Binomial Heaps

Last Week...

- Priority Queue Abstract Data Type

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  Object: Set S of elements with “keys” (“priority”) that can be compared

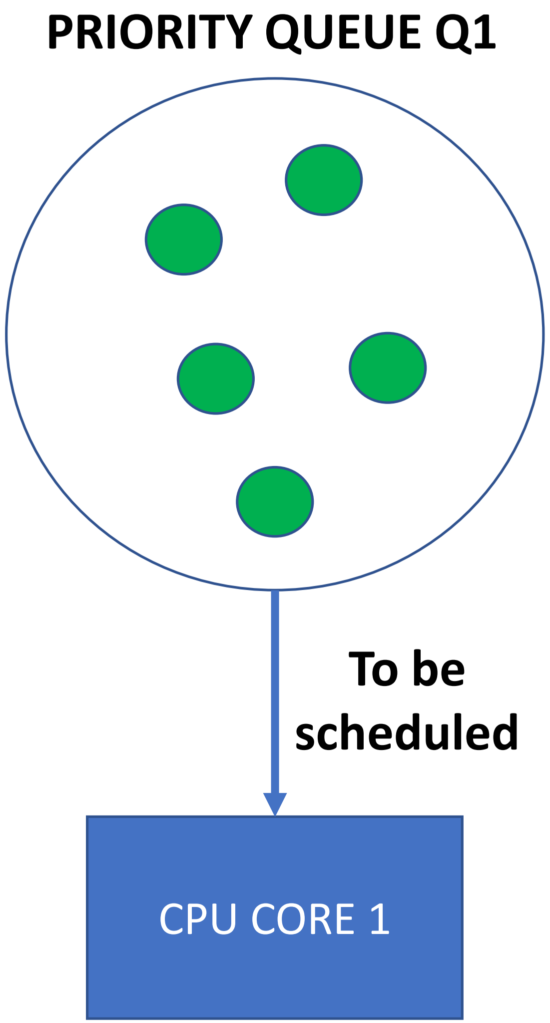
Last Week…

- Priority Queue Abstract Data Type

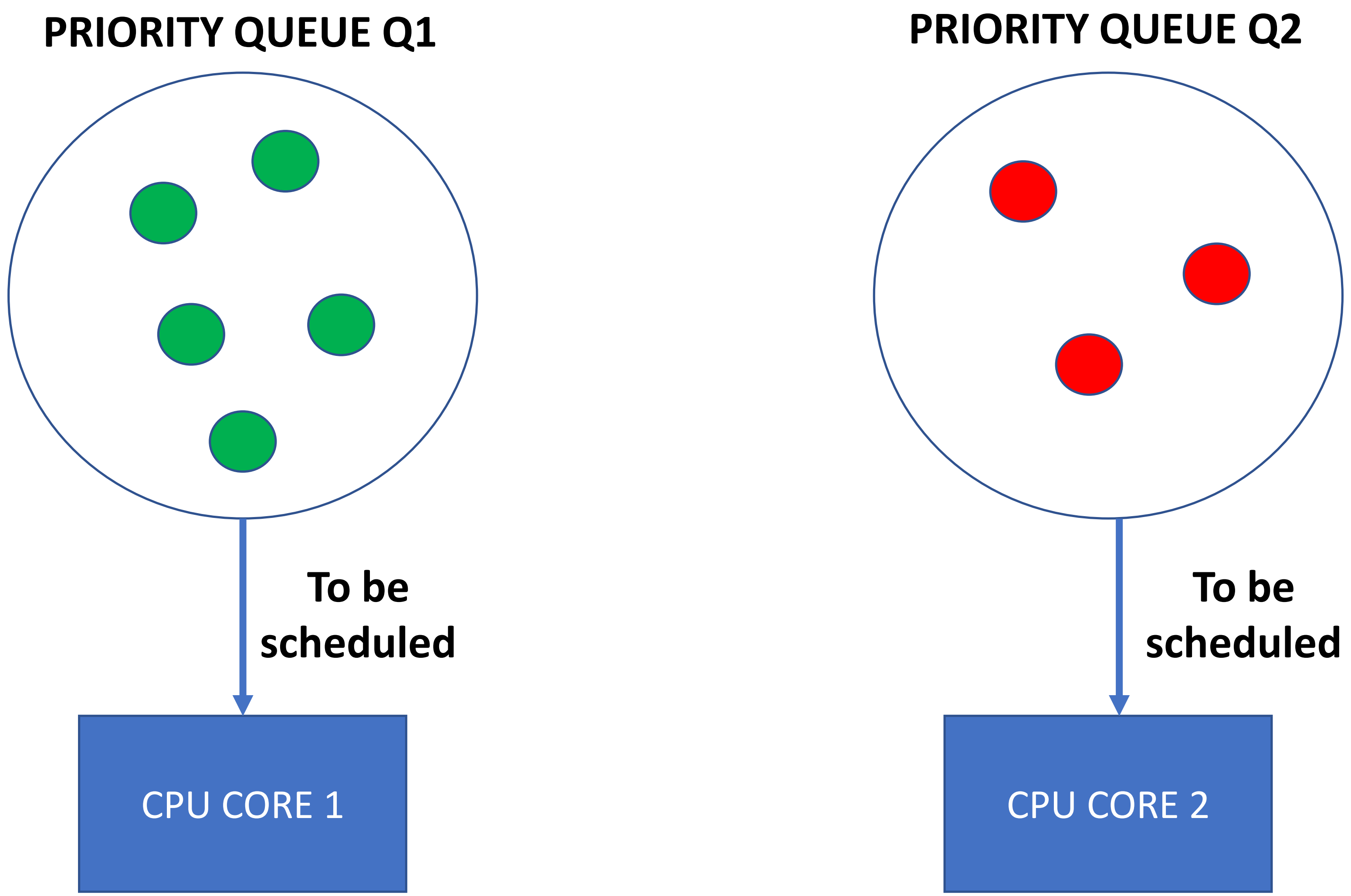
  Object: Set S of elements with “keys” (“priority”) that can be compared

  Operations: **Insert**(S,x), **Max**(S), **Extract_Max**(S)

Application: Job Scheduling by OS



Consider the following scenario



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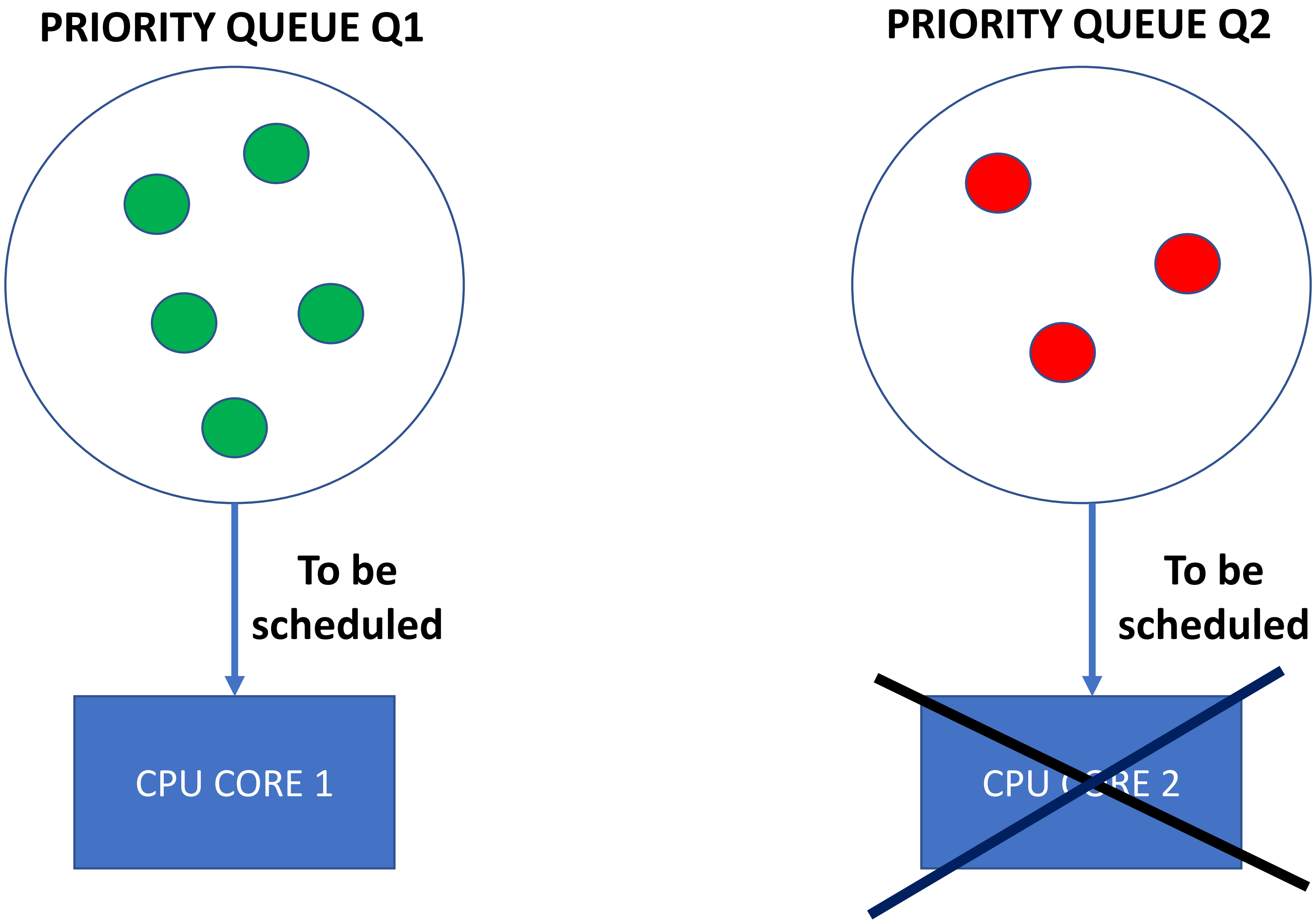

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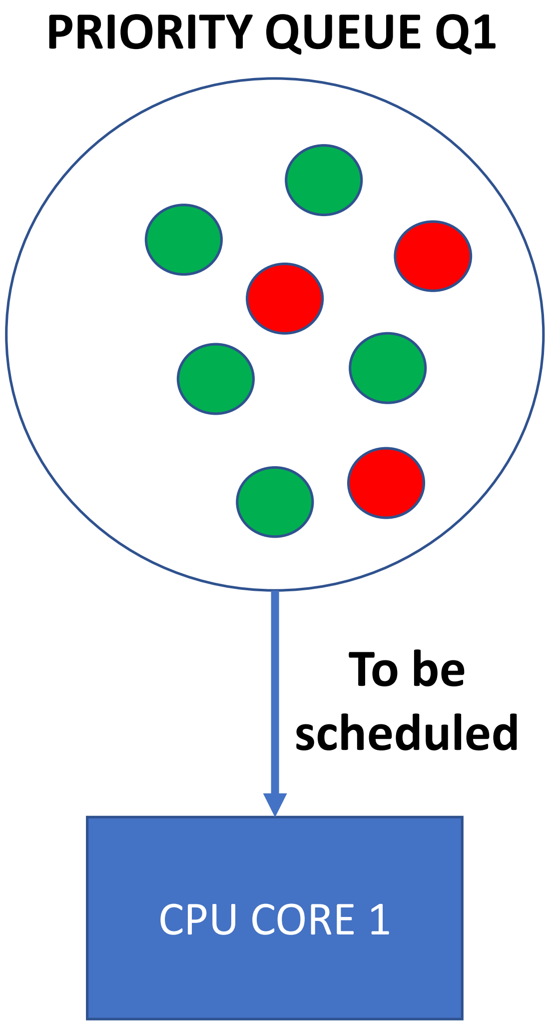

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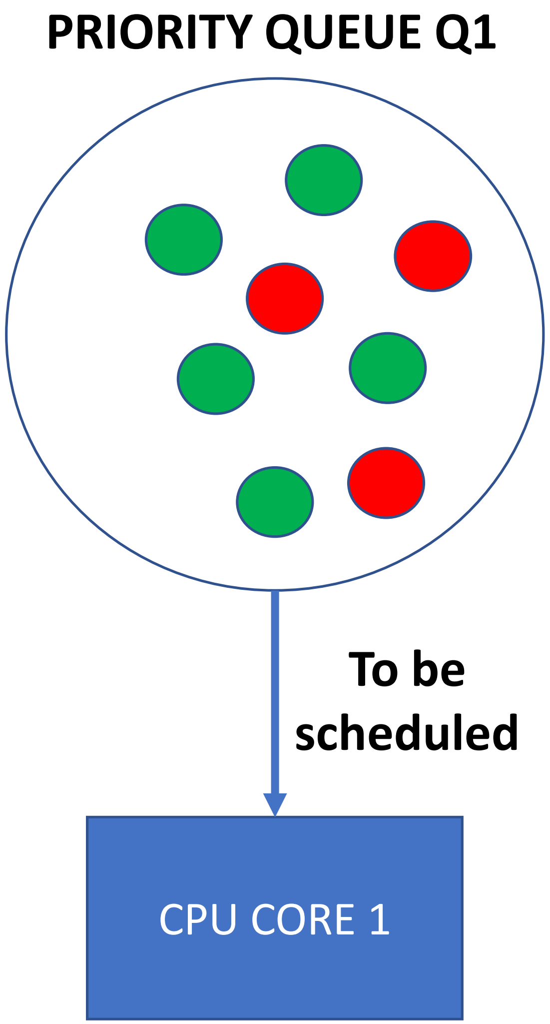

Goal: Implement **Union**(Q1, Q2)

Abstract Data Types

Abstract Data Types		Insert	Min	Extract_Min	Union
Priority Queues		✓	✓	✓	X

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Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	Min-Heap	✓	✓	✓	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

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Data Structures

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Data Structures

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Priority Queues	Min-Heap	$\Theta(\log n)$	$\Theta(1)$	$\Theta(\log n)$	X
Mergeable Priority Queues	Min Binomial Heap	✓	✓	✓	✓

Data Structures

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Priority Queues	Min-Heap	$\Theta(\log n)$	$\Theta(1)$	$\Theta(\log n)$	X
Mergeable Priority Queues	Min Binomial Heap	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$

Min Binomial Heaps

Visualizing Binomial Heaps

Elements are stored in a sequence of Binomial Trees.

Binomial Trees

B_k tree: defined recursively

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k = 0

Binomial Trees

B_k tree: defined recursively

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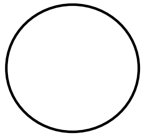

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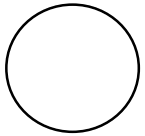

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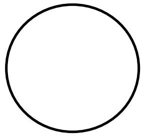

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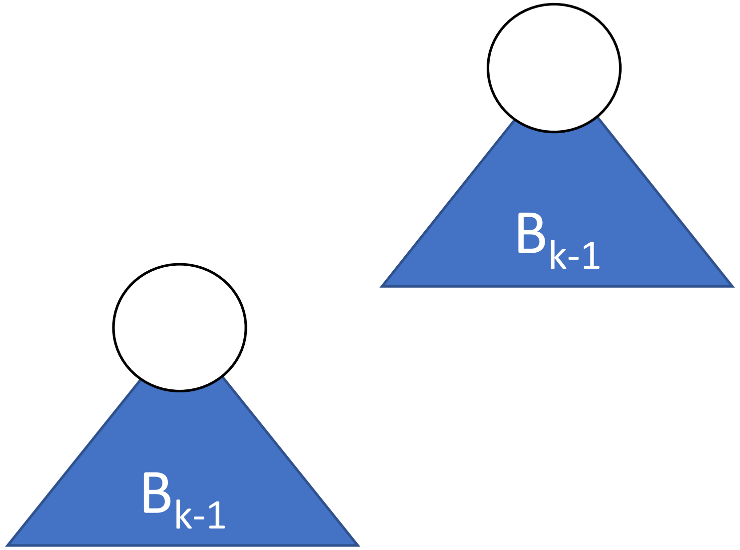

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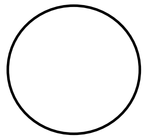
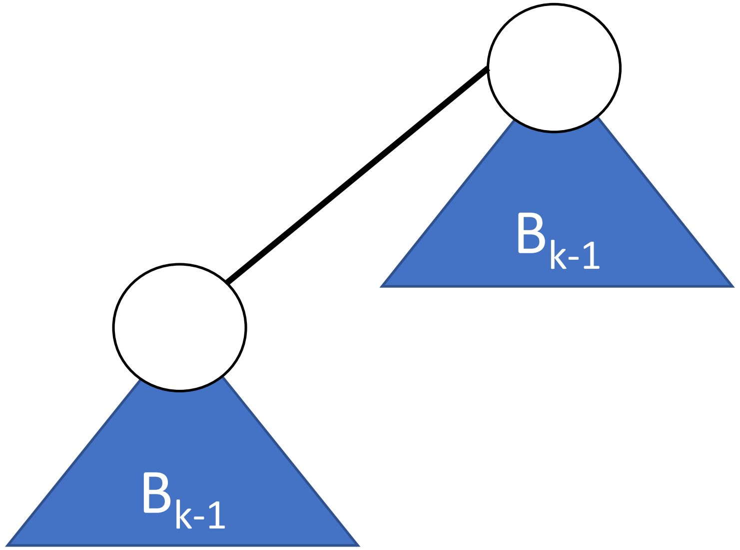


k > = 1 B_k :



Binomial Trees (for k = 0, 1, 2, 3)

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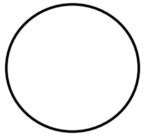

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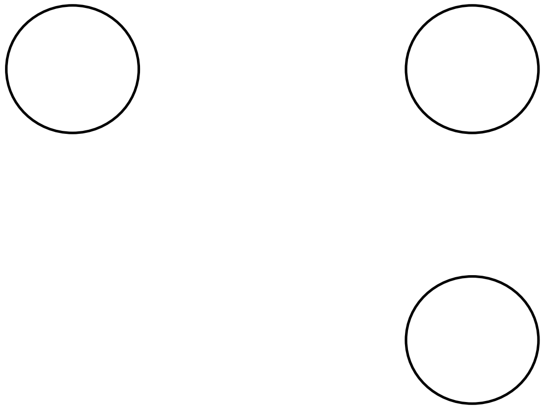

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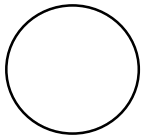
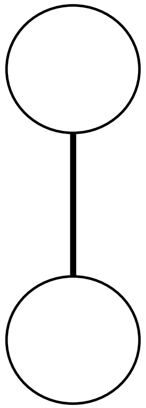


B_1



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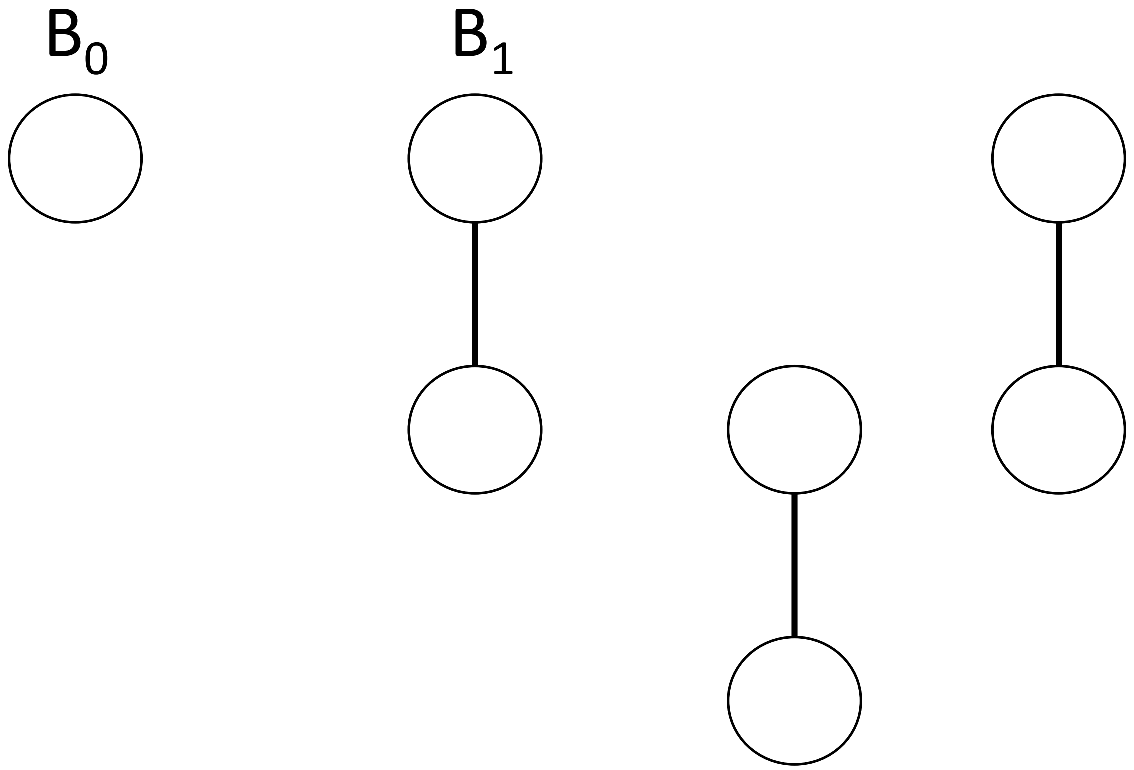

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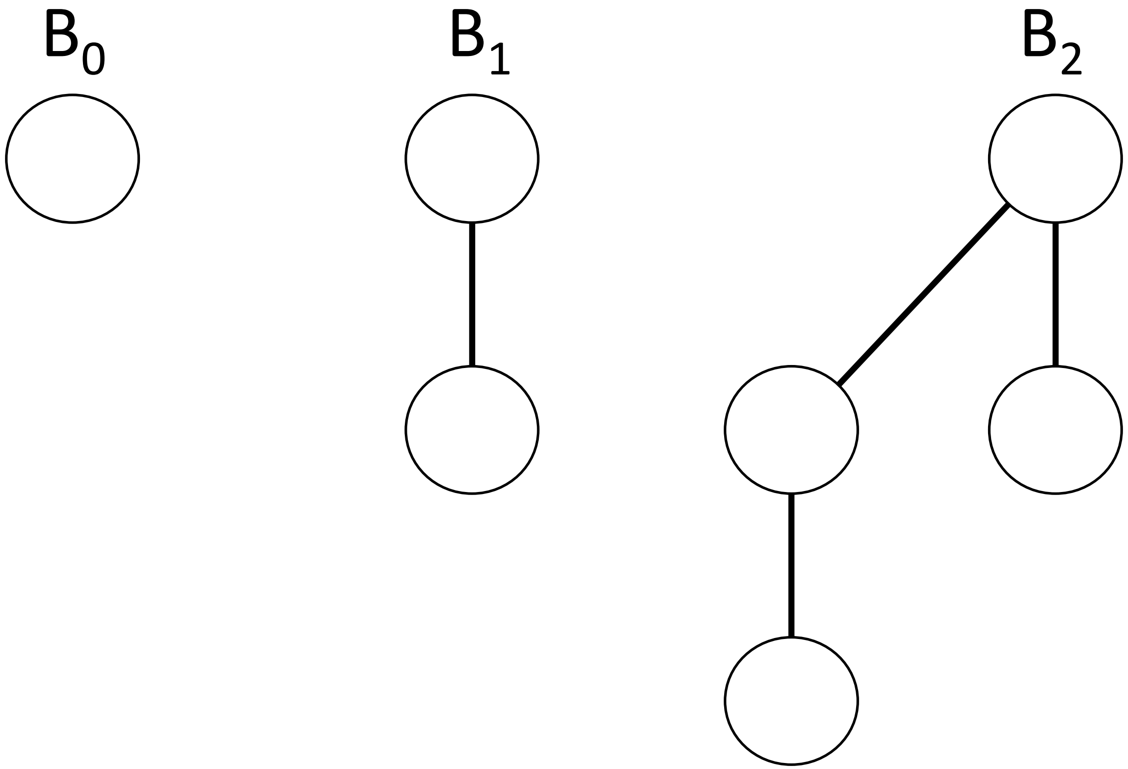

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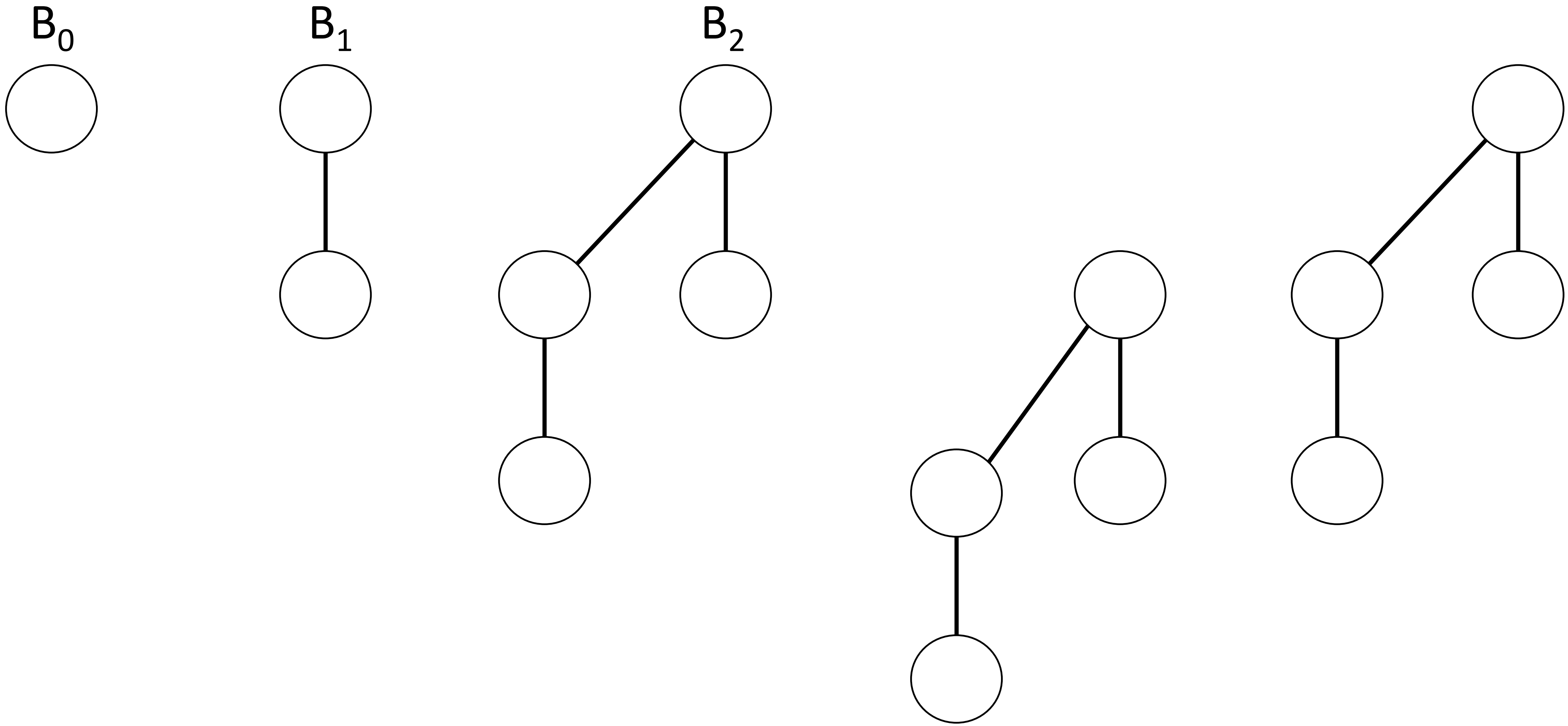

