Balanced BST

Balanced BST

- Goal: BST with $\Theta(\log n)$ height.

Balanced BST

- Goal: BST with $\Theta(\log n)$ height.
- Method: Balanced BSTs

Balanced BST

- Goal: BST with $\Theta(\log n)$ height.
- Method: Balanced BSTs
- Last lecture:
  - ✓ Defined AVL trees

Balanced BST

- Goal: BST with $\Theta(\log n)$ height.
- Method: Balanced BSTs
- Last lecture:
  - ✓ Defined AVL trees
  - ✓ Explained how to insert a key into an AVL tree:
    - ✓ High-level procedure
    - ✓ Examples to illustrate this procedure

Balanced BST

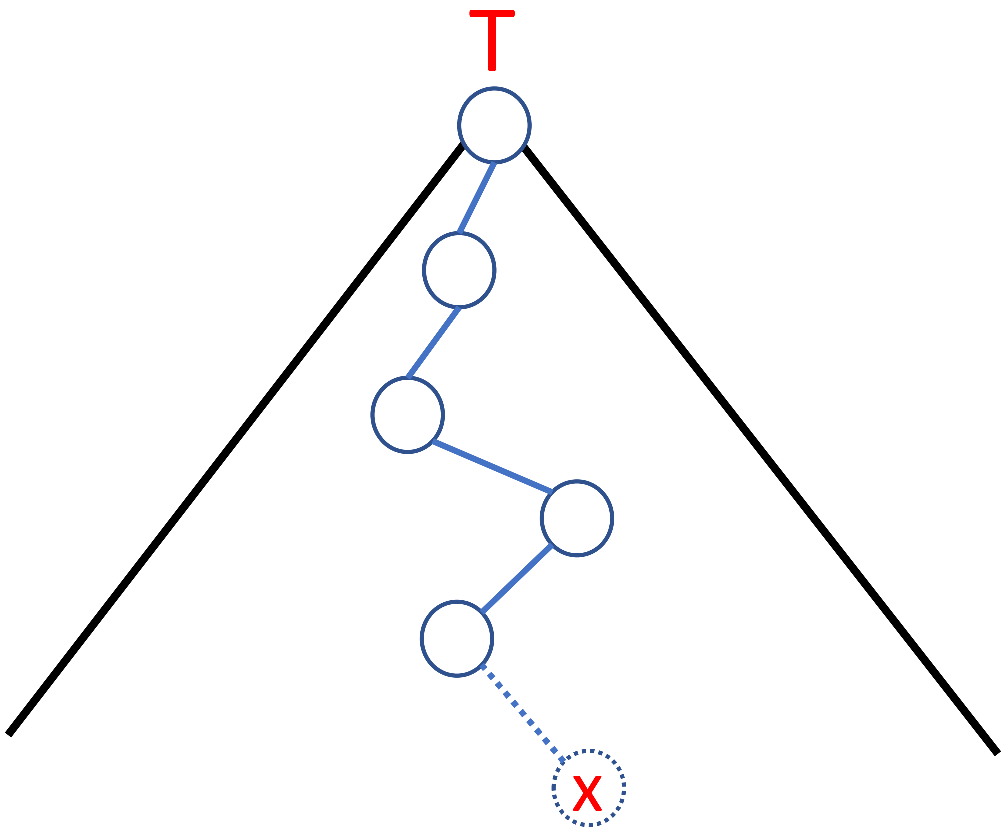
- Goal: BST with $\Theta(\log n)$ height.
- Method: Balanced BSTs
- Last lecture:
  - ✓ Defined AVL trees
  - ✓ Explained how to insert a key into an AVL tree:
    - ✓ High-level procedure
    - ✓ Examples to illustrate this procedure
  - ✓ From these examples we ``extracted'' an algorithm

Balanced BST

- Goal: BST with $\Theta(\log n)$ height.
- Method: Balanced BSTs
- Last lecture:
  - ✓ Defined AVL trees
  - ✓ Explained how to insert a key into an AVL tree:
    - ✓ High-level procedure
    - ✓ Examples to illustrate this procedure
  - ✓ From these examples we ``extracted'' an algorithm
- This lecture:
  - ✓ We show this algorithm works in all cases

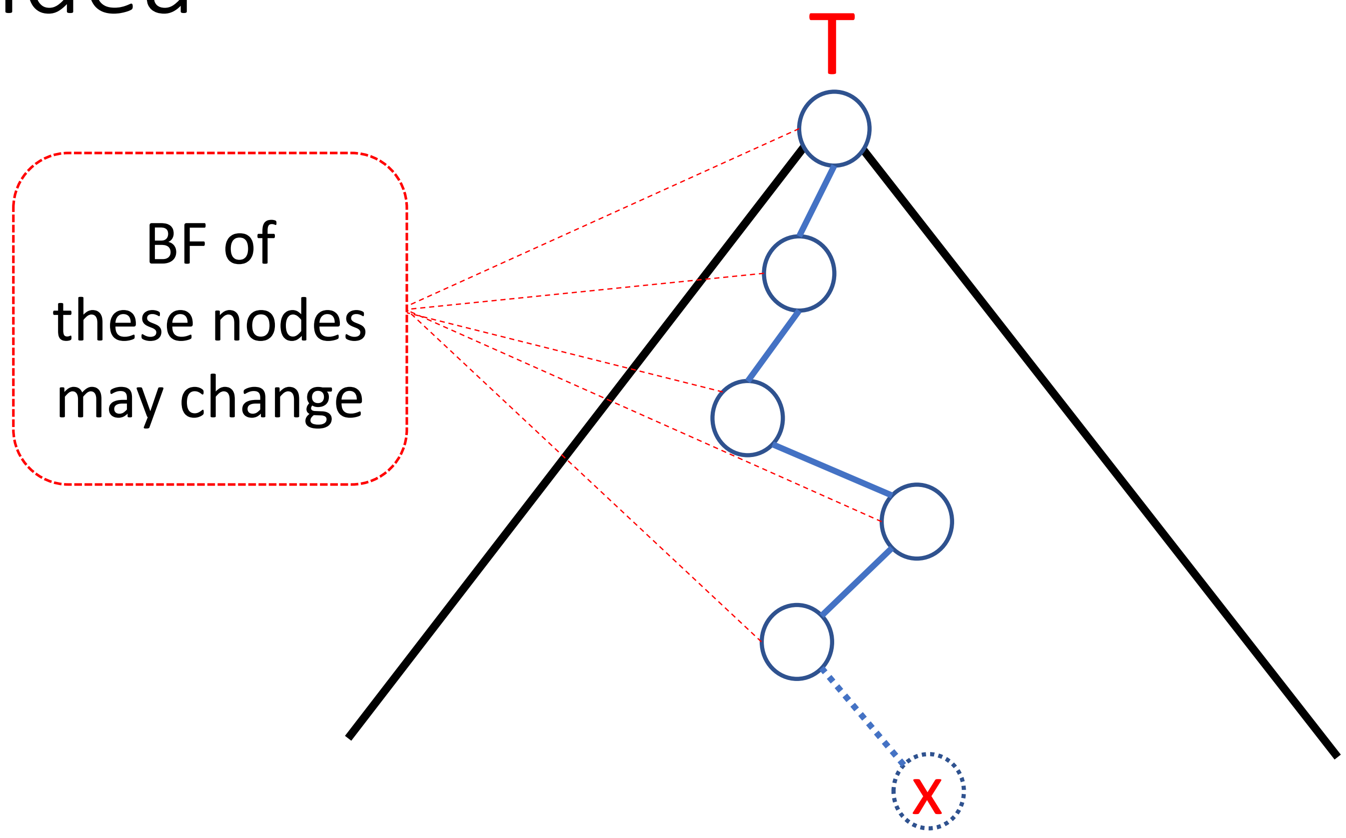
Insert(T, x) : General idea

- Insert x into T as in any BST :
  - x is now a leaf



Insert(T, x) : General idea

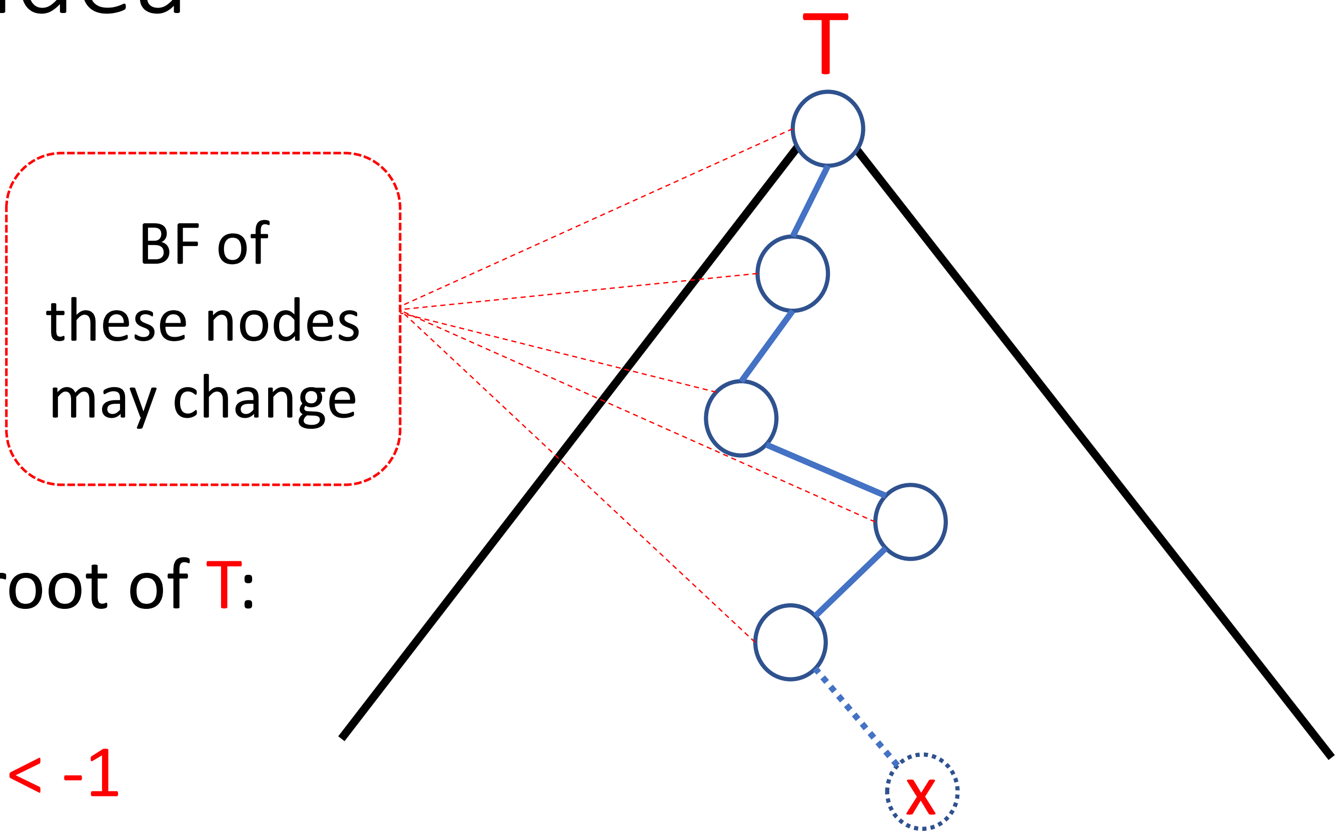
- Insert x into T as in any BST :
  - x is now a leaf



