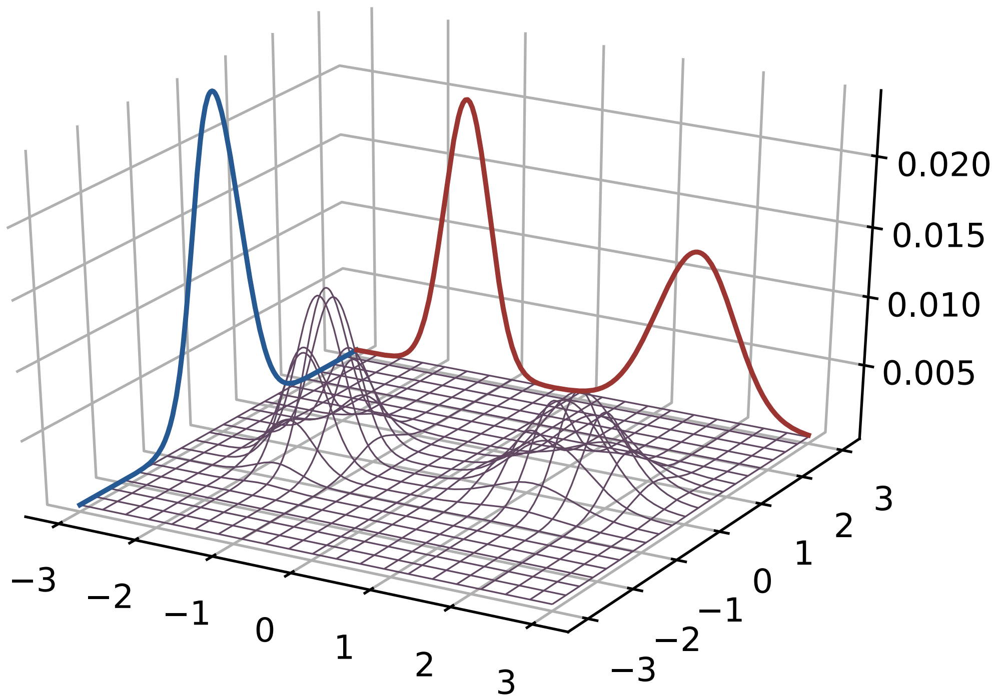


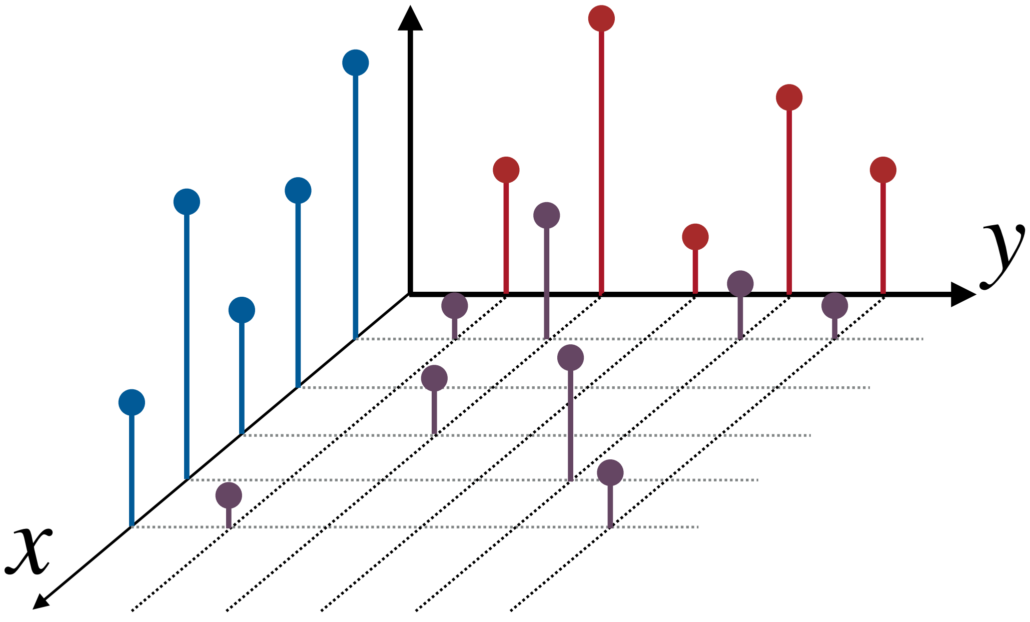
0.020
0.015
0.010
0.005
−3
−2
−1
0
1
2
3
3
2
1
0
−1
−2
−3