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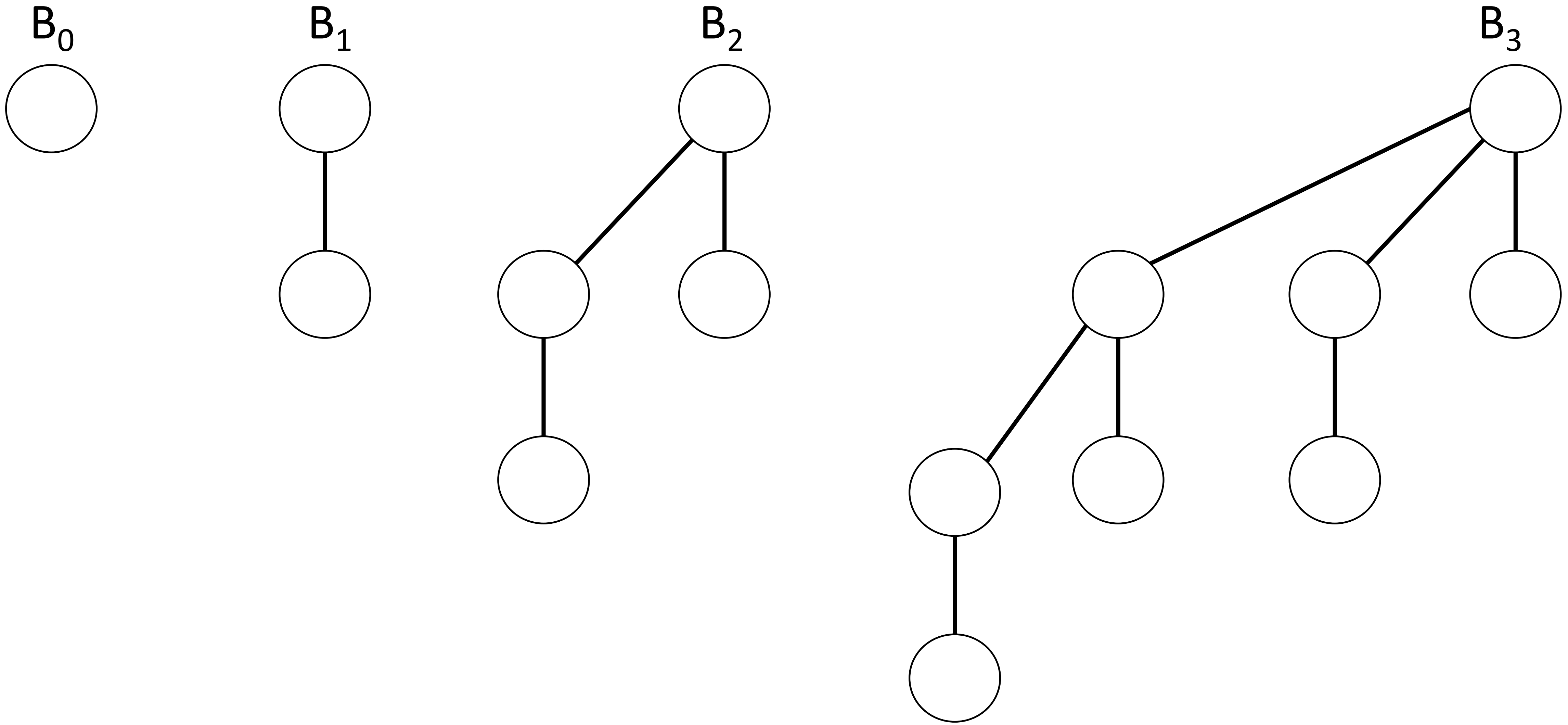
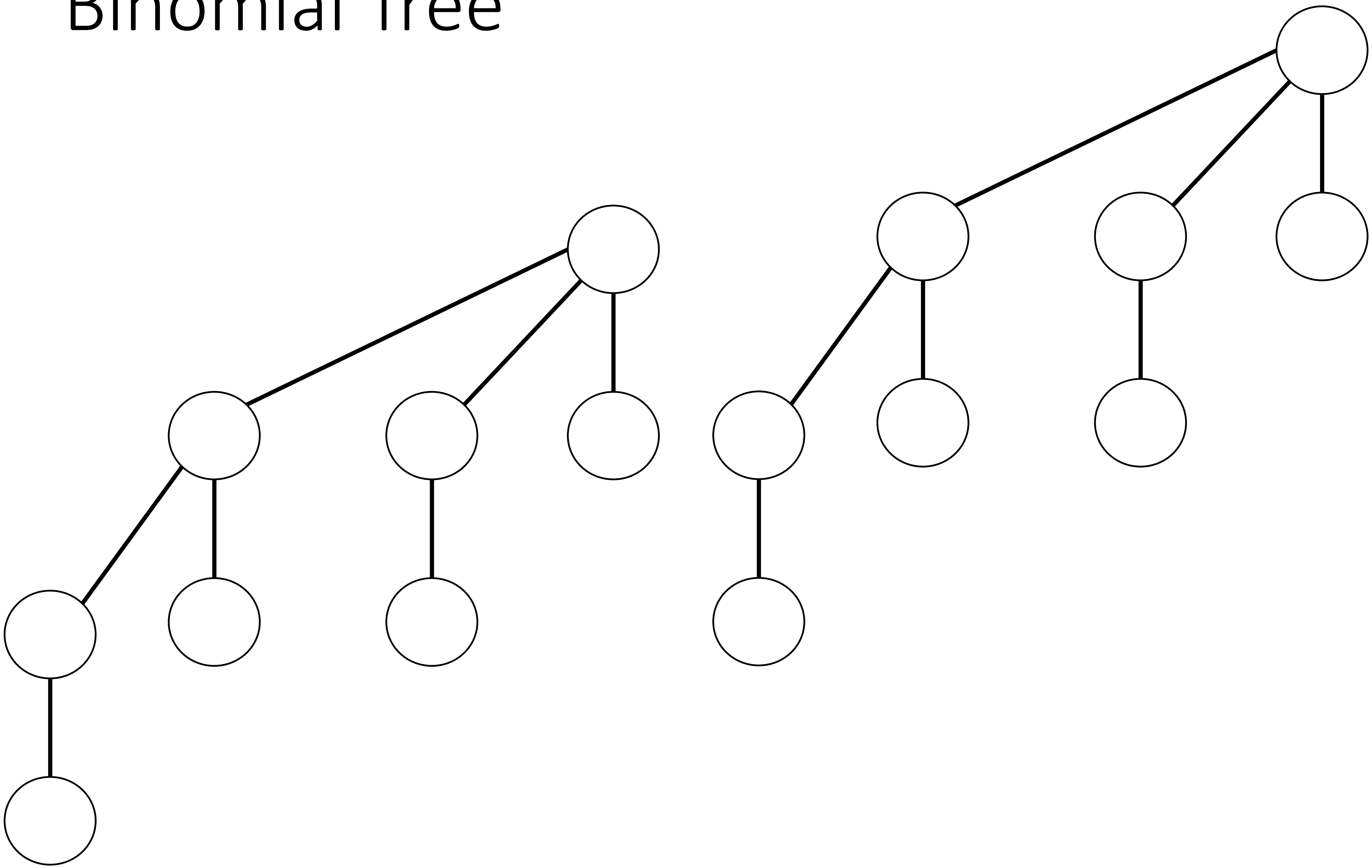
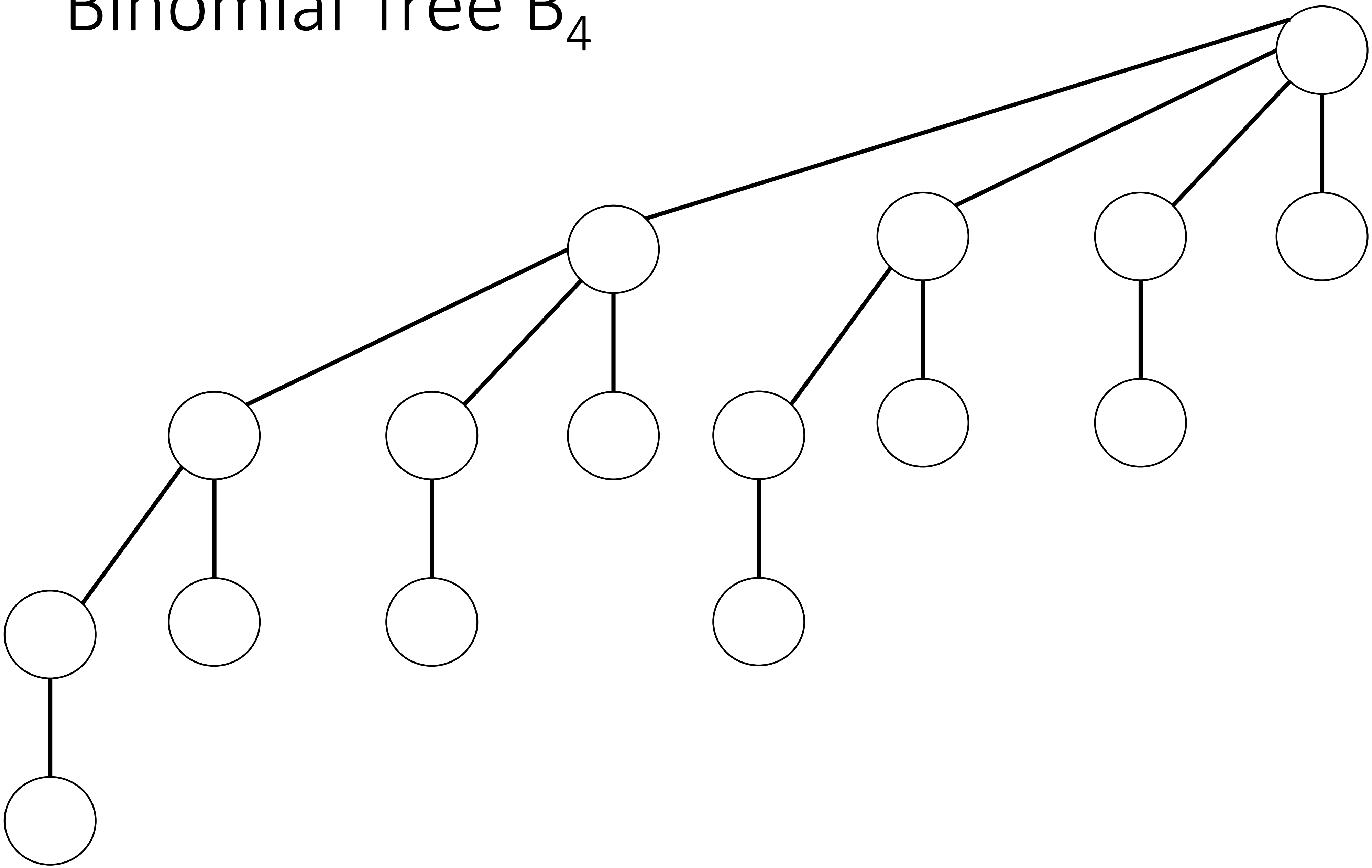


Binomial Tree



Binomial Tree B_4



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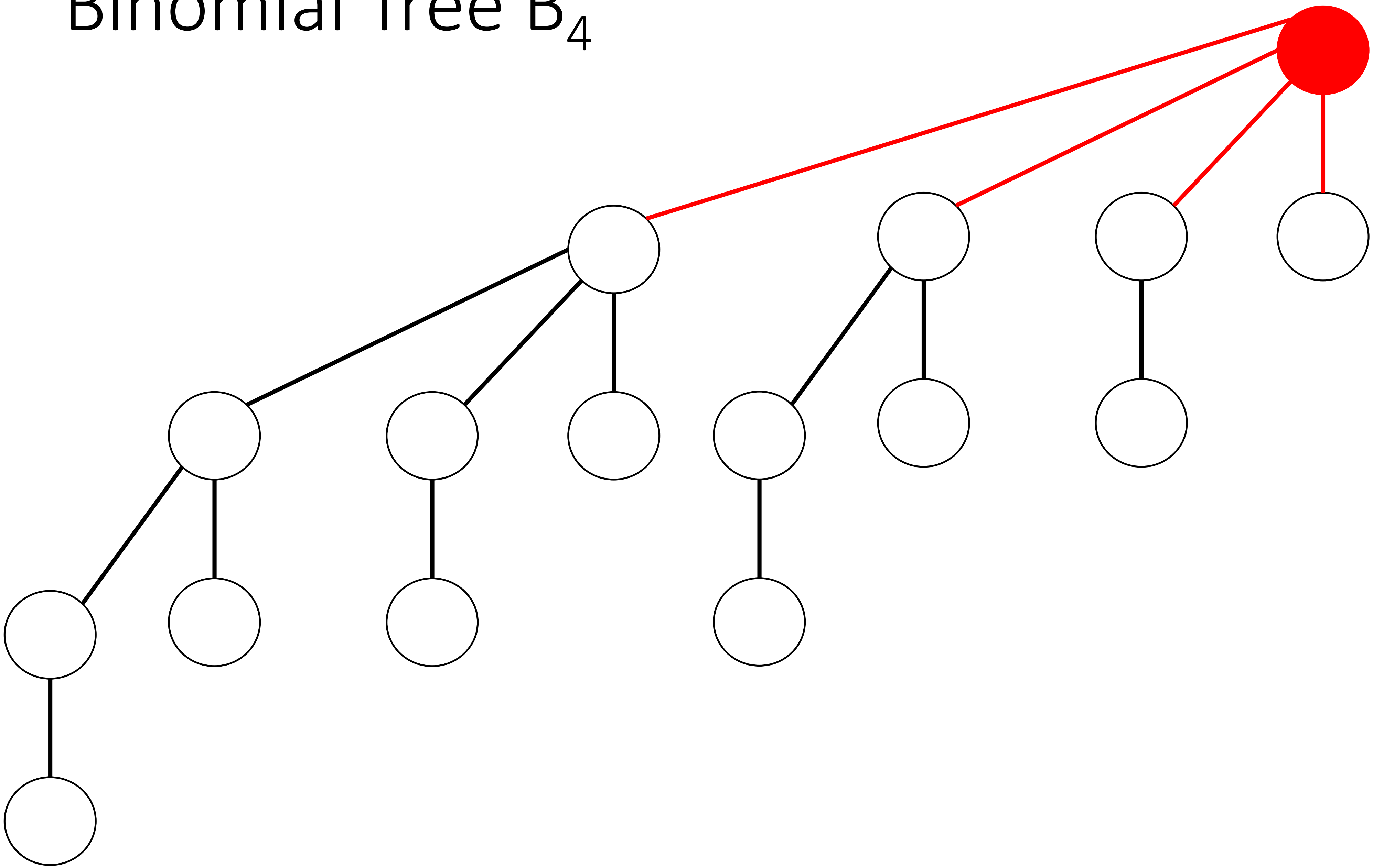

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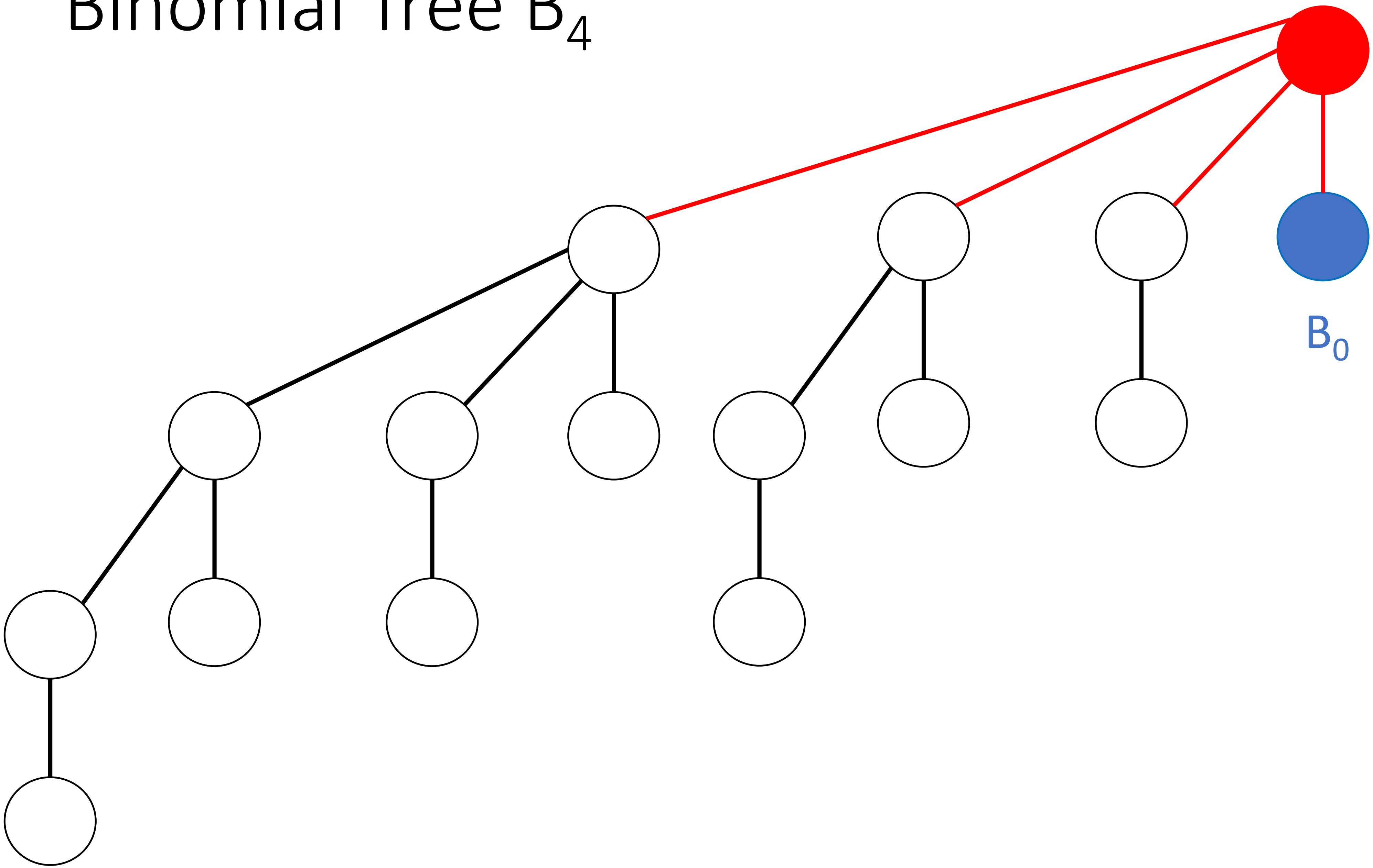

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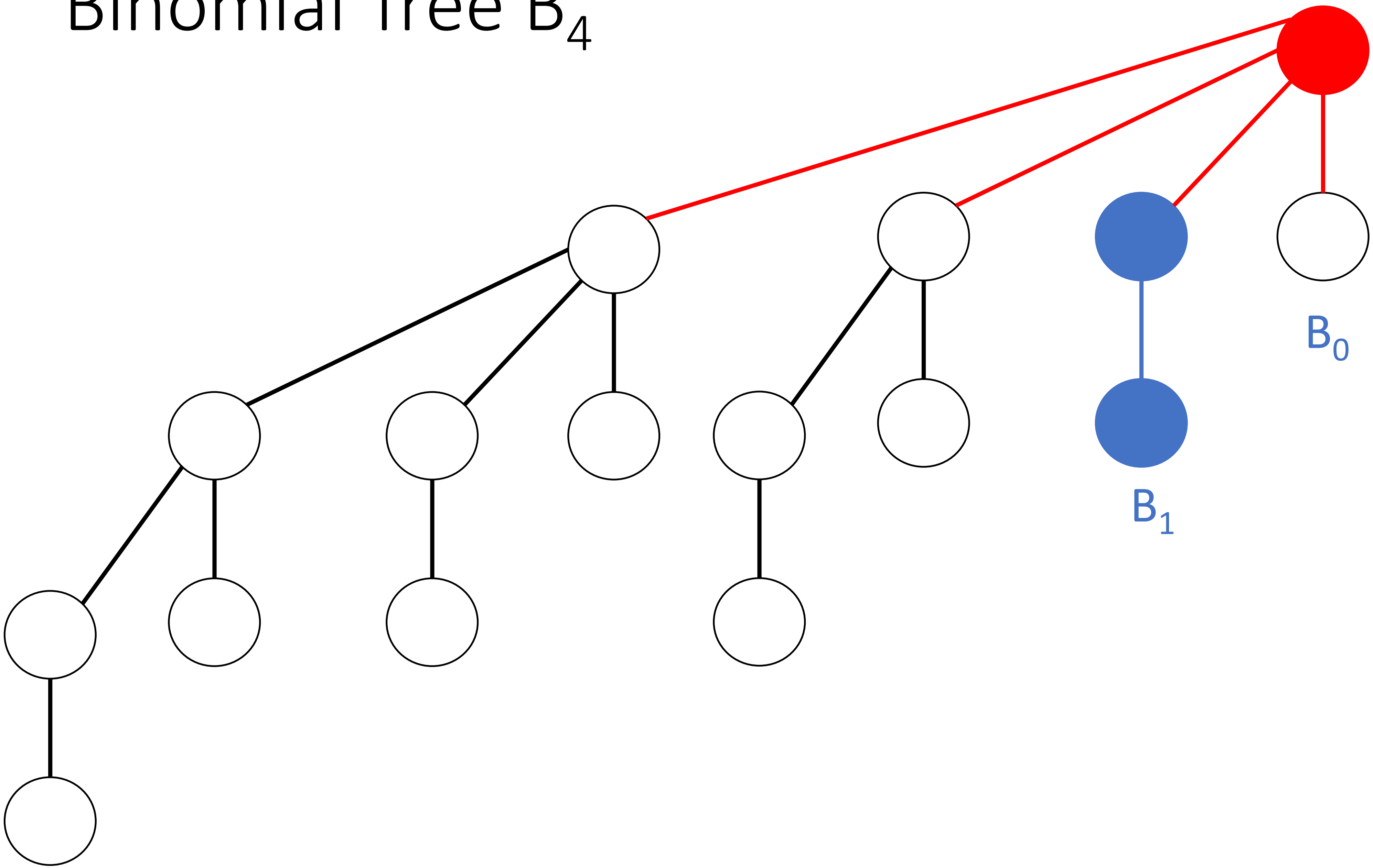

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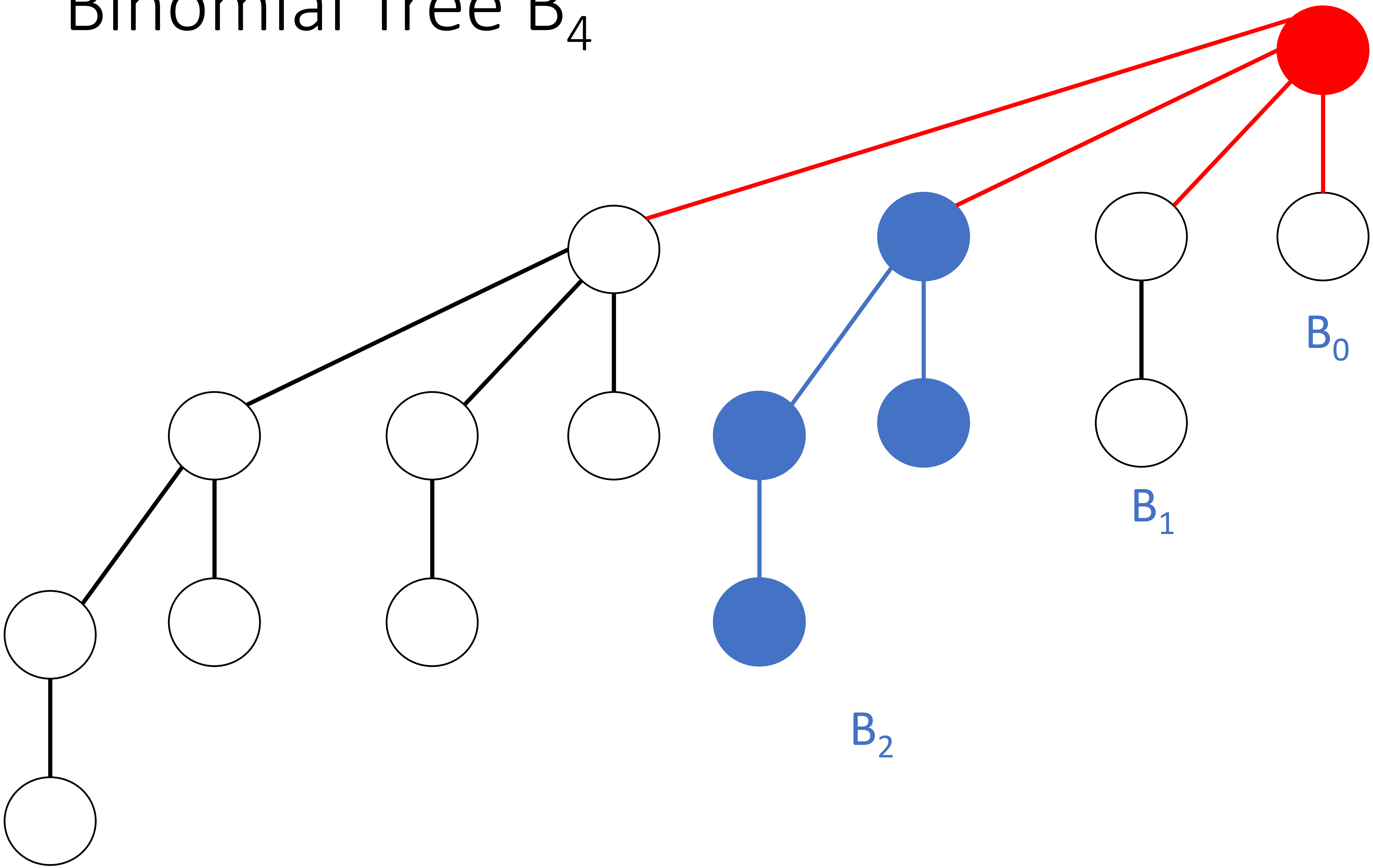

