore we know that $\mathcal{R}$ *must factor through* $Hom(E)$*.*

REMARK 26. *I do not really understand why it must factor through $Hom(E)$.*

Thus we get a map

$$\sigma(\mathcal{R}) \in C^\infty(M, \wedge^2 \otimes Hom(E))$$

Discussion. *Now we pick a connection on E and let Q_E equal to the curvature operator. We then have*

$$Q_E \in C^\infty(M, \mathbb{C} \wedge^2 \otimes Hom(E))$$

REMARK 27. *I am confused with the difference between $\sigma(\mathcal{R})$ and Q_E.*

Let $dg \in C^\infty(M, \wedge^{2m})$ be the Riemannian volume form in $C^\infty(M, T^*M \otimes \cdots T^*M)$, where we may regard

$$T^*M \otimes \cdots T^*M \subset Hom(E)$$

Then we have

$$\omega = i^m \sigma(dg) \in C^\infty(M, Hom(E))$$

Now let

$$S = Z \circ \omega^{-1}$$

We have

$$ch^1(E) = Tr(Se^{T/2\pi i}) \in C^\infty(M, \mathbb{C}\wedge^{even})$$

We finally reached Atiyah-Singer index theorem:

THEOREM 20. *Atiyah-Singer!*
We know that

$$ind(D^+) = 0, \dim M = 2k+1$$

and for $\dim M = 2k$:

$$IndD^+ = \int_M \hat{A}(\mathcal{R})ch(E) = \int_M (\det)^{1/2} \frac{\frac{\mathcal{R}}{4\pi i}}{\sinh(\mathcal{R}/4\pi i)} + r_{ru}(e^{Q_E \frac{1}{4}\sigma(Q)})$$

REMARK 28. *I suspect this formula needs some serious revision.*

We now try to prove this great theorem. We let

$$\tilde{Q} = Q_E + \frac{1}{4}\sigma(\mathcal{R})$$

where Q_E is the curvature of E and $\mathcal{R} \in C^\infty(M, \otimes_{i=1}^4 T^*M_i)$, $\sigma(\mathcal{R}) \in C^\infty(M, \wedge^2 \otimes Hom(E))$. We have

$$\omega = \frac{i^m}{m!}\sigma(dg) \in C^\infty(M, Hom(E))$$

REMARK 29. *Is there some typo at here or last page? The two definition seems different.*

We put

$$\tilde{Z} = Z \circ \omega$$

Execrise 3.

$$\omega \circ \omega = 1$$

REMARK 30. *This amounts to prove*

$$(\frac{i^m}{m!})^2 \sigma(dg)\sigma(dg) = 1$$

But I do not see, for example, why

$$i^{2m} = (-1)^m > 0, \sigma(dg) \circ \sigma(dg) > 0$$

since m is arbitrary.

We re-state Atiyah-Singer index Theorem again:

$$IndD^+ = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}})$$

Now let

$$D = d + d^* : C^\infty(M, \mathbb{C}\wedge^*) \to C^\infty(M, \mathbb{C}\wedge^*)$$

and

$$E = \mathbb{C} \wedge^{even} \oplus \mathbb{C}\wedge^{odd}, Z = 1 \oplus -1$$

We now have:

$$IndD^+ = \dim\ker D^+ - \dim\ker D^- = \chi(M)$$

Assume Atiyah-Singer for even dimensional manifolds, we have:

$$\chi(M) = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}})$$

Here

$$\tilde{Z} = Z \circ \omega, \omega = \frac{i^m}{m!} \sigma(dg), \sigma = \sigma_1(\mathcal{D}), \sigma(\xi) = i(\xi \wedge - \xi \lrcorner)$$

and

$$\tilde{Q} = Q_{\wedge^*} + \frac{1}{4}\sigma(\mathcal{R})$$

Note: If

$$\phi_1 \cdots \phi_n$$

is an oriented orthonormal basis of $T_p^*(M)$ and

$$dg_p = \phi_1 \wedge \cdots \wedge \phi_n = \sum sgn(P) \phi_{p(1)} \otimes \cdots \otimes \phi_{p(n)}$$

Then this implies

$$\omega = i^m \sigma(\phi_1) \cdots \sigma(\phi_m)$$

We also note that

$$\sigma(\xi)\sigma(\eta) + \sigma(\eta)\sigma(\xi) = \langle \eta, \xi \rangle$$

Therefore if $\eta \perp \xi$, we have

$$\sigma(\eta)\sigma(\xi) = -\sigma(\xi)\sigma(\eta)$$

and

$$\sigma(dg) = n!\sigma(\phi_1) \cdots \sigma(\phi_n)$$

This implies

$$\omega = i^m \sigma(\phi_1) \cdots \sigma(\phi_n)$$

We are heading on to the famous theorem of Gauss-Bonnet:

$$\xi(M) = \int_M Pf$$

where Pf is defined as follows: Recall that $\mathcal{R} \in C^\infty(M, \wedge^2 \otimes \wedge^2)$, so $\mathcal{R}^m \in C^\infty(M, \wedge^{2m} \otimes \wedge^{2m})$. But this is trivial, because dg gives a trivialization. Then we have

$$\frac{(-1)^m}{m!} \mathcal{R}^m = \alpha \otimes dg$$

where

$$\alpha \in C^\infty(M, \wedge^m), dg \in C^\infty(M, \wedge^m)$$

Then we have

$$Pf = \alpha \to \chi(M) = \int_M \alpha$$

Now, what is $Q_{\wedge^1}$? (recall we defined Q_E to be the curvature operator of E)

If ∇^{LC} is the Levi-Civita connection on TM equal to ∇^* the covariant derivative on forms. Then we have the following lemma:

Lemma 17.

$$\mathcal{R}^* \in C^\infty(M, \wedge^2 \otimes Hom(T^*M))$$

where $\mathcal{R}^*$ is the curvature operator corresponding to ∇^*.

Proof. We use the fact that

$$Hom(V) \cong Hom V^* : T \to T^t$$

Therefore if

$$\mathcal{R} \in C^\infty(M, \wedge^2 \otimes Hom(TM))$$

then this would imply

$$\mathcal{R}^t \in C^\infty(M, \wedge^2 \otimes Hom(T^*M))$$

Theorem 21. *We have*

$$\mathcal{R}^* = -\mathcal{R}^t$$

on T^*M.

2.15 Exterior covariant derivative

We still need a connection of $\wedge^p(E)$. *We know that*

$$\wedge^* \subset \oplus_{n=1}^n (T^*M)^{\otimes k}$$

We have a connection on T^*M*:*

Example 27. ∇^k *on* $(T^*M)^{\otimes k}$. *Here*

$$\nabla^k = \sum_{j=1}^k id \otimes \cdots id \otimes \nabla^j \otimes id \cdots \otimes id$$

So we have

$$\nabla^k : C^\infty(M, T^*M^{\otimes k}) \to C^\infty(M, \wedge^1 \otimes T^*M^{\otimes k})$$

Proof. We have

$$\begin{aligned} \nabla^k(\alpha_1 \wedge \cdots \wedge \alpha_k) &= \nabla^k(\sum sgn(s)\alpha_{s(1)} \otimes \cdots \otimes \alpha_{s(k)}) && (2.4) \\ &= \sum_{s,j} sgn(s)\alpha_{s(1)} \otimes \cdots \otimes \nabla\alpha_{s(j)} \otimes \cdots \alpha_{s(n)} && (2.5) \\ &= \sum_j \alpha_1 \wedge \cdots \wedge \nabla\alpha_j \cdots \wedge \alpha_k && (2.6) \\ &\in C^\infty(M, \wedge^1 \otimes \wedge^k) && (2.7) \end{aligned}$$

Thus, $\wedge^k$ is a connection on $\wedge^k$. Thus, we have

$$\oplus_{i=1}^n \nabla^k : C^\infty(M, \wedge^*) \to C^\infty(M, \wedge^*) \quad (2.8)$$

We let

$$Q_k = \text{curvature of } \nabla^k \text{on } \wedge^k \quad (2.9)$$

Then we have

$$Q_{\wedge^*} = \oplus_{k=1}^n Q_k \quad (2.10)$$

LEMMA 18. *We have*

$$Q_{\wedge^*} \in C^\infty(M, \wedge^2 \otimes Hom\wedge^*) \tag{2.11}$$

and

$$Q_{\wedge_k^*}(v,w)(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum \alpha_1 \wedge \cdots \wedge R(v,w)\alpha_j \wedge \cdots \alpha_k \tag{2.12}$$

Proof. Exercise.

2.16 Extension

Let V be a finite dimensional vector space and

$$A : V^* \to V^* \tag{2.13}$$

be a linear map. Let us define

$$\tilde{A} : \wedge^k V^* \to \wedge^k V^* \tag{2.14}$$

by

$$\tilde{A}(\alpha_1 \wedge \cdots \alpha_n) = \sum^k \alpha_1 \wedge \cdots \wedge A\alpha_j \wedge \cdots \wedge \alpha_k \tag{2.15}$$

THEOREM 22. *There exists a basis $\langle v_1 \cdots v_n \rangle$ of V with $\langle \alpha_1 \cdots \alpha_n \rangle$ being the dual basis. And we have*

$$\tilde{A} = \sum_{i,j=1}^{\dim V} A_{ij}(\phi_i \wedge) i_v \tag{2.16}$$

Here the $[A_{ij}]$ is the matrix of A.

Proof. The notation here is a bit garbled. I think this should follow from using an appropriate basis and plug in $\tilde{A}$ into the formula.

2.17 Towards Gauss Bonnet

Recall that

$$\mathrm{Ind} D^+ = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}}), m = \frac{1}{2}\dim M, \tilde{Z} = Z \circ \omega, \omega = \frac{i^m}{n!}\sigma(dg) \tag{2.17}$$

We have

$$\tilde{Q} = Q_E + \frac{1}{4}\sigma(\mathcal{R}) \tag{2.18}$$

Let $\epsilon = \mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd} = E^+ \oplus E^-$. Let $D = d + d^*$. Recall that

$$\frac{(-1)^m \mathcal{R}^m}{m!} \in C^\infty(M, \wedge^n \otimes \wedge^n) \tag{2.19}$$

We have

$$\frac{(-\mathcal{R})^m}{m!} = f dg \otimes dg \to Pf = f dg \tag{2.20}$$

Assuming this, we have

$$\chi(M) = \int Pf \tag{2.21}$$

Discussion. *Here is the local description of Pf: Let $\phi_1, \cdots \phi_n$ be a local orthonormal frame for T^*M, $v_1 \cdots v_n$. Let*

$$\mathcal{R} = \frac{1}{2}\sum_{j,k} \mathcal{R}_{jk}\phi_j \wedge \phi_k, R_{jk}(v,w) = \mathcal{R}(v,w,v_j,v_k) \tag{2.22}$$

Therefore we have

$$\mathcal{R} = \frac{1}{2}\sum_I R_I \phi_I, \phi_I = \phi_{i_1} \wedge \phi_{i_2} \tag{2.23}$$

So expanding out we have

$$\frac{(-\mathcal{R})^m}{m!} = \frac{1}{m!2^m}\sum_{I_1 \cdots I_m} \mathcal{R}_{I_1} \wedge \cdots \wedge \mathcal{R}_{I_m} \phi_{I_1} \cdots \phi_{I_m} \tag{2.24}$$

$$= \frac{1}{m!2^m}\sum_{I_1 \cdots I_m} sgn(I_1 \cdots I_m)\mathcal{R}_{I_1} \cdots \wedge \mathcal{R}_{I_m} \tag{2.25}$$

$$= Pf \tag{2.26}$$

Remark 31. *I am relatively lost with the equation (2.20). Can the professor check it?*

We now know that

1. *$\wedge^*$ has a LC connection.*

2. $$Q_{\wedge^*}(\alpha_1 \cdots \alpha_k) = \sum_{j=1}^k \alpha_1 \wedge \cdots \wedge \alpha_{j-1}\mathcal{R}^*(v,w)\alpha_j \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_k \tag{2.27}$$

   where as we know

   $$\mathcal{R}^*(v,w) \in Hom(T^*M)$$

Remark 32. *Something is fishy here, shouldn't the left hand operate on (v,w)? Otherwise the whole formula does not make sense.*

Lemma 19. *Let $A : V^* \to V^*$ be a linear map. Let $\phi_1 \cdots \phi_n$ be a basis of V^* with $v_1 \cdots v_n$ the basis for V. Let $[A_{ij}]$ be the matrix of A such that*

$$A\phi_i = A_{ij}\phi_j \tag{2.28}$$

Theorem 23.

$$D : \wedge^*(V^*) \to \wedge^* V^* \tag{2.29}$$

satisfying

$$D(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{j=1}^{n} \alpha_1 \wedge \cdots \wedge A\alpha_j \wedge \cdots \wedge \alpha_n \tag{2.30}$$

if and only if

$$D = \sum_{i,j} A_{ij}(\phi_i \wedge) \circ v_j \lrcorner \tag{2.31}$$

Here

$$\lrcorner$$

is the interior product symbol.

Proof. This is just a matter of definition.

Discussion. *Therefore we can write*

$$Q_{\wedge^*} = \sum_{j,k} R^*_{jk}(\phi_j \wedge) \circ v_k \lrcorner \tag{2.32}$$

Note, here $\phi_1 \cdots \phi_n$ are basis on T^*M, and $v_1 \cdots v_n$ are basis for its dual TM. We denote

$$\mathcal{R}^*(v,w)\phi_k = \sum_{j=1}^{n} \mathcal{R}^*_{jk}(v,w)\phi_j \tag{2.33}$$

in coordinates. Note that

$$\phi_k = \langle , v_k \rangle \tag{2.34}$$

We can interpret it as

$$\mathcal{R}^*_{jk}(v,w) = (\mathcal{R}^*(v,w)\phi_k)(v_j) \tag{2.35}$$
$$= -\langle \mathcal{R}(v,w)v_j, v_k \rangle \tag{2.36}$$
$$= -\mathcal{R}(v,w,v_j,v_k) \tag{2.37}$$

So now we can write

$$Q_{\wedge^*} = -\sum_{j,k} R_{jk}(\phi_j \wedge) \circ v_k \lrcorner \tag{2.38}$$

Recall that

$$\mathcal{R} = \frac{1}{2} \sum_{j,k} \mathcal{R}_{jk}\phi_j \wedge \phi_k = \sum_{j,k} \mathcal{R}_{j,k}\phi_j \otimes \phi_k \tag{2.39}$$

We know that

$$\sigma = \sigma_1(D) = \sigma_1(d + d^*) \tag{2.40}$$

This implies

$$\sigma(\mathcal{R}) = \mathcal{R}_{jk}\sigma(\phi_j)\sigma(\phi_k) \tag{2.41}$$

and since we know

$$\sigma(\xi) = i(\xi \wedge -v \lrcorner), \xi = \langle , v\rangle \tag{2.42}$$

We have

$$\sigma(\mathcal{R}) = -\sum \mathcal{R}_{jk}(\phi_j \wedge -v_j \lrcorner)(\phi_k \wedge -v_k \lrcorner) \tag{2.43}$$

Recall that

$$\sigma(\xi) = i(\xi \wedge -v \lrcorner)$$

Execrise 4. *:*

$$\sigma(\xi)^2 = |\xi|^2$$

Discussion. *We know that*

$$\tilde{\sigma}(\xi) = \sigma_1(\tilde{D})(\xi), \tilde{D} = -i(d - d^*)$$

So we have

$$Q_{\wedge^*} + \frac{1}{4}\sigma(\mathcal{R}) = -\frac{1}{4}\tilde{\sigma}(\mathcal{R})$$

where

$$\mathcal{R} = \sum_{j,k} \mathcal{R}_{jk}\varphi_j \otimes \varphi_k, \tilde{\sigma}(\mathcal{R}) = \sum_{j,k} \tilde{\sigma}(\varphi_j)\tilde{\sigma}(\varphi_k)$$

Now we need

$$\tilde{Z} = Z \circ \omega, \omega = \frac{i^m}{n!}\sigma(dg)$$

If

$$\varphi_1, \cdots, \varphi_n$$

are local orthonormal frame of TM, then we let

$$dg = \varphi_1 \wedge \cdots \varphi_n = \sum \mathit{sgn}(S)\varphi_{S(1)} \otimes \cdots \otimes \varphi_{S(n)}$$

This implies

$$\sigma(dg) = n!\sigma(\phi_1)\cdots\sigma(\phi_n)$$

Therefore

$$\omega = i^m \sigma(\varphi 1)\cdots\sigma(\varphi_n)$$

Now consider $\tilde{Z}$, we have:

Lemma 20.

$$\begin{aligned}\tilde{Z} = &= Z \circ \omega = i^m Z\sigma(\varphi_1)\cdots\sigma(\varphi_n) \\ &= (-1)^m i^m Z(a_1 - b_1)\cdots(a_n - b_n) \\ &= (-1)^m i^m Z(a_1 - b_1)\cdots(a_n - b_n)\end{aligned}$$

where we used the fact that

$$\sigma(\varphi_i) = i(a_i - b_i)$$

Let $c_j = (a_j - b_j)(a_j + b_j)$, where analogously we have

$$\tilde{\sigma}(\phi_j)$$

Lemma 21. *If*

$$\alpha = \varphi_{i_1} \cdots \wedge \varphi_{i_k}$$

then we have

$$c_j \alpha = \begin{cases} \alpha \textit{if} j \in \{i_1 \cdots i_l\} \\ -\alpha \textit{if} j \notin \{i_1 \cdots i_l\} \end{cases}$$

So we have

$$\begin{aligned} \tilde{Z} &= (-1)^m i^m Z(a_1 - b_1) \cdots (a_n - b_n) \\ &= (-1)^m i^m Z(a_1 - b_1)(a_1 + b_1)(a_1 + b_1) \cdots \\ &= (-1)^m i^m Z c_1 \tilde{\sigma}(\varphi_1) c_2 \tilde{\sigma}(\varphi_2) \cdots c_n \tilde{\sigma}(\varphi_n) = ? \end{aligned}$$

Execrise 5. *Using the lemma, prove that*

$$Z \circ c_1 \cdots c_n = Id \to \tilde{Z} = \tilde{\omega}$$

(Apply it to even, then to odd, then, done!)

Theorem 24. *In conclusion we have*

$$\mathrm{Tr}(\tilde{Z} e^{\tilde{Q}}) = \mathrm{Tr}(\tilde{\omega} e^{-\frac{1}{4}\tilde{\sigma}(Q)})$$

Note that

$$\tilde{\omega} = \frac{i^m}{n!} \tilde{\sigma}(dg)$$

2.18 Gauss-Bonnet

A review on the notation:

Let $n = \dim M = 2m$,

$$\mathcal{D} : C^\infty(M, E) \to (M, E)$$

is the Dirac operator on M associated with grading Z such that

$$D \circ Z = -Z \circ D$$

We now let

$$D^+ = D_{C^\infty(M,E^+)}, D \in \textit{Diff}^1(M,E), \sigma_1(\mathcal{D})(\xi)^2 = |\xi|^2$$

Then we have

$$IndD^+ = \frac{1}{(4\pi i)^m}\int_M \sqrt{\det(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)})}\,\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}})$$

here

$$\tilde{Z} = Z \circ \omega, \omega = \frac{i^m}{n!}\sigma(dg) = i^m\sigma(\varphi_1)\cdots\sigma(\varphi_n)$$

where φ_i is the local-framing on T^*M.

We also recall that

$$\tilde{Q} = Q + \frac{1}{4}\sigma(\mathcal{R})$$

where $\mathcal{R}$ is the Riemannian curvature tensor, a $(2,0)$ type tensor. And

$$\sigma(\mathcal{R}) \in C^\infty(M, \wedge^2 \otimes Hom(E))$$

Example 28. *We have*

$$E = \wedge^* = \wedge^0 \oplus \wedge^1 \cdots \wedge^n$$

and

$$Z(P) = 1, P \in \wedge^{even}, Z(P) = -1, P \in \wedge^{odd}$$

In this setting

$$\mathcal{D} = d + d^* : C^\infty(M, \wedge^{even}) \to C^\infty(M, \wedge^{odd})$$

and the index of $\mathcal{D}$ is equal to the Euler characteristic of the Dirac complex, hence equal to the Euler characteristic:

$$Ind(D^+) = \chi(M)$$

Proposition 1. *(linear algebra fact)*

Theorem 25. *Let V be a finite dimensional vector space and let $A_1 \cdots A_n \in Hom(V)$ be anti-commuting involutions. Let $I = (i_1 \cdots, i_n)$ where $n = 2m$ is even. and $i_1 \cdots i_k \in \{1, \cdots n\}, k \le n$. Then we have*

$$\mathrm{Tr}(A_1 \circ \cdots \circ A_n \circ A_I) = (-1)^m sgn(I)\dim(V)$$

where the sign is identical with the one from group theory if $i_1 \cdots i_k$ is a permutation of $\{1 \cdots n\}$.

Proof. Consider $I = 1$, then since n is even we have

$$\mathrm{Tr}(A_1 \cdots \circ A_n \circ A_1) = (-1)^{n-1}\,\mathrm{Tr}(A_1 \circ A_1 \circ A_2 \circ \cdots A_n) = -\,\mathrm{Tr}(A_2 \circ \cdots \circ A_n)$$

On the other hand since A_1 is an involution:

$$\mathrm{Tr}(A_1 \circ A_1) = \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_1) = 1, \to Tr(A_1) = \pm 1$$

Now if $\mathrm{Tr}(A_1) = 1$, then we have reduced the case to

$$\mathrm{Tr}(A_1 \circ \cdots A_1) = \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_2 \circ \cdots A_n)\,\mathrm{Tr}(A_1) = \mathrm{Tr}(A_2 \circ \cdots A_n)$$

and we reach a contradiction unless $\mathrm{Tr}(A_1 \circ \cdots A_1) = 0$. Similarly if $\mathrm{Tr}(A_1) = -1$ we can reach a contradiction as well. This settled the case for $I = 1$. Without loss of generality this proof can be generalized to all I where $|I|$ is odd.

For the case $|I| = k$ is even, we have

$$\mathrm{Tr}(\gamma A_{i_1} A_I A_{i_k}) = \mathrm{Tr}(A_{i_1} \gamma A_I A_{i_k}) = -\,\mathrm{Tr}(\gamma A_I), i \in n - \{i_1 \cdots I_k\}$$

and the proof for $k = n$ follows by:

$$\mathrm{Tr}(\gamma A_{i_1} \cdots A_{i_n}) = (-1)^m sgn(I)$$

This is called Patodi's lemma.

REMARK 33. *I think the* $\dim(V)$ *is a typo in the pdf file. The proof when* n *is even seems to be wrong, too.*

Recall that

$$\tilde{\sigma} : T^*M \to Hom(\wedge^*), \tilde{\sigma}(\xi) = \xi \wedge + \wedge^{\#} \lrcorner, \xi^{\#} = \langle , v \rangle$$

Fact:

$$\tilde{\sigma(\xi)}^2 = |\xi|^2$$

and

$$\tilde{\sigma}(\xi)\tilde{\sigma}(\eta) = -\sigma(xi)\tilde{\sigma}(\eta)$$

if

$$\xi \perp \eta$$

THEOREM 26. *We have*

$$\tilde{Q} = Q_{\wedge^*} + \frac{1}{4}\sigma(\mathcal{R}) = \frac{1}{4}\tilde{\sigma}(\mathcal{R})$$

Locally, if

$$\mathcal{R} = \sum R_I \otimes \varphi_I, \varphi_I = \varphi_{i_1} \wedge \varphi_{i_k}$$

Then

$$\tilde{Q} = -\frac{\mathcal{R}}{4}\sum_I \mathcal{R}_I\tilde{\sigma}(\varphi_{i_1})\tilde{\sigma}(\varphi_{i_2}) = -\frac{1}{2}\sum \mathcal{R}_I\tilde{\sigma}_I, \tilde{\sigma}_I = \prod_{k=1}^{n}\tilde{\sigma}(\varphi_{i_k})$$

Also recall that

$$\tilde{Z} = Z\circ\omega, \omega = i^m\sigma(\varphi_1)\cdots\sigma(\varphi_n)$$

Therefore

$$\tilde{Z} = (-1)^m i^m\tilde{\sigma}(\varphi_1)\cdots\tilde{\sigma}(\varphi_n)$$

The question we have now is

$$\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}}) = ?$$

First we recall that

$$e^{\tilde{Q}} = \sum\frac{1}{l!}\tilde{Q}^l = \sum\frac{1}{l!}(\frac{1}{i})^l\sum_{I_1\cdots I_l}\mathcal{R}_{I_1}\cdots\wedge\mathcal{R}_{I_l}\tilde{\sigma}_{I_1}\cdots\tilde{\sigma}_{I_l}$$

This implies

$$\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}}) = \sum_l\sum_{I_1\cdots I_l}\frac{1}{l!}\mathcal{R}_{I_1}\cdots\wedge\mathcal{R}_{I_l}\,\mathrm{Tr}(\tilde{Z}\tilde{\sigma}_{I_1}\cdots\tilde{\sigma}_{I_l}) \tag{2.44}$$

$$= i^m(\frac{-1^m}{m!})Z^m\sum_{I_1\cdots I_l} sgn(I_1\cdots I_l)\mathcal{R}_{I_1}\wedge\cdots\wedge\mathcal{R}_{I_l} \tag{2.45}$$

$$= Z^m i^m Pf \tag{2.46}$$

where Pf stands for the Ptaffian. Since we know that

$$\chi(M) = \frac{1}{(4\pi i)^m}\int_M\sqrt{\det(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)})}\,\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}})$$

We may conclude

$$\chi(M) = \frac{1}{(2\pi)m}\int_M Pf$$

because $\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}})$ is an n-form and we are only taking the highest power term from

$$\sqrt{\det(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)})}$$

Remark 34. *Not really clear to me...*

Chapter 3

Heat kernel asymptotics

3.1 Heat Kernel

Let us consider the simplest case $M = \mathbb{R}^n$*. Let* g *be a constant metric on* $\mathbb{R}^n$*:*

$$g(v, w) = \sum g_{ij} v_i w_j$$

Using this metric, the standard Laplacian can be written as

$$\Delta = -\sum g^{ij} \partial_i \partial_j$$

and

$$\sigma_2(\Delta)(\xi) = |\xi|^2$$

Discussion. *Given* $\varphi \in C_c^\infty(\mathbb{R}^n)$*, we want to find* $u(t,x) \in C^\infty([0,\infty) \times \mathbb{R}^n)$ *such that*

$$(\partial_t + \Delta) u(t,x) = 0, u(0,x) = \varphi(x), \forall x \in \mathbb{R}^n$$

Execrise 6. *: Show that*

$$u(t,x) = \frac{1}{(4\pi)^{n/2}} t^{-n/2} \int e^{-\frac{|x-y|}{4t}} \varphi(y) dg(y)$$

Now let

$$K(t,x,y) = t^{-n/2} \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$$

Definition 14. *This is called* ***the heat kernel****.*

Corollary 2. *We have*

$$u(t,x) = \int K(t,x,y) \varphi(y) dg(y)$$

DEFINITION 15. *The operator*

$$\mathcal{H} : C_c^\infty(\mathbb{R}^n) \to C^\infty([0,\infty) \times \mathbb{R}^n)$$

which operates by

$$H\phi(t,x) = \int_{\mathbb{R}^n} K(t,x,y)\varphi(x,y)dg(y)$$

*is called **the heat operator**.*

COROLLARY 3. *It is clear that K is the smoothing kernel of H.*

Execrise 7. *Show that*

$$u(t,x) = \int K(t,x,y)\varphi(y)dy \in C^\infty([0,\infty] \times \mathbb{R}^n)$$

LEMMA 22. *For all $\phi \in C_c^\infty(\mathbb{R}^n)$, we have $u : H\varphi \in C^\infty([0,\infty) \times \mathbb{R}^n)$ solves*

$$(\partial_t + \Delta)H\varphi = 0, \mathcal{H}\varphi|_{t=0} = \varphi$$

DEFINITION 16. *We can always write*

$$\mathcal{H} = e^{-t\Delta}$$

COROLLARY 4. *In conclusion we have*

$$e^{-t\Delta}\varphi = \int K(t,x,y)\varphi(y)dg(y)$$

satsifies

$$(\partial_t + \Delta)e^{-t\Delta} = 0, e^{-t\Delta}|_{t=0} = id$$

where $e^{-t\Delta}$ is a linear operator

$$C_c^\infty(\mathbb{R}^n) \to C^\infty([0,\infty] \times \mathbb{R}^n)$$

Discussion. *We can generalize this to the case of a Riemannian manifold:*

Let M be a compact, oriented Riemannian manifold. Let $\Delta \in Diff^2(M)$ be an operator such that

$$\sigma_2(\Delta)(\xi) = |\xi|^2$$

THEOREM 27. *There exist*

$$e^{-k\Delta} : C^\infty(M) : C^\infty(M) \to C^\infty([0,\infty) \times M)$$

such that for all $\varphi \in C^\infty(M)$, we have

$$(\partial_t + \Delta)(e^{-t\Delta}\varphi) = 0, e^{-t\Delta}\varphi|_{t=0} = \varphi$$

Moreover there exist $K \in C^\infty([0,\infty) \times M^2)$ such that

$$e^{-t\Delta}\varphi = \int_M K(t,x,y)\varphi(y)dg(y)$$

COROLLARY 5. *All the discussion at here can be transported to the case where*

$$L = -\sum g^{ij}\partial_{x_k}\partial_{x_j}$$

takes the place of the Laplacian.

REMARK 35. *For unknown reason the next page's content seems to be identical with this page except Δ is changed to a general second order elliptic operator L. So I skip.*

3.2 Smoothing operators

A map

$$K : C^\infty(M) \to C^\infty(M)$$

is a smoothing operator if there exist a function $K(x,y) \in C^\infty(M^2)$ such that

$$K\varphi = \int K(x,y)\varphi(y)dg(y), \forall \varphi \in C^\infty M$$

Example 29. *For all fixed $t > 0$,*

$$e^{-tL} : C^\infty(M) \to C^\infty(M)$$

is a smoothing operator over M.

DEFINITION 17. *We denote*

$$\Psi^{-\infty}(M)$$

be the set of all of all smothing operators over M.

THEOREM 28. *If $K \in \Psi^{-\infty}$ and $D \in Diff^1(M)$, then*

$$K \circ D \in \Psi^{-\infty}, D \circ K \in \Psi^{-\infty}$$

REMARK 36. *There should be a proof using trace class operators, I could not find it.*

Here is another property of the smoothing operator:

THEOREM 29. *Let*

$$A : C^\infty(M) \to C^\infty(M)$$

be a continuous linear map and $K \in \Psi^{-\infty}$, then

$$A \circ K \in \Psi^{-\infty}$$

Discussion. *It is clear that a key property of the smoothing operator is it is a 'trace'.*

DEFINITION 18.

$$\mathrm{Tr}(K) = \int K(x,x)dg(x)$$

THEOREM 30. *If $K \in \Psi^{-\infty}(M)$ and $D \in$ Diff$^2(M)$, then*

$$\mathrm{Tr}(D \circ K) = \mathrm{Tr}(K \circ D)$$

This would also work for $K \in \Psi^{-\infty}$ case.

REMARK 37. *I think there is a proof of this fact from Roe.*

3.3 Paramatrix

We will prove the Hodge Theorem. Let $\mathcal{D} \in$ Diff$^1(M)$ be elliptic and

$$D^* D + DD^*$$

are generalized Laplacians.

THEOREM 31. $\ker D, \ker D^*$ *are finite dimensional subspaces of $C^\infty M$ and*

$$C^\infty(M) = D(C^\infty(M)) \oplus \ker D^*$$

as well as

$$C^\infty(M) = D^*(C^\infty(M)) \oplus \ker D$$

In particular

$$D : C^\infty(M) \to C^\infty(M)$$

is Fredholm and

$$Ind(D) = \dim\ker(D) - \dim(\ker(D^*))$$

First we define:

DEFINITION 19. *A **Green's operator**:*

$$G : C^\infty(M) \to C^\infty(M)$$

is a linear operator such that

$$G\varphi = \Psi_1$$

where

$$\varphi = D\Psi_1 + \Psi_2$$

in the Hodge decomposition. Note that $G = D^{-1}$ in some sense off the object $\ker D^*$.

Execrise 8. *Let*

$$\pi : C^\infty(M) \to \ker(\mathcal{D})$$

be the orthogonal projection. Similarly let

$$\pi : C^\infty(M) \to \ker(\mathcal{D}^*)$$

Now let $\varphi_1 \cdots \varphi_n$ *be an orthonormal basis for* $\ker \mathcal{D}$. *We have:*

$$\begin{aligned}
(\pi\varphi)(x) &= \sum(\varphi\varphi_i)\varphi_i(x) && (3.1)\\
&= \sum\Big(\int \varphi(y)\overline{\varphi_j(y)}dg(y)\Big)\varphi_i(x) && (3.2)\\
&= \sum_i \int \varphi_i(x)\overline{\varphi_i(y)}\varphi(y)dg(y) && (3.3)\\
&= \sum_i \int \varphi_i(x)\overline{\varphi_i(y)}\varphi(y)dg(y) && (3.4)\\
&= \int \pi(x,y)\varphi(y)dg(y) && (3.5)
\end{aligned}$$

where

$$\pi(x,y) = \sum \varphi_i(x)\overline{\varphi_i(y)} \in C^\infty(M^2)$$

Therefore we conclude that

$$\pi : C^\infty(M) \to C^\infty(M) \in \Psi^{-\infty}(M)$$

Similarly

$$\pi' \in \Psi^{-\infty}(M)$$

Corollary 6. *We thus proved that there exist smoothing operator:*

$$G : C^\infty(M) \to C^\infty(M)$$

which satisfies

$$G \circ D = Id - \pi, D \circ G = Id - \pi'$$

Moreover, G *is the unique map on* $C^\infty M$ *that satisfies both equations. It is in fact a linear continuous map.*

Discussion. *We now define the paramatrix:*

Definition 20. *A continuous linear map*

$$B : C^\infty(M) \to C^\infty(M)$$

is called a paramatrix if

$$B \circ D = Id - R, D \circ B = Id - S, R, S \in \Psi^{-\infty}(M)$$

Example 30. *From what we proved, the Green's operator is a paramatrix for $\mathcal{D}$.*

PROPOSITION 2. *If B_1, B_2 are paramatrices for $\mathcal{D}$, then*

$$B_1 - B_2 \in \Psi^{-\infty}(M)$$

Proof. (the pdf file crossed it out)

Execrise 9. *Prove that G has an adjoint:*

$$(G\varphi, \Psi) = (\varphi, G^*\Psi), \forall \varphi, \Psi \in C^\infty(M)$$

and try to show that G^* is the Green's operator for D^*.

THEOREM 32. *(Hormander-Fedosov): For the $\mathcal{D}$ above, there exist a matrxmetric B of $\mathcal{D}$, and we have*

$$Ind\mathcal{D} = \mathrm{Tr}(Id - BD) - \mathrm{Tr}(Id - DB)$$

Note that

$$BD = Id - \mathcal{R} \in \Psi^{-\infty}(M), DB = Id - S \in \Psi^{-\infty}(M)$$

Proof. We have

$$\begin{aligned} Id - BD &= GD + \pi - BD && (3.6)\\ &= \pi + (G - B)D, K = (G - B) \in \Psi^{-\infty}(M) && (3.7) \end{aligned}$$

because of Proposition 2 we proved earlier. Now similarly we have

$$\begin{aligned} Id - DB &= \pi' + D(G - B) && (3.8)\\ &= \pi' + DK && (3.9) \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathrm{Tr}(Id - BD) - \mathrm{Tr}(Id - DB) &= \mathrm{Tr}(\pi + KD) - \mathrm{Tr}(\pi' - DK) && (3.10)\\ &= \mathrm{Tr}(\pi) - \dim(\ker D^*) && (3.11)\\ &= \dim\ker(\mathcal{D}) - \dim(\ker \mathcal{D}^*) && (3.12) \end{aligned}$$

3.4 Revisit Heat kernel

Finally, we consider generalized Laplacians of the form

$$L = D^*D, L^* = DD^*$$

then the heat kernel

$$e^{-tL}, e^{-tL'}$$

must exist for all fixed $t > 0$. And we know

$$e^{-tL}, e^{-tL'} \in \Psi^{-\infty}(M)$$

Definition 21. *We now fix $t > 0$ and define*

$$B : C^\infty(M) \to C^\infty(M)$$

by

$$B\varphi = D^* \int_0^t e^{-s\mathcal{D}\mathcal{D}^*} \varphi ds$$

where we note that

$$e^{-sDD^*}\varphi \in C^\infty([0,\infty) \times M)$$

and

$$\int_0^t e^{-s\mathcal{D}\mathcal{D}^*} \varphi ds \in C^\infty(M)$$

Proposition 3. *B is continuous and linear. Also we have*

- $$DB = Id - e^{-t\mathcal{D}\mathcal{D}^*}$$

- $$BD = Id - e^{-t\mathcal{D}^*\mathcal{D}}$$

Proof.

- We know the first part is true because

$$D \circ B\varphi = D \circ D^* \int^t (e^{-s\mathcal{D}\mathcal{D}^*\varphi}) ds \tag{3.13}$$

$$= \int_0^t \mathcal{D}\mathcal{D}^*(e^{-s\mathcal{D}\mathcal{D}^*}\varphi) ds \tag{3.14}$$

$$= e^{-s\mathcal{D}\mathcal{D}^*}\varphi|_{s=0}^{s=t} \tag{3.15}$$

$$= \varphi - e^{-t\mathcal{D}\mathcal{D}^*}\varphi \tag{3.16}$$

$$= (Id - e^{-t\mathcal{D}\mathcal{D}^*})\varphi \tag{3.17}$$

Similarly we have

$$BD = Id - e^{-t\mathcal{D}^*\mathcal{D}}$$

as desired.

COROLLARY 7. *By Hormander-Fedsov, we thus have*

$$Ind(D) = \mathrm{Tr}(Id - BD) - \mathrm{Tr}(Id - DB) \tag{3.18}$$

$$\to Ind(D) = \mathrm{Tr}(e^{-t\mathcal{D}^*\mathcal{D}} - e^{-t\mathcal{D}\mathcal{D}^*}) \tag{3.19}$$

Discussion. *Now the idea is to construct*

$$e^{-t\mathcal{D}^*\mathcal{D}}, e^{-t\mathcal{D}\mathcal{D}^*}$$

explicitly and compute the index.

3.5 Constructing the Heat kernel

We first construct heat kernel on $\mathbb{R}^n$*:*

Fix a Riemannian metric $g \in \mathbb{R}^n$ *and let* L *be a generalized Laplacian on* $\mathbb{R}^n$*:*

$$L = -\sum g^{ij}\partial_i\partial_j + \sum a_k\partial_k + \alpha(x)$$

Discussion. *We will find an approximation for* e^{-tL} *on* $\mathbb{R}^n$*. Our goal is give local approximations to see how to get a global approximation to* e^{-tL}*. In the case*

$$g = [g_{ij}]$$

and there is no first order terms, we have

$$L = -\sum g^{ij}\partial_i\partial_j$$

Therefore by discussion earlier we have

$$e^{-tL}(x) = \int \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} dy = \int K(t,x,y)dy$$

COROLLARY 8. *We note that we can write*

$$K(t,x,y) = t^{-n/2} q(\frac{x-y}{t})$$

where

$$q(\omega) = (4\pi)^{-n/2} e^{-\frac{1}{4}|\omega|_g^2} \in C^\infty(\mathbb{R}^n, \mathbb{C})$$

actually we may assert

$$q \in \mathbb{S}(\mathbb{R}^n)$$

Discussion. *Now let $g = g(x)$ be the metric on $\mathbb{R}^n$ and*

$$L = -\sum g^{ij}\partial_i\partial_j + \sum a_k(x)\partial_k + a(x)$$

If e^{-tL} exists, then we guess

$$e^{-tL}\varphi = \int h(t,x,y)\varphi(y)dg(y)$$

We would guess that

$$h(t,x,y) = t^{-n/2}q(t^{1/2}, x\frac{x-y}{t^{1/2}})$$

Remark 38. *This may be wrong.*

Discussion. *(continue) where*

$$q(s,x,\omega) \in C^{\infty}([0,\infty)_s \times \mathbb{R}^n \times \mathbb{R}^n)$$

and q is rapidly decreasing in ω. In other words:

$$\forall k, \alpha, \beta \in \mathbb{N}$$

and compact set $K \subset \mathbb{R}^n$ and $\forall l \in \mathbb{N}$ we have

$$\sup_{(s,x,n)\in K\times\mathbb{R}^n} |(1+|\omega|)^l \partial_s^k \partial_x^{\alpha} \alpha_{\omega}^{\beta} q(s,x,\omega)| < \infty$$

Further verification (in particular, why $t^{1/2}$?): If it solves

$$(\partial_t + \Delta)t = 0$$

then $\partial_t H$ also solves it as $\partial_t H$ should have a similar form as above:

If $L = \sum g^{ij}\partial_i\partial_j$, g independent of x. Then we have

$$e^{-tL}\varphi = \frac{t^{-n/2}}{(4\pi)^{n/2}} q(\frac{x-y}{t^{1/2}})\varphi(y)dy, q(\omega) = e^{-\frac{1}{2}|\omega|_g^2}$$

Remark 39. *Another typo?*

Discussion. *(continue) which implies*

$$\partial_t e^{-tL} = \frac{n}{2}\frac{t^{-n/2-1}}{(4\pi)^{n/2}}\int q(\frac{x-y}{t^{1/2}})\varphi(y)dy + \frac{t^{-n/2}}{(4\pi)^{n/2}}\sum\int(\partial_{u_i}q)(\frac{x-y}{t^{1/2}})\varphi(y)dy(x_i - y_i)(\frac{-1}{2})t^{-\frac{3}{2}}$$

(two typos are fixed)which after rearranging equals

$$\frac{t^{-n/2}}{(4\pi)^{n/2}}\int \tilde{q}()\varphi(y)dg(y)$$

where

$$\tilde{q}(x,y) = -\frac{n}{2}t^{-1}q(\frac{x-y}{t^{1/2}}) - \frac{1}{2}t^{-3/2}\sum(x_i - y_i)\partial_{u_i}q(\frac{x-y}{t^{1/2}})$$

This implies

$$\partial_t e^{-tL} = \frac{t^{-n/2-1}}{(4\pi)^{n/2}}\int \tilde{q}(\frac{x-y}{t^{1/2}})dg(y)$$

where

$$\tilde{q}(\omega) = -\frac{n}{2}q(\omega) - \frac{1}{2}\sum \omega_i \partial_i q(\omega)$$

and

$$\tilde{q} \in S(\mathbb{R}^n)$$

Discussion. *Now, our first guess of operators that would contain e^{-tL}. We define*

$$\Psi^p$$

as the space of operators

$$Q : C_c^\infty(\mathbb{R}^n) \to C^\infty((0,\infty)^t \times \mathbb{R}^n)$$

satisfying

$$(Q\varphi)(t,x) = t^{-\frac{n}{2}-\frac{p}{2}-1}\int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)dy, \forall \varphi \in C_c^\infty(\mathbb{R}^n), q(s,x,\omega) \in \mathbb{S}^\infty([0,\infty)_s \times \mathbb{R}^n \times \mathbb{R}^n)]$$

Remark 40. *What is the p at here?*

Discussion. *(continue) which means*

$$\forall k, \alpha, \beta \in \mathbb{N}$$

and compact set $K \subset \mathbb{R}^n$ and $\forall l \in \mathbb{N}$ we have

$$\sup_{(s,x,n)\in K\times\mathbb{R}^n} |(1+|\omega|)^l \partial_s^k \partial_x^\alpha \alpha_\omega^\beta q(s,x,\omega)| < \infty$$

Then we should have

$$e^{-tL} \in \Psi^{-\infty}(M)(\mathbb{R}^n)$$

This implies

$$(\partial_t + L)e^{tL} = 0$$

Remark 41. *I am confused - isn't this what we proved before? The (general) heat kernel is a smooth operator?*

Discussion. *The only problem left: We know, for $L = -\sum g^{ij}\partial_i\partial_j$, $e^{-tL}\varphi \in C^\infty([0,\infty)_t \times \mathbb{R}^n)$, for all $\varphi \in \mathbb{C}_c^\infty(\mathbb{R}^n)$. We need to re-define $\Psi^p(\mathbb{R}^n)$. We notice that in the fundamental equation*

$$e^{-tL}\varphi = t^{-n/2}\int q(\frac{x-y}{t^{1/2}})\varphi(y)dg(y)$$

where

$$q(\omega) = e^{-\frac{1}{2}|\omega|^2}$$

REMARK 42. *I think there is a typo inherited from earlier. It is clear it should be*

$$q(\omega) = e^{-\frac{1}{4}|\omega|^2}$$

instead!

Discussion. *And similarly we notice that in its derivative the function is even in ω!*

$$\partial_t e^{-tL} = \frac{t^{-n/2-1}}{(4\pi)^{n/2}}\int \tilde{q}(\frac{x-y}{t^{1/2}})dg(y)$$

DEFINITION 22. *With this in mind, We define*

$$\Psi^p$$

as the space of operators

$$Q : C_c^\infty(\mathbb{R}^n) \to C^\infty((0,\infty)^t \times \mathbb{R}^n)$$

satisfying

$$(Q\varphi)(t,x) = t^{-\frac{n}{2}-\frac{p}{2}-1}\int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)dy, \forall\varphi \in C_c^\infty(\mathbb{R}^n), q(s,x,\omega) \in \mathbb{S}^\infty([0,\infty)_s{\times}\mathbb{R}^n{\times}\mathbb{R}^n)]$$

where

$$q(s,x,\omega) = q_1|_{[0,\infty)\times\mathbb{R}^n_x\times\mathbb{R}^n_\omega}$$

We may define q_1 which satisfies the following two equations:

$$q_1(s,x,\omega) \in C^\infty(\mathbb{R}_s \times \mathbb{R}^n_x \times \mathbb{R}^n_\omega)$$

and

$$q_1(-s,x,-\omega) = (-1)^p q_1(s,x,\omega)$$

Discussion. *Why on earth did we choose odd and even?*

Look at e^{-tL} again, L is a generalized Laplacian for g with constant coefficients. We know that the fundamental equation claimed

$$e^{-tL}\varphi = t^{-n/2}\int q(\frac{x-y}{t^{1/2}})\varphi(y)dg(y)$$

In particular since $(\partial_t + L)e^{-tL} = 0$*, we have*

$$\partial_{x_i}(\partial_t + L)e^{-tL} = 0$$

This implies

$$(\partial_t + L)(\partial_{x_i} e^{-tL}) = 0$$

Therefore formally we have

$$\partial_{x_i} e^{-tL} = \frac{t^{-n/2-1/2}}{(4\pi)^{n/2}} \int (\partial_{w_i} q)(\frac{x-y}{t^{1/2}})(\partial\varphi(y)dg(y) \in \Psi^{-1}$$

If we use a different notation

$$\partial_{x_i} e^{-kL}, \tilde{q} = \partial_{w_i} q(\omega)$$

Then we know $\tilde{q}$ *has to be odd.*

First we prove the following technical theorem:

THEOREM 33. *If* $Q \in \Psi^p_n(\mathbb{R}^n)$*, then*

$$Q : C^\infty_c(\mathbb{R}^n) \to t^{-\frac{p}{2}-1} C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

In particular, if

$$Q \in \Psi^{-2}_n(\mathbb{R}^n)$$

then we have

$$Q : C^\infty_c(\mathbb{R}^n) \to C^\infty([0,\infty) \times \mathbb{R}^n)$$

Proof. Let $Q \in \Psi^p_n$, then we have

$$Q\varphi = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)dg(y)$$

by definition. We can re-write it as

$$Q\varphi = t^{-\frac{p}{2}-1} f(t,x)$$

where

$$f(t,x) = t^{-n/2} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)dg(y)$$

So to prove that $f \in t^{-\frac{p}{2}-1} C^\infty([0,\infty)_t \times \mathbb{R}^n)$, we compute:

$$\omega = \frac{x-y}{t^{1/2}}, y = x - t^{1/2}\omega, dy = t^{n/2} d\omega$$

Therefore

$$f(t,x) = \int q(t^{1/2}, x, \omega)\tilde{\varphi}(x, t^{1/2}, \omega)d\omega, \tilde{\varphi}(y) = \varphi(y)\sqrt{\det(g(y))} \in C^\infty_c(\mathbb{R}^n)$$

REMARK 43. *I am a little confused with the last step, how come* $\tilde{\varphi}$ *is related to* y*?*

3.6 The space Ψ_n^p and smoothing operators

Here are some Qs that satisfy $(\partial_t + L)Q = 0$:

- $$Q_1 = \partial_t^k \mathcal{H}, P = \sum_{|I| \leq m} a_I \partial_{X_I}$$
- $$Q_2 = P\mathcal{H}, P = \sum_{|I| \leq m} a_I \partial_{x_I}$$
- $$Q_3 = \mathcal{H}\varphi, \varphi \in C_c^\infty(\mathbb{R}^n)$$
- $$Q_4 = \varphi\mathcal{H}\Psi, \varphi, \Psi \in C_c^\infty(\mathbb{R}^n)$$
- $$Q_5 = \varphi\mathcal{H}$$

  Let us analyze them:

- $$Q_1(u) = t^{-\frac{n}{2}-K} \int q_k(\frac{x-y}{t^{1/2}})u(y)dg(y), q_k(\omega) \in \mathbb{S}(\mathbb{R}^n)$$
- $$Q_2 u = t^{-\frac{n}{2}} \int (\sum_{|I| \leq m} a_I t^{-\frac{|I|}{2}} (\partial w_I q)(\frac{x-y}{t^{1/2}})u(y)dg(y)$$

  This implies
  $$Q_2(u) = t^{-\frac{n}{2}-\frac{|I}{2}} \int q_p(t^{1/2}, (\frac{x-y}{t^{1/2}}))u(y)dg(y)$$
  where as
  $$q_p(s,\omega) = \sum_{|I| \leq m} a_I q^{m=|I|}(\partial_{W_I} q)(\omega)$$

- $$Q_3(u) = t^{-\frac{n}{2}} \int q(\frac{x-y}{t^{1/2}})\varphi(y)u(y)dg(y)$$

  and as we recall under the transformation
  $$\frac{x-y}{t^{1/2}} = \omega \to y = x - t^{1/2}\omega$$
  thus we can rewrite it as
  $$q_u(t^{1/2}, x, \frac{x-y}{t^{1/2}}), q_\varphi(s, x\omega) = q(\omega)\varphi)\varphi(x - s\omega)$$

- $$Q_5(\mu) = t^{-n/2} \int \varphi(x) q(\frac{x-y}{t^{1/2}}) u(y) dg(y)$$

Discussion. *What do $Q_1 \cdots \mathbb{Q}_5$ have in common? They are all looking like this:*

$$Qu = t^{-\frac{n}{2}+l} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}} u(y) dg(y), l \in \mathbb{Z}[\frac{1}{2}]$$

where

$$q(x, s, \omega) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$$

is Schwartz in ω and they have the same 'even-odd' property we discussed earlier:

For example

$$q_p(s, \omega) = (-1)^m q_p(s, \omega)$$

where m is the value involved in l.

We recall the following two definition (the third time)!

