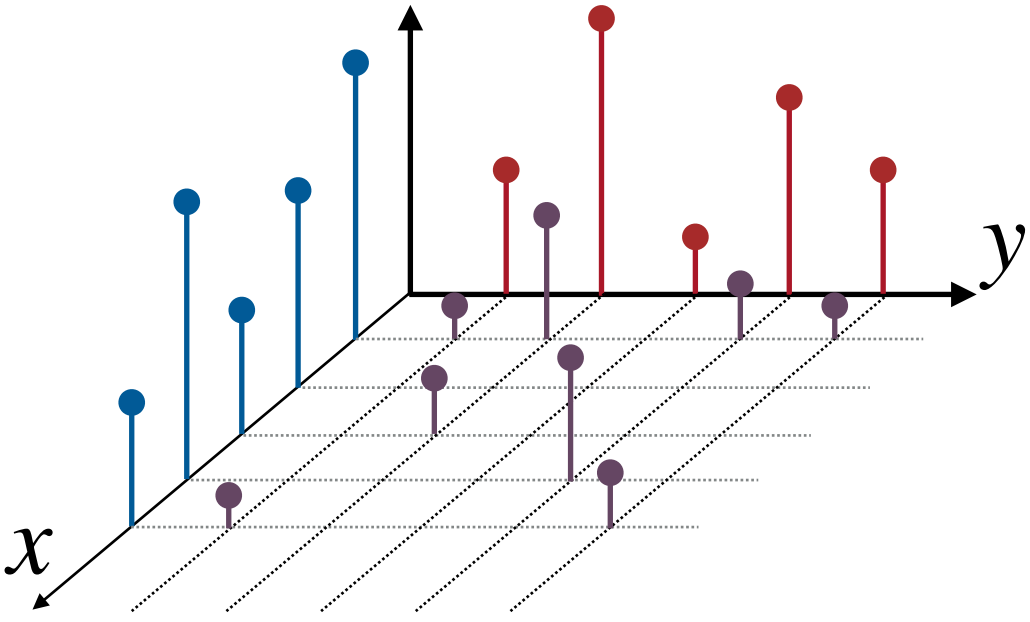


From mines to meadows

Optimal Transport

Discrete

continuous



$$\Gamma = \begin{array}{c} \begin{array}{c} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \end{array} \begin{array}{c} \begin{array}{c} b_1 \quad \cdots \quad b_j \quad \cdots \quad b_m \end{array} \\ \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} \end{array} \in \mathbb{R}^{n \times m}$$

The diagram illustrates a matrix Γ with dimensions $n \times m$. The rows are indexed by $a_1, \dots, a_i, \dots, a_n$ and the columns by $b_1, \dots, b_j, \dots, b_m$. A grid is shown with a large gray dot at the intersection of row a_i and column b_j . Smaller gray dots are located at the intersections of row a_1 and column b_m , row a_i and column b_1 , row a_i and column b_m , row a_n and column b_j , and row a_n and column b_2 .

$$\min_{\Gamma \in \Pi(\mathbf{a}, \mathbf{b})} \langle \Gamma, \mathbf{C} \rangle = \sum_{ij} \Gamma_{ij} c(x^{(i)}, y^{(j)})$$

$$\Pi(a,b) = \{\Gamma \mid \Gamma^{\top}1 = a, \Gamma 1 = b\}$$

Transpont Polytopre

Proriderm

constraints

Coupling

$$\min_{\gamma \in \Pi(a, \beta)} \int \mathcal{X} \times \mathcal{Y} \, c(x, y) d\gamma(x, Y)$$

Masurres

$$\Pi(\alpha,\beta)=\{\gamma\in\mathcal{P}(\mathcal{X}\times\mathcal{Y})\mid P_{x\#}\gamma=\alpha, P_{y\#}\gamma=\beta\}$$

$\alpha \in \mathcal{P}(\mathcal{P})$, $\beta \in \mathcal{P}(\mathcal{P})$

re x x

$$\alpha = \sum_i a_i \delta_{x^{(i)}}, \quad \sum_i a_i = 1$$

$$\beta = \sum_j b_j \delta_{y^{(i)}}, \quad \sum_j b_j = 1$$

Transport Coupling
Or "Plan"

Cost

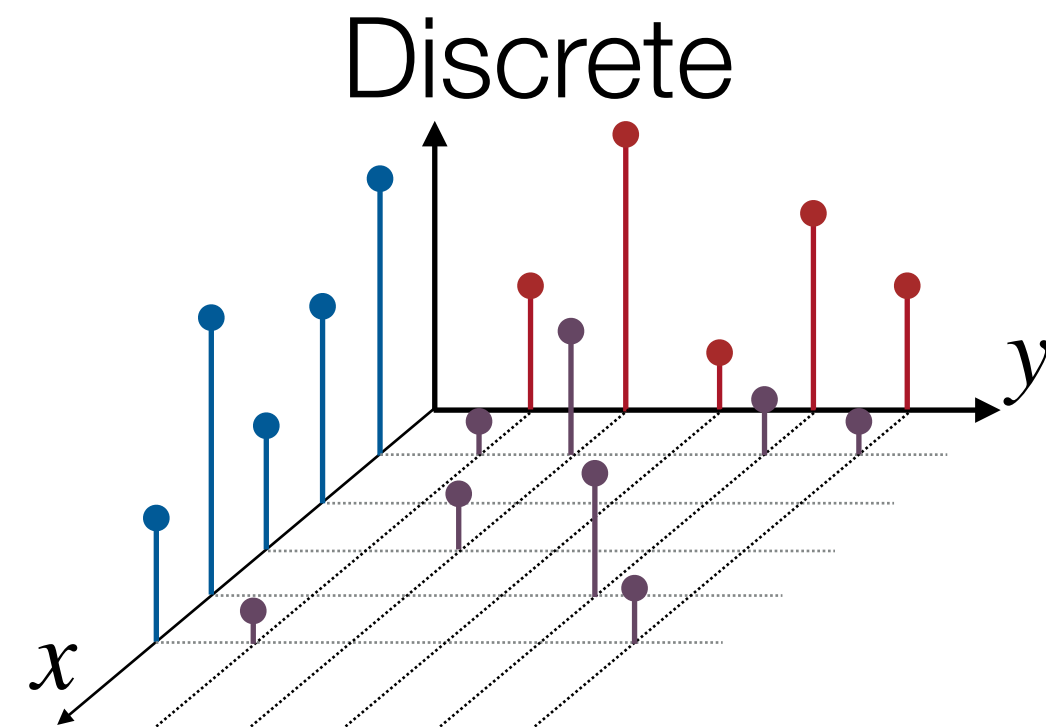
$$\mathbf{C} \in \mathbb{R}^{n \times m}, \quad \mathbf{C}_{ij} = c(x^{(i)}, y^{(j)})$$