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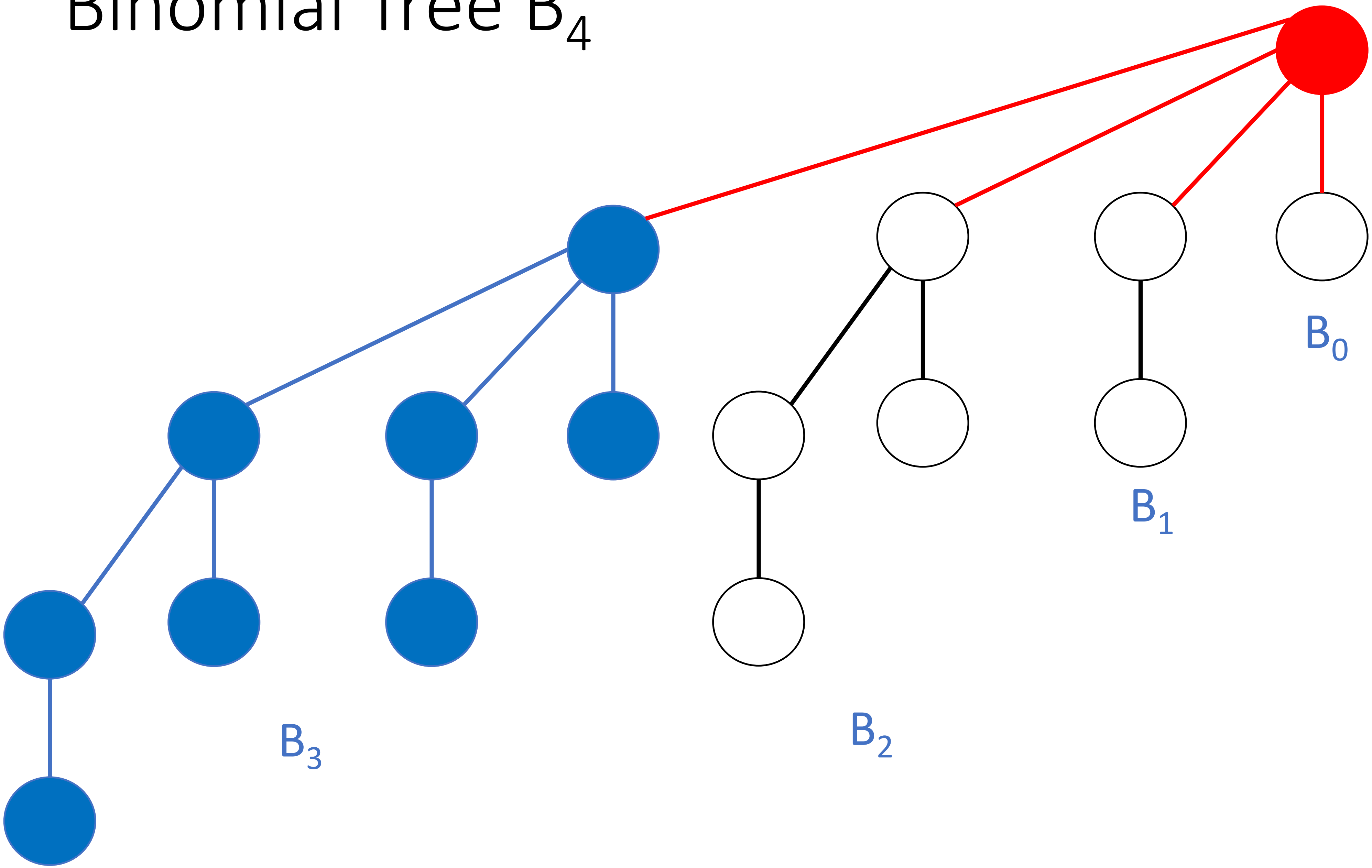
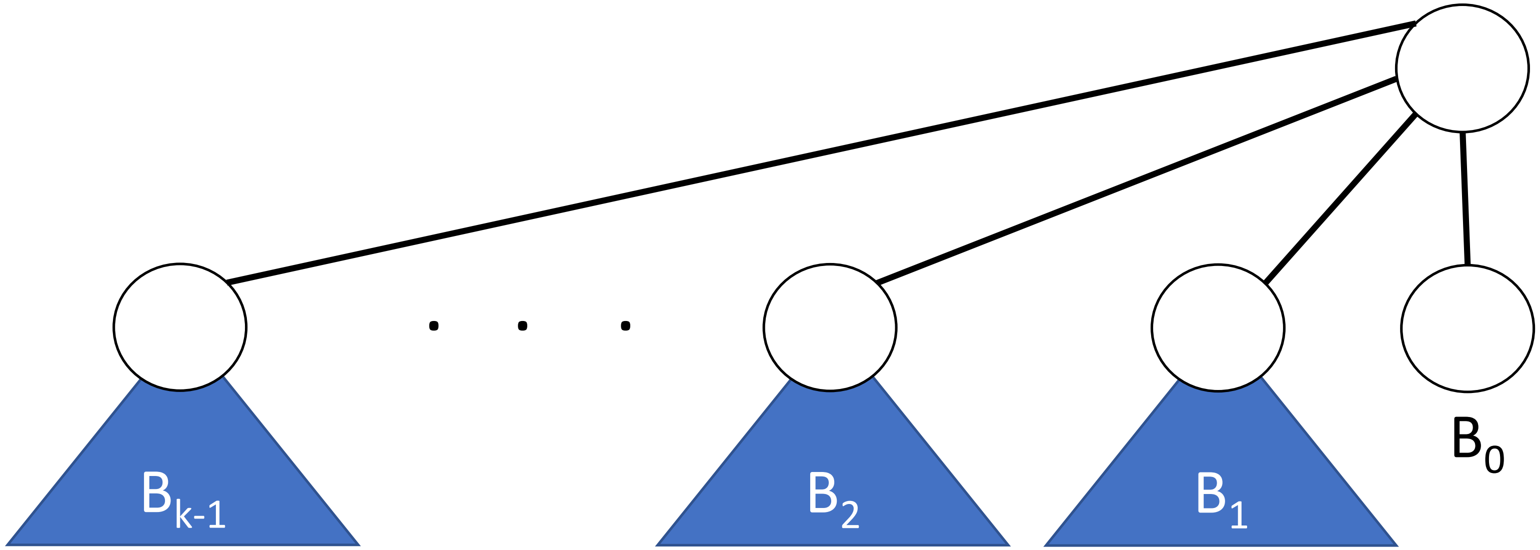
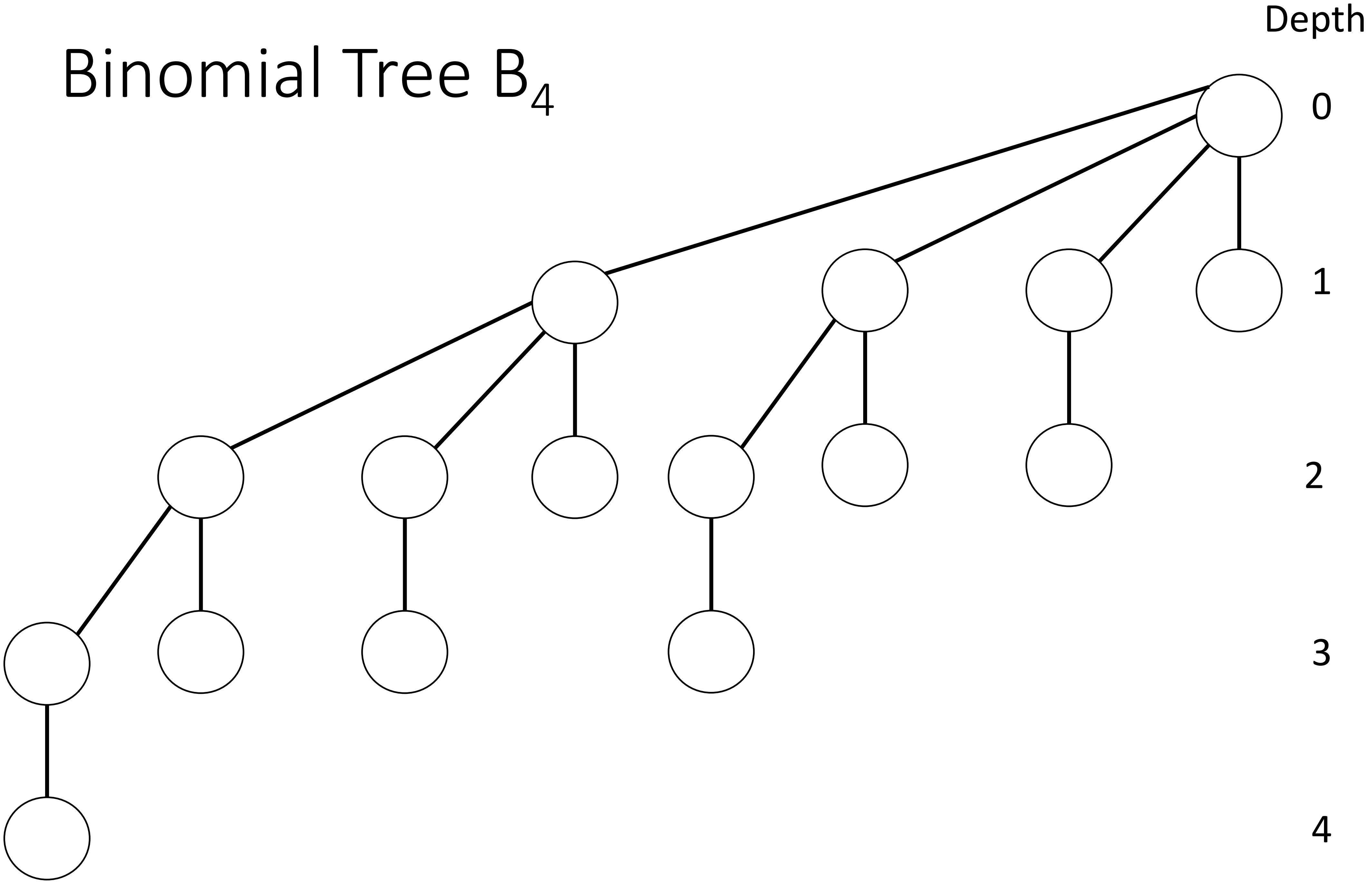


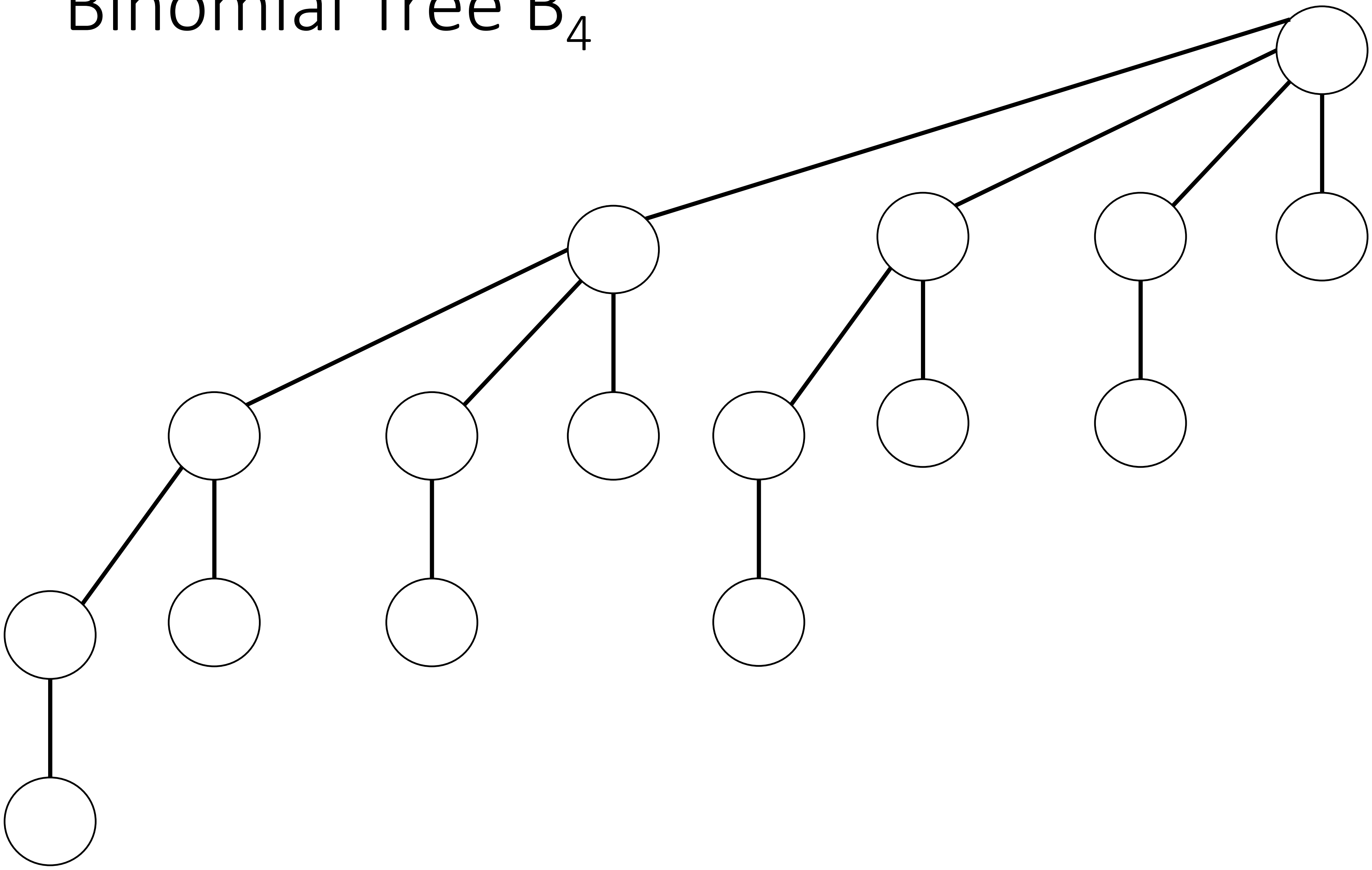
Binomial Tree B_k



Binomial Tree B_4

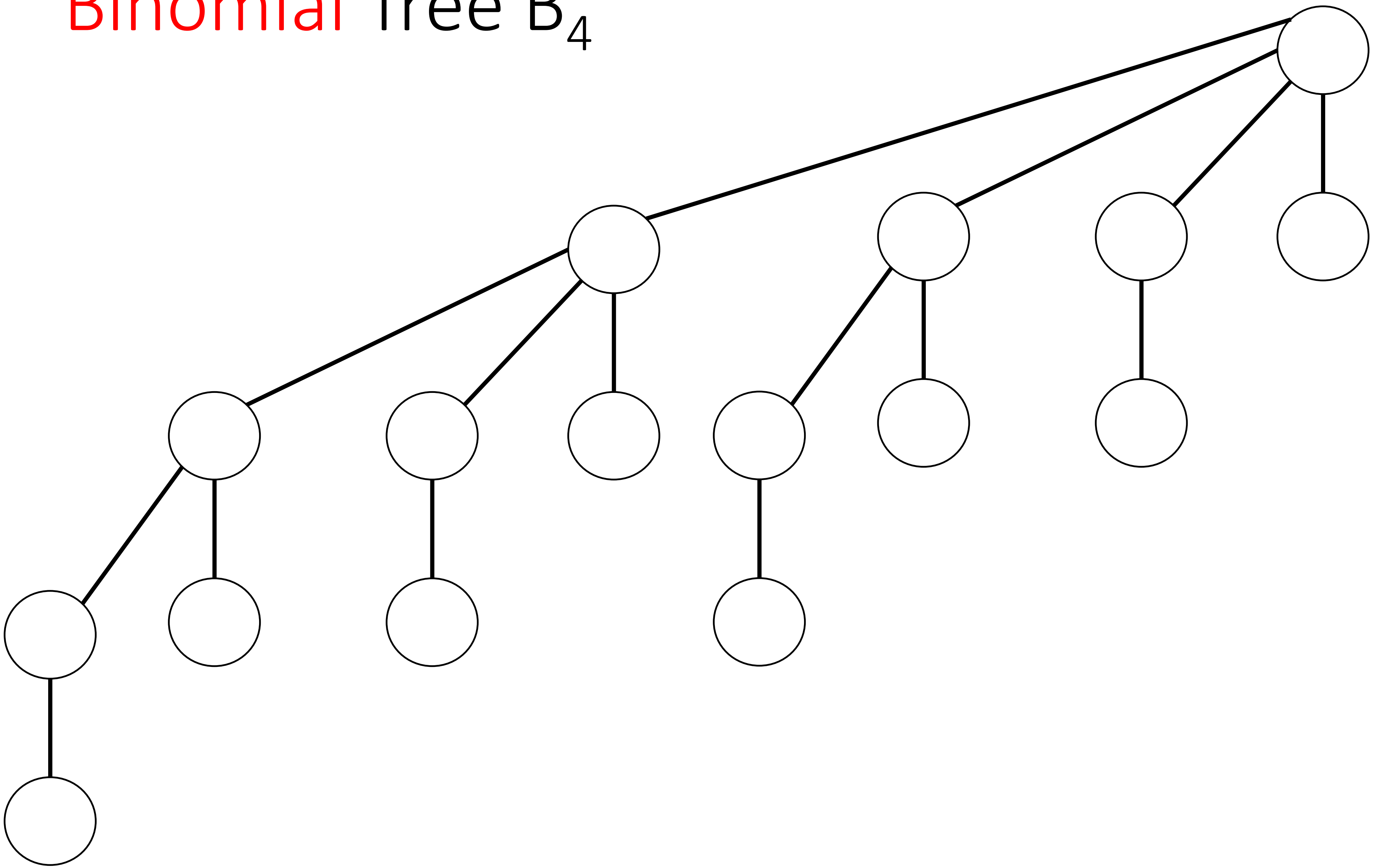


Binomial Tree B_4



Depth	# of nodes
0	1
1	4
2	6
3	4
4	1

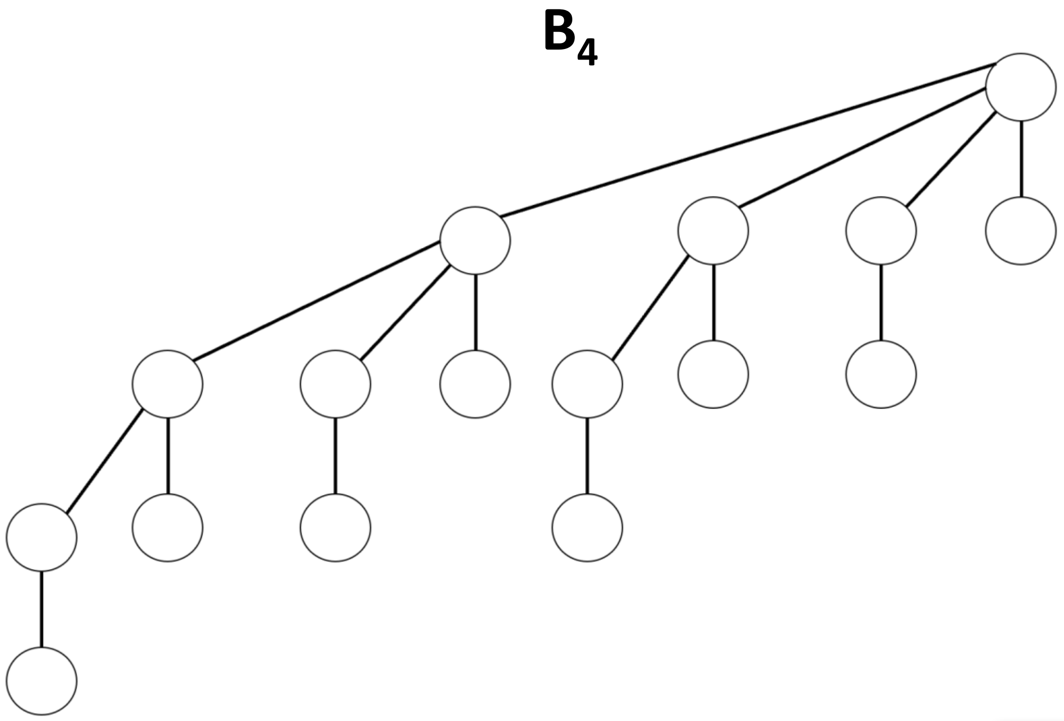
Binomial Tree B_4



Depth	# of nodes
0	$\binom{4}{0} = 1$
1	$\binom{4}{1} = 4$
2	$\binom{4}{2} = 6$
3	$\binom{4}{3} = 4$
4	$\binom{4}{4} = 1$

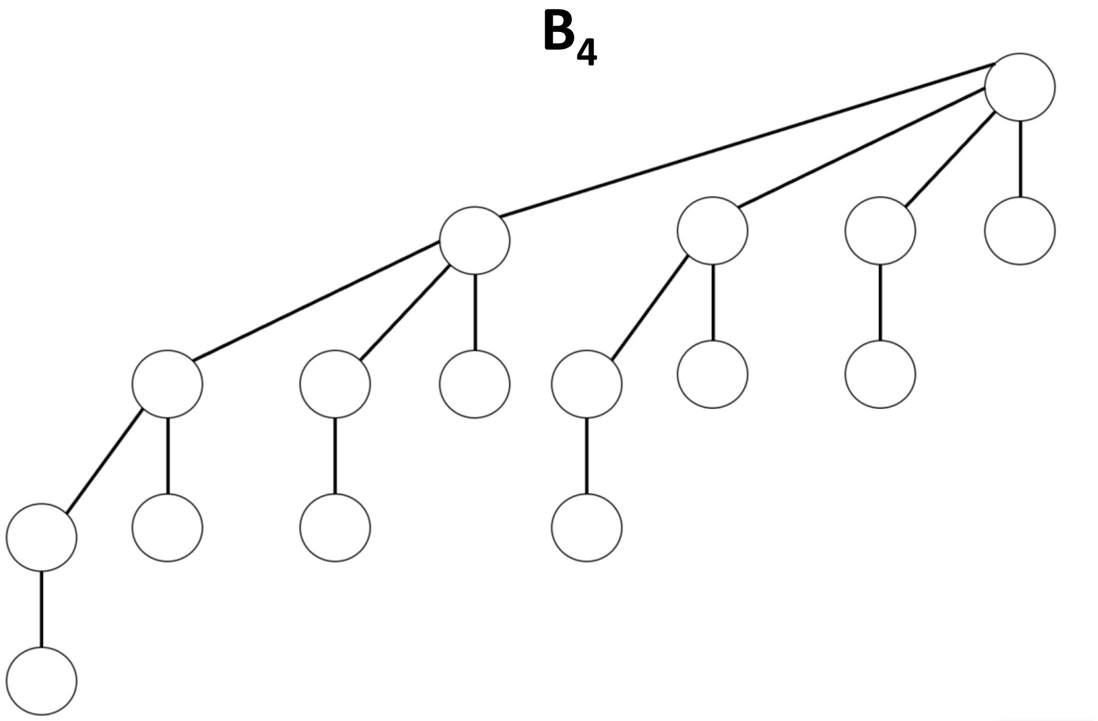
Binomial Tree properties

- B_k tree has:



Binomial Tree properties

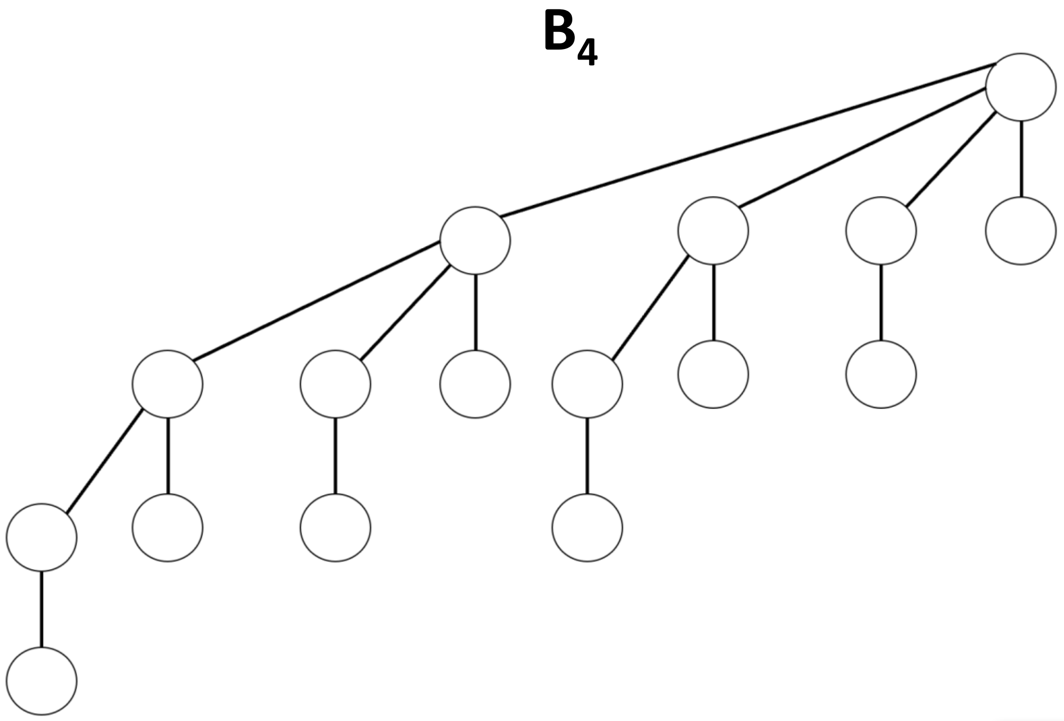
- B_k tree has:
  - Height k



Height = 4

Binomial Tree properties

- B_k tree has:
  - Height k
  - 2^k nodes

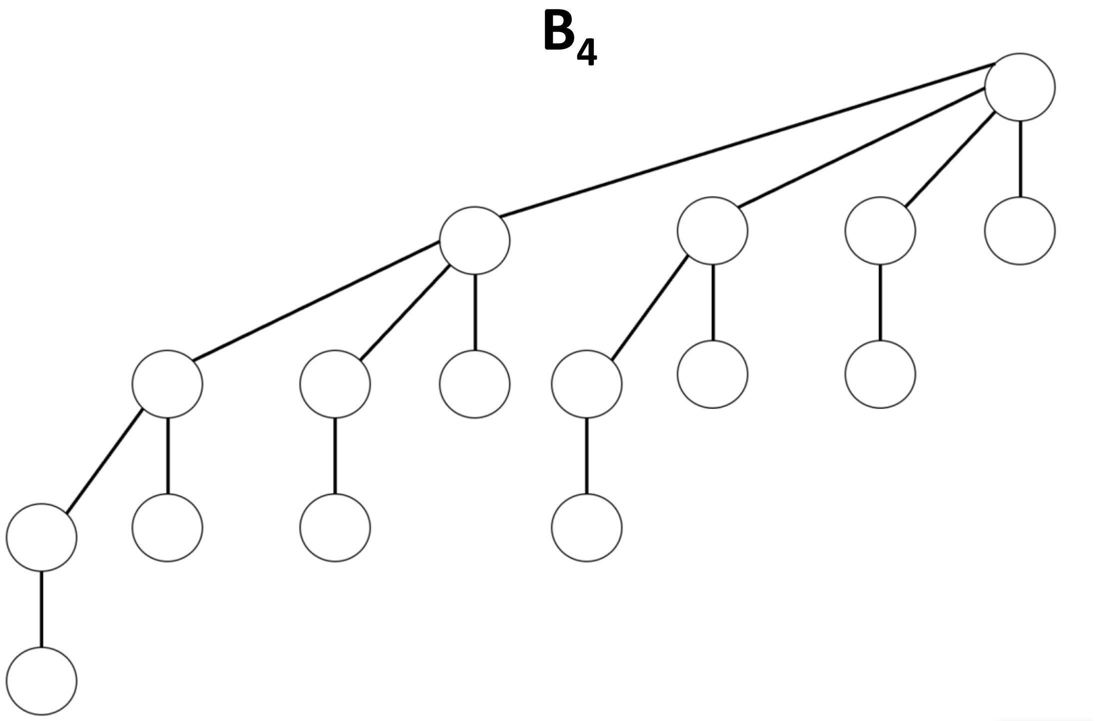


Height = 4

2^4 = 16 nodes

Binomial Tree properties

- B_k tree has:
  - Height k
  - 2^k nodes
  - $\binom{k}{d}$ nodes at depth d



Height = 4

2^4 = 16 nodes

$\binom{4}{2}$ = 6 nodes at level 2

Binomial Forest of size n (denoted F_n)

A sequence of B_k trees with *strictly decreasing* k's and a total of n nodes.

Example: Binomial Forest F_7 of n = 7 nodes

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$n = 7 = < 1\ 1\ 1 >_2 = 2^2 + 2^1 + 2^0$

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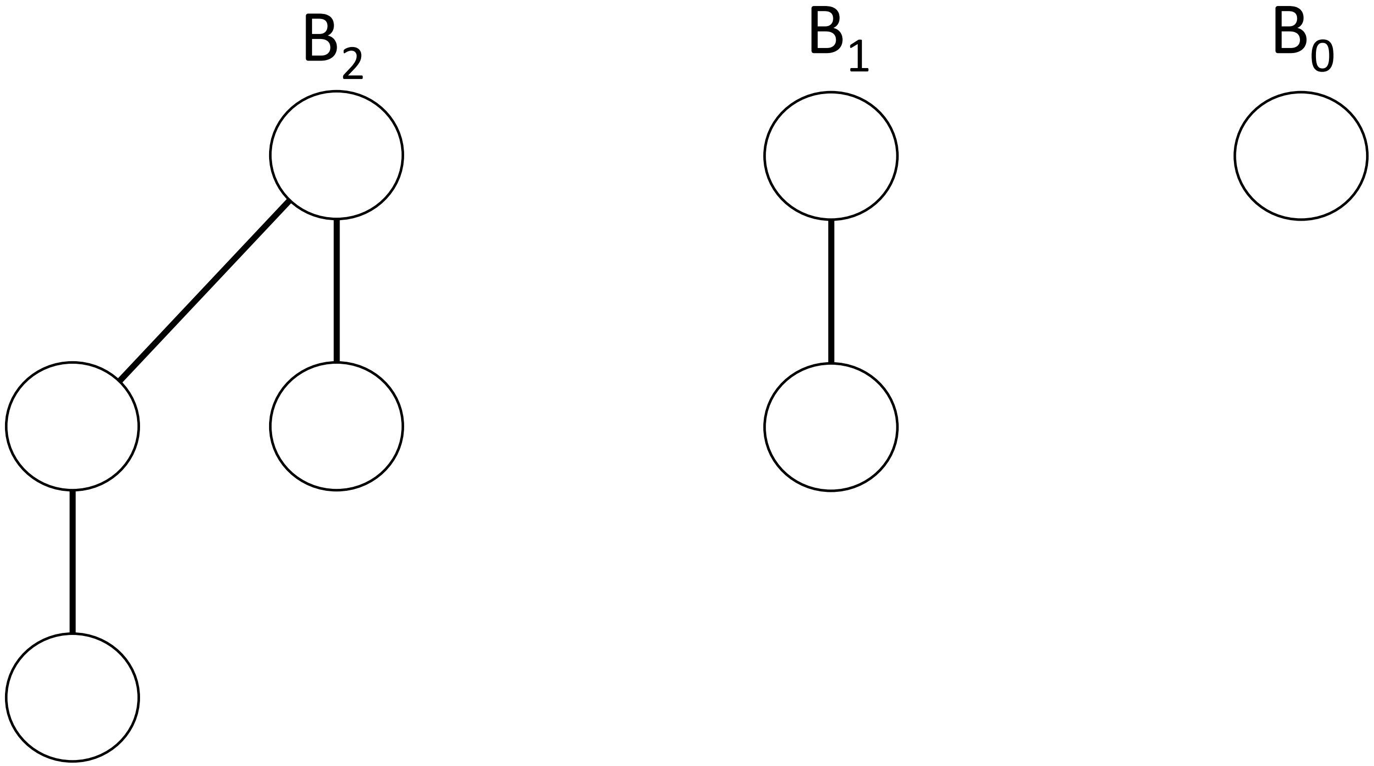

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Example: Binomial Forest F_9 of $n = 9$ nodes