DEFINITION 23. $q(x, s, \omega) \in \mathbb{S}^{-\infty}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ *if and only if*

$$\forall k, \alpha, \beta \in \mathbb{N}$$

and compact set $K \subset \mathbb{R}^n$ and $\forall l \in \mathbb{N}$ we have

$$\sup_{(s,x,n) \in K \times \mathbb{R}^n} |(1 + |\omega|)^l \partial_s^k \partial_x^\alpha \alpha_\omega^\beta q(s, x, \omega)| < \infty$$

DEFINITION 24. *We define*

$$\Psi^p$$

as the space of operators

$$Q : C_c^\infty(\mathbb{R}^n) \to C^\infty((0, \infty)^t \times \mathbb{R}^n)$$

satisfying

$$(Q\varphi)(t, x) = t^{-\frac{n}{2} - \frac{p}{2} - 1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}}) \varphi(y) dy, \forall \varphi \in C_c^\infty(\mathbb{R}^n), q(s, x, \omega) \in \mathbb{S}^\infty([0, \infty)_s \times \mathbb{R}^n \times \mathbb{R}^n)]$$

where

$$q(s, x, \omega) \in S^{-\infty}(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

and

$$q(x, -s, -\omega) = (-1)^p q(x, s, \omega)$$

The idea is to show that e^{-tL} should belong to

$$\Psi_n^{-2}(\mathbb{R}^n)$$

We recall the statement of the theorem proved in last lecture:

THEOREM 34. • *For p even, if $Q \in \Psi_n^p(\mathbb{R}^n)$, then*

$$Q : C_c^\infty(\mathbb{R}^n) \to t^{-\frac{p}{2}-1} C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

In particular, if

$$Q \in \Psi_n^{-2}(\mathbb{R}^n)$$

then we have

$$Q : C_c^\infty(\mathbb{R}^n) \to C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

- *For p odd, if $Q \in \Psi_n^p(\mathbb{R}^n)$, then*

$$Q : C_c^\infty(\mathbb{R}^n) \to t^{\frac{p-1}{2}} C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

In particular, if

$$Q \in \Psi_n^p(\mathbb{R}^n), p \leq -3$$

then we have

$$Q : C_c^\infty(\mathbb{R}^n) \to C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

and it vanishes when $t = 0$.

REMARK 44. *I think there is another typo here. $p \geq 3$ instead of $p \leq -3$.*

Proof. We review the proof from last time: Let $Q \in \Psi_n^p$, then we have

$$Q\varphi = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)dg(y)$$

by definition. We can re-write it as

$$Qu(x,t) = t^{-n/2-\frac{p}{2}-1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})u(y)dg(y)$$

Now we change the variable:

$$\omega = \frac{x-y}{t^{1/2}}, y = x - t^{1/2}\omega, dy = t^{n/2}d\omega$$

Therefore the above equals

$$t^{-\frac{p}{2}-1} \int q(x, t^{1/2}, \omega)u(x - t^{1/2}\omega)(\sqrt{\det g(x - t^{1/2}\omega)})d\omega$$

Therefore if we let

$$\tilde{q}(x,s,\omega) = q(x,s,\omega)u(x - s\omega)\sqrt{\det(g(x - s\omega))}$$

Then $\tilde{q}$ shares some property of q. Then we can write

$$Qu(x,t) = t^{-\frac{p}{2}-1} \int \tilde{q}(x, t^{1/2}, \omega)d\omega$$

PROPOSITION 4. *(Taylor expansion) For all $N \in \mathbb{N}$, we have*

$$\tilde{q}(x,s,\omega) = \sum_{k=1}^{N-1} s^k (\partial_s^k \tilde{q})(x,s,\omega) + s^{2N} \tilde{q}_{2N}(x,s,\omega)$$

Note that

$$(\partial_s^k \tilde{q})(x,s,-\omega) = (-1)^{p+k} (\partial_s^k \tilde{q})(x,s,\omega)$$

by the property of Ψ^{-p} and chain rule.

Proof. Should be trivial.

Discussion. *Some cases: If p is even, then*

$$\partial_s^k \tilde{q}(x,s,\omega)$$

has the same even/odd parity with k. We can write

$$Qu(x,t) = t^{-\frac{p}{2}-1} \sum_{k=0}^{N-1} t^{k/2} \int \partial_s^k \tilde{q}(x,s,\omega) d\omega + t^{-\frac{p}{2}-1} (t^{1/2})^{2N} \int \tilde{q}(x,t^{1/2},\omega)$$

where as the term

$$\partial_s^k \tilde{q}(x,s,\omega) d\omega = 0$$

if k is odd because of the even/odd parity. We note we can rewrite it as

$$t^{-\frac{p}{2}-1} \sum_{k=0}^{N-1} t^k f_{2k}(x) + t^{-p/2} t^N F_{2N}(x,t)$$

where

$$f_{2k}(x) = \int_{\mathbb{R}^n} \partial_s^{2k} (q(x,0,\omega)) d\omega \in C^\infty(\mathbb{R}_t^n), F_{2N}(x,t) = \int_{\mathbb{R}^n} \tilde{q}(x,t^{1/2},\omega)$$

REMARK 45. *I think there is some typo on the indices for the fourth equation.*

COROLLARY 9. *In conclusion we have*

$$Qu(t,x) \in t^{-\frac{p}{2}-1} \in C^\infty([0,\infty)_t \times \mathbb{R}^n)$$

Execrise 10. *Try to do the same thing for p is odd.*

3.7 More Properties of Ψ^p

THEOREM 35. *For all $p \in \mathbb{Z}, k \in \mathbb{N}$ we have*

$$\Psi^p \subset \Psi^{p+k}$$

and

- $$\partial_k : \Psi_h^p \to \Psi_h^{p+2}$$

- $$P \in Diff^m(\mathbb{R}^n) \to P : \Psi^p \to \Psi_h^{p+m}$$

Proof. We prove explicitly. Let $Q \in \Psi_n^p$. Then we have

$$Qu = (Q\varphi)(t,x) = t^{-\frac{n}{2}-\frac{p}{2}-1}\int q(t^{1/2},x,\frac{x-y}{t^{1/2}})\varphi(y)dy \tag{3.20}$$
$$= t^{-\frac{n}{2}-\frac{p+k}{2}-1}\int t^{k/2}q(t^{1/2},x,\frac{x-y}{t^{1/2}})\varphi(y)dy \tag{3.21}$$
$$= t^{-\frac{n}{2}-\frac{p+k}{2}-1}\int \tilde{q}(x,t^{1/2},\frac{x-y}{t^{1/2}})dy \tag{3.22}$$
$$\tag{3.23}$$

Check if it is okay with with even-odd parity:

$$\tilde{q}(x,s,\omega) = s^k q(x,s,\omega)$$

Remark 46. *I think this "proof" is completely false. This should follow straightforwardly from chain rule.*

Definition 25. *Let $Q \in \Psi_n^p$, such that its smoothing kernel is given by*

$$Q = t^{-\frac{n}{2}-\frac{p}{2}-1}\int q(x,t^{n/2},\frac{x-y}{t^{1/2}})dg(y)$$

Then we have

$$\sigma_p(Q)(x,\omega) = q(x,s,\omega) : \Psi_n^p \to \mathbb{S}^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$$

Theorem 36. *The following sequence is exact:*

$$0 \to \Psi_n^{p-1} \to \Psi_n^p \xrightarrow{\sigma_p} \mathbb{S}(\mathbb{R}^n,\mathbb{R}^n) \to 0$$

where

$$\sigma_p(Q) = 0 \leftrightarrow Q \in \Psi_n^{p-1}$$

Proof. Obvious.

Discussion. *Why is σ_p important?*

Ultimate goal: Find $Q \in \Psi_n^{-2}$ such that

$$(\partial_t + L)Q = 0$$

and

$$Qu|_{t=0} = 0$$

In other words we want find $Q \in \Psi_n^{-2}$ such that

$$(\partial_t + L)Q \in \Psi_n^p$$

and

$$q(x, s, \omega) = 0$$

In the end: We will find $Q \in \Psi_n^{-2}$ such that

$$(\partial_t + L)Q = t^{-\frac{n}{2}-1} \int q(x, s, \omega) dg(y)$$

where

$$q(x, s, \omega) = 0, s = 0$$

In other words:

$$(\partial_s^k)(x, 0, \omega) = 0, \forall k \in \mathbb{N}$$

and

$$Qu|_{t=0} = u$$

Note If we find $Q \in \Psi_n^{-2}$ such that $Qu|_{t=0} = u$ for all u, and $(\partial_t + L)Q = R$ where R has $g(x, s, \omega) = 0$ at $s = 0$, then this will imply Hodge theorem!

Remark 47. *I am a little confused why it would..*

Discussion. *After this, we (not readable) R such that RHS=0.*

What condition do we need on $Q \in \Psi_n^{-2}$ to get $Qu|_{t=0} = u$?

Theorem 37. *If $Q \in \Psi_n^p(\mathbb{R}^n), p \leq -2$ and $u \in C_c^\infty(\mathbb{R}^n)$, then*

$$Q(u)|_{t=0} = \begin{cases} c(x)u(x) & \textit{where} c(x) = \int \sigma_2(Q)(x, s, \omega) dx \\ 0 & \textit{if} p \leq -3 \end{cases}$$

We note that

$$Qu|_{t=0}(x) = [\int \sigma_{-2}Q(x, \omega)\sqrt{\det(g(x, s, \omega))} d\omega] u(x, t)\omega),$$

In particular we can make $Qu|_{t=0} = u$ by choosing Q such that

$$\int \sigma_{-2}(Q)(x, \omega)\sqrt{\det(g(x))} d\omega = 1$$

(normalizing condition)

Proof. Let $p = -2$, then we have

$$Qu(x, t) = \int q(x, t^{1/2}, \omega) u(x, t)\omega)\sqrt{\det(g(x, s, \omega))} d\omega$$

Therefore

$$Qu|_{t=0} = \int q(x, 0, \omega) u(x)\sqrt{\det(g(x, s, \omega))} d\omega = [\int \sigma_{-2}(Q)(x, \omega) dg(\omega)] u(x)$$

3.8 Towards Hodge Theorem

We now state a generalization of theorem 34:

THEOREM 38. *If $Q \in \Psi^p_n$ then*

- $$\sigma_{p+2}(\partial_t Q) = (-\frac{n}{2} - \frac{p}{2} - 1 - \frac{1}{2}\omega \cdot \partial_\omega)\sigma_p(Q)$$

- $$P = \sum_{|I|=m} a_I(x)\partial_{\omega_I} \to \sigma_{p+m}(P \circ Q) = \sum_{|I|=m} a_I(x)\partial_{\omega_I}\sigma_p(Q)$$

Proof. Do it as your homework!

Example 31. *For example, let*

$$L = -\sum g^{ij}(x)\partial_{x_i}\partial_{x_j} + L'$$

where L' is a lower order operator. Then we have

$$\sigma_{p+2}(LQ) = \Delta_{x_I}\sigma_p(Q)$$

where

$$\Delta_{X_I} = -\sum_{ij} g^{ij}(x)\partial_{w_i}\partial_{w_j}$$

In particular we have

$$\sigma_{p+2}((\partial_k + L)Q) = (-\frac{n}{2} - \frac{p}{2} - 1 - \frac{1}{2}\omega\partial_\omega + \Delta_{x_I})\sigma_p(Q)$$

LEMMA 23. *We recall theorem 37: If $Q \in \Psi^p_n(\mathbb{R}^n), p \leq -2$ and $u \in C^\infty_c(\mathbb{R}^n)$, then*

$$Q(u)|_{t=0} = \begin{cases} c(x)u(x) & \textit{where} c(x) = \int_{\mathbb{R}^n} \sigma_2(Q)(x,s,\omega)dx \\ 0 & \textit{if} p \leq -3 \end{cases}$$

We note that

$$Qu|_{t=0}(x) = [\int \sigma_{-2}Q(x,\omega)\sqrt{\det(g(x,s,\omega))}d\omega]u(x,t)\omega),$$

Proof. This was proved earlier.

DEFINITION 26. *We define*

$$\Psi_n^{-\infty}(\mathbb{R}^n) = \bigcap_{p \in \mathbb{Z}} \Psi_n^p(\mathbb{R}^n)$$

in the sense that

$$\cdots \Psi_n^{-2} \subseteq \Psi_n^{-1} \subseteq \Psi_n^0 \subseteq \cdots$$

This means

$$\forall Q \in \Psi_n^{-\infty} \leftrightarrow \forall p \in \mathbb{Z}, \exists q_p \in \mathbb{S}(\mathbb{R}^n, \mathbb{R}^n)$$

such that

$$Q = t^{-\frac{n}{2}-\frac{p}{2}-1} \int_{\mathbb{R}^n} q_p(x, t^{1/2}, \frac{x-y}{t^{1/2}}) dy$$

We now come to the characterization theorem:

THEOREM 39.

$$Q \in \Psi_n^{-\infty} \to Qu(x,t) = \int k(x,y) u(y) dg(y)$$

where

$$k(t,x,y) \in C^\infty([0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n)$$

and for all l we have

$$\partial_k^l|_{t=0} = 0$$

such that

$$\forall N \in N, k(t,x,y) = t^N k_N(t,x,y), k_N \in C^\infty((0,\infty) \times \mathbb{R}^n \times \mathbb{R})$$

Proof. The proof is clear by establishing the smoothing kernel is unique. We can consider the case where $u(y)$ is a step function, and since $k(x,y)$ is the same on the support of $u(y)$, we must have uniqueness. The rest is obvious.

REMARK 48. *We can see it more clearly by the fact that if*

$$f \in C^\infty, \partial_t^1 f(0) \cdots \partial_t^{N-1} f(0) = 0,$$

then we can write

$$f(t) = t^N F_N(t)$$

by Taylor expansion.

REMARK 49. *This is sloppy - how do we know the function is analytic, not just smooth?*

Discussion. *Our goal is to find*

$$\mathcal{H} \in \Psi_n^{-2}(\mathbb{R}^n), (\partial_t + L)\mathcal{H} = 0, \mathcal{H}_{t=0} = Id$$

Here is the trick: First find $Q \in \Psi_{\mathcal{H}}^{-2}$ *such that*

$$(\partial_t + L)Q = \mathcal{R} \in \Psi_n^{-\infty}(\mathbb{R}^n)$$

- *Then the idea is to move to compact manifolds and show that*

$$(\partial_t + L)Q = \mathcal{R} \in \Psi_n^{-\infty}(M)$$

- *Finally: We get rid of* $\mathcal{R}$ *and show that*

$$(\partial_t + L)H = 0$$

How to do Step 1? Our goal is to find

$$Q = t^{-n/2} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}}) dg(y)$$

which also require

$$Qu(x,t)|_{t=0} = u(x), \forall u \in C_c^\infty(\mathbb{R}^n)$$

Write out our (to be determined) $q(x, s, \omega)$ *as*

$$q(x, s, \omega) \sim \sum s^k q_k(x, \omega)$$

where

$$q_k(x, \omega) = \partial_s^k q(x, 0, \omega)$$

Thus we have

$$Q = Q_0 + Q_1 + Q_2 + \cdots$$

where

$$Q_k = \int t^{-\frac{n}{2}+\frac{k}{2}} \int q_k(x, \frac{x-y}{t^{1/2}}) dg(y)$$

We will find

$$q_0, q_1 \cdots q_n, \cdots$$

such that

$$(\partial_k + L)Q \in \Psi_n^{-\infty}, Q|_{t=0} = Id$$

3.9 The idea of approximating the heat kernel using paramatrix

Okay, let us do it as if it does exist, then see what comes!

We want

$$(\partial_t + L)(Q_0 + \cdots Q_n + \cdots) \equiv 0$$

at $t = 0$. *We distribute it and we have*

$$\underbrace{(\partial_t + L)Q_0}_{\in\Psi_n^0} + \underbrace{(\partial_t + L)Q_1}_{\in\Psi_n^1} + \cdots = 0$$

The idea is

- *Choose* Q_0 *such that*

$$\sigma_0(\partial_t + L) = 0$$

  So we have

$$R_0 = (\partial_t + L)Q_0 \in \Psi_n^{-1}$$

  Then we have

$$(\partial_t + L)(Q_0 + Q_1 + \cdots) = R_0 + \underbrace{(\partial_t + L)Q_1}_{\in\Psi_n^{-1}} + \underbrace{(\partial_t + L)Q_2}_{\in\Psi_n^{-2}}$$

  where R_0 *is now 'known'.* ***(Is there a typo in the formula above? It is different from the previous one!)***

- *Choose* Q_1 *such that*

$$\sigma_{-1}(R_0 + \overbrace{(\partial_t + L)Q_1}^{R_1}) = 0$$

  at $t = 0$. *Then this implies*

$$R_0 + R_1 \in \Psi_n^{-2}$$

  We now have

$$\therefore(\partial_t + L)(Q_0 + Q_1 + \cdots) = \underbrace{R_0 + R_1}_{\Psi_n^{-2}} + \underbrace{(\partial_t + L)Q_2}_{\Psi_n^{-2}} + \underbrace{(\partial_t + L)Q_3}_{\Psi_n^{-3}} \cdots$$

- *Choose* Q_2 *such that*

$$\sigma_{-2}(R_0 + R_1 + (\partial_t + L)Q_2) = 0$$

  at $t = 0$. *Then this implies*

$$R_0 + R_1 + (\partial_t + L)Q_2 \in \Psi_n^{-3}$$

- *Summary: For all $k \in \mathbb{N}$ we will find $Q_k \in \Psi_n^{-2-k}$ such that*

$$\sigma_{-k}(R_0 + R_1 + \cdots R_{k-1} + (\partial_t + L)\sigma_k) = 0$$

where

$$R_j = (\partial_t + L)Q_j$$

and

$$Q|_{t=0} = Id$$

3.10 Moving forward

Proof. Our goal is: Recall that we want

$$Q_0 \in \Psi_n^{-2}$$

such that

$$\sigma_0((\partial_t + L)Q_0) = 0 \tag{3.24}$$

and

$$(Q_0 u)|_{t=0} = u, \forall u \in C_c^\infty(\mathbb{R}^n) \tag{3.25}$$

We note that (3.24) holds if and only if

$$(-\frac{n}{2} - \frac{1}{2}\omega\partial_\omega + \Delta_{X,\omega})q_0(x,\omega) = 0$$

where

$$w\partial_\omega = \sum \omega_j \partial_{\omega_j}$$

and

$$\Delta_{x,\omega} = -\sum g^{ij}(x)\partial_{\omega_x}\partial_{\omega_j}, q_0(x,\omega) = \sigma_0(Q_0)$$

We note that (3.25) holds if and only if

$$(\int q_0(x,\omega)dg(\omega))u(x) = u(x), \forall u \in C_c^\infty(\mathbb{R}^n)$$

This is equivalent to

$$\int_{\mathbb{R}^n} q_0(x,\omega)dg(\omega) = 1, \forall x \in \mathbb{R}^n$$

Thus, we want $q_0(x,\omega)$ such that both

$$(-\frac{n}{2} - \frac{1}{2}\omega \cdot \partial_\omega + \Delta_{x,\omega})q(x,\omega) = 0, \int q_0(x,\omega)dg(\omega) = 1$$

are satisfied. Note that the first one is a PDE in ω.

Execrise 11. *Show that*

$$q_0(x,\omega) = \frac{1}{(4\pi)^{n/2}} e^{-|\omega|^2_{g_x}}$$

satisfies above.

Hint: Observe that $q_0(x,\omega)$ solves the above equation is equivalent to

$$\mathcal{H}u(\omega) = \int_{R^n} t^{-\frac{n}{2}} q_0(x, \frac{x-\omega}{t^{1/2}}) u(\omega) dg_x(\omega)$$

satisfying

$$(\partial_t + \nabla_{x,\omega})\mathcal{H} = 0, \mathcal{H}|_{t=0} = Id$$

Proof. (I think a classical proof using Fourier transform would work).

Discussion. *So let*

$$Q_0 = t^{-n/2} \int q_0(x, \frac{x-y}{t^{1/2}}) dg(y)$$

where

$$q_0(x,\omega) = \frac{e^{-|\omega|^2_{g_x}}}{(4\pi)^{n/2}}$$

Now we want to find $Q_1 \in \Psi^{-3}_{\mathcal{H}}$ such that

$$\sigma_{-1}(R_0 + (\partial_k + L)Q_1) = 0$$

where

$$R_0 = (\partial_t + L)Q_0 \in \Psi^{-1}_H$$

is already known. In other words, we want to find $Q_1 \in \Psi^{-3}_{\mathcal{H}}$ such that

$$\sigma_{-1}((\partial_t + L)Q_1) = -Q_{-1}(R_0)(x,\omega)$$

where by previous construction the left hand is known. We now re-write it as

$$\underbrace{(-\frac{n}{2} - \frac{1}{2}\omega\cdot\partial_\omega + \Delta_{x,\omega})\overbrace{q_1(x,\omega)}^{\in\sigma_{-3}(Q_1)}}_{\text{For all } x\text{, that is a PDE in } \omega\text{, so we can solve it by Fourier transform in } q_1} = r_0(x,\omega) \in \mathbb{S}(\mathbb{R}^n,\mathbb{R}^n)$$

We notice this is a PDE which we can solve explicitly. Thus there exists $q_1(x,\omega) \in \mathbb{S}(\mathbb{R}^n,\mathbb{R}^n)$ such that

$$Q_1 = t^{-\frac{n}{2}+\frac{1}{2}} \int q_1(x, \frac{x-y}{t^{1/2}}) dg(y)$$

Similarly, we can find $Q_2 \in \Psi^{-4}_{\mathcal{H}}$ such that

$$\sigma_{-2}(R_0 + R_1 + (\partial_t + L)Q_2) = 0$$

which is equivalent to

$$(-\frac{n}{2}+1-\frac{1}{2}\omega\cdot\partial_\omega+\Delta_{x,\omega})\underbrace{q_2(x,\omega)}_{\sigma_{-4}(Q_2)}=-\sigma_{-2}(R_0+R_1)(x,\omega)$$

Then we can again find $q_2(x,\omega)$ *via fourier transform. Thus we have*

$$Q_2=t^{-\frac{n}{2}+1}\int q_2(x,\frac{x-y}{t^{1/2}})dg(y)$$

The strategy is now clear: We continue by induction we get

$$q_0,q_1,q_2\cdots\in\mathbb{S}(\mathbb{R}^n,\mathbb{R}^n)$$

THEOREM 40. *(Easy!) There exist* $q(x,s,\omega)\in\mathbb{S}^{-\infty}(\mathbb{R}^{n+1},\mathbb{R}^n)$ *such that*

$$q\sim\sum_{k=0}^{\infty}s^k q_k(x,\omega)$$

and

$$q(x,-s,-\omega)=q(x,s,\omega)\forall x,s,\omega\leftrightarrow\forall k,\partial_s^k q(x,s,\omega)|_{s=0}=q_k(x,\omega)$$

Proof. The second one follows from definition. The first one should follow from definition of Schwartz class. But a proof of the first time is not immediately clear to me. It seems, for example we need to have a radius of convergence test for s.

Discussion. *Now, let*

$$Q=t^{-\frac{n}{2}}\int q(x,t^{1/2},\frac{x-y}{t^{1/2}})dg(y)$$

We have finally reached the theorem:

THEOREM 41.

$$(\partial_k+L)Q\in\Psi_{\mathcal{H}}^{-\infty}$$

3.11 Brief Review

We recall the basic setup:

DEFINITION 27.

$$Q \in \Psi^p_{\mathcal{H}} : C^\infty_c(\mathbb{R}^n) \to C^\infty(0, \infty \times \mathbb{R}^n)$$

means

$$Qu(x,t) = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}})u(y)dg(y)$$

and Q's grading is dependent on p:

$$Q : C^\infty_c(R^n) \to \begin{cases} t^{-\frac{p}{2}-1}C^\infty([0,\infty) \times \mathbb{R}^n) & p \text{ even} \\ t^{-\frac{p}{2}-\frac{1}{2}}C^\infty([0,\infty) \times \mathbb{R}^n) & p \text{ odd} \end{cases}$$

Its boundary value is given by

$$Qu|_{t=0} = \begin{cases} c(x)u(x) & p = -2, c(x) = \int \sigma_2(Q)(x,\omega)dg_x(\omega) \\ 0 & p \le -3 \end{cases}$$

We want to show that that given a generalized Laplacian, $L \in \mathit{Diff}^2(\mathbb{R}^n)$ there exist

$$Q \in \Psi^{-2}_{\mathcal{H}}(\mathbb{R}^n), (\partial_t + L)Q = \mathcal{R} \in \Psi^{-\infty}_{\mathcal{H}}(\mathbb{R}^n)$$

The idea is: For all $Q \in \Psi^{-2}_{\mathcal{H}}$, $\mathcal{R} = (\partial_t + L)Q \in \Psi^0_H(R^n)$. Now we choose Q such that

$$\sigma_0(\mathcal{R}) = 0 \to \mathcal{R} \in \Psi^{-1}_{\mathcal{H}}(\mathbb{R}^n)$$

similarly

$$\sigma_{-1}(\mathcal{R}) = 0 \to \mathcal{R} \in \Psi^{-2}_{\mathcal{H}}(\mathbb{R}^n)$$

$$\sigma_{-2}(\mathcal{R}) = 0 \to \mathcal{R} \in \Psi^{-3}_{\mathcal{H}}(\mathbb{R}^n)$$

We want to choose Q such that $\sigma_{-k}\mathcal{R} = 0, \forall k \in \mathbb{N}$. It is clear that we did it already last time!

REMARK 50. *We just need to solve the PDE!*

3.12 Move on to the manifolds

Let (M,g) be a compact, oriented, Riemannian manifold:

DEFINITION 28. $\Psi^{-\infty}_{\mathcal{H}}(M)$ *consists of operators*

$$\mathcal{R} : C^\infty(M) \to C^\infty([0,\infty)_t \times M)$$

of the form

$$(Ru)(x,t) = \int_M r(t,x,y)u(y)dg(y)$$

where

$$r(t,x,y) \in C^\infty([0,\infty)_t \times M \times M)$$

and for all k,

$$\partial_t^k r(k,x,y)|_{t=0} = 0$$

Definition 29. *Let $p \in \mathbb{Z}$. Then $\Psi_H^p(M)$ consists of operators $Q : C^\infty(M) \to C^\infty([0,\infty) \times M)$ such that there exists a coordinate cover $\{\mathcal{U}_i\}_{i=1}^N$ and a corresponding partition of unity $\{\varphi_i\}_{i=1}^N$ and $\Psi = \{\Psi_i\}_{i=1}^N$, where $\Psi_i \in C^\infty(M)$, $\operatorname{supp}\Psi \subseteq \mathcal{U}_i$, $\Psi_i \equiv 1$ on $\operatorname{supp}\varphi_i$. And $Q_i \in \Psi_{\mathcal{H}}^p(\mathbb{R}^n)$ such that*

$$Q = (\sum_{k=1}^N \Psi_i Q_i \varphi_i) + \mathcal{R}, \mathcal{R} \in \Psi_{\mathcal{H}}^{-\infty}(M)$$

Discussion. *Why it this definition natural? Let us look at*

$$\Psi_1 Q_1 \varphi_1 : C^\infty(M) \to C^\infty((0,\infty) \times M)$$

Let $u \in C^\infty(M)$, then we see

$$\varphi_1(u) \to \textit{cuts the support of } u$$

and

$$Q_1\varphi_1(u) \to \textit{maps to the correct space}$$

while

$$\Psi_1 Q_1 \varphi_1(u) \to \textit{cuts the support again!}$$

as the function is now on

$$C^\infty([0,\infty)_c, \mathcal{U}_1)$$

Of course, we would have to prove the definition is coordinate independent, but we do not need to use it!

Theorem 42. *Let $L \in \mathit{Diff}^2(M)$ be a generalized Laplacian,*

$$\sigma_2(L)(\xi) = |\xi|^2, \forall |\xi| \in T^*M$$

Then there exist $Q \in \Psi_{\mathcal{H}}^{-2}(M)$ such that

$$(\partial_k + L)Q = \mathcal{R} \in \Psi_{\mathcal{H}}^{-\infty}(M)$$

Proof. Let $\{\mathcal{U}\}_{i=1}^N$ be a coordinate cover of M. Let $\{\varphi_i\}_{i=1}^N$ be a corresponding partition of unity. . Let $\{\Psi_i\}$ be smooth functions with

$$\Psi_i \in C_c^\infty(\mathcal{U}_i), \Psi_i(m) = 1, m \in \operatorname{supp} \varphi_i$$

and let

$$Q_i \in \Psi_{\mathcal{H}}^{-2}(\mathbb{R}^n)$$

be such that

$$\forall i, (\partial_t + L_i)Q_i = \mathcal{R}_i \in \Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n), L_i = L|_{C^\infty(\mathcal{U}_i)} \in \text{Diff}^2(\mathbb{R}^n)$$

Now let

$$Q = \sum_{i=1}^N \Psi_i Q_i \varphi_i \in \Psi_{\mathcal{H}}^{-2}(M)$$

Execrise 12. *Show that*

$$(\partial_t + L)Q = \mathcal{R} \in \Psi_{\mathcal{H}}^{-\infty}(M)$$

Proof. This is obvious because the differential operator is only defined locally, and we have proved the statement already in $\mathbb{R}^n$!

Discussion. *We note that*

$$Qu|_{t=0} = u, \forall u \in C^\infty(M)$$

because

$$Qu|_{t=0} = \sum \Psi_i Q_i \varphi_i(u)|_{t=0} = \sum \Psi(\varphi_i(u) = \sum \varphi_i u = u$$

Also

$$Q : C^\infty(M) \to C^\infty([0,\infty) \times M)$$

This is because

$\Psi_{\mathcal{H}}^p(M)$ *has the same mapping properties as* $\Psi_{\mathcal{H}}^p(\mathbb{R}^n)$.

We will see next time how to get rid of $\mathcal{R}$ *so that*

$$(\partial_t + L)\mathcal{H} = 0$$

3.13 Proof of Hodge theorem

Now let us see how

$$\left\{(\partial_t + L)Q = R \in \Psi_{\mathcal{H}}^{-\infty}(M), Q|_{t=0} = Id\right\} \to \textit{Hodge theorem}$$

THEOREM 43.

$$C^\infty(M) = LC^\infty M \oplus \ker^* L$$

Execrise 13. *Apply the theorem to $L = \mathcal{D}^2$, where $\mathcal{D}$ is a Dirac operator. Recall that since D is self-adjoint, $\mathcal{D}^2$ is the generalized Laplacian.*

COROLLARY 10. *We thus conclude that*

$$C^\infty(M,E) = \mathcal{D}^2 C^\infty(M,E) \oplus \ker \mathcal{D}^2$$

Proof. We know that there exist $Q \in \Psi_{\mathcal{H}}^{-2}(M), \mathcal{R} \in \Psi_{\mathcal{H}}^{-\infty}(M)$ such that

$$(\partial_t + L) = \mathcal{R}, Q|_{t=0} = Id$$

Consider

$$\tilde{Q} : C^\infty(M) \to C^\infty(M)$$

such that

$$\tilde{Q}u(x) = \int_0^1 Qu(t,x)dt$$

Observe that

$$\int_0^1 (\partial_t Qu(t,x) + L(Qu)(t,x))dt = \int_0^1 \mathcal{R}u(t,x)dt$$

So we have

$$\underbrace{Qu|_0^1}_{(Qu)(1,x)-u(x)} + L\tilde{Q} = \tilde{\mathcal{R}}u \to L\tilde{Q}u(x) = u(x) - (Qu)(1,x) + (\tilde{R}u)(x)$$

where

$$(\tilde{Q}u)(x) = \int_0^1 Qu(t,x), (\tilde{\mathcal{R}}u)(x) = \int_0^1 \mathcal{R}u(t,x)dt$$

We can further rewrite the above equation as

$$L\tilde{Q} = Id_{C^\infty M} + K$$

where

$$Ku(x) = (Qu)(1,x) + \int_0^1 \mathcal{R}u(t,x)dt$$

and

$$\tilde{Q} : C^\infty M \to C^\infty M$$

Discussion. *What kind of operator is*

$$K : C^\infty M \to C^\infty M?$$

We recall that

$$\mathcal{R}u = \int_M r(t,x,y)u(y)dg(y) \to \tilde{\mathcal{R}}u(x) = \int r'(x,y)u(y)dg(y)$$

where

$$r'(x,y) = \int_0^1 r(t,x,y)dt \in C^\infty(M \times M)$$

Recall that we defined Q by

$$Q = \sum \Psi_i Q_i \varphi_i$$

and we know that

$$\Psi_i Q_i \varphi_i u(t,x) = t^{-\frac{n}{2}} \Psi_i(x) \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}})\varphi_i(y)u(y)dg(y)$$

Therefore

$$\Psi_i Q_i \varphi_i(u)(1,x) = \int \Psi_i(x)q(x,x-y)\varphi_i(y)u(y)dg(y), \Psi_i(x)q(x,x-y)\varphi_i(y) = f_i(x,y) \in C^\infty(M,M)$$

In conclusion we have

$$Ku(x) = \int K(x,y)u(y)dg(y), K(x,y) \in C^\infty(M \times M)$$

Definition 30. *Let I^∞ be the collection of all the 'integral operators', meaning*

$$F : C^\infty(M) \to C^\infty(M)$$

such that

$$Fu(x) = \int f(x,y)u(y)dg(y), f \in C^\infty(M,M)$$

PROPOSITION 5. *Then there exist operator*

$$B : C^\infty(M) \to C^\infty(M)$$

such that

$$L \circ B = Id_{C^\infty M} + K, K \in I^\infty$$

Proof. This is done from previous discussion.

PROPOSITION 6. *We can write*

$$K = K_1 + K_2$$

where

$$Id + K_1 : C^\infty(M) \to C^\infty(M)$$

is invertible with an inverse

$$(Id + K_1)^{-1} = I + F, F \in I^\infty$$

and

$$K_2 : C^\infty(M) \to C^\infty(M)$$

is of 'finite rank', which means

$$K_2 u = \sum_{i,j} (u, f_i) g_i, f_i, g_i \in C^\infty M$$

which equals

$$\int k_2(x,y)u(y)dg(y), k_2(x,y) = \sum_{i,j} g_i(x)\overline{f_i(y)}$$

Here we note that $C^\infty M$ is an (incomplete) inner product space, a dense subspace of $L^2(M)$.

REMARK 51. *There seems to be a typo from Maucrio, I think it should be f_i, g_j instead.*

PROPOSITION 7. *There exist*

$$A : C^\infty M \to C^\infty M$$

such that

$$L \cdot A = Id + \tilde{F}$$

where $\tilde{F} \in I^\infty$ is a finite rank operator, meaning

$$\tilde{F} = \sum_{i,j} \sum (, \alpha_i) \beta_j$$

Proof. Here we prove Proposition 7 by assuming Proposition 6 and 5. We calculate:

$$LB = Id + K \tag{3.26}$$
$$\leftrightarrow LB = (Id + K_1) + K_2 \tag{3.27}$$
$$\leftrightarrow L \cdot \underbrace{A}_{B\circ(Id+F)} = Id + \underbrace{K_2 \circ (Id + F)}_{\tilde{F}} \tag{3.28}$$

We note that

$$A \in C^\infty M \to C^\infty M$$

as it is the composition of two such operators. Further $K_2 \circ F$ is of finite rank. The idea is

$$K_2 \circ F = \sum (Fu, f_i) g_j = \sum (u, \tilde{f}_i) g_j, \tilde{f}_i \in C^\infty(M)$$

Remark 52. *Is $\tilde{f}_i = F^* f_i$? I think the proof is clear because the composition of a compact operator with a finite rank operator must be finite rank.*

Proposition 8. *We claim that*

$$W^\perp \subseteq ImageL$$

where $W = span\{\alpha_i\}$ be a finite dimensional subspace of $C^\infty M$.

Proof. If $u \perp W$, then we have

$$(u, \alpha_i) = 0, \forall i$$

which implies

$$\tilde{F}(u) = 0$$

But we know

$$L \circ A = Id + \tilde{F}$$

Therefore

$$(L \circ A)u = u$$

and

$$u \in LC^\infty M = \text{Image} L$$

as desired.

Definition 31. *Let V be an inner product space. A subspace $U \subseteq V$ is '**big**' if there exist finite dimensional subspace $W \subseteq V$ such that*

$$W^\perp \subseteq U$$

LEMMA 24. *If U is a big subspace of an inner product space V, then we have*

$$V = U \oplus U^{\perp}$$

Proof. Exercise!

REMARK 53. *I suppose if the space is a Hilbert space then this is trivial, and we do not need the 'big' at here. For the moment I do not know how to prove it.*

I think the problem is U may not be a closed subspace of V, and a vector orthogonal to U will be orthogonal to its closure. Thus a naive direct sum miss vectors in $\overline{U} - U$. But for 'big' subspaces this cannot happen. If we let $v \in \overline{U} - U$ such that $v * u = 0, \forall u \in U^{\perp}$.

Discussion. *Let $V = C^{\infty}M$, $W = span\{\alpha_i\}$, $U = LC^{\infty}(M)$, then we have*

$$C^{\infty}(M) = LC^{\infty}M \oplus (LC^{\infty}M)^{\perp}$$

Finally we have

$$LC^{\infty}M^{\perp} = \ker L^{*}$$

Then

$$C^{\infty}M = LC^{\infty}(M) \oplus \ker L^{*}$$

3.14 Finishing the proof of Hodge theorem

THEOREM 44. *Let V be an inner product space, let $A \subseteq V$ be 'big', such that there exist subspace $W \subseteq V, W^{\perp} \subseteq U$. Then we have*

- $$\dim(U^{\perp} < \infty$$
- $$V = U \oplus U^{\perp}$$

Proof. Here is a proof finished essentially by others:

Assume that W is finite-dimensional with $W^{\perp} \subseteq U$. The goal is to show that $U \oplus U^{\perp} = V$. All '$\oplus$' decompositions are orthogonal in what follows.

Because W is finite-dimensional then $V = W \oplus W^{\perp}$ which can be seen by choosing any basis of W and using Gram-Schmidt to find an orthonormal basis $\{e_1, \cdots, e_n\}$ of W. Then every $v \in V$ can be written as

$$v = \left(v - \sum_{j=1}^{n}(v, e_j)e_j\right) + \sum_{j=1}^{n}(v, e_j)e_j.$$

So $W^\perp \oplus W = V$. Assuming that $W^\perp \subseteq U$, it follows that every $u \in U$ can be written as $u = w_\perp + w$ where $w_\perp \in W^\perp$ and $w \in W$. Because $w_\perp \in W^\perp \subseteq U$, then $u - w_\perp = w \in U$, which gives the decomposition

$$U = W^\perp \oplus (U \cap W).$$

Because $U \cap W$ is a finite-dimensional subspace of W, then it follows that there exists a finite-dimensional subspace U' such that $(U \cap W) \oplus U' = W$; U' is found by completing an orthonormal basis of $U \cap W$ to one for W. Therefore $(U \cap W) \oplus U' = W$. Finally,

$$U \oplus U' = (W^\perp \oplus (U \cap W)) \oplus U' \tag{3.29}$$

$$= W^\perp \oplus ((U \cap W) \oplus U') \tag{3.30}$$

$$= W^\perp \oplus W = V. \tag{3.31}$$

This is enough to give $U' = U^\perp$ and $U \oplus U^\perp = V$.

Proof. (of Hodge theorem) Let $L \in \text{Diff}^k(M)$ be a generalized Laplacian on a compact, oriented, Riemannian manifold M. We showed that there exist $Q \in \Psi^{-2}(M)$, $\mathcal{R} \in \Psi^\infty_{\mathcal{H}}(M)$ such that

$$(\partial_t + L)Q = \mathcal{R}, Q|_{t=0} = Id$$

Recall that $\Psi^p_{\mathcal{H}}(M)$ consisting of linear maps

$$Q : C^\infty(M) \to C^\infty((0,\infty) \times M)$$

which locally are of the form

$$Qu(t,x) = t^{-\frac{n}{2}-\frac{p}{2}} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}})u(y)dg(y)$$

such that $q(x,s,\omega)$ is C^∞ in $(x,s) \times \mathbb{R}^{n+1}$ and Schwartz in $\omega \in \mathbb{R}^n$.

Similarly we recall that we defined $\Psi^{-\infty}(M)$ as the space of operators

$$\mathcal{R} : C^\infty(M) \to C^\infty([0,\infty) \times M)$$

of the form

$$Ru(t,x) = \int r(t,x,y)u(y)dg(y) \in C^\infty([0,\infty)_t \times M \times M)$$

such that

$$\partial_t^k|_{t=0} r(t,x,y) = 0, \forall k$$

To prove the Hodge theorem:

$$C^\infty(M) = LC^\infty M \oplus \ker L^* = LC^\infty M \oplus [LC^\infty M]^\perp$$

Proof. We review what we did earlier. Let

$$\tilde{Q} : C^\infty M \to C^\infty M$$

be defined by

$$(\tilde{Q})u(x) = \int^1 (Qu)(t,x)dt$$

Then we know

$$(\partial_t + L)_Q = \mathcal{R}$$

So if we integrate both sides we get

$$L\tilde{Q} = Id + K, K = -Q|_{t=1} + \int_0^1 \mathcal{R}dt$$

and here

$$Q|_{t=1} = \int q(x,1,x-y)dg(y), q(x,1,x-y) \in C^\infty(x,y)$$

Here

$$\int_0^1 \mathcal{R}dt = \int \tilde{r}(x,y)dg(y), \tilde{r}(x,y) = \int_0^1 r(t,x,y)dt$$

We have shown that with this we have the following decomposition:

$$L \circ \tilde{Q} = Id + K, K = \int f(x,y)dg(y), f(x,y) \in C^\infty(M \times M)$$

We also proved the following theorem:

Theorem 45.

$$K = K_1 + K_2$$

where

$$Id + K_1 : C^\infty(M) \to C^\infty(M)$$

is invertible. And

$$(Id + K_1)^{-1} = Id + \tilde{K}_1, K_2 = \int \tilde{f}_2(x,y)dg(y), \tilde{f}_2(x,y) = \sum_{i,j} \varphi_i(x)\Psi_j(y), \varphi_i, \Psi_j \in C^\infty M$$

Thus put them together we proved the Hodge theorem:

We have

$$L \circ \tilde{Q} = (Id + K_1) + K_2$$

and

$$L \circ \underbrace{\tilde{Q}(Id + \tilde{K}_1)}_{A} = Id + \underbrace{K_2(Id + \tilde{K}_1)}_{F}$$

where

$$F = \int \sum_{i,j} \alpha_i \beta_j(y)dg(y)$$

Observe that

$$Fu = \sum_{i,j} \int \alpha_i(x)\beta_j(y)u(y)dg(y) = \sum \alpha_i(x)\langle u, \overline{\beta_j}\rangle$$

This implies

$$F = \sum_{i,j} \langle , \overline{\beta}_j\rangle \alpha_i$$

Now let

$$W = \textit{span}\{\overline{\beta}_i\} \subset \mathbb{C}^\infty(M)$$

is finite dimension. Then we have

$$W^\perp \subseteq LC^\infty M$$

Because if $u \perp W$*, we have*

$$Fu = 0 \to LAu = u \to L(Au) = u \to u \in LC^\infty M$$

Thus we have

$$C^\infty(M) = LC^\infty M \oplus \ker L^* = LC^\infty M \oplus [LC^\infty M]^\perp$$

as desired.

Discussion. *We now prove the decomposition theorem of the operators:*

Proof. Let

$$I^0 = C^0(M \times M)$$

with

$$|f|_\infty = \max_{x,y} |f(x,y)|$$

is complete. For $f \in I^0$, we let $F : C^\infty M \to C^\infty M$ be the map:

$$u \to \int f(x,y)u(y)dg(y)$$

Now we define

$$I^\infty = C^\infty(M \times M) \subseteq I^0$$

Note that, If

$$F, G \in I^0 \to F \circ G \in I^0$$

and we can also define a product

$$C^0(M \times M) \times C^\infty(M \times M) \to C^0(M \times M)$$

by the map

$$(f \circ g)(x,y) = \int f(x,z)g(z,y)dg(z)$$

which is essentially the composition law for trace class operators.

THEOREM 46. *We have*

$$I_f \circ I_g = I_{f \circ g}$$

REMARK 54. *What is the definition in this case? The objects are not defined!*

LEMMA 25. *Let* $f \in C^0(M \times M)$*, with*

$$|f|_\infty < \frac{1}{vol(M)}$$

Then

$$\sum_{k=1}^{\infty} f^k$$

converges to an element of $C^\infty(M \times M)$*. In other words, the function*

$$g_N = \sum_{k=1}^{N} f^k \in C^\infty(M \times M)$$

is a Cauchy sequence in $C^0(M \times M)$ *and converges in* $C^0(M \times M)$*.*

LEMMA 26. *If* $h_1, h_2 \in C^\infty(M \times M)$*, then*

$$|h_1 \circ h_2|_\infty \le vol(M)|h_1|_\infty |h_2|_\infty$$

Proof. We have

$$|h_1 \circ h_2|(x, y) = \int h_1(x, z) h_2(y, z) dg(z)$$

and the rest is obvious.

We now prove Lemma 25:

Proof. Recall that if V is a Banach space, $v_i \in V$, then if

$$\sum_{i=1}^{\infty} |v_i| < \infty$$

then

$$\sum_{i=1}^{\infty} v_i$$

converges in V. This is clear since

$$g_N = \sum_{i=1}^{N} v_i$$

is a Cauchy sequence in V, so it must converge to $\sum_{i=1}^{\infty} v_i$ by completeness.

In our case, we have

$$\sum |f^k|_\infty < \infty$$

Therefore

$$\sum_{k=1}^{\infty} f^k$$

converges in $C^0(M \times M)$.

Remark 55. *Note that*

$$g : \sum_{k=1}^{\infty} \underbrace{f \circ \cdots \circ f}_{k} = f + f \circ g = f + f \circ f + f \circ g \circ f$$

We now go back to the proof of the original theorem:

By (very old?) definition, if $K \in I^\infty \leftrightarrow K = \int f(x,y) dg(y)$, then we can decompose it as

$$K = K_1 + K_2, Id + K_1 : C^\infty \to C^\infty M, (Id + K_1)^{-1} = Id + \tilde{K}_1$$

and

$$K_2 = \int \sum f_i(x) g_i(y) dg(y)$$

Consider the collection of finite sums of products:

$$S \subset C^0(M \times M) \leftrightarrow K(x,y) = \sum_{i,j} f_i(x) g_j(y), f_i, g_j \in C^\infty(M)$$

Then S is an algebra, separates points, and it is also closed under complex conjugation. Therefore by the Stone-Weistrauss theorem in functional analysis we have

$$\overline{S} = C^0(M \times M)$$

Now recall that

$$K = I_f$$

where

$$f \in C^\infty(M \times M) \subseteq C^0(M \times M)$$

Therefore by Stone-Weistrass theorem, there exist some element in S such that

$$\frac{|f(x,y) - \sum f_i(x) g_j(y)|_\infty}{|f(x,y)|_\infty} < \frac{1}{\mathit{vol} M}$$

Now observe that

$$K = I_f + I_{\tilde{f}} + I_n$$

where

$$h = \sum_{i,j} f_i(x) g_j(y)$$

Now let $K_1 = I_{\tilde{f}}, K_2 = I_K$. Finally we must show that

$$Id + K_1 = Id + I_{\tilde{f}} : C^\infty(M) \to C^\infty(M)$$

is invertible, and $()^{-1}$ is of the same form.

The idea is simple: We want to have:

$$(Id + K_1)^{-1} = Id + \sum_{j=1}^{\infty}(-K_1)^j$$

To make this precise, observe that by our construction

$$|-\tilde{f}|_\infty = |\tilde{f}|_\infty < \frac{1}{volM}, \tilde{f} \in C^\infty(M \times M) \subseteq C^\infty(M \times M)$$

Therefore we know

$$g = \sum_{j=1}^{\infty}(-\tilde{f})^j \in C^\infty(M \times M)$$

But we know that

$$g = -\tilde{f} - g \circ \tilde{f} = -\tilde{f} - \tilde{f} \circ g = -\tilde{f} + \tilde{f} \circ \tilde{f} + \tilde{f} \circ g \circ \tilde{f}$$

Here by definition we have

$$g(x,y) - \tilde{f}(x,y) + \int \tilde{f}(x,z)\tilde{f}(z,y)dg(z) + \int\int \tilde{f}(x,t)g(t,s)\tilde{f}(s,y)dg(t)dg(s)$$

Therefore

$$g(x,y) \in C^\infty(M \times M)$$

as well.

Lemma 27. *We have*

$$(Id + K_1)(Id + G) = Id, (Id + G)(Id + K_1) = Id$$

on $C^0(M)$, where G is the operator associated with $g(x,y)$.

Let us check it now:

$$(Id + K_1)(Id + G) = Id + K_1 + G + K_1G$$

But we know

$$g + \tilde{f} + \tilde{f}g = 0$$

Therefore

$$(Id + K_1)(Id + G) = Id$$

as desired.

3.15 Explicit construction of Heat kernel

Recall that L is a generalized Laplacian in Diff$^2(M)$, there is

$$H : C^\infty M \to C^\infty([0,\infty) \times M)$$

such that

$$(\partial_t + L)H = 0, H_{t=0} = Id$$

We know there exist

$$Q : C^\infty M \to C^\infty([0,\infty) \times M)$$

such that

$$(\partial_t + L)Q = \mathcal{R} \in \Psi_H^{-\infty}(M)$$

We want to get rid of $\mathcal{R}$.

We observe that the operator

$$(\partial_t + L) : C^\infty([0,\infty) \times M) \to C^\infty([0,\infty) \times M)$$

is a linear operator. Further both Q and $\mathcal{R}$ maps $C^\infty(M)$ to $C^\infty([0,\infty)\times M)$. We want to modify it so that it maps on $C^\infty([0,\infty) \times M)$ instead of $C^\infty(M)$.

Definition 32. *Here is a natural way to change the domain to $C^\infty([0,\infty) \times M)$: We can use 'convolution':*

$$\tilde{T} : C^\infty([0,\infty) \times M) \to C^\infty([0,\infty) \times M)$$

with

$$(\tilde{T}u)(t,x) = \int_0^t T(t-s)u(s)ds$$

where for fixed $s \in [0,\infty)$, we have $u(s) = u(s,) \in C^\infty M$.

$$T(t-s)u(s) = (Tu(s))(t-s,x)$$

Here we use convolution because of the classical Duhamel's principle:

Theorem 47. *(Duhamel's principle): We have the existence of an operator H is equivalent to the operator G. In formal language we have:*

$$\exists H : C^\infty(M) \to C^\infty([0,\infty) \times M) : \begin{cases} (\partial_t + L)H = 0 \\ H|_{t=0} = Id \end{cases}$$

is equivalent to

$$\exists G : C^\infty([0,\infty) \times M) \to C^\infty([0,\infty) \times M) : \begin{cases} (\partial_t + L)G = Id \\ G|_{t=0} = 0 \end{cases}$$

This arise out of the concern that, for any $g(t,x) \in C^\infty([0,\infty) \times M)$, we let

$$f(t,x) = G(g)(t,x)$$

Then we have

$$\begin{cases} (\partial_t + L)f(t,x) = g(t,x) \\ f(0,x) = 0 \end{cases}$$

In other words, f solves the in homogeneous heat equation!