$c(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}^+$



Optimal Transport

From mines to measures



$$\alpha = \sum_i a_i \delta_{x^{(i)}}, \quad \sum_i a_i = 1$$

$$\beta = \sum_j b_j \delta_{y^{(j)}}, \quad \sum_j b_j = 1$$

$$\mathbf{C} \in \mathbb{R}^{n \times m}, \quad \mathbf{C}_{ij} = c(x^{(i)}, y^{(j)})$$

$$\Gamma = \begin{matrix} & \begin{matrix} b_1 & \dots & b_j & \dots & b_m \end{matrix} \\ \begin{matrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{matrix} & \begin{matrix} \bullet & & \bullet & & \bullet \\ \vdots & & \vdots & & \vdots \\ \bullet & & \bullet & & \bullet \\ \vdots & & \vdots & & \vdots \\ \bullet & & \bullet & & \bullet \end{matrix} \end{matrix} \in \mathbb{R}^{n \times m}$$

Transport Coupling
Or "Plan"

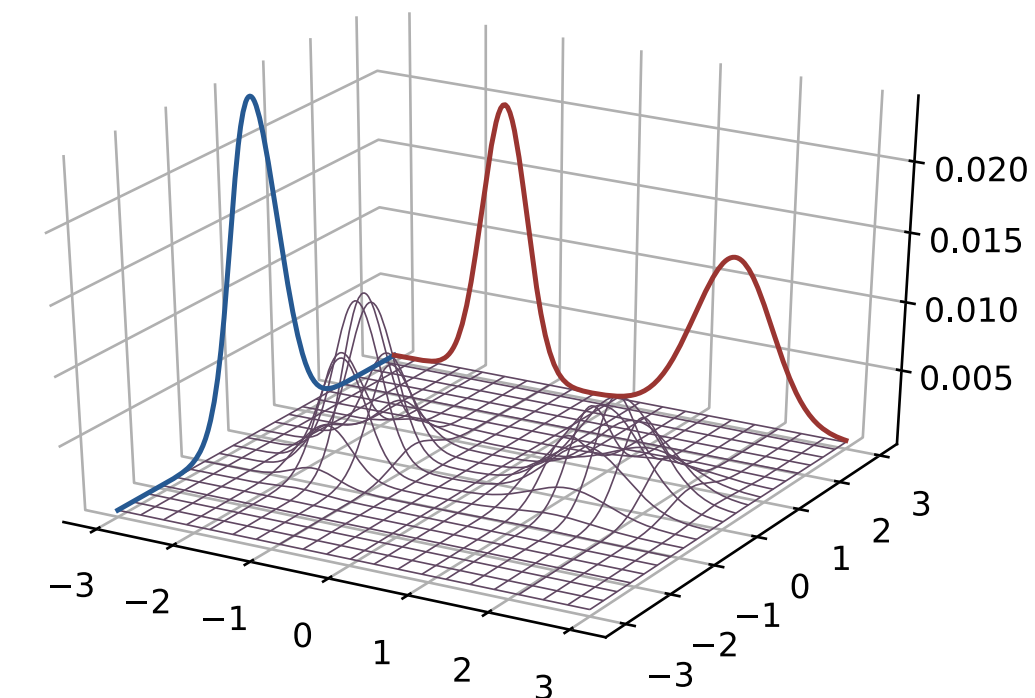
$$\min_{\Gamma \in \Pi(\mathbf{a}, \mathbf{b})} \langle \Gamma, \mathbf{C} \rangle = \sum_{ij} \Gamma_{ij} c(x^{(i)}, y^{(j)})$$

$$\Pi(\mathbf{a}, \mathbf{b}) = \{\Gamma \mid \Gamma^\top \mathbf{1} = \mathbf{a}, \Gamma \mathbf{1} = \mathbf{b}\}$$

Transport Polytope



Continuous



$$\alpha \in \mathcal{P}(\mathcal{X}), \beta \in \mathcal{P}(\mathcal{Y})$$

$$c(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}^+$$

$$\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$$

$$\min_{\gamma \in \Pi(\alpha, \beta)} \int_{\mathcal{X} \times \mathcal{Y}} c(x, y) d\gamma(x, Y)$$

$$\Pi(\alpha, \beta) = \{\gamma \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \mid P_{x\#} \gamma = \alpha, P_{y\#} \gamma = \beta\}$$

Measures

Cost

Coupling

Problem

Constraints

Optimal Transport in Machine Learning