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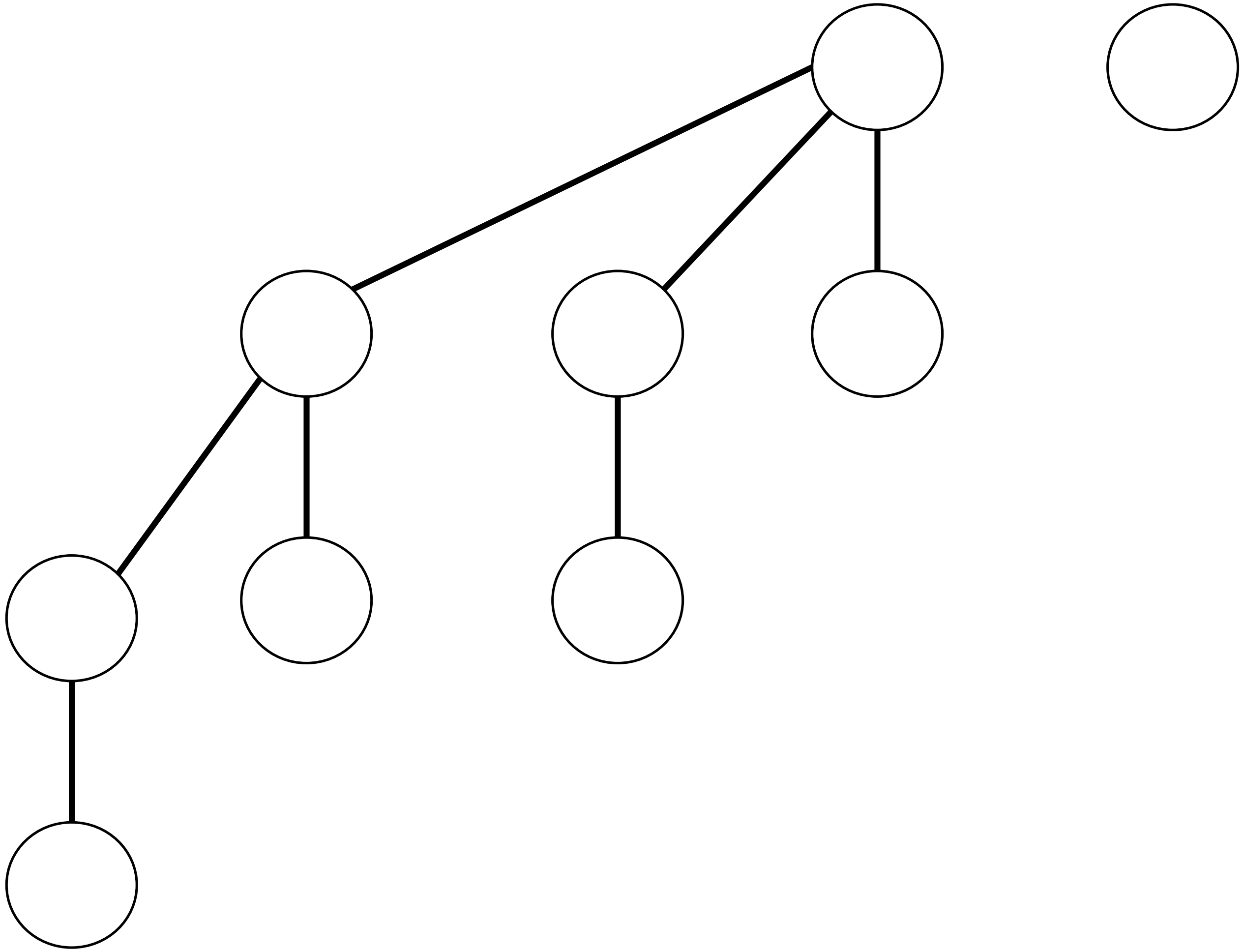
F_9 : $\qquad$ B_3 $\qquad$ B_0



Binomial Forest F_n

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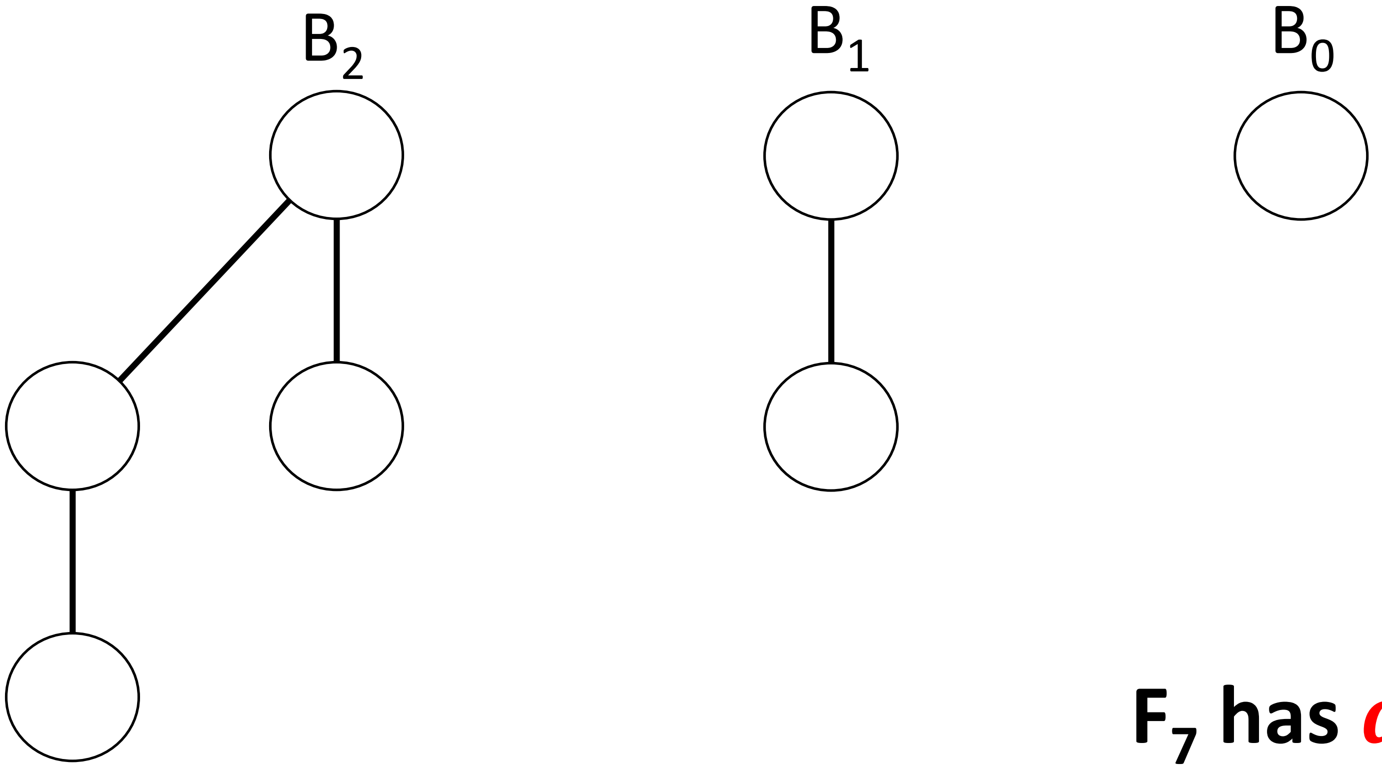

F_7 has $\alpha(7) = 3$ trees

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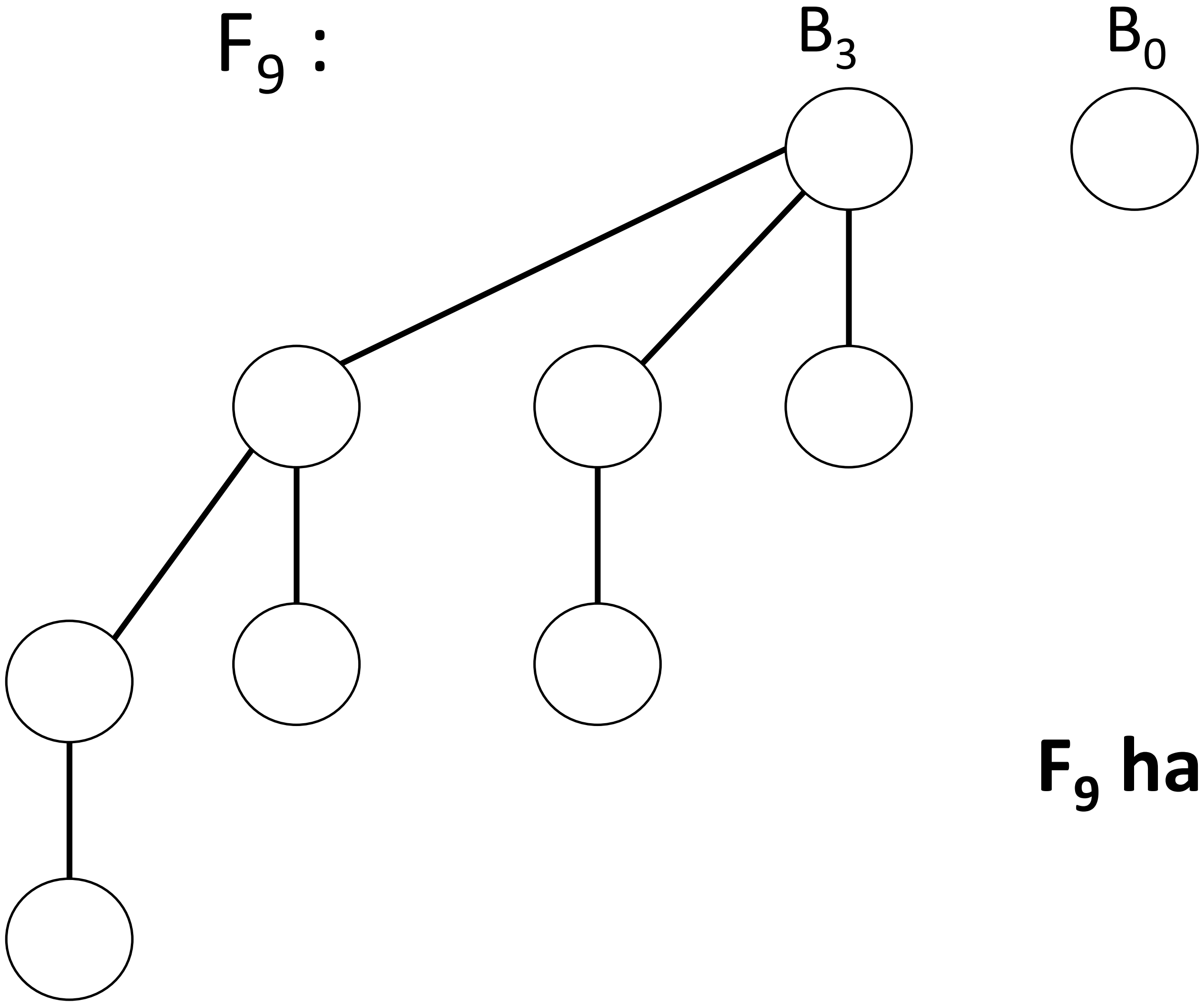

F_9 has $\alpha(9) = 2$ trees

Binomial Forest F_n

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- Let α(n) = # of 1's in binary representation of n
  - F_n has α(n) trees
  - F_n has n - α(n) edges

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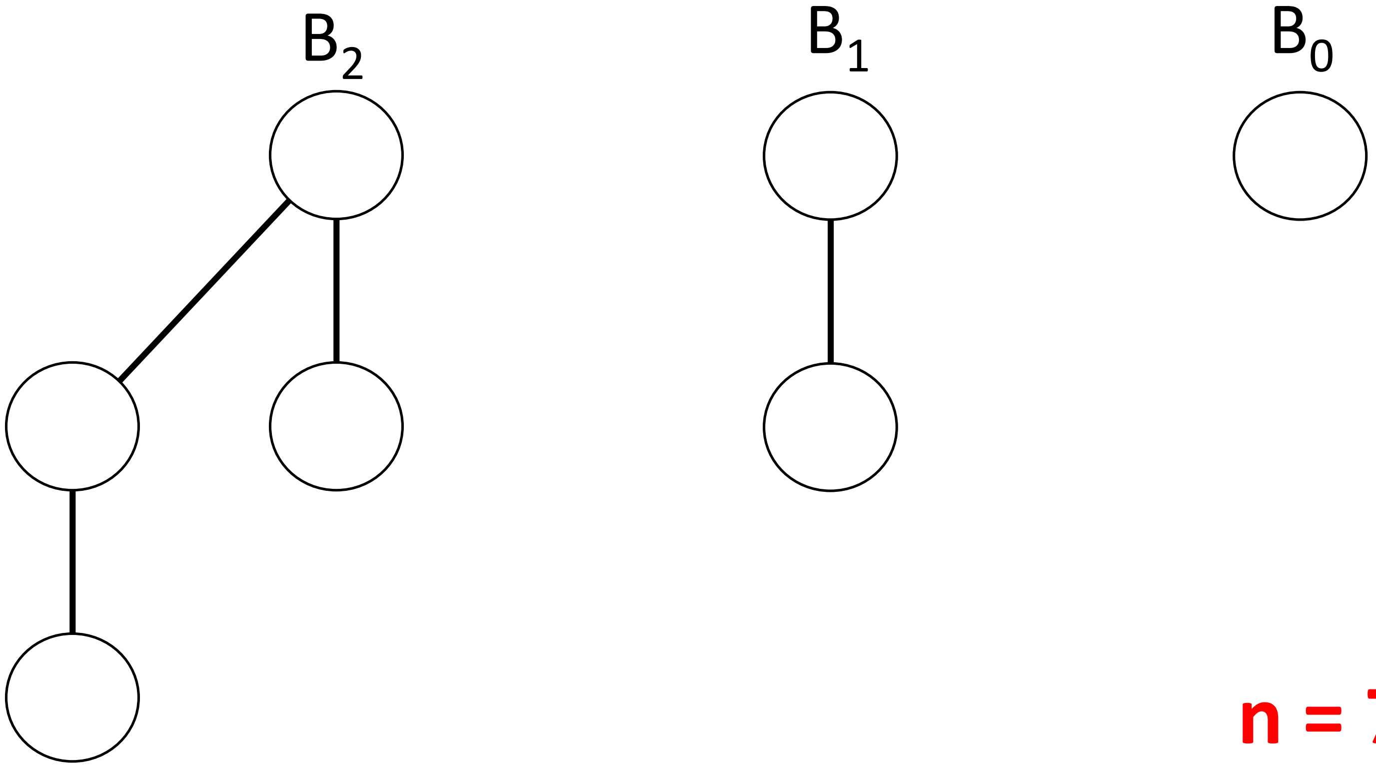

n = 7, $\alpha(7) = 3$

Example: Binomial Forest F_7 of n = 7 nodes

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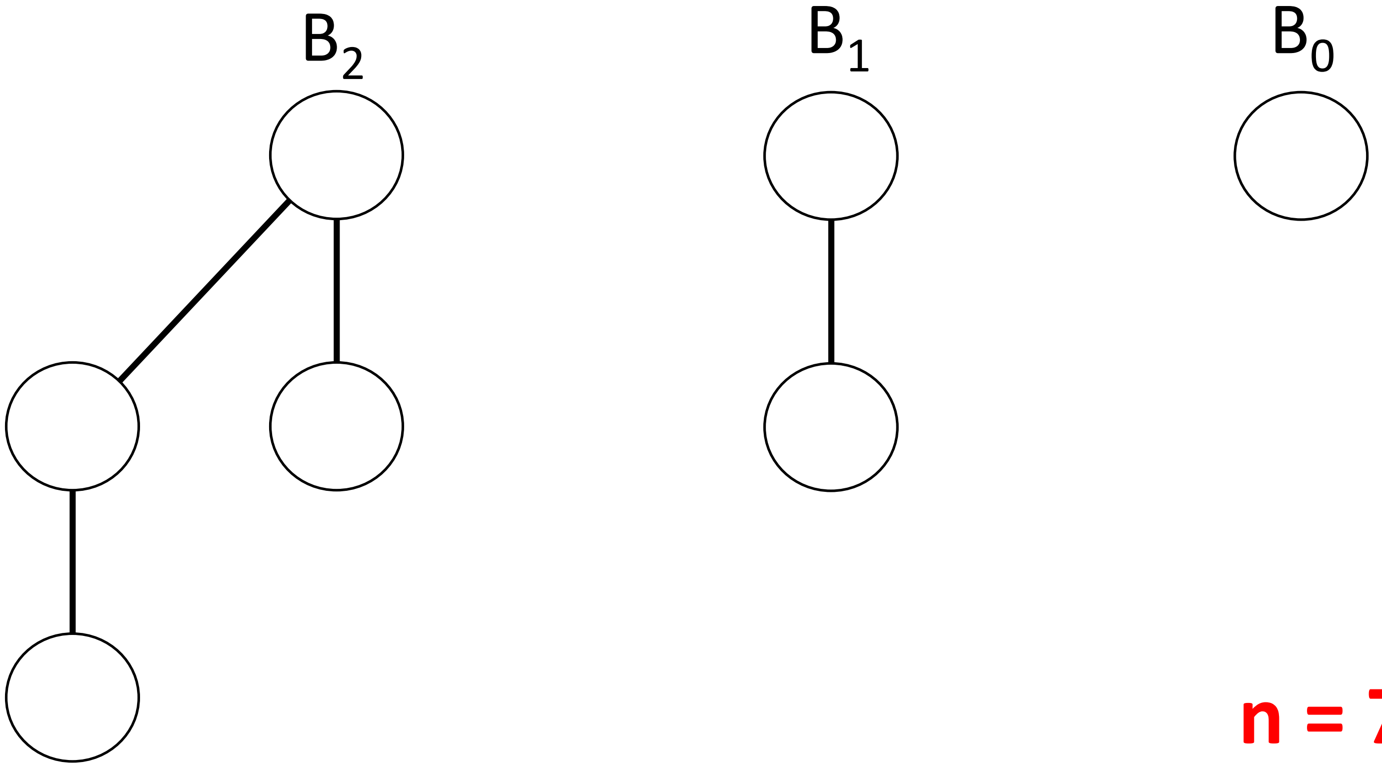

$n = 7, \alpha(7) = 3$

F_7 has $7-\alpha(7) = 4$ edges

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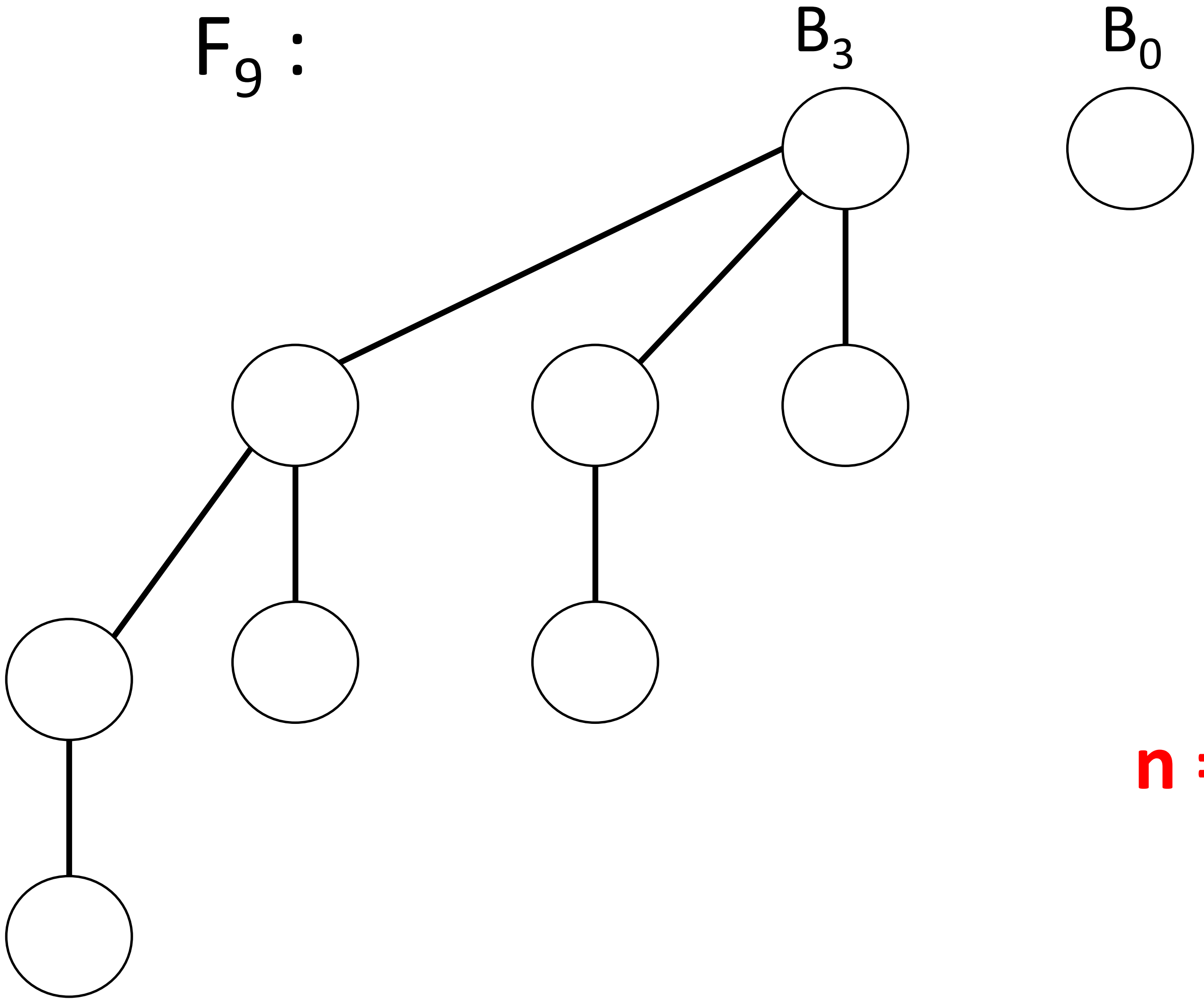
F_9 :



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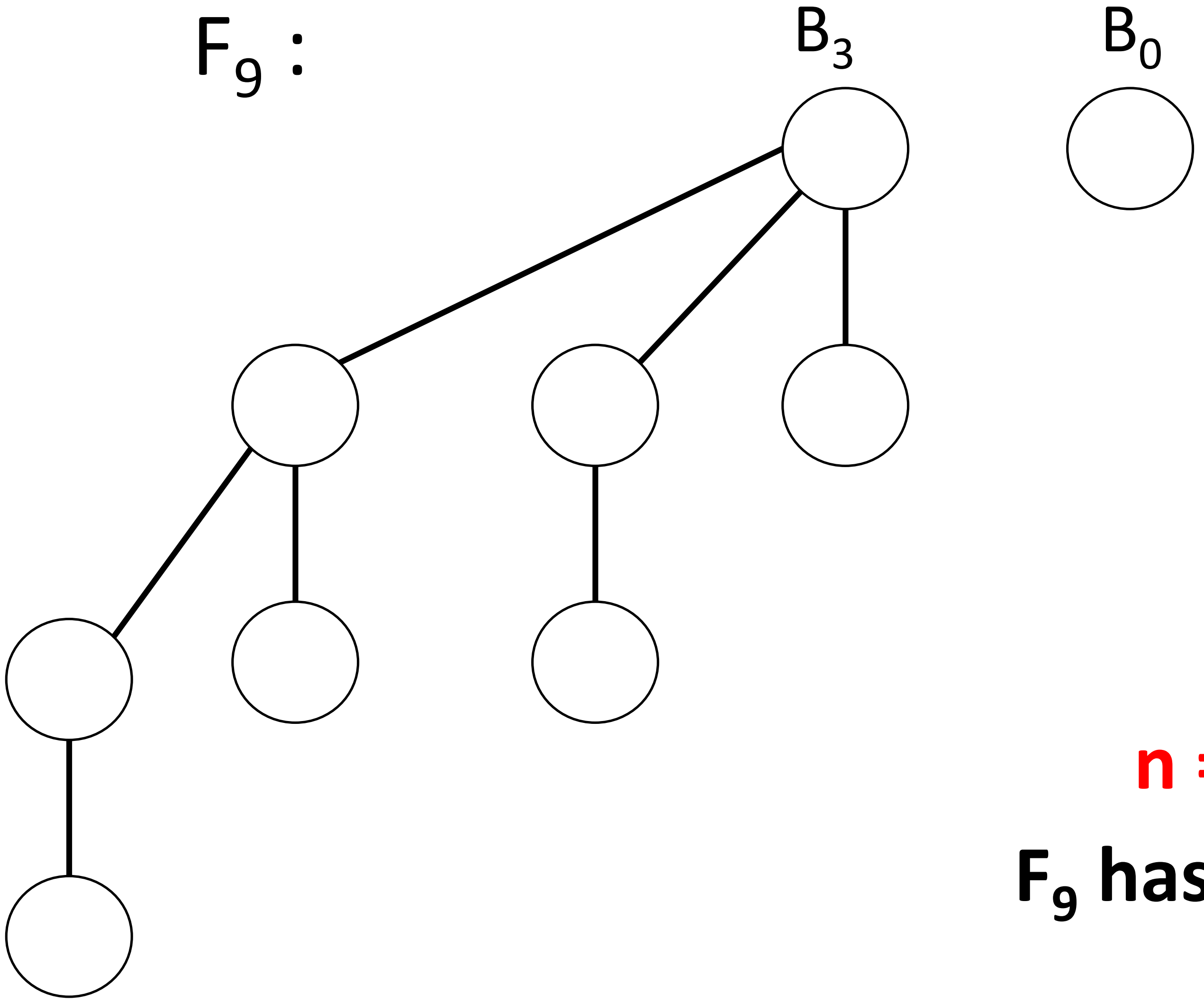

$n = 9$, $\alpha(9) = 2$

F_9 has $9-\alpha(9) = 7$ edges

Binomial Forest F_n

- F_n has n nodes.
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- F_n : <all trees B_i such that bit $b_i = 1$>

- Let α(n) = # of 1's in binary representation of n
  - F_n has α(n) trees
  - F_n has n - α(n) edges

A Min Binomial Heap of n elements is a Binomial Forest F_n such that

1. Each node of F_n stores one element

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2. Each B_k tree of F_n is Min-Heap ordered,
   i.e. each tree satisfies the Min-Heap property

Example of a Min Binomial heap

S = {10, 13, 1, 3, 8, 18, 7}   n = 7 elements

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Put S in $F_7 = < B_2\ B_1\ B_0 >$

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B_0

10

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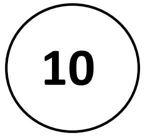
B_0



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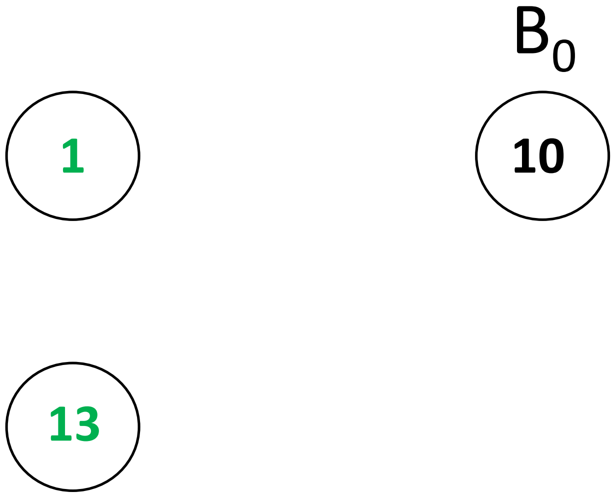

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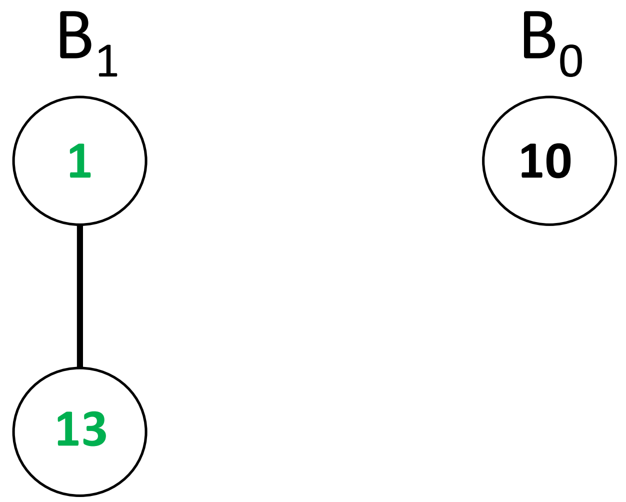
Put S in $F_7 = < B_2\ B_1\ B_0 >$



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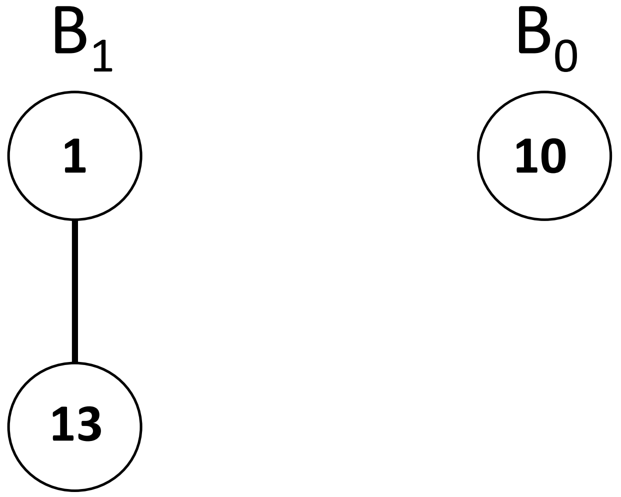