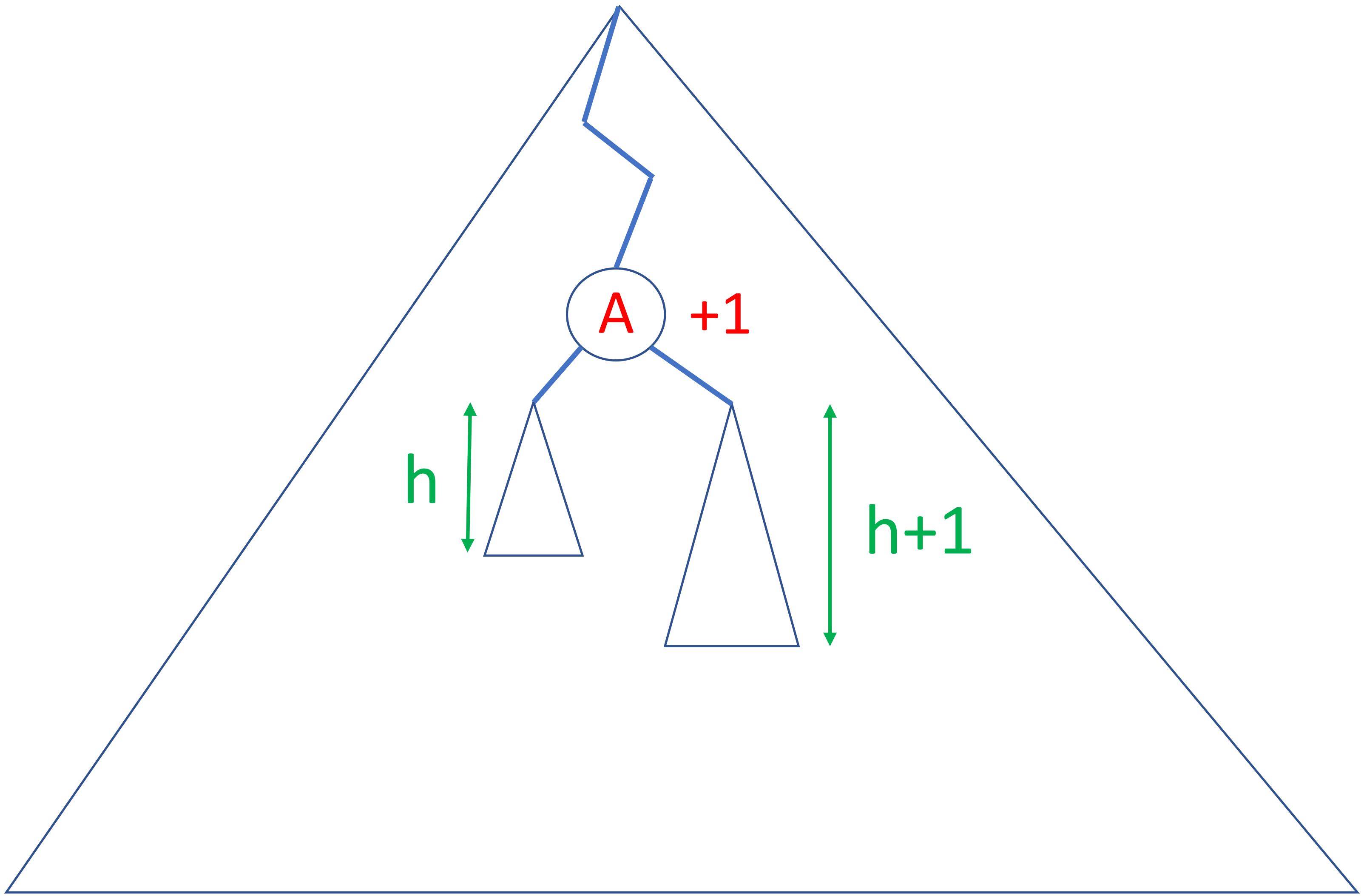
Insert(T, x) : General idea

- Insert x into T as in any BST :
  - x is now a leaf

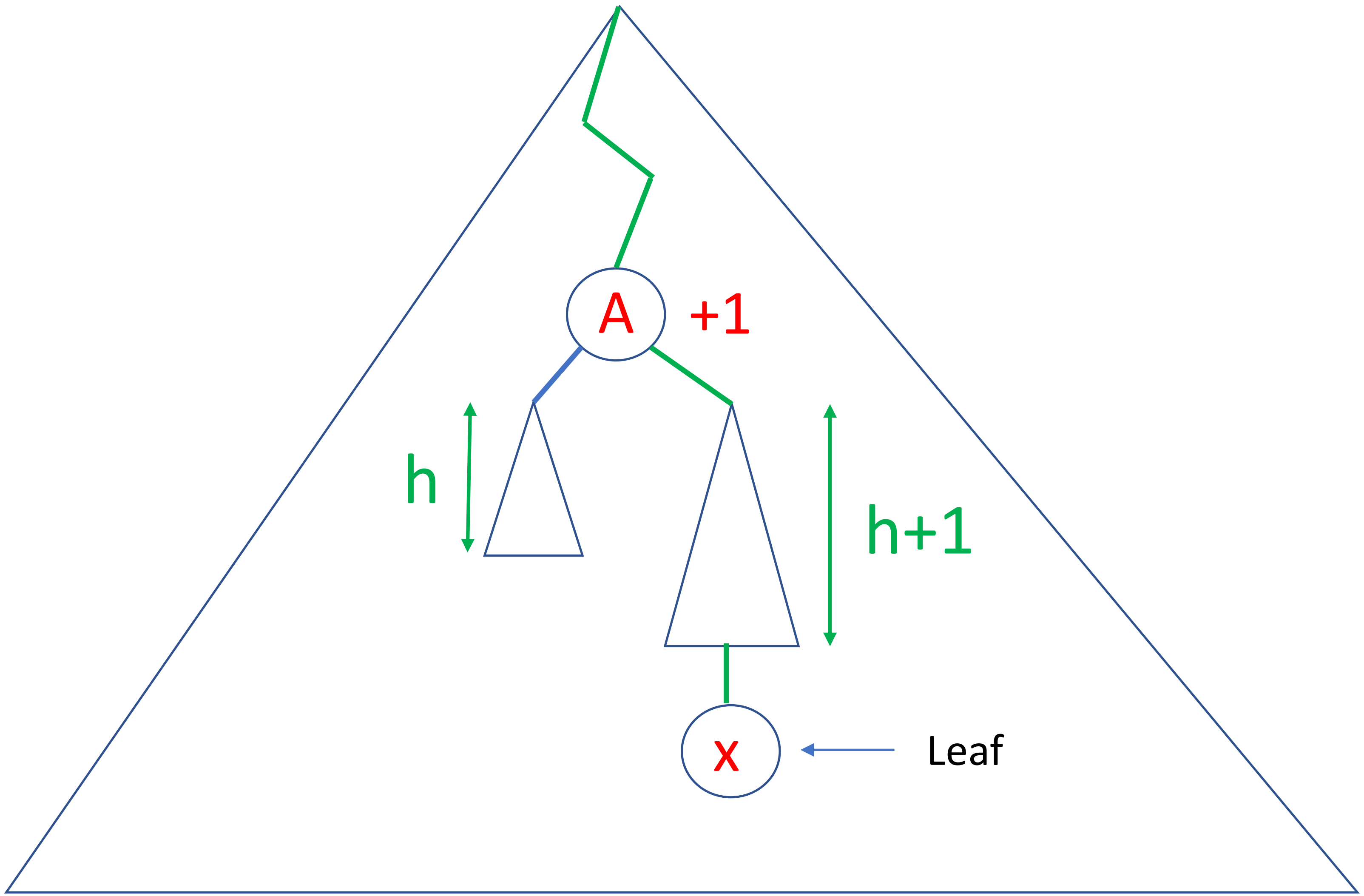
- For each node v from x to the root of T:
  - Adjust BF(v)
  - Rebalance if BF(v) > 1 or BF < -1