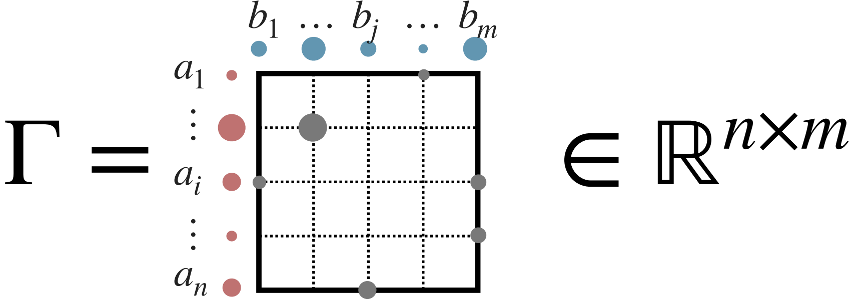
From mines to measures

Optimal Transport

Discrete

Continuous


x
y



$\Gamma =$
$b_1 \ \dots \ b_j \ \dots \ b_m$
a_1
a_i
a_n
$\in \mathbb{R}^{n\times m}$

$$\min_{\Gamma\in\Pi(\mathbf{a},\mathbf{b})} \langle \Gamma, \mathbf{C} \rangle = \sum_{ij} \Gamma_{ij}\, c(x^{(i)}, y^{(j)})$$

$$\Pi(\mathbf{a}, \mathbf{b}) = \{\Gamma \mid \Gamma^\top \mathbf{1} = \mathbf{a}, \Gamma \mathbf{1} = \mathbf{b}\}$$

Transport Polytope

Problem

Constraints

Coupling

$$\min_{\gamma \in \Pi(\alpha,\beta)} \int_{\mathcal{X}\times\mathcal{Y}} c(x, y) d\gamma(x, Y)$$

Measures

$$\Pi(\alpha, \beta) = \{\gamma \in \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \mid P_{\mathscr{X}\sharp}\gamma = \alpha, P_{\mathscr{Y}\sharp}\gamma = \beta\}$$

$$\alpha \in \mathscr{P}(\mathscr{X}), \beta \in \mathscr{P}(\mathscr{Y})$$

$$\gamma \in \mathscr{P}(\mathscr{X} \times \mathscr{Y})$$

$$\alpha = \sum_i a_i \delta_{x^{(i)}}, \quad \sum_i a_i = 1$$
$$\beta = \sum_j b_j \delta_{y^{(i)}}, \quad \sum_j b_j = 1$$