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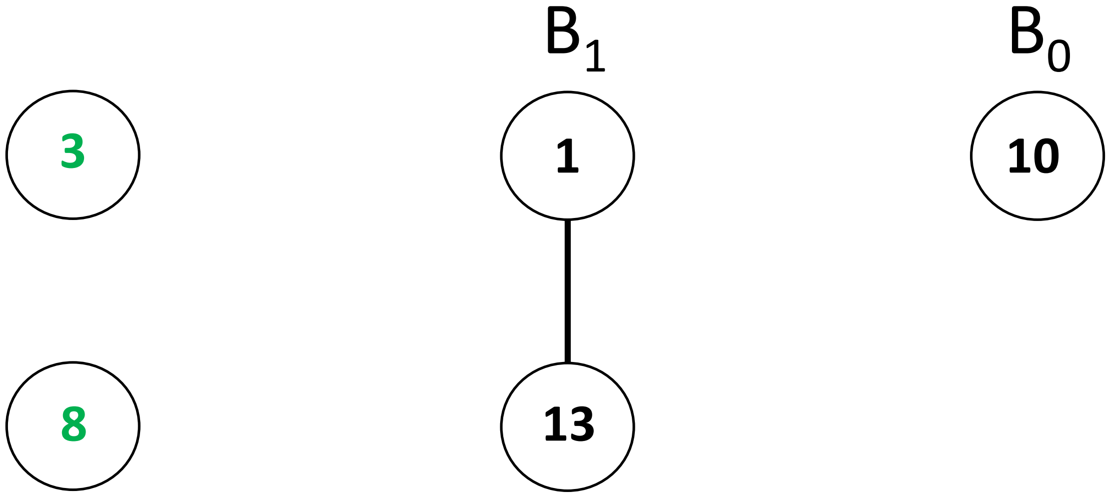

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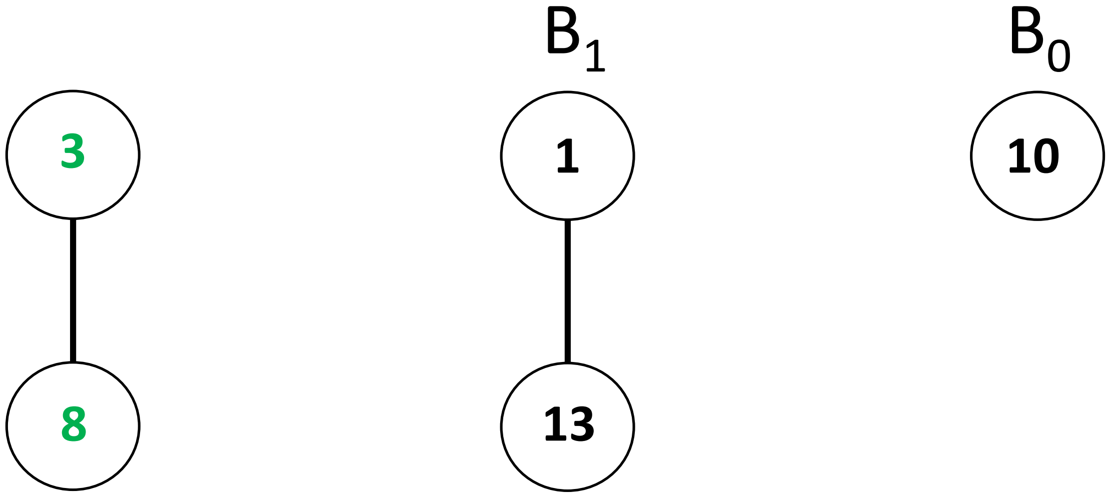

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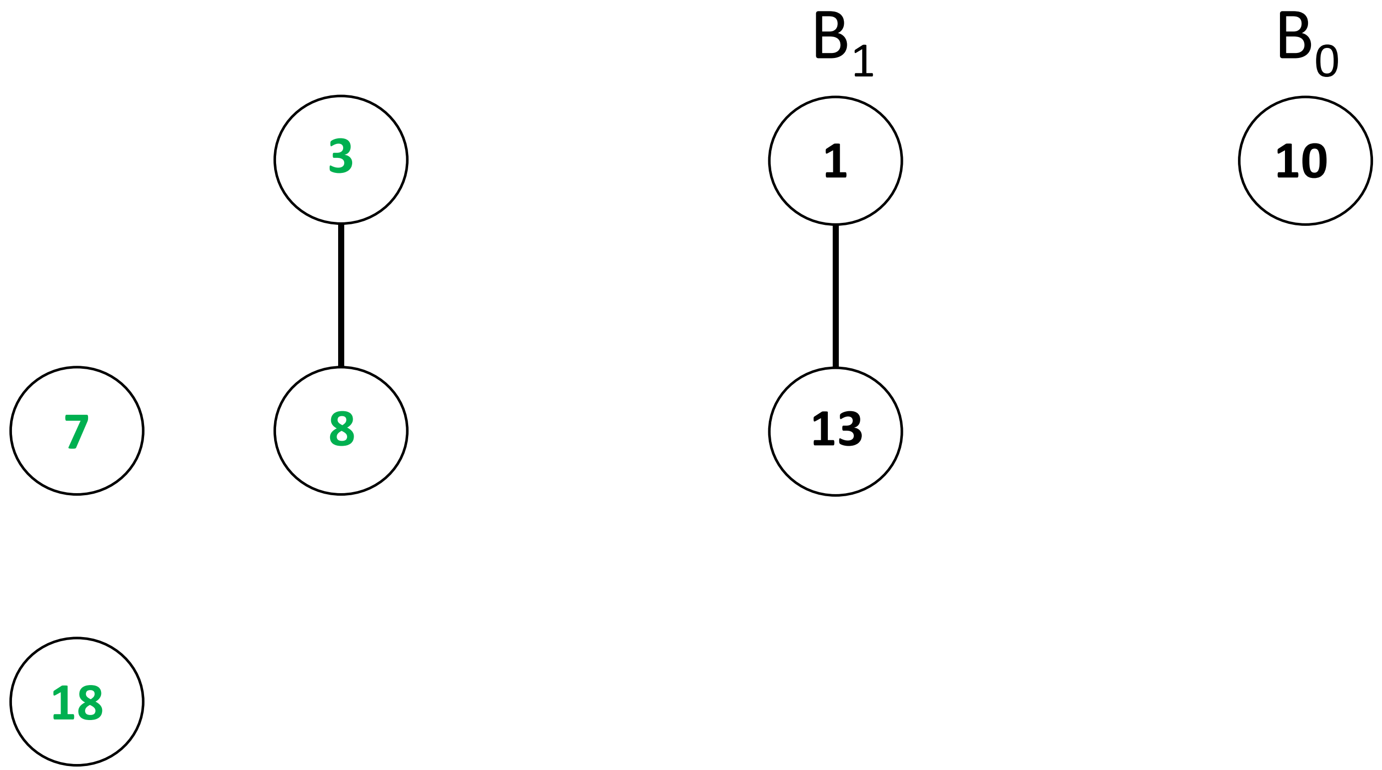

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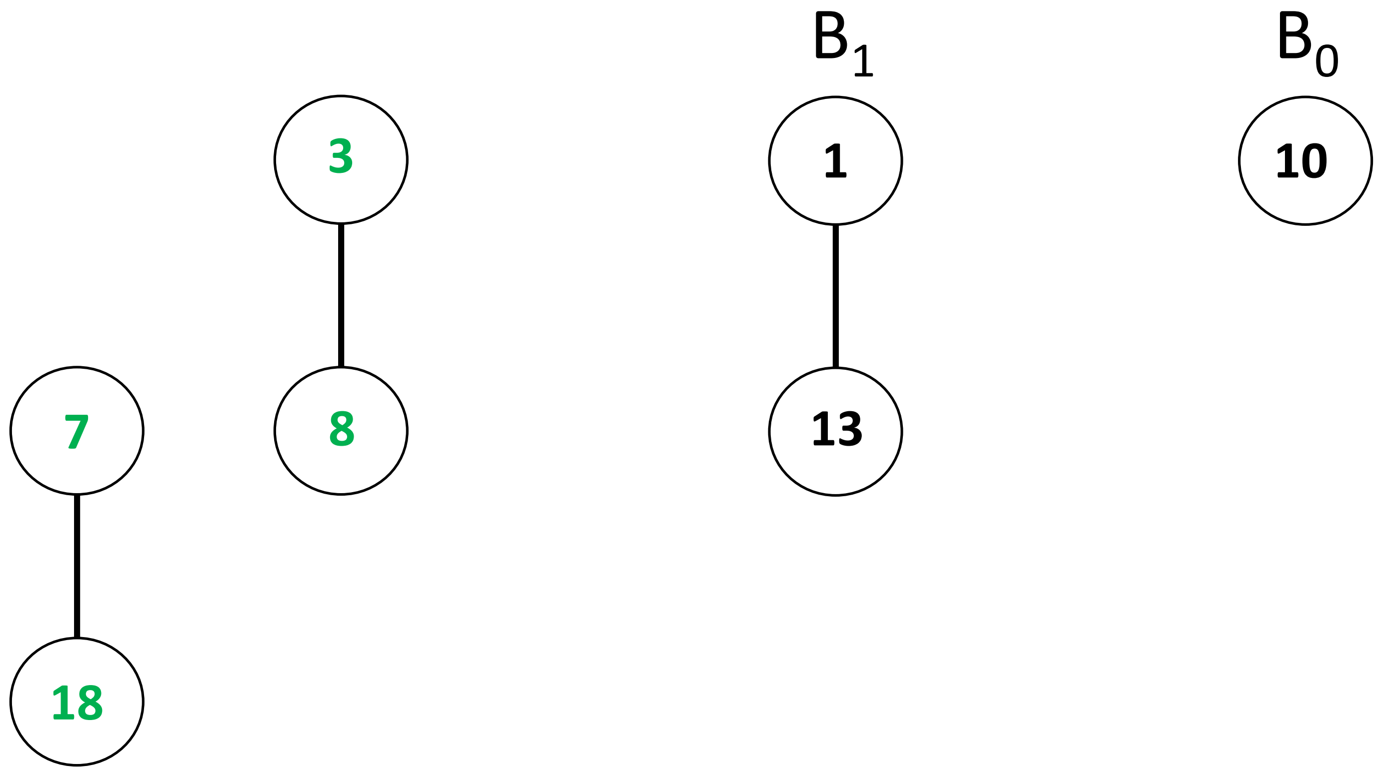

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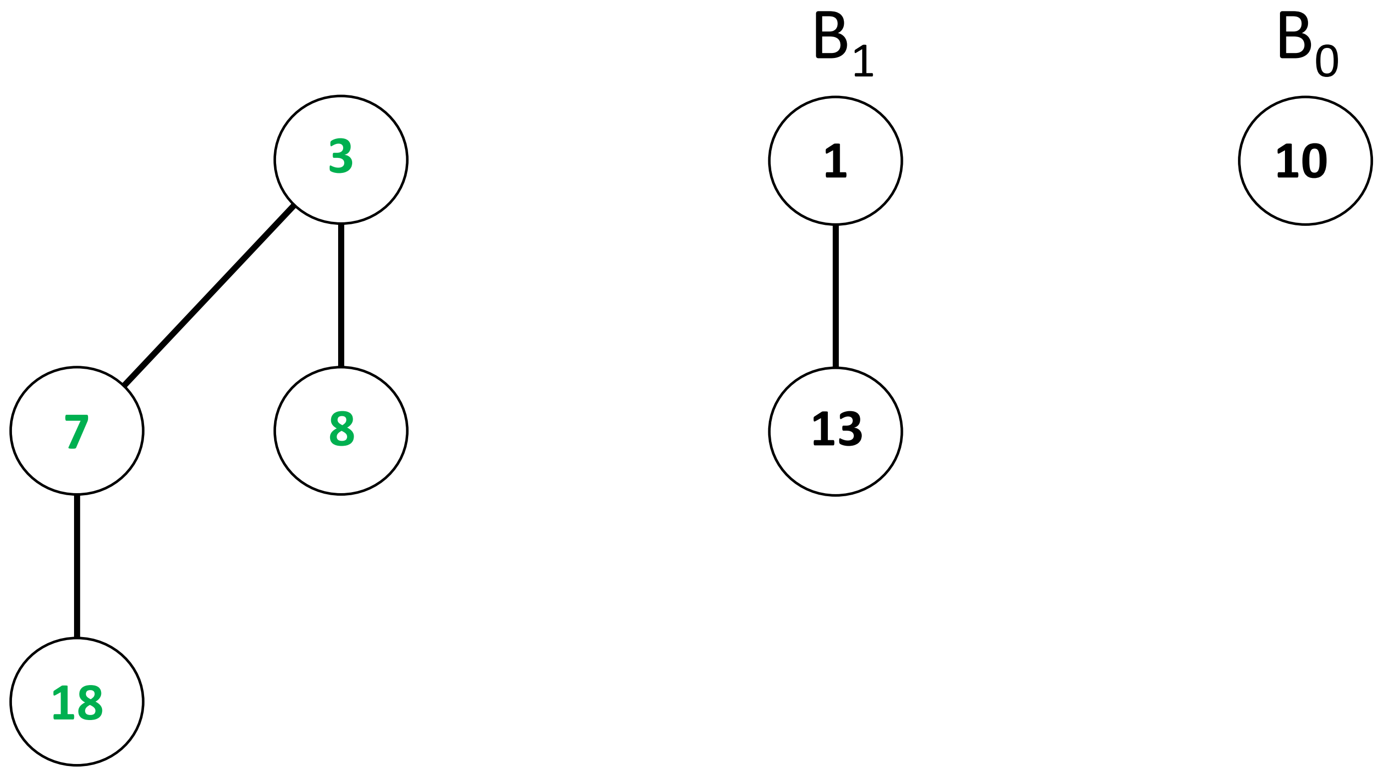

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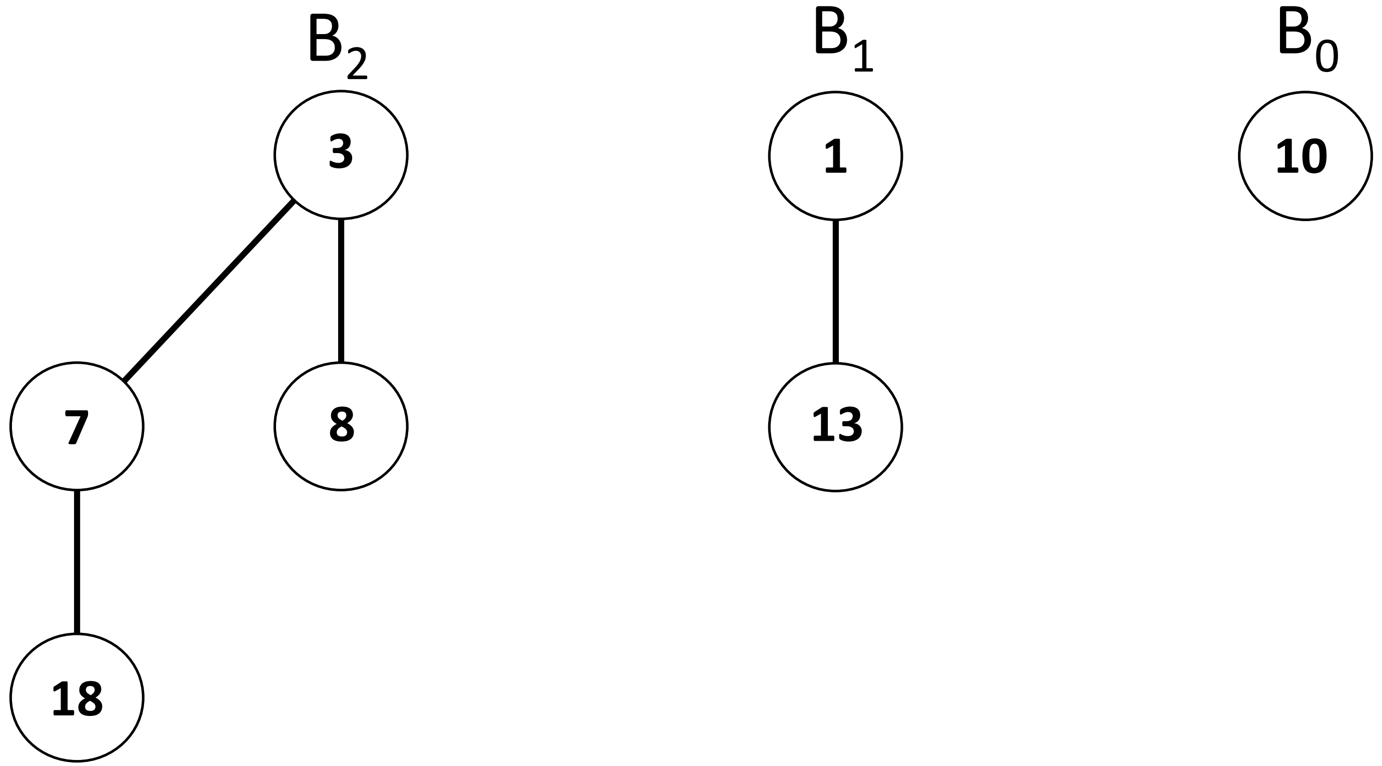

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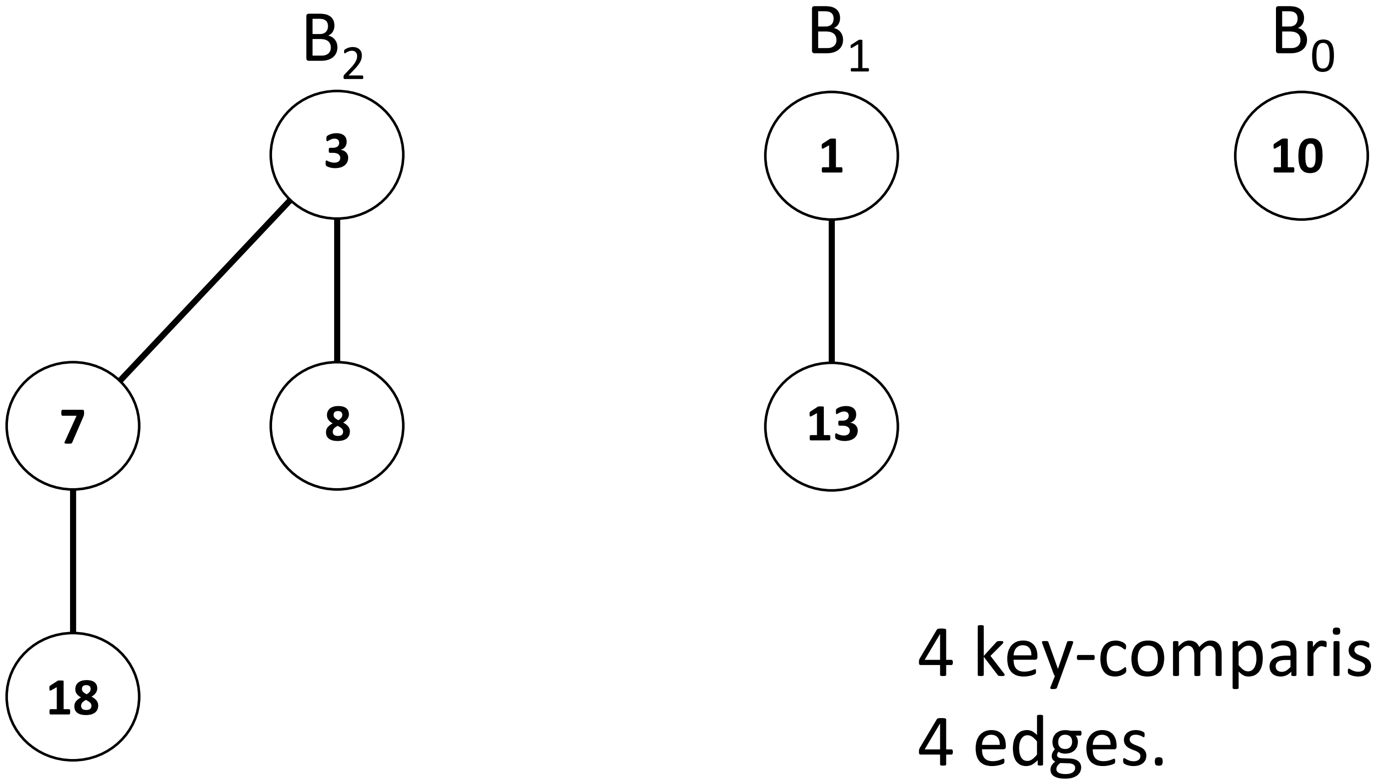

4 key-comparisons.
4 edges.

FACTS:

- One key-comparison per Binomial Heap edge.

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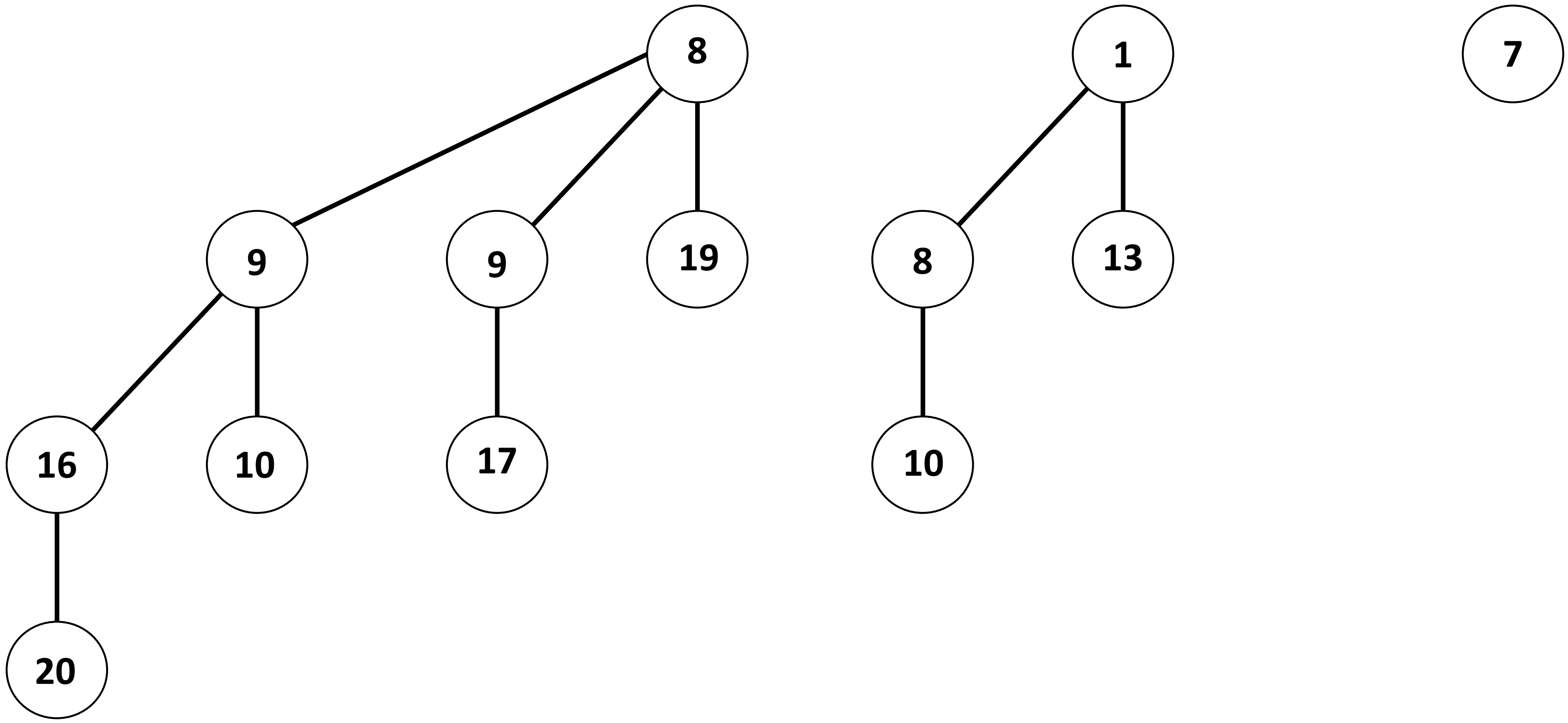
- A Binomial Heap for n elements can be built in O(n) key-comparisons.

Storing Binomial Heaps in Memory

Binomial Heap (Visually)



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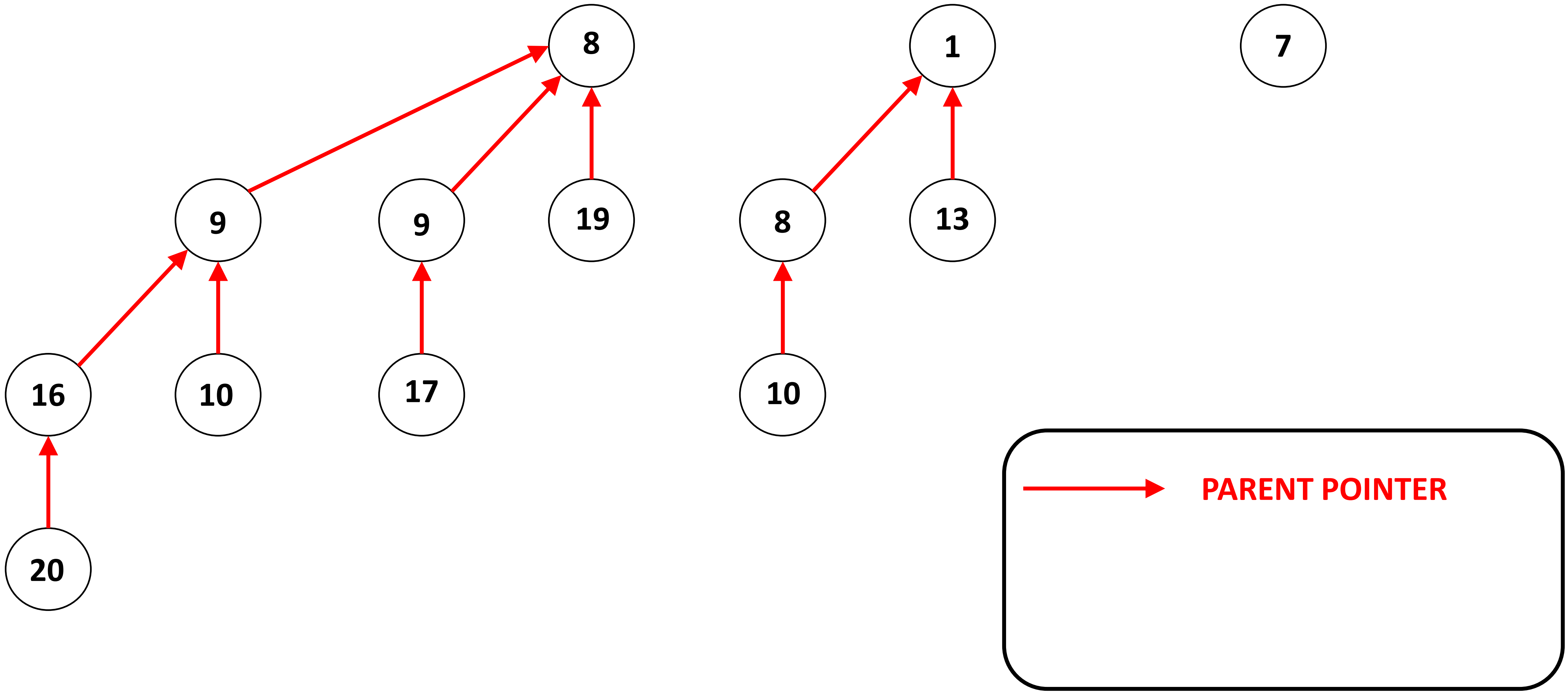