Inserting x into an AVL Tree T

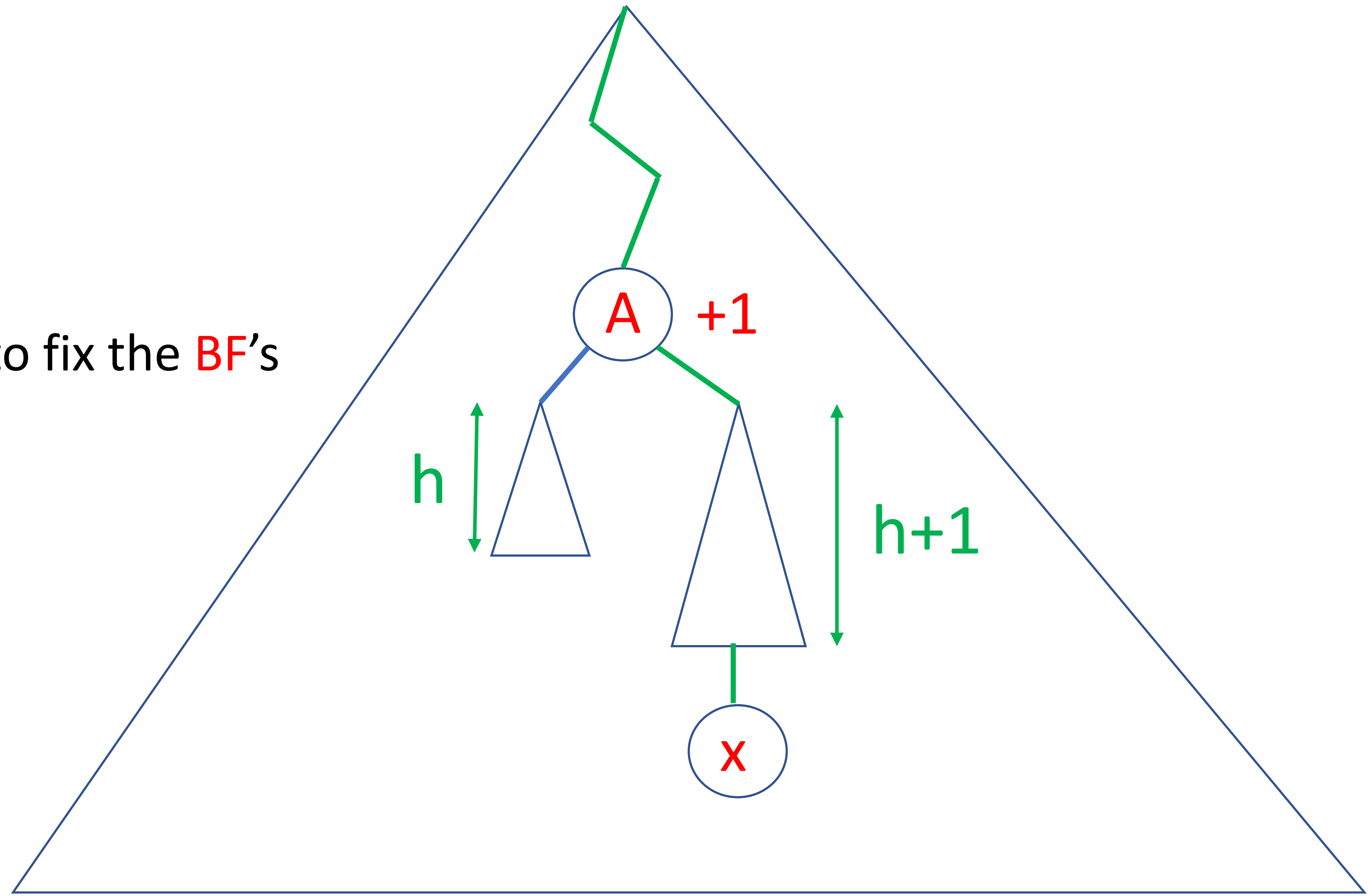


Inserting x into an AVL Tree T



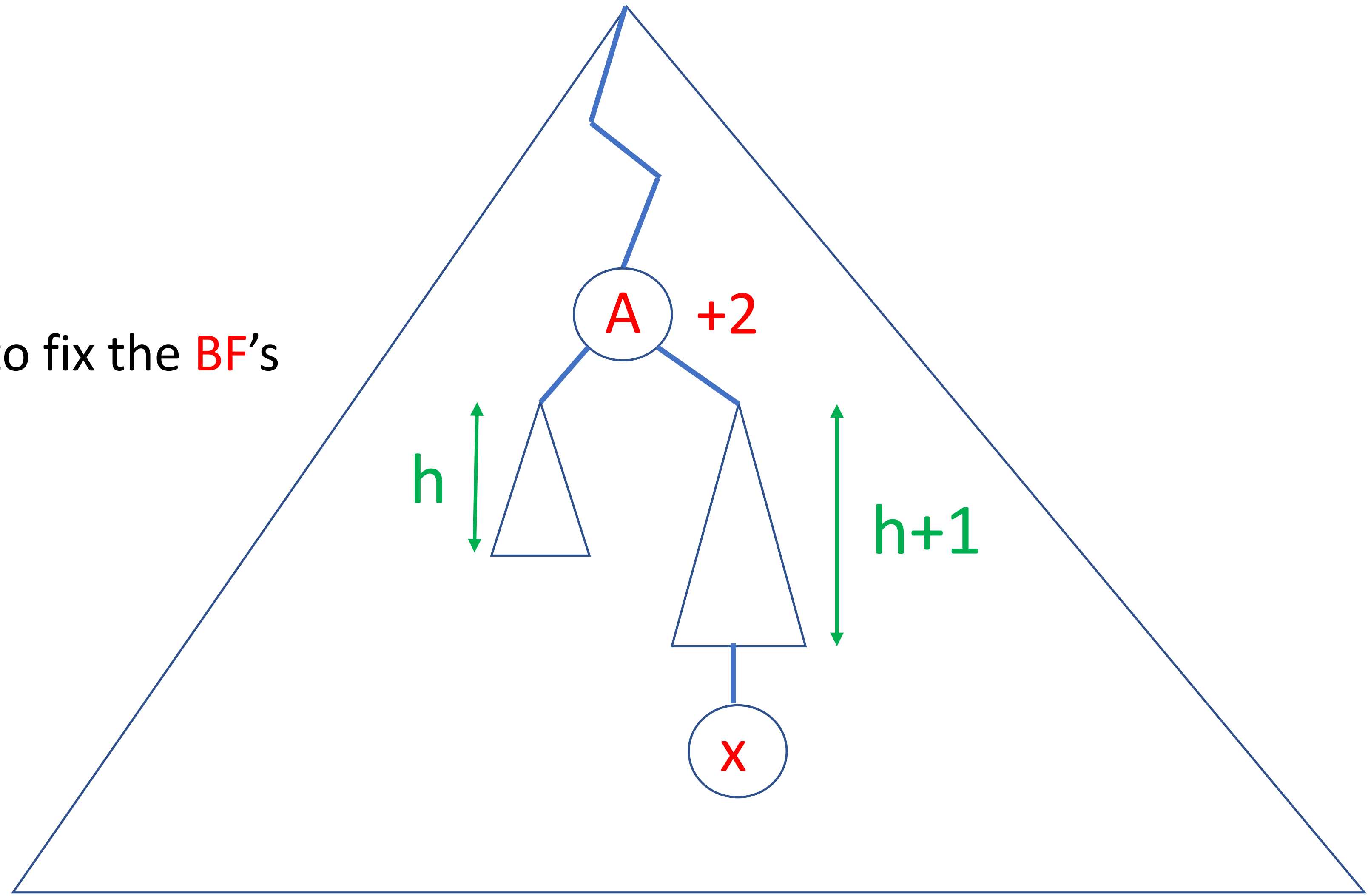
Inserting x into an AVL Tree T

Go up from x to the root of T to fix the BF's

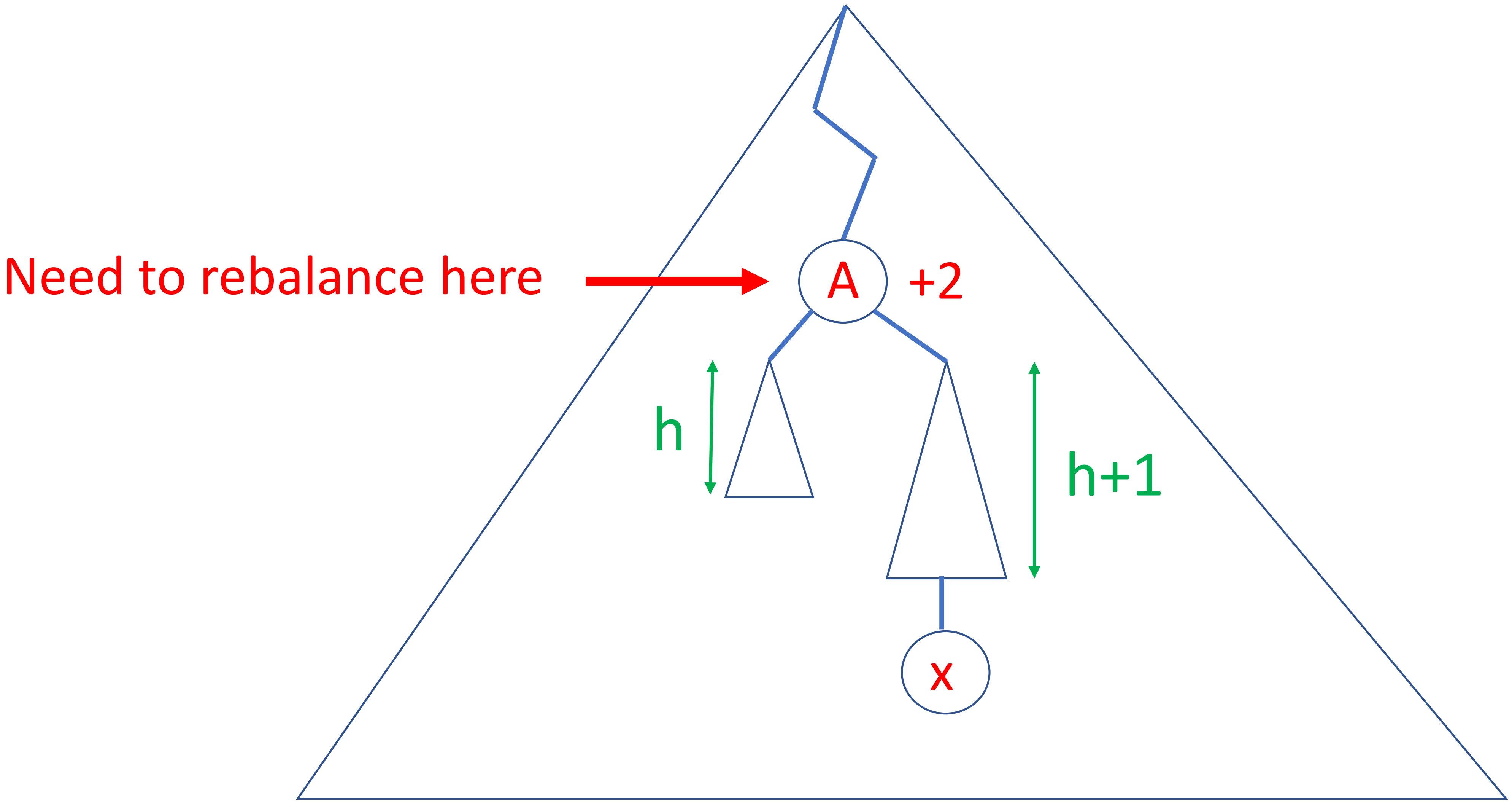


Inserting x into an AVL Tree T

Go up from x to the root of T to fix the BF's

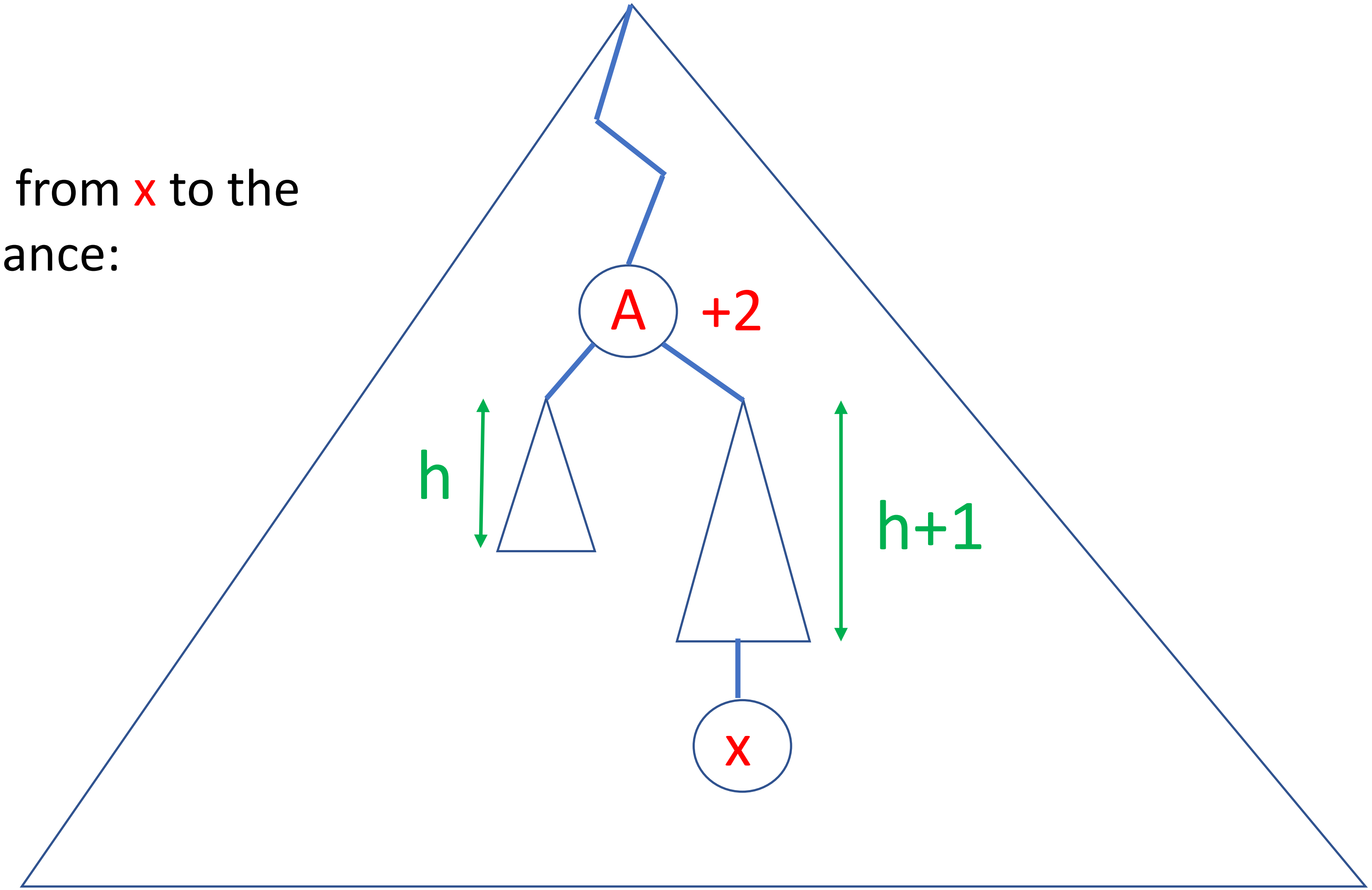


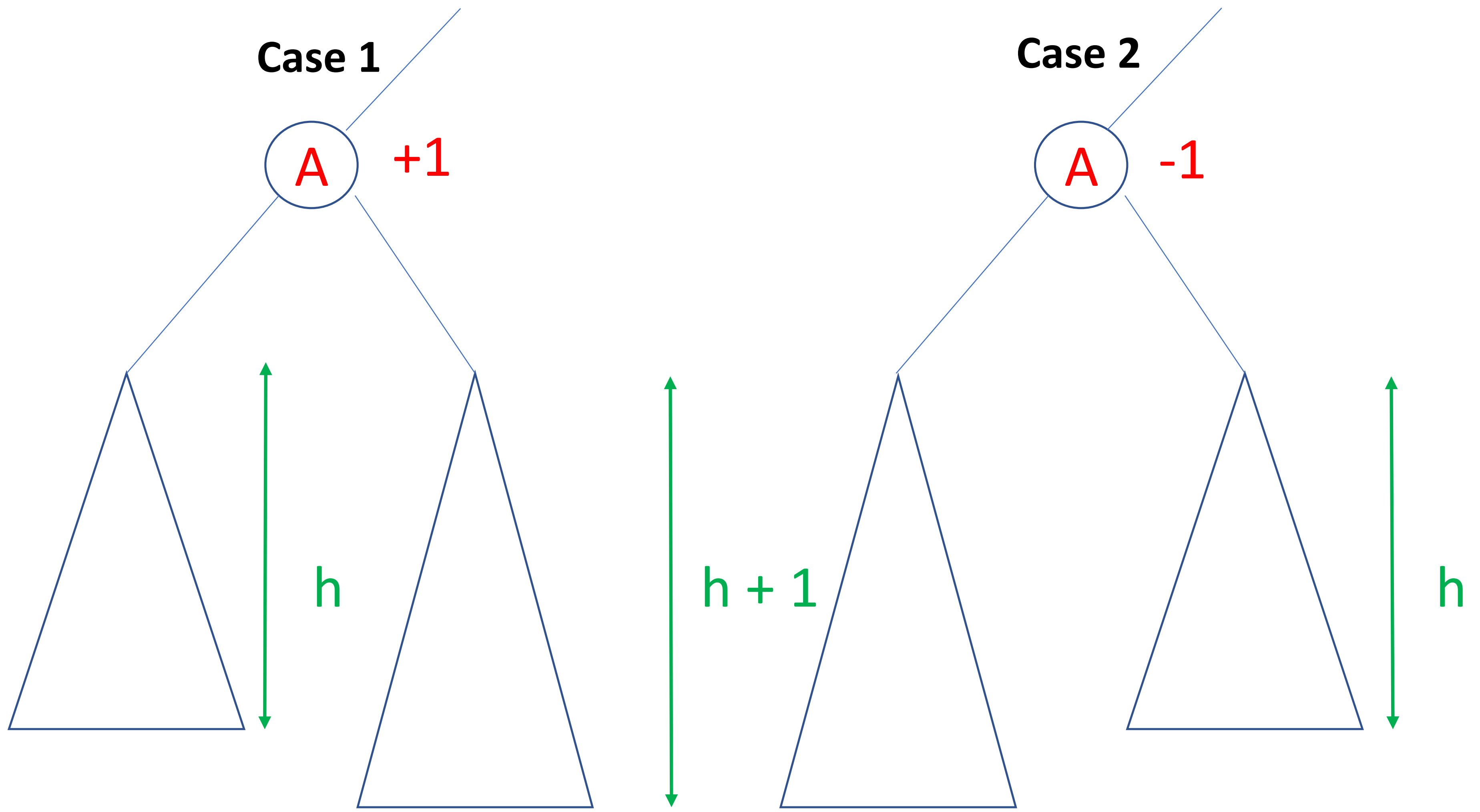
Inserting x into an AVL Tree T



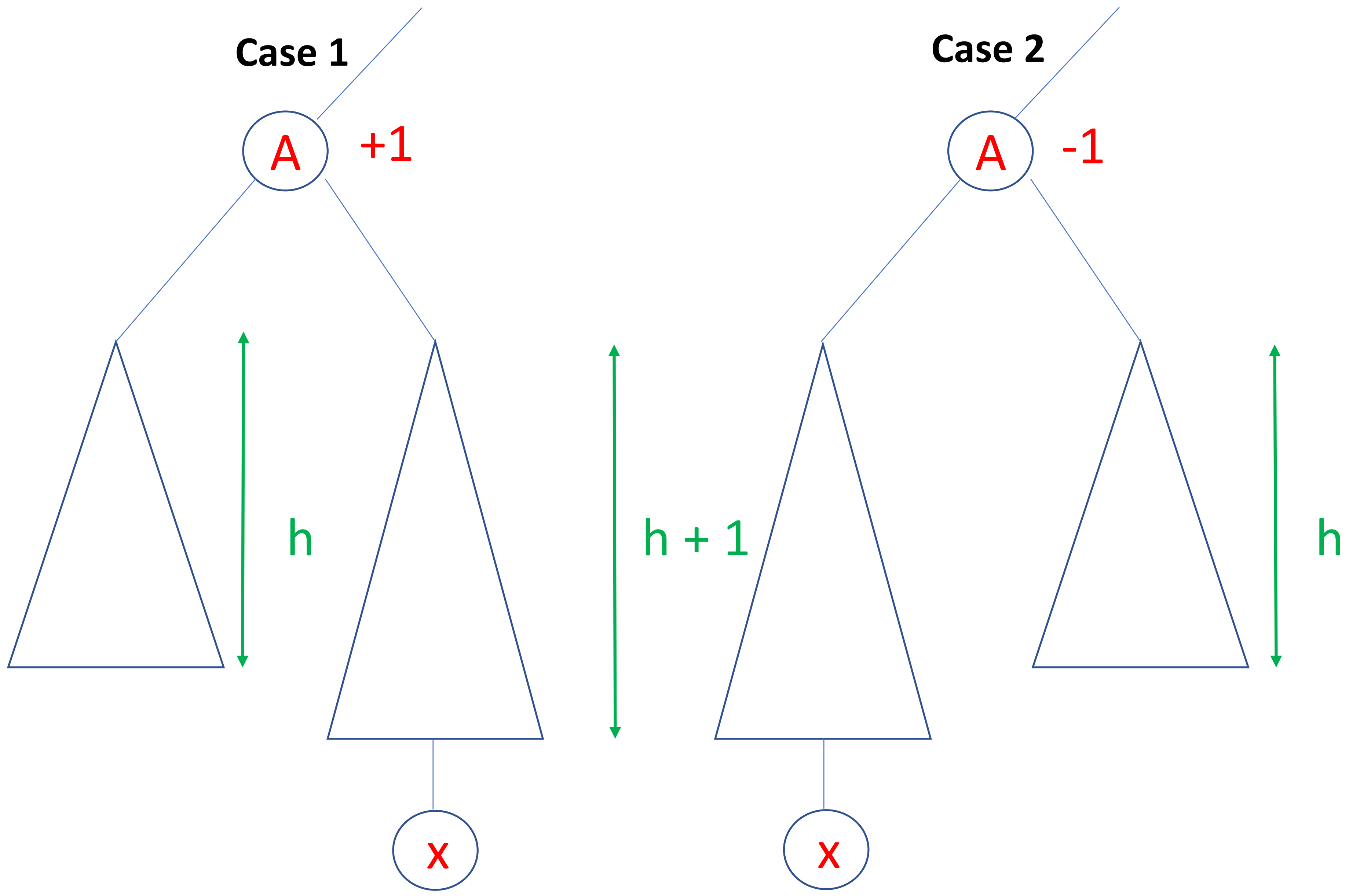
Inserting x into an AVL Tree T

A is the first node on the path from x to the root of T that loses its AVL balance:
BF(A) becomes +2 or -2.

