Relaxing the Objective

Structured OT

27

- So far: **deterministic** matches
- Want: **soft**, fractional assignments

?

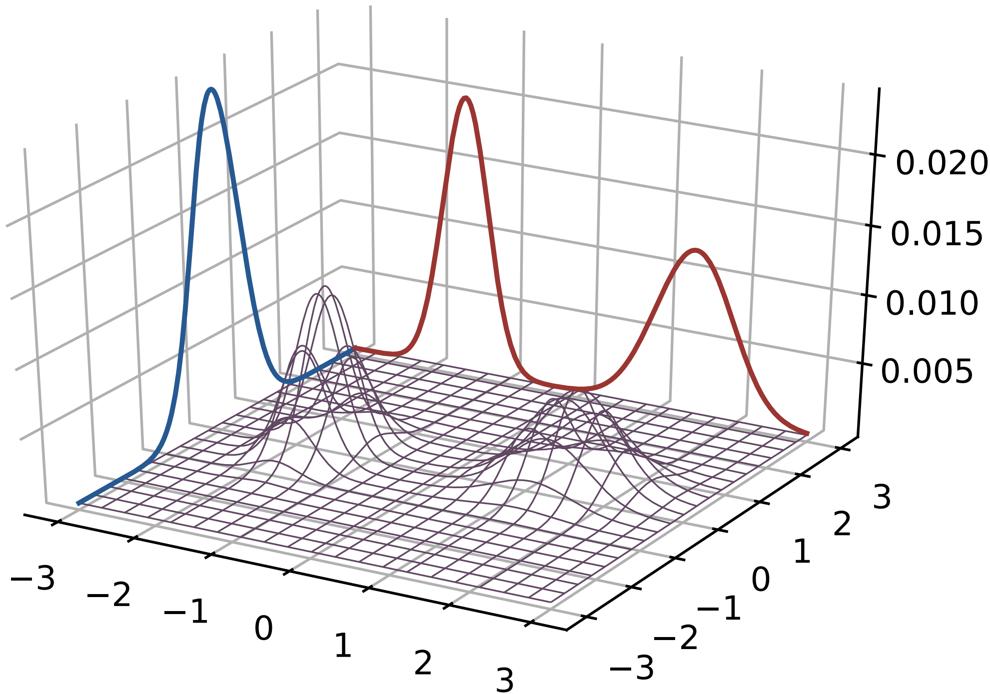
Submodular Optimal Transport Problem

[redacted]

$$\min_{\Gamma \in \Pi(\mathbf{a},\mathbf{b})} f(\Gamma)$$

$$\min_{\Gamma \in \Pi(\mathbf{a},\mathbf{b})} \max_{\kappa \in \mathscr{B}_F} \langle \Gamma, \kappa \rangle$$

$\longrightarrow$ Lovász Extension!!


0.020
0.015
0.010
0.005
−3
−2
−1
0
1
2
3
−3
−2
−1
0
1
2
3

Theorem. *If the singleton cost function is a metric, this yields a semi-metric between* **a** *and* **b**.

	Classic OT	Submodular OT

Relaxed Formulation

Strict Formulation

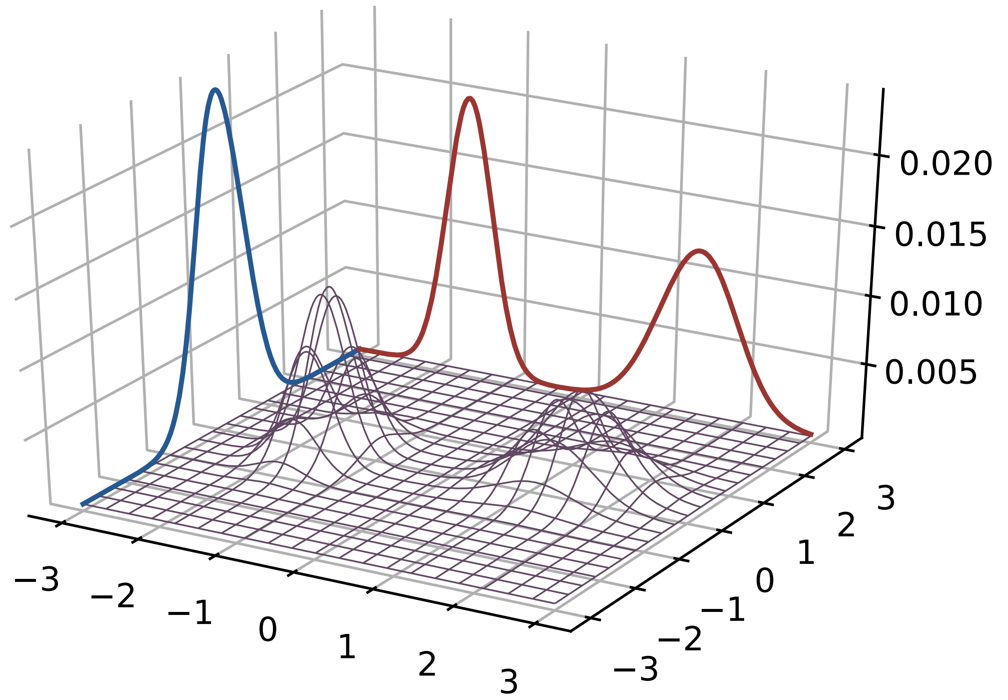
$$\min_{M} \sum_{(i,j)\in M} C_{ij} \qquad \min_{M} F(M)$$

$$\min_{\Gamma \in \Pi(\mathbf{a},\mathbf{b})} f(\Gamma)$$

$$\min_{\Gamma \in \Pi(\mathbf{a},\mathbf{b})} \sum_{(i,j)} C_{ij} \Gamma_{ij}$$

Structured OT
Relaxing the Objective

- So far: **deterministic** matches
- Want: **soft**, fractional assignments



	Classic OT	Submodular OT
Strict Formulation	$\min_{M} \sum_{(i,j)\in M} C_{ij}$	$\min_{M} F(M)$
Relaxed Formulation	$\min_{\Gamma\in\Pi(\mathbf{a},\mathbf{b})} \sum_{(i,j)} C_{ij}\Gamma_{ij}$	$\min_{\Gamma\in\Pi(\mathbf{a},\mathbf{b})} f(\Gamma)$ $\longrightarrow$ Lovász Extension!!

$$\boxed{\min_{\Gamma\in\Pi(\mathbf{a},\mathbf{b})} f(\Gamma) \equiv \min_{\Gamma\in\Pi(\mathbf{a},\mathbf{b})} \max_{\kappa\in\mathscr{B}_F} \langle \Gamma, \kappa \rangle}$$

Submodular Optimal Transport Problem

Theorem. *If the singleton cost function is a metric, this yields a semi-metric between **a** and **b**.*

Structured OT

Game Theoretic Interpretation

$$\min_{\Gamma \in \Pi} \max_{\kappa \in \mathscr{B}_F} \langle \Gamma, \kappa \rangle$$