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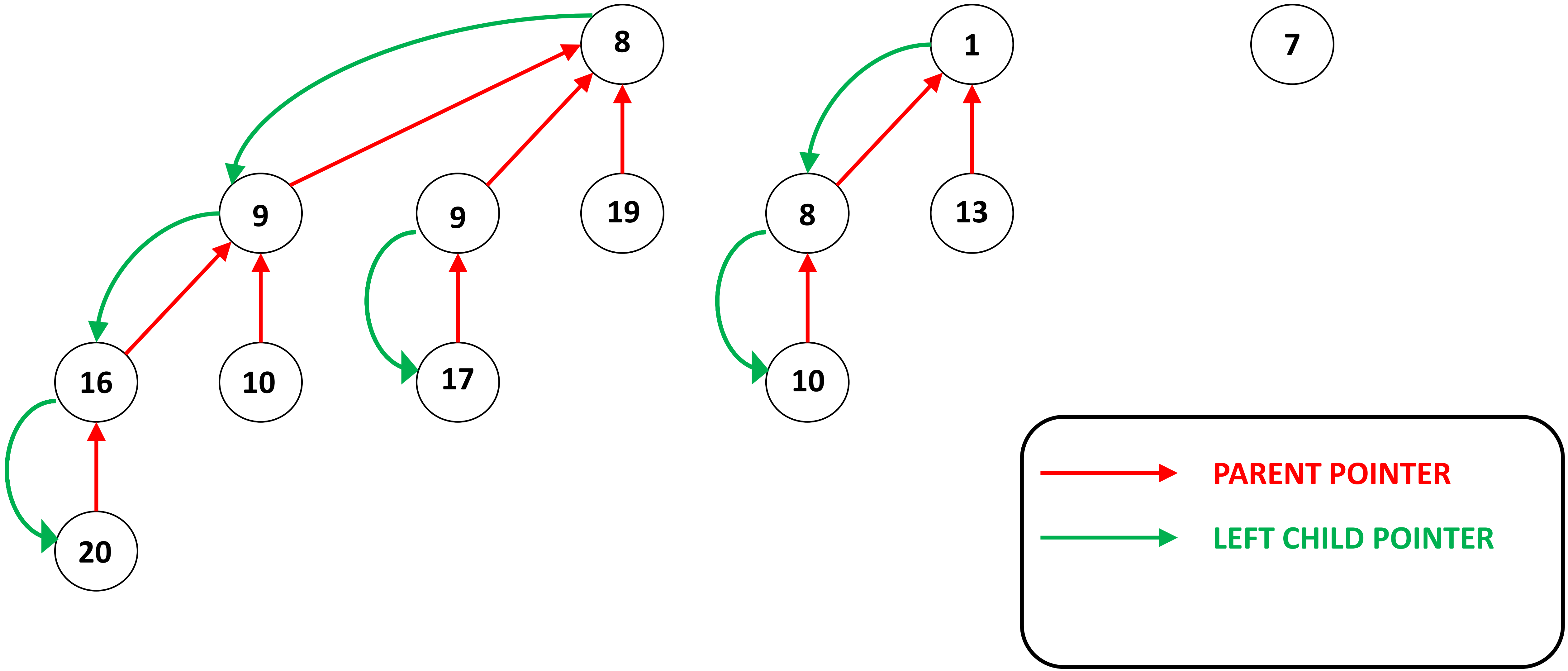

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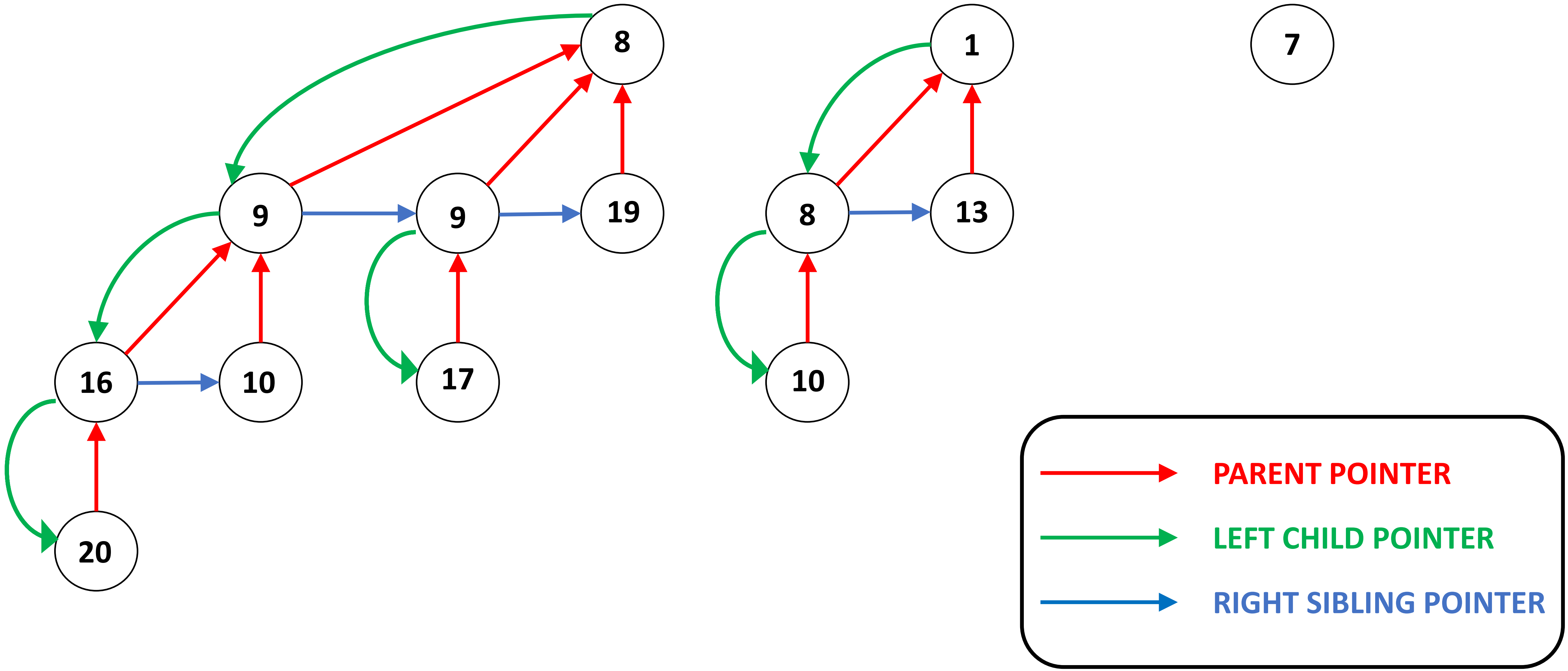
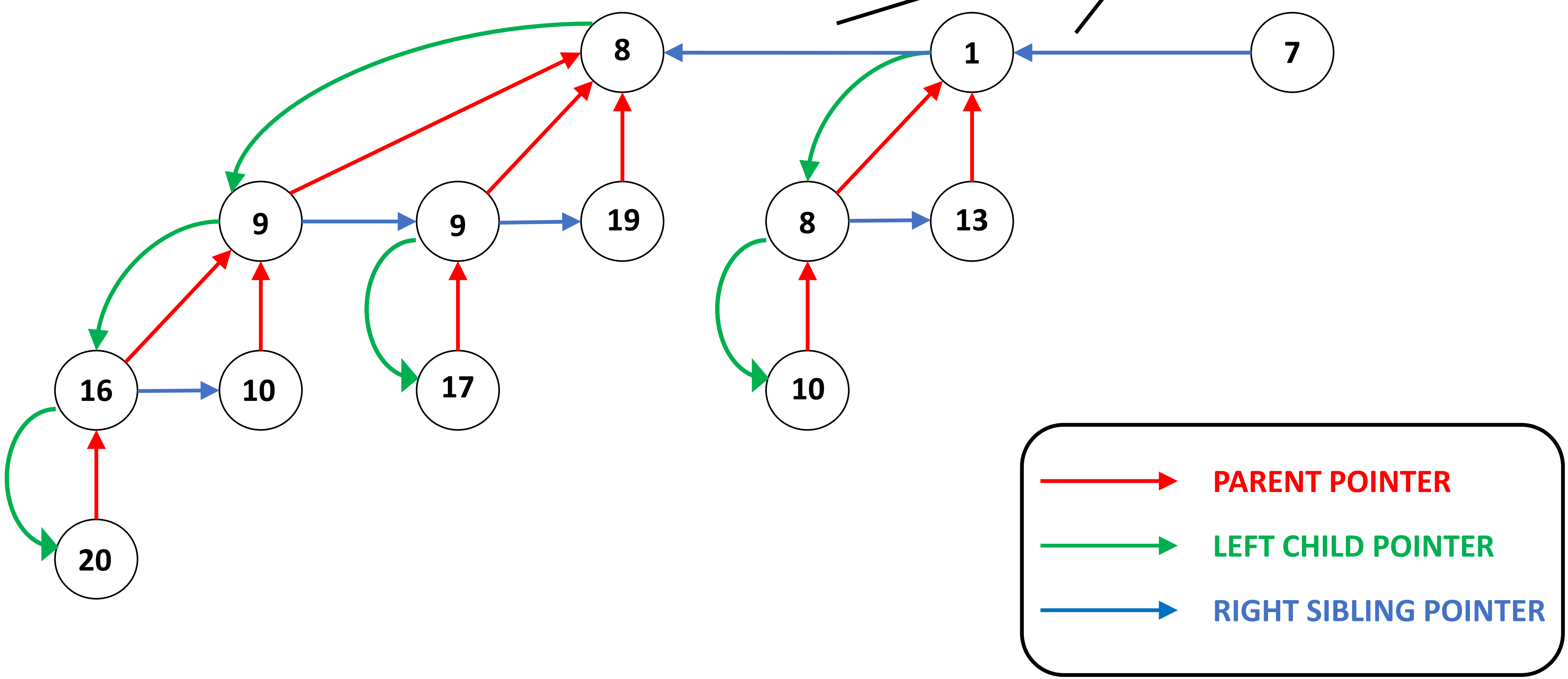


Binomial Heap (in Memory)

Use Sibling Pointer to Connect Trees



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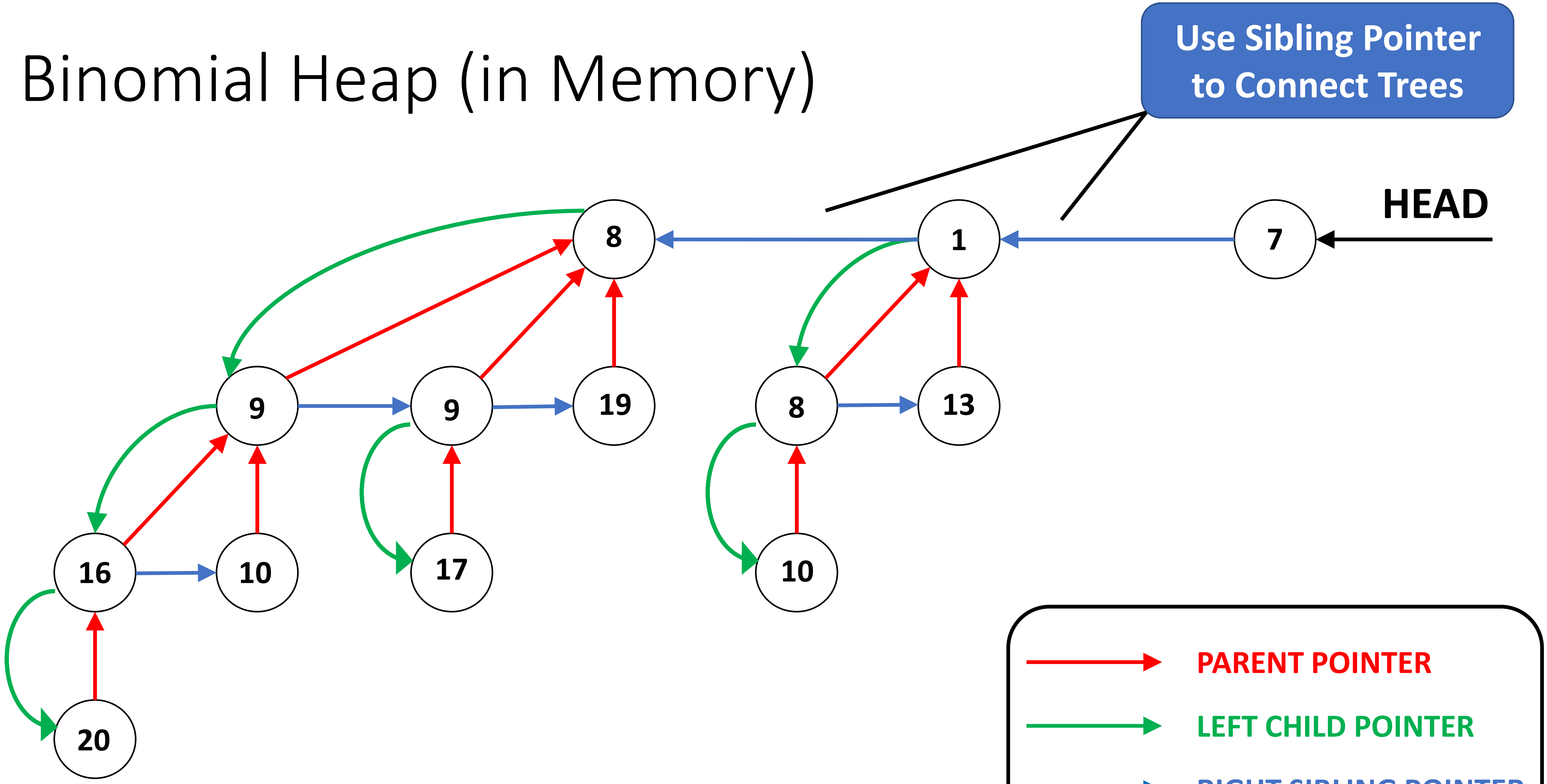

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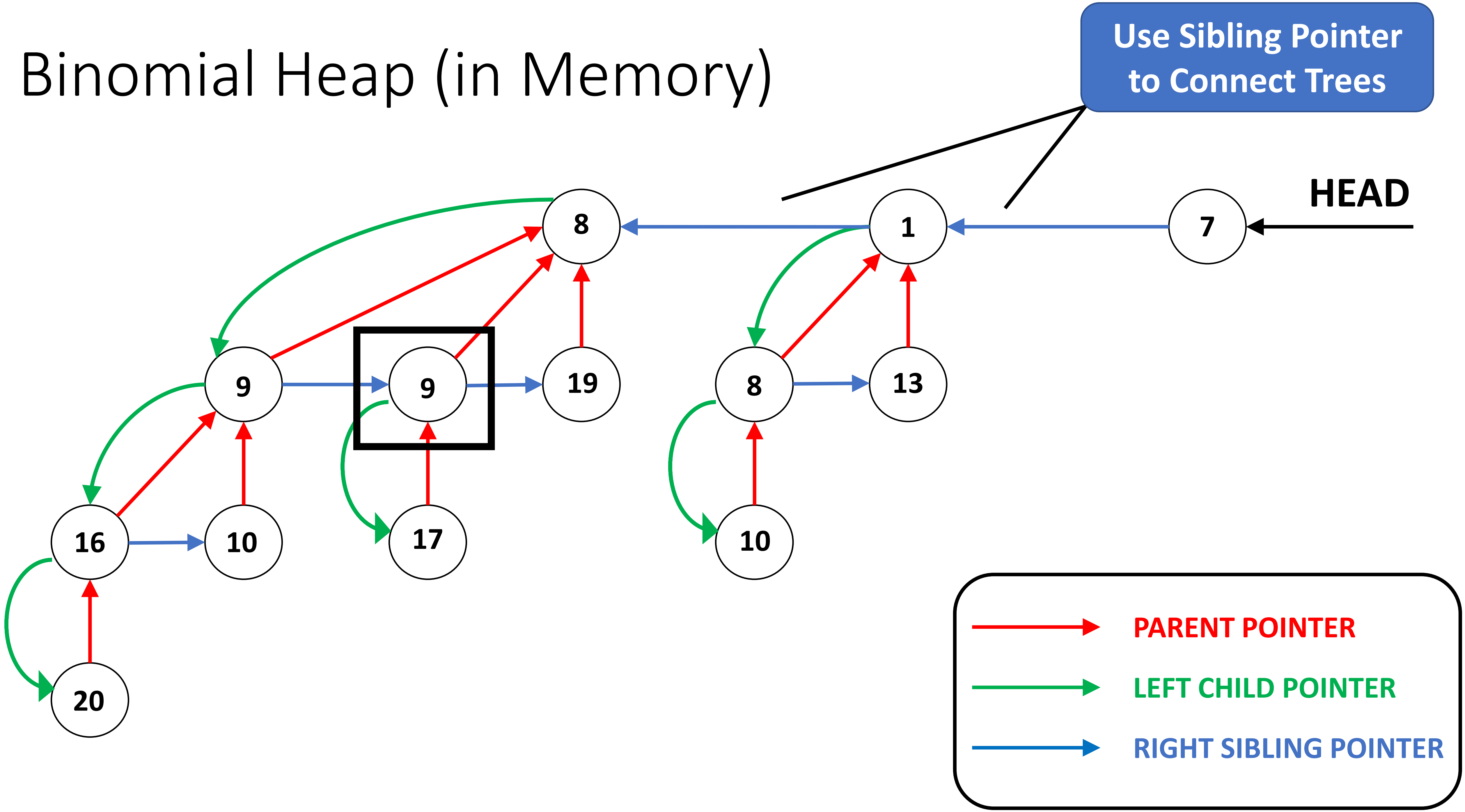

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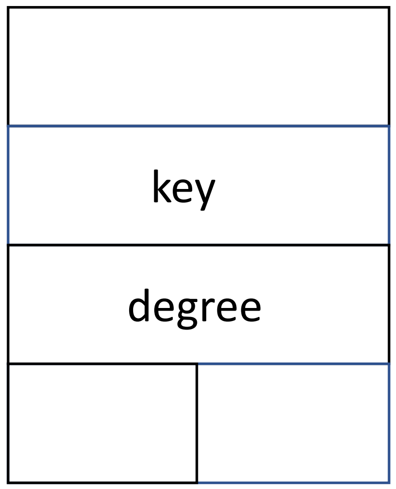

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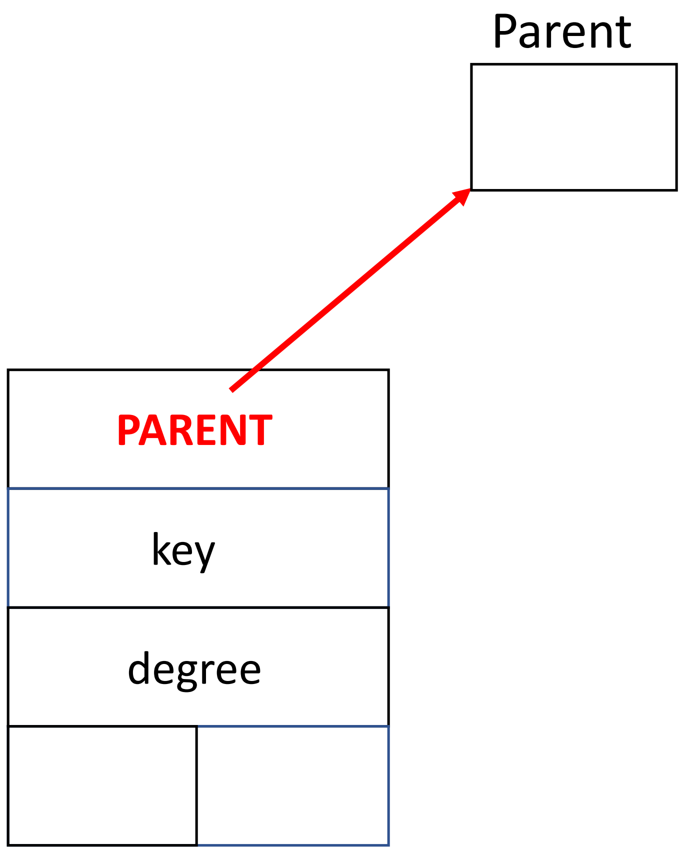

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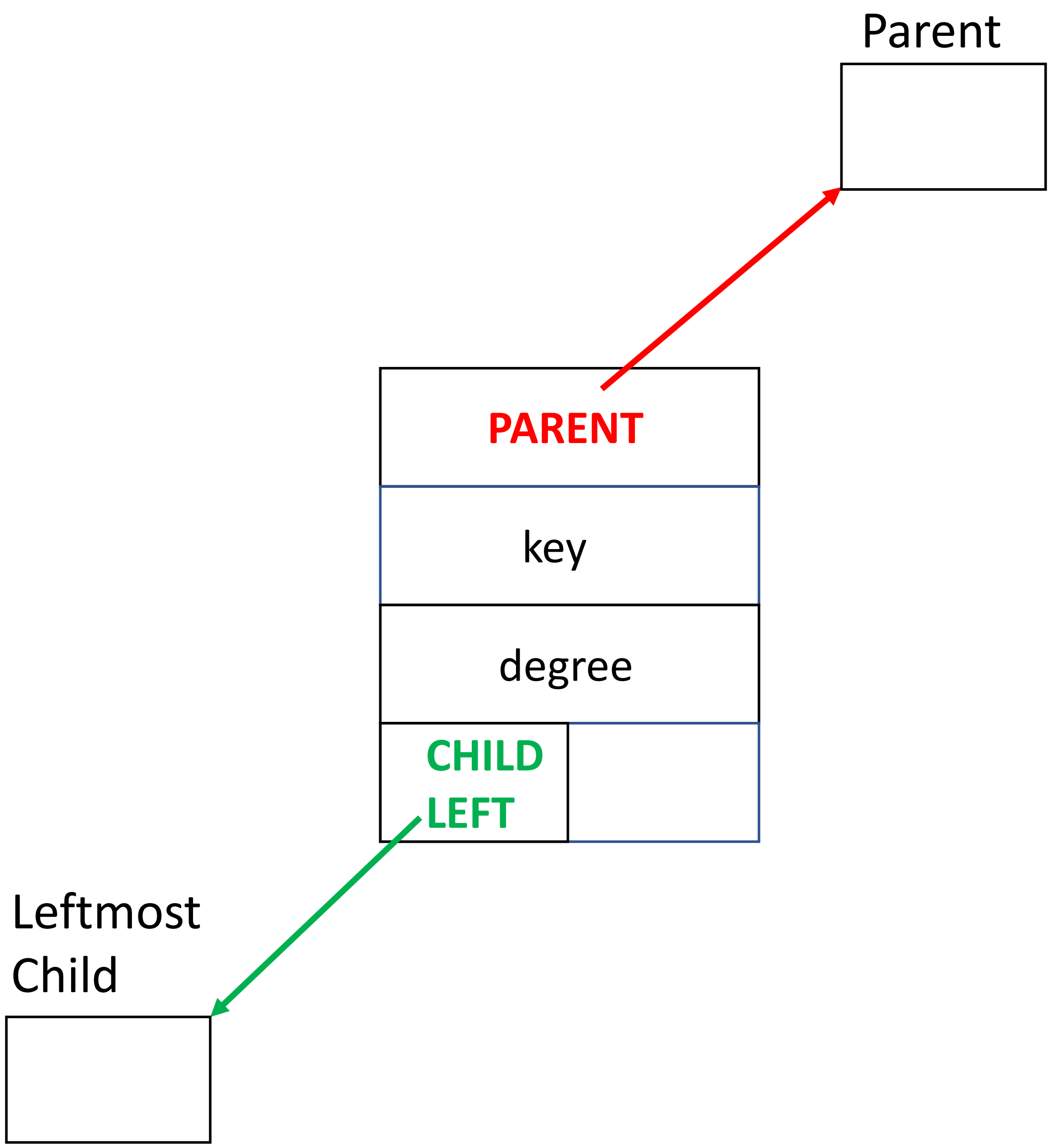

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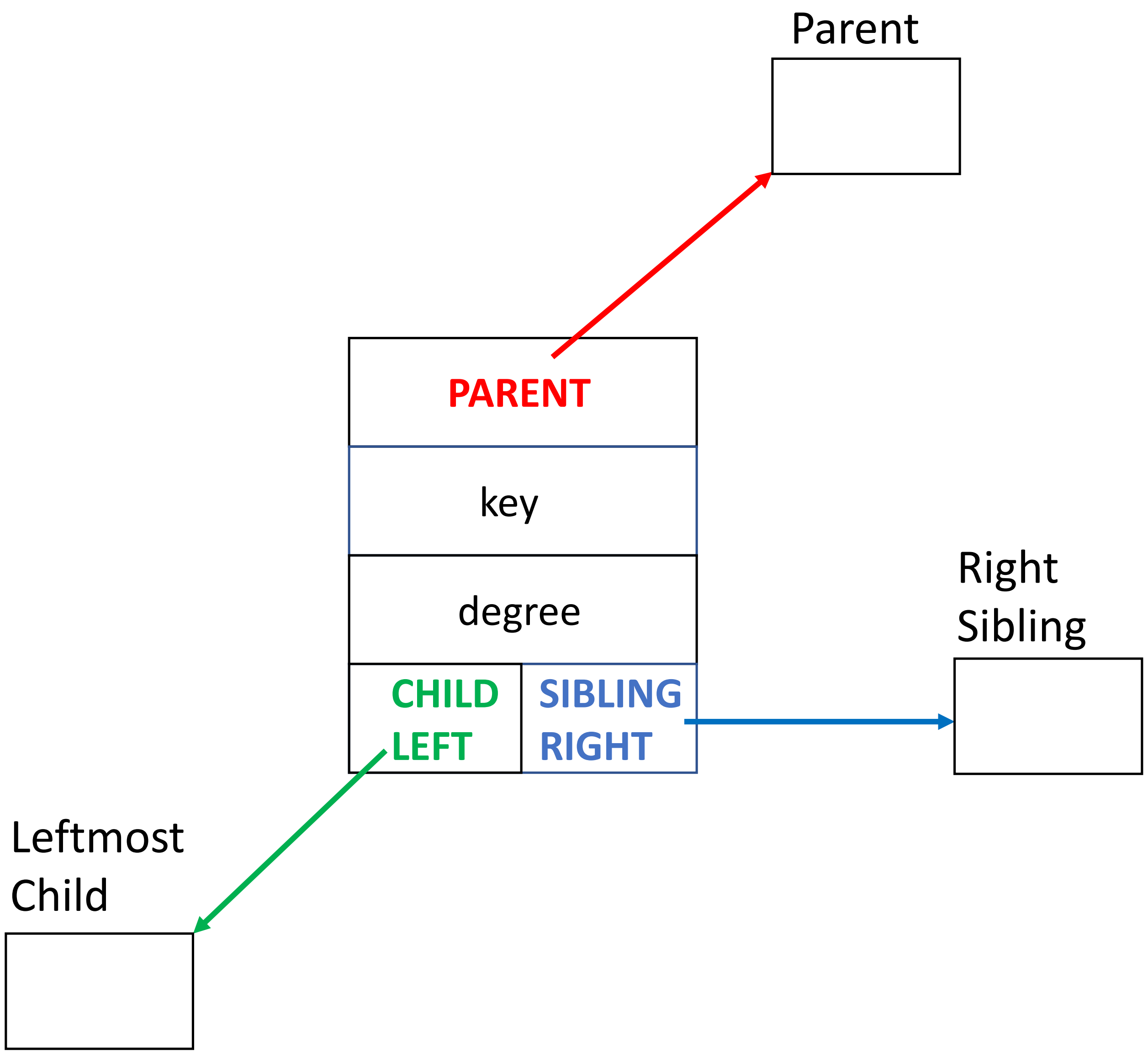
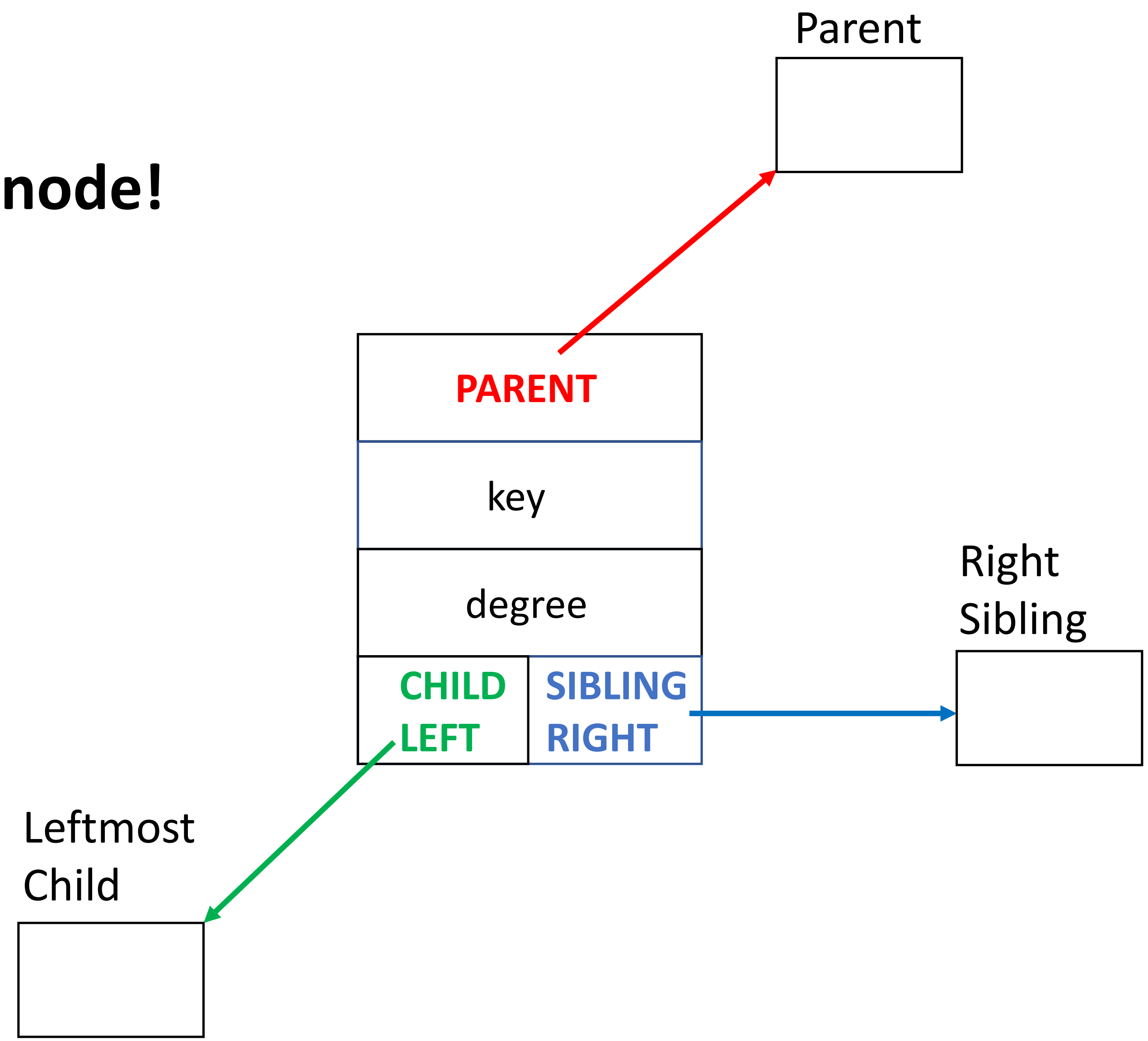


Binomial Heap (in Memory)

3 pointers per node!