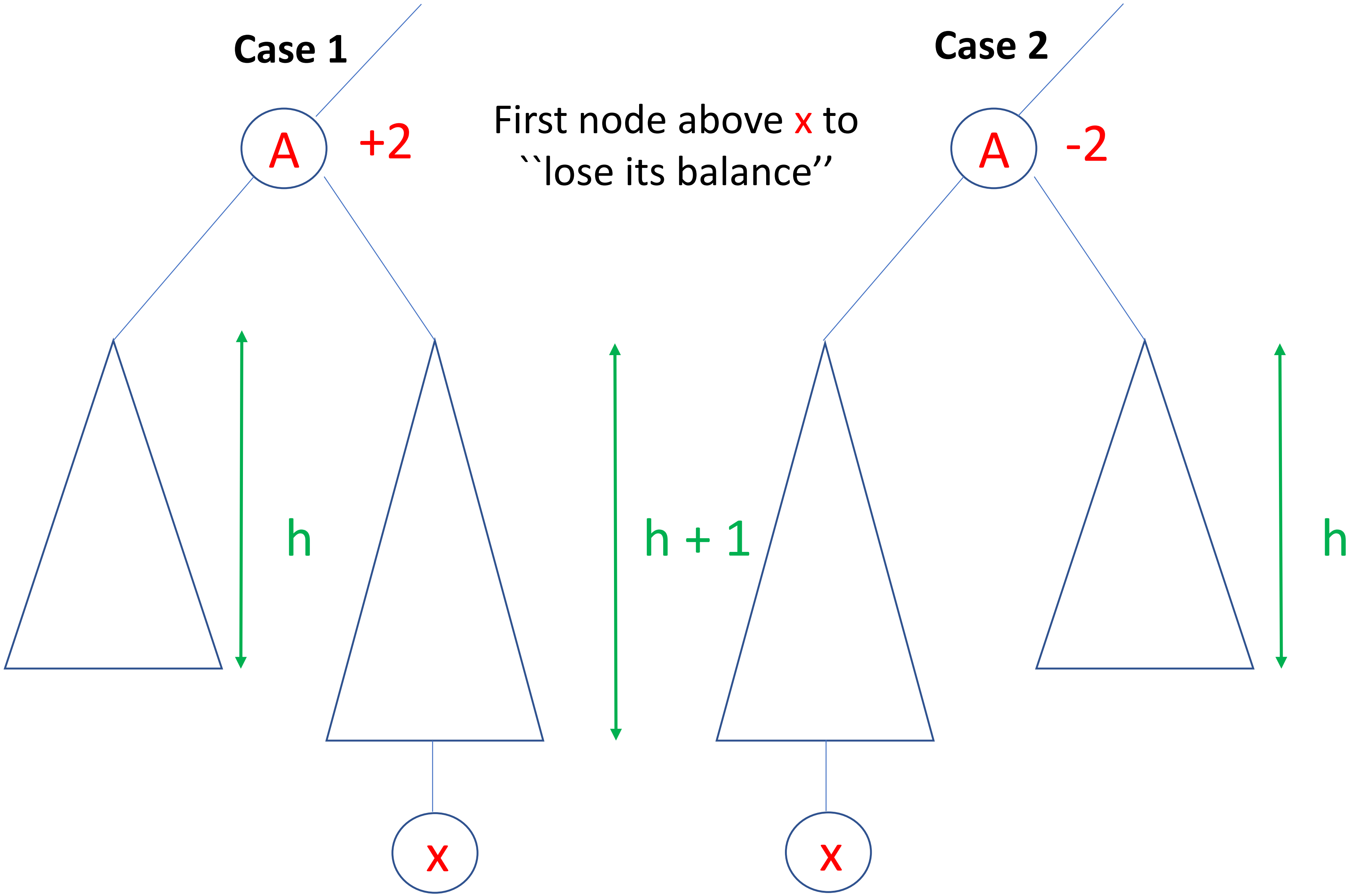




Case 1
A
+1
h
h + 1
Case 2
A
-1
h



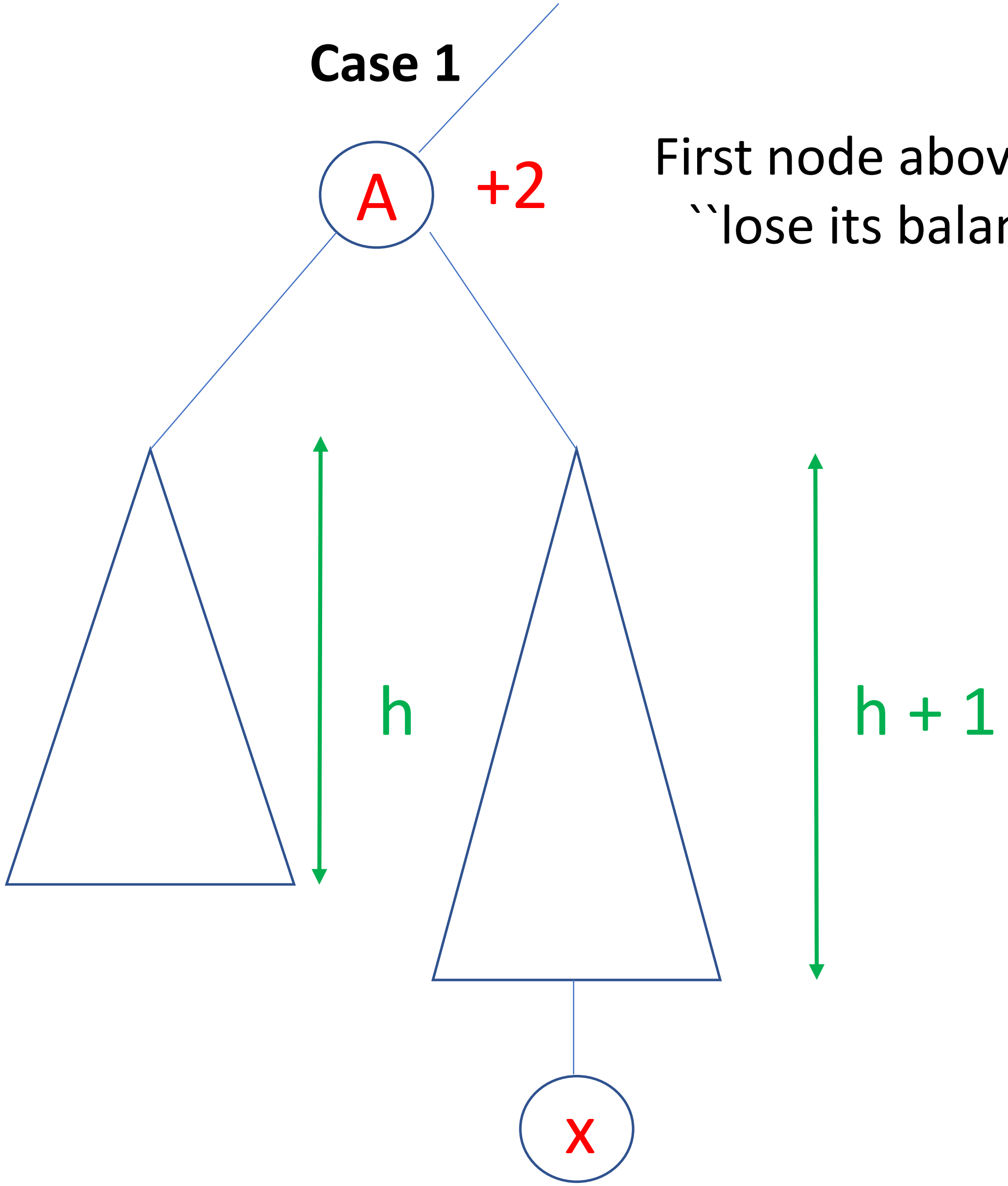
Case 1
A
+1
h
h + 1
x
Case 2
A
-1
h
x



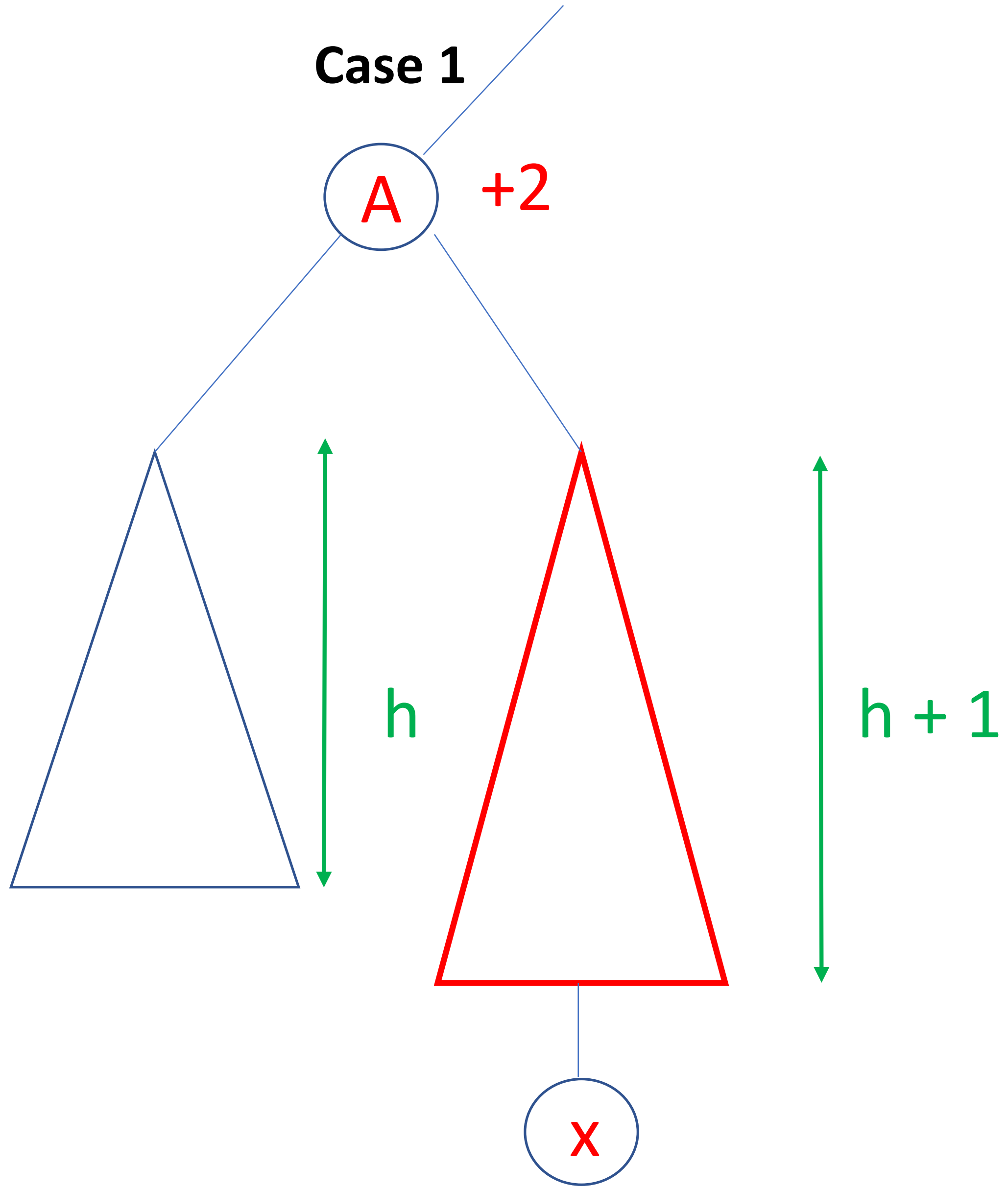
Case 1
A
+2
First node above x to
``lose its balance''
Case 2
A
-2
h
h + 1
h
x
x

Case 1

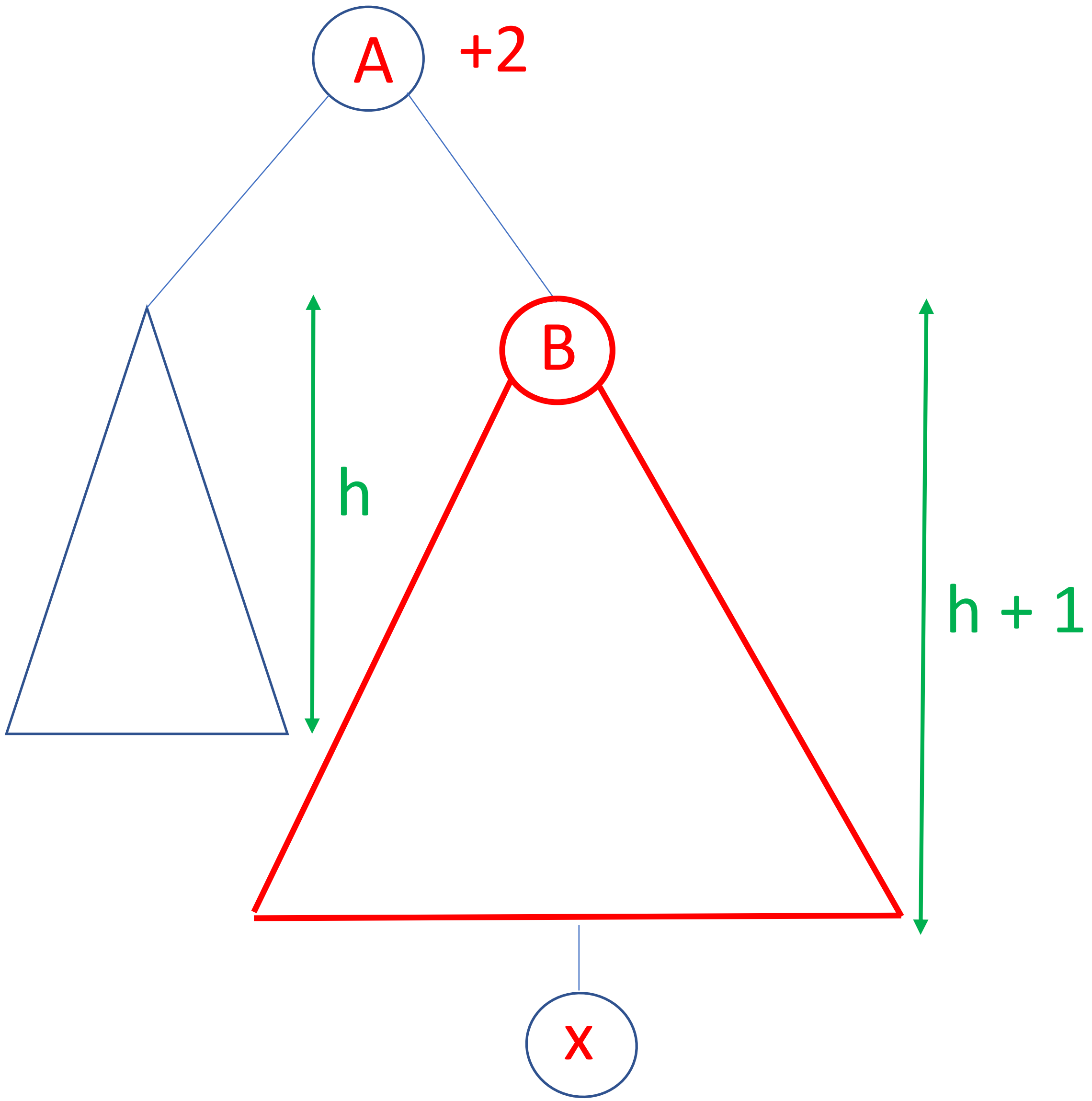
First node above x to ``lose its balance''