Proof. Given H, let $G = \tilde{H}$. Given G, let $H = \partial_t G$. We now prove both claims:

- Assume H exists, define

  $$f = Gg = \tilde{H}g = \int_0^t H(t-s)g(s)ds, H(t-s)g(s)(x) = (Hg(s))(t-s,x)$$

  Let us show that

  $$\begin{cases} (\partial_t + L)f = g \\ f|_{t=0} = 0 \end{cases}$$

  We have

  $$(\partial_t + L)f = H(0)g(t,x) + \int_0^t \partial_t H(t-s)g(s)ds + \int_0^t LH(t-s)g(s)ds = g(t,x)$$

  Remark 56. *I think $H(0)g(t,x)$ is confusing, it should be $(Hg(t))(0,x) = g(t,x)$, where $g(t)$ is understand as fixed t function on M, and $Hg(t)$ maps it to a function on $[0,\infty) \times M$.*

- Assume there exists $G : C^\infty([0\times\infty) \times M) \to C^\infty([0,\infty) \times M)$ such that

  $$(\partial_t + L)G(g) = g, \forall g \in C^\infty([0,\infty) \times M), G(g)|_{t=0} = 0$$

  Define

  $$H : \underbrace{C^\infty M}_{u(x)} \to C^\infty([0,\infty) \times M)$$

  by

  $$(Hu)(t,x) = \partial_t G(u) \in C^\infty([0,\infty) \times M)$$

  We claim that

  $$(\partial_t + L)(Hu) = 0, Hu|_{t=0} = u$$

  because

  $$(\partial_t + L)(Hu) = (\partial_t + L)\partial_t G(u) = \partial_t((\partial_t + L))G(u) = \partial_t u = 0$$

  Using this we have

  $$(\partial_t + L)G(u)|_{t=0} = \partial_t G(g)|_{t=0} + LG(u)|_{t=0} = \partial_t G(u)|_{t=0} + G(u)|_{t=0} = \partial_t G(u)|_{t=0} = u$$

  because $u(x,t)$'s value is not dependent on t.

REMARK 57. *I think it is a bit odd that $u(x,t) = u(x)$ at all times. I do not know if I understand it wrong.*

Discussion. *With this in mind, we will have to show that there exist*

$$\exists G : C^\infty([0,\infty)\times M) \to C^\infty([0,\infty)\times M) : \begin{cases} (\partial_t + L)G = Id \\ G|_{t=0} = 0 \end{cases}$$

Go back to what we know: We know that

$$\exists Q, \mathcal{R} : C^\infty(M) \to C^\infty([0,\infty)\times M) \in \Psi_{\mathcal{H}}^{-\infty}$$

such that

$$(\partial_t + L)Q = \mathcal{R}$$

Now let

$$\tilde{Q}, \tilde{\mathcal{R}} \in C^\infty([0,\infty)\times M) \to C^\infty([0,\infty)\times M)$$

be the corresponding operators.

LEMMA 28. *We have*

$$(\partial_t + L)\tilde{Q} = Id_{C^\infty((0,\infty)\times M)} + \tilde{R}$$

Thus, we have

$$G = \tilde{Q} \circ (Id + \tilde{\mathcal{R}})^{-1}$$

if

$$(Id + \tilde{\mathcal{R}})^{-1}$$

exists.

Proof. We calculate like last time:

$$(\partial_t + L)\underbrace{\int_0^t Q(t-s)u(s)ds}_{\tilde{Q}u(t,x)} = \underbrace{Q(0)}_{Id}u(t) + \int_0^t (\partial_t + L)Q(t-s)u(s)ds \tag{3.32}$$

$$= u(t,) + \int_0^t R(t-s)u(s)ds \tag{3.33}$$

$$= u + \tilde{R}(u) \tag{3.34}$$

$$= (Id + \tilde{R})u \tag{3.35}$$

THEOREM 48. *We have*

$$Id + \tilde{\mathcal{R}} : C^\infty([0,\infty)\times M) \to C^\infty([0,\infty)\times M)$$

is invertible. Further we have

$$(Id+\tilde{\mathcal{R}})^{-1} = Id + \tilde{S}$$

where $\tilde{S} \in \Psi_{\mathcal{H}}^{-\infty}$.

COROLLARY 11. *Heat kernel exists!*

Proof. We have

$$\begin{aligned} G &= \tilde{Q} + \tilde{Q}\circ\tilde{S} && (3.36)\\ \to H &= \partial_t(\tilde{Q}+\tilde{Q}\circ\tilde{S}) && (3.37)\\ &= \partial_t\tilde{Q} + \partial_t\tilde{Q}\circ\tilde{S} + \tilde{S}\circ\partial_t\tilde{S} && (3.38)\\ &= Q + \underbrace{Q\circ\tilde{S} + \tilde{Q}\circ S}_{\in\Psi_{\mathcal{H}}^{-\infty}(M)} && (3.39)\end{aligned}$$

REMARK 58. *I think there is one item missing at here, we have*

$$G(g)|_{t=0} = 0, \forall g \in C^\infty([0,\infty)\times M)$$

It follows trivially from the construction, but one need this detail.

Proof. (of Theorem) Consider $\mathcal{L}$ be the collection of all convolution operators, where one identifies the convolution with a continuous function. So we have

$$(\tilde{F}u)(t,x) = \int_0^t F(t-s)u(s)ds, \tilde{\mathcal{R}} \in \mathcal{L}$$

Therefore

$$\tilde{\mathcal{R}} = C_f$$

where C_f is convolution with f. Then we have

$$\begin{aligned} (Id+\tilde{\mathcal{R}})^{-1} &= Id + \sum(-\tilde{\mathcal{R}})^k && (3.40)\\ &= Id + \sum(-C_f)^k && (3.41)\\ &= Id + \sum_{k=1}^{\infty} C_{(-f)^k} && (3.42)\end{aligned}$$

We still need to address the issue of convergence. For this we have the following theorem:

THEOREM 49. *(About convergence) Let $h \in C^0([0,\infty) \times M \times M)$, then the series*

$$j(t,x,y) = \sum (-h)^k (t,x,y)$$

converges point-wise and for any $t_0 > 0$ it converges uniformly on

$$[0,t_0] \times M \times M$$

In particular, we have

$$j \in C^\infty([0,\infty) \times M \times M)$$

Proof. Let $t_0 > 0$. We want to show that j converges uniformly. Claim: Let $|\cdot|_\infty$ to be the sup norm on $C^0([0,t_0) \times M \times M')$, so it is complete. Then for all $f_1, f_2 \in C^\infty([0,t_0]) \times M \times M)$ we have

$$|f_1 \circ f_2(t,x,y)|_\infty \leq t\text{vol}M|f_1|_\infty |f_2|_\infty$$

This is proved earlier in Lemma 26. We now claim that

$$|h \times h \times h \cdots (t,x,y)|_\infty \leq \frac{t^{k-1} vol(M)^{k-1} (|h|_\infty)^k}{(k-1)!}$$

Execrise 14. *: Consider $k = 3$, then use induction.*

REMARK 59. *This is not trivial. I do not really know where the $\frac{1}{(k-1)!}$ term coming from. Is it because we are integrating over a simplex? For example, for $k = 3$ we have*

$$h^3(x,y) = \int_M (\int_M h(x,z)h(w,z)dg(z))h(y,w)dg(w)$$

which equals

$$\int_{M\times M} h(x,z)h(w,z)h(y,w)dg(z)dg(w) \leq |h|_\infty^3 \int_{M\times M} dg(w)dg(z)$$

and it is hard to simplify this integral as ω, z should have nothing to do with each other. So we can only bound it by vol$(M)^2$. But in general we need a better estimate.

THEOREM 50. *We have*

$$|h^k|_\infty \leq \frac{t_0^{k-1} vol(M)^{k-1} |h|_\infty^k}{(k-1)!}$$

Discussion. *Observe that the right hand side equal*

$$|h|_\infty e^{t_0} vol(M) < \infty$$

Therefore

$$\sum (-h)^k$$

converges in $C^0([0,t_0]\times M\times M)$. Therefore, since t_0 is arbitrary, it converges pointwise to all t_0. We note that

$$j = -h - j * h = -h - h * j = -h + h * h^2 + h * j * h$$

Thus, $(\underbrace{Id + C_f}_{\tilde{\mathcal{R}}})^{-1}$ exists. Now by the previous lemma, we have

$$g(t,x,y) = \sum_{k=1}^{\infty} (-f)^k \in C^\infty([0,\infty)\times M\times M)$$

and converges uniformly on $[0,t_0]\times M\times M$, for all $t_0 > 0$.

Lemma 29. *Now we claim that*

$$g(t,x,y) \in C^\infty([0,\infty)\times M\times M), \partial_t^k g(t,x,y) = 0, \forall k$$

Proof. Easy!

Remark 60. *I think the first one should follow directly. The second one is proved earlier in a different setting.*

Lemma 30. *We have*

$$(Id + C_f)(Id + C_g) = Id$$

on $C^\infty([0,\infty)\times M)$.

Proof. We have

$$Id + C_{f+g+f*g}$$

but $g = -f - f * g$. We are done!

3.16 asympototic expansion

We let

$$E = E^+ \oplus E^- \to M$$

be a Hermitian vector bundle over a compact oriented Riemannian manifold. Let

$$\mathcal{D} : C^\infty(M,E) \to C^\infty(M,E)$$

be a $\mathbb{Z}_2$ graded generalized Dirac operator, such that $\mathcal{D}$ is adjoint and

$$\sigma(\mathcal{D})(\xi)^2 = |\xi|_p^2, \xi \in T^*M_p$$

and we have $\mathcal{D}$ anti-commutes with the grading:

$$\mathcal{D} : C^\infty(M,E^\pm) \to C^\infty(M,E^\mp)$$

Example 32. *We let*

$$E = \mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd}$$

Discussion. *Given any generalized Dirac operator, we have proved that*

$$\mathcal{H}_t = e^{-t\mathcal{D}^2}$$

exists such that

$$\begin{cases} (\partial_t + \mathcal{D})\mathcal{H}_t = Id \\ \mathcal{H}_t|_{t=0} = Id \end{cases}$$

Also, if

$$P : C^\infty(M, E_1) \to C^\infty(M, E_2)$$

is such that

$$P^*P, PP^*$$

are generalized Laplacians, then we have

$$Ind(P) = \mathrm{Tr}(e^{-tP^*P}) - \mathrm{Tr}(e^{-tPP^*})$$

Remark 61. *A typo here in the notes.*

Discussion. *Apply this formula to*

$$P = \mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$$

we have

$$Ind(\mathcal{D}^+) = \mathrm{Tr}(e^{-t\mathcal{D}^-\mathcal{D}^+} - \mathrm{Tr}(e^{-t\mathcal{D}^+\mathcal{D}^-}), \mathcal{D}^- = (\mathcal{D}^+)^*$$

Now we will see how to use heat operator in constructing Ind$(\mathcal{D}^+)$.

Theorem 51. *For any general Laplacian operator:*

$$L : C^\infty(M) \to C^\infty(M)$$

as $t \to 0$ *we have*

$$\mathrm{Tr}(e^{-tL}) \sim \sum_{k=1}^{\infty} t^{-\frac{n}{2}+k} a_k, a_k \in \mathbb{C}$$

More precisely, for all $N \in \mathbb{N}$, *there exist* C_N *such that*

$$|\,\mathrm{Tr}(e^{-tL}) - \sum_{k=0}^{N} t^{-\frac{n}{2}+k} a_k| \le t^{-\frac{n}{2}+N+1} C_N, \forall t \in (0,1]$$

In particular, the following stronger result holds:

COROLLARY 12. *The function*

$$f(t) = t^{n/2}\operatorname{Tr}(e^{-tL})$$

extends over $(0,\infty)$ *to an* C^∞ *function on* $[0,\infty)$.

Proof. Recall that

$$e^{-tL} = \underbrace{Q}_{\in\Psi_{\mathcal{H}}^{-2}(M)} + \underbrace{T}_{\in\Psi_{\mathcal{H}}^{-\infty}M}$$

Therefore by linearity we have

$$\operatorname{Tr}(e^{-tL}) = \operatorname{Tr}(Q) + \operatorname{Tr}(T)$$

Now we have

$$Tu = \int r(t,x,y)u(y)dy$$

where

$$r(t,x,y) \in C^\infty([0,\infty), M\times M)$$

and

$$\partial_t^k r(t,x,y) = 0, \forall k$$

This implies, for all $N\in\mathbb{N}$, we have

$$r(t,x,y) = t^N r_N(t,x,y)$$

Therefore

$$\operatorname{Tr}(T) = \int_M r(t,x,x)dg(x) \to t^{N/2}\operatorname{Tr}(T_N(t)) \in C^\infty([0,\infty))$$

and it is 0 in Taylor expansion at $t=0$. Therefore it has no contribution to $\operatorname{Tr}(e^{-tL})$!

REMARK 62. *I think there is another typo, the* $N/2$ *is* $n/2$ *in the notes.*

Proof. (continue) Now to compute $\operatorname{Tr}(Q)$, first we need to look at $\Psi_{\mathcal{H}}^{-2}(\mathbb{R}^n)$. We know that on $\mathbb{R}^n$ we have

$$Q(u)(x,t) = t^{-\frac{n}{2}}\int q(x,t^{1/2},\frac{x-y}{t^{1/2}})u(y)dy \tag{3.43}$$

$$= \int K_Q(t,x,y)u(y)dy \tag{3.44}$$

$$\to \operatorname{Tr}(Q) = \int K_Q(t,x,x)dx \tag{3.45}$$

$$= t^{-\frac{n}{2}}\int q(x,t^{1/2},0)dx \tag{3.46}$$

Recall that

$$\underbrace{q(x,s,\omega)}_{\text{Schwartz}} \in C^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^n)$$

and it satisfies

$$q(x,-s,\omega) = (-1)^p q(x,s,\omega) = q(x,s,\omega), p = -2$$

Therefore $q(x,,0)$ is even, and we have the Taylor expansion:

$$q(x,s,0) \sim \sum_{k=0}^{\infty} s^{2k}\tilde{a}_k(x)$$

where

$$\tilde{a}_k(x) = \frac{\partial_s^{2k} q(x,0,0)}{k!}$$

So we have

$$q(x,t^{1/2},0) \sim \sum_{k=0}^{\infty} t^k \tilde{a}_k(x) \tag{3.47}$$

Therefore

$$t^{n/2}\operatorname{Tr}(Q) = \sum_{k=0}^{\infty} t^k a_k, a_k \in \mathbb{C}$$

REMARK 63. *A notation issue - why $\tilde{a}$ becomes a now?*

Proof. (continue) Thus we have

$$\operatorname{Tr}(e^{-tL}) = \sum_{k=0}^{\infty} t^{-\frac{n}{2}+k} a_k, a_k \in \mathbb{C}$$

COROLLARY 13. *We have*

$$\operatorname{Tr}(e^{-t\mathcal{D}^-\mathcal{D}^+}) \xrightarrow{t\to 0} \sum_{k=1}^{\infty} t^{-\frac{n}{2}+k} a_k^+$$

as well as

$$\operatorname{Tr}(e^{-t\mathcal{D}^+\mathcal{D}^-}) \xrightarrow{t\to 0} \sum_{k=0}^{\infty} t^{-\frac{n}{2}+k} a_k^-$$

because we used Taylor expansion at the origin. In fact we have

$$a_k^{\pm} = (\frac{d}{dt})^k t^{\frac{n}{2}} \operatorname{Tr}(e^{-\mathcal{D}^{\mp}\mathcal{D}^{\pm}})|_{t=0}$$

REMARK 64. *Is there a typo here? I think there should be an $\frac{1}{(k)!}$ factor involved.*

COROLLARY 14. *We have*

$$\mathit{Ind}\mathcal{D}^+ \sim \sum_{k=0}^{\infty} t^{-\frac{n}{2}+k}(a_k^+ - a_k^-)$$

that is

$$t^{\frac{n}{2}}\mathit{Ind}\mathcal{D}^+ \in C^\infty([0,\infty))$$

COROLLARY 15. *Therefore if n is odd, then $t^{-\frac{n}{2}+k}$ would be zero when $t = 0$. Since the index is not dependent on k, we have*

$$\mathit{Ind}\mathcal{D}^+ = 0$$

for all odd dimensional manifolds.

REMARK 65. *It seems to me there is quantum jump from $\mathbb{R}^n$ to manifolds. Perhaps more detail is needed.*

Discussion. *Henceforth we assume $\dim M = 2m$. Then we have*

$$t^m\mathit{Ind}\mathcal{D}^+ \sim \sum_{k=0}^{\infty} t^k(a_k^+ - a_k^-)$$

Therefore

$$a_k^+ - a_k^- = 0, \forall k, k \neq m$$

and for $k = m$ we have

$$\mathit{Ind}\mathcal{D}^+ = m!(a_m^+ - a_m^-)$$

THEOREM 52. *We have*

$$\mathit{Ind}\mathcal{D}^+ = m!(a_m^+ - a_m^-)$$

where the difference is the m-th term in the Taylor series of $t^m(\mathrm{Tr}(e^{t\mathcal{D}^-\mathcal{D}^+}))-\mathrm{Tr}(e^{t\mathcal{D}^+\mathcal{D}^-}))$.

REMARK 66. *With all respect I think the $m!$ term is not needed. This is either a typo or a mistake.*

Discussion. *So we have reduced the topological problem to an analytical problem of Taylor series!*

Discussion. *Now the idea is to get into the Taylor expansion of the heat traces to compute the index. To do this we need to understand the geometry of $\mathcal{D}^+$ and $\mathcal{D}^-$.*

Chapter 4

Clifford Modules

4.1 Background review

Recall $E = E^+ \oplus E^-$. For all $p \in M$, we have a map

$$\sigma = \sigma(\mathcal{D}) : V \to Hom(W)$$

where $V = \mathbb{C}T_p^*M, W = E_p$ and

$$\sigma : \xi \to \sigma(\mathcal{D})(\xi)$$

Note: Since W is a Hermitian inner product space, it has orthogonal decomposition of two isotropical subspaces of equal dimension with respect to j. Hence we have the decomposition $W^\pm = E_p^\pm$.

We know that $V = \mathbb{C}T_p^*M$ is an even dimensional inner product space. Now σ satisfies the following properties:

- *Ellipticity:*

  $$\sigma(\xi)^2 = |\xi|^2$$

  (the top order term is elliptic)

- *Self-adjoint:*

  $$\forall \xi \in V, \sigma(\xi) : W \to W$$

  satisfies:

  $$\sigma(\mathcal{D})(\xi)^* = \sigma(\mathcal{D}^*)(\xi) = \sigma(\mathcal{D})(\xi)$$

- *Grading:*

  $$\forall \xi \in V, \sigma(\xi) : W^\pm \to W^\mp$$

DEFINITION 33. *Let V be a real inner product space. Let W be a complex Hermitian inner product space. Let*

$$\sigma : V \to Hom(W)$$

be a linear map. We call the pair (σ, V, W) a Clifford module if

- *For all $\xi \in V, \sigma(\xi)^2 = |\xi|^2$.*
- *For all $\xi \in V$, the endomorphism*

  $$\sigma(\xi) : W \to W$$

  is self-adjoint.
- *We call this Clifford module $\mathbb{Z}_2$ graded if*

  $$W = W^+ \oplus W^-$$

  such that the two subspaces are orthogonal to each other, and they have the same dimension. Further, we want

  $$\sigma : W^\pm \to W^\mp$$

Example 33. *(This is the only one we will need) We recall that*

$$(\sigma, V, W) = (\sigma(\mathcal{D}), T_p^*(M), Hom(E))$$

is a Clifford module, where $\mathcal{D}$ is a $\mathbb{Z}_2$ graded generalized Dirac operator.

LEMMA 31. *If $\xi, \eta \in V, \xi \perp \eta$, then we have*

$$\sigma(\xi)\sigma(\eta) + \sigma(\eta)\sigma(\xi) = 0$$

More generally we have

$$\sigma(\eta)\sigma(\xi) = -\sigma(\eta)\sigma(\xi) + 2\langle \xi, \eta \rangle Id$$

Proof. We have

$$\begin{aligned} \sigma(\xi+\eta)^2 &= |\xi+\eta|^2 && (4.1)\\ &= \langle \xi, \xi \rangle + 2\langle \xi, \eta \rangle + \langle \eta, \eta \rangle && (4.2)\\ &= \langle \xi, \xi \rangle + \langle \eta, \eta \rangle && (4.3)\\ &= \sigma(\xi)^2 + \sigma(\eta)^2 && (4.4) \end{aligned}$$

On the other hand by linearity

$$\begin{aligned} \sigma(\xi+\eta)^2 &= (\sigma(\xi)+\sigma(\eta))(\sigma(\eta)+\sigma(\xi)) && (4.5)\\ &= \sigma(\xi)^2 + \sigma(\eta)^2 + \sigma(\xi)\sigma(\eta) + \sigma(\eta)\sigma(\xi) && (4.6) \end{aligned}$$

Comparing (4.4) and (4.6) give the desired result.

Discussion. *Let us look at the structure of Clifford modules in relation to* $\wedge^* V$. *Observe that* $\mathbb{C} \wedge^* V$ *has a basis*

$$\{1, \varphi_{i_1} \wedge \varphi_{i_2} \cdots\}, i_1 < i_2 < \cdots$$

Also, we know

$$\varphi_i \wedge \varphi_j = -\varphi_j \wedge \varphi_i, i \neq j$$

while

$$\sigma(\varphi_i)\sigma(\varphi_j) = -\sigma(\varphi_j)\sigma(\varphi_i)$$

Theorem 53. *We let* $Cl(V)$ *be the algebra generated by* $\sigma(\xi) \in Hom(W), \xi \in V$. *Then we have the isomorphism:*

$$Cl(V) \cong \mathbb{C} \wedge^* V$$

To prove this, we do it by hand.

Execrise 15. *Do it!*

Remark 67. *I think this should follow directly by extending the map and Patodi's lemma.*

Definition 34. *We let*

$$c_k = i\sigma(\varphi_{2k-1})\sigma(\varphi_{2k}), A_k = \sigma(\varphi_{2k}), c_1 = i\sigma(\varphi_1)\sigma(\varphi_2), c_2 = i\sigma(\varphi_3)\sigma(\varphi_4) \in Hom(W), k = 1 \cdots m$$

Then we have

$$c_1 c_2 = c_2 \circ c_1, c_1^* = c_1$$

and

$$A_1 \cdots A_m$$

are anti-commuting involutions. Further we have

$$A_l c_k = \begin{cases} -c_k \circ A_l & if k = l \\ c_k \circ A_l & if k \neq l \end{cases}$$

Remark 68. *I think this is the so called 'spin representation' in other books.*

Remark 69. *The last time in the page is not readable.*

Theorem 54. *For all* $\lambda = (\lambda_1, \cdots \lambda_m) \in \mathbb{Z}_2^m$, *let*

$$W_\lambda = \{w \in W | c_k w = \lambda_k w\}$$

Then we have

$$W = \oplus_{\lambda \in \mathbb{Z}_2^m} W_\lambda, \forall \lambda \in \mathbb{Z}_2^m, \dim W_\lambda = \frac{\dim W}{2^m}$$

Proof. For the first part, do it. For the second part, for all $\lambda \in \mathbb{Z}_2^m$ and $1 \leq k \leq m$, in fact we have

$$A_k : W_{(\lambda_1 \cdots \lambda_m)} \to W_{(\lambda_1 \cdots \lambda_k \cdots \lambda_m)}$$

and it is an isomorphism. Please fill in details.

REMARK 70. *The fact that A_k satisfies the properties should be straightforward, since it anti-commutes or commutes with c_k depending on the grading. Therefore their eigenspaces must be the same. Since A_k is an involution, it must be an isomorphism.*

For the statement itself, it suffice to prove the following:

- $$\dim(Hom(W)) = 2^{2m}$$

  This follows directly from definition.

- $$\forall \lambda^1 \neq \lambda^2, W_{\lambda^1} \cap W_{\lambda^2} = \varnothing$$

  This should also follow directly, as different eigenspaces have empty intersection.

- $$\dim(W_\lambda) = 2^m, \forall \lambda \in \mathbb{Z}_2^m$$

  Obviously once we establish this, then by dimension counting we would prove

  $$W = \bigoplus W_\lambda, \lambda \in \mathbb{Z}_2^m$$

  We consider the base case where $|m| = 1$, so we want to show

  $$W = W_1 \oplus W_{-1}, \lambda = 1, -1$$

  for $c_1 = i\varphi_1\varphi_2$. We have

  $$c_1c_1 = i\sigma(\varphi_1)\sigma(\varphi_2)i\sigma(\varphi_1)\sigma(\varphi_2) = \sigma(\varphi_1)^2\sigma(\varphi_2)^2 = 1$$

  as φ_1, φ_2 are orthogonal basis vectors. Therefore if λ is an eigenvalue for c_1, we have

  $$c_1c_1w_\lambda = \lambda^2 w_\lambda \to \lambda^2 = 1 \to \lambda = \pm 1$$

  Now we only need to prove that

  $$\langle w_1, w_{-1}\rangle = 0, \forall w_1 \in W_1, w_{-1} \in W_{-1}$$

  But we have

  $$\langle w_1, w_{-1}\rangle = \langle c_1^2 w_1, w_{-1}\rangle = \langle c_1w_1, c_1w_{-1}\rangle = -\langle w_1, w_{-1}\rangle$$

Therefore

$$W_1 \perp W_{-1}$$

Further, any vector w in W is a unique combination of some vector in W_1 and some vector in W_{-1}. To see this we have

$$c_1(\frac{w_1 + c_1 w_1}{2}) = (\frac{w_1 + c_1 w_1}{2})$$

and similarly

$$c_1(\frac{w_1 - c_1 w_1}{2}) = -(\frac{w_1 - c_1 w_1}{2})$$

It is clear that our discussion can be extended for any subspace of W. Therefore using induction we can prove the statement.

REMARK 71. *I think there is a typo here. The $\mathbb{Z}_2$ grading can only have eigenvalue $-1, 1$, not $0, 1$. Otherwise c_1 would not be an isomorphism. Therefore the only eigenvalue for c_1 is ± 1.*

REMARK 72. *I think this is the so called 'half spin representation' in Roe's book. However the construction seems very different.*

DEFINITION 35. *Let*

$$Cl = \langle 1, (i_1, \cdots i_k), 1 \le k \le n, 1 \le i_1 < \cdots < i_k \le n \rangle$$

If $I \in Cl$, put

$$\sigma_I = \begin{cases} 1 & if I = 1 \\ \sigma(\varphi_{i_1}) * \cdots * \sigma(\varphi_{i_k}) & if I = (i_1 \cdots i_k) \end{cases}$$

REMARK 73. *Is there a typo here? I think Maucrio took $1 = \phi$ or something....*

THEOREM 55. *We claim that*

$$\{\sigma_I | I \in Cl\}$$

is a basis for $Cl(V) \subseteq Hom(W)$.

REMARK 74. *This theorem will be used in the future to prove that*

$$W = S \otimes W^l$$

where

$$S \subseteq W, W^l \subseteq W, Hom(S) \cong Cl(V) \subseteq Hom(V)$$

REMARK 75. *I think this fact is proved in Roe's book, Chapter 10.*

Proof.

Execrise 16. *We claim that* $\{\sigma_I | I \in Cl\}$ *spans* $Cl(V)$*, which is the algebra generated by* $\sigma(\varphi), \varphi \in V$*. The proof is trivial.*

Now consider

$$\sum_{I \in Cl} a_I \sigma_I = 0 \in Hom(W)$$

- *Step 1: If not all* $a_I = 0$*, then there exist a sum*

$$1 + b_I \sigma_I = 0$$

  If $a_\phi \neq 0$*, then just divide by* a_ϕ*.*
  If $a_\phi = 0$*, we have*

$$\sum_{|I|>0} a_I \sigma_I = 0$$

  choose I_0 *such that* $a_{I_0} \neq 0$*, such that*

$$\sigma_{I_0} = \sigma(\varphi_{i_1}) \cdots \sigma(\varphi_{i_k})$$

  Now multiply both side by

$$a_{I_0}^{-1} \sigma(\varphi_{i_k}) \cdots \sigma(\varphi_{i_1})$$

  we have

$$1 + \sum_{|J|>0} b_J \sigma_J = 0$$

  for some new coefficient b_J*.*

- *Step 2: If* $|J| < n$ *and* $b_J \neq 0$*, we can write Step 1 as*

$$1 + \sum_{|I|>0} c_I \sigma_I = 0$$

  Let $J = (j_1, \cdots j_k)$*. We claim that there exist* l *such that*

$$\sigma_J \circ \sigma_l = -\sigma_l \circ \sigma_J$$

  We divided it into two cases:

- *Case 1: We know* k *is even,* $k < n$*, then* $l = j_1$ *works!*

- *Case 2:* k *is odd,* $k < n$*. Then we let*

$$l \in \{1 \cdots n\} / \{j_1, \cdots j_k\}$$

  Check and we see it works!

- *Now we multiply both sides of*

$$1+\sum_{|I|>0} b_I\sigma_I = 0$$

by σ_l. Then we have

$$\sigma_l^2 - b_J\sigma_J + \sum_{I\neq J,|I|>0} b_I \overbrace{\sigma_l\sigma_I\sigma_l}^{\pm\sigma_J} = 0$$

adding this to the previous equation and divide by 2 we have

$$1+\sum_{I\neq J,|I|>0} c_I\sigma_I = 0$$

- *Now we return to Step 2. We either end up with $1=0$, or end up with*

$$1+a_I\sigma_I = 0, |I| = n$$

But neither can hold. The later one because we have

$$(-a)c_1\cdots c_m = i^m, c_k = i\sigma(\varphi_{2k-1})\sigma(\varphi_{2k})$$

This gives

$$a = -i^m$$

So the only possible case is all a_I are zero to begin with.

Remark 76. *I think this proof does not really work, as it assumes $c_1\cdots c_m = 1$, which has no reason to be true. I think a proof may be multiplying*

$$a_{I^\perp}\sigma_{I^\perp}$$

on both sides and discuss the parity of m. This can be subtle. For example for

$$1+a\sigma(x)\sigma(y)\sigma(z) = 0$$

by multiplying $\sigma(x)$ on both sides we change it to

$$\sigma(x) = -a\sigma(y)\sigma(z)$$

Therefore plug it in we have

$$1-a^2\sigma(y)\sigma(z)\sigma(y)\sigma(z) = 0 \to 1+a^2 = 0, a=\pm i \to \sigma(x)\sigma(y)\sigma(z) = a$$

But this does not really help us resolve anything. So in general it might be difficult.

Remark 77. *I think there is a typo in the notes, as it is*

$$\sigma_l\sigma_I\sigma_J$$

for no reason.

4.2 Brief overview

We now take a brief overview of the isomorphism

$$Cl(V) \to \mathbb{C} \wedge^* V$$

which maps

$$\prod_{i=1}^{k} \sigma(\varphi_i) = \wedge_{i=1}^{k} \varphi_i$$

Here σ *is the map*

$$\sigma : V \to Hom(W, W)$$

such that

- *For all* $\xi \in V, \sigma(\xi)^2 = |\xi|^2$.
- *For all* $\xi \in V$*, the endomorphism*

  $$\sigma(\xi) : W \to W$$

  is self-adjoint.
- *We call this Clifford module* $\mathbb{Z}_2$ *graded if*

  $$W = W^+ \oplus W^-$$

  such that the two subspaces are orthogonal to each other, and they have the same dimension. Further, we want

  $$\sigma : W^{\pm} \to W^{\mp}$$

Now let $\{e_i\}$ *be a basis for* V*. Let* $Cl(V)$ *denote the algebra generated by* $\{\sigma(e_i)\}$*. Then we have*

$$Cl(V) = span\{\sigma(e_i) | i = 1 \cdots n\}$$

So an element in $Cl(V)$ *would be*

$$\sum_{k=1}^{n} \prod_{i=1}^{k} \sigma(e_i), k \leq n \to \dim(Cl(V)) = 2^n$$

Now define a map

$$Cl(V) \to \mathbb{C} \wedge^* V : \sigma(e_i) \to \varphi_i$$

extending it we get the isomorphism from the beginning.

4.3 Index and the heat operator

Let M be a compact, oriented, Riemannian manifold.

THEOREM 56. *Let $D_r \in Diff^1(M, \sum)$ be a smooth family of Z_2 graded generalized Dirac operators (they are self adjoint). We have*

$$\sigma(\mathcal{D}_r)(\xi)^2 = |\xi|^2$$

And Ind$(\mathcal{D}_r^+)$ is a continuous function of r.

Proof. The corresponding heat operators

$$e^{-t\mathcal{D}_r^- D_r^+}, e^{-t\mathcal{D}_r^-\mathcal{D}_r^+}$$

depend smoothly on r, where

$$e^{-t\mathcal{D}_r^-\mathcal{D}_r^+}u(x) = \int h_r(t,x,y)u(y)dg(y), h_r(t,x,y) \in C^\infty[0,\infty)_t \times M \times M$$

We know that $h_r(t,x,y)$ is actually C^∞ in $(r,t,x,y) \in [a,b] \times (0,\infty) \times M \times M$, and $\mathrm{Tr}(e^{-t\mathcal{D}_r^-\mathcal{D}_r^+}), \mathrm{Tr}(e^{-t\mathcal{D}_r^-\mathcal{D}_r^+})$ are smooth functions of $(r,t) \in [a,b]_r \times (0,\infty)_t$. But we know

$$\underbrace{\mathrm{Ind}\mathcal{D}_-^+}_{\in\mathbb{Z}} = \mathrm{Tr}(e^{-t\mathcal{D}_r^-\mathcal{D}_r^+}) - \mathrm{Tr}(e^{-t\mathcal{D}_r^-\mathcal{D}_r^+})$$

Therefore it is a constant because $[a,b]$ is connected!

REMARK 78. *This statement is correct, but the proof is dubious, because a similar proof in Roe's book showed one need supersymmetry cancellation and compute the trace's derivative. For a related post, see*
http://math.stackexchange.com/questions/904488/a-technical-step-in-the-proof-of-atiyah-bott-fixed-point-formula

Discussion. *This theorem helps as follows:*

LEMMA 32. *Let $\mathcal{D}_0$ and $\mathcal{D}_1$:*

$$C^\infty(M,E) \to C^\infty(M,E)$$

be any two $\mathbb{Z}_2$-graded Dirac operators such that

$$\sigma(\mathcal{D}_0) = \sigma(\mathcal{D}_1)$$

Then there exists a smooth family $\mathcal{D}_r$ of $\mathbb{Z}_2$-graded generalized Dirac operators, $r \in [0,t]$ such that

$$\mathcal{D}_r|_{r=0} = \mathcal{D}_0, \mathcal{D}_r|_{r=1} = \mathcal{D}_1$$

Proof. We let

$$\mathcal{D}_r = (1-r)\mathcal{D}_0 + r\mathcal{D}_1 \to \sigma(\mathcal{D}_r) = \sigma(\mathcal{D}_0) = \sigma(\mathcal{D}_1)$$

Discussion. *Now fix a $\mathbb{Z}_2$ graded generalized Dirac operator:*

$$\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$$

Our goal is to compute the index of $\mathcal{D}^+$. This is hard in general. Instead we will find a 'nice' general Dirac operator $\mathcal{D}_1$ with the same symbol and compute Ind$\mathcal{D}_1^+$. Here 'nice' means $\mathcal{D}_1^+$ involves in a heavy of the geometry of E, for example the curvature of E.

We will show that there exist a unitary, $\mathbb{Z}_2$-graded, Clifford connection:

$$\nabla : C^\infty(M, E) \to C^\infty(M, \mathbb{C}T^*M \otimes E)$$

where unitary means for all $e, f \in C^\infty(M, E), v \in C^\infty(M, TM)$ we have

$$v\langle e, f\rangle = \langle \nabla e, f\rangle + \langle e, \nabla_v f\rangle$$

Remark 79. *I think this is another way of saying the Clifford action is skew-adjoint. In Roe's notation we have*

$$(v \cdot s_1, s_2) + (s_1, v \cdot s_2) = 0$$

But I am quite confused with Prof. Loya's definition at here. It does not seem to be symmetrical, for example. I suspect Prof.Loya meant the following instead:

$$v\langle e, f\rangle = \langle \nabla_v e, f\rangle + \langle e, \nabla_v f\rangle$$

But I have never seen this in literature.

Discussion. *Here $\mathbb{Z}_2$ graded means:*

$$\nabla : C^\infty(M, E^\pm) \to C^\infty(M, \mathbb{C}T^*M \otimes E^\mp)$$

In other words the connection keeps the grading.

Here Clifford means:

$$\forall v \in C^\infty(M, TM), \alpha \in C^\infty(M, T^*M), e \in C^\infty(M, T^*E)$$

we have

$$\nabla_v^E(\sigma(\alpha)e) = \sigma(\nabla_v \alpha)e + \sigma(\alpha)\nabla_v^E e$$

where ∇_v denotes the Levi-Civita connection on T^*M.

Remark 80. *I think this is a restatement of Roe's definition:*

$$\nabla_X(Ys) = (\nabla_X Y)s + Y(\nabla_X s)$$

where Prof Loya implicitly assumed the Clifford action commutes with the connection.

Discussion. *Let*

$$\mathcal{D}_1 = \frac{1}{i}\sigma \circ \nabla$$

Here $\sigma(\mathcal{D})$ is the principle symbol of $\mathcal{D}$, where $\mathcal{D}$ is the original Dirac operator. We know that we have

$$\sigma : C^\infty(M, \mathbb{C}T^*M \times E) \to C^\infty(M, E)$$

Observe that

$$\mathcal{D}_1 = \frac{1}{i}\sigma \circ \nabla \in \textit{Diff}^1(M, E)$$

REMARK 81. *I think the net arrow works as*

$$C^\infty(M, E) \xrightarrow{\nabla} C^\infty(M, \mathbb{C}T^*M \times E) \xrightarrow{\sigma} C^\infty(M, E) \xrightarrow{\frac{1}{i}} C^\infty(M, E)$$

and by characterization theorem we know it is a first order differential operator. In fact I think it is a generalized version of d on De Rham complex.

Discussion. *Therefore we have*

$$\sigma(\mathcal{D}_1) = \frac{1}{i}\sigma \circ i\xi \otimes = \sigma(\xi)$$

and we have proved

THEOREM 57.

$$\mathcal{D}_1 = \frac{1}{i}\sigma \circ \nabla, \sigma_1(\mathcal{D}_1) = \sigma_1(\mathcal{D})$$

Discussion. *Moreover, we have*

$$\mathcal{D}_1 : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$$

and finally (not obvious) we have

$$\mathcal{D}_1^* = \mathcal{D}_1$$

So $\mathcal{D}_1$ is a $\mathbb{Z}_2$ graded, generalized Dirac operator and a previous theorem tells us

$$Ind(\mathcal{D}^+) = Ind(\mathcal{D}_1^+)$$

We will now find Ind$(\mathcal{D}_1^+)$.

REMARK 82. *I think this is Theorem 56.*

Discussion. *Now we need to find a ∇:*

REMARK 83. *I skip the discussion here as it has been covered by earlier discussions. We are now at page 117.*

4.4 The spin representation

Recall: fix φ_i be an orthonormal basis on V. Then we have

$$c_k = i\sigma(\varphi_{2k-1})\sigma(\varphi_{2k}), A_k = \sigma(\varphi_{2k})$$

Let

$$W_0 = W_{(1\cdots 1)}$$

Observe that given

$$\lambda = (\lambda_1, \cdots, \lambda_m)$$

If $I = (i_1, \cdots, i_k)$ where $i_1, \cdots, i_k$ are the indices in $\lambda_1, \cdots \lambda_k$ where they are -1 entries in λ. Then we have

$$A_I = A_{i_1} \circ \cdots \circ A_{i_k} : W_0 \to W_\lambda$$

is an isomorphism. Therefore we have

$$W \cong \underbrace{W_0 \oplus \cdots W_?}_{2^m copies} \cong W_0 \otimes \mathbb{C}^{2^m}$$

We will show

$$W = S \otimes W', S \subseteq W, W' \subseteq W'$$

and the isomorphism w.r.t a natural 'Clifford' multiplication.

Remark 84. *What does the symbol 'w.r.t' stand for?*

Remark 85. *The proof is straightforward and better than my proof, for we have*

$$c_k A_k = -A_k c_k$$

so the anti-commute property made it obvious that we change the eigenvalue.

Discussion. *We now fix a unit vector*

$$w_0 \in W_0 \leftrightarrow c_k w_0 = w_0, \forall k$$

Now let

$$S = span_I\{A_I w_0\}, I \in Cl = \{1, (i_1, \cdots i_k)\}$$

Lemma 33.

$$\forall \xi \in V, \sigma(\xi) : S \to S$$

Therefore $Cl(V)$ acts on S and in particular,

$$Cl(V)_S = Hom(S)$$

Discussion. *We skip the discussion as it is largely identical. Now we reach this theorem at page 120:*

THEOREM 58. *If $w_1 \cdots w_l$ is an orthonormal basis of*

$$\tilde{W} = W_{(1,\cdots 1)} = \{w \in W | c_k w = w\}$$

Then we have

$$\{A_I w_j | I \in Cl, j = 1 \cdots p\}$$

is an orthonormal basis of W.

Proof. : Observe that

$$A_I = A_{i_1} \cdots A_{i_k} : \tilde{W} \xrightarrow{\infty} W_{\lambda_I}$$

and the map is an isomorphism. Here

$$\lambda_I = \{\lambda_1, \cdots, \lambda_m\}$$

and

$$\lambda_j = \begin{cases} 1 & \text{if} j \notin \{i_1, \cdots, i_k\} \\ -1 & \text{if} j \in \{i_1, \cdots, i_k\} \end{cases}$$

Your job is to finish it!

LEMMA 34.

$$\varphi \in I \leftrightarrow \lambda_I \in \mathbb{Z}_2^m$$

where we map

$$(i_1, \cdots, i_k) \leftrightarrow (\lambda_1 \cdots \lambda_m)$$

THEOREM 59. *Fix an element $w_0 \in \tilde{W} = W_{(1 \cdots 1)}$ and let*

$$S = \textit{span}\{A_I w_0 | I \in Cl\}$$

We know that

$$W = W_{\lambda^0} \oplus \cdots \oplus W_{\lambda^{2m}}$$

Then we claim that

$$Cl(V) : S \to S, Hom(S) = Cl(V)|_S$$

The later follows because

$$\dim S = 2^m \to \dim Hom(S) = 2^{2m} = \dim Cl(V)|_S$$

as the later acts transitively on S.

Proof. To prove $Cl(V) : S \to S$, first we prove

$$\sigma(\varphi_j) : S \to S$$

Here is a fact: For all j and I, we have

$$\sigma(\varphi_j) \circ A_I = aA_J, a = \pm i, \pm 1$$

for some $J \in Cl$.

Proof. If $j = 2k$, then by definition we have

$$\sigma(\varphi_j) = A_k \to A_k A_I = aA_J, J \in Cl, a = \pm 1$$

on the other hand, if $j = 2k - 1$ we have

$$\sigma(\varphi_{2k-1})A_I = \frac{i}{i}\sigma(\varphi_{2k-1})\sigma(\varphi_{2k})\sigma(\varphi_{2k})A_I \tag{4.7}$$

$$= \pm i c_k A_J \tag{4.8}$$

$$= \pm i A_J c_k \tag{4.9}$$

$$= \pm i A_J \tag{4.10}$$

Thus we have

$$\sigma(\varphi_j)A_I\omega_0 = aA_J\omega_0 \in S, a = \pm i, \pm 1$$

Theorem 60. *There exist subspaces $S, \tilde{W} \subseteq W$ and a unitary isomorphism*

$$W \cong S \otimes \tilde{W}$$

such that

$$\forall \xi \in V, \sigma(\xi) \in Hom(W)$$

correspond to

$$\sigma(\xi) \otimes Id_{\tilde{W}}$$

Proof. We consider the map

$$A_I w_0 \otimes w \to A_I w, w \in \tilde{W}$$

This gives a map

$$S \otimes \tilde{W} \to W$$

So if

$$\{w_1 \cdots w_p\}$$

is an orthonormal basis of $\tilde{W}$, then

$$A_I w_0 \otimes w_j \to A_I w_j$$

where

$$A_I w_0 \otimes w_j$$

is an orthonormal basis for $S \otimes \tilde{W}$, and $A_I w_j$ is an orthonormal basis for W by Theorem 58.

We know that by the fact we proved earlier, we have

$$(\sigma(\varphi_j) A_I w_0) \otimes w_j = \pm a(A_J w_0) \otimes w_j$$

Therefore we have

$$a A_J w_j = \sigma(\varphi_j)(A_I w_j), A_I w_j \in W$$

Remark 86. *Prof. Loya said that 'another proof of this fact would be appreciated'. I think there is a proof of this fact on Roe's book in Chapter 10 or Chapter 11. But the proof used some elementary representation theory.*

Remark 87. $T \in Hom(W)$ *commute with* $Cl(V) \subset Hom(V)$ *if and only if* $T = Id_S \otimes \tilde{T}$ *for some* $\tilde{T} \in Hom(\tilde{W})$. *Note that*

$$Hom(W) = Hom(S) \otimes Hom(\tilde{W})$$

Execrise 17. *Prove it!*

Remark 88. *I think the proof is trivial using the result above. But I really like the statement.*

4.5 Clifford and forms

We recall that

$$Cl^k = span(1, \sigma(\xi_1) \cdots \sigma(\xi_j)), \xi_i, \xi_j \in V, i \leq j \leq k$$

Observe that

$$Cl^j \subseteq Cl^k, j \leq k$$

as well as

$$Cl^j Cl^k \subseteq Cl^{j+k}$$

DEFINITION 36. *We define*

$$Cl_G(V) = \oplus_{k=1}^{n} Cl^k / Cl^{k-1}$$

THEOREM 61. *We claim that*

$$\mathbb{C}\wedge^* \cong Cl_G(V)$$

Proof. We define a map

$$V \times V \cdots V \xrightarrow{\hat{\sigma_k}} Cl^k / Cl^{k-1}$$

such that

$$(\varphi_1 \wedge \cdots \wedge \varphi_k) \to [\sigma(\varphi_{i_1}) \cdots \sigma(\varphi_{i_k})]$$

We claim that $\hat{\sigma_k}$ is an isomorphism.

Execrise 18. *: We have*

$$\tilde{\sigma} : \mathbb{C}\wedge^k \to Cl_G(V) : \alpha \wedge \beta \to \hat{\sigma}(\alpha) \circ \hat{\sigma}(\beta)$$

4.6 Clifford algebra and manifolds

We have the following data:

- (M, g) *a compact, oriented Riemannian manifold.*
- $E \to M$ *is a* $\mathbb{Z}_2$ *graded Hermitian vector bundle.*
- $\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$ *is a* $\mathbb{Z}_2$ *graded self-adjoint map. We have*

$$\sigma(\mathcal{D})(\xi)^2 = |\xi|^2$$

Now fix $x_0 \in M$*, we have*

$$\sigma(\mathcal{D}) : T^*_{x_0} M \to Hom(E)$$

Now we have a Hermitian $\mathbb{Z}_2$ *graded Clifford module!*

Therefore, if $\mathcal{U}$ *is a coordinate patch and* $\{\varphi_1 \cdots, \varphi_n\}$ *is an orthonormal trivialization of* $T^*M|_{\mathcal{U}}$*. Then by Theorem 60*

$$E|_{\mathcal{U}} \cong S \otimes \tilde{E}$$

where $S, \tilde{E}$ *are vector bundles over* M*. We know that by Theorem 69 we have*

$$Cl(T^*M) = Hom(S)$$

because the action of $\sigma(\xi)$ *on* E *is the same as* $\sigma(\xi) \otimes Id_{\tilde{E}}$*.*

REMARK 89. *Please check that* $\sigma(\varphi_i)$ *are of constant entries, such that the entries are* $\pm 1, \pm i, 0$.

REMARK 90. *I am confused with this remark....*

Chapter 5

Proof of Atiyah-Singer index theorem

5.1 Reveal the secret of Atiyah-Singer index theorem

As above, let us work in a local coordinate patch. Recall that

$$\Gamma = i^m \sigma(\varphi_1)\cdots\sigma(\varphi_k), \tilde{\Gamma} = Z \circ \Gamma^{-1} = Z \circ \Gamma$$

Remark 91. *I do not think this is defined earlier.*

Discussion. *We note that*

$$\tilde{\Gamma}\sigma(\varphi_j) = \sigma(\varphi_j)\tilde{\Gamma} \to \tilde{\Gamma} = Id_S \otimes \tilde{\Gamma}$$

because of Remark 87. Thus we have

$$\tilde{\Gamma} \in Hom(\tilde{E})$$

Remark 92. *I think this is awkward, because $\tilde{\Gamma}$ is assumed to be an element in $Hom(W)$. The appropriate way to say it is $\tilde{\Gamma} = Id_S \otimes \Gamma'$, with $\Gamma' \in Hom(\tilde{E})$.*

Discussion. *Therefore*

$$Z = \Gamma \circ \tilde{\Gamma}, \Gamma \in Cl(T^*M)$$

Lemma 35. *We have the following eigenspace decomposition:*

$$\tilde{E} = \tilde{E}^+ \oplus \tilde{E}^-$$

because $\tilde{\Gamma}^2 = 1$. Similarly

$$S = S^+ \oplus S^-$$

because $\Gamma^2 = 1$. Thus we have the decomposition

$$E^+ = (S^+ \otimes \tilde{E}^+) \oplus (S^- \otimes \tilde{E}^-), E^- = (S^+ \otimes \tilde{E}^-) \oplus (S^- \otimes \tilde{E}^+)$$

Proof. I think the proof is straightforward since E^+, E^- are unique.

We now reached the following theorem:

THEOREM 62. *If at a point $m \in M$, we have $A \otimes B \in HomS \otimes Hom\tilde{E}$. Then we have*

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$$

COROLLARY 16. *We have*

$$\mathrm{Tr}(ZA \otimes B) = Tr(\Gamma A \otimes \tilde{\Gamma} B) = \mathrm{Tr}(\Gamma(A))\,\mathrm{Tr}(\tilde{\Gamma} B), \forall A \otimes B \in Hom(E) = Hom(S) \otimes Hom(\tilde{E})$$

LEMMA 36. *If $A = \sigma_I = \sigma(\varphi_{i_1}) \cdots \sigma(\varphi_{i_k})$, $i_1 < \cdots < i_k$. Then we have*

$$\mathrm{Tr}(Z\sigma_I \otimes B) = \begin{cases} 0 & \text{if} |I| < n \\ 2^m(-i)^m \,\mathrm{Tr}(\tilde{\Gamma} B) \end{cases}$$

by Patodi's lemma.

REMARK 93. *I think this is the same as Lemma 11.5 from page 143 of Roe's book. However Roe considered a generalized case, where at here we worked with the Clifford algebra basis elements. Otherwise there is no difference.*

Discussion. *Now we reveal the secret:*

PROPOSITION 9. *To prove the Atiyah-Singer index theorem we just need to compute*

$$Ind\mathcal{D}^+ = \mathrm{Tr}(e^{-t\mathcal{D}^-\mathcal{D}^+}) - \mathrm{Tr}(e^{-t\mathcal{D}^+\mathcal{D}^-}) = \mathrm{Tr}(Ze^{-t\mathcal{D}^2})!$$

Proof. This is because we know

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix} : C^\infty(M, E^+ \oplus E^-) \to C^\infty(M, E^- \oplus E^+)$$

Therefore

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}^-\mathcal{D}^+ & 0 \\ 0 & \mathcal{D}^+\mathcal{D}^- : \end{pmatrix} C^\infty(M, E^+ \oplus E^-) \to C^\infty(M, E^+ \oplus E^-)$$

and

$$e^{-t\mathcal{D}^2} = \begin{pmatrix} e^{-\mathcal{D}^-\mathcal{D}^+} & 0 \\ 0 & e^{-t\mathcal{D}^+\mathcal{D}^-} \end{pmatrix}$$

By the eigenspace decomposition we have

$$Ze^{-t\mathcal{D}^2} = \begin{pmatrix} e^{-t\mathcal{D}^-\mathcal{D}^+} & 0 \\ 0 & e^{-\mathcal{D}^+\mathcal{D}^-} \end{pmatrix}$$

Therefore

$$\mathrm{Ind}\mathcal{D}^+ = \mathrm{Tr}(e^{-t\mathcal{D}^-\mathcal{D}^+}) - \mathrm{Tr}(e^{-t\mathcal{D}^+\mathcal{D}^-}) = \mathrm{Tr}(Ze^{-t\mathcal{D}^2})$$

as desired!