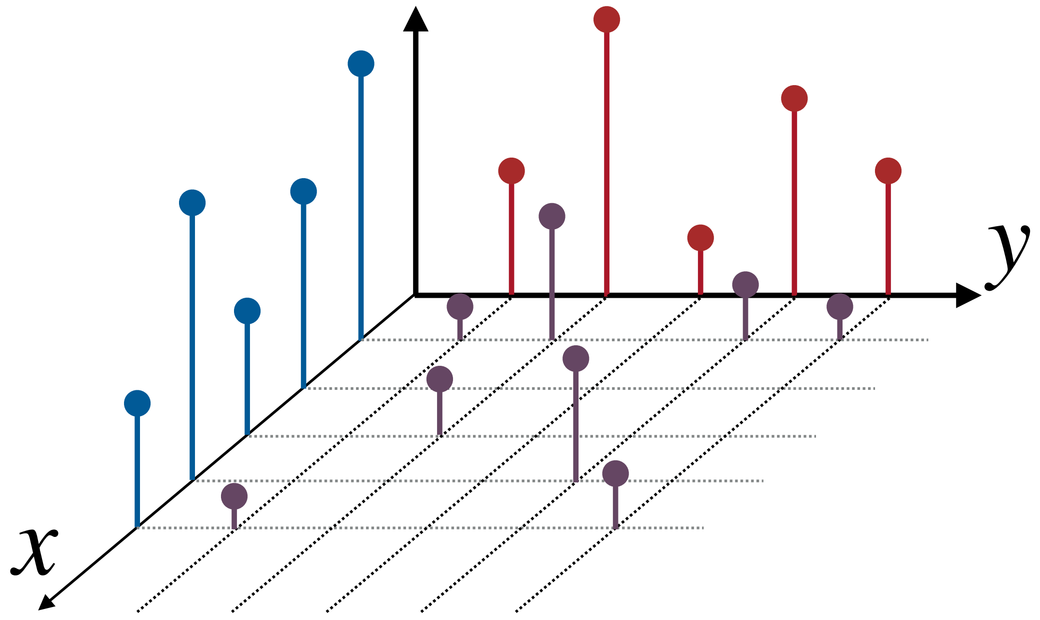
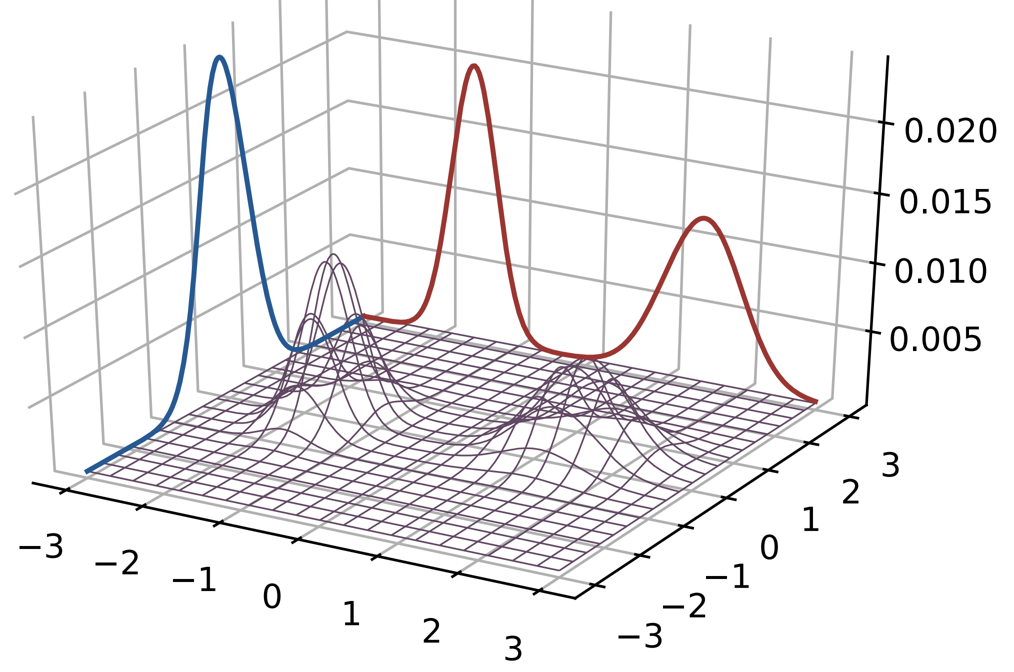
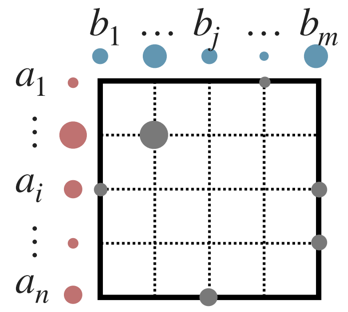
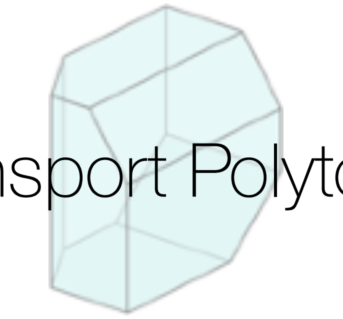
Transport Coupling Or "Plan"

Cost

$$\mathbf{C} \in \mathbb{R}^{n \times m}, \quad \mathbf{C}_{ij} = c(x^{(i)}, y^{(j)})$$

$$c(\cdot,\cdot):\mathscr{X}\times\mathscr{Y}\to\mathbb{R}^+$$

Optimal Transport
From mines to measures

	Discrete		Continuous
			
Measures	$\alpha = \sum_i a_i \delta_{x^{(i)}}, \quad \sum_i a_i = 1$ $\beta = \sum_j b_j \delta_{y^{(i)}}, \quad \sum_j b_j = 1$		$\alpha \in \mathscr{P}(\mathscr{X}), \beta \in \mathscr{P}(\mathscr{Y})$
Cost	$\mathbf{C} \in \mathbb{R}^{n \times m}, \quad \mathbf{C}_{ij} = c(x^{(i)}, y^{(j)})$		$c(\cdot\,,\cdot) : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}^+$
Coupling	$\Gamma =$  $\in \mathbb{R}^{n \times m}$	Transport Coupling Or "Plan"	$\gamma \in \mathscr{P}(\mathscr{X} \times \mathscr{Y})$
Problem	$\min_{\Gamma \in \Pi(\mathbf{a},\mathbf{b})} \langle \Gamma, \mathbf{C} \rangle = \sum_{ij} \Gamma_{ij}\, c(x^{(i)}, y^{(j)})$		$\min_{\gamma \in \Pi(\alpha,\beta)} \int_{\mathscr{X} \times \mathscr{Y}} c(x,y) d\gamma(x, Y)$
Constraints	$\Pi(\mathbf{a},\mathbf{b}) = \{\Gamma \mid \Gamma^\top \mathbf{1} = \mathbf{a}, \Gamma \mathbf{1} = \mathbf{b}\}$	Transport Polytope	$\Pi(\alpha,\beta) = \{\gamma \in \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \mid P_{\mathscr{X}\sharp}\gamma = \alpha, P_{\mathscr{Y}\sharp}\gamma = \beta\}$

Optimal Transport
in Machine Learning