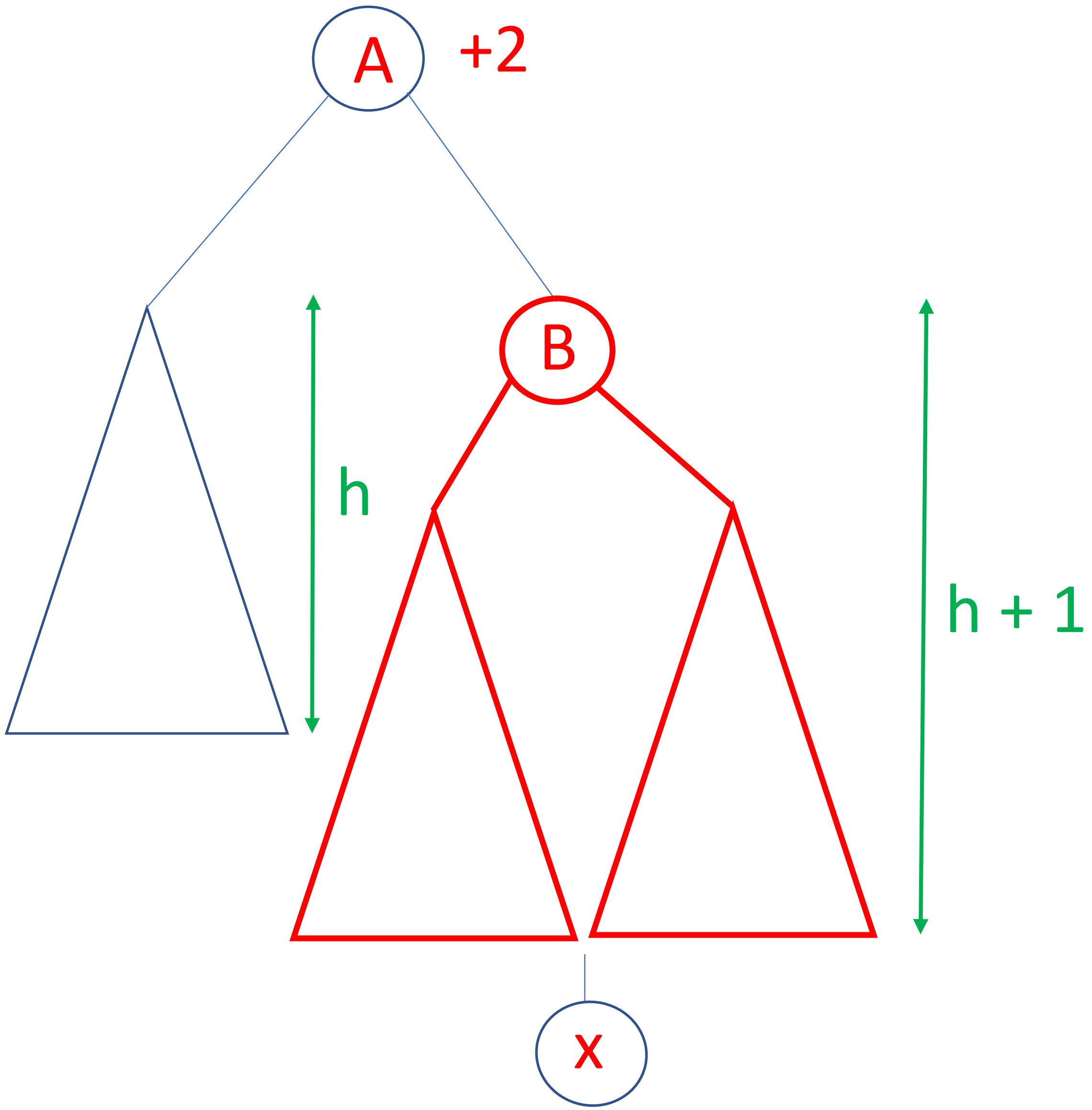
Case 1



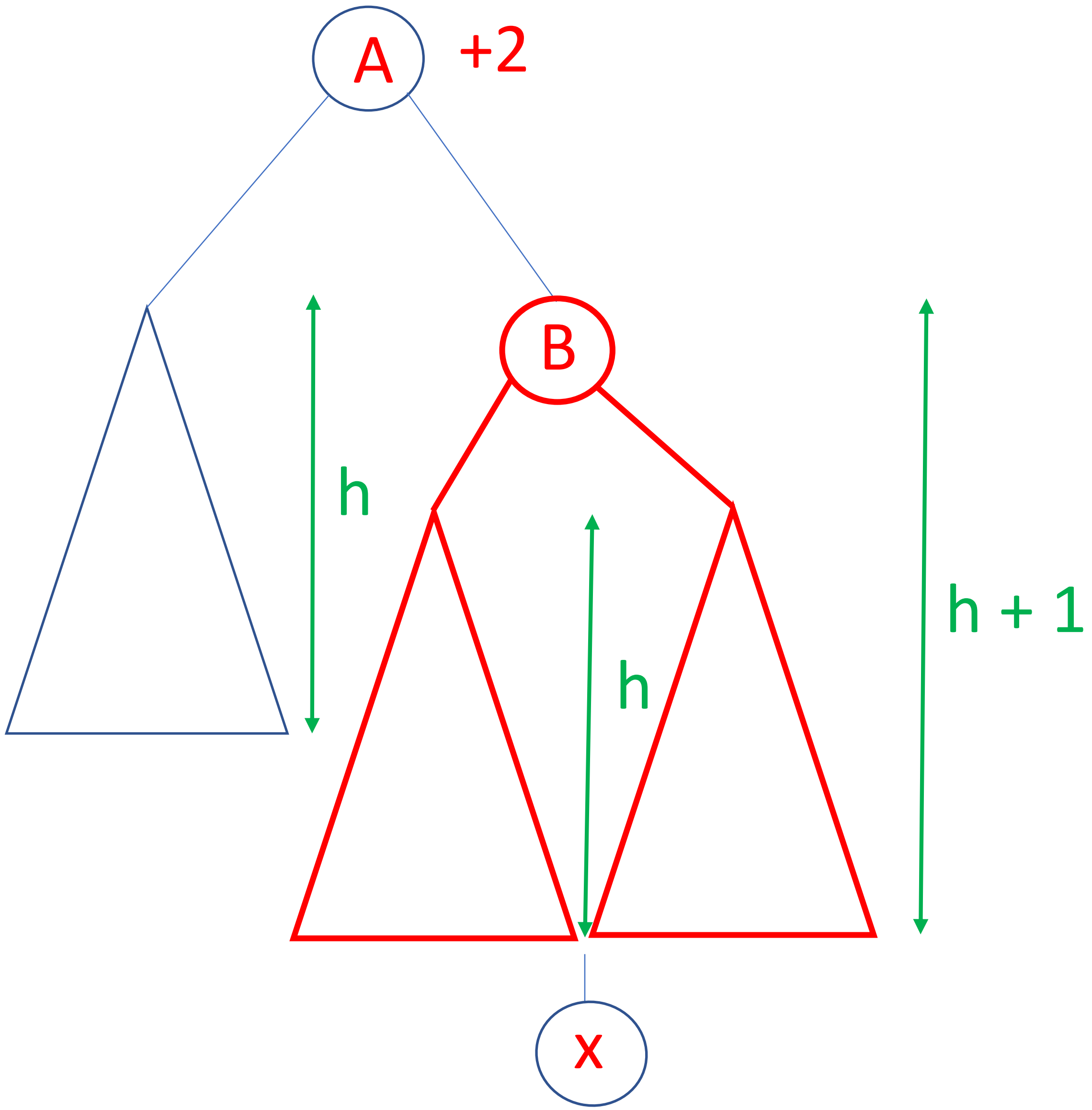
Case 1



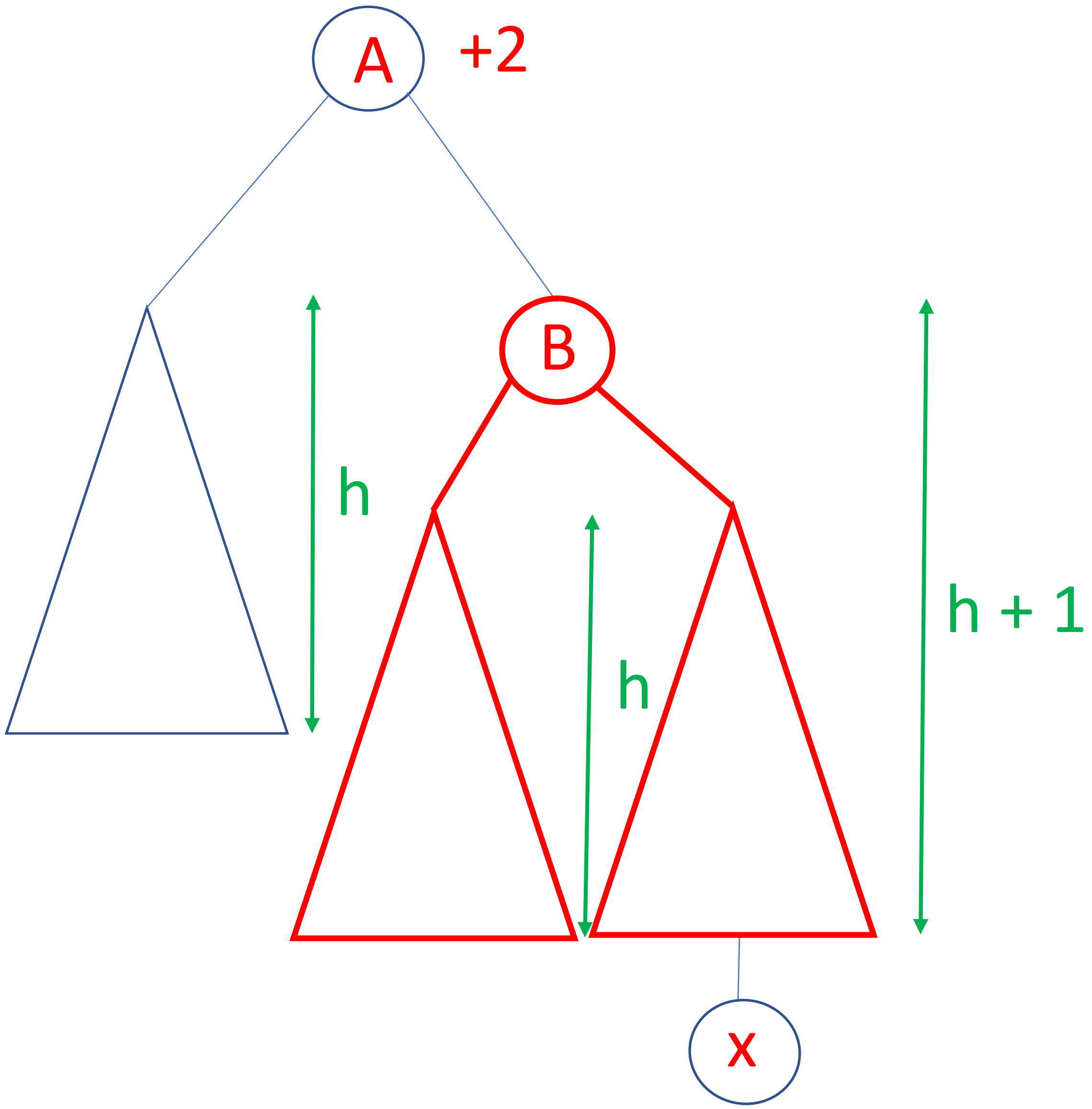
Case 1



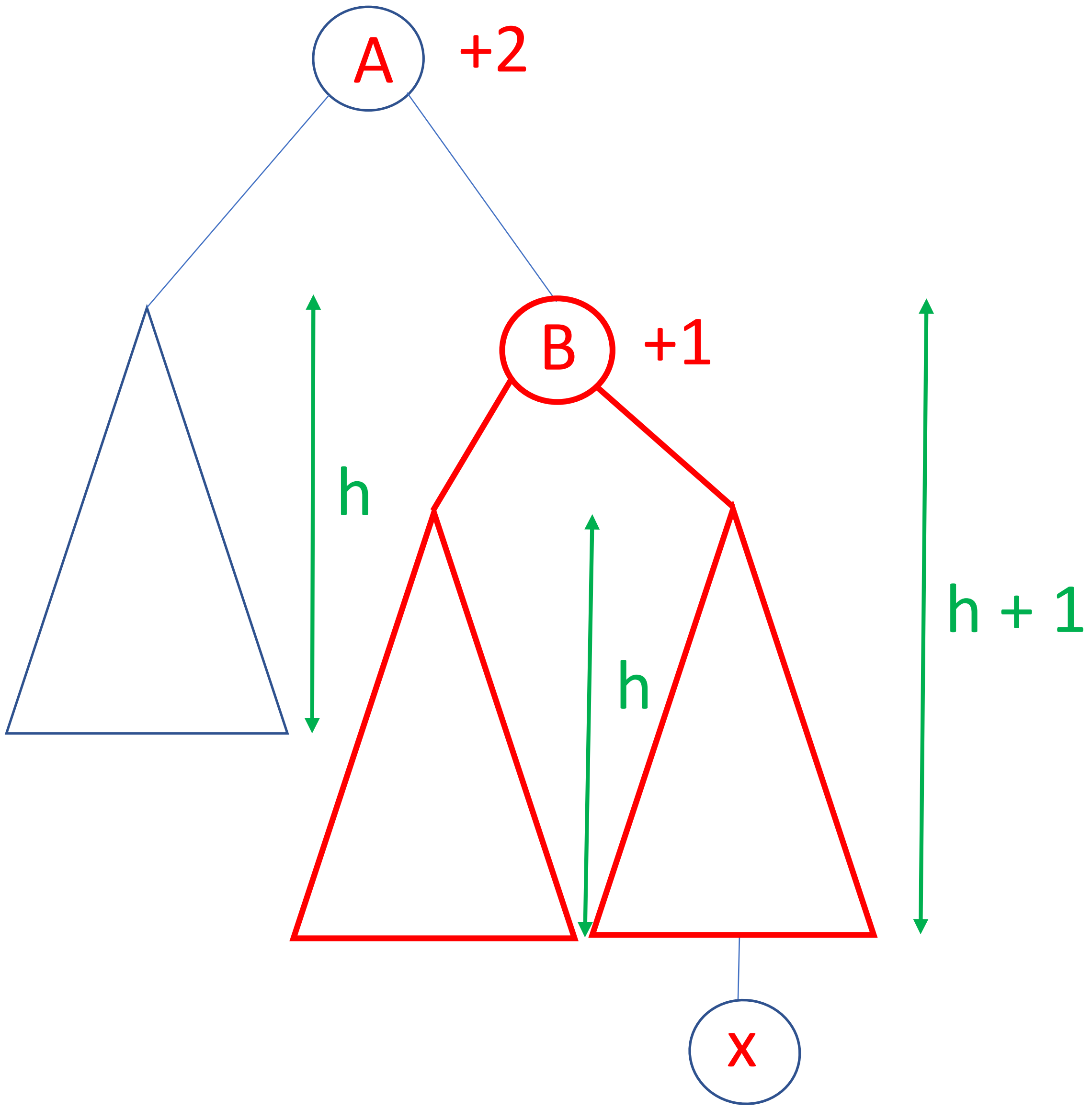
Case 1