Discussion. *We note that*

$$e^{-t\mathcal{D}^2}u = \int h(t,x,y)u(y)dg(y)$$

where $h(t,x,y) \in C^\infty[0,\infty)_t \times M \times M$*, and locally* $h(t,x,y) \in C^{N*N} = Hom(E)$*. We thus have*

$$\mathrm{Tr}(e^{-t\mathcal{D}^2}) = \int_M \mathrm{Tr}(h(t,x,x))dg(x), h(t,x,x) \in Hom(E_x)$$

Therefore

$$\mathit{Ind}\mathcal{D}^+ = \int_M \mathrm{Tr}(Zh(t,x,x)dg(x)$$

Here is the secret: We know

$$h(t,x,x) \in Hom(E_x) = Hom(S_x) \otimes Hom(\tilde{E}_x)$$

Therefore we can decompose it into

$$h(t,x,x) = \sum q_{I,j}(t,x,0)\sigma_I \otimes \Gamma_j$$

where

$$q_{I,j} \sim t^{-\frac{n}{2}}\tilde{q}(t,x,\frac{x-y}{t^{1/2}})$$

and Γ_j *are the basis elements for* $Hom(\tilde{E})$*. Therefore*

$$\mathrm{Tr}(Zh(t,x,x)) = \sum_j q_{(1\cdots n),j}(t,x)(-1)^m 2^m \mathrm{Tr}(\tilde{\Gamma}\Gamma_j)$$

by invoking Lemma 36.

REMARK 94. *Is there a typo here? Should not it be* t^{-n} *instead?*

Discussion. *Next time we will discuss the following: If* ∇ *is a unitary* $\mathbb{Z}_2$*-graded Clifford connection and*

$$\mathcal{D} = \frac{1}{i}\sigma \circ \nabla = \sigma(\mathit{old}\mathcal{D})$$

Then we have

$$h(t,x,x) = t^{-n}\sum_{I,j} t^{|I|}q_{I,j}(t^{1/2},x)\sigma_I \otimes \Gamma_j$$

Therefore we have

$$\mathrm{Tr}(Zh(t,t,x)) = \sum q_{(1,\cdots n)}(t^{1/2},x)(-i)^m 2^m \mathrm{Tr}(\sigma\Gamma_j)$$

Therefore

$$\mathit{Ind}\mathcal{D}^+ = \int = \sum q_{(1\cdots n)}(0,x)(-i)^m 2^m \mathrm{Tr}(\sigma\Gamma_j)$$

REMARK 95. *I think Mauricio's notes has a typo in the line second to the last line.*

Discussion. *We proved that*

$$W \cong S \otimes \tilde{W}$$

and in particular

$$Cl(V) \cong Cl(V) \otimes Id_{\tilde{W}}, Hom(S) = Cl(V)$$

REMARK 96. *I think this is bad notation. I think Prof.Loya meant Theorem 60 instead. Otherwise the second statement does not really state anything as the isomorphism is obvious.*

Discussion. *Let (M, g) be a compact oriented Riemannian manifold, $v \in C^\infty(M, TM)$.*

THEOREM 63. *There exist a function $div(v) \in C^\infty(M)$ such that*

$$\int vfdg = -\int \text{div}(v)fdg, \forall f \in C^\infty M$$

Proof. We let

$$v : C^\infty(M) \to C^\infty(M) : f \to vf = df(v), v \in \text{Diff}^1(M)$$

let

$$v^* : C^\infty(M) \to C^\infty(M)$$

be the adjoint of $v : C^\infty M \to C^\infty M$. So locally we choose $\{\mathcal{U}, x\}$, and $v = \sum a_j \partial_{x_j}$. Then we calculate:

$$\int vfdg = \int \sum a_j \partial_{x_j} f \sqrt{G} dx, G = \det[g_{ij}] \tag{5.1}$$

$$= -\int f \sum \partial_{x_j}(a_j \sqrt{G}) dx \tag{5.2}$$

$$= -\int f\Big(\sum_j \frac{\partial_{x_j}(a_j\sqrt{G})}{\sqrt{G}}\Big) \overbrace{\sqrt{G}dx_j}^{=dg} \tag{5.3}$$

$$= -\int f\text{div}(v)dg \tag{5.4}$$

where we have

$$\text{div}(v) = \sum \frac{\partial_{x_j}(a_j\sqrt{G})}{\sqrt{G}}, G = \det(g) = \sum \partial_{x_j} a_j + \sum \frac{a_j \partial_{x_j}\sqrt{G}}{\sqrt{G}}$$

COROLLARY 17. *For all $v \in C^\infty(M, TM)$ we have*

$$v^* = -v - \textit{div}(v)$$

Proof. We have

$$\begin{aligned}
\int (v^* f)hdg &= \int f(vh)dg && (5.5)\\
&= \int (v(fh) - (vf)h)dg && (5.6)\\
&= -f \int \mathrm{div}(v) fhdg - \int vfhdg && (5.7)\\
&= \int (-\mathrm{div}(v) - v)fhdg && (5.8)
\end{aligned}$$

Therefore

$$v^* = -v - \mathrm{div}(v)$$

as desired.

THEOREM 64. *For all $v \in C^\infty(M, TM)$, we have $div(V) = \mathrm{Tr}(L_v)$, where $L_v : TM \to TM$ is the map*

$$w \to \nabla_w v$$

Proof. If $v = \sum a_j \partial_{x_j}$, then we have

$$\mathrm{div}(v) = \sum(\partial_{x_j} a_j + \frac{a_j \partial_{x_j} \sqrt{G}}{G})$$

Now observe that

$$\begin{aligned}
\frac{\partial_{x_j} \sqrt{G}}{\sqrt{G}} &= \partial_{x_j} \log \sqrt{G} && (5.9)\\
&= \frac{1}{2} \partial_{x_j} \log(\det(g)) && (5.10)\\
&= \frac{1}{2} \mathrm{Tr}(g \partial_{x_j} g) && (5.11)\\
&= \frac{1}{2} \sum_{j,k,l} g^{kl} \partial_{x_j} g_{kl} && (5.12)\\
&= \sum dx_k (\nabla_{\partial_{x_j}} \partial_{x_k}) && (5.13)\\
&= \sum dx_k (\nabla_{\partial_{x_k}} \partial x_j) && (5.14)
\end{aligned}$$

and the theorem follows from here.

REMARK 97. *The computation at here is not so clear, I think Prof. Loya used this identity implicitly:*

$$\det(e^X) = e^{\mathrm{Tr}(X)}$$

for a proof see Brian Hall's book, Lie groups, Lie algebras and represenations page 47. Using this formula we have

$$\log(\det(e^X)) = \mathrm{Tr}(X)$$

so plugging it in we have

$$\frac{1}{2}\partial_j \log(\det(g)) = \frac{1}{2}\partial_{x_j} \mathrm{Tr}(\log(g))); \log(g) = X, g = e^X$$

We know that

$$\partial_j \mathrm{Tr}(\log(g)) = \mathrm{Tr}(\partial_j \log(g)) = \mathrm{Tr}(\partial_j g)g^{-1}) = \mathrm{Tr}(g^{-1}(\partial_j g))$$

Therefore after expanding we have

$$\frac{1}{2}\sum_{j,k,l} g^{kl}\partial_{x_j} g_{kl}$$

which coincide with Prof. Loya's computation. However I am having trouble to see that

$$\frac{1}{2}\sum_{j,k,l} g^{kl}\partial_{x_j} g_{kl} = \sum dx_k(\nabla_{x_j}\partial_k)$$

because the later would involve Christoffel symbols. By definition we have

$$\nabla_j \partial_k = \sum \Gamma^l_{j,k}\partial_l$$

and it is not clear to me that this gives us what we want after simplification.

Remark 98. *In particular, let $v_1 \cdots v_n$ be a local orthonormal frame.*

Execrise 19. *We have*

$$\div(v_k) = \mathrm{Tr}(L_{v_k}) = -\sum_l (\nabla_{v_l}\varphi_l)v_k$$

where φ_i is the dual basis and ∇ is the Levi-Civita connection on T^*M.

And further we have

$$\nabla_{\varphi_l} = \sum \gamma_{kl}\varphi_k$$

Therefore

$$-div(v_k) = -\sum_l \Gamma_{k,l}(v_l) \to div(v_k) = \sum_l \gamma_{lk}(v_l)$$

Theorem 65. *There exist a $\mathbb{Z}_2$ graded unitary connection on E.*

Proof. We fix an orthonormal frame $\varphi_1 \cdots \varphi_n$ of T^*M, then we get an orthonormal frame $e_1 \cdots e_N$ of E^+, and $e_{N+1} \cdots e_{2N}$ of E^- such that $\sigma(\varphi_i)$ are the constant matrices with entries $\pm 1, \pm i, 0$. We want to find ω, which is a matrix of 1 forms such that

$$\nabla = d + \omega$$

is $\mathbb{Z}_2$-graded, unitary, and further Clifford:

$$\nabla(\sigma(\alpha)e) = \sigma(\nabla\alpha)e + \sigma(\alpha)\nabla(e)$$

Execrise 20. *: We claim that ∇ satisfies above and the unitary is equivalent to*

$$\omega = \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix}$$

Therefore $\omega^* + \omega = 0$. Let us see:

$$\nabla(\sigma(\varphi_k)e_j) = \sigma(\nabla_{\varphi_k})e_j + \sigma(\varphi_k)\nabla e_j$$

as well as

$$\omega(\sigma_k e_j) = \sum \gamma_{k,l}\sigma(\varphi_k)e_j + \sigma(\varphi_k)\omega e_j$$

Therefore

$$\omega\sigma_k = \sigma_k\omega + \sum \gamma_{lk}\sigma_l$$

The following works:

$$\omega = \frac{1}{4}\sum_{l,k} \gamma_{l,k}\sigma_l\sigma_k$$

which equals

$$\frac{1}{2}\sum_{k<l} \gamma_{kl}\sigma_k\sigma_l$$

Remark 99. *Frankly I am quite confused with the construction. Is the γ_{lk} the Pauli matrices, for example?*

Definition 37. *We have*

$$\not{\partial} = \frac{1}{i}\sigma \circ \nabla$$

We know

$$\sigma(\not{\partial}) = \sigma(\mathcal{D})$$

and

$$\not{\partial}: C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$$

THEOREM 66. *We claim that $\not{\partial}$ is self-adjoint.*

Proof. We have

$$\nabla = \sum \varphi_k \otimes \nabla_{v_k} \tag{5.15}$$
$$= \sum \varphi_k \otimes (v_k + \omega(v_k)) \tag{5.16}$$
$$\tag{5.17}$$

Therefore we have

$$\not{\partial} = \frac{1}{i}\sigma \circ \nabla = \sum \frac{1}{i} \sum (\sigma_k v_k + \sigma_k \omega(v_k))$$

and its adjoint is given by

$$\not{\partial}^* = -\frac{1}{i} \sum (v_k^* \sigma_k + \omega(v_k)^* \sigma_k)$$

we know that

$$v_k^* = -v_k - \mathrm{div}(v_k)$$

So plug it in we have

$$\not{\partial}^* = \not{\partial} + \frac{1}{i} \sum_k (\mathrm{div}(v_k) + \sum \gamma_{k,l}(v_l))\sigma_k$$

We note that the terms in the bracket cancel out each other, therefore we have

$$\not{\partial}^* = \not{\partial}$$

as desired.

REMARK 100. *I could not really follow the computation, unfortunately.*

5.2 Clifford Connections

Discussion. *If ∇, ∇' are two Clifford connections on E. Then we have*

$$\nabla - \nabla' \in C^\infty(M, \mathbb{C} \wedge^1 \otimes \tilde{Hom}(E))$$

Here $\tilde{Hom}(E)$ are homomorphisms on E are commute with the Clifford action by $Cl(T^*M) \subseteq Hom(E)$.

Proof. We have

$$\nabla - \nabla' = \omega - \omega'$$

We compute

$$\omega(\sigma(\varphi_i)) - w'(\sigma(\varphi_i)) = \sigma(\varphi_i)\omega - \sigma(\varphi_i)\omega'$$

Therefore

$$(w - w')\sigma_i = \sigma_i(\omega - \omega')$$

REMARK 101. *This is not a proof to me. I think some step is missing.*

Discussion. *Thus, in the decomposition $E \cong S \otimes \tilde{E}$, locally we have*

$$\nabla - \nabla' = Id_S \otimes T, T \ni \mathbb{C} \wedge^* \otimes Hom(\tilde{E}), HomE = \hom S \otimes HomT$$

THEOREM 67. *In a local decomposition we have*

$$E \cong S \otimes \tilde{E}$$

So we can write

$$\nabla^E = \nabla^S \otimes Id_{\tilde{E}} + Id_S \otimes \nabla^{\tilde{E}} = \nabla^S \otimes \nabla^{\tilde{E}}$$

where

$$\nabla^S = d + \omega_s, \omega_s = \frac{1}{4} \sum_{j,k} \gamma_{j,k} \sigma_j \sigma_k$$

Proof. Exercise.

REMARK 102. *Not clear to me!*

5.3 Connection and curvature

We have the following theorem:

THEOREM 68. *Let V be a vector bundle with connection ∇, let $x_0 \in M$, and let $e^0 \in V_{x_0}$. Then there exist $e \in C^\infty(M, V)$ such that $e(x_0) = e^0$ and $(\nabla e)|_{x_0} = 0$.*

Proof. This is left as an exercise. We need to use 'xxx trick' together with 'radial transport'.

Discussion. *We now come to the following fact:*

PROPOSITION 10. *If V is a Hermitian vector bundle and $e_i^0 \cdots e_N^0 \in V_{x_0}$ are orthonormal vectors. Then there exists local orthonormal frame of V near x_0 with $e_1 \cdots e_N$ such that*

$$e_i(x_0) = e_i^0, (\nabla e_i)|_{x_0} = 0$$

Discussion. *This has the following application:*

$$\not{\partial} = \frac{1}{i} \sigma \circ \nabla$$

THEOREM 69. *(Another proof) We have*

$$\not{\partial}^* = \not{\partial}$$

REMARK 103. *I think there is a typo at here again.*

Proof. We fix $x_0 \in M$ W. T. S

$$(\not{\partial}^* e)|_{x_0} = (\not{\partial} e)_{x_0}, \forall e \in C^\infty(M, E)$$

Let $\varphi_1 \cdots \varphi_n$ be a local orthonormal frame for T^*M near x_0 such that

$$\nabla \varphi_i|_{x_0} = 0$$

Then we have

$$\not{\partial} = \frac{1}{i}\sigma(\sum_k \varphi_k \nabla_{v_k}) = \frac{1}{i}\sum_k \sigma_k(v_k + \omega_k)$$

Therefore

$$\not{\partial}^* = -\frac{1}{i}(\sum_k v_k^* \sigma_k - w_k \sigma_k) = \frac{1}{i}\sum_k \sigma_k(v_k + \omega_k) = \not{\partial}$$

where we note that

$$v_k^* = -v_k - \text{div}(v_k), \text{div}(v_k) = 0$$

and

$$\omega_k \sigma_k = \sigma_k \omega_k$$

by the choice of our frame!

5.4 Curvature

Fix a point $x_0 \in M$, our goal is to compute a formula for

$$Q_{x_0}^E \in \mathbb{C} \wedge^2 \otimes Hom E = Hom(S) \otimes Hom(\tilde{E})$$

We pick a local frame φ_i such that $\nabla \varphi_i = 0$ at x_0. Then we have

$$E \cong S \otimes \tilde{E}$$

and

$$\nabla^E = \nabla^S \otimes Id_{\tilde{E}} + Id_S \otimes \nabla^{\tilde{E}}$$

where the first one equals

$$d + \frac{1}{4}\sum_{j,k} \gamma_{j,k} \sigma_j \sigma_k$$

Therefore we decompose it as

$$Q_{x_0}^E = Q^s + Q^{\tilde{E}}$$

Note

$$Q^s_{x_0} = dw_s|_{x_0} + w_s \wedge w_s|_{x_0}$$

The later is zero because $\gamma_{jk} = 0$ at our coordinate system. Therefore the above equal to

$$\frac{1}{4} d\gamma_{jk}\sigma_j\sigma_k = \frac{1}{4}\sum \mathcal{R}_{jk}\sigma_j\sigma_k \quad (5.18)$$

$$= -\frac{1}{4}\sum \mathcal{R}(v_j, v_k)\sigma_j\sigma_k \quad (5.19)$$

$$= -\frac{1}{4}\sigma(\mathcal{R}) \quad (5.20)$$

where we knote that

$$\gamma_{jk} = \mathcal{R}_{jk}$$

at x_0 in our coordinate system. Here $\mathcal{R}_{jk}$ is the curvature form in the cotangent bundle, and

$$\sigma(\mathcal{R}) \in T^*M \otimes T^*M \otimes T^*M \otimes T^*M$$

We thus come to the following theorem:

THEOREM 70.

$$\tilde{Q} = Q^E + \frac{1}{4}\sigma(\mathcal{R}) \in C^\infty(M, \mathbb{C} \wedge^2 \otimes \tilde{Hom}(E))$$

5.5 Ricci Curvature

We review the notation:

- *$\mathcal{R}$ is the Riemannian curvature tensor in*

$$C^\infty(M, T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$$

- *g the metric on TM. It lives in $C^\infty(M, T^*M \otimes T^*M)$.*
- *g' lives in $C^\infty, TM \otimes TM$ is the metric on the cotangent bundle induced by the metric.*

Now we define the Ricci curvature:

DEFINITION 38. *Locally, if v_i is a local orthonormal basis for TM, then we have*

$$Ric(v, u) = \sum_{k=1}^{n} \mathcal{R}(v, v_k, v_k, \omega)$$

REMARK 104. *It seems there is another obvious typo here!*

DEFINITION 39. *Now we define the scalar curvature:*

$$S = Ric(g') = \sum_{i,k} \mathcal{R}(v_i, v_k, v_k, v_i) \in C^\infty(M)$$

because $Ric \in C^\infty(M, T^*M \otimes T^*M), g' \in C^\infty, TM \otimes TM$.

DEFINITION 40. *Now locally we have*

$$\mathcal{R} = \sum \mathcal{R}_{ijkl}\varphi_i \otimes \varphi_j \otimes \varphi_k \otimes \varphi_l$$

and its symbol is given by

$$\sigma_{\text{full}}(\mathcal{R}) = \sum \mathcal{R}_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$$

LEMMA 37. *We have*

$$\sigma_{\text{full}}(\mathcal{R}) = 2S$$

Proof. Observe that if i, j, k, l are all distinct, then

$$\sigma_i\sigma_j\sigma_k = \sigma_j\sigma_k\sigma_i = \sigma_k\sigma_i\sigma_j$$

On the other hand we know that

$$\mathcal{R}_{ijkl} + \mathcal{R}_{jkil} + \mathcal{R}_{kijl} = 0$$

Thus, in $\sigma_{textfull}(\mathcal{R})$ we must have $i = j$ or $j = k$, or $j = k$. Since otherwise they would pairwise cancel. Now we have

$$\begin{aligned}
\sigma_{full}(\mathcal{R}) &= \sum \mathcal{R}_{ijjl} - \sum \mathcal{R}_{ijil}\sigma_j\sigma_l && (5.21)\\
&= 2\sum \mathcal{R}_{ijjl}\sigma_i\sigma_l && (5.22)\\
&= 2\sum \mathcal{R}_{ijji} && (5.23)\\
&= 2S && (5.24)
\end{aligned}$$

where we used the fact that the term involving $\sigma_i\sigma_l$ is anti-symmetric for $i \not l$, and $\mathbb{R}_{ijjl}$ is symmetric for i and l.

REMARK 105. *I feel the proof skipped a few steps.*

5.6 Connection Laplacian

We recall the following definitions:

- $$\nabla : C^\infty(M, E) \to C^\infty(M, \wedge^1 \otimes E)$$

- *and its dual:*
$$C^\infty : C^\infty(M, \mathbb{C}^1 \otimes E) \to C^\infty(M, E)$$

Definition 41. *We now define the connection Laplacian as*

$$\Delta^\nabla = \nabla^* \nabla$$

Lemma 38. *If $v_1 \cdots v_n$ is the local orthonormal frame for TM, then we have*

$$\Delta^\nabla = -\sum_{k=1}^{n} (\nabla^2_{v_k} + div(v_k \nabla_{v_k})$$

Proof. We have

$$\nabla = \sum \varphi_k \otimes \nabla_{v_k} \tag{5.25}$$
$$= \sum \varphi_k (v_k + \omega_k) \tag{5.26}$$

Therefore

$$\nabla^* \nabla = \sum (v_k^* + \omega_k) i_{v_k} \sum_l \varphi_l \otimes (\varphi_l + \omega_l) \tag{5.27}$$
$$= \sum (-v_k - \mathrm{div}(v_k) - \omega_k)(v_k + \omega_k) \tag{5.28}$$

where we note the pair
$$-\mathrm{div} v_k(v_k) = \sum \nabla^2_{v_k}$$
and the rest would cancel.

We are now ready to prove the following important theorem:

Theorem 71. *(Lichnerowicz formula) We have*

- $$\not{\partial} = \frac{1}{i} \sigma \circ \nabla$$

- $$\not{\partial}^2 = \Delta^\nabla - \frac{1}{2}\sigma(\tilde{Q}) + \frac{1}{4} S$$

Proof. We want to show that

$$(LHS)e|_x = (RHS)e|_x, \forall x \in M$$

We now fix $x_0 \in M$.

Proof. We let $\varphi_1 \cdots \varphi_n$ be a local orthonormal frame of T^*M such that $\nabla\varphi_i = 0$ at x_0.

We have

$$\partial\!\!\!/ = \frac{1}{i}\sigma(\sum \varphi_k \otimes \nabla_{v_k}) = \frac{1}{i}\sum \sigma_k(v_k + \omega_k), \omega_k = \omega(v_k)$$

Therefore

$$\begin{aligned}
\partial\!\!\!/^2 &= -\sum \sigma_k(v_k+\omega_k)\sigma_l(v_l+\omega_l) && (5.29)\\
&= -\sum \sigma_k\sigma_l(v_k+\omega_k)(v_l+\omega_l) && (5.30)\\
&= -\sum_{k=l} - \sum_{k\neq l} && (5.31)\\
&= \sum_{k=l} \nabla_k^2 && (5.32)\\
&= \sum_{k\neq l} \sigma_k\sigma_l(v_kv_l + v_k\omega_l + w_lv_k + w_kv_k + w_kv_l) && (5.33)
\end{aligned}$$

where (5.32) holds because the divergence at x_0 equal to zero, and (5.33) holds because the connection is torsion free and symmetric.

Now by symmetry the above equals

$$\begin{aligned}
&= \Delta^\nabla - \frac{1}{2}_{k\neq l}\sigma_k\sigma_l(v_k\omega_l - v_l\omega_k + \omega_k\omega_l - \omega_l\cdots) && (5.34)\\
&= \Delta^\nabla - \frac{1}{2}\sum_{k\neq l}\sigma_k\sigma_l(d\omega(v_k,v_l) + (\omega\wedge\omega)(v_k,v_l)) && (5.35)\\
&= \Delta^\nabla - \frac{1}{2}\sum_{k,l} Q^E(v_k,v_l)\sigma_k\sigma_l && (5.36)\\
&= \Delta^\nabla - \frac{1}{2}\sum_{k,l}\tilde{Q}(v_k,v_l)\sigma_k\sigma_l + \underbrace{\frac{1}{8}\sum_{k,l}\sigma(\tilde{\sigma}(v_k,v_l))\sigma_k}_{\sigma_{\mathrm{full}\mathcal{R}}} && (5.37)\\
&= \Delta^\nabla - \frac{1}{2}\sigma(\tilde{Q}) + \frac{1}{4}S && (5.38)
\end{aligned}$$

5.7 Local index theorem

The idea is:

$$\partial\!\!\!/^2 = -\sum(\nabla_{v_k}^2 + \mathrm{div}(v_k)\nabla_{v_k}) - \frac{1}{2}\sigma(Q') + \frac{1}{4}S$$

We want to compute $e^{-t\eth^2}$. To do that we recall, in the past in order to solve the equation

$$(\partial_t + \eth)e^{-t\eth} = 0$$

We found an $Q \in \Psi_{\mathcal{H}}^{-2}(\mathbb{R}^n)$.

Here will be our plan:

- First we let ∇_V be the curvature on 'parallel sections'.
- Second we want to develop an analogous heat calculus $\tilde{\Psi}_{\mathcal{H}}^P$ involving this 'parallel sections'.

Now let V be a vector bundle with connection. Let $\mathcal{U}$ be a coordinate patch over $V \cong \mathbb{C}^?$ if $y \in \mathcal{U}$, $\sigma_y = (x-y)\cdot\partial_x$.

THEOREM 72. *Let $y \in \mathcal{U}$. Then forall $e^0 \in V \in V_\partial$, there exists sections $e \in C^\infty(U,V)$, which are the radial transport of e^0 such that*

- $$e(y) = e^0$$
- $$\nabla_{\vec{r_y}} e \equiv 0, at\mathcal{U}$$

If V is Hermitian and ∇ is unitary and if $e_1^0 \cdots e_v^0 \in V_{x_0}$ are orthonormal frames, then $e_1 \cdots e_N$ equal to the radial transports are orthonormal on $\mathcal{U}$.

Execrise 21. *(ODE) hange to(r,ω), $r = \sqrt{\sum(x_i-y_i)^2}$, $\omega = \frac{x-y}{|x-y|}$. Then*

$$\vec{r_y} = (x-y)\cdot\partial_x \tag{5.39}$$
$$= \sum_i (x_i - y_i)\partial_{x_i} \tag{5.40}$$
$$= \sigma\partial_\sigma \tag{5.41}$$

REMARK 106. *The writing is barely legible, so maybe there is some typo here!*

Discussion. *Above equation have solution*

$$e = P(x,y)e^0$$

where

$$P(x,y) = P_{ij}(x,y)$$

and

$$P_{ij}(x,y) = \exp(-\int_0^1 \omega_{ij}((x-y),\partial_x)(s(x-y))$$

In particular, $P(x,y)$ can be expressed in the radial terms from y to x. It is an element in $C^\infty(\mathcal{U}\times\mathcal{U},\mathbb{C}^{N\times N})$

THEOREM 73. *(100 years old fact). In the above setting, fix a point y and $\vec{r} = \vec{r}_y$. If $e \in C^\infty(\mathcal{U}, V), \nabla_{\vec{r}} e \equiv 0$, then*

$$\nabla_v e = -\frac{1}{2}Q(v, \vec{r})e + O|x|^2$$

something vanishing (at) origin of coordinate on μU is the point.

Thus we have

$$\nabla = -\frac{1}{2}Q(,\vec{r})$$

on radial transports, similarly

$$\nabla = d - \frac{1}{2}Q(,\vec{r}) + O|x|^2$$

Proof. Assume $v = \partial_{x_k}$, then

$$\nabla_{\partial_{x_k}} e = \sum a_i(x)e_i$$

where we choose $e_1 \cdots e_N$ to be a radially transported frame for V on $\mathcal{U}$ for some $a_i \in C^\infty(\mathcal{U})$. Therefore

$$\nabla_{\vec{r}}\nabla_{x_k} e = \sum \vec{r}a_r(x)e_i + a_i \underbrace{\nabla_{\vec{r}} e_i}_{0}$$

We know that

$$\nabla_{\partial_{x_k}}\nabla_{\vec{r}} e = 0$$

Therefore

$$\nabla_{[\vec{r},\partial_{x_k}]} e = -\nabla_{\partial_{x_k}} e = -\sum a_i(x)e_i$$

where

$$[\vec{r}, \partial_{x_k}] = \sum x_j \partial_{x_j}\partial_{x_k} - \partial_{x_k}(\sum x_j \partial_{x_j})$$

Therefore

$$Q(\vec{r}, \partial_{x_k}) = \sum(a_i + \vec{r}a_i)e_i$$

where we know

$$\vec{r} = \sum x_j \partial_{x_k}$$

But we know left hand side equals zero at $x = 0$, therefore the right hand side also equals zero at $x = 0$. This implies

$$a_j(0) = 0$$

Therefore

$$a_i = \sum_j x_j b_{ij}(x) \in C^\infty$$

Because we know $a_i(0) = 0$, we can Taylor expand it with a remainder. Now we have

$$\vec{r}a_i = \sum x_k \partial_{x_k} a_i \tag{5.42}$$
$$= \sum_j x_j b_{ij} + O|x|^2 \tag{5.43}$$
$$= a_i + O(|x|^2) \tag{5.44}$$

which implies

$$a_i + \vec{r}a_i = 2a_i + O(|x|^2)$$

Therefore

$$Q(\vec{r}, \partial_{x_k})e = 2\nabla_{\partial_{x_k}} e + O|x|^2$$

Next time:

We want to show locally we have

$$e^{-t\partial\!\!\!/^2} \in C^\infty((0,\infty) \times \mathcal{U} \times \mathcal{U}, Hom(E))$$

where

$$Hom(E) = Cl \otimes Hom\tilde{E}, Hom(E) \cong \mathbb{C}^{N\times N}$$

5.8 The heat kernel

We review the definition:

- $\partial\!\!\!/$ is the Dirac operator expressed in terms of $\triangledown$.
- $\triangledown$ is the Clifford Connection, which can be expressed via curvature via radial transport.
- We should use radial transport to bring it to E!

Discussion. *Here will be our plan: Build this into $Q_{\mathcal{H}} \in \Psi^P_{\mathcal{H}}$ then find*

$$\mathrm{Tr}(ZQ_{\mathcal{H}})$$

Now working over a local coordinate patch, we may assume that

$$(M, g) = (\mathbb{R}^n, g)$$

and

$$E = S \otimes \tilde{E}$$

where over the local coordinate patch E trivializes. Now let $\varphi_1 \cdots \varphi_n$ be an orthonormal basis for $T^*\mathbb{R}^n$ such that

$$\sigma(\varphi_1)\cdots\sigma(\varphi_n)$$

are constant matrices. Then from earlier discussion we have

$$\nabla^E = \nabla^S \otimes Id_{\tilde{E}} + Id_S \otimes \nabla^{\tilde{E}}$$

as well as

$$\partial\!\!\!/ = \frac{1}{i}\sigma \circ \nabla$$

We know on M we have the following heat kernel expansion:

$$e^{-t\partial\!\!\!/^2} = t^{-\frac{n}{2}} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}})u(y)dg(y) \tag{5.45}$$

$$= \int h(t,t,y)u(y)dg(y) \tag{5.46}$$

Here $h(t,x,y) \in C^\infty(0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, Hom(E))$, where

$$Hom(E) \cong \mathbb{C}^{N\times N} = Cl(T^*M) \otimes Hom(\tilde{E}), N = \textit{rank}(E)$$

and the elements in $Cl(T^*M), Hom(\tilde{E})$ are constant coefficient matrices.

Now we want to find radically transported sections of

$$Hom(E) \to \mathbb{R}^n \times \mathbb{R}^n$$

- *Let $P_S(x,y)$ be the radial transport from y to x in S.*
- *We have*

  $$Hom(E) = Cl \otimes Hom(\tilde{E})$$

  We want to consider from the point of view of Patodi's lemma. We have

  $$hom^{(l)} = Cl^{(l)} \otimes Hom(\tilde{E})$$

  where $Cl^{(l)}$ is the sub Clifford algebra generated by products of at most l Clifford multiplications.
- *We know*

  $$P_S(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n, Hom(S)), Hom(S) \cong Cl$$

  Consider

  $$Hom^l = \textit{span}\{(P_s(x,y) \otimes Id_{\tilde{E}}) \circ A | A \in Cl^l \otimes Hom(\tilde{E})\}$$

  Note that if $A = B \otimes T, B \in Cl^l$, then

  $$(P_S(x,y)B) \otimes T = \nabla_{\vec{r}} P_S(x,y)B = 0, P_S(y,y) = Id_S$$

- *There is another possibility. It would be to use* $C^\infty(\mathbb{R}^n \times \mathbb{R}^n), Hom^{(l)}$ *to filter*

$$C^\infty(\mathbb{R}^n \times \mathbb{R}^n, Hom(E))$$

  but it does not work...

Lemma 39. *If*

$$A \in Hom^{(l)} = Cl^{(l)} \otimes Hom(\tilde{E}), B \in Hom^{(k)}$$

Then

$$A \circ B \in Hom^{(l+k)}$$

Proof. Exercise!

Lemma 40. *Let* $A \in Hom^{(l)} \subseteq C^\infty(\mathbb{R}^n \times \mathbb{R}^n, Hom(E))$*. Now forall* $v \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$*, we have*

$$\nabla_V^E A = \frac{1}{8}\sigma(\mathcal{R}(v, \vec{r}_y))A + A_1 + A_2$$

where $A_1 \in Hom^{(l)}, A_2 \in O(|x-y|^2)Hom^{(l+2)}$.

Proof. Consider

$$P_S(x,y) \cdot B \otimes T, B \in Cl^{(l)}$$

where

$$P_S(x,y) = (P_S(x,y)B \otimes Id_{\tilde{I}}) \circ (Id_S \otimes T)$$

and we know

$$\nabla_V^E = \nabla_V^S \otimes Id_{\tilde{E}} + Id_S \otimes \nabla^{\tilde{E}}$$

where the second term $Id_S \otimes \nabla^{\tilde{E}}$ disappears given A_1. Now we have

$$\begin{aligned}
(\nabla_V^S \otimes Id_{\tilde{E}})A &= (\nabla^S P_S(x,y)B) \otimes T && (5.47)\\
&= -\frac{1}{2}Q_S(v, \vec{r}_y)P_S(x,y)B \otimes T + O(|x-y|^2) && (5.48)\\
&= \frac{1}{8}\sigma(\mathcal{R}(v, \vec{r}_y))P_s(x,y)S \otimes T + O(|x-y|^2) && (5.49)
\end{aligned}$$

Discussion. *Why is the last thing useful?*

5.9 Heat symbols

Recall that

$$Q_{\mathcal{H}}u = t^{-\frac{n}{2}} \int q_{\mathcal{H}}(x, t^{1/2}, \frac{x-y}{t^{1/2}} u(y) dg(y), q_{\mathcal{H}}(x, t^{1/2}, \frac{x-y}{t^{1/2}}) \in Hom(E)$$

where the heat kernel can be expressed as complex matrices in $\bigcup_l Hom^{(l)}$. We now that $Q_{\mathcal{H}}$ is the local paramatrix of the heat kernel operator.

Now incorporate $Hom^{(l)}$ spaces into the definition we define

Definition 42. *$S^{-\infty,l}(\mathbb{R}^{n+1}, \mathbb{R}^n) \subseteq S^{-\infty}(\mathbb{R}^{n+1} \times \mathbb{R}^n, Hom(E))$ is equivalent to*

$$q(x, t^{1/2}, \frac{x-y}{t^{1/2}}) = \sum_{\textit{finite sum}} q_k(x, t^{1/2}, \frac{x-y}{t^{1/2}}) A_k(x,y)$$

where

$$q_k(x, s, \omega) \in S^{-\infty}(\mathbb{R}^{n+1}, \mathbb{R}^n, \mathbb{C}), A_k \in Hom^{(l)}$$

In particular

$$q(x, s, \omega) = \sum_{k=0} q_k(x, s, \omega) A_k(x, x - s\omega)$$

How does the connection related to these spaces?

Lemma 41. *Let $p \in \mathbb{Z}$, $q \in S^{-\infty,l}$, then*

$$\nabla_{\partial_k} t^{-\frac{n}{2}-\frac{p}{2}-1} \underbrace{q(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{\in \Psi^P_{\mathcal{H}}} = t^{\frac{n}{2}-\frac{p+1}{2}-1} (\underbrace{q^{(l)}}_{S^{-\infty,l}} (x, t^{1/2}, \frac{x-y}{t^{1/2}} + t \underbrace{q^{(l+2)}}_{S^{-\infty,l+2}} (x, t^{1/2}, \frac{x-y}{t^{/2}})$$

Also:

$$q^{(l)}(x, 0, \omega) + q^{(l+2)}(x, 0, \omega) = [\partial_{\omega_k} + \frac{1}{8}\sigma(\mathcal{R}(\partial_{x_k}, \omega\partial_{x_k}))q(x, t^{1/2}, \omega)]$$

Proof. We assume

$$q = \tilde{q}(x, t^{1/2}, \frac{x-y}{t^{1/2}}) A(x,y), A(x,y) \in Hom^{(l)}$$

and $\frac{x-y}{t^{1/2}}$ is $\mathbb{C}$-valued. Now

$$t^{1/2}\nabla_{\partial_{x_k}} \tilde{q}(x, t^{1/2}, \frac{x-y}{t^{1/2}}) A(x,y) = t^{1/2}([(\partial_{x_k}\tilde{q})(x, t^{1/2}, \frac{x-y}{t^{1/2}}) + t^{-1/2}\partial_{x_k}\tilde{q}] A(x,y) + t^{1/2}\tilde{q}() \nabla_{\partial_{x_k}} A(x,y)) \tag{5.50}$$

$$= sq^{(l)}\text{term} + q^{(l)}\text{term} + t^{1/2}\tilde{q}\frac{1}{8}\sigma(\mathcal{R}(\partial_{x_k})(x-y)\partial_x)A(x,y) + t^{1/2}\tilde{q}Hom^{(l)} + t^{1/2}\tilde{q}(\tag{5.51}$$

where the last term could not fit in the last line is

$$t \cdot t^{1/2}\tilde{q}O|\frac{x-y}{t^{1/2}}|^2 Hom^{(l+2)}$$

and after rearranging we have

$$sHom^{(l)} + Hom^{(l)} + sHom^{(l)} + tHom^{(l+2)} + t * sHom^{(l+2)}$$

Execrise 22. *Let $p \in \mathbb{Z}, q \in S^{-\infty,l}$, then*

- $$\partial_t t^{-\frac{n}{2}-\frac{p}{2}-1} q(x, t^{1/2}, \frac{x-y}{t^{1/2}}) = t^{-\frac{n}{2}-\frac{p+2}{2}-1} q^{(l)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})$$

  where the last term is in $S^{-\infty,l}$ and we note this does not change Clifford multiplication.

- *We have*

  $$\not{\partial}\, 2t^{-\frac{n}{2}-\frac{p}{2}-1} q = -(\sum_{k=1}^{n} \nabla^2_{v_k} + div(v_k)\nabla_{v_k}) - \frac{1}{2}\sigma(\tilde{Q}) + \frac{1}{4}S$$

  which then equals

  $$t^{-\frac{n}{2}-\frac{p+2}{2}-1}[\underbrace{q^{(l)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{\in S^{\infty,l}} + t\underbrace{q^{(l+2)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{\in S^{-\infty,l+2}} + t^2\underbrace{q^{(l+4)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{\in S^{-\infty,l+4}}]$$

This motivates the following definition:

Definition 43.

$$q(x, s, \omega) \in \tilde{S}^{-\infty}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \leftrightarrow q(x, s, \omega) = \sum_{k=0}^{n} s^k q^{(k)}(x, s, \omega), q^{(k)} \in S^{-\infty,k}$$

and

$$q^{(k)}(x, -s, -\omega) = (-1)^{p+k} q(x, -s, -\omega)$$

Discussion. *We can also consider*

$$q \in \tilde{S}^{-\infty,l} \leftrightarrow q(x, s, \omega) = \sum_{k=0}^{n} s^k q^{(l+k)}(x, s, \omega), q^{(l+k)} \in S^{-\infty,l+k}$$

And then

$$\Psi^{p,l}_{\mathcal{H}} \in Q : C^\infty_c(\mathbb{R}^n) \to C^\infty((0,\infty) \times \mathbb{R}^n, E)$$

where the right hand side is a trivial bundle. In particular we have

$$Qu = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(x, s, \omega) u(y) dg(y), q \in \tilde{S}^{-\infty,l}$$

and

$$q^{l+k}(x, -s, -\omega) = (-1)^{p+k} q(x, s, \omega)$$

DEFINITION 44. *We now define the heat symbol:*

$$\sigma_p(Q) = q(x,0,\omega) = q^{(l)}(x,0,\omega)$$

such that there exist a sequence similar to the one before and

$$\partial_t : \Psi^{p,l} \to \Psi^{p+2,l}$$

as well as

$$\not{\partial}: \Psi^{p,l} \to \Psi^{p+2,l}$$

Therefore

$$(\partial_t + \not{\partial}^2) : \Psi^{p,l} \to \Psi^{p+2,l}$$

Now go back through the proof word by word, one can prove that there exist $Q_H \in \Psi^{-2,0}$ *such that*

$$(\partial_t + \not{\partial}^2) Q_{\mathcal{H}} \in \Psi_{\mathcal{H}}^{-\infty}, Q_{\mathcal{H}}|_{t=0} = Id$$

Further we can expand $Q_{\mathcal{H}}$ *by*

$$Q_{\mathcal{H}} = Q_0 + Q_1 + \cdots$$

where

$$Q_0 = \frac{t^{-n/2}}{(4\pi)^{n/2}} \int e^{\frac{|x-y|^2}{4t}} u(y) dg(y)$$

We want to reconstruct via these new heat spaces to prove the existence of $Q_{\mathcal{H}}$.

Back to

$$q(x,s,\omega) \in \tilde{S}^{-\infty} = \sum_{k=0}^{n} s^k q^{(k)}$$

DEFINITION 45. *We define*

$$\tilde{\Psi}_{\mathcal{H}}^p \subset \Psi_{\mathcal{H}}^p$$

to be the set of $q \in \Psi_{\mathcal{H}}^p$ *such that*

$$Qu = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}}) u(y) dg(y)$$

where

$$q(x,s,\omega) \in \tilde{S}^{-\infty}, q^{(k)}(x,-s,-\omega) = (-1)^{p+k} q^{(k)}(x,s,\omega)$$

THEOREM 74. *There exist* $Q_{\mathcal{H}} \in \tilde{\Psi}_{\mathcal{H}}^{-2} = \Psi_{\mathcal{H}}^{-2,0}$ *of 'project' such that*

$$(\partial_t + \not{\partial}^2) Q_{\mathcal{H}} \in \Psi_{\mathcal{H}}^{-\infty}, Q_{\mathcal{H}}|_{t=0} = Id$$

Proof. This can be proven by observing in the old construction.

REMARK 107. *I feel this sentence is quite incomprehensible.*

Conclusion 1. *On the original Riemannian manifold M locally we have*

$$e^{-t\partial\!\!\!/^2}u = \int h(t,x,y)u(y)dg(y) \pmod{\textit{something equal to zero in the Taylor expansion at time zero}}$$

such that

$$h(t,x,y) = t^{-\frac{n}{2}}\sum_{k=0}^{n} t^{\frac{k}{2}} \underbrace{q^{(k)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{S^{-\infty,k}}$$

and

$$q^{(k)}(x,-s,\omega) = (-1)^k q^{(k)}(x,s,\omega)$$

This could be called the 'local index theorem' because

$$\mathit{Ind}(\partial\!\!\!/^+) = \mathrm{Tr}(Ze^{-t\partial\!\!\!/^2}) \tag{5.52}$$

$$= \textit{sum of the kernel}\ .t^{-\frac{n}{2}}\int\sum_{k=0}^{n} t^{k/2}\,\mathrm{Tr}(Zq^{(k)}(x,t^{1/2},0)dg(x)) \tag{5.53}$$

$$\tag{5.54}$$

Recall that

$$A \in Hom(E) = Cl \otimes \tilde{E}$$

Then we have

$$\mathrm{Tr}(\underbrace{ZA}_{\in\Gamma\otimes\tilde{\Gamma}}) = (-1)^m\,\mathrm{Tr}_E(\tilde{\Gamma}A_{top})$$

where

$$A_{top} = Id_S \otimes Hom(\tilde{E})$$

is the coefficient of $(\sigma(\varphi_1)\cdots,\sigma(\varphi_n))\otimes Id_{\tilde{E}}$. Also, the $q^{(l)}$ are sums like

$$q(x,s,\omega)A_j(x,x) \in Cl^{(k)} \otimes Hom(\tilde{E})$$

Now back to the $k=n$ part we have

$$\mathit{Ind}\ \partial\!\!\!/^+ = \int 2^n(-i)^m\,\mathrm{Tr}_E(\tilde{\Gamma}q_?^{(n)}(x,t^{1/2},0)dg(x)$$

Next time: We are going to compute $q_{top}^{(n)}$ to find

$$\sqrt{\det(\frac{\mathcal{R}/2}{\sinh})}\,\mathrm{Tr}(\tilde{\Gamma}e^{-\overline{\partial}2})$$

Then

$$\mathit{Ind} = \lim_{t\to\infty}\int q^{(n)}(x,t^{1/2},0)$$

5.10 Finishing the proof of Atiyah-Singer index theorem

By discussion in last section we know that

$$q(x,s,\omega) \in \tilde{S}^{-\infty}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \leftrightarrow q(x,s,\omega) = \sum_{k=0}^{n} s^k q^{(k)}(x,s,\omega) \in S^{-\infty,k}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$$

where the later is (letters not legible) equal to

$$S^{-\infty}(\mathbb{R}^{n+1} \times \mathbb{R}^n)(P_s(x,y)B) \otimes T, B \subseteq Cl^{(k)}, T \in Hom(\tilde{E})$$

and we know

$$Q \in \tilde{\Psi}^p_{\mathcal{H}} \subset \Psi^p_{\mathcal{H}} \leftrightarrow Qu = t^{\frac{n}{2}-\frac{p}{2}-1} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}} u(y) dg(y), q \in \tilde{S}^{-\infty}$$

Further

$$q^{(k)}(x,-s,\omega) = q^{(k)}(x,s,\omega)(-1)^{p+k}$$

If you study the construction of the heat kernel for the usual heat spaces you will prove the following theorem:

Theorem 75. *There exists $Q_u \in \tilde{\Psi}^{-2}_{\mathcal{H}}$ such that*

$$(\partial_t + \eth^2) Q_{\mathcal{H}} \in \Psi^{-\infty}_{\mathcal{H}}, Q_{\mathcal{H}}|_{t=0} = Id_{C^\infty_c(\mathbb{R}^n,E)}$$

This, locally give us

$$e^{-t\eth^2} u = \int h(t,x,y) u(y) dg(y), h(t,x,y) = t^{-\frac{n}{2}} \sum t^{\frac{l}{2}} q^l(x, t^{1/2}, \frac{x-y}{t^{1/2}})$$

Discussion. *But why?*

Recall that

$$A \in Hom(E) = Cl \otimes Hom(\tilde{E}) = \bigcup_l Cl^{(l)} \otimes Hom(\tilde{E})$$

and

$$\mathrm{Tr}_E(ZA) = (-i)^m \,\mathrm{Tr}_E(\tilde{\Gamma} A_{top}), ZA = \Gamma \otimes \tilde{\Gamma}$$

where $A_{top} \in Hom(\tilde{E}) = Id_S \otimes Hom(\tilde{E})$ are the homomorphism on E communing with the Clifford action. We know that A_{top} is the coefficient of

$$\sigma(\varphi_1) \cdots \sigma(\varphi_n)$$

So we can write

$$A = \sigma(\varphi_1) \cdots \sigma(\varphi_n) A_{top} \pmod{Cl^{(n-1)} \otimes \tilde{E}}$$

Alternatively:

$$A \in Cl \otimes Hom(\tilde{E}) \xrightarrow{\wedge} \underbrace{\mathbb{C}\wedge^*}_{\tilde{A}} \otimes Hom(\tilde{E})$$

where we used the isomorphism between Clifford modules and exterior algebra. Here

$$A = A_0 + \cdots A_n$$

which implies

$$\mathrm{Tr}(ZA) = (-i)^m \,\mathrm{Tr}_E(\tilde{\Gamma}\tilde{A}_{top})$$

Thus we have

$$\begin{aligned}
Ind\mathcal{D}^+ &= Ind\, \not{\partial}^+ && (5.55)\\
&= \mathrm{Tr}(Ze^{-t\not{\partial}^2}) && (5.56)\\
&= \int \mathrm{Tr}(Zh(t,x,x))dg && (5.57)\\
&= (-i)^m \int \mathrm{Tr}(\tilde{\Gamma}, \underbrace{\tilde{q}^{(n)}}_{top})\varphi_1 \wedge \cdots \wedge \varphi_n && (5.58)\\
&= (-i)^m \int \mathrm{Tr}(\tilde{\Gamma}\tilde{q}^{(n)})\varphi_1 \wedge \cdots \wedge \varphi_n && (5.59)
\end{aligned}$$

where

$$\varphi_1 \wedge \cdots \wedge \varphi_n$$

only integrate the top degree part. In particular we have

$$\begin{aligned}
Ind\mathcal{D}^+ &= \lim_{t\to 0} \int \mathrm{Tr}(\tilde{\Gamma}\tilde{q}^{(n)}(x, t^{1/2}, 0) && (5.60)\\
&= \int \mathrm{Tr}(\tilde{\Gamma}\tilde{q}^{(n)}(x, 0, 0) && (5.61)
\end{aligned}$$

Discussion. *Here is our goal: Find $\tilde{q}^{(n)}$. We know that the principal symbol is the most significant symbol of an operator $Q \in \tilde{\Psi}^p_{\mathcal{H}}$.*

$$Qu = t^{-\frac{n}{2}-\frac{p}{2}-1} \int q(x, t^{1/2}, \frac{x-y}{t^{1/2}})u(y)dg(y)$$

where

$$= \sum_{i=0}^{n} t^{i/2} q^i(t, t^{1/2}, \frac{x-y}{t^{1/2}})$$

Old principal symbol:

$$q^{(0)}(x, 0, \omega) \in Cl^{(0)} \otimes Hom(E), Cl^{(0)} = \mathbb{C}$$

Not good enough presently as we want $\tilde{q}^{(n)}$. However, we want

$$q^{(n)}(x,0,\omega)$$

Recall that

$$q^{(l)}(x,t^{1/2},\frac{x-y}{t^{1/2}} = q_l(x,t^{1/2},\frac{x-y}{t^{1/2}})P_s(x,y)B\otimes T$$

and

$$q^{(l)}(x,s,\omega) = q_{(l)}(x,s,\omega)P_s(x,x,\omega)B\otimes T$$

such that at $s=0$ we have

$$q^{(l)}(x,0,\omega) = q(x,0,\omega)B\otimes T$$

because $P_s = Id$.

Now to get both we define:

DEFINITION 46.

$$\tilde{\sigma}_P(Q)(x,\omega) = \sum_{l=0}^{n}[\tilde{q}^{(l)}(x,0,\omega)], \tilde{q}^{(l)} \in Cl^{(l)}/Cl^{(l-1)}$$

so we get

$$\tilde{\sigma}_p : \tilde{\Psi}^P_{\mathcal{H}} \to S^{-\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_\omega, \mathbb{C}\wedge^* \otimes Hom(\tilde{E}))$$

THEOREM 76. *Show that the following sequence is exact:*

$$0 \to \tilde{\Psi}^{p-1}_{\mathcal{H}} \to \tilde{}$$

Proof. Exercise.