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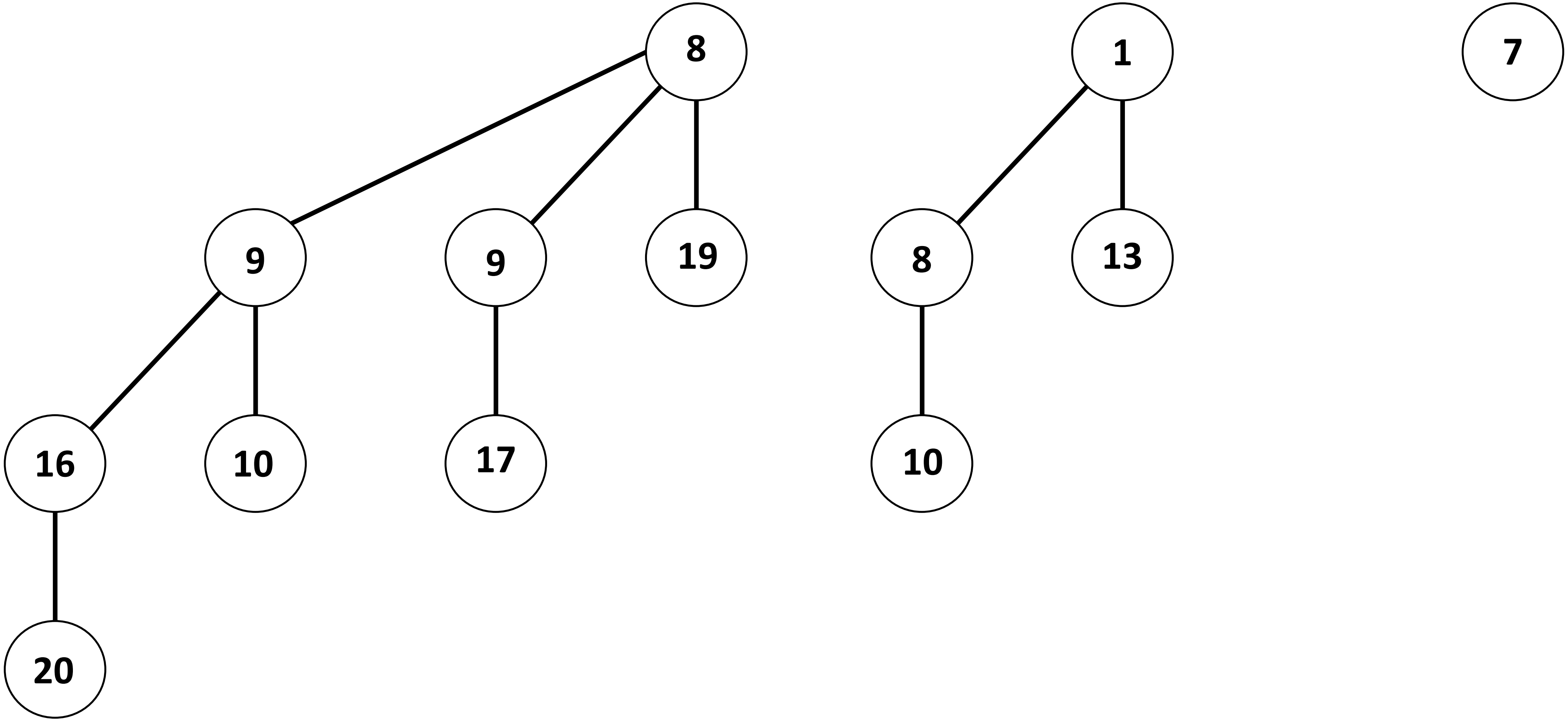

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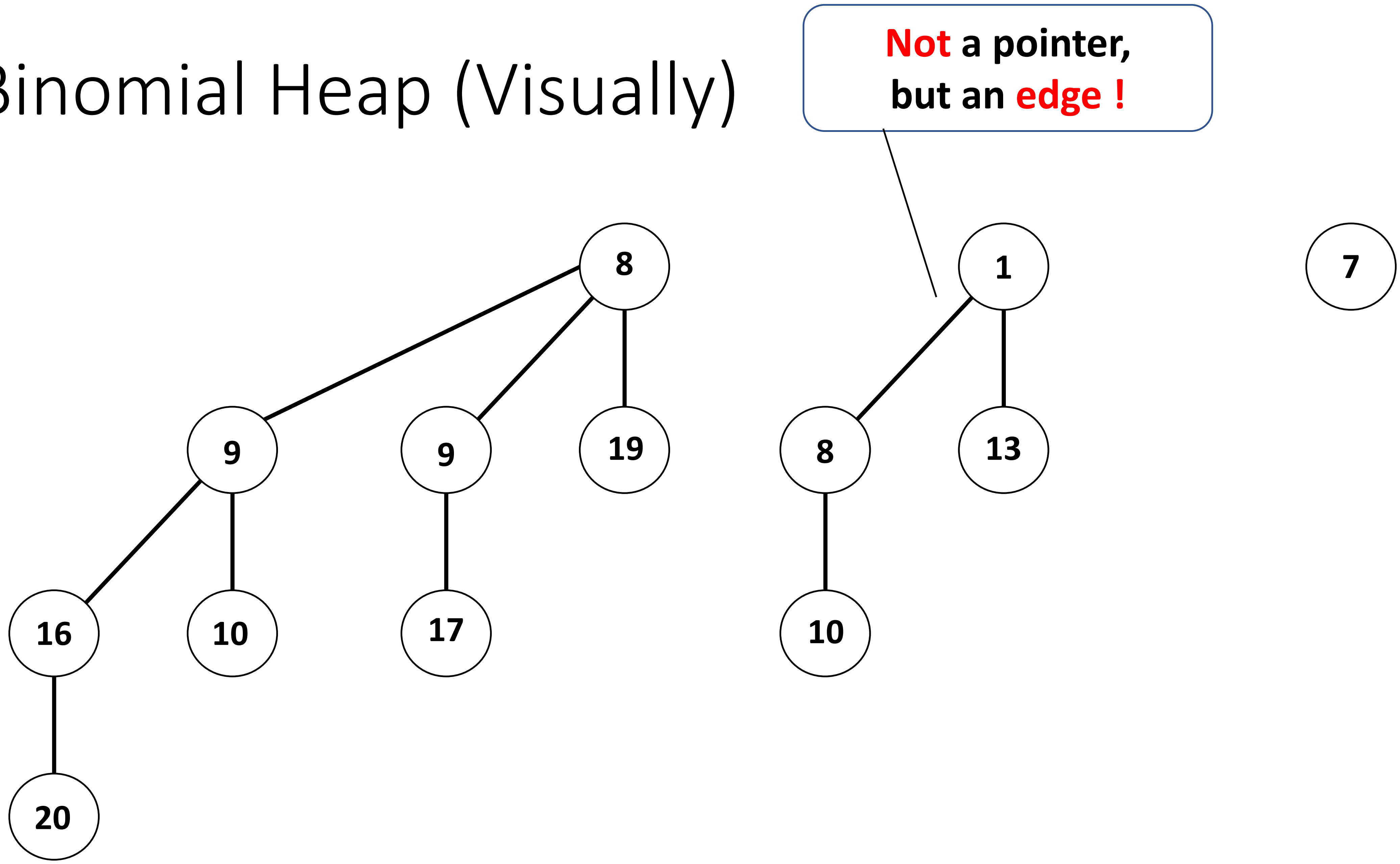

Binomial Heap Operations

Must implement the following operations:

- **Union(T, Q)**
- **Insert(T, x)**
- **Min(T)**
- **Extract_Min(T)**

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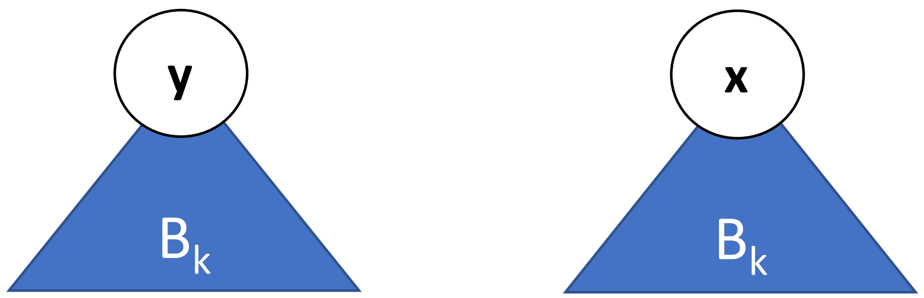

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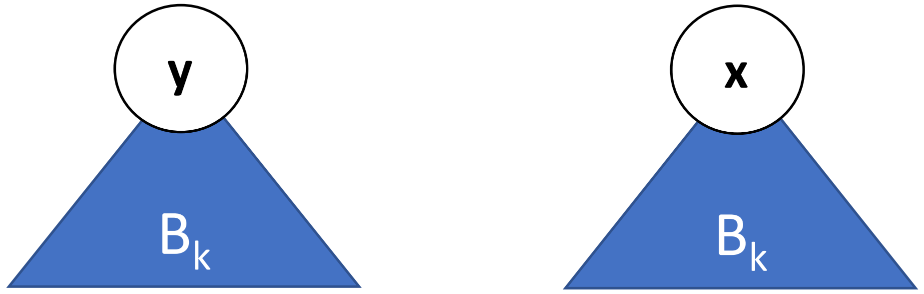

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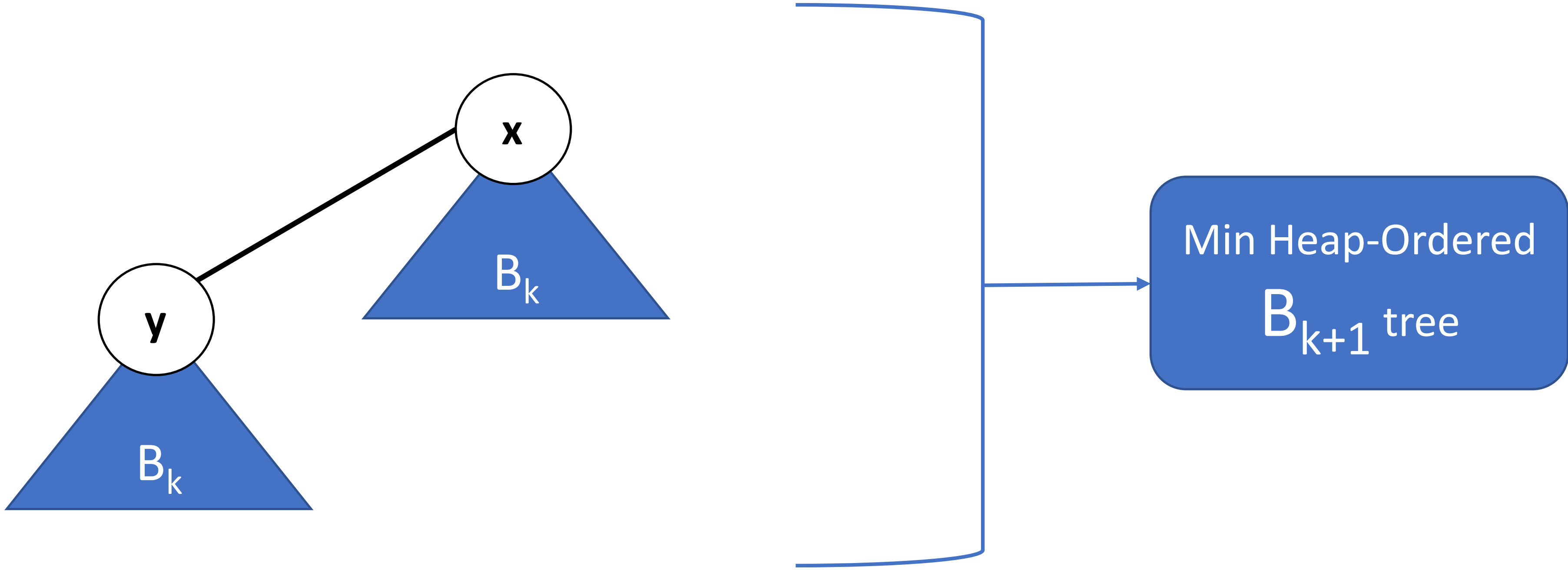

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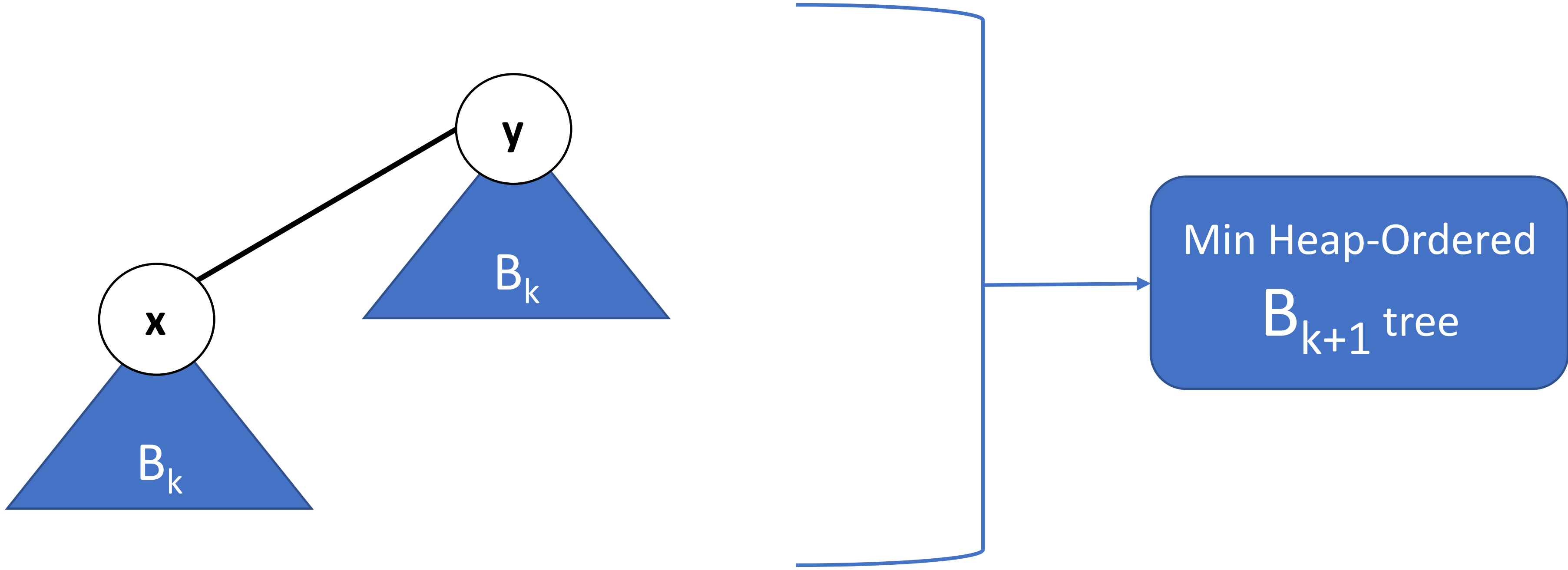

High Level Ideas

Cannot merge two max-heaps So easily !

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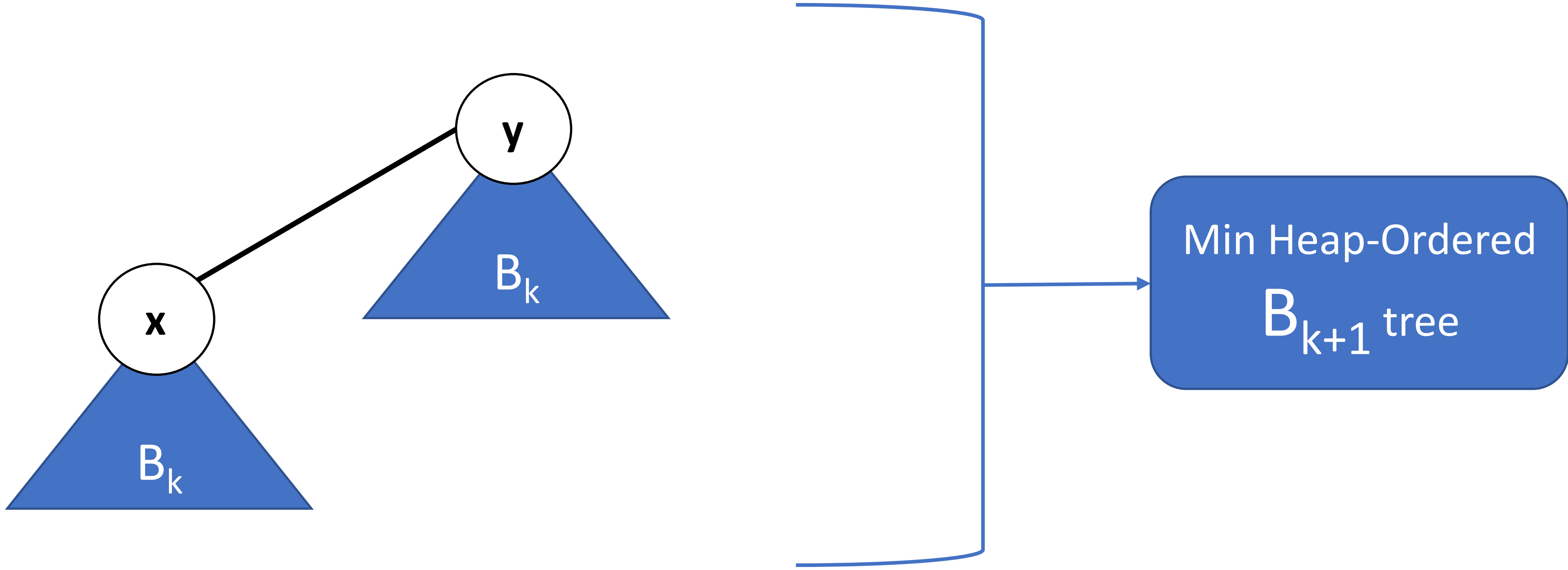

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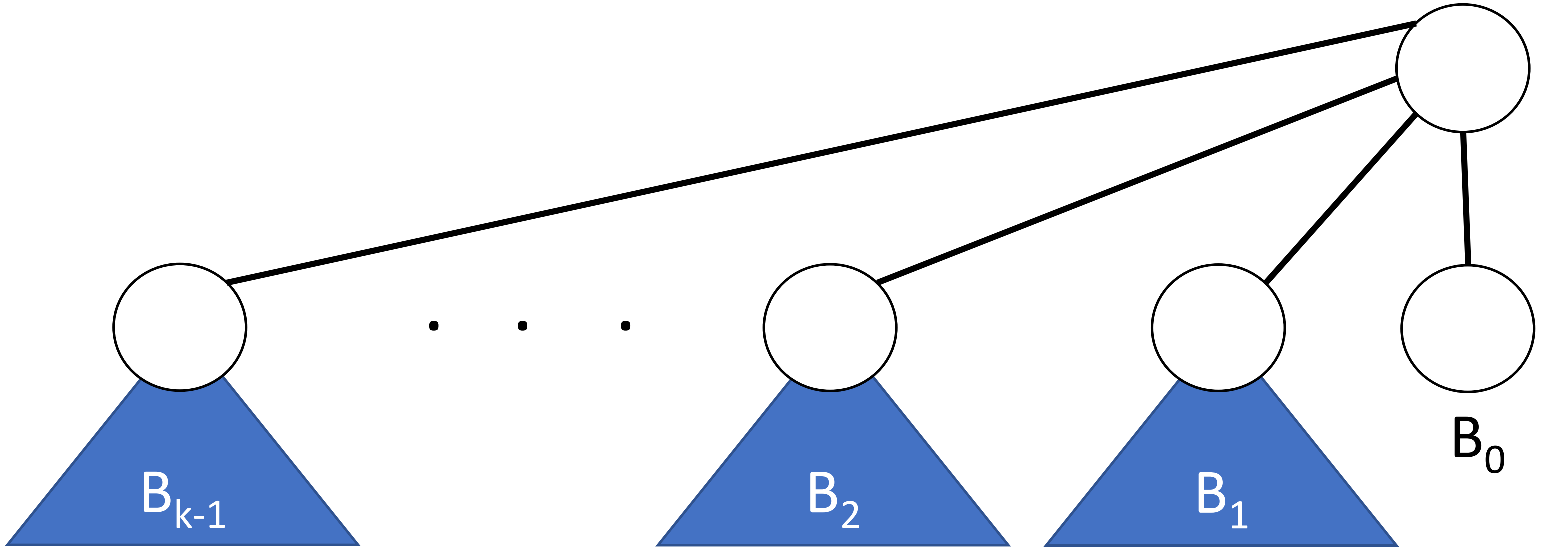

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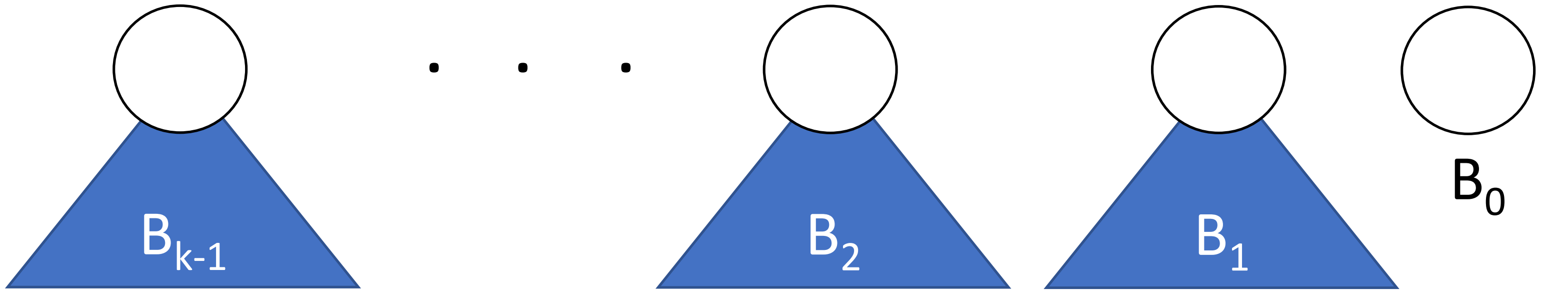

Union(T, Q)

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T is a Binomial Heap of size $n = 3 = \langle \; 1\;1 \rangle_2$

1	1	B_1	B_0	:	T

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	1	1		B_1	B_0	:	T
1	1	1	B_2	B_1	B_0	:	Q

Union(T, Q)

T is a Binomial Heap of size n = 3 = <   1 1>$_2$

Q is a Binomial Heap of size n = 7 = < 1 1 1>$_2$

	1	1	+		B_1	B_0	: T	+
1	1	1		B_2	B_1	B_0	: Q	

Union(T, Q)

T is a Binomial Heap of size $n = 3 = <\ \ 1\ 1>_2$

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	1					B_1		**Carry**
	1	1	**+**			B_1	B_0	: T **+**
1	1	1			B_2	B_1	B_0	: Q
		0					X	**Result**

Union(T, Q)

T is a Binomial Heap of size $n = 3 = <\ \ 1\ 1>_2$

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1	1			B_2	B_1		**Carry**	
	1	1			B_1	B_0	: T	
			+					**+**
1	1	1		B_2	B_1	B_0	: Q	
	1	0			B_1	X	**Result**	

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		1	1				B_1	B_0	: T
				+					**+**
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		1	1				B_1	B_0	: T
				+					+
	1	1	1			B_2	B_1	B_0	: Q
1	0	1	0		B_3	X	B_1	X	**Result**

How many new edges were added?

Union(T, Q)

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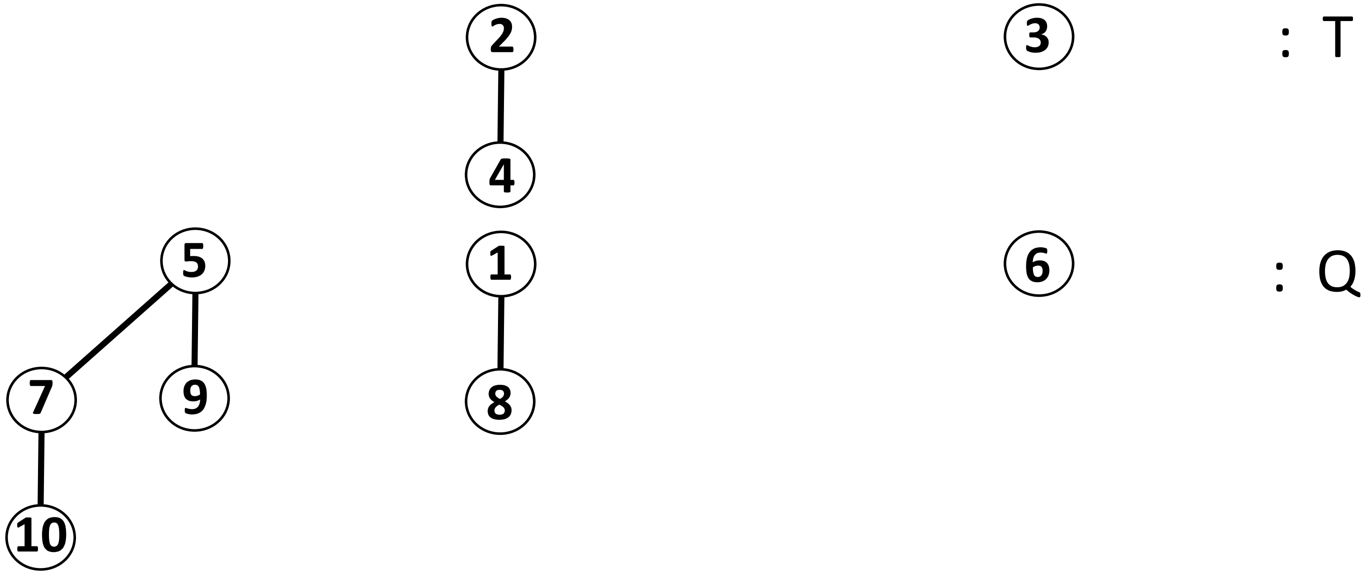
1	1	1			B_3	B_2	B_1		**Carry**
		1	1				B_1	B_0	: T
				+					**+**
	1	1	1			B_2	B_1	B_0	: Q
1	0	1	0		B_3	X	B_1	X	**Result**

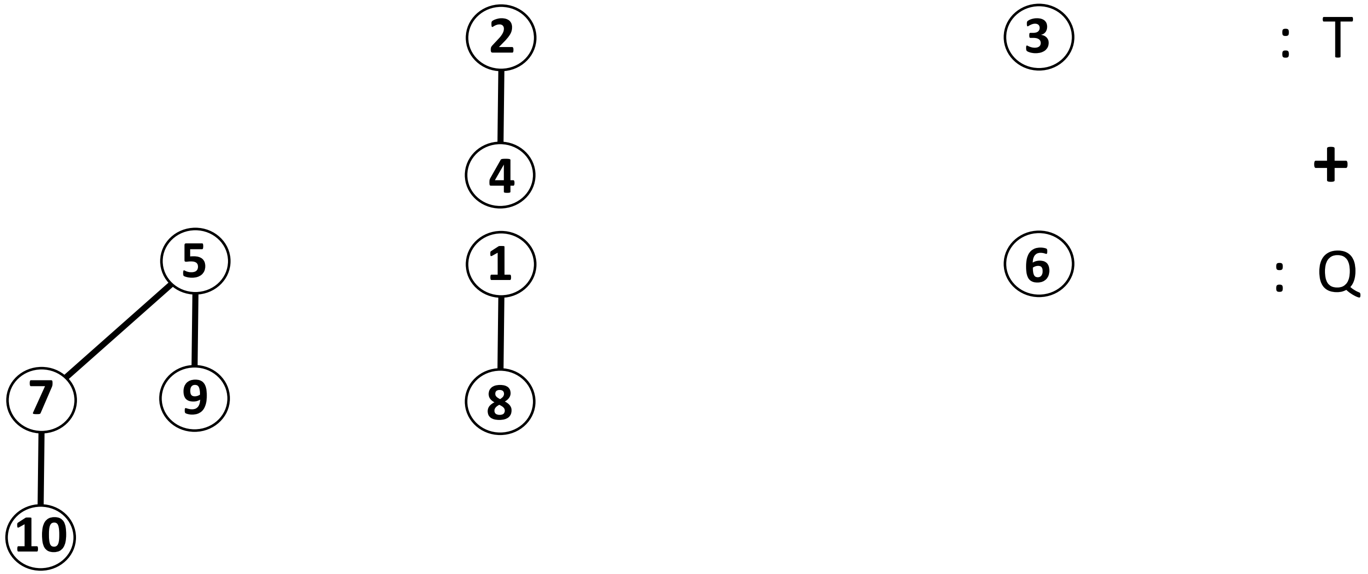
How many new edges were added?