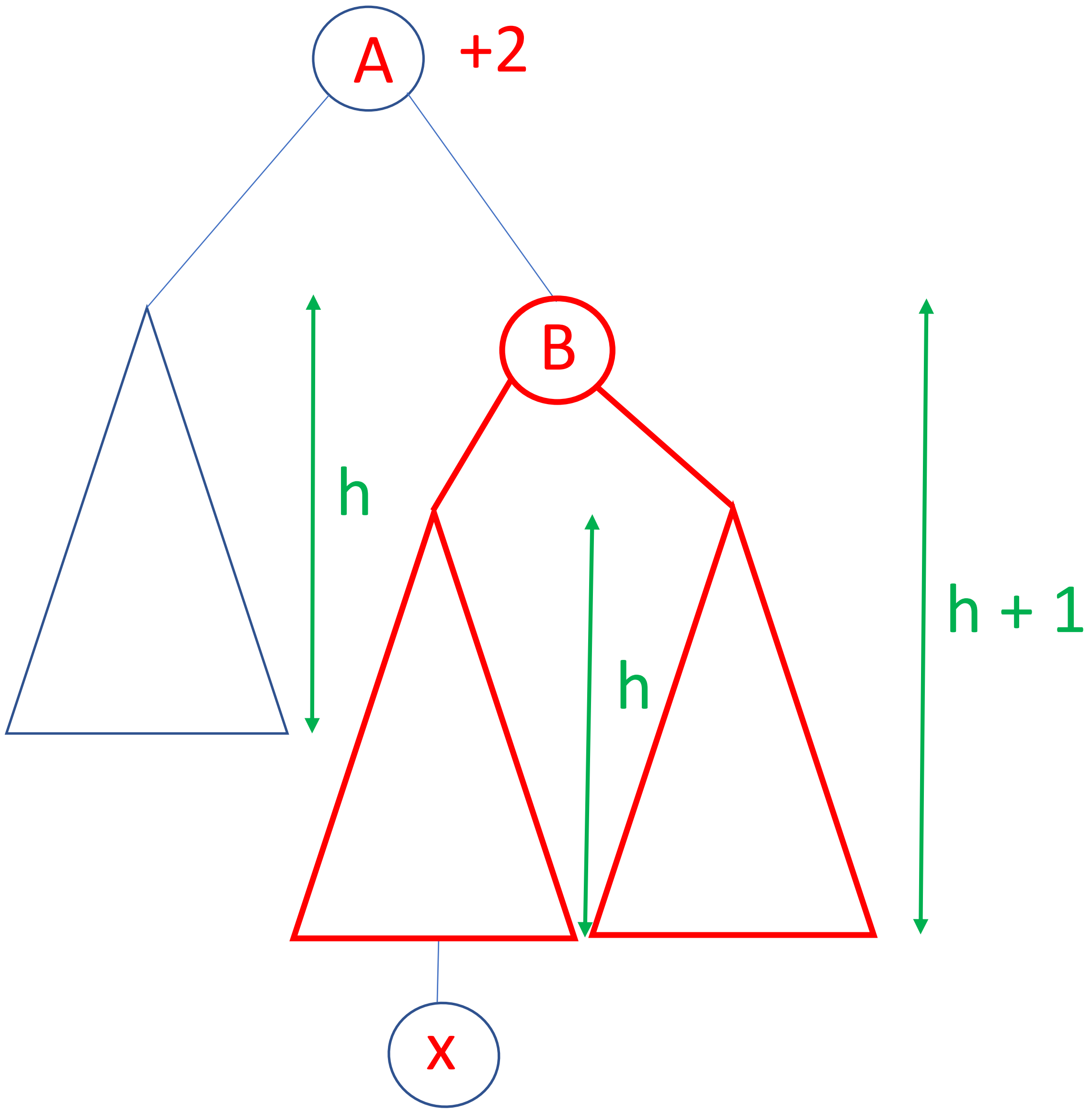
Case 1 (a)



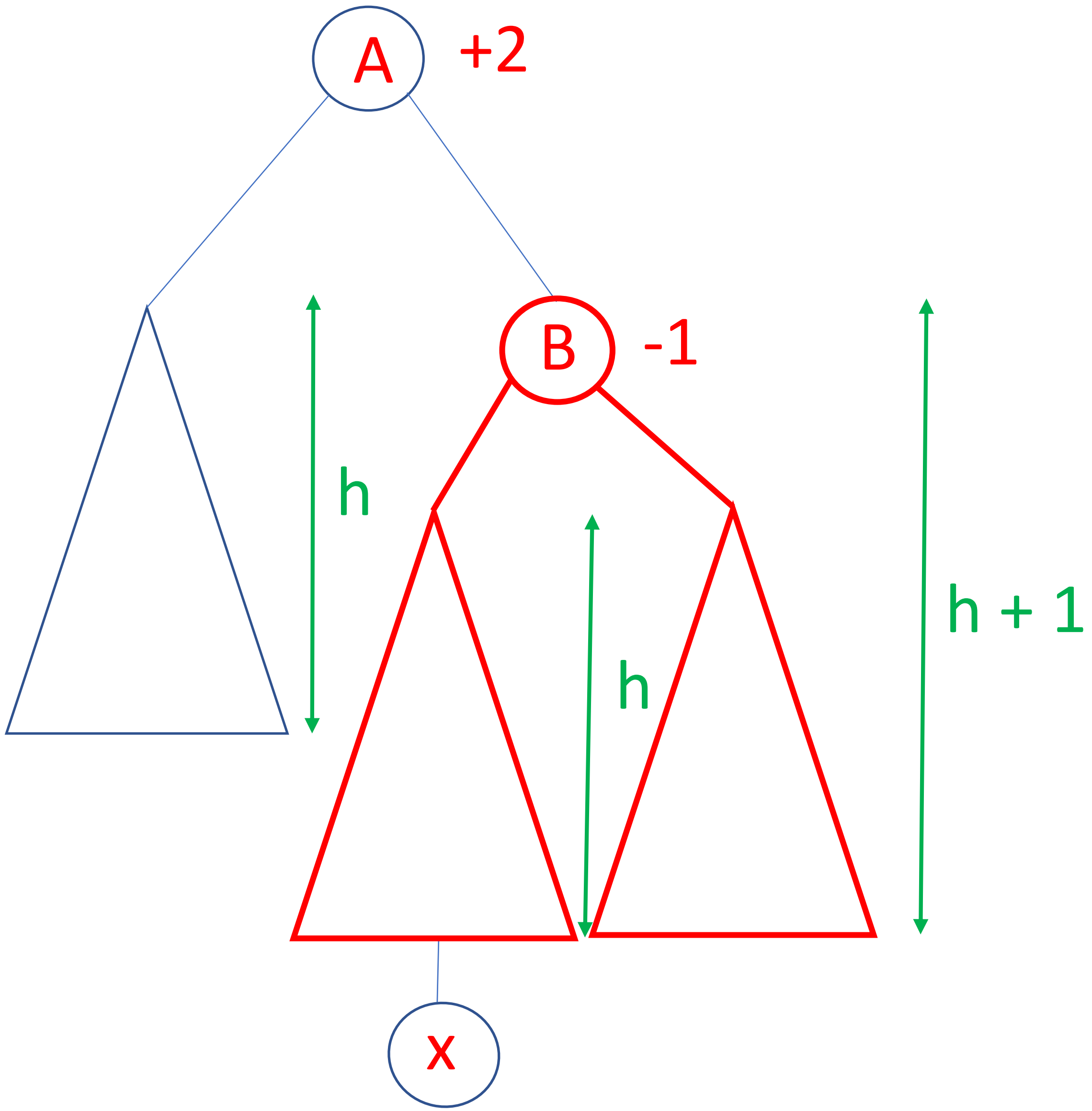
Case 1 (a)



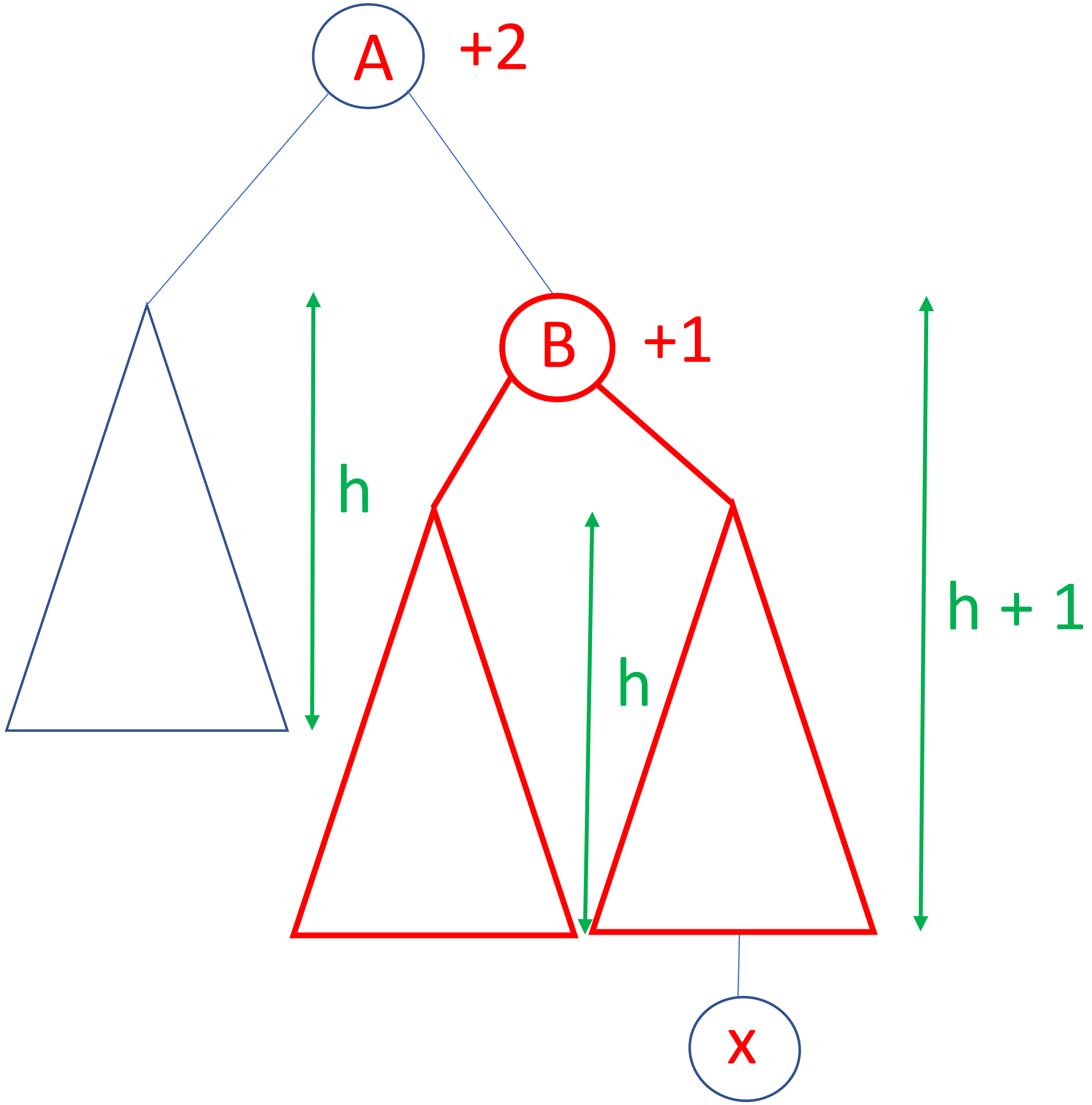
Case 1 (b)



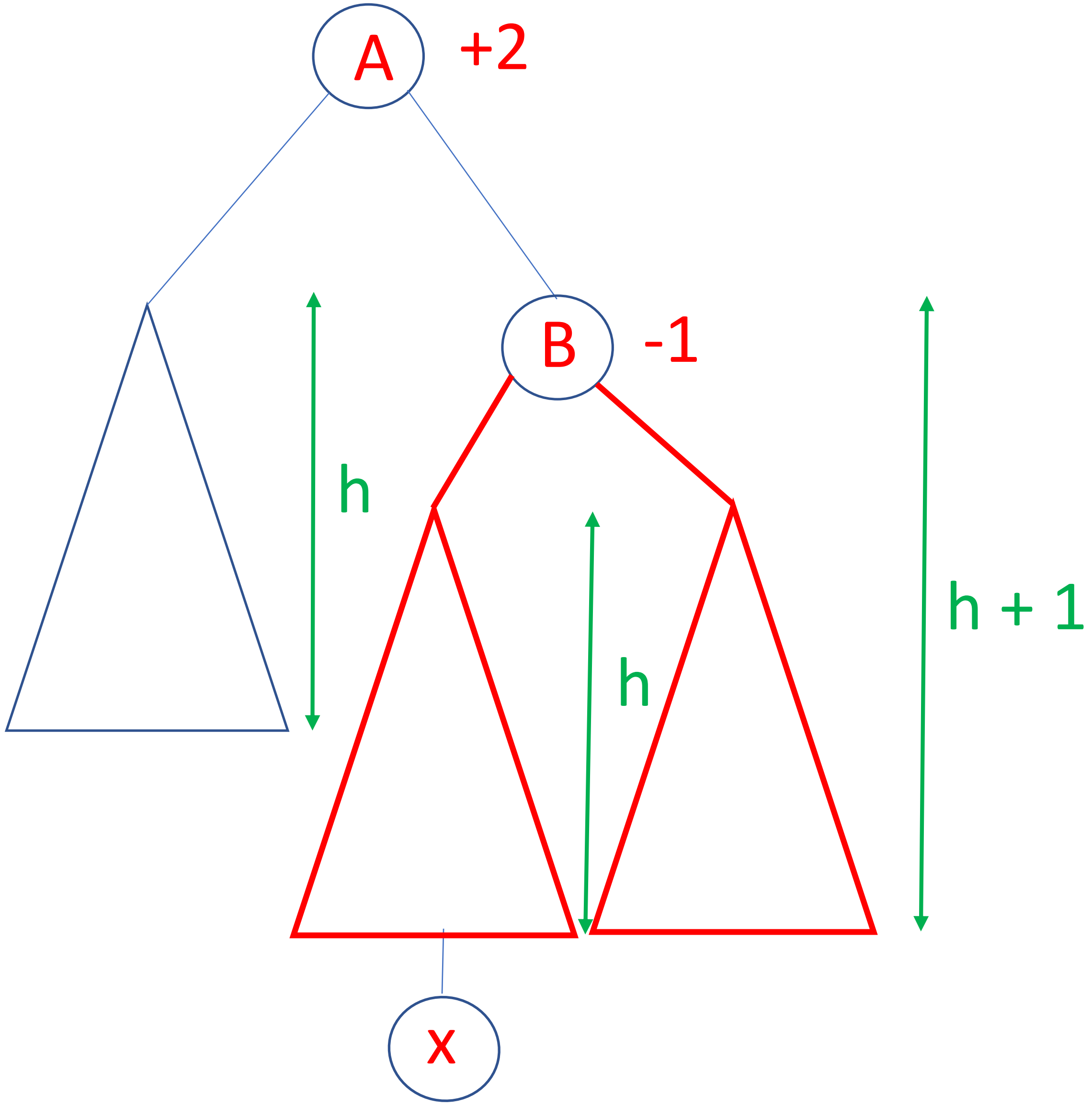
Case 1 (b)



Case 1 (a)