Discussion. *Observe*

$$IndD^{+} = (-i)^m\int \mathrm{Tr}(\tilde{\Gamma}\tilde{q}^m(x,0,0)) = (-i)^m\int \mathrm{Tr}(\tilde{\Gamma}) \tag{5.62}$$

$$= (-i)^m\int \mathrm{Tr}(\tilde{\Gamma}\tilde{\sigma}_{-2}(Q_{\mathcal{H}})) \tag{5.63}$$

where by convention we only integrate the top form. We will prove:

$$Q_{\mathcal{H}} \in \tilde{\Psi}^{-2}_{\mathcal{H}}$$

Here is our new goal: Find $\tilde{\sigma}_{-2}(Q_{\mathcal{H}})$. How to do that? We reconstruct the heat kernel. Assume that

$$Q_{\mathcal{H}} = Q_0 + Q_1\cdots Q_n + \cdots$$

where

$$Q_k \in \tilde{\Psi}^{-2-k}, \sigma_{-2}(Q_k) = 0, \forall k \geq 1$$

Therefore

$$\tilde{\sigma}_{-2}(Q_{\mathcal{H}}) = \tilde{\sigma}_{-2}(Q_0)!$$

Now to construct $Q_{\mathcal{H}}$, we let

$$A = \partial_t + \not{\partial}^2$$

Then

$$AQ_{\mathcal{H}} = \underbrace{AQ_0}_{\in \tilde{\Psi}^0_{\mathcal{H}}} + \underbrace{AQ_1}_{\in \tilde{\Psi}^1_{\mathcal{H}}} + \underbrace{AQ_2}_{\in \tilde{\Psi}^2_{\mathcal{H}}}$$

We want

$$AQ_{\mathcal{H}} \in \Psi^{-\infty}_{\mathcal{H}}$$

to vanish to ∞ order at $t = 0$. This means we want that

- $$0 = \tilde{\sigma}_0(AQ_{\mathcal{H}}) = \tilde{Q}_0(AQ_0)$$

  and we can choose Q_0.

- *Similarly*

  $$0 = \tilde{\sigma}_1(AQ_{\mathcal{H}}) = \sigma_{-1}(AQ_0) + \tilde{\sigma}_{-1}(AQ_1)$$

  and we want to choose Q_1.

- *Similarly*

  $$0 = \tilde{\sigma}_{-2}(AQ_{\mathcal{H}}$$

  and so on.

Discussion. *Our new goal is to find $\tilde{\sigma}_{p+2}(AQ)$ where $Q|in\tilde{\Psi}^p_{\mathcal{H}}$.*

Theorem 77. *Let $Q \in \tilde{\Psi}^P_{\mathcal{H}}$, then*

- *We claim*

  $$\partial_t Q \in \tilde{\Psi}^{p+2}_{\mathcal{H}}$$

  and

  $$\tilde{\sigma}_{p+2}(\partial_t Q) = (-\frac{n}{2} - \frac{p}{2} - 1 + \frac{G}{2} - \frac{1}{2}\omega \cdot \partial_\omega)\tilde{\sigma}_p(Q)(x, \omega)$$

  where

  $$G : \alpha \to K\alpha, \alpha \in \mathbb{C}\wedge^*$$

Proof. Exercise.

- *If* $v \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n), \nabla_v Q \in \tilde{\Psi}^{p+1}$ *and*

$$\tilde{\sigma}_{p+1}(\nabla_v Q)(x,\omega) = (\overline{v} + \frac{1}{4}\mathcal{R}(v, \omega \circ \partial_x))\tilde{\sigma}_p(Q)(x,\omega)$$

where

$$v = \sum_{i=1}^n a_i(x)\partial_{x_i}, \overline{v} = \sum a_i(x)\partial_{\omega_i}$$

and

$$\mathcal{R}(v,\omega) = \textit{Riemannian curvature tensor, a 2n form}$$

Recall that

$$\mathcal{R} \in C^\infty(M, T^*M \otimes T^*M \otimes T^*M \otimes T^*M) = C^\infty(M, \wedge^2 \otimes \wedge^2)$$

Proof. This will be an exercise! Look at yesterday's class.

Theorem 78. *If* $Q \in \tilde{\Psi}^p_{\mathcal{H}}, \not{\partial}^2\, Q \in \tilde{\Psi}^{p+2}_{\mathcal{H}}$ *and*

$$\tilde{\sigma}_{p+2}(\not{\partial}^2\, Q)(x,\omega) = [-\sum(\nabla_k + \frac{1}{2}\mathcal{R}(v_k, \omega, \partial_x))^2 - \tilde{Q}]\tilde{\sigma}_p(Q)(x,v), \tilde{Q} = Q_E + \frac{1}{4}\sigma(\mathcal{R})$$

and $v_1 \cdots v_n$ *are duals of* $\varphi_1 \cdots \varphi_n$.

Proof. We have

$$\not{\partial}^2 = -\sum_{k=1}^n \sum_{v_k}^2 - \sum_{k=1}^n \text{div}(v_k)\nabla_{v_k} - \frac{1}{2}\sigma(\tilde{Q}) + \frac{1}{4}S$$

Therefore

$$\not{\partial}^2\, Q = -\sum_{i=1}^n \nabla_{v_k}(\nabla_{v_k}Q) - \sum_{k=1}^n \text{div}(v_k)\nabla_{v_k}Q - \frac{1}{2}\sigma(\tilde{Q})Q + \frac{1}{4}SQ$$

But we notice the term below are zero because of our choice of local frame:

$$\sum_{k=1}^n \text{div}(v_k)$$

and the other term is zero too because of grading:

$$\frac{1}{4}SQ$$

Observe that:

$$-\frac{1}{2}\sigma(\tilde{Q})t^{-\frac{n}{2}-\frac{p}{2}-1}q(x, t^{1/2}, \frac{x-y}{t^{1/2}}) = t^{-\frac{n}{2}-\frac{p+2}{2}-1}\sum_{t=0}^n t^{k/2}_{-\frac{1}{2}}\underbrace{\sigma(\tilde{Q})q^{(l)}(x, t^{1/2}, \frac{x-y}{t^{1/2}})}_{S^{-\infty, l+2}} \in \tilde{\Psi}^{p+2}_{\mathcal{H}}$$

Moreover

$$\tilde{\sigma}_{p+2}(-\frac{1}{2}\sigma(\tilde{Q})Q) = -\tilde{Q}\cdot\tilde{\sigma}_p(Q)(x,\omega)$$

Finally

$$\tilde{\sigma}_{p+2}(\nabla_{v_k}(\nabla_{v_k}(Q))) = (\overline{v}_k + \frac{1}{4}\mathcal{R}(v_k,\omega,\partial_k))\tilde{\sigma}_{p+1}(\nabla_{v_k}Q) \tag{5.64}$$

$$= (\overline{v}_k + \frac{1}{4}\mathcal{R}(v_k,\omega,\partial_x)^2\tilde{\sigma}_p(Q) \tag{5.65}$$

We now reached the following theorem:

THEOREM 79. *For all $Q \in \tilde{\Psi}^p_{\mathcal{H}}$,*

$$\tilde{\sigma}_{p+2}((\partial_t + \not{\partial}^2)Q) = [-\frac{n}{2} - \frac{p}{2} - 1 + \frac{G}{2} - \sum(\overline{v_k} + \frac{1}{4}\mathcal{R}(v_k,\omega,\partial_x))^2 - Q]\tilde{\sigma}_p(Q)(x,\omega)$$

where

$$v_k = \sum a_{ik}(x)\partial_{x_k}, \tilde{v}_k = \sum a_{ik}(x)\partial_{v_k}$$

REMARK 108. *In the one hundred years old fact we did not have the last two terms. So we just got a PDE to solve in ω!*

Discussion. *Let*

$$A = [a_{ij}(x)] \in M_{n\times n}$$

Let

$$Z = A^{-1}\omega$$

be a linear change of variables, which change with coordinate system to Z coordinates.

Execrise 23. *In the Z-coordinates:*

$$RHS = [-\frac{n}{2} - \frac{p}{2} - 1 + \frac{G}{2} - \frac{1}{2}z\cdot - \sum(\partial_{z_k} + \frac{1}{4}(\mathcal{R}z))^2_k - \tilde{Q}]\tilde{\sigma}_p(Q)\mathcal{R}(v_i,v_j)$$

Discussion. *We now go back to heat kernel: Recall that we want to find $Q_0 \in \tilde{\Psi}^{-2}_{\mathcal{H}}$ such that*

$$\tilde{\sigma}_0[(\partial_t + \not{\partial}^2)Q_0] = 0, Q_0|_{t=0} = Id_{C^\infty(\mathbb{R}^n,\mathbb{R}^n)}$$

Hence we want

$$[-\frac{n}{2} + \frac{G}{2} - \frac{1}{2}z\cdot\partial_z - \sum(\partial_{Z_n} + \frac{1}{4}(\mathbb{R}Z)_k)^2(-\tilde{Q})]\tilde{\sigma}_{-2}(Q)_0(x,\omega) =$$

In other words: Find $q_0(x,\omega)$ such that

$$[\cdots]q_0(x,\omega) = 0$$

Using reparation of variables it is straightforward to show a solution q_0 *is*

$$q_0(x,z) = \frac{1}{(4\pi)^{n/2}} \overset{1/2}{\det}\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right) e^{-\frac{1}{4}\langle \mathcal{R}/2 \coth(\mathcal{R}/2)Z, Z\rangle} e^{\tilde{Q}}$$

Finally, we reached the end of the proof: We recall:

$$Ind\mathcal{D}^{+} = \int (-i)^m \operatorname{Tr}(\tilde{\Gamma}\tilde{\sigma}_{-2}(Q_0)(x,0) = \frac{1}{(4\pi i)^m}\int_M (\det)^{1/2}\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right) Tr(\tilde{Z}e^{\tilde{Q}})$$

Chapter 6

Proof of signature theorem

6.1 Lecture 1

Recall

$$E = E^+ \oplus E^-$$

is a $\mathbb{Z}_2$ *graded Hermitian vector bundle over a compact oriented Riemannian manifold.*

Let

$$\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$$

be a first order elliptic differential operator such that

$$\sigma_1(\sigma) = \sigma_1(\mathcal{D})(\xi) = g(\xi, \xi)$$

Now if $Z = \pm 1$ *on* $E^\pm$*, then we assume*

$$Z \circ \mathcal{D} = -\mathcal{D} \circ Z$$

Therefore

$$\mathcal{D} : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$$

DEFINITION 47. *We let* $\mathcal{D}^+$ *to be the restriction of* $\mathcal{D}$ *on* E^+.

THEOREM 80. *We have*

$$Ind\mathcal{D}^+ = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}})$$

where $\mathcal{R}$ *is the Riemmanian curvature tensor,* $\tilde{Z} = Z \circ \omega$*. Here*

$$\omega = \frac{i^m}{n!} \sigma(dg) = i^m \sigma(\phi_1) \cdots \sigma(\phi_n)$$

such that locally

$$dg = \phi_1 \wedge \cdots \wedge \phi_n$$

and

$$\tilde{Q} = Q_E + \frac{1}{4}\mathcal{R}$$

Discussion. *Our goal is to find $E, Z, \mathcal{D}$ such that*

$$Ind(\mathcal{D}^+) = sgn(M) = \int e(TM)$$

DEFINITION 48. *We first define the signature of M. Let V be a n-dimensional $\mathbb{R}$ vector space with orthonormal basis $\{v_i\}$. We now define the **Hodge star** to be*

$$* : \wedge^k(V) \to \wedge^{n-k}(V)$$

DEFINITION 49. *Let $I = \{i_1 \cdots i_k\}$ be a subset of $I_n = \{1 \cdots n\}$. Let $I^* = I - \{I_n\}$. Assume I^* is ordered. Then $*(v_I)$ is defined by*

$$*(v_I) = *(v_{i_1} \cdots \wedge v_{i_k}) = sgn(I \cup I^*) v_{I^*}$$

LEMMA 42. *We need to prove that this is independent of the choice of the basis. We claim that if $\alpha, \beta \in \wedge^k$, then*

$$\alpha \wedge *(\beta) = \langle \alpha, \beta \rangle \mathit{vol}$$

as well as

$$* \circ * = (-1)^{k(n-k)}$$

Proof. Exercise!

LEMMA 43. *Suppose $*$ is a linear map satisfying both properties, then we have*

$$\tilde{*} = *$$

Proof. Let $* \in \wedge^k$. Then we have

$$\langle *\alpha, \beta \rangle \mathrm{vol} = \langle \beta, *\alpha \rangle \mathrm{vol}$$

But this by definition is the same as

$$\beta \wedge * * \alpha = \beta \wedge (-1)^{k(n-k)} \alpha$$

Discussion. *Let M be a compact, oriented Riemannian manifold. Let T_pM be the tangent space at $p \in M$. Then it has a hodge star operator*

$$* : \wedge^k(T_pM) \to \wedge^{n-k}(T_pM)$$

The question is:

What is the Hodge Star operator good for?

To answer this we need to see some examples.

Example 34. *Consider $\mathbb{R}^2$ with Euclidean inner-product. Then we have*

$$d : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2, \wedge^2)$$

Let

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Recall that by definition we have

$$d(adx + bdy) = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})vol$$

Now by definition we have

$$\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle$$

Execrise 24. *Show that*

$$d^*(\alpha dx + \beta dy) = -(\frac{\partial \alpha}{\partial x} + \frac{\partial b}{\partial y})$$

as well as

$$d^*(\alpha dx \wedge dy) = -\frac{\partial \alpha}{\partial x}dx - \frac{\partial \alpha}{\partial y}dy$$

Note that we have

$$*dx = dy, *dy = -dx, *(1) = vol, *(vol) = 1$$

Thus we have

$$*(adx + bdy) = ady - bdy, d*(adx + bdy) = \frac{\partial a}{\partial x}dx \wedge dy + \frac{\partial b}{\partial x}dx \wedge dy$$

Therefore

$$*d*(adx + bdy) = -d^*(adx + bdy) \to d^* = -*d*$$

This fact can be properly generalized for all $C^\infty(M, \wedge^*)$:

THEOREM 81. *On any Riemannian manifold we have*

$$d^* : C^\infty(M, \wedge^{k+1}) \to C^\infty(M, \wedge^k)$$

given by

$$d^* = (-1)^{nk+1} * d*$$

REMARK 109. *This expression is cool because we finally get rid of local coordinates!*

REMARK 110. *To prove the theorem we need two lemmas:*

LEMMA 44. *For all $\alpha, \beta \in C^\infty(M, \wedge^k)$, such that one of them has compact support, we have*

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

LEMMA 45. *For all $\gamma \in C_c^\infty(M, \wedge^{n-1})$, we have*

$$\int_M d\gamma = 0$$

Proof. Let $\alpha \in C^\infty(M, \wedge^k)$, $\beta \in C^\infty(M, \wedge^{k+1})$. Then we have

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^k \alpha \wedge (d^* \beta)$$

Therefore

$$0 = \int d\alpha \wedge *\beta + (-1)^k \int \alpha \wedge (d * \beta)$$

by Lemma 45. Now note that

$$d * \beta = * * (d * \beta)(-1)^{k(n-k)}$$

Therefore we have

$$\int d\alpha \wedge *\beta = \langle d\alpha, \beta \rangle$$

as well as

$$(-1)^k \int \alpha \wedge (d * \beta) = (-1)^k (-1)^{k(n-k)} (\alpha, *d * \beta)$$

Therefore

$$(d\alpha, \beta) = (-1)^{nk+1} (\alpha, *d * \beta) \to d^* \beta = (-1)^{nk+1} * d * \beta$$

Discussion. *For other applications, the Hodge star can be used to prove Poincare duality:*

$$* : H^k_{dR}(M) \to H^{n-k}_{dR}(M)$$

In particular we have

THEOREM 82. *(Poincare duality):*

$$H^k_{dR}(M) * H^{n-k}_{dR}(M)$$

by consider the map

$$P : H^k_{dR} \times H^{n-k}_{dR} \to \mathbb{R}$$

which is given by

$$P([\alpha, \beta]) = \int \alpha \wedge \beta \in \mathbb{R}$$

Example 35. *Consider the special case that n is divisible by 4. Then P is a symmetric bilinear map on $H^{2d}_{dR} \times H^{2d}_{dR}$. If we take any basis of V and consider the matrix we get by P, then the signature is defined by*

$$\text{signature} = \#\text{positive eigenvalue} - \#negativeeigenvalue$$

6.2 Lecture 2

THEOREM 83. *Let*

$$H^k_{Ho} = \ker(d + d^*) \in C^\infty(M, \wedge^k)$$

Then we have

$$* : H^k_\Delta \to H^{n-k}_\Delta$$

is an isomorphism.

THEOREM 84. *Poincare duality: Let*

$$P : H^k \times H^{n-k} \to \mathbb{R}$$

given by

$$P([\alpha], [\beta]) = \int \alpha \wedge \beta \in C^\infty(M, \wedge)$$

LEMMA 46. *P is not degenerate.*

Proof. Given $\alpha \in H^k$, we need to show that there exist $\beta \in H^{n-k}$ such that

$$P([\alpha], [\beta]) \neq 0$$

But it is clear that if we pick up a harmonic representative of α, then

$$\int \alpha \wedge *\alpha \neq 0$$

since by definition

$$a \wedge *(\beta) = \langle \alpha, \alpha \rangle vol, \langle \alpha, \alpha \rangle \neq 0$$

Discussion. *Assume $n = 2m$ where m is even. Then P gives a blinear form which is nondegenerate and symmetric.*

DEFINITION 50. *The* ***the signature*** *of P is given by*

$$\textit{signature} = \#\textit{positive eigenvalue} - \#negativeeigenvalue$$

THEOREM 85. *The signature of the manifold coincide with the signature of P.*

LEMMA 47. *We know $** = 1$. So it is an involution of a vector space. This means we have decomposition of the type*

$$H^m_M = A^+ \oplus A^-$$

Proof. If

$$\alpha_1 \cdots \alpha_p$$

is a basis for A^+, and

$$\beta_1 \cdots \beta_q$$

is a basis for A^-. Then via Hodge theorem we have

$$\langle \alpha_i, \beta_i \rangle$$

being a basis for H^m_M. Then we have

$$p(\alpha_i, \alpha_j) = \int \alpha_i \wedge \alpha_j = \int \alpha_i \wedge *\alpha_j = \langle \alpha_i, \alpha_j \rangle = \delta_{ij}$$

where we used the fact that

$$*\alpha_j = \alpha_j$$

similarly

$$p(\alpha_i, \beta_j) = 0, p(\beta_i, \beta_j) = -\delta_{ij}$$

because

$$*\beta_j = -\beta_j$$

Therefore we have

$$sgn(P) = p - q$$

Discussion. *It is a fact that for a map*

$$T : V \to V$$

We can decompose V into even part and odd part, for example

$$V^+ = \{v : \frac{v + T(v)}{2}\}, V^- = \{v : \frac{v - T(v)}{2}\}$$

Now let $E = E^+ \oplus E^-$, with M a compact manifold. Then with the same set up as lecture 1, we have

$$Ind\mathcal{D}^+ = sgn(Z : \ker \mathcal{D} \to \ker D)$$

Remark 111. *because we know that*

$$Ind\mathcal{D}^+ = \dim(\ker) - \dim(coker)$$

But

$$\dim(coker\mathcal{D}^+) = \dim(\ker \mathcal{D}^-)$$

This is because the Dirac operator is self-adjoint:

$$(\mathcal{D}^+)^* = \mathcal{D}^-$$

In other words

$$\alpha \in \ker(\mathcal{D}^-) \leftrightarrow (\beta, \mathcal{D}^-\alpha) = 0 \leftrightarrow (\mathcal{D}^+\beta, \alpha) = 0 \leftrightarrow \bigcup \alpha \cong coker(\mathcal{D}^+)$$

by dimension arguments. Therefore we have

$$\textit{Ind}\mathcal{D}^+ = \dim(\ker \mathcal{D}^+) - \dim(\ker \mathcal{D}^-)$$

But this is the same as

$$\textit{Ind}\mathcal{D}^+ = sgn(Z : \ker \mathcal{D} \to \ker D)$$

which proved the claim.

Our main job left is the following: We need to construct E, D, Z explicitly such that the index is the same as the signature of M. For example we may guess that

$$E = \mathbb{C}\wedge^*, D = d + d^*, Z = (-1)^{m/2+\frac{k(k-1)}{2}}$$

Discussion. *It is in fact not clear what Z should be. We may conjecture that*

$$Z = (-1)^{f(k)} *$$

for some function f. But we need to find what $f(k)$ is. So we need to verify the properties one by one:

- $$Z^2 = 1$$

- $$\mathcal{D} \circ Z = -Z \circ \mathcal{D}$$

- *We want that*

  $$sgn(Z : \ker \mathcal{D} \to \ker \mathcal{D}) = sgn(P)$$

  after all. Let us do it step by step:

- *Let* $\alpha \in \wedge^k$. *Then we want*

$$Z^2\alpha = \alpha \leftrightarrow (-1)^{f(n-k)+f(k)} * *\alpha = \alpha$$

But we know that

$$* * \alpha = (-1)^{k(n-k)}\alpha$$

since $4|n$, *this is the same as*

$$f(n-k) + f(k) + k = 0 \in \mathbb{Z}_2$$

- *Now we try to use the second condition. We have*

$$\mathcal{D} \circ Z + Z \circ \mathcal{D} = 0$$

So we have

$$D = -ZDZ, d + d^* = -ZdZ - Zd^*Z$$

Thus

$$d + d^* = -ZdZ - Zd^*Z$$

Instead of let

$$Z : \wedge^k \to \wedge^k$$

we want

$$Z : \wedge^{m+k} \to \wedge^{m-k}$$

Then we would have

$$d = -Zd^*Z, d^* = -ZdZ$$

On $\mathbb{C}\wedge^{k+1}$ *we have*

$$d^* = (-1)^{nk+1} * d* = - * d*, 4|n$$

which we proved a long time ago. Thus we have

$$-Zd^*Z = Z * d * Z, -ZdZ = -Zd(-1)^{f(k)}* = (-1)^{f(k)+f(n-k+1)} * d*$$

This now (for unknown reason)implies

$$f(n-k) + f(k+1) \equiv 0 \pmod 2$$

So we know that

$$f(n-k) + f(k+1) = 0, f(k) + f(n-k) + k = 0$$

Therefore it follows from induction that

$$f(k) = \frac{k(k-1)}{2} + f(0)$$

- *We want* $Z \circ * = *$ *on* $\mathbb{C}\wedge^m$ *would be natural. Thus we have*

$$(-1)^{f(m)} = 0 \to f(m) = 0 \mod 2$$

- *Combine together we have*

$$f(m) = f(0) + \frac{k(k-1)}{2}, f(m) = 0 \mod 2$$

In particular

$$f(0) + \frac{2l(2l-1)}{2} \equiv 0 \to f(0) + l(2l-1) = 0 \to f(0) = l \mod 2$$

Therefore

$$f(m) \equiv \frac{m}{2} \pmod 2$$

and

$$f(k) = \frac{k(k-1)}{2} + \frac{m}{2}$$

Theorem 86. *We now claim that*

$$sgn(Z : \ker\mathcal{D} \to \ker\mathcal{D}) = sgn(Z : H^m_\Delta \to H^m_\Delta)$$

thus since we proved the later is equal to $sgn(M)$ *and the former is equal to the index of* $\mathcal{D}^+$*, we have the signature theorem.*

Remark 112. *The second formula after wards is illegible.*

Proof. Sketch To fix notation let

$$E = \mathbb{C}\wedge^*, \dim(M) = 2m = 4l, Z : \mathbb{C}\wedge^{n-k} \to \mathbb{C}\wedge^{n+k}, M_k = \mathbb{C}\wedge^{m-k}\oplus\wedge^{m+k}, Z : M_k \to M_k, Z_k = Z|_{M_k}$$

Using these notation we have

$$\ker\mathcal{D} = \bigoplus_{k\le n}\ker(1_k + Z_k) \cap H^*(M)$$

This implies

$$\text{Ind}\mathcal{D}^+ = \dim\ker(1_0 - Z_0) - \dim(1_0 + Z_0) + \cdots = sgn(M)$$

6.3 Lecture 3

We have

$$sgn(M) = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}})$$

Note that

$$\tilde{Z} = Z \circ \omega$$

such that locally

$$\omega = Z^m \sigma(\phi_1) \cdots \sigma(\phi_n)$$

and recall that

$$\tilde{Q} = Q_E + \frac{1}{4}\sigma(\mathcal{R})$$

First we need to compute $\tilde{Z}$. Recall that we found through some guess work that

$$Z = (-1)^{\frac{m}{2} + \frac{k(k-1)}{2}}$$

works. But what is ω? We claim that

Lemma 48.

$$\omega = Z$$

Therefore we have the corollary:

Corollary 18. *:*

$$\tilde{Z} = Z \circ \omega = Z^2 = 1!$$

Proof. To prove the lemma recall that

$$\omega = i^m \sigma_1(\phi_1) \cdots \sigma(\phi_k)$$

in $\wedge^k$, where

$$\sigma_1(\mathcal{D}(\xi)) = i(\xi \wedge -\xi \lrcorner)$$

Thus we have

$$\omega = i^m i(\phi_1 \wedge -v_1 \lrcorner) i(\phi_2 \wedge -v_2 \lrcorner) \cdots i(\phi_3 \wedge -v_3 \lrcorner) \cdots i(\phi_k \wedge -v_k \lrcorner)$$

Therefore we have

$$i^{m+k} \prod_{i=1}^{k} (\phi_1 \wedge -v_1 \lrcorner)$$

and by 'easy' argument (induction?) we have

$$\omega = (-1)^{\frac{k(k-1)}{2} + m}$$

REMARK 113. *I do not really get what is the 'easy argument' at here.*

Discussion. *In conclusion we have*

$$\mathrm{Tr}(\tilde{Z}e^{\tilde{Q}}) = \mathrm{Tr}(e^{\tilde{Q}})$$

and recall that we defined

$$\tilde{Q} = -\frac{1}{2}\tilde{\sigma}(\mathcal{R})$$

Hence we really have to compute

$$\mathrm{Tr}(e^{-\frac{1}{4}\tilde{\sigma}(\mathcal{R})})$$

REMARK 114. *I think there is a typo, it should be $-\frac{1}{4}$. But then what is Q_E at here? Why it vanished?*

6.4 Lecture 4, The proof

We were about to prove the following lemma:

LEMMA 49. *If* $4|n$*, then*

$$\omega = i^m \sigma(\phi_1)\cdots\sigma(\phi_n)$$

and

$$Z = (-1)^{\frac{k(k-1)}{2}+\frac{m}{2}} *$$

Proof. Let

$$\phi_I = \phi_{i_1}\cdots\wedge\phi_{i_k}, \phi_J = \phi_{j_1}\cdots\wedge\phi_{j_{n-k}}$$

such that

$$i_1 \le i_2 \cdots \le i_k, I \cup J = \{1\cdots n\}$$

Observe that

$$\begin{aligned}
i^m\sigma(\phi_1)\cdots\sigma(\phi_n)\phi_I &= i^m sgn(I,J)\sigma(\phi_I)\sigma(\phi_J)\phi_I && (6.1)\\
&= i^{m+n-k} sgn(I,J)\sigma(\phi_I)\phi_J\wedge\phi_I && (6.2)\\
&= i^{m+n-k} sgn(I,J)\sigma(\phi_I)\phi_I\wedge\phi_J(-1)^{k(n-k)} && (6.3)\\
&= i^{m+n+kn-k^2} sgn(I,J)\sigma(\phi_I)\phi_I\wedge\phi_J && (6.4)\\
&= i^m sgn(I,J) i^k \prod_{s=1}^{k}((\phi_{i_s})\wedge) - (v_{i_s}\lrcorner))\phi_I\wedge\phi_J && (6.5)\\
&= i^m(-1)^{\frac{k(k-1)}{2}} sgn(I,J)\phi_J && (6.6)\\
&= (-1)^{\frac{m}{2}+\frac{k(k-1)}{2}} * \phi_I && (6.7)
\end{aligned}$$

REMARK 115. *The last three steps is clear to me, but might be unclear to the reader. It took me some time to understand.*

Discussion. *From last lecture it is clear that we want to compute*

$$\mathrm{Tr}(e^{-\frac{1}{4}\tilde{\sigma}(\mathcal{R})})$$

Recall that

$$sgn(M) = \frac{1}{(4\pi i)^m}\int_M (\det)^{1/2}(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)})Tr(\tilde{Z}e^{\tilde{Q}})$$

LEMMA 50. *Recall that*

$$\mathcal{R} = \sum \mathcal{R}_{ij}\phi_i\otimes\phi_j, \mathcal{R}_{ij}(v,w) = \langle\mathcal{R}(v,w)v_i, v_j\rangle$$

Therefore we have

$$\tilde{\sigma}(\mathcal{R}) = \sum\mathcal{R}_{ij}\tilde{\sigma}(\phi_i)\tilde{\sigma}(\phi_j) = \sum\langle\mathcal{R}v_i, v_j\rangle\tilde{\sigma}_i\tilde{\sigma}_j$$

Remark 116. *Prof.Loya claimed that we may think $\mathcal{R}$ as a matrix of two-forms, thus $\mathcal{R}v_i$ is thought of as a matrix multiplication.*

Definition 51. *Let $\mathcal{A}_0$ be a nilpotent, commutative algebra with finite dimension.*
Let

$$\mathcal{A} = \mathbb{C} \oplus \mathcal{A}_0$$

Example 36.

$$\mathcal{A} = \mathbb{C}\wedge^{even}$$

is a good example.

Discussion. *Let V be a complex vector space. Let $P : hom(V) \to \mathcal{A}$ be a power series with positive radius of convergence. To more precise, we let*

$$P(A) = \sum P_\alpha a^\alpha, P_\alpha \in \mathcal{A}, a^\alpha = \prod_{i,j\le n} a_{\alpha_{ij}}^{\alpha_{ij}}$$

We want to extend P from $A_0 \otimes Hom(V)$ to $\mathcal{A}$. We use the definition

$$P(A) = \sum P_\alpha a^\alpha$$

where P is defined on all of $\mathcal{A}_0 \oplus hom(V)$.

Example 37. *Recall that the Chern-Weil theory definition of $\hat{A}$-genus is defined in these terms.*

Remark 117. *To clarify notation. Note that for every matrix A, we have*

$$A = \sum A_{ij}E_{ij}, P(A) = \sum P_\alpha a_{11}^{\alpha_{11}} \cdots a_{nn}^{\alpha_{nn}}$$

Now recall that we want to compute

$$\mathrm{Tr}(e^{-\frac{1}{4}\tilde{\sigma}(\mathcal{R})}), \mathcal{R} \in \mathbb{C} \wedge^{even} \otimes hom(T_pM)$$

We may think of this as follows:

$$P : hom(T_pM) \to \mathcal{A}, \mathcal{A} = \mathbb{C}\wedge^{even} = \mathbb{C} \oplus \mathbb{C} \wedge^2 \oplus \ldots$$

and

$$P(A) = \mathrm{Tr}(e^{-\frac{1}{4}\tilde{\sigma}(\mathcal{R})})$$

Remark 118. *Is there a typo here? This clearly does not define P!*

Discussion. *I believe Prof. Loya really wanted to define it by*

$$P(A) = \mathrm{Tr}(e^{-\frac{1}{4}\tilde{\sigma}(A)})$$

instead! Here P is the extended one:

$$P = \mathcal{A}_0 \oplus hom(T_pM) \to \mathcal{A}$$

Our goal is to find a close form formula for $P(A)$ such that it is related to the other part in the index:

$$\frac{1}{(4\pi i)^m}\int_M (\det)^{1/2}(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)})Tr(\tilde{Z}e^{\tilde{Q}})$$

The reason that we want to use $P(A)$ is this: If we let

$$\alpha = \{A \in hom(T_pM) | A^T = -A\}$$

then we have

$$\mathcal{R} \in \mathbb{C} \wedge^2 \otimes \alpha$$

This is the key for the proof. In fact we just need a nice formula for $P(A), A \in \alpha$ in terms of determinants.

Theorem 87. *We do have a nice formula for $P(A), A \in \alpha$. Let $A \in \alpha, A^T = -A$, then there exists a basis of T_pM such that*

$$A = \begin{bmatrix} J_1 & 0 & 0 & \cdots \\ 0 & J_2 & 0 & \cdots \\ 0 & 0 & J_3 & \cdots \\ \cdots & & & \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{bmatrix}, \lambda_i \in \mathbb{R}$$

In other words

$$Av_{2j-1} = \lambda_j v_{2j}, Av_{2j} = -\lambda_j v_{2j-1}$$

Therefore we have

$$\begin{aligned} \tilde{\sigma}(A) &= \sum_{i,j} \langle Av_i, v_j\rangle \tilde{\sigma}(\phi_i)\tilde{\sigma}(\phi_j) && (6.8) \\ &= \sum \langle Av_{2j-1}, v_{2j}\rangle \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}) + \sum \langle Av_{2j}, v_{2j-1}\rangle \tilde{\sigma}(\phi_{2j})\tilde{\sigma}(\phi_{2j-1}) && (6.9) \\ &= 2\sum \lambda_j \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}) && (6.10) \end{aligned}$$

Thus put into the formula

$$-\frac{1}{4}\tilde{\sigma}(A) = -\frac{1}{2}\sum \lambda_j \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}) \tag{6.11}$$

$$\to e^{-\frac{1}{4}\tilde{\sigma}(A)} = e^{-\frac{1}{2}\sum \lambda_j \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j})} \tag{6.12}$$

Notice that if we let

$$\mathcal{A}_j = \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}) \to \mathcal{A}_i\mathcal{A}_j = \mathcal{A}_j\mathcal{A}_i$$

Thus the above term is equal to

$$e^{-\frac{1}{4}\tilde{\sigma}(A)} = \prod_{i=1}^{m} e^{-\frac{1}{2}\lambda_j \mathcal{A}_j}$$

Let us consider the single term of the form

$$e^{-\frac{1}{2}\lambda_j \tilde{\sigma}(\phi)\tilde{\sigma}(\psi)} = \sum(even) + \sum(odd)$$

Because

$$\tilde{\sigma}(\phi)\tilde{\sigma}(\psi)\tilde{\sigma}(\phi)\tilde{\sigma}(\psi) = 1$$

Use the same argument as the classical Euler's formula we have

$$e^{-\frac{1}{2}\lambda_j \tilde{\sigma}(\phi)\tilde{\sigma}(\psi)} = \cos(-\frac{\lambda}{2}) + \sin(-\frac{\lambda}{2})\tilde{\sigma}(\phi)\tilde{\sigma}(\psi)$$

Substitute this into above formula we have

$$\prod_{j=1}^{m} \tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}) = \prod_{j=1}^{m}(\cos(\frac{\lambda_j}{2}) - \sin(\frac{\lambda_j}{2})\tilde{\sigma}(\phi_{2j-1})\tilde{\sigma}(\phi_{2j}))$$

Thus after multiplying out we get

$$\prod_{j=1}^{m} \cos(\frac{\lambda_j}{2}) + *$$

where $*$ *is a big term involving terms like the product of* $\tilde{\sigma}(\phi)$*. Now use Patodi's lemma we only need to extract the top order term, thus we have*

$$P(A) = \mathrm{Tr}(\cos(\frac{\lambda_j}{2}) + *) = 2^n \prod_{j=1}^{m} \cos(\frac{\lambda_j}{2}) = 2^n \overset{1/2}{\det} \cosh(\frac{A}{2})$$

where A *is the original matrix. To see why the last step is true it suffice to verify it in a block matrix* J_i *such that*

$$\cos(\frac{\lambda_j}{2}) = \sqrt{\det(\cosh(\frac{J_j}{2}))}$$

by noticing that

$$\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} * \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix} = \begin{bmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{bmatrix}$$

and the definition that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Thus the power series on both side coincide!

REMARK 119. *This term makes sense because all the eigenvalues are close to* 0*. So the whole product converges.*

Conclusion 2. *We have*

$$P(A) = 2^n \sqrt{\det(\cosh(\frac{A}{2}))}$$

substitute this into our formula we have

$$sgn(M) = \frac{1}{(4\pi i)^m} \int_M (\det)^{1/2} (\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}) Tr(\tilde{Z} e^{\tilde{Q}}) = \frac{1}{(2\pi i)^m} \int \sqrt{\det} (\frac{\frac{\mathcal{R}}{2}}{\tanh(\mathcal{R}/2)})$$

which completed the proof of the signature theorem.

Chapter 7

Complex structures

The purpose is to analyze complex manifolds using real manifold ideas. We don't want to waste all the work we have done, we want to work on complex manifolds using real methods.

Remember that V is a real vector space. A complex structure is a linear map

$$V \to V : J^2 = -I$$

REMARK 120. *Given such a complex structure, $-J$ is also a complex structure. The canonical example is:*

$$\mathbb{R}^{2n} with J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ & \end{bmatrix}$$

Natural question: Given (V, J), how is this related to $\mathbb{C}V = \mathbb{C} \otimes_{\mathbb{R}} V$?

Answer: $J : \mathbb{C}V \to \mathbb{C}V$ hsa $\pm i$ eigenvalues.

$$\mathbb{C}V = V^{1,0} \otimes V^{0,1}$$

where the first one has eigenvalue i and second one has egienvalue $-i$. The first one has generator $(1 - iJ)$ and second one has $(1 + iJ)$. The map

$$V^J \subset v \to v - iJv \in V^{1,0}$$

is a $\mathbb{C}$-linear isomorphism. You can check this maps into with no kernel, easily be a complex isomorphism, because they have the same dimension.

REMARK 121. *Given any complex number a, the map*

$$v \in V^J \to a()v - iJv)$$

is an isomorphism.

Let's see, how can we prove this? We know V^J has half the dimension of $\mathbb{C}V$, the complex dimension of V is half of the real dimension of V.

Let's prove it, if $v = iJv$, then we have

$$Jv + iv = 0$$

we want to show $v = 0$. The above implies $v = iJv$, so $Jv = -iv$. Let's see $JV = iv$ by taking complex conjugate. However we know J cannot be 0. So we conclude v must be 0 itself. Or even clearer by multiplying both sides by J:

$$Jv = 0 \leftrightarrow -v = 0$$

or:

$$v = iJv \to \overline{v} = \overline{iJv} \to v = -iJv \to v = 0$$

REMARK 122. *We have $V^J \cong V^{1,0}$. If $\widetilde{J} = -J$, then*

$$V^{\widetilde{J}} \cong V^{0,1}$$

by

$$v \to v + iJv$$

Let's have an example: Consider $\mathbb{R}^{2n}$ with J and bases

$$e_1, e_2, e_3 \cdots e_{2n-1}, e_{2n}$$

the standard basis.

Let's see: just the odd ones, $e_1, e_3 \cdots, e_{2n-1}$ can be called as the 'standard basis' of $(\mathbb{R}^{2n})^J$.

REMARK 123. *We iterate that we regard $V^J = 1 \otimes v$ sitting in $\mathbb{C}V$. The above complex conjugate really happens in $\mathbb{C}V$. So the above formula should be:*

$$v \to a(1 \otimes v - iJ(1 \otimes v)) \in V^{1,0}$$

and so on and so on...

Okay, we know u_j just corresponds to the jth standard basis with 1 in the jth spot. We have:

$$u_j \to \frac{1}{\sqrt{2}}[e_{2j-1} - ie_{2j}] \in (\mathbb{R}^{2n})^{0,1}$$

We claim the map is unitary, because u_i is an orthonnormal basis of $\mathbb{R}^{2n^J}$. While the corresponding ones has length 1.

Let $g : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to R$ be the standard Euclidean metric. Then g defines a map: $\mathbb{C}\mathbb{R}^{2n} \to \mathbb{C}\mathbb{R}^{2n} \to \mathbb{C}$ as follows:

Taking

$$g(\alpha \otimes v, \beta \otimes v) = \alpha\overline{\beta}g(v, w)$$

then this is perfectly well defined. This is a genuine **Hermitian metric***. It is non-degenerate, linear in the first factor, complex linear in the second factor, etc.*

Then we claim

$$\frac{1}{\sqrt{2}}(e_{2j-1} \pm ie_{2j})$$

form an orthonormal basis of $\mathbb{R}^{2n}$. We have

$$\frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$$

orthonormal basis for $(\mathbb{R}^{2n})^{1,0}$ and

$$\frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$$

orthonormal basis for $(\mathbb{R}^{2n})^{0,1}$

The trick is the old one:

$$\omega \in \mathbb{C}V \to \frac{1}{2}(\omega - iJ\omega) + \frac{1}{2}(\omega + iJ\omega)$$

It is also common to put the scaling factor $a = \frac{1}{2}$. Now if we label $e_i = x_i$, how should we think about $e_{2j-1} - e_{2j}$? You can think about them as taking partials. What does it tell you?

We have the correspondence:

$$e_{2j-1} = \frac{\partial}{\partial x_j}, e_{2j} = \frac{\partial}{\partial y_j}$$

Here $x_1y_1, x_2y_2, \cdots, x_ny_n)$ are the coordinates on $\mathbb{R}^{2n}$. We recall that

$$\frac{1}{2}(\frac{\partial}{\partial x_j}) \pm i\frac{\partial}{\partial y_j} = \frac{\partial}{z_j}, \frac{\partial}{\partial \overline{z}_j}$$

This showes up in the Dirac operator $\sqrt{2}(\partial + \overline{\partial}^*)$, for example.

7.1 Almost complex structure

Let M be a real manifold. An almost complex structure on M is an endomorphism:

$$J \in C^{\infty}(M, EndTM)$$

such that $\forall x \in M, J_x^2 = -I$ on T_xM.

DEFINITION 52. *An almost complex manifold is a manifold with an almost complex structure is a pair (M, J) where J is an almost complex structure on M.*

Discussion. *The easiest example is the following: Any orientable surface admits an almost complex structure. We are going to see that orientatablity is very important.*

Proof. The proof is very 'easy'. Take any Riemannian metric on M, choose any orientation. Given $x \in M$, define $J_x : T_xM \to T_xM$ as follows:

$$J_x v_1 = v_2, J_x v_2 = -v_1$$

where v_1, v_2 is an oriented orthonormal basis of TM_x. You can prove this is defined independent of the representative of v_1, v_2. All you do is consider the chart (v_1, v_2) and J is just the rotation by $\frac{\pi}{2}$.

Question: If we want to define J, do we need the metric?
A: we need the metric to define an orthonormal basis.
Q: An orientation does not depend on the metric.
A: That's right, this is the interpretation of rotation counterclosewise by $\frac{\pi}{2}$. Otherwise we cannot do it.
Q: Can we define J without introducing the metric? A: Forget the metric....
Q: When you get the complex structure on $\mathbb{R}^2$.
A: We have chosen an orthonormal basis already...If you have another set of orthonormal basis using the other set, then define J' accordingly we have

$$J = J'$$

But by your method it would not be invariantly defined. So this is important. This would be a nice execrise. For example if we have $v_1' = v_1, v_2' = v_1 + v_2$, then J would not properly defined.

If v_1', v_2' is another oritented orthonormal basis of T_xM, then J' defined this way would not change anything.

Thus we get a genuine map: $J : T_xM \to T_xM, \forall x \in M$ such that

$$J^2 = -I$$

on all tangent spaces. So $J \in C^\infty(M, EndTM)$ is an almost complex structure on M.

Q: But...this is still depend on the metric....how could it be...anyway...how could this one better than the basis dependent one?
A: I agree...but how are you going to define J instead?
Q: The conclusion is...if M is a manifold, then there is no way to put an a canonical one unless you fix a metric?
A: I mean, what we really proved is the following one...:
A: The examples we are going to talk about are manifolds which are like complex manifolds...like $\mathbb{S}^6$.
Q: Can we use the same idea to prove for 4,6 dimensional surfaces?
A: No...2 dimensional surfaces are very special. The rotation...there are too many rotations on $\mathbb{R}^4$.
A: For example we can fix the other vector by fixing v' such that $\langle v', v'\rangle = 0$ and $|v'| = 1$.

Here is another Execrise for you:

$$\forall v, w \in I_x M$$

, we have

$$g(Jv, Jw) = g(v, w)$$

THEOREM 88. *: Given any almost complex manifold such that J is given, there exists a metric g on TM that the previous formula holds.*

REMARK 124. *Such a metric which has the property is called* **Hermitian metric***.*

Execrise: Drop the orthonormal condition, define $J'v_1 = -v_2, J'v_2 = -v_1, v_1 = e_1, v_2 = e_1 + e_2$. such that $e_1 \wedge e_2 > 0$. Show

$$J \not\cong J'$$

REMARK 125. *An n-form in $\wedge^p V$ is positive if and only if it is a constant times the volume form.*

Here is a theorem:

THEOREM 89. *All almost complex manifolds are orientable.*

Proof. Take any point $x \in M$, let $w_1 \cdots w_n$ be a basis.

REMARK 126. *For any $x \in M$, we know $\mathbb{C}T_xM = TM_x^{1,0} \oplus TM_x^{0,1}$. Recall this from the beginning of the lecture, etc. So we have $TM^{1,0}$ as a subbundle. Given any local basis, v_1, etc. If we look at $TM_x^J = T_x^{1,0}M$. All we need to show $TM^{1,0}$ a smooth subbundle.*

Locally, there exists a basis...how do I say it? I want to give you an explicit basis.....oh yeah yeah...that's ok. Locally locally there exists a basis, label it by

$v_1, v_2, v_3 \cdots, v_{2n-1}, v_{2n}$.*such that* $v_{2j} = Jv_{2j-1}, j = 1 \cdots n$.

Then $w_j = v_{2j-1} - iv_{2j}, j = 1 \cdots n$. *We have* $(w_1, \cdots w_n)$ *is a smooth basis for* $T^{1,0}M$. *We simply define*

$$W_j = v_{2j-1} - Jv_{2j}$$

so w_j *form a nice frame for* $TM^{1,0}$. *Okay for anyways...let's go back to the proof.*

Proof. Let $w_1, \cdots, w_n$ be a basis of $T_x^{1,0}(M)$ as above. Here $\forall j \le n$,

$$w_j = v_{2j-1} - iv_{2j}$$

where v_j is a basis for TM_x.

What is the obvious way to orient M?

Let us define an orientation of M at x by declaring

$$v_1 \wedge v_2 \cdots \wedge v_{2n}$$

to be the orientation form to be positive.

Let $w'_1 \cdots w'_n$ be any other basis of $TM_x^{1,0}$, and write

$$w'_j = v'_{2j-1} - iv'_{2j}$$

where we relabeled v_{2j} already. Then of course $v'_1, v'_2 \cdots v'_{2n-1}, v'_{2n}$ is a basis for T_xM.

We need to show:

$$v'_1 \wedge v'_2 \cdots v'_{2n-1} \cdots v'_{2n} = cv_1 \wedge v_2 \cdots v_{2n-1} \wedge v_{2n}, c > 0$$

We thus defined an orientation. Here is how we do it:

$$\langle w_1, w_2 \cdots w_n \rangle, \rangle w'_1, w'_2 \cdots w'_n$$

are related by a matrix

$$w'_j = \sum_{j=}^{n} c_{kj} w_k, C = \mathbb{C}^{n \times n}$$

Then

$$\wedge^n(\mathbb{C}TM) = \mathbb{C} \otimes \wedge^n(TM)$$

So we have

$$w_1' \wedge \cdots \wedge w_n' = \det(C) w_1 \wedge \cdots w_n$$

Also if we take the conjugate

$$\overline{w_1'} \wedge \overline{w_2'} \wedge \cdots \overline{w_n'} = \overline{\det}(C)\overline{w_1} \wedge \cdots \overline{w_n}$$

Here of course

$$w_1' \wedge \cdots \wedge w_n' \overline{w_1'} \wedge \overline{w_2'} \wedge \cdots \overline{w_n'} = \det(C)^2 w_1 \wedge \cdots w_n \overline{w_1} \wedge \cdots \overline{w_n}$$

This implies:

$$w_1' \wedge \overline{w_1'} \wedge w_2' \overline{w_2}' \wedge \cdots \wedge w_n' \wedge \overline{w}_n' = \det(C)^2 w_1 \wedge \overline{w}_1 \wedge w_2 \wedge \overline{w}_2 \wedge \cdots w_n \wedge \overline{w}_n$$

But observe

$$w_j \wedge \overline{w_j} = (v_{2j-1} - iv_{2j}) \wedge (v_{2j-1} + iv_{2j}) = 2iv_{2j-1} \wedge \wedge v_{2j}$$

So we have

$$(2i)^n v_1' \wedge v_2 \wedge \cdots v_{2n-1}' \wedge v_{2n}' = \det(C)^2 (2i)^n v_1 \wedge v_2 \cdots \wedge v_{2n-1} \wedge v_{2n}$$

Therefore the two parts defines the same orientation in $\wedge^{2n} T_{x_0} M$.

7.2 Complex and Kahler manifolds

Discussion. *Today's goal is to understand the following question:*

What exactly is a Hermitian metric?

Theorem 90. *A* **Hermitian metric** *Let $h : W \times W \to \mathbb{C}$ be a Hermitian metric on a complex vector space W. Let $V = W$ as a $\mathbb{R}$ vector space with J equal to multiplication by $i : V \to V$. Then $W = V^J$.*

Remark 127. *We know $g = \Re h : V \times V \to \mathbb{R}$. And $\Omega = -\Im h : V \times V \to \mathbb{R}$. Here g is a Riemannian metric and Ω is a 2-form. So we can write:*

$$h = g - i\Omega$$

Proof. : We have:

$$\begin{aligned} g(v,w) &= \Re h(v,w) && (7.1)\\ &= \Re \overline{h(v,w)} && (7.2)\\ &= \Re h(w,v) \qquad = g(w,v) && (7.3) \end{aligned}$$

and we have:

$$g(v,v) = \Re h(v,v) = h(v,v)$$

We also have $g(av,w) = ag(v,w), a \in \mathbb{R} > 0$ unless $v = 0$. On the other hand we have:

$$\begin{aligned} \Omega(v,w) &= -\Im h(v,w) && (7.4)\\ &= +\Im \overline{h(v,w)} && (7.5)\\ &= \Im h(w,v) && (7.6)\\ &= -\Omega(w,v) && (7.7) \end{aligned}$$

So Ω is alternating.

We also have:

$$\begin{aligned} \Omega(v,w) &= -\Im h(v,w) && (7.8)\\ &= +\Im ih(iv,w) && (7.9)\\ &= +\Re h(iv,w) && (7.10)\\ &= g(iv,w) && (7.11)\\ &= g(Jv,w) && (7.12) \end{aligned}$$

Hence

$$\Omega(v,w) = g(Jv,w)$$

We have

$$h(v,w) = g(v,w) - ig(Jv,w)$$

Also

$$h(v,w) = \Omega(Jv,w) - i\Omega(v,w)$$

Remark 128. *A hermitian metric on a complex vector space can be written in terms h or a 2-form.*

Remark 129. *Here Ω is called a fundamental form. For example if we assume on a manifold Ω is closed, then that manifold is* **Kahler**.

7.3 Review

Let V, J be a vector space with almost complex structure.

We have

$$\mathbb{C}V = V^{1,0} \otimes V^{0,1}$$

Note

$$V^J \cong V^{1,0}$$

Here for all $a \in \mathbb{C}$, we have

$$v \to a(v - iJv)$$

this map is an isomorphism, which is not canonical.

Remark 130. *However if we choose a metric it will be canonical.*

Example 38. *Here is an example: Any almost complex space is isomorphic to $\mathbb{R}^{2n}$ with J the canonical complex structure on $\mathbb{R}^{2n}$.*

$$J(e_{2j-1}) = e_{2j}$$

Proof. : Completely obvious! Pick a basis of V^J by $v_1, v_3, v_5, \cdots, v_{2n-1}$. Then $v_1, v_2, \cdots, v_{2n-1}, v_{2n}$ where for any j, we have:

$$v_{2j} = Jv_{2j-1}$$

is a basis of V. Here if we pick a basis for the complex vector space and multiply each basis element by i, then the multiplied new basis elements are what we needed. Then

$$v_j \to e_j$$

gives an isomorphism

$$(V, J) \to (\mathbb{R}^{2n}, J)$$

Remark 131. *Next, we will discuss manifolds.*

Example 39. *Recall an almost complex structure on a manifold M is a section*

$$J \in C^\infty(M, TM)$$

such that $J^2 = -I$.