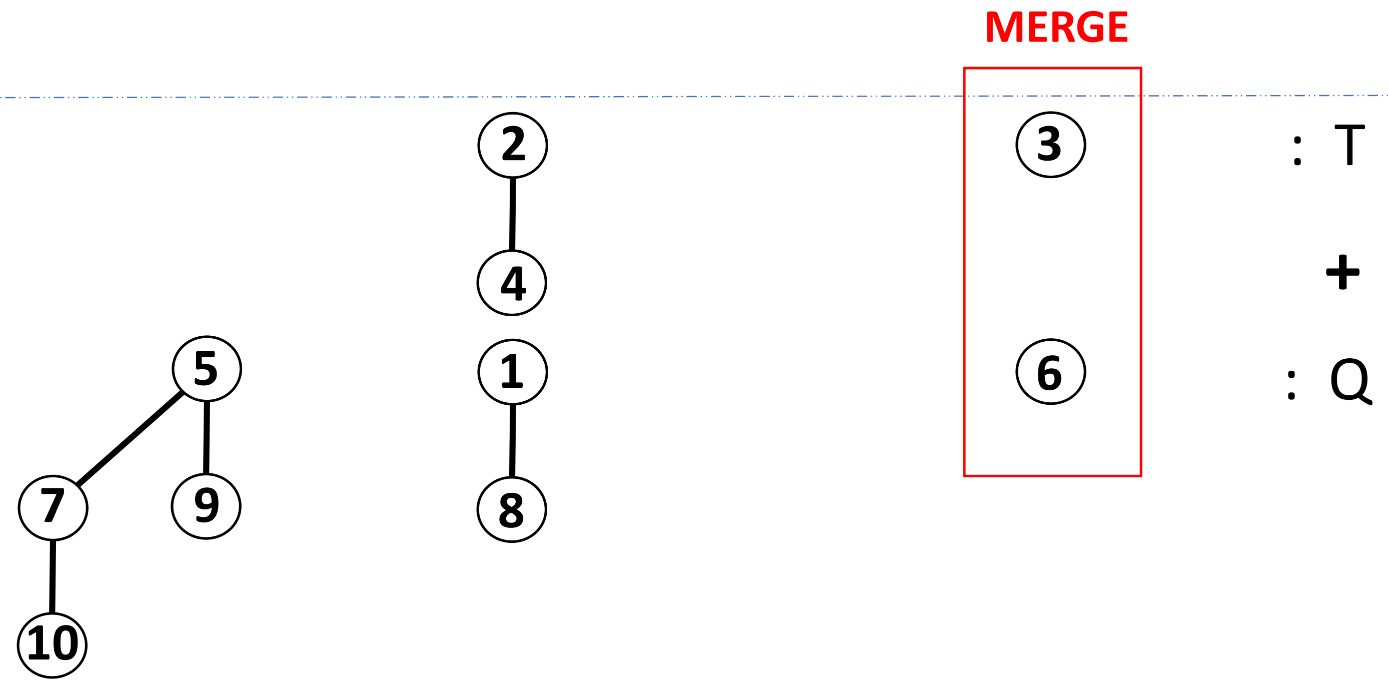
3

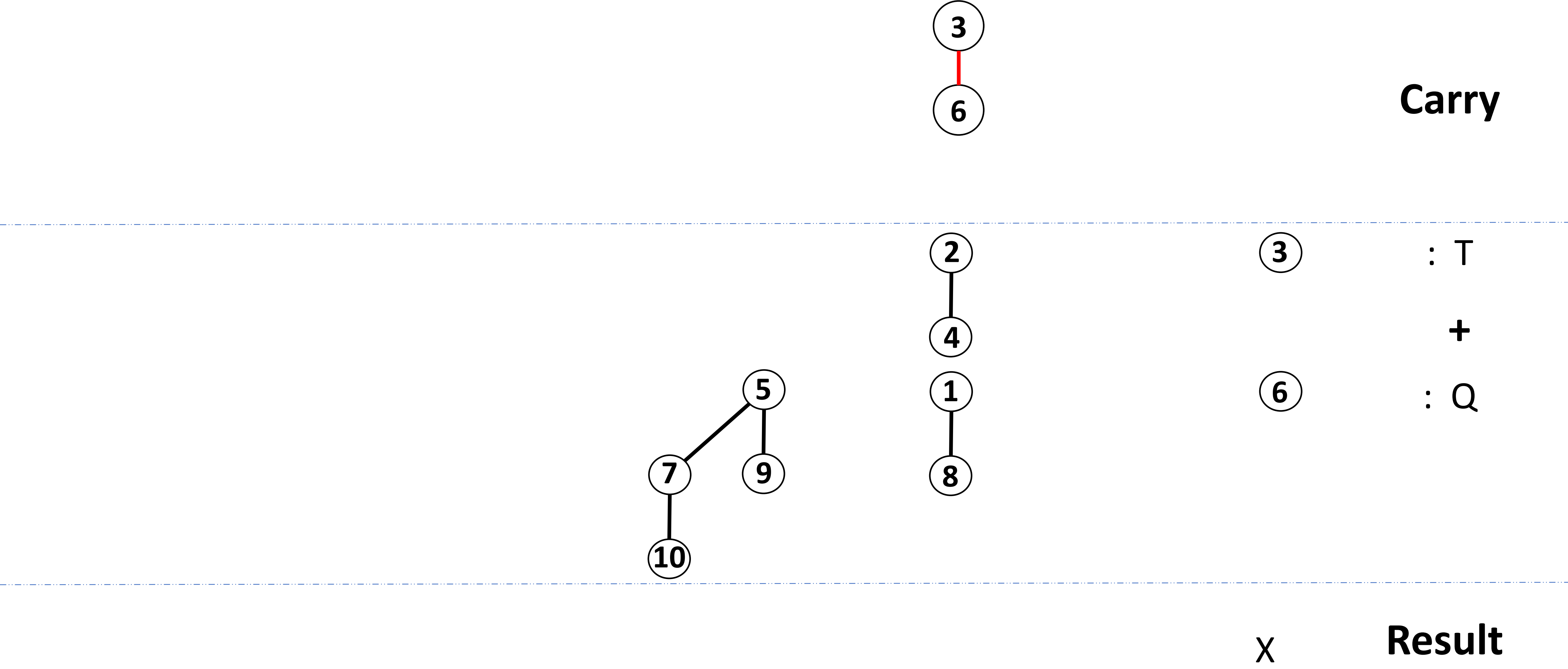
: T

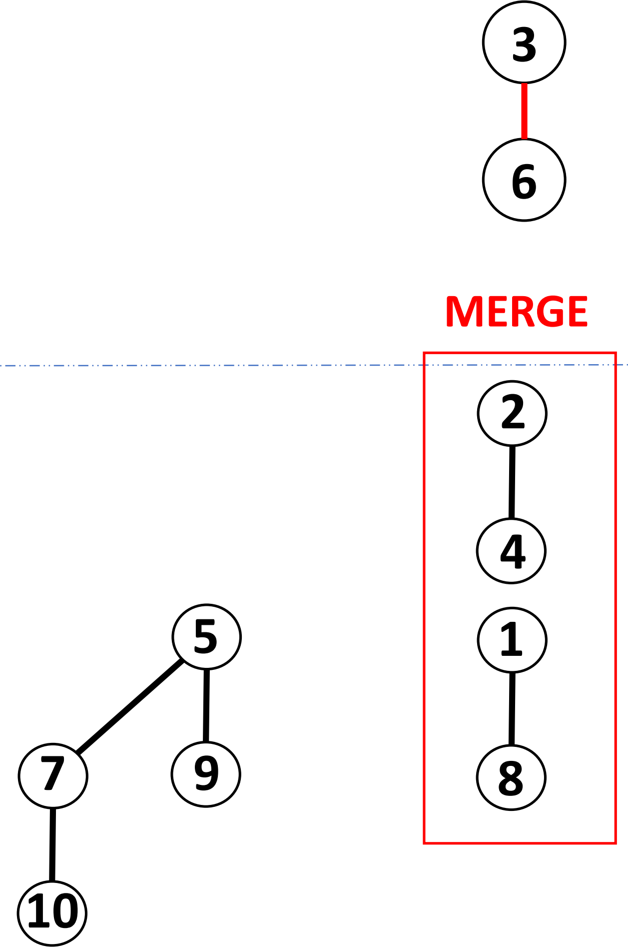


2
3
: T
4
5
1
6
: Q
7
9
8
10


: T
+
: Q


MERGE
3
6
: T
+
: Q
2
4
1
8
5
7
9
10


Carry
3 : T
+
6 : Q
X
Result



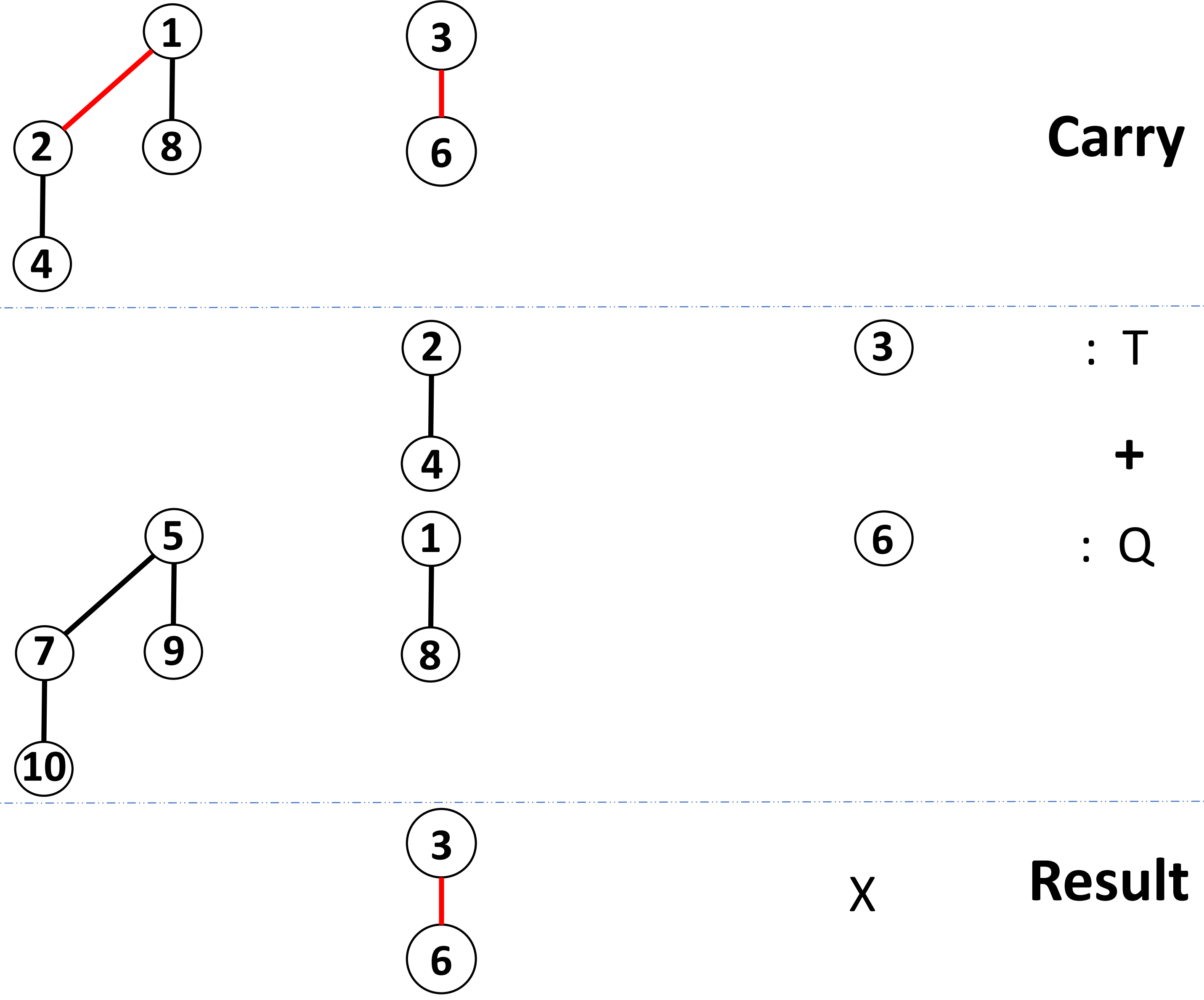
Carry

3 : T

+

6 : Q

X **Result**



Carry
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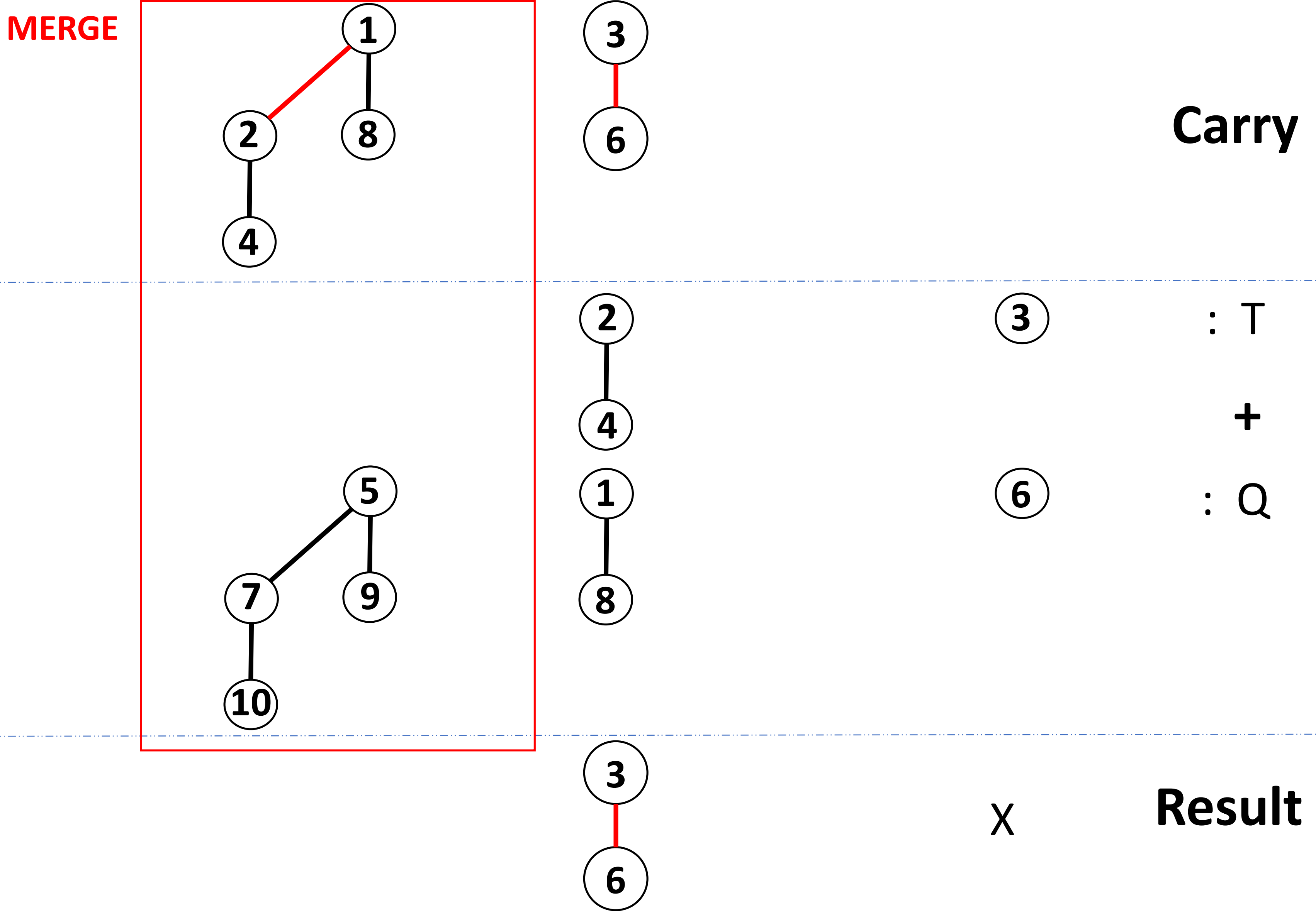

MERGE
1
2
8
4
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9
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Carry
2
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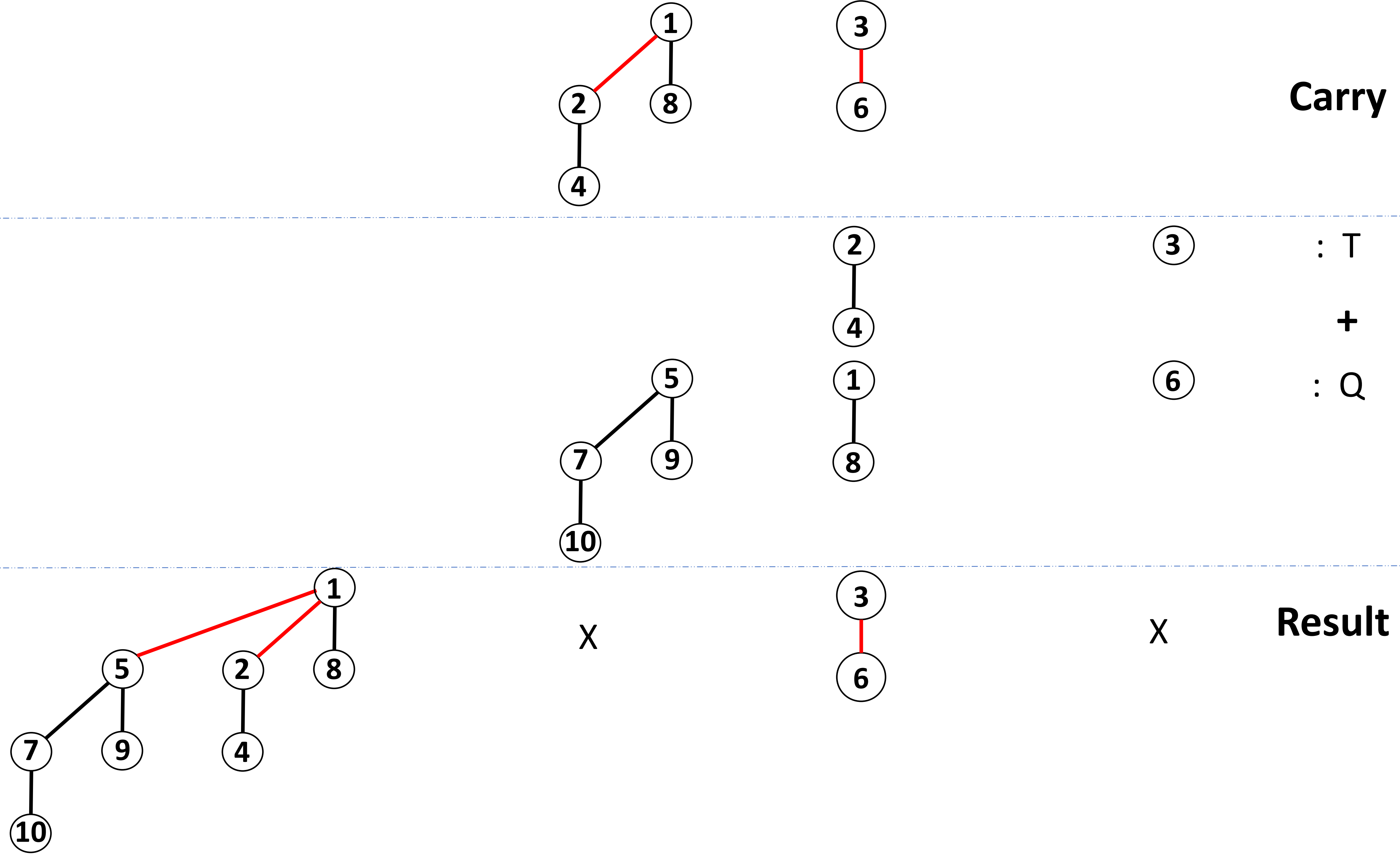
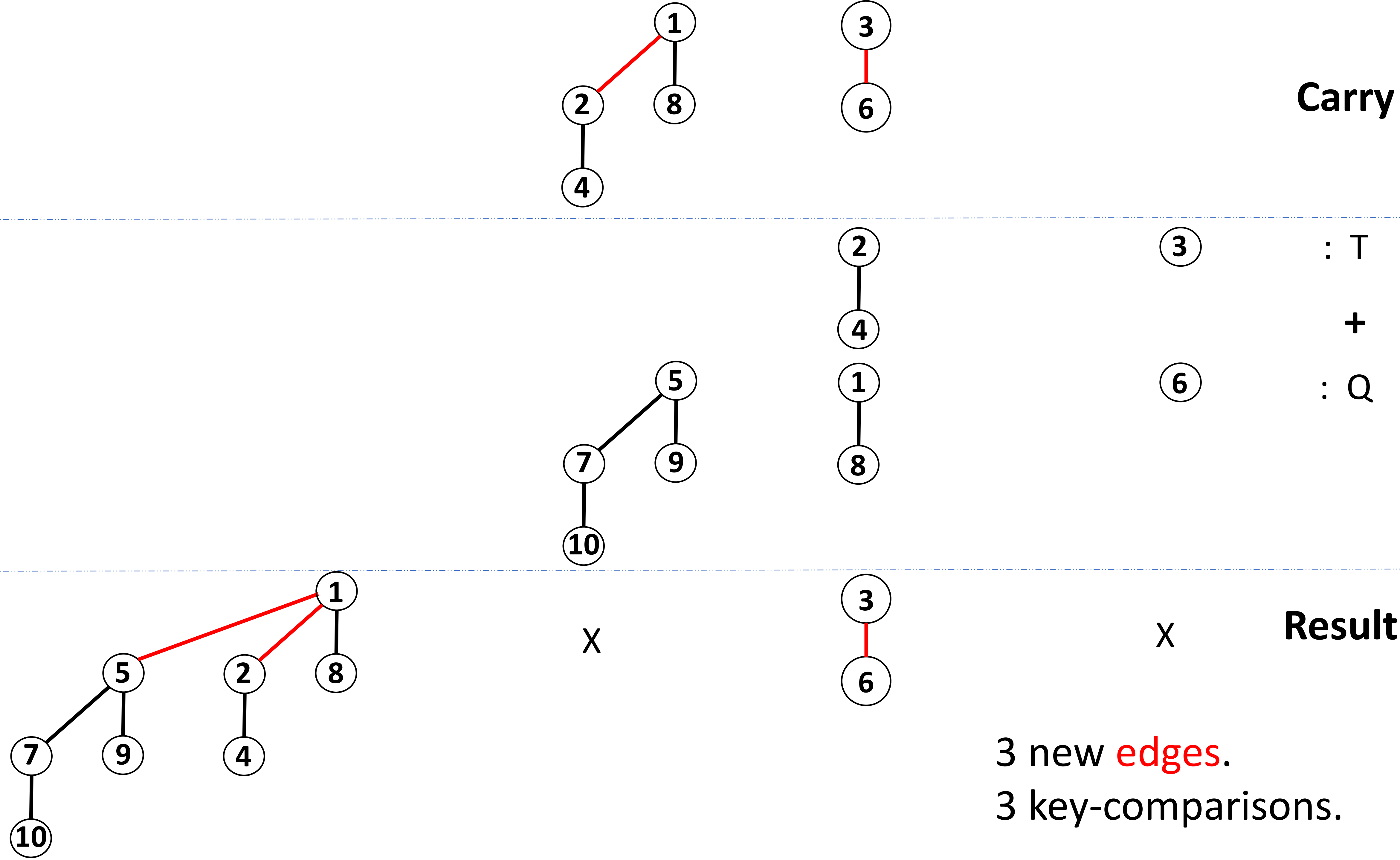
6
X
Result

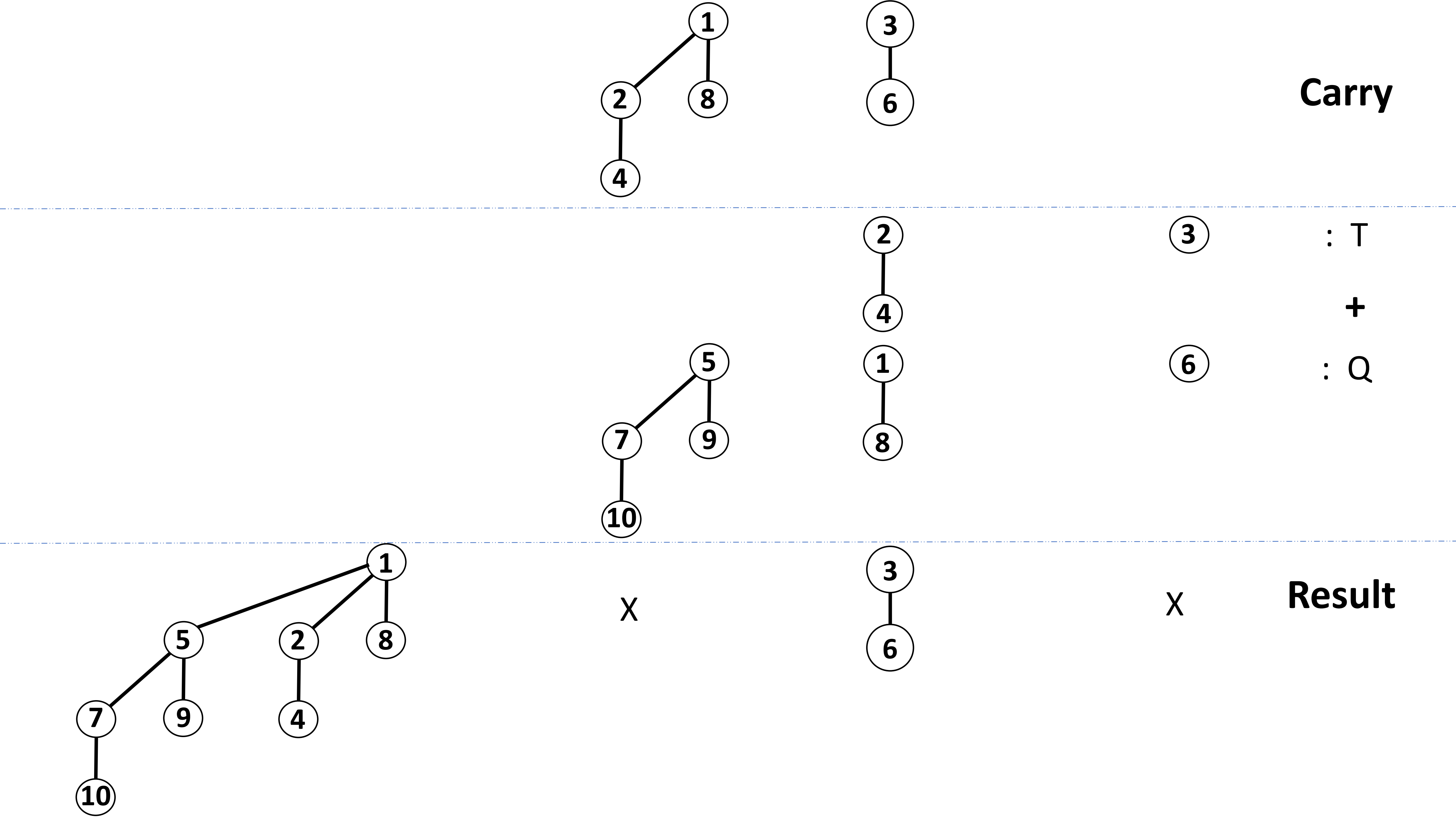


Carry
: T
+
: Q
X
Result



3 new edges.

3 key-comparisons.


Carry
: T
+
: Q
X
X
Result

Worst-Case Complexity of **Union(T, Q)**

Say |T| <= n and |Q| <= n (i.e. each contains at most n elements)

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$\Rightarrow$ Each of T, Q have O(log n) B_k trees.

$\Rightarrow$ **Union**(T, Q) takes at most O(log n) key-comparisons