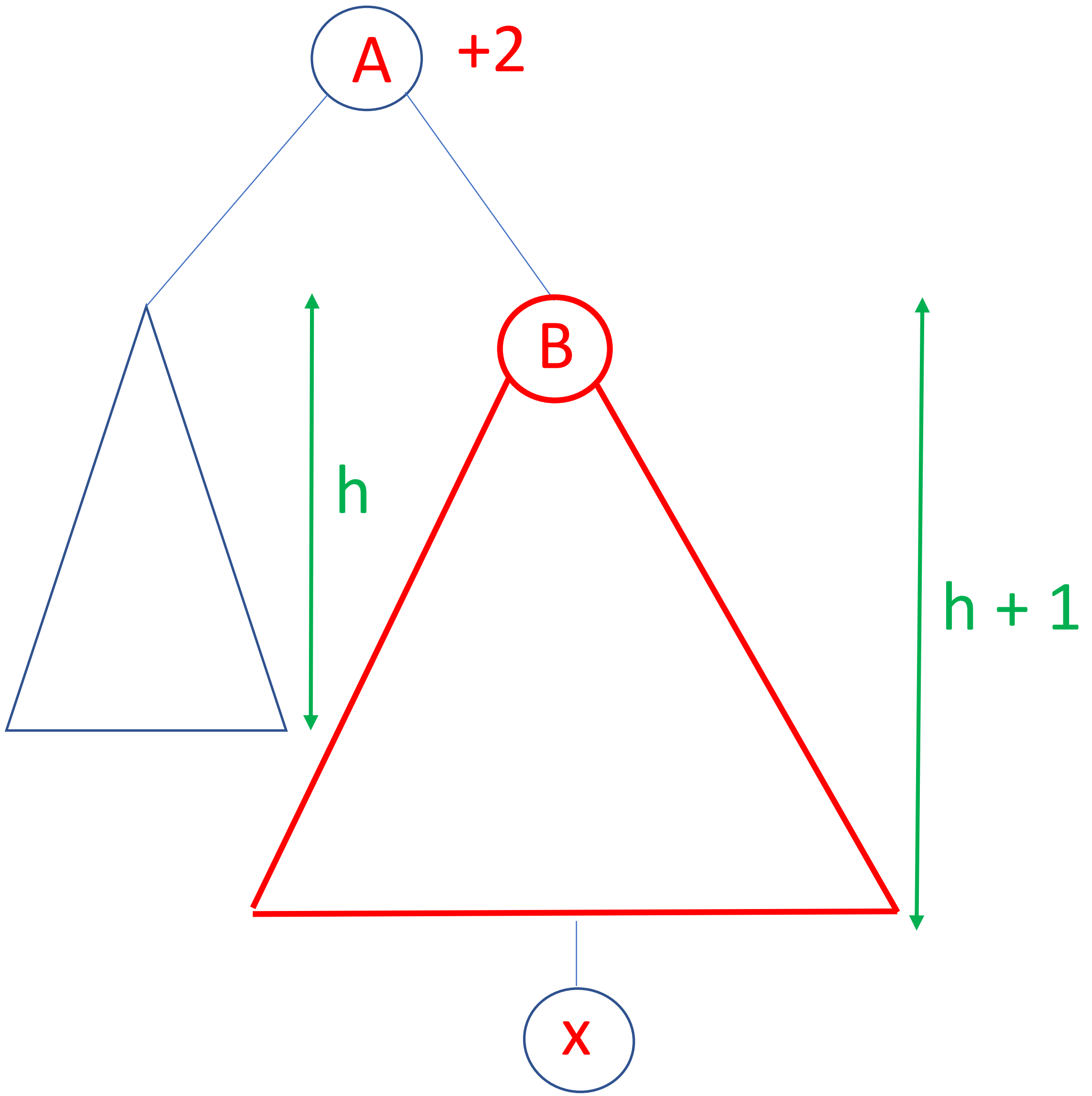
Case 1 (b)



We (sneakily :-) skipped a step in the argument!

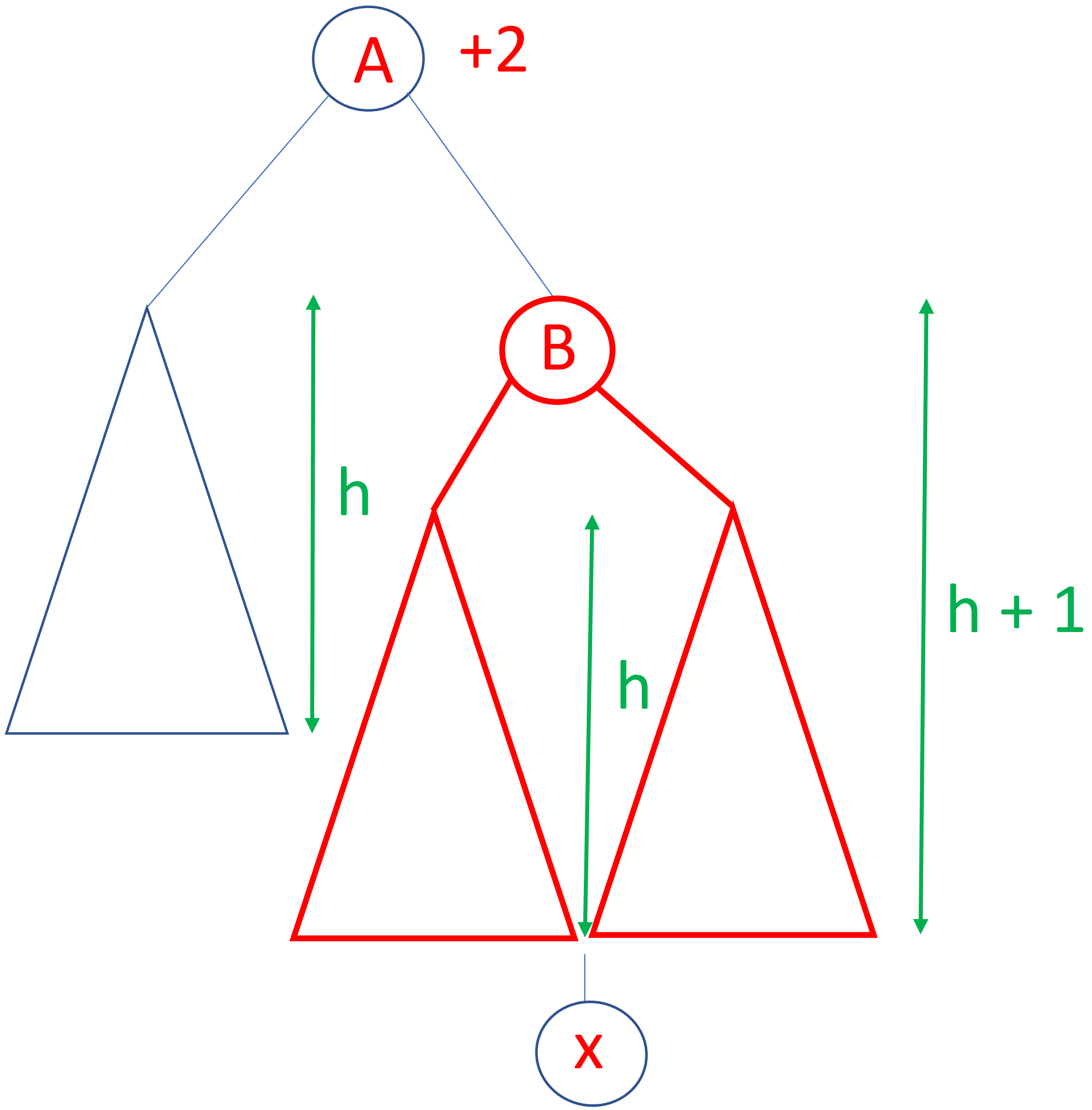
We (sneakily :-) skipped a step in the argument!

Case 1

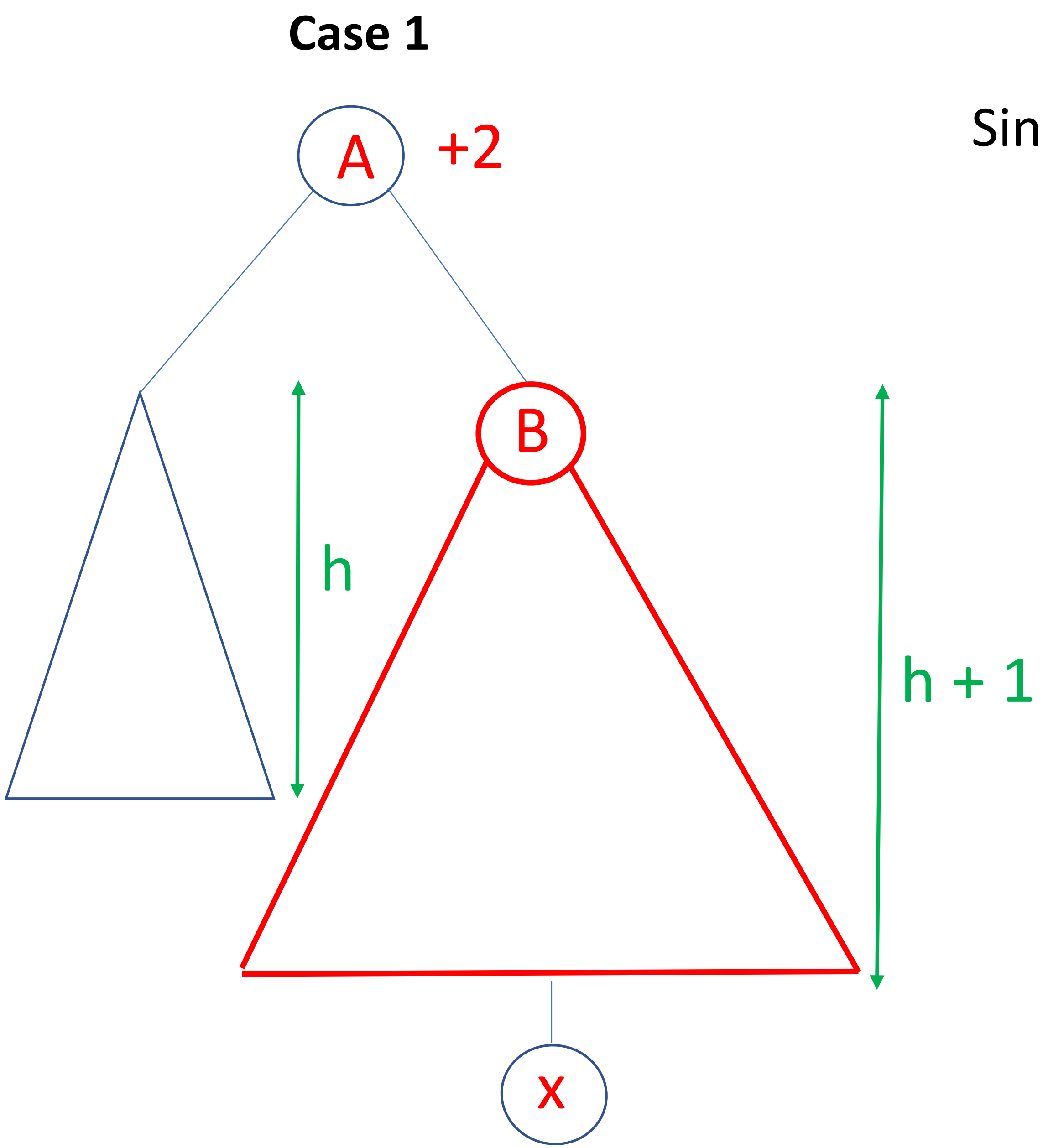


We (sneakily :-) skipped a step in the argument!

Case 1



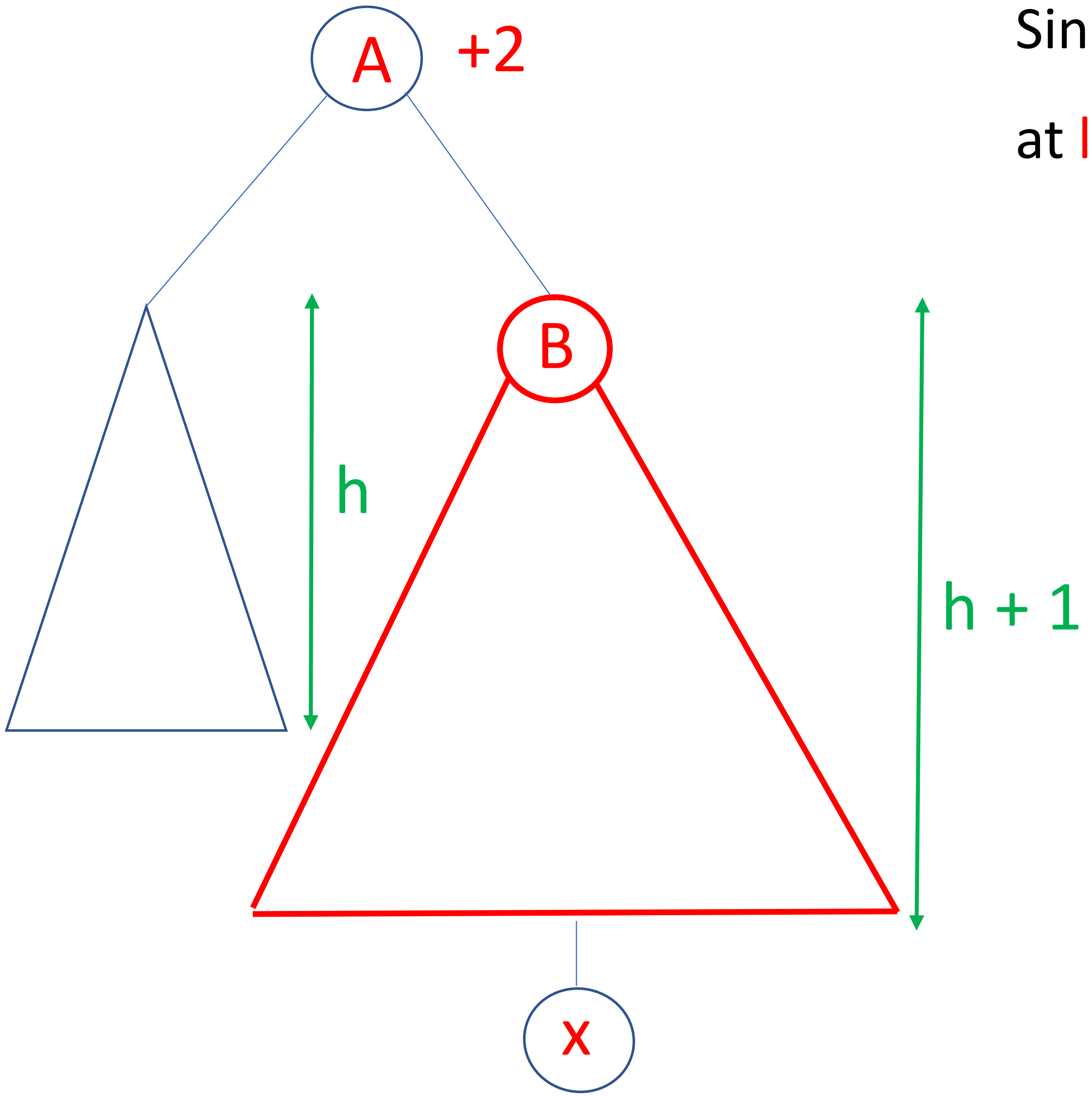
We (sneakily :-) skipped a step in the argument!



Since B has height h+1:

We (sneakily :-) skipped a step in the argument!