Execrise 25. *Prove that $(TM)^J$ is a smooth vector bundle over M.*

Remark 132. *for each $p \in M$, we have $TM_p^J = (T_pM)^J$.*

Proof. Hint: If $v_1, v_3 \cdots v_{2n-1}$ is a smooth local trivialization of TM^J, then $v_1, v_2 \cdots v_{2n-1}, v_{2n}$ where $v_{2j} = Jv_{2j-1}$ is a local trivilization of TM.

Remark 133. *Also recall*

$$\mathbb{C}TM = \mathbb{C} \otimes TM = T^{1,0}M \oplus T^{0,1}M$$

As a consequence of the the execrise, we have

$$T^{1,0}M \subset \mathbb{C}TM$$

is a smooth sub-bundle.

Example 40. *Let $E \to M$ be a complex vector bundle and let $F \subset \mathbb{C}^\infty(M, HomE)$. Suppose that $\dim \Im F_p = CT$. Then*

$$\Im F$$

is a smooth sub-bundle of E.

Remark 134. *This implies $T^{1,0}M$ is a smooth subbundle of $\mathbb{C}TM$. In fact, $T^{1,0}M$ is the image of $i + J \in \mathbb{C}^\infty(M, End(\mathbb{C}TM))$.*

7.4 p, q forms

Let (V, J) be a real almost complex vector space. Define $\wedge^{1,0} = (V^{1,0})^*$, $\wedge^{0,1} = (V^{0,1})^*$. Then we have

$$\mathbb{C}V^* = (CV)^* = \wedge^{1,0} \oplus \wedge^{0,1}$$

This implies

$$\mathbb{C}(\wedge V^*) = \wedge(\mathbb{C}V)^* = \oplus_{p,q} \wedge^{p,q}$$

Here

$$\wedge^{p,q} = \wedge^p(\wedge^{1,0}) \wedge \wedge^q(\wedge^{0,1})$$

Also

$$\wedge^k(\mathbb{C}V^*) = \oplus_{p+q=k}\wedge^{p,q}$$

Remark 135. *If $\alpha \in \wedge^k(\mathbb{C}V^*)$, then $\alpha \in \wedge^{p,q}$ if and only if $\alpha = 0$ on $(V^{1,0})^{p'} \times (V^{0,1})^{q'}$ if $(p,q) \neq (p',q')$.*

Example 41. *As always, let $v_1, v_2, \cdots, v_{2n-1}$ be as basis of V^J, where $v_i, 1 \leq i \leq 2n$ is a basis of V. We have*

$$w_j = \frac{1}{2}(v_{2j-1} - v_{2j})$$

is a basis of $V^{1,0}$, and their conjugates is a basis for $V^{0,1}$.

Lemma 51. *If $\phi_1 \cdots \phi_{2n}$ is the dual basis of $v_1 \cdots v_{2n}$. Then*

$$\Psi_j = \phi_{2j-1} + i\phi_{2j}$$

is a basis for $\wedge^{1,0}$ and

$$\overline{\Psi_j} = \phi_{2j-1} - i\phi_{2j}$$

is a basis for $\wedge^{0,1}$. Here

$$\alpha^{p,q} \leftrightarrow \alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ}\phi_I \wedge \overline{\phi}_J$$

Proof. We have

$$\phi_j(w_k) = (\phi_{2j-1})(\frac{1}{2}(v_{2k-1} - iv_{2k})) = \frac{2\delta_{jk}}{2} = \delta_{jk}$$

and

$$\phi_j(\overline{w_k}) = 0, \overline{\phi}(w_k) = 0, \overline{\phi_j}(\overline{w_k}) = \delta_{jk}$$

Definition 53. *We introduce*

$$\partial_{Z_j} = \frac{1}{2}(\partial_{X_j} - i\partial_{Y_j})$$

and

$$\overline{\partial_{Z_j}} = \frac{1}{2}(\partial_{X_j} + i\partial_{Y_j})$$

here

$$dz_j = dx_j + idy_j, d\overline{z}_j = dx_j - idy_j$$

We claim that

$$\alpha \in \wedge^{p,q}$$

if and only if

$$\alpha = \sum_{|I|=p,|J|=q} a_{IJ} = dz_I \wedge d\overline{z}_J$$

Remark 136. *Ω is a $1-1$ form.*

7.5 Hermitian Metrics and almost complex structures

Let (V, J) be an almost complex space.

THEOREM 91. *If $h : V^J \times V^J \to \mathbb{C}$ is a Hermitian metric. Then $g : \Re h : V \times h \to \mathbb{R}$ is an i.p and $\Im h V \times h \to \mathbb{R}$ is a 2-form. Moreover both g and Ω preserves J. In otherwords we have*

$$g(Jv, Jw) = g(v, w), \Omega(Jv, Jw) = \Omega(v, w)$$

for all $v, w \in V$.

Proof.

$$\begin{aligned} g(Jv, Jw) &= \Re h(Jv, Jw) && (7.13) \\ &= \Re h(iv, iw) && (7.14) \\ &= \Re h(v, w) && (7.15) \\ &= g(v, w) && (7.16) \end{aligned}$$

THEOREM 92. *If $g : V \times V \to \mathbb{R}$ is an inner product preserves J. Then there exists a unique Hermitian metric*

$$V^J \times V^J \to \mathbb{C}$$

such that $g = \Re h$.

Proof. $\forall v, w \in V^J$, define

$$h(v, w) = g(v, w) = ig(v, w)$$

Execrise 26. *Easy to show that h is a hermitian metric.*

DEFINITION 54. *An inner product on V is called Hermitian if it preserves J.*

Example 42. *(R^{2n}, J) is the standard example. Here $(R^{2n})^J \cong \mathbb{C}^n$.*

LEMMA 52. *g the standard Euclidean metric is Hermitian. Hint: This is true because this holds for basis vectors.*

LEMMA 53. *h is the standard inner product on $\mathbb{C}^n$. We want to show*

$$h_g(e_{2j-1}, e_{2k-1}) = \delta_{jk}$$

We can show it by

$$h(v, w) = g(v, w) - ig(Jv, w)$$

Noticing that $Je_{2j-1} = e_{2j}$. *So we have*

$$h_g(e_{2j-1}, e_{2k-1}) = \delta_{jk}$$

Let $\phi_1, \phi_2, \cdots, \phi_{2n}$ *be the dual basis of* $e_1, e_2, \cdots e_{2n}$.

Execrise 27. *Prove that*

$$\Omega = \sum \phi_{2j-1} \wedge \phi_{2j} = \sum dx_j \wedge dy_j$$

Execrise 28. *Prove that*

$$\Omega = \frac{1}{4i} \sum dz_j \wedge \overline{dz_j}$$

Hint:

$$dz_j = dx_j + idy_j, d\overline{z}_j = dx_j - idy_j$$

Discussion. *we shall discuss Kahler manifolds in future.*

7.6 $\overline{\partial}$-complex

Discussion. *Let* (V, J) *be an almost complex vector space. Recall that the map* $h : V^J \times V^J \to \mathbb{C}$ *is a Hermitian metric. This implies* $g = \Re(h) : V \times V \to \mathbb{R}$ *is an ip and is compatible with* J. *This also applies to* $\Omega = -\Im(h)$. *Here* $g(J-, J-) = g$.

Conversely, if $g : V \times V \to \mathbb{R}$ *is an complex inner product compatible with* J, *then there exist* $h_g : V^J \times V^J \to \mathbb{C}$ *Hermitian metric such that*

$$g = \Re(h_g), h_g(v, w) = g(v, w)$$

and

$$-g(Jv, w) = \Omega(v, w)$$

DEFINITION 55. g *is Hermitian if it is compatible with* J.

Execrise 29. *Let* g *be an Hermitian innerproduct on* V, *let* v_1, v_3, v_{2n-1} *be an orthonormal basis of* V^J. *Put* $v_{2i} = Jv_{2i-1}$. *Then* $v_1 \cdots v_n$ *is an orthonormal basis for* V.

Example 43.

$$\Omega = g(J, ..) = \sum_{j=1}^{n} \phi_{2j-1} \wedge \phi_{2j} \tag{7.17}$$

Discussion. *Observe that* $\Omega = \tau^T J\tau$, *where* $\tau = [\phi_1 \cdots \phi_n]^T$. *And* $\{\phi_1, \cdots, \phi_n\}$ *is the dual basis of* $\{v_1 \cdots v_{2n}\}$. *Here* $\tau : V \to \mathbb{R}^{2n}$ *is a unitary linear transformation. Here we regard* τ *as a column vector of one forms, and* Ω *is a 2-form. Here* J *is the classical almost complex structure on* $\mathbb{R}^{2n}$.

To clarify we reiterate we have

$$\Omega = \tau^T \wedge J \wedge \tau \tag{7.18}$$

that's why we get a 2-form.

7.7 Inner product on manifolds

Definition 56. $h : \mathbb{C}V \times \mathbb{C}V \to \mathbb{C}$ *by (here* $g : V \times V \to \mathbb{R}$*)*

$$h(v, u) = g(v, \overline{u}) \tag{7.19}$$

Consider $w_j = \frac{1}{\sqrt{2}}(v_{2j-1} - iv_{2j}), j = 1 \cdots n$ *and* $\overline{w}_j = \frac{1}{\sqrt{2}}(v_{2j-1} + iv_{2j}), j = 1 \cdots n$.

Execrise 30. $\{w_1, w_n, \overline{w}_1 \cdots \overline{w}n\}$ *is an orthonormal basis of* $\mathbb{C}V$. *In particular we have*

$$V^{1,0} \perp V^{0,1}$$

Remark 137. *Also we have*

$$v \in V^J \to \frac{1}{2}(v - iJv) \in V^{1,0} \tag{7.20}$$

is a unitary isomorphism of Hermitian vector spaces. Thus, in terms of metrics, we have

$$v \in V^J \cong a(v - iJv) \in V^{1,0}$$

is most natural $\mathbb{C}$ *choosing* $a = \frac{1}{\sqrt{2}}$.

Remark 138. *Also note*

$$\psi_j = \frac{1}{\sqrt{2}}(\phi_{2j-1} + i\phi_{2j}) \tag{7.21}$$

Then $\{\overline{\psi_j}, \psi_k\}$ *is a dual basis of* $\{w_1, \overline{w_j}\}$. *and moreover*

$$\Omega = \sum_{j=1}^{n} \phi_{2j-1} \wedge \phi_{2j} = \frac{1}{2}\sum_{j=1}^{n} \psi_j \wedge \overline{\psi}_j \tag{7.22}$$

Thus we claim

$$\Omega \in \wedge^{1,1} \to \Omega \in \wedge^2 \in \mathbb{C}\wedge^2 = \wedge^{2,0} \oplus \wedge^{1,1} \oplus \wedge^{0,2} \tag{7.23}$$

Discussion. *Let M be a manifold, g is Riemmanian metric on TM. Let J be an almost complex structure on M.*

Definition 57. *We call (M, g, J) an Almost Hermitian Manifold if g is compatible with J:*

$$g(Ja, Jb) = g(a, b) \tag{7.24}$$

If (M, J) is an almost complex manifold, then there always exists a Riemmanian metric g compatible with J.

Proof. Pick any Riemannian metric g. We define

$$g(v, w) = g_0(v, w) + g_0(Jv, Jw) \tag{7.25}$$

is compatible with J.

Definition 58. $\Omega \in C^\infty(M, \wedge^2)$ *such that*

$$\Omega(v, w) = g(Jv, w) \tag{7.26}$$

Here Ω is called **the fundamental 2-form***. A word of reminder, we fixed a hermitian manifold (M, g, J) already.*

Definition 59. *We define*

$$h : \mathbb{C}TM \times \mathbb{C}TM \to \mathbb{C} \tag{7.27}$$

by

$$h(v, w) = g(v, \overline{w}) \tag{7.28}$$

Theorem 93. *It now follows from above that*

$$T^{1,0}M \perp T^{0,1}M \tag{7.29}$$

If $w_1 \cdots w_n$ is a local orthonormal basis of $T^{1,0}M$, then there exist an orthonormal basis $v_1, v_2, \cdots v_{2n-1}, v_{2n}$ on TM such that

$$Jv_{2j-1} = v_{2j}, w_j = \frac{1}{\sqrt{2}}(v_{2j-1} - iv_{2j}), j = 1 \cdots n \tag{7.30}$$

Execrise 31. *For all* $\wedge \in T^{1,0}$*, define*

$$\phi_w : \mathbb{C}TM \to \mathbb{C} \tag{7.31}$$

by $\phi_w(v) = \langle v, w\rangle$*. We prove that* $\forall w, \phi_w \in \wedge^{1,0}$ *and show* $w \in T^{1,0} \to \langle *, \overline{w}\rangle \in \wedge^{1,0}$ *is an isomorphism. Similarly we have*

$$w \in T^{0,1} \to \langle *, \overline{w}\rangle \in \wedge^{0,1} \tag{7.32}$$

is an isomorphism.

By the way, we can easily prove this by a basis.

Let (M, g, J) *is a Riemanian manifold, it has a Levi-Civita connection* ∇*. We have*

$$\nabla : C^\infty(M, TM) \to C^\infty(M, \wedge^1 \otimes TM) \tag{7.33}$$

after complexification we have

$$\nabla_{\mathbb{C}} : \mathbb{C}^\infty(M, \mathbb{C}TM) \to C^\infty(M, \mathbb{C}\wedge^1 \otimes \mathbb{C}TM) \tag{7.34}$$

mind that $T^{1,0}M \oplus T^{0,1}M$ *is a* $\mathbb{Z}_2$*-grading on* $\mathbb{C}TM$*. Here we have* $Z = \frac{1}{2}(i + J)T^{1,0}$ *and equal to* $= \frac{1}{2}i - JT^{0,1}$*. The natural question we ask is:*

Is ∇ $\mathbb{Z}_2$ *graded? (In the sense we have* ∇ *split into* ∇_v *on each piece of* $T^{1,0}M, T^{0,1}M$*)*

Remark 139. *The answer is yes, but if and only if* Ω *is a closed. We know:*

- *2-form*
- *non-degenerate in the sense*

$$\Omega(*, v) = 0 \leftrightarrow v = 0 \tag{7.35}$$

We conclude that ∇ *is* $\mathbb{Z}_2$ *is graded if and only if* $d\Omega = 0$*.*

Definition 60. *Given any manifold* M*, a 2-form* ω *on* M *is called a symplectic form if* ω *is non-degenerate and closed.*

Remark 140. *On an almost complex Hermitian manifold, then L-C connection is* $\mathbb{Z}_2$ *graded if and only if* Ω *is* **symplectic***. Now* (M, ω) *is called a symplectic manifold.*

Definition 61. (M, g, J) *is almost Kahler if and only if* Ω *is symplectic.*

Theorem 94. *Let* (M, g, J) *be an almost Hermitian manifold, the following is true(TFAE):*

- $d\Omega = 0$
- $\nabla J = 0$

- ∇ *is* $\mathbb{Z}_2$ *graded if*

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \tag{7.36}$$

Proof. Pick a local orthonormal basis $v_1, v_2 \cdots v_{2n-1}, v_{2n} \in M$ such that $Jv_{2j-1} = v_{2j}$. Then we have

$$\Omega = \tau^T J \tau \tag{7.37}$$

Here τ is the vector of dual basis of 1-forms equal

$$[\phi_1 \cdots \phi_n]^T$$

as before. Here J is the almost complex structure.

To compute $d\Omega$ we recall $\nabla = d + \omega$, where ω is the connection 1-form. Here ω satisfies

$$\omega^T = -\omega \tag{7.38}$$

and

$$d\tau = -\omega\tau \tag{7.39}$$

Observe

$$\begin{aligned}
d\Omega &= d\tau^T J\tau && (7.40)\\
&= (d\tau)^T J\tau - \tau^T J d\tau && (7.41)\\
&= -(\omega\tau)^T J\tau + \tau^T J\omega\tau && (7.42)\\
&= -(-\tau^T\omega^T)J\tau + \tau^T J\omega\tau && (7.43)\\
&= -\tau^T\omega J\tau + \tau^T J\omega\tau && (7.44)\\
&= \tau^T(-\omega J + J\omega)\tau && (7.45)
\end{aligned}$$

This showed that

$$d\Omega = 0 \leftrightarrow \omega J = J\omega \tag{7.46}$$

Now let us consider

$$\begin{aligned}
(\nabla J)(v) &= \nabla(Jv) - J(\nabla v) && (7.47)\\
&= (d+\omega)Jv - J[(d+\omega)v] && (7.48)\\
&= Jdv + \omega Jv - Jdv - J\omega v && (7.49)\\
&= (\omega J - J\omega)V && (7.50)
\end{aligned}$$

This implies

$$\nabla J \equiv 0 \leftrightarrow \omega J = J\omega \tag{7.51}$$

For the third one we claim that ∇ preserves $T^{1,0}M$ if and only if

$$\nabla \frac{1}{2}(i+J) = \frac{1}{2}(i+J)\nabla \tag{7.52}$$

$$=> \nabla J = J\nabla \tag{7.53}$$

$$=> (d+\omega)J = J(d+\omega) \tag{7.54}$$

$$=> \omega J = J\omega \tag{7.55}$$

$$=> d\Omega = 0 \leftrightarrow \nabla J = 0 \tag{7.56}$$

Recall here we used

$$d\Omega = \tau^T(-\omega J + J\omega)\tau \tag{7.57}$$

and if $\tau^T A\tau = 0$, then $A = 0$.

THEOREM 95. *Let (M, g, J) be an almost Kahler manifold, let $R \in C^\infty(M, \wedge^2 \otimes HomTM)$ is a Riemannian curature operator.*

Execrise 32. *Prove that*

$$R \circ J = J \circ R$$

and:

$$R(Jv, Jw) = R(v, w), \forall v, w \in TM \tag{7.58}$$

The same holds for $R \in C^\infty(M, \wedge^2, \otimes Hom\wedge^*)$.

Here we already defined $J : TM \to TM$, so we extend it to the cotangent bundle. J is a derivation, etc.

REMARK 141. *We can define Levi-Civita connection on forms:*

$$\nabla^{LC} : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \mathbb{C}\wedge^1 \otimes \wedge^{0,*})$$

We want to define in the end of the semester a clifford algebra structure on $\wedge^{0,*}$. We can then define a Dirac operator $D = \frac{1}{i}\sigma \circ \nabla$. We may then ask what is the index of it, which will turned out to be given by something called Todd class. More generally we want to talk about complex manifolds. Last semester we took the exterior deriative and consider d^*. For almost Kahlet manifold we do not have this available, because the exterior derivative is not related to the metric. But for a complex manifold it turns out the operator we have is essentially

$$\frac{1}{\sqrt{2}}[\overline{\partial} + \partial^*]$$

The question is, what is this? The operator will be complex version of $d + d^*$, this will be what will do later on.

7.8 Next class

Adam ask how to put a Hermitian inner product on the complex vector space $\mathbb{C}V$. He protest the original notation does not make sense unless one extend by scalars.

Discussion. *We have*

$$g : V \times V \to \mathbb{R} \tag{7.59}$$

which extends to

$$g : \mathbb{C}V \times \mathbb{C}V \to \mathbb{R} \tag{7.60}$$

by

$$g(av, bw) = abg(v, w) \tag{7.61}$$

Then we have

$$h(v, w) = g_{\mathbb{C}}(v, \overline{w}), \forall v, w \in \mathbb{C}V \tag{7.62}$$

Remark 142. *We shall not define*

$$A(av, bw) = a\overline{b}A(v, w) \tag{7.63}$$

7.9 An error

Review: Let (M, g, J) be an almost hermitian manifold if J is compatible with g. It is almost Kahler if J is closed. This means

$$\Omega(v, w) = g(Jv, w) \tag{7.64}$$

Remark 143. *Error from last time: If $v_1, v_2 \cdots v_{2n}$ is an orthonormal basis of TM, where $Jv_{2j-1} = v_{2j}$, then*

$$\Omega = -\frac{1}{2}\tau^T J\tau \tag{7.65}$$

where τ is $[\phi_1 \cdots \phi_n]$, and J is the classical J. Let us check this when $n = 1$, where $\Omega = \phi_1 \wedge \phi_2$. Then we have

$$-\frac{1}{2}[\phi_1, \phi_2][-\phi_2, \phi_1] = \frac{1}{2}\phi_1 \wedge \phi_2 - \frac{1}{2}\phi_2 \wedge \phi_1 = \phi_1 \wedge \phi_2 \tag{7.66}$$

Remark 144. *Another mistake:*

We have

$$\tau^T(wJ - Jw)\tau = 0 \to wJ - Jw = 0 \tag{7.67}$$

Here $A^T = -A$. Questionable lemma:

Lemma 54. *If $\tau^T A\tau = 0$ when $A^T = -A$, then $A = 0$.*

Theorem 96. *(M, g, J) is almost kahler is equivalent to:*

1. $\nabla^{LC} J = 0$
2. ∇^{LC} *is $\mathbb{Z}_2$-graded such that $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$.*
3. *(M, g, J) is almost Kahler.*

Remark 145. *Goal: Next week, find a natural index theorem for AKM. Next week, the special case then AKM is a true Kahler manifold.*

7.10 Twisted index theorem

AlCM is a special case of the above.

Discussion. *Let $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$ graded Hermitian vector bundle. We have $D : C^\infty : C^\infty(M, E) \to C^\infty : C^\infty(M, E)$ is $\mathbb{Z}_2$-graded. Such that*

$$D \circ Z = -\mathbb{Z} \circ D \tag{7.68}$$

and $Z = \pm 1$ on $E^\pm$. Here we assume D is self adjoint and elliptic. We let $\sigma = \sigma(D)$, such that $\sigma(\psi)^2 = |\psi|^2, \forall \psi \in TM$.

Then we have

$$IndD^+ = \frac{1}{(4\pi i)^m}\int \overline{A}(M)Tr(\overline{Z}e^{\overline{Q}}) \tag{7.69}$$

Here $m = \frac{n}{2}, \overline{Z} = Z \circ w$, where $w = i^m\sigma(\phi_1)\cdots\sigma(\phi_m)$. Then we have

$$\overline{Q} = Q_E + \frac{1}{4}\sigma(\mathcal{R}) \tag{7.70}$$

Definition 62. *We say E is 'twisted' if $\overline{Z} = Id$, for example if $Z = \omega$.*

Discussion. *Why twisted?*

REMARK 146. *Locally $E = S \otimes \overline{w}$, where S is a 2^m-dim vector bundle. We have σ on E equal to $\sigma_K \otimes Id_{\overline{w}}$. So σ acts only on S. So if $\mathbb{Z} = \omega$, then with*

$$S^{\pm} = \pm 1 \tag{7.71}$$

which are the ± 1 eigenspaces of ω on S. Then we have

$$E^{\pm} = S^{I} \otimes \overline{w} \tag{7.72}$$

locally the $\mathbb{Z}_2$ graded Hermitian vector bundle E is exactly the $\mathbb{Z}_2$ graded Hermitian bundle S tensored with $\overline{\omega}$.

Terminology:V, W are vector bundles, then we say $V \otimes W$ is "twisted" by W.

REMARK 147. *Now we assume E is twisted.*

Then we have

$$Q_E = Q_S + \overline{Q} \tag{7.73}$$

Here

$$Q_S = -\frac{1}{4}\sigma(\mathcal{R}) \tag{7.74}$$

remrember Q_S commute with all clifford actoins, we have

$$e^{QE} = e^{Qs} \circ e^{\overline{Q}} \tag{7.75}$$

Recall for

$$A : V \to V, B : W \to W \tag{7.76}$$

We have

$$A \otimes B : V \otimes W \to V \otimes W \tag{7.77}$$

given by

$$(A \otimes B)(V \otimes W) = Av \otimes Bw \tag{7.78}$$

THEOREM 97. $Tr(A \otimes B) = Tr(A) \times Tr(B)$.

LEMMA 55.

$$Tr(e^{Q_E}) = Tr_S(e^{Qs}) \times Tr_{\overline{u}}(e^{\overline{Q}})$$

DEFINITION 63. *We have*

$$ch(E) = ch(S)2^{-m|}Tr_E(e^{\overline{Q}}) \tag{7.79}$$

Therefore

$$Tr(e^{\overline{Q}})) = 2^m ch(S)^{-1} * ch(E) \tag{7.80}$$

LEMMA 56. *we have*

$$ch(S) = Tr_S(e^{-\frac{1}{4}\sigma(\mathbb{R})}) = 2^m \overset{\frac{1}{2}}{\det}(\cosh\frac{\mathcal{R}}{2}) \tag{7.81}$$

REMARK 148. *What is* $chS)^{-1}$*?*

Execise hint: See proof of signature theorem. Hint: see proof of signature theorem.

7.11 Atiyah-Singer index theorem

Recall in the proof of signature theorem, we have

$$E = \mathbb{C}\wedge^* = \wedge^+ \oplus \wedge^-, d = d + d^* \tag{7.82}$$

Recall $\mathbb{Z} = \omega$*. We have*

$$syn(M) = \frac{2^m}{(4\pi i)^m}\int \overline{A}(M)ch(S)^{-1}ch(\mathbb{C}\wedge^*) \tag{7.83}$$

Execrise 33. *Prove*

$$IndD^+ = \frac{1}{(4\pi i)^m}\int Td_{\mathcal{R}}(M)ch(E) \tag{7.84}$$

Here the curvature term is not on E *but only on* $T^{1,0}$*. Here* $Td_{\mathbb{R}}(M)$ *is a Todd form equal to*

$$\overset{\frac{1}{2}}{\det}(\frac{R/2}{1-e^{-R/2}}) \tag{7.85}$$

To prove this signature theorem, all we need is $ch(\mathbb{C}\wedge^*)$*.*

THEOREM 98.

$$ch(\mathbb{C}\wedge^*) = 2^n\det(\cosh(\frac{\mathcal{R}}{2})) \tag{7.86}$$

Recall that

$$R_{\mathbb{C}\wedge^*}(v,w|\phi_1\wedge\cdots\phi_k)=\sum_{l=1}^{k}\phi_1\wedge\cdots\wedge g\mathcal{R}_{\mathbb{C}\wedge_1}\partial_l\cdots\partial_k \tag{7.87}$$

In terms of power series, if $A:V\to V$ and $A_{\wedge^*}:\wedge^*V\to\wedge^*V$ is the associated derivation, consider

$$P(A)=Tr(e^{A_{\wedge^*}}),P:Hom(V)\to\mathbb{C} \tag{7.88}$$

Then we define

$$ch(\mathbb{C}\wedge^*)=Tr(e^{\mathcal{R}_{\wedge^*}})=P(\mathcal{R}_{\wedge^*}) \tag{7.89}$$

Lemma 57. *$\forall A\in Hom(V)$, we have*

$$P(A)=\det(Id+e^A) \tag{7.90}$$

which equal

$$\det(e^{A/2}+e^{-A/2})\det e^{A/2} \tag{7.91}$$

Remark 149. *As Adam pointed out we have the space of A operated on is*

$$\mathcal{C}\oplus V\oplus\wedge^2V\oplus\wedge^3V\oplus\cdots \tag{7.92}$$

Execrise 34. *If $A=\begin{bmatrix}\lambda_1 & \cdots 0\\ \cdots & \lambda_N\end{bmatrix}$ Then We have*

$$A_{\wedge^*}(e_{i_1}\wedge\cdots e_{i_k})=(\lambda_{i_1}+\cdots\lambda_{i_k})*e_{i_1}\wedge\cdots e_{i_k} \tag{7.93}$$

And we have

$$Tr(e^{A_{\wedge^*}})=1+\sum_{1\le i_1<i_2<\cdots<i_k\le n}e^{\lambda_{i_1}+\cdots\lambda_{i_k}}$$

equal to

$$(1+e^{\lambda_1})(1+e^{\lambda_2})\cdots(1+e^{\lambda_N}) \tag{7.94}$$

which equal to

$$\det(Id+e^A) \tag{7.95}$$

So we conclude that

$$Tr(e^{A_{\wedge^*}}) = \det(Id + e^A) \tag{7.96}$$

Using the fact diagonal matrices is dense in all matrices we proved the result. Thus

$$ch(\mathcal{R}_{\mathbb{C}\wedge^*}) = \det(e^{\frac{\mathcal{R}_{\wedge^1}}{2}} + e^{-\frac{\mathcal{R}_{\wedge^1}}{2}}) \cdot \det(e^{-\frac{\mathcal{R}_{\wedge^1}}{2}}) \tag{7.97}$$

This is equal to

$$2^n(\det\cosh(\frac{\mathcal{R}_{\wedge^1}}{2})) \tag{7.98}$$

Therefore we conclude

$$\overline{A}(M)ch(S)^{-1}ch(\mathbb{C}\wedge^*) = \overset{\frac{1}{2}}{\det}\frac{(\frac{\mathcal{R}}{2})}{\sinh(\mathcal{R}/2)}\frac{1}{2^m\det^{1/2}\cosh(\frac{\mathcal{R}}{2})}2^n\det(\cosh(\frac{\mathcal{R}}{2})) \tag{7.99}$$

which is equal to

$$2^m\overset{1/2}{\det}(\frac{\mathcal{R}/2}{\tanh(\mathcal{R}/2)}) \tag{7.100}$$

which implies signature theorem. Hence we are done.

7.12 Error

Error: "Thm" For (M, g, J) almost Hermite manifold. The following are equivalent:

- $d\Omega = 0$
- $\nabla^{LC}J = 0$
- ∇^{LC}*preserves*$T^{1,0}, T^{0,1}$

Here $(2) \leftrightarrow (3)$ is true. Also $(2) \to (1)$ is also true.
We know $\nabla^{LC}J = \leftrightarrow \omega J - J\omega =$ since $d\Omega = -\frac{1}{2}\tau^T \wedge (\omega J - J\omega) \wedge \tau$. So we have

$$\nabla^{LC}J \to d\Omega = 0 \to (M, g, J)\textit{is almost Kahler} \tag{7.101}$$

However

$$d\Omega = 0 \to \nabla^{LC}J = 0 \leftrightarrow \textit{M is Kahler} \tag{7.102}$$

when the manifold itself is complex. The remarkable fact is:
In 2-dimension almost complex manifold is complex.

7.13 $\overline{\partial}$ complex

Recall $f : U \subset \mathbb{R}^2 \to \mathbb{C}$ if holomorphic iff

$$\overline{\partial} f = 0 \tag{7.103}$$

here

$$\overline{\partial}_z = \frac{1}{2}(\partial_x - i\partial_y) \tag{7.104}$$

We can express this in terms of forms

$$df = \partial_x f dx + \partial f dy \tag{7.105}$$
$$= (\partial_z + \overline{\partial}_z) f \tag{7.106}$$
$$= \partial_z f dz + \overline{\partial}_z f d\overline{z} \tag{7.107}$$

Definition 64. *We define*

$$\partial f = \partial_z f dz, \overline{\partial} f = \overline{\partial_z} f d\overline{z} \tag{7.108}$$

Remark 150. *We have*

$$\partial : C^\infty(U) \to C^\infty(U, \wedge^{1,0}); \overline{\partial} : C^\infty(U) \to C^\infty(U, \wedge^{0,1}) \tag{7.109}$$

Now f is holomorphic if and only if $\overline{\partial} f = 0$, which if and only if f is $\overline{\partial}$-closed.

Definition 65. *f is holomorphic means*

$$\overline{\partial_{z_j}} f = 0, \forall j = 1 \cdots m \tag{7.110}$$

Note we have

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \overline{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y) \tag{7.111}$$

As above we can express this via forms. Recall we have

$$d : C^\infty(U) \to C^\infty(U, \wedge^1) \tag{7.112}$$

We have

$$df = \sum \partial x_j f dx_j + \sum \partial_{y_j} f dy_j \tag{7.113}$$

In above notation we have

$$\sum \partial_{z_j} f dz_j + \sum \overline{\partial_{z_j}} d\overline{z}_j = \partial f + \overline{\partial} f \tag{7.114}$$

Remark 151. *It is natural to look at forms, in particular p, q forms.*

DEFINITION 66. *If $\alpha \in C^\infty(M, \wedge^{p,q})$, α is holomorphic if*

$$\overline{\partial}\alpha = 0 \tag{7.115}$$

Here

$$\overline{\partial}\alpha \in C^\infty(u, \wedge^{p,q+1})$$

REMARK 152. *We recall*

$$d : C^\infty(u, \mathbb{C}\wedge^k) \to C^\infty(U, \mathbb{C}\wedge^{k+1}) \tag{7.116}$$

Therefore

$$d = \sum \partial_{x_j} + \partial_{y_j} dy_j \tag{7.117}$$

$$= \sum \partial_{z_j} dz_j + \sum \overline{\partial}_{z_j} d\overline{z}_j \tag{7.118}$$

Then we have

$$\partial : C^\infty(U, \wedge^{p,q}) \to C^\infty(U, \wedge^{p+1,q}) \tag{7.119}$$

and

$$\overline{\partial} : C^\infty(U, \wedge^{p,q}) \to C^\infty(U, \wedge^{p,q+1}) \tag{7.120}$$

Execrise 35. *If $k = p + q$, we may define*

$$\pi_{p,q} : C^\infty(U, \mathbb{C}\wedge^k) \to C^\infty(U, \wedge^{p,q}) \tag{7.121}$$

to be the projection onto $\wedge^{p,q}$. Prove that

$$\partial = \pi_{p+1,q} \tag{7.122}$$

on $C^\infty, \wedge^{p,q}$. Similarly we have

$$\overline{\partial} = \pi_{p,q+1} \tag{7.123}$$

on $C^\infty, \wedge^{p,q}$. Therefore we have

$$0 \to C^\infty(U) \to^{\overline{\partial}} C^\infty(U, \wedge^{0,1}) \to^{\overline{\partial}} C^\infty(U, \wedge^{0,2}) \to^{\overline{\partial}} C^\infty(U, \wedge^{0,3}) \tag{7.124}$$

REMARK 153. *This is a very famous* ***Dolbeault complex****. Recall that for M complex manifold, this Dolbeault complex is the complex version of the deRham cohomology. Here we also have*

DEFINITION 67. *The k-th Dolbeault cohomology is given by*

$$H_D^k(M) = \frac{\ker \overline{\partial}}{Im\overline{\partial}} \tag{7.125}$$

REMARK 154. *Recall that by de Rham cohomology we have*

$$H_{d\mathbb{R}}^k(M) = \frac{\ker d}{Imd} \tag{7.126}$$

And by Hodge THeorem we have

$$H_{d\mathbb{R}}^k \cong \ker(d + d^*) \tag{7.127}$$

DEFINITION 68. *The **arithemetic genus** is given by*

$$\sum_{k=0}^{n}(-1)^k \dim H_D^k(u) \tag{7.128}$$

This is the complex version of

$$\chi(M) = \sum_{k=0}^{n}(-1)^k \dim H_{d\mathbb{R}}^k(M) = ind(D_{GB}^+) \tag{7.129}$$

where

$$D_{GB} = d + d^*, C\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd} \tag{7.130}$$

7.14 Almost Dolbeault operator on an ACM

Discussion. *Let (M, g, J) be an almost Hermitian manifold. Let us define*

$$\partial : C^\infty(M, \wedge^{p,q}) \to C^\infty(M, \wedge^{p+1,q}) \tag{7.131}$$

and

$$\overline{\partial} : C^\infty(M, \wedge^{p,q}) \to C^\infty(M, \wedge^{p,q+1}) \tag{7.132}$$

Absent of complex structure, we define it by

$$\partial = \pi_{p+1,q} \circ d : C^\infty(M, \wedge^{p,q}) \to C^\infty(M, \wedge^{p+1,q}) \tag{7.133}$$

and

$$\overline{\partial} = \pi_{p,q+1} \circ d : C^\infty(M, \wedge^{p,q}) \to C^\infty(M, \wedge^{p,q+1}) \tag{7.134}$$

REMARK 155. *Before, we had*

$$d = \partial + \overline{\partial} \tag{7.135}$$

on U^{open} in $\mathbb{R}^{2n}$. On an AHM, the = is no longer true.

THEOREM 99. *On an AHM, we have*

$$d - (\partial + \overline{\partial}) \in Diff^0 \in C^\infty(M, End(C\wedge^*)) \tag{7.136}$$

REMARK 156. *Question: How do you calculate this in local coordinates anyway?*

A: We calculate it by

$$d = \sum_{j=1}^{2n} \partial_{x_j} dx_j \tag{7.137}$$

REMARK 157. *If we can find a J such that*

$$J\partial_{x_j} = -\partial_{y_j}, J\partial_{y_j} = \partial_{x_j} \tag{7.138}$$

*then this manifold is indeed **complex**. But for general AHM this is not correct.*

Proof. When $k = 0$ we have

$$d = \partial + \overline{\partial} : C^\infty(M) \to C^\infty(M, \mathbb{C}\wedge^1) \tag{7.139}$$

This is true for trivial reasons because

$$\mathbb{C}\wedge^1 = \wedge^{1,0} \oplus \wedge^{0,1} \tag{7.140}$$

and

$$d = \pi_{1,0} \circ d + \pi_{0,1} \circ d = \partial + \overline{\partial} \tag{7.141}$$

When $k = 1$ we have

$$d : C^\infty(M, \mathbb{C}\wedge^2) \to C^\infty(M, \mathbb{C}\wedge^2) \tag{7.142}$$

whereas we have

$$\mathbb{C}\wedge^2 = \wedge^{2,0} \oplus \wedge^{1,1} \oplus \wedge^{0,2} \tag{7.143}$$

work locally there exist an orthonormal basis

$$v_1, v_2 \cdots v_{2n-1}, v_{2n} \tag{7.144}$$

such that

$$\frac{1}{\sqrt{2}}(v_{2j-1} - \pm i v_{2j}) \tag{7.145}$$

is a basis of $T^{1,0}M$ and $T^{0,1}M$.

Let $\phi_1, \phi_2, \cdots \phi_{2n}$ be the duals of $v_1, \cdots v_{2n}$. Then we have

$$\psi_j : \frac{1}{\sqrt{2}}(\phi_{2j-1} + i\phi_{2j}) \tag{7.146}$$

is a basis for $\wedge^{1,0}$. Similarly we have

$$\overline{\psi}_j = \frac{1}{\sqrt{2}}(\phi_{2j-1} - i\phi_{2j}) \tag{7.147}$$

is a basis for $\wedge^{0,1}$. We conclude $\psi_j, \overline{\psi}_k$ is a basis for $\mathbb{C}\wedge^1$. Now we have

$$\begin{aligned} d\psi_j \in \wedge^{1,0} &= \pi_{2,0}d\psi_j + \pi_{1,1}d\psi_j + \pi_{0,2}d\psi_j &(7.148)\\ &= \partial\psi_j + \overline{\partial}\psi_j + \delta_j &(7.149)\\ & &(7.150)\end{aligned}$$

Similarly computation holds for $\overline{\psi}_j$. Now let $\alpha \in C^\infty(M, \mathbb{C}\wedge^1)$. Then we have

$$\alpha = \sum a_j\psi_j + \sum b_j\overline{\psi}_j \tag{7.151}$$

This implies in particular

$$\begin{aligned} d\alpha &= \sum d_j \wedge \psi_j + \sum a_j d\psi_j + \sum db_j \wedge \overline{\psi}_j + \sum b_j d\overline{\psi}_j &(7.152)\\ &= \sum(\partial a_j + \overline{\partial}a_j) \wedge \psi_j + \sum a_j\partial\psi_j + a_j\overline{\partial}\psi_j + a_j\delta_j + \sum(\partial b_j + \overline{\partial}b_j) \wedge \overline{\partial}\psi_j + \sum b_j\overline{\psi_j} + b_j\overline{\partial\psi}_j + b_j\epsilon + j &(7.153)\\ &= \sum \partial(a_j\psi_j) + \overline{\partial}(a_j\psi_j) + \sum\partial(b_j\overline{\psi_j}) + \overline{\partial}(b_j\psi_j) &(7.154)\\ &= \partial\alpha + \overline{\partial}\alpha + T\alpha &(7.155)\end{aligned}$$

Here if $T = 0$ then M is a complex manifold. We note

$$T(\sum a_j\psi_j + b_j\overline{\psi_j}) = \sum a_j\delta_j + b_j\epsilon_j \tag{7.156}$$

Here $T \in Diff^0$, which implies

$$d \in Diff^0 = \partial \pmod{Diff^0} \tag{7.157}$$

Discussion. *Recall the Deboult complex. In the case of AHM, it is natural to consider*

$$0 \to C^\infty(M) \to^{\overline{\partial}} C^\infty(M, \wedge^{0,1}) \to^{\overline{\partial}} C^\infty(M, \wedge^{0,2}) \to^{\overline{\partial}} \tag{7.158}$$

We want to define the 'almost arithemetic genus'.

THEOREM 100. *The assumption that above is a complex is equivalent to M is a complex manifold.*

This implies concepts like almost holomorphic cohomology and existing arithmetic genus defined by cohomology. We can try to define the arithemetic genus via elliptic theory instead of cohomology.

Recall $D_{GB} = d + d^* : C^\infty(M, \mathbb{C}\wedge^*)$ where $\mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd}$. We have

$$\chi(M) = indD^+_{GB} \tag{7.159}$$

Consider

$$\overline{\partial} : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \wedge^{0,*}) \tag{7.160}$$

and its adjoint

$$\overline{\partial}^* : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \wedge^{0,*}) \tag{7.161}$$

THEOREM 101.

$$\overline{\partial} + \overline{\partial}^* : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \wedge^{0,*}) \tag{7.162}$$

is $\mathbb{Z}_2$-graded with

$$\wedge^{0,*} = \wedge^{0,even} \oplus \wedge^{0,odd} \tag{7.163}$$

LEMMA 58. *We have $\overline{\partial} + \overline{\partial}^*$ is an elliptic 1st order differential operator. We have*

$$\overline{\sigma}(\delta)^2 = \frac{1}{2}|\delta|^2 \tag{7.164}$$

THEOREM 102. *We have*

$$D_d = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) \tag{7.165}$$

is a Dirac operator. on $\wedge^{0,*} = \wedge^{0,even} \oplus \wedge^{0,odd}$.

Here

$$Ind(D^+) = \textit{almost arithemetic genus of } M \tag{7.166}$$

REMARK 158. *Next time we have*

$$\frac{1}{(4\pi)^n} \int_M Td(M)$$

where the Todd class is given by

$$\det\left(\frac{K/2}{1 - e^{-K/2}}\right)$$

7.15 almost Riemann-Roch and characteristic classes

Recall that (M, g, J) is an AHM. Here

$$\overline{\partial} : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \wedge^{0,*})$$

THEOREM 103. *Let $D_d = \sqrt{2}(\partial + \partial^*)$ is a $\mathbb{Z}_2$ graded Dirac operator, with grading*

$$\wedge^{0,*} = \wedge^{0,even} \oplus \wedge^{0,odd} \tag{7.167}$$

DEFINITION 69. *If D is a $\mathbb{Z}_2$ graded dirac operator. Let us make a new definition. A $\mathbb{Z}_2$ graded vector bundle is* ***twisted*** *if the $\mathbb{Z}_2$ grading on E equal $\pm\omega$, where $\omega = i^m \sigma(D)(\psi_i)$.Locally the grading made*

$$E \cong S \otimes \overline{W} \tag{7.168}$$

or we have

$$E^{\pm} = S^{\pm} \otimes \overline{W}, Z = \omega \tag{7.169}$$

or

$$E^{\pm} = S^{-\pm} \otimes \overline{W}, Z = -\omega \tag{7.170}$$

THEOREM 104. *We claim*

$$\wedge^{0,*} \text{ is twisted!} \tag{7.171}$$

In fact

$$Z = (-1)^m \omega = \pm\omega \tag{7.172}$$

Discussion. *Thus, by the twisted index theorem, we have*

$$IndD_d^+ = \frac{(-1)^m}{(4\pi i)^m} \int \overline{A}(M) ch(S)^{-1} ch(\wedge^{0,*}) \tag{7.173}$$

THEOREM 105. *Pick any curvature K on $\mathbb{C}T^{1,0}M$, then*

$$IndD_d^+ = c \int Td(M) \tag{7.174}$$

where

$$Td(M) = \det\left(\frac{K/2}{1 - e^{-K/2}}\right) \tag{7.175}$$

equal to the Todd class of M.

Discussion. *We note the curvature term equals the Levi-Civita connections on the tangent bundle. The trick is we can choose any connection we want, so if we choose a special connection on $T^{1,0}$, then we can... This is why we need theory of characteristic classes. In particular if we have a connection on $\mathbb{C}T^{1,0}M$, we have a connection on $\wedge^{1,0}M$, which is due to $T^{0,1}M$. Therefore one get a connection on $\mathbb{C}TM$.*

Here we have

$$\overline{A}(M) = \overset{1/2}{\det}(\frac{\mathcal{R}/2}{\sinh(R/2)}) \tag{7.176}$$

$$ch(S)^{-1} = 2^m ... \tag{7.177}$$

whereas the last term $ch(\wedge^{0,*})$ depends on an connection compatible with the Clifford action.

We get the generalized Riemann Roch theorem:

$$IndD_d^+ = \frac{1}{\pi^m}\int_M \det(\frac{K/2}{1-e^{-K/2}}) \tag{7.178}$$

7.16 Characteristic classes

Discussion. *Here is our goal. Let E be a rank N complex vector bundle over a manifold M. Let $f : \mathbb{C}^N \to \mathbb{C}$ be an invariant polynomial. In other words f satisfies*

$$f(A) = f(B^{-1}AB) \tag{7.179}$$

forall $A \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}, B \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. Then we have

$$f(x) = \sum_I a_I x_I, x = [x_{ij}] \tag{7.180}$$

and

$$X_I = X_{i_1,j_1} \cdots X_{i_k,j_k} \tag{7.181}$$

Remark 159. *$f(x)$ is essentially a polynomial. It is the sum of products of entries in matrices.*

$$f(x) = \sum_I a_I X_{I_1} \cdots X_{I_k}, I = (I_1, \ldots, I_k), I_k \in \{1 \cdots N\}^2 \tag{7.182}$$

f is a linear combination of products of entries of a matrix X.

Example 44.

$$f(x) = C_N \overset{1/2}{\det}(\frac{X/2}{\sinh(X/2)}) \tag{7.183}$$

where

$$C_N(\sum_{|I|=0}^{\infty} a_I x_I) = \sum_{|I|=0}^{N} a_I x_I \tag{7.184}$$

Here

$$f(x) = C_n(\overset{1/2}{\det}(\cosh(\frac{X}{2}))), f_X = C_N(Tr(e^X)) \tag{7.185}$$

because of trace properties

$$f(A) = f(B^{-1}AB) \tag{7.186}$$

Here we view as f as

$$Hom(V) \to X \tag{7.187}$$

THEOREM 106. *If Q is the curvature of a connection on E and $f : C^{N\times N} \to \mathbb{C}$ is an invariant polynomial, then $f(Q) \in C^\infty(M, \mathbb{C}\wedge^*)$ is a clifford differential form.*

THEOREM 107. *If Q_2 and Q_1 are two curvature operators, then*

$$[f(Q_0)] = [f(Q_1)] \in H^*_{dR}M \tag{7.188}$$

In paricular, if M is compact and oriented we have

$$\int_M f(Q_0) = \int_M f(Q_1) \tag{7.189}$$

LEMMA 59.

- *If $Tr(A_1, B) = Tr(A_2, B)$ for any $B \in \mathbb{C}^{N\times N}$. Then $A_1 = A_2$.*
- *If B ia an $\mathbb{C}$-algebra, such that $A \subset B$ is a common subalgera st all elements A cannot be all element*
- *Then we have*

$$Tr(AB) = Tr(BA), \forall A \in A \otimes \mathbb{C}^{N\times N} \tag{7.190}$$

and

$$Tr(AB) = Tr(BA), \forall A \in \mathbb{C}^N = C^{eve} \otimes \mathbb{C}^{N\times N} \tag{7.191}$$

- *In paricular we have*

$$Tr(AB) = Tr(BA), \forall A \in \mathbb{C}\wedge^{ev} \otimes \mathbb{C}^{N\times N}, B \in \mathbb{C}\wedge^{even} \otimes \mathbb{C}^{N\times N} \tag{7.192}$$

Proof. To prove Thm1, recall f is a polynomial, so

$$f(X) = \sum a_I X_I \tag{7.193}$$
$$= \sum_I a_I X_{I_1} X_{I_2} \cdots X_{Ik}, I_j \in \{1, \cdots N\}^2 \tag{7.194}$$

Then

$$df(x) = \sum_{I,l} a_I x_{I_1} x_{I_2} \overline{x_{I_2}} \cdots x_{I_k} dx_{I_l} \tag{7.195}$$
$$= \sum_{i,j} \sum_{I,l} a_I \delta_{I,\{i,j\}} x_{I_1} x_{I_2} \overline{x_{Il}} x_{I_k} dx_{ij} \tag{7.196}$$
$$= \sum_{i,j} g_{ij}(X) dx_{ij} \tag{7.197}$$

Here

$$g_{ji}(X) = \sum a_I \delta_{I,\{i,j\}} X_{I_1} \cdots \overline{x}_{I_l} \cdots x_{I_k} \tag{7.198}$$

is a polynomial in X.

Note

$$df(x) = Tr(g(x)dx) \tag{7.199}$$

where we have

$$g(x) = [g_{ij}(x)] \tag{7.200}$$

$$Tr(g(x)dx) = \sum_i [g(x)dx]_{ii}, dx = [dx_{ij}] \tag{7.201}$$
$$= \sum_i \sum_j g_{ij}(x) dx_{ji} \tag{7.202}$$
$$= \sum_{i,j} g_{ji}(x) dx_{ji} \tag{7.203}$$

Notice we proved this lemma:

Lemma 60.

$$df(X) = Tr(g(x)dx) \tag{7.204}$$

Here is the proof of theorem 1

Proof. $f(Q) = \sum_I a_I Q_I, Q = [Q_{ij}]$. If we look back of the argument then we replace x_{I_j} with the Qs, because technically the ds and Qs commute. If you look at this argument, the reason I did this crazy algebra is because after substituting we get an even form, and we get no difference in the end! This is why I hate this proof, because the algebra is 100 percent identical as we did everything so explicitly and everything just pass by substituted...I should have made another lemma.

Then we have

$$df(Q) = Tr(g(Q)d(Q)) \tag{7.205}$$

recall that by definition

$$Q = d\omega + \omega \wedge \omega \tag{7.206}$$

So we have

$$dQ = d\omega \wedge \omega - \omega \wedge d\omega = Q \wedge \omega - \omega \wedge \mathbb{Q} \tag{7.207}$$

where we used

$$d\omega = Q - \omega \wedge \omega \tag{7.208}$$

We want to show the left hand side is 0. Therefore

$$df(Q) = Tr(g(Q) \wedge Q \wedge \omega) - Trg((Q) \wedge \omega \wedge Q) \tag{7.209}$$

$$= Tr(g(Q) \wedge Q \wedge \omega) - Tr(Qg(Q) \wedge \omega) \tag{7.210}$$

Here we used the fact Q is an even form so we can move it around as it commutes with other elements. We want to move Q around so eventually we get 0.

Lemma 61. *For all $A \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$, we have*

$$g(A)A = Ag(A) \tag{7.211}$$

Proof. Consider 2 curves with $A, B \in C^{\mathbb{N}\times N}$

$$c_1(t) = A + tAB = A(Id + tB), c_2(t) + tBA = (Id + tB)A \tag{7.212}$$

Since f is invariant

$$f(C_1(t)) = f(C_2(t)) \tag{7.213}$$

this implies

$$\frac{d}{dt}_{t=0} f(c_1 t) = \frac{d}{dt}_{t=0} f(c_2(t)) \tag{7.214}$$

So we have

$$df_A(AB) = df_A(BA) \to Tr(g(A)AB) = Tr(g(A)BA) = Tr(Ag(A)B) \tag{7.215}$$

Holds for any A, B implies

$$g(A)A = Ag(A) \forall A \tag{7.216}$$

by the lemma. This finished the proof.

Recall if $c(0) = P, c'(0) = v$, $\frac{d}{dt}|_{t=0} f(c(t)) = df_p(v)$. Therefore $[f(Q)] \in H^*_{dR}(M)$ is defined.

7.17 New lecture

Last time we have $E \to M$ be a complex vector bundle of rank N, $\triangledown$ a connection on E, Q the curvature form of $\triangledown$, and $f : \mathbb{C}^{\mathbb{N}\times\mathbb{N}} \to \mathbb{C}$ is an invariant polynomial. Further $f(Q) \in C^\infty(M, \mathbb{C}\wedge^*)$ is closed. Today we will show that for any two $\triangledown_0, \triangledown_1$.

$$f(Q_0) = f(Q_1) \tag{7.217}$$

Remark 160. *Kunal thinks this follows from the fact that connections is an affine space.*

Remark 161. *Let $\alpha \in C^\infty(\mathbb{R} \times M, \wedge^*(R \times M)$. Then we can always write*

$$\alpha = dt \wedge \beta + \gamma \tag{7.218}$$

where $\beta, \gamma \in C^\infty(\mathbb{R} \times M, \wedge^*(M))$.

Lemma 62.

$$d\alpha = 0 \leftrightarrow \frac{d}{dt}\alpha = d_M \beta \tag{7.219}$$

because

$$d\alpha = -dt \wedge d_M\beta + dt \wedge \frac{d}{dt}\gamma + d_M\gamma \tag{7.220}$$

which equals

$$dt \wedge (-d_M\beta + \partial_t\gamma) + d_M\gamma = 0 \tag{7.221}$$

In particular we have

$$\gamma \text{is closed} \to \forall t_0, t_1 \in \mathbb{R}, \gamma_{t_0} = \gamma_{t_1} \text{mod an exact form on M} \tag{7.222}$$

In fact we have

$$\partial_t \gamma = d_M \beta \to \int_{t_0}^{t_1} \partial_t \gamma = \int_{t_0}^{t_1} d_M \beta \qquad = \gamma_{t_1} - \gamma_{t_0} = d_M \overline{\gamma} \tag{7.223}$$

here

$$\overline{\gamma} = \int_{t_0}^{t_1} \beta dt \tag{7.224}$$

Remark 162. *this is just homotopy invariance of De Rham cohomology.*
More stronger we have

$$[\gamma_t] \in H^*_{dR}(M) \tag{7.225}$$

is independent of M

Discussion. *Let* ∇_0, ∇_1 *be a connection on* $E \to M$*. The trick is to prove*

$$f(Q_0) = f(Q_1) \tag{7.226}$$

by using a form on $\mathbb{R} \times M$ *and check both ends. Let us construct*

$$\nabla^t = (1-t)\nabla_0 + t\nabla_1, \forall t, \nabla^t \text{is a connection on M} \tag{7.227}$$

Definition 70. $\overline{\nabla} + \nabla^t$ *is a connection on* $E \to \mathbb{R} \times M$*. This is actually just the pull back of* ∇^t *from* $\mathbb{R} \times M \to M$ *on* $E \times M \to \mathbb{R} \times M$*.*

Discussion. *Therefore we conclude if* $\overline{Q}$ *is the curvature of* $\overline{\nabla}$*, then*

$$f(\overline{Q}) \in C^\infty(\mathbb{R} \times M, \wedge^*(\mathbb{R} \times M)) \tag{7.228}$$

is closed. Observe that $X = [x_{ij}]$

$$f(x) = \sum a_I x_{I_1} x_{I_2} \cdots x_{I_k} \tag{7.229}$$

so

$$f(\overline{Q}) = \sum a_I \overline{Q}_{I_1} \wedge \overline{Q}_{I_2} \wedge \cdots \overline{Q}_{I_k} \tag{7.230}$$

Note

$$\overline{Q} = d\overline{\omega} + \overline{\omega} \wedge \overline{\omega} \tag{7.231}$$

Notice that locally we have

$$\overline{\nabla} = \tau^{-1}(d_{\mathbb{R}\times M} + \omega_t)\tau \tag{7.232}$$

where

$$\omega_t = (1-t)\omega_0 + t\omega_1 \tag{7.233}$$

Here the $d_{\mathbb{R}}$ *earlier is* $dt \wedge \partial_t$*, and as we know*

$$\nabla^t = d_M + (1-t)\omega_0 + t\omega_1 \tag{7.234}$$

and

$$d_{\mathbb{R}\times M} = d_{\mathbb{R}} + d_M \tag{7.235}$$

Discussion. *So we have*

$$\begin{aligned}\overline{Q} &= d_{\mathbb{R}\times M}\omega_t + \omega_t \wedge \omega_t &(7.236)\\ &= dt \wedge \partial_t\omega_t + d_M\omega_t + \omega_t \wedge \omega_t &(7.237)\\ &= dt \wedge \partial_t\omega_t + Q_t &(7.238)\\ &= dt \wedge \beta + Q_t, \beta = \partial_t\omega &(7.239)\end{aligned}$$

Here we used Q_t *is the curvature form of* ∇^t*.*

Therefore we have

$$\begin{aligned}f(\overline{Q}) &= \sum a_I\overline{Q}_{I_1} \wedge \overline{Q}_{I_2} \wedge \cdots \overline{Q}_{I_k} \quad = \sum a_I(dt \wedge \beta + Q_t)_{I_1} \wedge \cdots \wedge (dt \wedge \beta + Q_t)_{I_k} &(7.240)\\ &= dt \wedge \overline{\beta} + f(Q_t) &(7.241)\end{aligned}$$

Therefore because we know $f(\overline{Q})$ *is closed, therefore* $\forall t_0, t_1 \in \mathbb{R}$ *we have*

$$[f[Q_{t_0}]] = [f(Q_{t_1})] \in H^*_{dR}(M) \tag{7.242}$$

In particular for $t_0 = 0, t_1 = 1$

$$[f(Q_0)] = [f(Q_1)] \in H^*_{dR}(M) \tag{7.243}$$

Execrise 36. *Here is a big execrise:*

Let us consider $D_d = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$*, then we can prove*

- $$D_d : C^\infty(M, \wedge^{0,*}) \to C^\infty(M, \wedge^{0,*}) \tag{7.244}$$
  is $\mathbb{Z}_2$ graded.

- *In fact, prove the $\mathbb{Z}_2$ grading on the bundle is equal to*
  $$Z = (-1)^m \omega \tag{7.245}$$

- *By the twisted index theorem we have the almost arithemetic genus*
  $$indD_d^+ = \textit{almost holomorphic Euler characteristic} = K \tag{7.246}$$
  Here
  $$K = \frac{1}{(2\pi i)^m} \int_M \overset{1/2}{\det}\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right) ch(S)^{-1} ch(\wedge^{0,*}) \tag{7.247}$$
  and
  $$ch(S) = 2^m \overset{1/2}{\det}(\cosh(\mathcal{R}/2)) \tag{7.248}$$
  and
  $$ch(\wedge^{0,*}) = Tr(e^{Q_{\wedge^{0,*}}}) \tag{7.249}$$

Pick any unitary connection $\nabla^{1,0}$ on $T^{1,0}M$, then we have $\nabla^{0,1} = \overline{\nabla^{1,0}}$ on $T^{0,1}M$. This is by for any $v \in C^\infty(M, T^{0,1}M)$. So we have

$$\nabla^{0,1} V = \overline{\nabla^{1,0}\overline{v}} \tag{7.250}$$

Then we have

$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1} \tag{7.251}$$

We can use tehe curvature of $\nabla^{1,0} \oplus \nabla^{0,1}$ in place of R in the index formula.

Can use $\nabla^{0,1}$ to get a connection on $\nabla^{0,*}$.

Mind the curvature on the dual bundle is the negative transpose of the original bundle. So the curvature on $T^{1,0}$ is minus the transpose of the curvature on $T^{0,1}$.

The answer is

$$IndD_d^+ = \frac{1}{(\pi i)^m} \int \det\left(\frac{K/2}{1 - e^{-K/2}}\right) \tag{7.252}$$

Hint: we have:

$$\overset{1/2}{\det}\left(\frac{\mathcal{R}/2}{\sinh \mathcal{R}/2}\right) \equiv \overset{1/2}{\det}\left(\frac{K/2 \oplus -K^T/2}{\sinh(K/2 \oplus K^T/2)}\right) \tag{7.253}$$

Here $\mathcal{R}$ is the curvature on $\mathbb{C}TM$. Here the last equality only equal up to an exact form because the integral of an exact with something closed is always 0:

$$\int (d\alpha) \wedge \beta = 0 \tag{7.254}$$

Here implicitly used the fact that $f(Q)$ and $f(\overline{Q})$ differ by an exact form.

REMARK 163. *This is usually proved by using $spin^c$ structure and construct an associated Dirac operator in literature.*

REMARK 164. *Here we suspect the authors did not realized the form bundle is somehow twisted or did not know the twisted index theorem. The trick is not to use Levi-Civita connection, just choose a connection and compute it by one connection, etc.*

REMARK 165. *An almost complex manifold may not have a spin structure, but it must have a spin-c structure. Any spin-c manifold can be endowed a spin-c dirac operator. The index can be computed. Probably the experts in spin-c manifolds just assume the manifolds are spin. And the computation can be done easily. And then they can avoid proving ...? We don't know. Probably to them they think spin is somehow natural....*

7.18 Dirac complexs

Recall: If M is a compact Riemannian manifold, we have

$$C^\infty(M) \to^d C^\infty(M, \mathbb{C}\wedge^1) \to^d \to C^\infty(M, \mathbb{C}\wedge^2) \to^d$$

We have the Hodege theorem which says $d + d^*$ is Dirac operator, $H^k_{DR}(M, \mathbb{C}) = kerd/imd$ is isomorphic to the kernel of $d + d^*$ on $C^\infty(M, \mathbb{C}\wedge^k)$. More over $d + d^*$ is $\mathbb{Z}_2$ graded with respect to grades

$$\mathbb{C}\wedge^* = \mathbb{C}\wedge^{even} \oplus \mathbb{C}\wedge^{odd}$$

Therefore we have the index theorem:

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M, \mathbb{C}) = Ind(d + d^*)^+$$

We can generalize this more general operators. Let $E_0, E_1 \cdots E_p$ be Hermitian complex vector bundles over M, suppose we have operators

$$C^\infty(M, E_0) \to^{\delta_1} C^\infty(M, E_1) \to^{\delta_2} C^\infty(M, E_2) \cdots$$

such that

$$\delta_{n+1} \circ \delta_n = 0$$

DEFINITION 71. *We define*

$$H^k)(\delta) = \frac{\ker \delta_{k+1}}{Im\delta_k}$$

We define the δ-Euler characteristic:

$$\chi(\delta) = \sum_{k=0}^{p} (-1)^k \dim H^k(\delta)$$

We call this complex as a ***Dirac complex*** *if $\delta + \delta^*$ is a Dirac operator. We mean if*

$$E = E_0 \oplus E_1 \oplus E_2 \cdots E_p$$

then $\delta : C^\infty(E) \to C^\infty(E)$ has an adjoint δ^*. We require $\delta + \delta^*$ to be a Dirac operator. We let

$$E = (\oplus_{keven})E_k \oplus (\oplus_{kodd} E_k)$$

and

$$\chi(\delta) = In(\delta + \delta^*)^+$$

For example on E_2 we have

$$\delta + \delta^* = \delta_1^* + \delta_2$$

REMARK 166. *The above sum only make sense if all dimensions are finite.*

THEOREM 108. *$\forall k$, we have*

$$H^k(\delta) \textit{is finite dimensional}$$

and

$$H^k(\delta) = \ker(\delta + \delta^*)$$

REMARK 167. *The proof is 100 percent identical to what we did last semester.*

7.19 Almost Hermitian Manifolds

Let (M, J, g) be an almost complex oriented Hermitian manifold. Recall we have operators

$$\overline{\partial} : C^\infty(\wedge^{p,q}) \to C^\infty(\wedge^{p,q+1})$$

when $p = 0$, we set maps

$$C^\infty(M) \to^{\overline{\partial}} C^\infty(\wedge^{0,1}) \to^{\overline{\partial}} C^\infty(\wedge^{0,2}) \to^{\overline{\partial}} C^\infty(\wedge^{0,3}) \to^{\overline{\partial}}$$

This is generally ***not*** *a complex unless the manifold is complex. So the above result has no meaning as we cannot define cohomology. However we can cheat a bit.*

THEOREM 109.
$$\sqrt{2}(\overline{\partial}+\overline{\partial}^*):C^\infty(\wedge^{0,*})\to C^\infty(\wedge^{0,*})$$

So even if the left hand side is not properly defined, the right hand side can be defined. We define the ***Almost Euler Characteristic*** *as*

$$AX(\wedge^{0,*}):=Ind(D_d)$$

where $D_d=\sqrt{2}(\overline{\partial}+\overline{\partial}^*)$ *is the "almost" Dolbeault operator. Goal: Prove this theorem explicitly by finding a formula for the above index.*

REMARK 168. *For complex manifolds this is the genuine Euler characteristic because it is the true index, etc.*

Discussion. *What is the symbol of* D_d*?. Consider*

$$\overline{\partial}:C^\infty(\wedge^{p,q})\to C^\infty(\wedge^{p,q+1})$$

Recall that
$$\overline{\partial}=\pi_{p,q+1}\circ d$$

Given $\epsilon\in T^*(M)$*, we have*

$$\sigma(\overline{\partial})(\epsilon)=\sigma(\pi_{p,q+1})\sigma(d)(\epsilon)$$

which equals

$$\pi_{p,q+1}(i\epsilon\wedge)=\pi_{p,q+1}(i\epsilon^{0,1}\wedge)=i(\epsilon^{0,1}\wedge)=\frac{i}{2}((\epsilon+iJ\epsilon)\wedge)$$

The computation stems from the fact that

$$\epsilon\in T^*M\in\mathbb{C}T^*M=\wedge^{1,0}\oplus\wedge^{0,1}$$

So
$$\epsilon=\frac{1}{2}(\epsilon-iJ\epsilon)+\frac{1}{2}(\epsilon+iJ\epsilon)$$

Therefore we have

$$\sigma(\overline{\partial}^*)(\epsilon)=\sigma(\overline{\partial})(\epsilon)^*=-\frac{i}{2}c(\epsilon-iJ\epsilon)=-ic(\epsilon^{1,0})$$

Here the c *denotes the contraction operator, and + becomes − because we conjugated.*

We thus have

$$\delta(\overline{\partial} + \overline{\partial}^*)(\delta) = i(\epsilon^{0,1} - c\epsilon^{0,1})$$

Here c denotes the Riemannian dual of a $(1,0)$ co-vector, which is actually a vector. So we have

$$\sigma(D_d)(\epsilon) = i\sqrt{2}(\epsilon^{0,1} \wedge -c(\epsilon^{1,0}))$$

We have

$$g(\gamma \wedge \alpha, \beta) = g(\alpha, c(\gamma)\beta)$$

We need to show

$$\sigma(D_d)(\epsilon)^2 = Id * |\epsilon|^2$$

In fact let us compute it:

$$\sigma(D_d)(\epsilon)^2 = -2(\epsilon^{0,1} \wedge -c(\epsilon^{1,0}))(\epsilon^{0,1} \wedge -c(\epsilon^{1,0}))$$

which equals

$$-2(-\epsilon^{0,1} \wedge c(\epsilon^{1,0}) - c(\epsilon^{1,0})(\epsilon^{0,1} \wedge)$$

which further equals

$$2(\epsilon^{0,1} \wedge c(\epsilon^{1,0}) + c(\epsilon^{1,0}(\epsilon^{0,1} \wedge))$$

Let $\beta \in \wedge^{p,q}$, such that

$$\sigma(D_d(\epsilon^2)(\beta) = |\epsilon|^2\beta$$

We know that β can be written in linear combination of $\epsilon^{0,1} \wedge \alpha$, where $c(\epsilon^{1,0})\alpha = 0$. Similarly we have $\overline{\alpha}$ when $c(\epsilon^{1,0})\overline{\alpha} = 0$.

REMARK 169. *Due to aesthetic considerations, we declare the above approach is too abstract. Adam further raised aesthetic considerations from real manifolds that might simplify the argument. So we abandon the above proof.*

REMARK 170. *So this might be a good exercise for you to prove that*

$$2(\epsilon^{0,1} \wedge c(\epsilon^{1,0}) + c(\epsilon^{1,0}(\epsilon^{0,1} \wedge)) = |\epsilon|^2$$

REMARK 171. *Adam insist the because "the same property should hold true for real". Professor objected for handwaving and propose to work this out by fixing a basis. Adam then suggests one can work in the interior product and work out the contractions, etc.*

Remark 172. *Professor now claim Adam is right:*

$$2(\epsilon^{0,1}\wedge c(\epsilon^{1,0})+c(\epsilon^{1,0}(\epsilon^{0,1}\wedge))=2(\epsilon^{0,1}\wedge(c\epsilon^{1,0}))+c(\epsilon^{1,0})\epsilon^{0,1}-\epsilon^{0,1}\wedge(c(\wedge^{0,1}))$$

which after simplification equals to

$$2(c\epsilon^{1,0}\epsilon^{1,0})\times Id$$

Theorem 110. *Recall a $\mathbb{Z}_2$ graded Dirac operator $D:C^\infty(E)\to C^\infty(E)$ is **twisted** if*

$$\mathbb{Z}=\pm\omega$$

where we have

$$\omega=i^m\sigma(\phi_1)\cdots\sigma(\phi_n)$$

Theorem 111. *Using the Z_2-grading on*

$$\wedge^{0,*}=\wedge^{0,even}\oplus\wedge^{0,odd}$$

so

$$D_d:C^\infty(\wedge^{0,*})\to C^\infty(\wedge^{0,*})$$

is Z_2 graded. In fact we have

$$Z=\omega$$

Proof. Let $v_1\cdots v_n$ be an oriented orthonormal basis of T_pM. We have $Jv_{2j-1}=-v_{2j}$. We want $i^m\sigma(\phi_1)$(omit)

Let $\phi_1,\phi_2\cdots\phi_n$ be the dual basis of $T_p^*(M)$. Observe that ϕ_{2j-1} with an odd index...we need to look up this term:

$$2(\epsilon^{0,1}\wedge c(\epsilon^{1,0})+c(\epsilon^{1,0}(\epsilon^{0,1}\wedge))=|\epsilon|^2$$

and figure out the $(0,1)$ part and $(1,0)$ part, etc. Observe

$$\psi_j=\frac{1}{\sqrt{2}}(\phi_{2j-1}+i\phi_{2j})\in\wedge^{1,0},\overline{\psi}_j=\frac{1}{\sqrt{2}}(\phi_{2j-1}-i\phi_{2j})\in\wedge^{0,1}$$

REMARK 173. *Adam objected the minus should be a plus because of the dual relation. So we fixed the above typos. Just think of the transpose of the matrix J must be $-J$, etc.*

Similarly we have

$$\phi_{2j-1} = \frac{1}{\sqrt{2}}(\psi_j + \overline{\psi_j}), \phi_{2j-1}^{0,1} = \frac{1}{\sqrt{2}}\overline{\psi_j}$$

and

$$\phi_{2j} = \frac{1}{i\sqrt{2}}(\psi_j - \overline{\psi}_j), \phi_{2j-1}^{1,0} = \frac{1}{\sqrt{2}}\psi_j$$

Observe

$$i\sigma(D_d)(\phi_{2j-1})\sigma(D_d)(\phi_{2j}) = i * i\sqrt{2}(\frac{1}{\sqrt{2}}\overline{\psi_j} \wedge -C(\frac{1}{\sqrt{2}}\psi_j))i\sqrt{2}(-\frac{1}{\sqrt{2}}\overline{\psi_j} - c(\frac{1}{\sqrt{2}}\psi_j))$$

Further we have

$$+(\overline{\psi_j} \wedge -c(\psi_j))(\overline{\psi_j} \wedge +c(\psi_j))$$

which equals

$$(\psi_j \wedge)c(\psi_j) - c(\psi_j)(\overline{(\psi_j)}\wedge) = -Id + 2(\overline{\psi_j \wedge})c(\wedge_j)$$

REMARK 174. *The professor consider he to be wrong, but adam think he is right. In the end he also think he is right. Therefore we have*

$$\omega(2(\overline{(\psi_1)}\wedge)c(\phi_1) - 1)(2(\overline{\psi}_2)c(\psi_2 - 1)\cdots$$

Our goal is $\omega = (-1)^m Id$ on $\mathbb{C}\wedge^{0,even}$, or $\omega = (-1)^m Id$ on $\mathbb{C}\wedge^{0,odd}$. We want to know fix j, how does

$$2(\overline{\psi_j})c(\psi_j) - 1$$

act on an element of $\wedge^{0,*}$?

We have two cases:

Case 1. $\overline{\psi_j} \wedge \beta$, where $c(\psi_j)\beta = 0$, so we have

$$w_j(\overline{\psi_j} \wedge \beta) = \overline{\psi_j} \wedge \beta, w_j(\beta) = -\beta$$

So if $\gamma \in \wedge^{0,k}$, then

$$\omega = \omega_1 \circ \omega_2 \circ \cdots \circ \omega_m)(\gamma) = (-1)^{m-k}\gamma = (-1)^m) - 1)^k\gamma = (-1)^m Z\gamma$$

So we have

$$Z = (-1)^m \omega$$

Remark 175. *Professor think this proof is ugly.*

Recall for D twistd, $Z = \pm\omega$, we have

$$IndD^+ = \frac{\pm 1}{(2\pi i)m}\int \overline{A}(M)ch(S)^{-1}ch(E)$$

Here as before we have

$$ch(S) = 2^m \overset{1/2}{\det}[cosh(R/2)]$$

and

$$ch(E) = Tr(e^{Q_E})$$

Therefore we have

$$AX(\wedge^{0,*}) = Ind(D_d^+) = \frac{(-1)^m}{(2\pi i)^m}\int \overline{A}(M)ch(S)^{-1}ch(\wedge^{0,*})$$

Remark 176. *Idea for proof: Each term as a differential form is independent of the connection chosen. They are defined up to exact forms, independent of the choice of connection on the vector bundles.*

Discussion. *Therefore we have*

$$\overline{A}(M) = \overset{1/2}{\det}(\frac{R/2}{\sinh(R/2)}), ch(S)^{-1}$$

are defined up to exact forms independent of choice of connection on $\mathbb{C}TM$. Therefore

$$ch(\wedge^{0,*}) = Tr(e^{Q_{\wedge^{0,*}}})$$

is defined up to exact form independent of connection on $\wedge^{0,*}$. Pick any unitary connection on $T^{1,0}M$:

$$\nabla^{1,0} : C^\infty(T^{1,0}M) \to C^\infty(C\wedge^1 \otimes T^{1,0}M)$$

Then define

$$\nabla^{0,1} : C^\infty(T^{0,1}M) \to C^\infty(\mathbb{C}\wedge^1 \wedge T^{0,1}M)$$

by $\nabla^{0,1}v = \overline{\nabla^{1,0}\nabla}$.

Then

$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$$

is a connection on $\mathbb{C}TM$. Let K be the curvature of the $\nabla^{1,0}$ part.

Observe

$$\nabla^{1,0} = d + \omega$$

is unitary. So $\omega^* + \omega = 0$. Note that this $\nabla^{0,1}$ equal $d + \overline{\omega}$.

REMARK 177. *I will leave you to think about it.*

$$K_{\nabla_{0,1}} = d\overline{\omega} + \overline{\omega} \wedge \overline{\omega} = d(-\omega^T) + \omega^T \wedge \omega^T = -[d\omega + \omega \wedge \omega]^T = -K^T_{\nabla^{1,0}}$$

Therefore the curvature of ∇ *on the full tangent bundle is* $K_\nabla = K \oplus -K^T$. *So we have*

$$\overline{A}(M) = \overset{1/2}{\det}\left(\frac{K \oplus -K^T/2}{\sinh(K/2) \oplus -\frac{K^T}{2}}\right)$$

using the fact the matrix is in block form on $\mathbb{C}TM$ *we have the above to be equal to*

$$\overset{1/2}{\det}\left(\frac{K/2}{\sinh(K/2)}\right) \overset{1/2}{\det}\left(\frac{-K^T/2}{\sinh(-K^T/2)}\right)$$

REMARK 178. *This approach is partly stems from our efforts to avoid Kahler manifolds and the notion of Kahler manifolds.*

The above continue to be

$$\det\left(\frac{K/2}{\sinh(K/2)}\right) = \det\left(\frac{K}{e^{K/2} - e^{-K/2}} = \overline{A}(M)\right)$$

Also

$$ch(S) = 2^m \overset{1/2}{\det}\left(\cosh(K/2 \oplus \frac{-K^T}{2}\right)$$

which equals by the same argument to be

$$2^m \det\left(\cosh(K/2 \oplus \frac{-K^T}{2}\right) = \det(2\cosh(K/2))$$

which is equal to

$$\det(e^{K/2} + e^{-K/2})$$

Finally we need

$$ch(\wedge^{0,*}) = Tr(e^{Q_{\wedge^{0,*}}})$$

First, let $\overline{\nabla}^{0,1}$ *to be the dual of the connection on* $T^{0,1}(M)$, *which equal to connection on* $\nabla^{0,1}$.

Recall $K_{\nabla^{0,1}} = -K^T_{T^{0,1}} = -(-K^T_{T^{1,0}})^T = K_{T^{1,0}}$. *Therefore we have*

$$K_{\nabla^{0,*}} = \text{ derivation associated to K}$$

So we have

$$ch(\wedge^{0,*}) = Tr(e^{K_{\wedge^{0,*}}}) = \det(Id + e^{K_{\wedge^{0,1}}}) = \det(e^{K/2} + e^{-K/2}) \det(e^{K/2})$$

REMARK 179. *Adam has some objections and Professor think it might be confusing. I am really confused. Professor think the second line is really important.*

Putting all together we have

$$\overline{A}(M)ch(S)^{-1}ch(\wedge^{0,*}) = \det(\frac{K}{e^{K/2} - e^{-K/2}}) * \frac{1}{\det(e^{K/2} + e^{-K/2})} * ?$$

Here $? = \det(e^{K/2} + e^{-K/2})\det(e^{K/2})$. *So the above simplifies to be*

$$\det(\frac{K}{e^{K/2} - e^{-K/2}}) * \det(e^{K/2}) = \det(\frac{K}{Id - e^{-K}})$$

REMARK 180. *Adam raised up another concern whether the above proof can be simplified in the real setting and result be 1. Professor considered this to be sweeping complex structure under the rug, and started to discuss the compatibility of complex structure with the underlying real vector bundle.*

Finally we get the theorem:

THEOREM 112. *Almost Hirzebruch-Riemann-Roch theorem. For any almost Hermitian manifold,*

$$AX(\wedge^{0,*}) = \frac{(-1)^m}{(2\pi i)^m} \int Td(M)$$

here $Td(M) = \det(\frac{K}{Id - e^{-K}})$, K *is any curvature on* $T^{1,0}M$. *We have* $Td(M)$ *is called the Tood character of* M.

REMARK 181. *What we are going to do on Wednesday is we are going to discuss genuine complex manifolds and the corresponding theorem on genuine complex manifolds. We shall prove the Riemann-Roch theorem!*

Take a 1-dimensional complex manifold, we shall define something call a divisior and the space of meromorphic functions on the manifold, the relations, etc...we can write it in terms of the Euler characteristic of the underlying real manifold.

We want to show the dimension of meromorphic functions on a 1-D complex surface is equal to the dimension mero $\wedge^{1,0}$ *forms with divisor* $-D$ *plus degree of* D *and one half of* $\chi(M)$ *plus* $1 - g$.

7.20 Adam's claim

Professor asked if anyone proved $Z = (-1)^m\omega$. *Adam claimed he has proved it. He suggested to use the dual relation from* $\mathbb{C}T^*M$ *and* $\mathbb{C}TM$. *Such that for* $\delta \in \mathbb{C}T^*M$ *there is a* $v \in \mathbb{C}TM$ *such that* $v^* = \delta$. *Then we have*

$$\pi_{0,1}\delta = (\pi_{0,v})^*$$

REMARK 182. *Professor claimed that if*

$$J : TM \to TM$$

then this implies

$$J^T : T^*M \to T^*M$$

LEMMA 63.

$$\wedge^{1,0} = i\text{eigenvalue space of } J^T \text{on} \mathbb{C}T * M$$

and the other similar relationship holds as well.

7.21 Complex manifolds

Recall a complex function $f : U \to \mathbb{C}$ is holomorphic if

$$(\partial_x + i\partial_y)f = 0$$

More generally, if

$$f : U \to \mathbb{C}$$

is C^α, where $U \subset \mathbb{C}^m = \mathbb{R}^{2n}$, we say that f is holomorphic if and only if for all j, we have

$$(\partial_{x_j} + i\partial_{y_j})f = 0$$

Even more generally, if $f : U \subset \mathbb{C}^n \to \mathbb{C}^q$ is smooth, we say it's holomorphic if $f(z) = (f_1(z), \cdots, f_q(z)$ such that $f_k(z)$ is holomorphic.
Finally a biholomorphic map is C^∞ bijective map

$$f : U \to V$$

such that $U, V \subset \mathbb{C}^m$ open, such that f is holomorphic and $f^{-1} : V \to U$ is holomorphic as well.
Let us fix $m \in \mathbb{N}$ throughout the following. Let M be a set, a complex coordinate chart is a pair (U, F), where $U \subset M$ and $F : U \to V$ where F is a bijection, where $V \subset \mathbb{C}^m$ is an open set.

REMARK 183. *Here M is just a set with no topology at all, M is a set and U is..(inaudible)*

We say that two complex patches $(U, F), (\overline{U}, \overline{F})$ are holomorphically compatible if $F(U \cap \overline{U}) \subset \mathbb{C}^m$ is open, and $\overline{F}(U \cap \overline{U})$ is open, so we have

$$F \circ \overline{F}^{-1} : \overline{F}(U \cap \overline{U}) \to F(U \cap \overline{U})$$

is biholomorphic, where both sets are open. This means it is bijection and its inverse is also holomorphic.

Remark 184. *In particular we have $(U,F) \subset (\overline{U},\overline{F})$ are C^∞ compatible.*

Remark 185. *Holomorphic compatible implies complex compatible.*

An atlas on M is a collection $A = \{(U_\alpha, F_\alpha)\}$. Complex coordinate patches are pairwise compatible. Such that we have

$$M = \bigcup_\alpha U_\alpha$$

The atlas is complete if every complex manifold patch that is compact such that every complex coordinate patch on A is in fact in $\mathcal{A}$.

Definition 72. *Let M be a set with a maximal atlas.*

Remark 186. *Professor there is no need for the atlas to be maximal.*

Definition 73. *Then M inherits the natural topology, namely, the topology generated by new subbases $\{U_\alpha\}$.*

Remark 187. *Professor now claim he needs completeness again.*

Remark 188. *We can get of the lettice, then add by the remainders.*

Definition 74. *An m-dimensional complex manifold is a pair (M,A) where M is a set, A is a maximal atlas, s.t the "induced topology" is Hausdauff and 2nd order counterable.*

Remark 189. *It is kind of interesting that there are some...*

Example 45. *Let*

$$M = \mathbb{C} \cup \{\delta\}\{\times\infty\}$$

where ∞ is any object that is not on a complex surface.

Remark 190. *Note that an $m - \dim$ complex manifold is a also a $Z_m \dim C^*$ manifold. AC any manifold of any hol-atlas a $\mathbb{Z}_{nm} : CP\infty$ manifold.*

Example 46. *Let us define an atlas $U \subset M, U = \mathbb{C}$, $F : U \to \mathbb{C}$, $F(z) = z$. The surface is homeomorphic to ^{2}. Then F is a bijection!*

We have

$$\overline{U} = (\mathbb{C} - \{0\}) \cup \{\infty\}$$

We have

$$\overline{F} : \overline{U} \to \overline{V}$$

with $\overline{V} = \mathbb{C}$. Then we let $\overline{F}(p)$ be the map

$$\overline{F}(p) = \frac{1}{o}, \forall p \in \mathbb{C}^{\{0\}}$$

*So we have the classical **Riemann Sphere**. This is clearly a bijection, so $\overline{F}$ is a bijection. We still need to show that (U, F) and $(\overline{U}, \overline{F})$*

So we have

$$F(U \cap \overline{U}) = F(\mathbb{C} - \{0\}) = \mathbb{C} - \{0\}$$

and

$$F(U \cap \overline{U}) = \overline{F}(, \mathbb{C}\ 0) = \mathbb{C} - \{0\}$$

So we have

$$F \circ \overline{F}^{-1} = F(\frac{1}{z}) = \frac{1}{z}$$

and we have

$$F \circ \overline{F}^{-1} : \mathbb{C}\{0\} \to \mathbb{C} - \{0\}$$

is biholomorphic. Then we have

$$A = \{(U, F), (\overline{U}, \overline{F})\}$$

is an atlas. The 1-dim complex manifold of M is called the Riemann Sphere.

Example 47. $\mathbb{CP}^n$. *Define an equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ by $z \equiv \omega$ if $z = \lambda\omega$. Let us define*

$$\mathbb{C}P^m = (\mathbb{C}^{m+1} - \{0\})/$$

Let us make $\mathbb{C}P^m$ into an m-dimensional complex manifold. Let $U_k \subset \mathbb{C}P^m$ be the set

$$U_k = \{[z_0, z_1, z_2 \cdots z_n | z_k \neq 0]\} \subset \mathbb{C}P^m$$

Define

$$F_k : U_k \to V_k, U_k = \mathbb{C}^m$$

by

$$F_k([(z_0, z_1 \cdots z_m)]) = (\frac{z_1}{z_k}, \frac{z_1}{z_k}, \cdots \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \cdots, \frac{z_n}{z_k})$$

F_k is a bijection, this showed that (U_k, F_k) is complex coordinate patch. Let us check

$$F_0 : U_0 \to V_0 : F_1 : U_1 \to V_1$$

are holomorphic and compatible.

By the above formula we have F_0 just skip z_0, etc. F_1 just skip z_1, etc.

So we have

$$F_0(U_0 \cap U_1) = \{(w_1, w_2 \cdots w_m) | w_1 \pm 0\}$$

similarly we have

$$F_1(U_0 \cap U_1) = \{(w_1, w_2 \cdots w_m) | w_1 \pm 0\}$$

both are open subsets of $\mathbb{C}^m$. Then we have

$$F_0 \circ F_1^{-1}(w_1, w_2 \cdots, w_m) = F_0[(\omega_1, 1, \omega_2, \cdots \omega_m)]$$

Since ω_1 is not zero the following map is clearly a homeomorphism:

$$(\frac{1}{w_1}, \frac{w_2}{w_1}, \frac{w_3}{w_1} \cdots \frac{w_m}{w_1})$$

We thus have

$$A = \{(U_0, F_0), (U_1, F_1) \cdots (U_m, F_m)\}$$

is atlas. $\mathbb{C}P^m$ is complex m-dimensional projective space.

7.22 almost complex structures

Every complex manifold is, of course an almost complex manifold. So every theorem we talked about in the past 3 weeks can automatically be translated here.

REMARK 191. *If M is an m-dimensional complex manifold, M is an n-dimensional C^∞ manifold where $n = 2m$. A one dimensional complex manifold is called a complex surface. The "Riemann-Roch" theorem about Riemann Surfaces is related.*

I think this is more important at the moment. Hence being a C^∞ manifold, T^*M and $C^\infty(M)$, etc all the C^∞ objects exist for complex manifolds.

Discussion. *To elaborate on this, let M be an m dimensional complex manifold, let (U, F) be a complex coordinate patch on M. This means that U is a subset of M*

$$F : U \to V \subset \mathbb{C}^m$$

is open.Actually also we have

$$\mathbb{C}^m = \mathbb{R}^n : F : U \to V$$

is a bijection from open subset to open subset.
Let us talk about coordinates. Let us use the coordiniates on $\mathbb{R}^n$, in the form

$$(x_1, y_1, x_2, y_2 \cdots x_n, y_n)$$

Remember the equality is what? We have m copies of $\mathbb{R}^2$, so when we write equal that is what we mean. That is the coordiniates we use to study the C^∞ structure. Or consider V to be a subset of $\mathbb{C}^m$, the complex coordinates

$$(z_1, z_2 \cdots z_m) \in \mathbb{C}^m$$

Then the guys here

$$\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}, \cdots \partial_{x_n}, \partial_{y_n}$$

form the local basis for TM and their duals

$$dx_1, dy_1 \cdots dx_n, dy_n$$

form a basis for T^*M.

In other words we have....what do I want to say. Let me define an almost complex structure on M. Recall that for $\mathbb{R}^n = \mathbb{C}^m, n = 2m$ the complex structure was given by

$$J(\partial x_j) = \partial y_j, J\partial y_j = -\partial x_j$$

This is almost god given, it is there. So in this way we have $J : T\mathbb{R}^n \to T\mathbb{R}n$.

Let M be an m-dim complex manifold. Let us define

$$J : TM \to TM$$

such that $J^2 = -Id$. The trick to choose any complex coordinate patch and it gives us x_n, y_n, etc. So it is very easy.

Easy: Just copy the $\mathbb{R}^n$ case. We have

$$\omega(U, F) = (U, x, y, x_2, y_2 \cdots x_m, y_m)$$

be a coordinate patch on M. Define

$$J : TM_U \to TM_U$$

by

$$J(\partial x_j) = \partial y_j, J(\partial y_j) = -\partial x_j$$

The only problem is, how do we prove this is well defined (with respect to complex coordinate transformations?) Let

$$(\overline{U}, \overline{F}) = (\overline{U}, u_1, v_1, u_2, v_2 \cdots u_m, v_m)$$

be another complex coordinate system. Here $U \cap \overline{U} \neq \varnothing$. This coordinate system give a map

$$\overline{J} : TM|_{\overline{U}} \to TM_{\overline{U}}, \overline{J}(\partial_{u_j}) = \partial_{v_j}, \overline{J}(\partial_{v_j}) = -\partial_{u_j}$$

We need to show

$$J = \overline{J}$$

on $TM|_{U\cap\overline{U}}$.

Let us prove that

$$J(\partial_{u_j}) = \partial_{v_j}, J(\partial_{v_j}) = -\partial_{u_j}$$

Let

$$H = F \circ \overline{F}^{-1}$$

We have

$$H(u_1, v_2, u_2, v_2 \cdots u_m, v_m)$$

This one takes coordinates in $\overline{F}$ *and gives one* $H(\omega) = (H_1(\omega), H_2(\omega), \cdots H_m(\omega)$.

By definition of compatibleness we have all $H_k(u)$*'s are holomorphic. Here the holomorphicness means*

$$(\partial_{u_j} + i\partial_{v_j})(H_k(\omega)) = 0, \forall j$$

Recall we are in a real tangent bundle, we have to break it up into the real components to verify. So if

$$H_k(\omega) = H_k'(\omega) + iH_k''(\omega)$$

which satisfies the Cauchy-Riemann equation(Adam's suggestion). Then we have

$$\partial_{u_j} H_k'(\omega) = \partial_{v_j} H_k''(\omega)$$

and similarly

$$\partial_{v_j} H_k'(\omega) = -\partial_{u_j} H_k''(\omega)$$

Now let us go ahead to prove this. By the chain rule we have

$$\partial_{u_j} = \sum \frac{\partial x_k}{\partial u_j}\frac{\partial}{\partial x_k} + \sum \frac{\partial y_k}{\partial u_j}\frac{\partial}{\partial y_k}$$

which then equals

$$\sum \frac{\partial H_k'}{\partial u_j}\frac{\partial}{\partial x_k} + \sum \frac{\partial H_k''}{\partial u_j}\frac{\partial}{\partial y_k}$$

We forgot to mention the obvious:

$$H(u_1, v_2, u_2, v_2 \cdots u_m, v_m) = (x_1, y_1, x_2, y_2 \cdots x_m, y_m)$$

similarly we have

$$\partial_{v_j} = \sum \frac{\partial H_k'}{\partial v_j}\frac{\partial}{\partial x_k} + \sum \frac{\partial H_k''}{\partial v_j}\frac{\partial}{\partial y_k}$$

So we have

$$J(\partial u_j) = \sum \frac{\partial H'_k}{\partial v_j}\partial_{y_k} + \sum -\frac{\partial H''_k}{\partial u_j}\partial_{x_k}$$

this then equals

$$J(\partial u_j) = \sum \frac{\partial H''_k}{\partial v_j}\partial_{y_k} + \sum -\frac{\partial H'_k}{\partial v_j}\partial_{x_k}$$

But this equals ∂_{v_j}. Also we have

$$J(\partial_{v_j}) = -\partial_{u_j}$$

REMARK 192. *In the same way we can prove complex manifolds are orientable. We can split the even and odd parts of the tangent bundle as usual, but these things are in general not vector fields for almost complex manifolds.*

However, on a complex manifold J is defined via coordinate vector fields.

THEOREM 113. *Every complex manifold is an almost complex manifold.*

REMARK 193. *However for an ACM, J is not necessarily defined via coordinate vector fields.*

Execrise 37. *Let (M, J) be an almost complex manifold. Suppose you can cover M by coordinate patches, we have $(U, x, y, x_1, y_1, x_2, y_2), x_m, y_m$. such that*

$$J\partial_{x_j} = \partial_{y_j}, J\partial_{y_j} = -\partial_{x_j}$$

Then we may conclude that the coordinate patches must be holomorphically compatible. Then M is actually a complex manifold.

More precisely, show that assume ... can show J has this property, and the following are coordinate patches

$$(U, F), (\overline{U}, \overline{F})$$

such that j has the above coordinate form, then

$$F \circ \overline{F}^{-1} \in C^\infty(M)$$

by definition. But we claim it is even bi- holomorphic between...This holomorphic meaning that

$$\partial_{u_j} + i\partial_{v_j}(H'_k + iH''_k) = 0$$

here of course we can write

$$H = (H'_1, H''_1, H'_2, H''_2 \cdots H'_n, H''_n)$$

This is very important because the exterior deriative using coordinates, etc. For complex manifolds it is explicitly written, for almost complex manifolds it is not true, and we always have $d + \overline{d} + error$*, and we always have this error.*

Now we want to talk about the real $\overline{\partial}$ *complex since we have the real* $\overline{\partial}$ *now, etc.*

Chapter 8

Riemann-Roch

8.1 $\overline{\partial}$ on complex manifolds

Discussion. *Recall an complex manifold M is a $2m$-dimensional C^∞ manifold such that you can cover it with coordinate patches that are holomorphicaly compatible.*

We know every complex manifold is almost complex. Let $(x_1, y_1, x_2, y_2 \cdots, x_n, y_n)$ be holomorphic coordinates. Then

$$J(\partial_{x_j}) = \partial_{y_j}, J(\partial_{y_j}) = -\partial_{x_j}$$

such that

$$J : TM \to TM$$

is well-defined and it defines an almost complex manifold on M.

Conversely, if an almost complex manifold has a cover by coordinate patches such that J is as above, then M is complex.

Remark 194. *Recall, for an almost complex manifold (M, J), we can always locally pick a basis $\langle v_1, v_3, v_5 \cdots, v_{n-1}\rangle$ of TM^J. Then we have*

$$v_{2j} = Jv_{2j-1}$$

and

$$\frac{1}{2}[v_{2j-1} - iv_{2j}], \frac{1}{2}[v_{2j-1} + iv_{2j}]$$

are basis for $T^{1,0}M$ and $T^{0,1}M$ respectively.

If M is complex and $(x_1, y_1 \cdots x_m, y_m)$ are holomorphic coordinates, then we have

$$\partial_{x_1}, \partial_{x_2} \cdots, \partial_{x_n}$$

are a basis for TM^J.

Hence we have

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$$

form a local basis for $T^{1,0}M$*, similarly*

$$\overline{\partial}_{z_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$$

form a local basis for $T^{0,1}M$.

If $z_j = x_j + iy_j, \overline{z}_j = x_j - iy_j$*, and we have*

$$dz_j = dx_j + idy_j, d\overline{z}_j = dx_j - idy_j$$

Execrise 38. *Prove that*

$$dz_j(\partial_{z_k}) = \delta_{jk}, d\overline{z}_j(\overline{\partial}_{z_k}) = \delta_{jk}, dz_j(\overline{\partial}_{z_k}) = 0, d\overline{z}_j(\partial_{z_k}) = 0$$

Such that

$$\{dz_j\}, \{d\overline{z}_j$$

form local basis for $\wedge^{1,0}$ *and* $\wedge^{0,1}$.

Discussion. *In particular we have*

$$dz_I \wedge dz_J, |I| = p, |J| = q$$

form a local basis for $\wedge^{p,q}$.

Observe that

$$d = \sum dx_j \wedge \partial_{x_j} + \sum d_{y_j} \wedge \partial_{y_j}$$

Since $dx_j = \frac{1}{2}(dz_j + d\overline{z}_j)$ *and* $dy_j = \frac{1}{2}(dz_j - d\overline{z}_j)$*, as well as*

$$\partial_{x_j} = \partial_{z_j} + \overline{\partial}_{z_j}, \partial_{y_j} = \partial_{z_j} - \overline{\partial}_{z_j}$$

Now substituting these formulas into d*, we obtain*

$$d = \sum_{j=1}^{m}(dz_j \wedge *)\partial_{z_j} + \sum_{j=1}^{m}(d\overline{z}_j \wedge *)\overline{\partial}_{z_j}$$

Remark 195. *Binbin raised concerns about possible problems in coordinate patches and whether these formulas applies to almost complex manifolds.*

Recall we introduced ∂ *and* $\overline{\partial}$*. We have*

$$\partial : C^\infty(\wedge^{p,q}) \to C^\infty(\wedge^{p+1,q}), \overline{\partial} : C^\infty(\wedge^{p,q}) \to C^\infty(\wedge^{p,q+1})$$

be

$$\partial = \pi_{p+1,q} \circ d, \overline{\partial} = \partial_{p,q+1} \circ d$$

For an almost complex manifold, we have

$$d = \partial + \overline{\partial} + \textit{order terms}$$

THEOREM 114. *For a complex manifold, we have $d = \partial + \overline{\partial}$. Where in holomorphic coordinates*

$$x_1, y_1, \cdots x_m, y_m$$

we have

$$\partial = \sum (dz_j \wedge)\partial_{z_j}, \overline{\partial} = \sum (d_{\overline{z}_j} \wedge)\overline{\partial}_{z_j}$$

REMARK 196. *In general manifold we have the differential form to have the form*

$$\phi = g_{I,J} dx_I \wedge dx_J$$

and its differential is not necessarily 0.

THEOREM 115. *For a complex manifold, we have*

$$\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} = -\overline{\partial}\partial$$

Proof. Let $\alpha \in C^\infty(\wedge^{p,q})$,then

$$d^2\alpha = 0 = (\partial + \overline{\partial})(\partial + \overline{\partial})\alpha$$

Then we have

$$\partial^2\alpha + (\partial\overline{\partial} + \overline{\partial}\partial)\alpha + \overline{\partial}^2\alpha$$

and the rest is obvious.

DEFINITION 75. *Let $U \subset M$ be an open set, a C^∞ function $f : U \to \mathbb{C}$ is holomorphic if for all holomorphic coordinate patches $(x_1, y_1 \cdots x_m, y_m)$ on U, we have*

$$\overline{\partial}_{z_j} f \equiv 0, j = 1 \cdots m$$

such that

$$\frac{1}{2}(\partial_{x_j} + i\partial_{y_j})f \equiv 0$$

In particular, a C^∞ function $f : M \to \mathbb{C}$ is holomorphic if $\overline{\partial}_{z_j} f \equiv 0$ in all holomorphic coordinate patches.

We can also discuss holomorphic (p,q) forms:

$$\overline{\partial} : C^\infty(\wedge^{p,q}) \to C^\infty(\wedge^{p,q+1})$$

we call a C^∞ form $\alpha \in C^{\alpha(\wedge^{p,q})}$holomorphic if

$$\overline{\partial}\alpha \equiv 0$$

Execrise 39. *Prove that*$\alpha \in C^{\alpha}(\wedge^{p,q})$*is holomorphic if and only if for all hlomorphic coordinate patches,* $U, (x_1, y_1 \cdots, x_m, y_m)$*, we can write*

$$\alpha = \sum_{|I|=p,|J|=q} a_{IJ} dz_I \wedge d\overline{z}_J$$

where $a_{IJ} \in \mathbb{C}^{\infty}(U)$ *is hlomorphic.*

In particular we have $\ker \overline{\partial}$ *on* $C^{\infty}(\wedge^{p,q})$ *are by definition holomorphic* (p,q) *forms.*

Remark 197. *Adam found the above statement is totally wrong.*

Remark 198. *Professor consider the notion of holomorphic is not well defined considering Adam's criticism.*

8.2 Dolbeault complex

Recall, we have a sequence

$$0 \to C^{\infty}(M) \to^{\overline{\partial}} C^{\alpha}(\wedge^{0,1}) \to^{\overline{\partial}} C^{\infty}(\wedge^{0,2}) \to^{\overline{\partial}} C^{\alpha}(\wedge^{0,3}) \to^{\overline{\partial}}$$

Since $\overline{\partial}^2 = 0$*, this sequence is a complex! The* k*-th Dolbeault cohomology of* M *is*

$$\frac{\ker \overline{\partial}}{Im \overline{\partial}}$$

The holomorphic Euler characteristic of M *is*

$$\chi(M) = \sum_{k=0}^{m} (-1)^k \dim H^k_{DR}(M)$$

From our work of almost complex manifolds we know that

Theorem 116. *If* M *is a compact complex manifold, we have a formula*

$$\chi(M) = (\frac{i}{2m})^m \int Td(M)$$

where

$$Td(M) = \det(\frac{K}{Id - e^{-K}})$$

where K *is the curvature form on* M*.*

Proof. By our theorem on Dirac complexes, the Dolbeault cohomology

$$H^k_{DR}(M) = \dim \ker(\overline{\partial} + \overline{\partial}^*)$$

on $C^\alpha(\wedge^{0,k})$. Here ∂^* is the formal adjoint of ∂ with respect to any Riemannian metric. Pick any Hermitian metric on M, which means any Riemannian metric compatible with the J. We know that

$$\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$$

are Dirac operators. Further the bundle is $\mathbb{Z}_2$ graded as

$$\wedge^{0,even} \oplus \wedge^{0,odd}$$

which is twisted.

We thus have

$$\chi(M) = IndD_d^+ = (\frac{i}{2\pi})^m \int Td(M)$$

This is the Riemann-Roch-Hirezubech theorem. This the direct consequence of our work before, the left hand side is not defined, and the right hand side is defined, etc. So the Euler characteristic being the alternating sum equals the above expression.

Remark 199. *Adam asked if this would simplify under the assumption that it is Kahler, professor answered it would not really simply anything except now we can choose the curvature form to be the Levi-Civita connection.*

8.3 $\overline{\partial}$ on holomorphic vector bundles

We want to generalize this proof to twisted holomorphic vector bundles.

Definition 76. *Let M be a complex manifold, a complex vector bundle is holomorphic if there exists a cover U_α of M by holomorphic patches which trivilize E, such that the transition maps*

$$E|_{U_\alpha} \cong U_0 \times \mathbb{C}^N, E|_{U_\beta} = U_\beta \times \mathbb{C}^{\mathbb{N}}$$

are holomorphic. Under local trivializations the above definition means the follows:

$$U_\alpha, e_1, e_2 \cdots, e_N$$

trivializes E_{U_α},

$$U_\beta, f_1, \cdots, f_N \cdots E|_{U_\beta}$$

then we have

$$f_j = \sum_{i=1}^N a_{ij} e_i$$

the functions $a_{ij} \in C^\infty(U)$ are holomorphic. So we have

$$\overline{\partial} a_{ij} \equiv 0, \forall i, j$$

Example 48. *For all $p = 0, \cdots m$, we have $\wedge^{p,0}$ is a holomorphic vector bundle with $dz_I, |I| = p$.*

Given any holomorphic vector bundle $E \to M$, we can defined an operator

$$\overline{\partial} : C^\infty(E) \to C^\infty(E \otimes \wedge^{0,1})$$

as follows: Let $e \in C^\infty(M)$, locally we can write

$$e = \sum a_j e_j$$

form a local holomorphic trivilization of E. We define

$$\overline{\partial}_E e|_U = \sum_{j=1}^{N} (\overline{\partial} a_j) e_j$$

This defines

$$\overline{\partial}_E(e) \in C^\infty(E \otimes \wedge^{0,1})$$

So why? This defines independent of the holomorphic trivializations.

Proof. If $V, f_1 \cdots f_N$ is another holomorphic trivilization of E, we need to show that $e|_v = \sum b_j f_j$, such that

$$\sum_j (\overline{\partial} b_j) f_j = \sum (\overline{\partial} a_j) e_j$$

on $U \cap V$. Here is a quick check: But we know that

$$f_j = \sum a_{ij} e_i$$

for some holomorphic function a_{ij}. Hence

$$e|_U = \sum_j b_j f_j = \sum_j (a_{ij} b_j) e_i$$

This implies

$$a_i = \sum_j a_{ij} b_j$$

Hence, on $U \cap V$, we have

$$\sum (\overline{\partial} a_i) e_i = \sum a_{ij} \overline{\partial} b_j e_i = \sum (\overline{\partial} b_j) f_j, \partial a_{ij} \equiv 0$$

I claim that we can define $\overline{\partial}$ on $E \otimes \wedge^{0,q}$. Here is the claim:

Moreover, we can define

$$\overline{\partial}_E : C^\infty(\wedge^{0,k} \otimes E) \to C^\infty(\wedge^{0,k+1} \otimes E)$$

REMARK 200. *My dream was shattered by Adam and I am a bit worried now. Let us pick a local coordinate, we have:*

Take a section: If $e \in C^\infty(\wedge^{0,k} \otimes E)$ and U is a holomorphic coordinate patch on M over which existing holomorphic trivialization $e_1 \cdots e_N$ on E, we can write

$$e|_U = \sum_{I,j} a_I \otimes e_j$$

where $\alpha_I \in C^\infty(U, \wedge^{0,k})$. We put

$$\overline{\partial}_E e|_U = \sum_j (\overline{\partial}\alpha_j) \otimes e_j$$

Note here

$$\sum_{I,j} a_{I,j} d\overline{z_I} \otimes e_j, \alpha_j = \sum_I \alpha_{I,j} d\overline{z}_I \in C^\alpha(U, \wedge^{0,k})$$

and thus we have the

$$\sum_j \alpha_j \otimes e_j$$

REMARK 201. *Adam claiming the transition functions may be a problem, Professor claim it might not.*

Proof. If $f_1 \cdots f_N$ is another holomorphic trivilization of E, then

$$f_j = \sum a_{ij} e_i, a_{ij}$$

are holomorphic functions. Hence $e = \sum \beta_j \otimes f_j = \sum (a_{ij}\beta_j) \otimes e_i$. This implies

$$\alpha_i = \sum_j a_{ij}\beta_j$$

This implies

$$\sum_i (\overline{\partial}\alpha_i) \otimes e_i = \sum_i [\sum_j \overline{\partial} a_{ij} \wedge \beta_j + a_{ij} \wedge \overline{\partial}_j \beta_j] \otimes e_i$$

Here we assume Leibniz's rule:

$$\overline{\partial}(fg) = \overline{\partial} f * g + f * \overline{\partial} g$$

Therefore we have

$$\sum_i \sum_j \alpha_{ij} \overline{\partial}\beta_j \otimes e_i = \sum_j \overline{\partial}\beta_j \otimes f_j$$

REMARK 202. *Adam raised up the issue when the transitional functions are not holomorphic we might end up in trouble.*

REMARK 203. *Professor propose to define $\overline{\partial}$ by*

$$\sum(\partial_{\overline{\partial} z_j} *) \wedge d_{\overline{\partial}_{z_j}}$$

In other words we really have

$$\overline{\partial}_E = \overline{\partial}_{\wedge^{p,q}} \otimes Id_E$$

For all p, q we have an operator

$$C^\infty(\wedge^{p,q} \otimes E) \to^{\overline{\partial}} C^\infty(\wedge^{p,q+1} \otimes E)$$

So we get a complex by definition. So the k-th Dolbeault cohomomlogy is defined by the following notation:

$$H^k(O(E)) = \frac{\ker \overline{\partial}_E}{Im\overline{\partial}_E}$$

on $C^\infty(\wedge^{0,k} \otimes E)$. The holomorphic Euler characteristic is now given by

$$\chi(E) = \sum_{k=0}^{m} (-1)^m \dim H^k(O(E))$$

THEOREM 117. *Hirzebruch-Riemann-Roch We have*

$$\chi(E) = (\frac{i}{2\pi})^m \int_M Td(M) \wedge ch(E)$$

where

$$ch(E) = Tr(e^{Q_E})$$

is the Chern character.

REMARK 204. *Here we literally twist the complex by E. If we take a vector bundle and the Dirac operator, then the twisted index should involve the original Todd class and the vector bundle we are considering.*

Execrise 40. *We have*

$$(\frac{i}{2\pi})^m \int \overline{A}(M) ch(S)^{-1} ch(\wedge^{0,p}) \wedge ch(E) = (\frac{i}{2\pi})^m \int Td(M) \wedge ch(E)$$

8.4 Adam's question

Let

$$H^1_{hol}(M) = \frac{f|\overline{\partial} f = 0}{\partial_z g | g \text{holol morphic}} \cong H^1_{dR}(\mathbb{R}^n)$$

Here is the question, can we prove the above (below) relation for the Riemann Surface?

$$H^1_{dR}(M) = \frac{\text{holomorphic} \wedge^{1,0} \text{forms}}{\partial f \in C(\wedge^{1,0}, \overline{\partial} f = 0)}$$

8.5 $\overline{\partial}E$, E holomorphic vector bundles

Very quick review:

$$\partial = \pi_{p+1,q} \circ d : C^\infty(\wedge^{p,q}) \to C^\infty(\wedge^{p+1,q})$$

and similar formula for $\overline{\partial}$.

Let M *be a complex manifold. We have the following: Then* $d = \partial + \overline{\partial}$. *Locally, choose holomorphic coordinates* $(z_1, \cdots z_m)$, $z_j = (x_j + iy_j)$. *Then we have*

$$\{dz_I \wedge d\overline{z}_J, |I| = p, |J| = q$$

Point:

$$d(dz_I \wedge d\overline{z}_J) = 0$$

Thus, if

$$\alpha \in C^\infty(\wedge^{p,q}), \alpha = \sum a_{IJ} dz_I \wedge d\overline{z}_J$$

then we have

$$d\alpha = \sum (d\alpha_{IJ}) \wedge dz_I \wedge d\overline{z}_J + \sum \overline{\partial}_{z_k} a_{IJ} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

REMARK 205. *The same does not apply to almost complex manifolds as there are general* 0*-order term available.*

REMARK 206. *The above now implies*

$$\partial = \sum (dz_n \wedge) \partial_{z_k}, \overline{\partial} = \sum (d\overline{z}_n \wedge) \overline{\partial}_{z_k}$$

A holomorphic vector bundle is a C^∞ *vector bundle with a cover by trivilazations whose transition functions are holomorphic.*

Example 49. *We noticed from Adam's objection last time that* $\wedge^{p,0}$ *is holomorphic vector bundle, but* $dz_I \sim dw_I$ *with holomorphic coefficients.*

Given any holomorphic vector bundle, we can define

$$\overline{\partial}_E : C^\alpha(\wedge^{0,k} \otimes E) \to C^\infty(\wedge^{0,k+1} \otimes E)$$

locally, if $e \in C^\infty(\wedge^{0,k} \otimes E)$, *then we have*

$$e = \sum \alpha_j \otimes e_j, \alpha_j \in C^\infty(\wedge^{0,k})$$

Such that

$$e_1 \cdots e_N$$

is a holomorphic trivialization of E.

Therefore we have

$$\overline{\partial}_E(e) = \sum(\overline{\partial}\alpha_j) \otimes e_j$$

That is

$$\overline{\partial}_E = \overline{\partial}_{\wedge^{0,*}} \otimes Id_E$$

We have for $k = 0$,

$$\ker(\overline{\partial}_E : C^\infty(E) \to C^\infty(\wedge^{0,1} \otimes E)$$

are exactly the holomorphic sections of E.

We get a complex

$$0 \to C^\infty(E) \to^{\overline{\partial}_E} C^\infty(\wedge^{0,1} \otimes E) \to^{\overline{\partial}_E} C^\infty(\wedge^{0,2} \otimes E) \to^{\overline{\partial}E}$$

We thus define

$$H^k(O(E)) = \frac{\ker \overline{\partial}_E}{Im\overline{\partial}_E}$$

$$\chi(O(E)) = \sum_k (-1)^k \dim H^k(O(E))$$

By our theorem we constructed $D_d = \sqrt{2}(\overline{\partial}_E + \overline{\partial}_E^*)$ is a Z_2 graded Dirac operator on $\wedge^{0,*} \otimes E$ for the Dolbeault complex. It is Z_2 graded with the grading

$$(\wedge^{0,even} \otimes E) \oplus (\wedge^{0,odd} \otimes E)$$

Finally

$$(D_d, \wedge^{0,*} \otimes E)$$

is twisted. In fact we have

$$\overline{Z} = (-1)^m$$

where m equal the complex dimension of M.

Putting a Hermitian Riemannian metric on M. Our question is what is the principal symbol of $\overline{\partial}_d$?

Proof. We have

$$\sigma(\overline{\partial}_E)(\delta) = \sigma(\overline{\partial}_{\wedge^{0,*}})(\delta) \otimes Id_E$$

similarly we have

$$\sigma(\overline{\partial}_E^*)(\delta) = \sigma(\overline{\partial}_{\wedge^{0,*}})(\delta)^* \otimes Id_E$$

Therefore we have

$$\sigma(D_d)(\delta) = \sigma(\text{standard Dol. Dirac operator})(\delta) \otimes Id_E$$

which equals

$$\sigma(D_d)(\delta)^2 = |\delta|^2$$

So we have

$$\omega = \frac{i^m}{n!}\sigma(D_d)(dg)$$

which equals

$$\frac{i^m}{n!}\sigma(\text{standard Dol})(dg) \otimes Id_E = (-1)^m \cdot (-1)^k = (-1)^m Z \to \overline{Z} = (-1)^m$$

Theorem 118. *The Dol E-complex is a Dirac complex, such that*

$$IndD_d = \chi(O(E))$$

What is

$$IndD_d?$$

By our twisted index theorem, we have

$$IndD_d = (\frac{-1}{2\pi i})^m \inf \overline{A}(M)ch(S)^{-1}ch(\wedge^{0,*} \otimes E)$$

Theorem 119. *For all complex vector bundles E_1, E_2, we have*

$$ch(E_1 \otimes E_2) = ch(E_1) \wedge ch(E_2)$$

Hence we have

$$\chi(O(E)) = (\frac{-1}{2\pi i})^m \int \overline{A}(M)ch(S)^{-1}ch(\wedge^{0,*})ch(E)$$

Here K is any curvature form on $T^{1,0}M$. If we let

$$Td(M) = \det(\frac{K}{I - e^{-K}})$$

Theorem 120. *(Hirzebruch-Riemann-Roch Theorem) We have*

$$\chi(O(E)) = (\frac{-1}{2\pi i})^m \int Td(M)ch(E)$$

And K is the curvature on $T^{1,0}$ equal to

$$(\frac{-1}{2\pi i})^m \int \det(\frac{K}{I - e^{-K}})Tr(e^{Q_E})$$

Execrise 41. *Let E be the holomorphic vector bundle, pick a Hermitian metric on E. Prove that there exist a unique connection*

$$\nabla : C^\infty(E) \to C^\infty(\mathbb{C}\wedge^1 \otimes E = \wedge^{1,0} \oplus \wedge^{0,1})$$

Such that

- *The connection ∇ is unitary, such that ∇^* is $-\nabla$.*
- $$\pi_{0,1} \circ \nabla = \overline{\partial}_E : C^\infty(E) \to C^\infty(\wedge^{0,1} \otimes E)$$

  We have to prove that there exist a unique connection with zero $\wedge^{0,1}$ part.

Here is the proof of theorem 37:

Proof. Pick any connection on ∇^1onE_1, ∇^2onE_2, then we have

$$\nabla = \nabla^1 \otimes Id_{E_2} + Id_{E_1} \otimes \nabla^2$$

Locally, $\nabla^1 = d + \omega^1, \nabla^2 = d + \omega^2$.Therefore $\nabla = d + \omega$, where ω is equal $\omega_1 \otimes Id_{E_2} + Id_{E_2} \otimes \omega^2$.

Now finally the curvature

$$Q_{E_1 \otimes E_2} = d\omega + \omega \wedge \omega = (d\omega') \otimes Id_{E_2} + Id_{E_1} \otimes d\omega + L$$

Where the L is the sum of four entries:

$$(\omega' \otimes Id_{E_2}) \wedge (\omega' \otimes Id_{E_2}) + (\omega' \otimes Id_{E_2}) \wedge (Id_{E_1} \otimes \omega^2) + (Id_{E1} \otimes \omega^2) \wedge (\omega^1 \otimes Id_{E_2}) + (Id_{E_1} \otimes \omega^2) \wedge (Id_{E_1} \otimes \omega^2)$$

In any case we have this to be equal to

$$d\omega' \otimes Id_{E_2} + Id_{E_1} \otimes d\omega^2 + \omega^1 \wedge \omega^1 \otimes Id_{E_2} + Id_{E_1} \otimes \omega^2 \wedge \omega^2$$

In particular we have

$$Q_{E_1 \otimes E_2} = Q_{E_1} \otimes Id_{E_2} + Id_{E_1} \otimes Q_{E_2}$$

Thus since two forms commute with each other, we have

$$e^{Q_{E_1 \otimes E_2}} = e^{Q_{E_1} \otimes Id_{E_2} + Id_{E_1} \otimes Q_{E_2}}$$

This equals

$$e^{Q_E} \otimes Id_{E_2} \wedge e^{Id_{E_1}} \otimes Q_{E_2} = e^{Q_{E_1}} \otimes e^{Q_{E_2}}$$

Hence we have

$$ch(E_1 \otimes E_2) = Tr(e^{Q_{E_1}} \otimes e^{Q_{E_2}}) = Tr(e^{Q_{E_1}}) \wedge Tr(e^{Q_{E_2}}) = ch(E_1) \wedge ch(E_2)$$

Remains to under multiplication using tensor field. Let v_i be a basis of E_1, w_i a basis of E_2, then we have $v_i \otimes w_j$ is a basis for $E_1 \otimes E_2$. Let $A : E_1 \to E_1$, $B : E_2 \to E_2$. Let $Av_j = \sum a_{ij} v_i$, $Bw_j \sum b_{ij} w_j$

Let us look at $A \otimes B$!

So we have

$$(A \otimes B)(v_p \otimes w_q) = Av_p \otimes Bw_q = \sum a_{rp} b_{sq} v_r \otimes w_s = \sum (A \otimes B)_{rs,pq} v_r \otimes w_s$$

which implies

$$(A \otimes B)_{rs,pq} = a_{rp} b_{sq}$$

This implies

$$[(A \circ B) \circ (C \otimes D)])_{rs,pq} = \sum_{k,l} (A \circ B)_{rs,kl} \cdot (C \otimes D)_{kl,pq}$$

which equals

$$\sum_{k,l} a_{rk} b_{sl} c_{kp} d_{lq}$$

So we conclude that $(A \otimes B) \otimes (C \otimes D)$ is a matrix of (rs, pq) entry

$$\sum_{k,l} a_{rk} b_{kl} c_{kp} d_{lq}$$

This implies if A, B, C, D are matrices of forms with degrees a, b, c, d respectively, then

$$(A \otimes B) \wedge (C \otimes D)$$

is a matrix with (rs, pq) entry

$$\sum_{k,l} a_{rk} b_{sl} c_{kp} d_{lq} = (-1)^{bc} \sum_{k,l} a_{rk} c_{kp} b_{sl} d_{lq} = (-1)^{bc} ((AC)_{rp} \cdot (BD)_{sq})$$

That is

$$(A \otimes B) \wedge (C \wedge D)$$

is the matrix for

$$(-1)^{bc} (AC) \otimes (BD)$$

Remark 207. *We used the fact that*

$$(\omega' \otimes Id_{E_2}) \wedge (Id_{E_1} \otimes \omega^2) = \omega^1 \wedge \omega^1 \otimes Id_{E_2}$$

similarly we have

$$(Id_{E1} \otimes \omega^2) \wedge (\omega^1 \otimes Id_{E_2}) = -\omega^1 \otimes \omega^2$$

But this follows from what we proved earlier.

8.6 Line bundles and Riemann Surfaces

We thus have proved the Hirzebruch-Riemann-Roch Theorem. Now we want to restrict the case M is a surface and E is a line bundle.

Now we are going to specialize to M a Riemann-Surface. M is a one dimensional complex manifold, and $E = L$ is a rank one holomorphic line bundle.

Now we have

$$0 \to C^\infty(L) \to^{\overline{\partial}} C^\infty(\wedge^{0,1} \otimes L)$$

The H-R-R is the following: The index of $\sqrt{2}(\overline{\partial}_L + \overline{\partial}_L^*)^+$ is

$$\frac{-1}{2\pi i} \int \det(\frac{K}{Id - e^{-K}}) ch(L)$$

Because M is a Riemann surface with two real dimensions, the formula simplifies considerably.

The left hand side is the dimension of the kernel minus the dimension of the cokernel of the operator. This is the dimension of the harmonic section of L.

LEMMA 64.

$$Ind\sqrt{2}(\overline{\partial}_L + \overline{\partial}_L^*)^+ = \dim \ker \overline{\partial}_L \to$$

This equals

$$\dim(\ker \overline{\partial}_L : C^\infty(L) \to C^\infty \otimes L) - \dim(\ker \overline{\partial}_L^* : C^\infty(\wedge^{0,\wedge} \otimes L) \to C^\infty(L))$$

REMARK 208. *The professor discovered a typo that he forgot the plus, so the operator is by definition from $E^+ = \wedge^{0,even} \otimes L$ to $E^- = \wedge^{0,odd} \otimes L$.*

What is our vector bundle E? It is $\wedge^{0,*} \otimes L$. But the only $\wedge^{0,even}$ forms is the 0-order forms. So the index is the same as the index of

$$\sqrt{2}\overline{\partial}_L : C^\infty(L) \to C^\infty(\wedge^{0,1} \otimes L)$$

REMARK 209. *Similarly $E^{-1} = \wedge^{0,1} \otimes L$.*

Finally we conclude by

$$D_d = \sqrt{2}(\overline{\partial}_L + \overline{\partial}_L^*), D_d^+ = \sqrt{2}\overline{\partial}_L, D_d^- = \sqrt{2}\overline{\partial}_L^*$$

This implies

$$IndD_d^+ = \ker \overline{\partial}_L - \dim \ker \overline{\partial}_L^* = \dim O(L) - \dim \ker \overline{\partial}_L^*$$

We want to show that this equals the dimension of holomorphic sections of the vector bundle.

THEOREM 121.

$$\ker(\overline{\partial}_L^* : C^\infty(\wedge^{0,1} \otimes L) \to C^\infty(L)) \cong \ker \overline{\partial}_{\wedge^{1,0}\otimes L^*} : C^\infty(\wedge^{1,0} \otimes L^*) \to C^\infty(\wedge^{0,1} \otimes \wedge^{1,0} \otimes L^*)$$

In particular we have

$$\dim \ker \overline{\partial}_L^* = \dim O(\wedge^{1,0} \otimes L^*)$$

Therefore for a line bundle the index is given by holomorphic information:

$$X(O(L)) = IndD_d^+ = \dim O(L) - \dim(O(\wedge^{1,0} \otimes L^*))$$

What is the dimension of $T^{1,0}$? It is 1! So we have

$$\det(\frac{K}{Id - e^{-K}}) = \frac{K}{1 - e^{-K}} = 1 + \frac{K}{2} + ?$$

But $? = 0$ because we are in 1-dimensional complex manifold. So the Todd class equal to $1 + \frac{K}{2}$. Now we have

$$ch(L) = Tr(e^{Q_L}) = e^{Q_L} = 1 + Q_L$$

Therefore we have

$$Td(M)ch(L) = (1 + \frac{K}{2}_{)}(1 + Q_L) = 1 + \frac{K}{2} + Q_L$$

Therefore

$$X(O(L)) = \frac{-1}{2\pi i}\int \frac{K}{2} - \frac{1}{2\pi i}\int Q_L = \frac{-1}{4\pi i}\int K + \frac{i}{2\pi}\int Q_L$$

Thus we have

$$[\frac{iQ_L}{2\pi}] = \text{1st Chern class of} L = c_1 L$$

So the above equation gives the theorem

THEOREM 122.

$$-\frac{1}{4\pi i}\int K = \frac{1}{2}\chi(M)$$

Proof. Hint: The Levi-Civita connection preserves $T^{1,0}$. It preserves the local trivialization. We have

$$\nabla^{LC}(v) = \omega, \omega = \cdots ?$$

THEOREM 123. *General Riemann-Roch Theorem.*

For any holomorphic line bundle over a compact Riemann Surface, we have

$$\dim O(L) - \dim O(\wedge^{1,0} \otimes L^*) = \frac{1}{2}\chi(M) + \int_M c(L)$$

where $c(L)$ is the 1st Chern class of L.

REMARK 210. *A very neat fact of line bundles is a line bundle is completely determined by ...if and only if the integral of the chern class of L_1 equal the integral of the chern class of L_2.*

REMARK 211. *We are going to prove Riemann-Roch by using the divisor bundle. One more lecture to prove other things like degree, etc.*

8.7 HRR

Last time let M to be a compact Riemannian manifold with Hermitian metric. Let L be a holomorphic line bundle with Hermitian metric.

Let

$$\overline{\partial}_L : C^\infty : C^\infty(L) \to C^\infty(\wedge^{0,1} \otimes L)$$

THEOREM 124. *we have*

$$\dim\ker\overline{\partial}_L - \dim\ker\overline{\partial}_L^* = \frac{-1}{2\pi i}\int K + \int c_1 L$$

where K any curvature form on $T1,0M$, $c_1 L = \frac{i}{2\pi}Q_L$, where Q_L is any curvature form on L. This $c_1 L$ is called the Chern class.

REMARK 212. *Adam ask if we need a unitary connection. Professor said that we did took it unitary then we define the curvature accordingly. Adam ask if we need unitary connection or just any connection. Professor said he is not sure. So by recalling the detail the curvature on $T^{1,0}$ is the conjugate transpose of the curvature on $T^{0,1}$. Here we remark that Todd class is a general characteristic class and has nothing particular about it.*

Discussion. *Today we are going to identify the left hand side and right hand side. The left hand side is the holomorphic section of L. We have*

$$LHS = \dim O(L) - \dim\ker\overline{\partial}_L^*$$

THEOREM 125.

$$\dim \ker \overline{\partial}_L = \dim O(\wedge^{1,0} \otimes L^*)$$

Here L^* is the dual bundle of L, which is also a line bundle. Then L^* is holomorphic if s is a local holomorphic trivilization on U. Then $\phi : U \to L^*$, with $|phi(s) = 1$ is a holomorphic trivilization of L^*.

LEMMA 65. *If s, t are overlapping local holomorphic trivialization of L, we have $s = at$, such that $a \neq 0$ is a holomorphic, then we havev $\phi = \frac{1}{a}\psi$, where $\phi(s) = 1, \psi(t) = 1$, and $\frac{1}{a}$ is holomorphic as well. For this reason we usually denote L^* by L^{-1}.*

Anyway, the point we are trying to make is that in other words, if

$$\overline{\partial}_{\wedge^* \wedge L^*} : C^\infty(\wedge^{1,0} \otimes L^*) \to C^\infty(\wedge^{0,1} \otimes \wedge^{1,0} \otimes L^*)$$

is the canonical $\overline{\partial}$ operator on $\wedge^{1,0} \otimes L^*$. Then we have

$$\dim \ker \overline{\partial}_{\wedge^{1,0} \otimes L^*}$$

Strategy: Work locally, see what the theorem tells us.

Proof. All we do is at a local formula for OP_L^* let $(U, z) = (u, (x, y))$ be a local holmorphic trivlization on M over wihch L is trivial and let s be a local trivization of L over U. Let $\phi : U \to L^*$ be the dual of $s : \phi(s) = 1$. Finally let

$$\phi = \langle *, v \rangle$$

where $v \in C^\infty(M, L)$ is another trivialization of L over u.

Recall

$$\overline{\partial}_L : C^\infty(L) \to C^\infty(\wedge^{0,1} \otimes L)$$

this implies

$$\overline{\partial}_L^* : C^\infty(\wedge^{0,1} \otimes L) \to C^\infty(L)$$

satisfying

$$\int_M \langle w, \overline{\partial}_L^*(\alpha) \rangle dg = \int_M \langle \overline{\partial}_L w, \alpha \rangle dg$$

Over U, any section of $\wedge^{0,1} \otimes L$ can be written as

$$\alpha = a d\overline{z} \otimes v$$

for some $a \in C^\infty(M)$. Similarly we have any section of L over U can be written as

$$w = bs, b \in C^\infty(U)$$

To use integration by parts, let us assume b has compact support. Let us compute!

We have

$$\int_M \langle w, \overline{\partial}_L \alpha\rangle dg = \int_M \langle \overline{\partial}_L w, \alpha\rangle dg = \int \langle (\overline{\partial}_Z b) d\overline{z} \otimes s, \wedge d\overline{z} \otimes v\rangle dg$$

which is

$$\int_M (\overline{\partial}_Z b)\overline{a}|d\overline{z}|^2 \langle s, v\rangle dg$$

Using $\langle s, v\rangle = \phi(s) = 1$. So we have

$$\int_M \frac{1}{2}(\partial_x + \partial_y) b(x,y)\overline{(a(x,y))}|d\overline{z}| dg$$

Execrise 42. *:*

Using the fact that the metric is Hermitian we have

$$|d\overline{z}|^2 dg = 2dx \wedge dy, J(\partial_x) = \partial_y$$

So the above integral simplified to

$$\int \frac{1}{2}(\partial_x + i\partial_y) b(x,y)\overline{a(x,y)} 2dxdy = -\int b(x,y)\frac{1}{2}(\partial_x + i\partial_y)\overline{a(x,y)} 2dxdy$$

which further equates

$$\overline{\partial}_L^*(d\overline{z} \otimes v) = -(|d\overline{z}|^2 \partial_z a) v$$

With this in mind let us consider the conjugate linear isomorphism, which let us call it $F : \wedge^{0,1} \otimes L \to \wedge^{1,0} \otimes L^*$. The map is given by

$$F : \beta \otimes l \to \overline{l} \to \overline{\beta} \otimes \langle *, l\rangle$$

Claim: $\alpha \in C^\infty(\wedge^{0,1} \otimes L)$ *such that* $\overline{\partial}_L^*(\alpha) \equiv 0$ *if and only if there is* $F(\alpha) \in C^\infty(\wedge^{1,0} \otimes L^*)$ *such that*

$$\overline{\partial}_{\wedge^{1,0} \otimes L^*}(F\alpha) \equiv 0$$

such that

$$\ker \overline{\partial}_L^* \cong^F O(L^{1,0} \otimes L^*)$$

Indeed,

$$\alpha \in C^\infty(\wedge^{0,1} \otimes L)$$

such that $\overline{\partial}_L^* \alpha = 0$*, if and only if exist local trivializations with holomorphic coordinates, such that we have*

$$\alpha = a\overline{z} \otimes v$$

where

$$\overline{\partial}_z a = 0$$

What do we need to construct such α? We have

$$F(\alpha) = \overline{\alpha} dz \otimes \phi$$

satisfies

$$\partial_z \alpha =\equiv 0 \leftrightarrow \overline{\partial}_z \overline{\alpha} \equiv 0$$

which is equivalent to

$$\overline{\partial}_{\wedge^{1,0} \otimes L^*}(F(\alpha)) \equiv 0$$

This is equivalent to

$$F(\alpha) \in C^\infty(\wedge^{1,0} \otimes L^*)$$

such that

$$\overline{\partial}_{\wedge^{1,0} \otimes L^*}(F\alpha) = 0$$

We used the fact that for E, with $\gamma = ae$, $\overline{\partial}\alpha \equiv 0$, we have

$$\overline{\partial}_E \gamma = (\overline{\partial}_Z \alpha)(e), \overline{\partial}_E \gamma = 0$$

Discussion. *Therefore the left hand side of the index formula is very nice:*

$$\dim O(L) - \dim O(\wedge^{1,0} \otimes L^*) = -\frac{1}{4\pi i} \int K + \int c_1(L)$$

Proof. Since

$$\int c_1(L_1) = \int c_1(L_2) \leftrightarrow L_1 \cong L_2$$

which implies

$$\int c_1(L)$$

is an integer, it lives in $H^2(M, \mathbb{Z})$.

Theorem 126.

$$\chi(M) = \frac{-1}{2\pi i} \int K$$

Execrise 43.

$$\chi(M) = \dim(H^0(m)_{dR} \oplus H^2(M)_{dR}) - \dim H^1_{dR}(M)$$

Prove $H^0_{dR} \cong H^2_{dR}$, Prove that $\dim H^1_{dR}(M)$ is even dimensional using Poincare duality.

Remark 213. *Maurioccio claimed that this is trivial.*

Discussion. *Thus we have*

$$\chi(M) = 2(\textit{number of connected components}) - 2g, g \in \{0,1\}$$

where g is the genus of M. In particular we have

$$\frac{1}{2}\chi(M)$$

THEOREM 127. *The Riemann-Roch theorem for line bundles:*

$$\dim O(L) - \dim O(\wedge^{1,0} \otimes L^*) = \frac{1}{2}\chi(M) + \int c_1 L$$

Proof. What is K? Claim the LC connection preserves $T^{1,0}$.

Proof. Let v_1 and v_2 be local orthonormal oriented vector basis for TM with $Jv_1 = v_2$.

Recall that

$$\nabla v_1 = \omega_{21} v_2 = -\omega_2 \otimes v_2, \nabla^{LC} \otimes v_2 = \omega_{12} \otimes v_1$$

Let

$$w = v_1 - iJv_1 = v_1 - iv_2$$

trivilize $T^{1,0}M$. Then we have

$$\nabla^{LC} w = -w_{12} \otimes v_2 - iw_{12} \otimes v_1 = -iw_{12}(v_1 - iv_2) = -iw_{12}w$$

Thus we have

$$\nabla^{LC} w = \omega \otimes w, \omega = -iw_{12}$$

Therefore

$$K = d\omega + \omega \wedge \omega = -id\omega_{12}$$

Recall that

$$\omega^{LC} = \begin{bmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \\ & \end{bmatrix}$$

Therefore $R^{LC} = d\omega^{LC} + \omega \wedge \omega^{LC}$. The result is

$$\omega^{LC} = \begin{bmatrix} 0 & d\omega_{12} \\ -d\omega_{12} & 0 \\ & \end{bmatrix}$$

Therefore we have

$$K = -iR^{LC}_{12} = -i \times (1,2)$$

entry of the Riemannian curvature operator. We conclude that

$$-\frac{1}{2\pi i}\int K = \frac{1}{2\pi}\int R^{LC}_{12}$$

Execrise 44. *Use the Gauss-Bonnet to show*

$$\frac{1}{2\pi}\int R_{\mathbb{R}}^{LC} = \chi(M)$$

Hint:

$$(-\frac{\mathcal{R}^m}{m!}) = \alpha \otimes dg$$

and the $Pf = \alpha$. For a surface, if we write $-\mathcal{R} = \alpha \otimes \phi_1 \wedge \phi_2$, we shall find

$$\alpha = R_{12}^{LC}$$

So we have

$$\chi(M) = \frac{1}{2\pi}Pf = \frac{1}{2\pi}\int R_{12}^{LC}$$

You apply the curvature operator to v_2 and inner product with v_1, that's how you get this. So recall $\mathcal{R} \in C^\infty(\wedge^2 \otimes \wedge^2)$. We have

$$\mathcal{R}(v,w,u,v) = \langle R(v,w)u, v\rangle$$

8.8 To Riemannian-Roch

We have

$$\dim O(L) - \dim O(\wedge^{1,0} \otimes L) = \frac{1}{2}\chi(M) + \int c_1 L = 1 - g + \int c_1(L)$$

to prove Classical R-R we let L be a holomorphic line bundle. We know that $s \in C^\infty$ means. What about a meromorphic section of L?

Definition 77. *A function $s : M \to L$ is meromorphic section of L if L is a holomorphic section of $L|_{M\ D}$ where D is a finite subset such that the following is true: For all points in D and for all holomorphic coordinates centered at p and for all holomorphic trivializations of L around p, we have*

$$s = av$$

where a is a meromorphic function on U with a possible pole at p and v.

Example 50. *:*

The Riemann Sphere with $\alpha = \frac{1}{z}$. We have

$$\alpha : M - \{0\} \to \mathbb{C}$$

is a meromorphic section on M.

Definition 78. *A* **Divisor** *on M is finite subset of $M \otimes (\mathbb{Z} - \{0\})$ of the form*

$$\{(p_1, m_1), (p_1, m_2) \cdots, (p_N, M_N)\}$$

Remark 214. *If $p \in D$, we say that s has a zero of order m at p if*

$$a(z) = z^m \times b(z)$$

where $b(z)$ is holomorphic on U and $b_p \neq 0$. Similarly we say s has a pole of order $m > 0$ if

$$a(z) = z^{-m} b(z)$$

where $b(z)$ holomorphic on U with $b(0) \neq 0$.

We should check these definitions do not depend on the choice charts involved.

Definition 79. *We say a meromorphic section of L is said to have divisor $\mathfrak{D}$ (at least $\mathfrak{D}$ or greater than or equal $\mathfrak{D}$) if the following is true:*

For each section $s \in H\infty(L, M - \{p_1, \cdots p_n\})$, for each p_i local holomorphic coordinates we have

$$S = z^{m_i} b(z)$$

where $b(z)$ is holomorphic on a neighborhood of p_i.

We usually write this as

$$div(s) \geq \mathfrak{D}$$

In other words, if $m_i > 0$, we say s has a zero of order m_i or greater, if $m_i < 0$, s has a pole of order at most m_i.

Discussion. *Here is the classical Riemann-Roch theorem:*

The collection of such meromorphic sections of L having the above property are denoted by $M_{\mathfrak{D}}(L)$ if we include the zero function.

The theorem states:

Theorem 128. *$\forall \mathfrak{D}$ on M,*

$$\dim M_{\mathfrak{D}}(\mathbb{C}) - \dim_{-\mathfrak{D}}(\wedge^{1,0}) = \frac{1}{2}\chi(M) + \deg \mathfrak{D}$$

Remark 215. *Adam tried to use $\mathfrak{D} = \emptyset$. In this case we have*

$$1 - \dim M_0(\wedge^{1,0}) = \frac{1}{2}\chi(M)$$

which means

$$\dim O(\wedge^{1,0}) = \frac{1}{2} \dim H^1_{dR}(M)$$

8.9 Proof of Riemann-Roch

Review:

DEFINITION 80. *Let M be a compact Riemann Surface, let L be a holomorphic line bundle over M. A* **meromorphic section** *s of L is a holomorphic section of M/DL where $D = \{P_1 \cdots P_N\} \subset M$ is finite such that there exist (U_i, z) centered at p_1 and locally trivialization v_i over U_i, we have*

$$s = a(z)v(z)$$

where a has a Laurent expansion on v_i

DEFINITION 81. *A* **divisor** *$\mathfrak{D}$ is finite subset $\mathfrak{D} \subset M \times \{\mathbb{Z}/\{0\}$. Such that $deg\mathfrak{D} = \sum_i m_i$ if*

$$\mathfrak{D} = \{(p_1, m_1) \cdots (P_N, m_N)\}$$

A meromorphic section of L has 'divisor $\mathfrak{D}$' or 'has divisor a multiple of $\mathfrak{D}$' if

$$s : M/\{p_1 \cdots p_N\} \to \mathbb{C}$$

is holomorphic and has holomorphic coordinate patch (U_i, z) centered over P_i. Such that for all i, we have

$$s = z^{m_i} t(z)$$

where $t(z)$ is holomorphic functions on P_i. We denote

THEOREM 129. *$\forall$divisor $\mathfrak{D}$ we have*

$$\dim M_{-\mathfrak{D}}(\mathbb{C}) - \dim M_{\mathfrak{D}}(\wedge^{1,0}) = \frac{1}{2}\chi(M) + deg(\mathfrak{D})$$

Idea: Given $\mathfrak{D}$, find $L(\mathfrak{D})$. Such that

$$M_{-\mathfrak{D}}(\mathbb{C}) = O(L(-\mathfrak{D}))$$

So we have

$$M_{\mathfrak{D}}(\wedge^{1,0}) = O(\wedge^{1,0} \otimes L(-\mathfrak{D})^*)$$

So we want to apply HRR to $L(-\mathfrak{D})$.

REMARK 216. *To do this, given a divisor $\mathfrak{D}$ and a holomorphic line bundle L, the question is:*

Does there exist $L(\mathfrak{D})$ such that

$$M_{\mathfrak{D}}(L) = O(L \otimes L(\mathfrak{D}))?$$

We get

$$M_{-\mathfrak{D}}(\mathbb{C}) = O(\mathbb{C} \otimes L(-\mathfrak{D})) = O(L(-\mathfrak{D}))$$

Note this in theorem we let $L = \mathbb{C}$.

Similarly we have

$$M_{\mathfrak{D}}(\wedge^{1,0}) = O(\wedge^{1,0} \otimes L(\mathfrak{D}))$$

Discussion. *'Obviously' we want* z^{m_i} *to be local holomorphic trivializations of the line bundle* $L(\mathfrak{D})$. *One way popular in literature is to use transition functions.*

Here is a cool way I learned from Richard Melrose to make z^{m_i} *trivializations of a vector bundle:*

Example 51. *Let* $E = \mathbb{R} \times \mathbb{R}$. *We can define* E *from its smooth sections. Let* $V = C^{\infty}(\mathbb{R})$. *Given a point* $p \in \mathbb{R}$, *find two functions* $f, g \in V$ *are equivalent We have*

$$f \sim_p g \leftrightarrow f(p) = g(p) \leftrightarrow f = g + O(x - p)$$

The claim is

$$V_p = V / \sim p \cong \mathbb{R}$$

The proof is obvious as every function is equivalent to the constant function $f(p)$. *We have*

$$[f]_p = [f_p]_p = f(p)[1]_p$$

by the map

$$F_p : V_p \to \mathbb{R}$$

So we have

$$f - f(p) = O(x - p)$$

So we have F *defines a trivializatio of* V:

$$F : V \to \mathbb{R} \times \mathbb{R}$$

by

$$F : [f]_p \to (p, f(p))$$

F *is a trivialization as* $[1]_p$ *is a global trivialization of* V.

Remark 217. *This method defined the so called b-cotangent bundle. And there are other kinds of vector bundles using similar ideas.*

Discussion. *Does there exist a line bundle over* $\mathbb{R}$ *such that* $s(x) = x^2$ *is a global trivialization of* L?

REMARK 218. *'Obviously' standard trivial line bundle does not work.*

As a section of V, we have $s(x) = x^2[1] = (x, x^2)$. But this is not a global trivialization since at 0 it degenerates. So the standard trivial line bundle does not work!

Here is the trick!There are a couple of ways we can make it work. Let us make up some notation. Let

$$C = \{f \in C^\infty(\mathbb{R}) | f(x) = a(x)x^2\}$$

for some $a \in C^\infty(\mathbb{R})$. Given $p \in \mathbb{R}$, $f, g \in \mathbb{C}$, we have

$$f \sim g \leftrightarrow a(p) = b(p)$$

assuming

$$f(x) = a(x)x^2, g(x) = b(x)x^2$$

Let $\forall p \in \mathbb{R}, L_p = C/\sim_p$ is a one dimensional real vector space. Why is this true? Take an element $[f] = [a(x)x^2]_p$, and p could be 0. So

$$[f]_p = [a(x)x^2]_p = [a(p)x^2]_p = a(p)[x^2]_p \to a(p) \in \mathbb{R}$$

So we have

$$F_p : L_p \to \mathbb{R}, F_p([f]_p) = a(p)$$

is an isomorphism and for $p \neq 0$ we have $L_p \cong V_p$. Then we have

$$F : L = \bigcup_p L_p \to \mathbb{R} \times \mathbb{R}$$

such that

$$F : [a(x)x^2]_p \to (p, a(p))$$

is a divisor of L such that

$$s(p) = [x^2]_p$$

is a global trivialization of L.

Alternatively we can define the subspace $(x - p)C = \{f(x) = a(x)x^2\}, a \in C^\infty, a(p) = 0$.

Then we have

$$L_p = C/(x - p)C$$

and similarly

$$V_p = C^\infty(\mathbb{R})/(x - p)C^\infty(\mathbb{R})$$

Example 52. *Let $M = [0, \infty]$, does there exist a vector bundle $V \to M$ such that $x\partial_x$ is a trivialization of V?*

REMARK 219. *Since this vector field vanishes, $x\partial_x$ clearly would not work. Let C be the set of all vector fields on TM:*

$$C = \{v \in C^\infty(TM) | v(x) = a(x)x\partial_x$$

for some $a \in C^\infty(M)$.Given any $p \in M$, $v \sim_p w \leftrightarrow a(p) = b(p)$ with $v = a(x) \times \partial_x, w = b(x) \times x\partial_x$. Then we have

$${}^bTM_p = C / \sim_p$$

So we have

$${}^bT_pM \cong \mathbb{R}, [a(x) \times \partial_x](p) \to a(p)$$

So we have

$$s(x) = [x\partial_x]_p$$

is a global trivialization of ${}^bTM$ called the b-tangent bundle.

Let $p \in M$, let $C = \{(U, f) | f$is meromorphic on V with divisor$\mathfrak{D}|_V\}$. So C_p is a vector space where

$$(v_1, f_1) + (v_2, f_2) = (v_1 \cap v_2, f_1 + f_2)$$

Define

$$\overline{C_p} = \{(V, h), h = gf, , f \in C_p, g\text{is holomorphic}, g(p) = 0\}$$

DEFINITION 82. *We have*

$$L(\mathfrak{D})_p = C_p / \overline{C}_p, L(\mathfrak{D}) = \bigcup L(\mathfrak{D})_p$$

LEMMA 66.

$$\forall p \in M, L(\mathfrak{D})_p\text{is a one dimensional}\mathbb{C}\text{vector space}$$

Let $p = p_i$ and let $[(v; f)]_p \in L(\mathfrak{D})_p$.

Observe, in holomorphic coordinate centered at P_i, $f(z) = z^{m_i}b(z)$ $b(z)$ is holomorphic at P_i.

Therefore

$$f(z) = z^{m_i}h(0) + (b(z) - b(0))z^{m_i}$$

with

$$(b(z) - b(0))z^{m_i} \in \overline{C}_p$$

With this observation we have

$$[(V, f)]_p = [(U, b(0)z^m)]_p = b(0)[(U, z^{m_i})]$$

The map

$$F_p : L(\mathfrak{D})_p \to \mathbb{C}, [(U, f)_{p_i}] \to b(0)$$

where $f(z) = b(z)z^{m_i}$ is $1 - 1$. If $p \neq \{p_1 \cdots p_N\}$, then we leave it as an

Execrise 45.

$$L(\mathfrak{D})_p \cong \mathbb{C}$$

Discussion. *We can define a trivialization*

$$F : L(D)|_{U_i} \to U_i \times \mathbb{C}$$

by

$$F_i : [(U_i, b(z)z^{m_i})]_p \to (p, b(p))$$

is a trivialization. And the $[(U_i, z^{m_i})] = \{z^{m_i}\}$ *are local trivializations of* $L(\mathfrak{D})$ *over* U_i.

We can thus cover $L(\mathfrak{D})$ *by* U_i. *Let* $U_0 = M / \bigcup_{i=2}^{n} U_i$. *Let* $\psi_i = z^{m_i} = (U_i, z^{m_i}), i = 1 \cdots N$. *Let* $\psi_1 = 1$ *over* U_0.

Then ψ_i *is a local trivialization for each* $i = 0 \cdots N$.

Execrise 46. *Prove that* $\forall L$ *which is a holomorphic line bundle*

$$M_{\mathfrak{D}}(L) \cong O(L \otimes L(\mathfrak{D}))$$

Such that near the p_i *we have*

$$s(z) = t(z) * z^{m_i} = t(z) \times \psi_i(z)$$

Similarly

$$M_{-\mathfrak{D}}(L) = O(L \otimes L(\mathfrak{D})^*) = O(L \otimes L(-\mathfrak{D}))$$

This result require a further execrise:

Execrise 47. *:*

Over $U_i \cap U_j, i, j \in \{0, 1, \cdots N\}, \psi_i = (\psi_i \cdot \psi_j^{-1})\psi_j$. *So we conclude* $L(\mathfrak{D})$ *is holomorphic!*

8.10 Final Proof

Let $\mathfrak{D}$ *be a divisor, let* $L(\mathfrak{D})$ *be the divisor bundle.*

REMARK 220. *In many books, the* $L(\mathfrak{D})$ *is in fact* $L(\mathfrak{D}^*) = L(-\mathfrak{D})$. *Then by the Riemann-Roch Hirebruch theorem we have*

$$\dim O(L(-\mathfrak{D})) - \dim O(\wedge^{1,0} \otimes L(-\mathfrak{D}^*) = \frac{1}{2}\chi(M) + \int_M c_1(L(-\mathfrak{D}))$$

By what we did earlier we have

$$\dim M_{-\mathfrak{D}}(\mathbb{C}) - \dim M_{\mathfrak{D}}(\wedge^{1,0}) = \frac{1}{2}\chi(M) + \int_M c_1(L(-\mathfrak{D}))$$

Now we just have to compute

$$\int_M c_1(L(-\mathfrak{D}))$$

LEMMA 67.

$$\int_M c_1(L(-\mathfrak{D})) = deg\mathfrak{D} = \sum_{i=1}^{N} m_i$$

Proof. Construct a cover on $L(-\mathfrak{D})$, let $\psi_0 = 0, \psi_i = z^{-m_i}, i = 1, \cdots N$ be the cover of $L(-\mathfrak{D})$ be the local trivializations over U_0, U_i, $i = 1 \cdots N$. Define a connection on $L(-\mathfrak{D})|_{U_i}, i = 0 \cdots N$ by

$$\nabla^i \psi_i = 0$$

Define ∇ on all of $L(-\mathfrak{D})$ by picking a partition of unity $\rho_1 \cdots \rho_N$ over M such that $\rho_i \subset U_i$ and $\rho_i \equiv 1$ near $P_i, i = 1 \cdots N$. Let

$$\nabla = \sum_i \rho_i \nabla^i$$

i.e $\alpha \in C^\infty(L(-\mathfrak{D}))$. So we have

$$\nabla \alpha = \sum_{i=0}^{N} \rho_i \nabla^i (\alpha|_{U_i})$$

and

$$\nabla : C^\infty(L(-\mathfrak{D})) \to C^\infty(\mathbb{C} \wedge^1 \otimes L(-\mathfrak{D}))$$

Fix i. We shall compute the curvature form and integrate it to compute.

$$\nabla \psi_j = \omega_j \otimes \psi_j, \exists \omega_j$$

where ω_j is a 1-form So we have

$$Q|_{U_j} = d\omega_j$$

Observe we have

$$\nabla \psi_j = \sum_k \rho_k \rho^k \nabla^k (\psi_j|_{u_k}) = \sum_k \rho_k \nabla^k [(\psi_j \psi_k^{-1}) \psi_k]$$

this by definition is

$$\sum_k \rho_k d(\rho_j \rho_k^{-1}) \rho_k = \sum_k \rho_k d(\psi_j \psi_k^{-1})(\psi_k \psi_j) \psi_j$$

Therefore

$$\omega_j = \sum_{k=0}^{N} \rho_k d(\psi_j \psi_k^{-1})(\psi_j^{-1} \psi_k)$$

Therefore

$$Q|_{U_j} = d\omega_j = \sum_k d\rho_k \wedge d(\psi_j \psi_k^{-1})(\psi_j^{-1}\psi_k)$$

As we can check

$$d[d\psi_j \psi_k^{-1})(\psi_j^{-1}\psi_k)] = 0$$

Therefore we have

$$\sum_k d\rho_k \wedge (\frac{d\psi_j}{\psi_k} - \frac{\psi_j d\psi_k}{\psi_k^2})\psi_j^{-1}\psi_k$$

which equals

$$\sum_k d\rho_k (\frac{d\psi_j}{\psi_j} - \frac{d\psi_k}{\psi_k})$$

And we note

$$\sum d(\rho_k) \wedge \frac{d\psi_j}{\psi_j} = d(1) \wedge \frac{d\psi_j}{\psi_j} = 0$$

So the above now equals

$$-\sum_k d\rho_k \wedge \frac{d\psi_k}{\psi_k}$$

Therefore we have

$$Q|_{U_j} = -\sum_k d\rho_k \wedge \frac{d\psi_k}{\psi_k} \in C^\infty(\mathbb{C}\wedge^2)$$

where we can omit the term $k = 0$ since $\psi_0 = 0$. We thus have

$$\int_M c_1(L(-\mathfrak{D})) = -\sum_{k=1}^N \frac{i}{2\pi i} \int_M d\rho_k \wedge \frac{d\psi_k}{\psi_k}$$

Consider a disk centered at p_i such that $\rho_i \equiv 1$ on the disk. Then we can use stoke's theorem since the integral inside the disk is 0:

$$\sum_{k=1}^N \frac{1}{2\pi i} \int_{M/D_k} d\rho_k \wedge \frac{d\psi_k}{\psi_k},$$

Using Stoke's theorem:

$$\sum_{k=1}^N -\frac{1}{2\pi i} \int_{\partial D_k} \psi_k \frac{d\psi_k}{\psi_k}$$

Using assumption $\psi_k = z^{-m_k}$:

$$\sum_{k=1}^N -\frac{1}{2\pi i} \int_{\partial D_k} -m_k \frac{dz}{z^{-m_k}} z^{m_k - 1} = \sum_{k=1}^N \frac{1}{2\pi i} m_k \int \frac{dz}{z} = \sum_{k=1}^N m_k = deg(\mathfrak{D})$$