Case 1

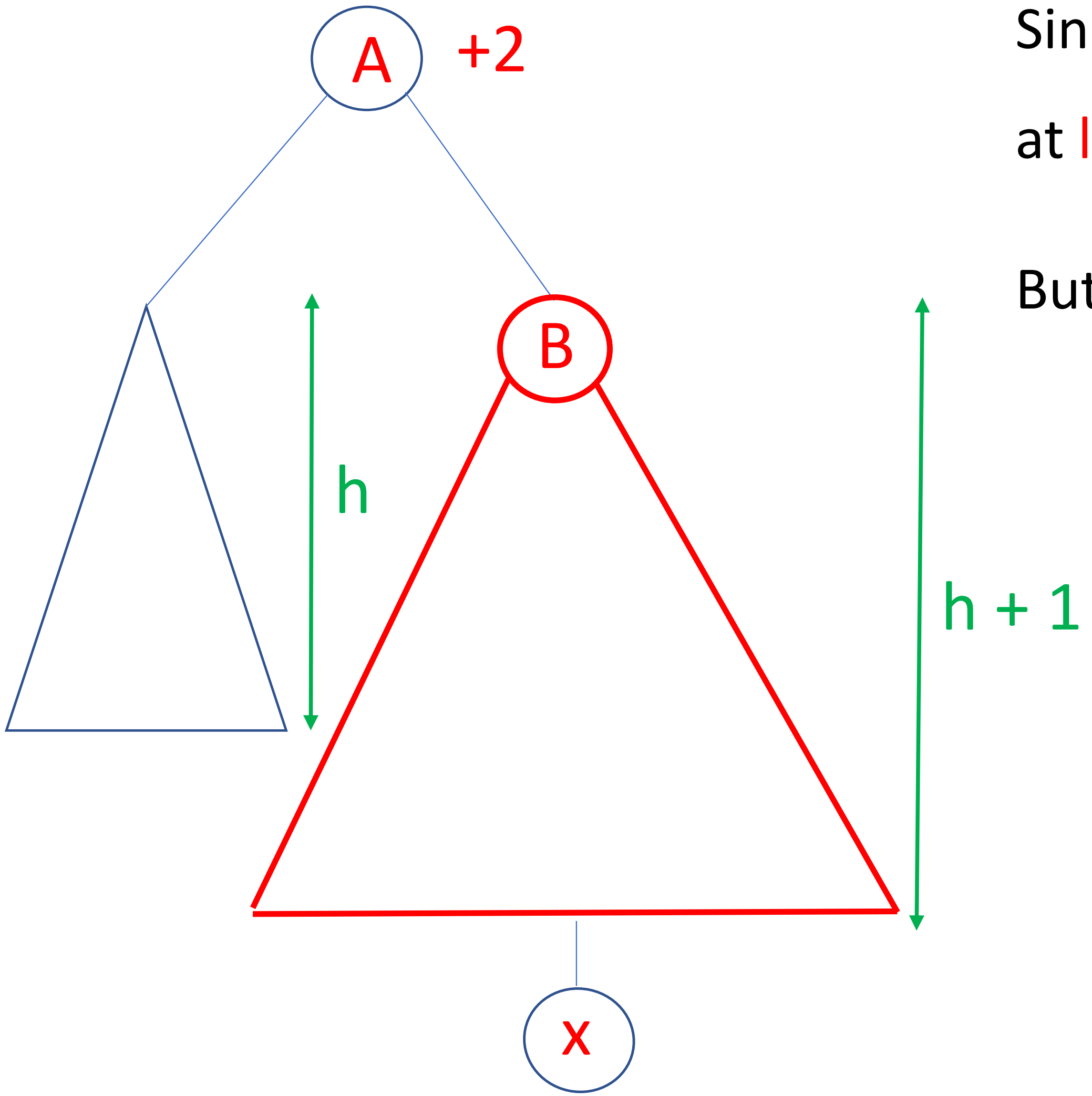


Since B has height h+1:

at least one subtrees of B has height h.

We (sneakily :-) skipped a step in the argument!

Case 1

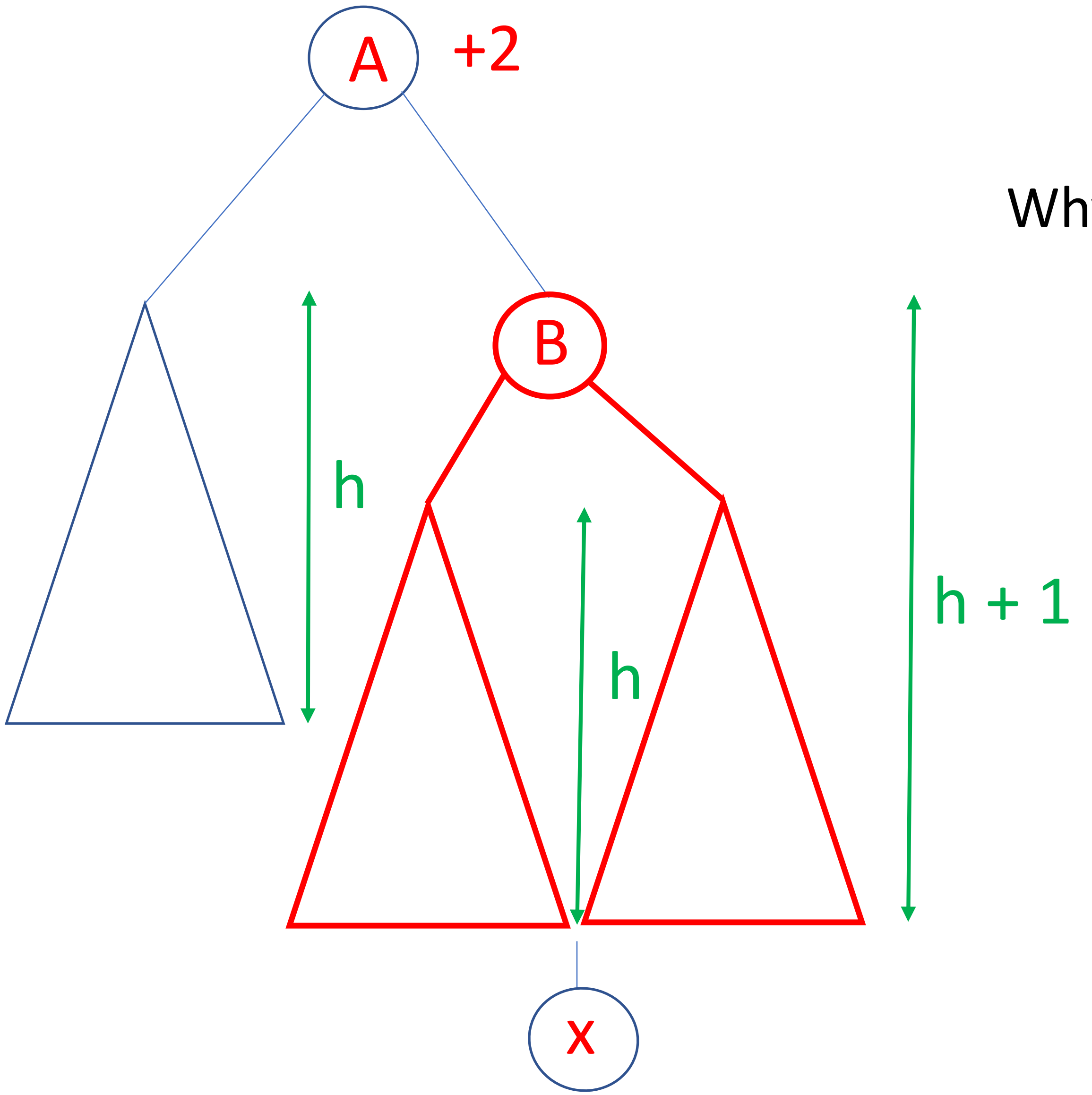


Since B has height h+1:

at least one subtrees of B has height h.

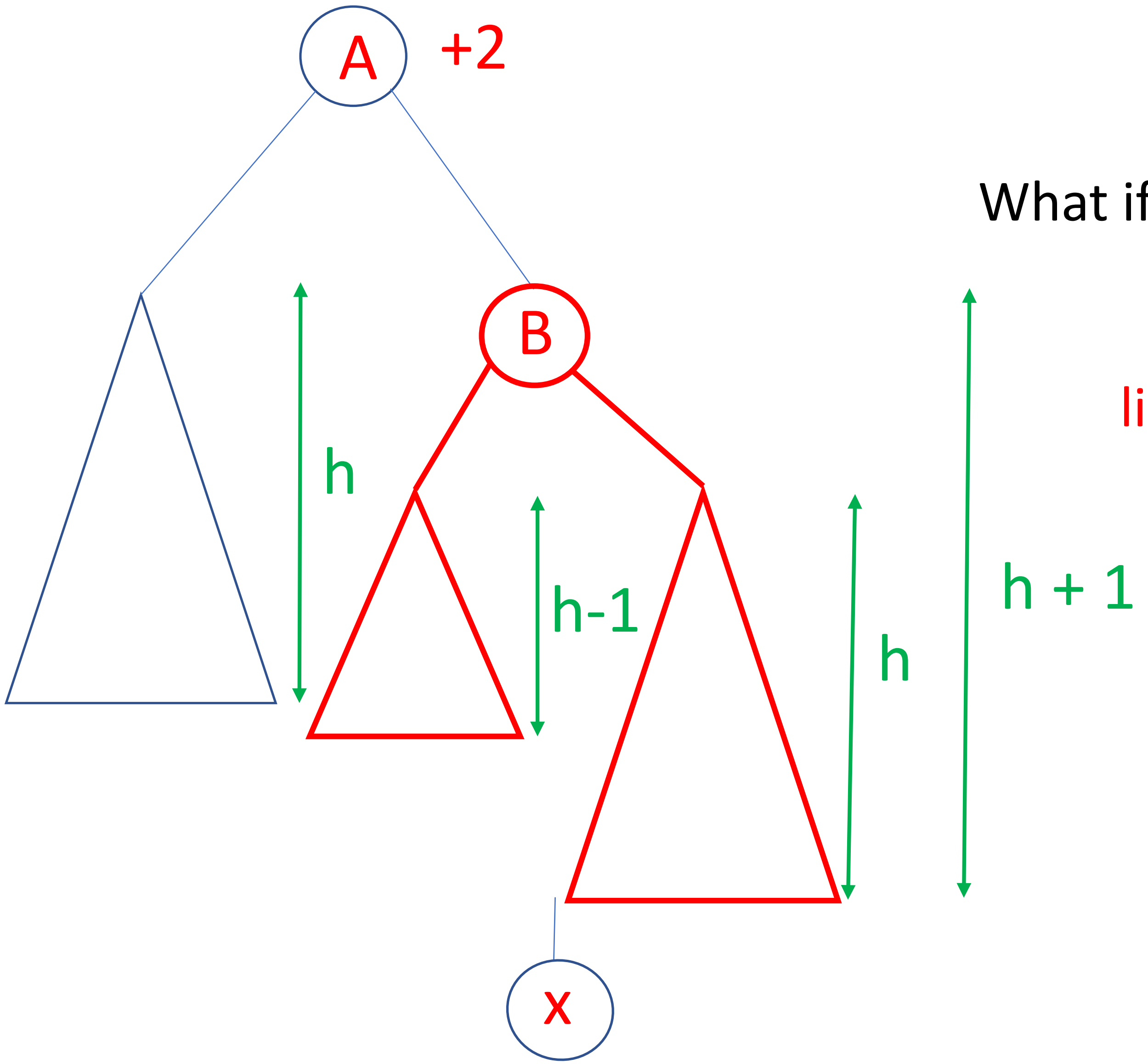
But why both subtrees of B have the same height h ?

Case 1



Why both subtrees of B have the same height h?

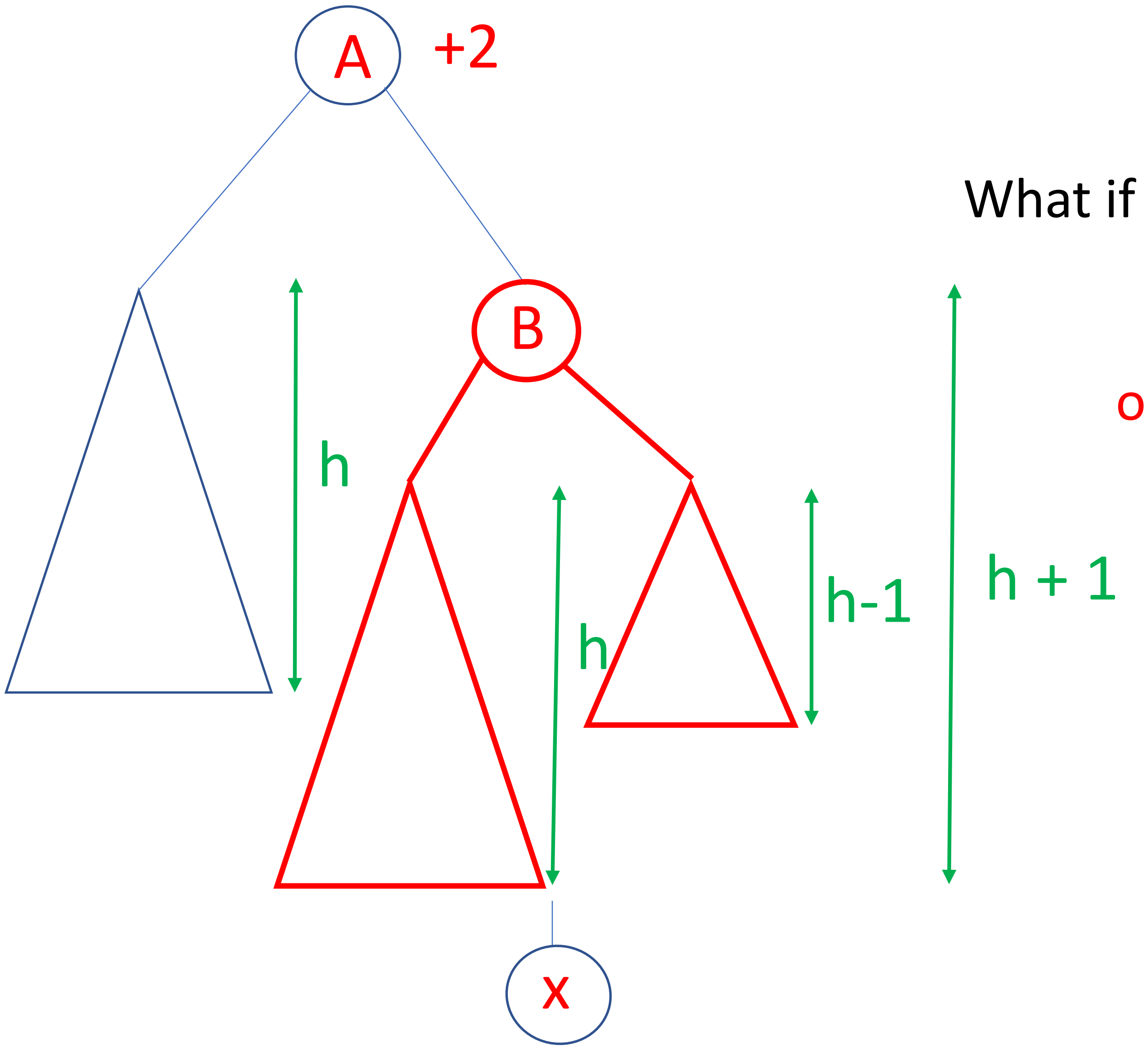
Case 1



What if only one of them has height h ?

like this

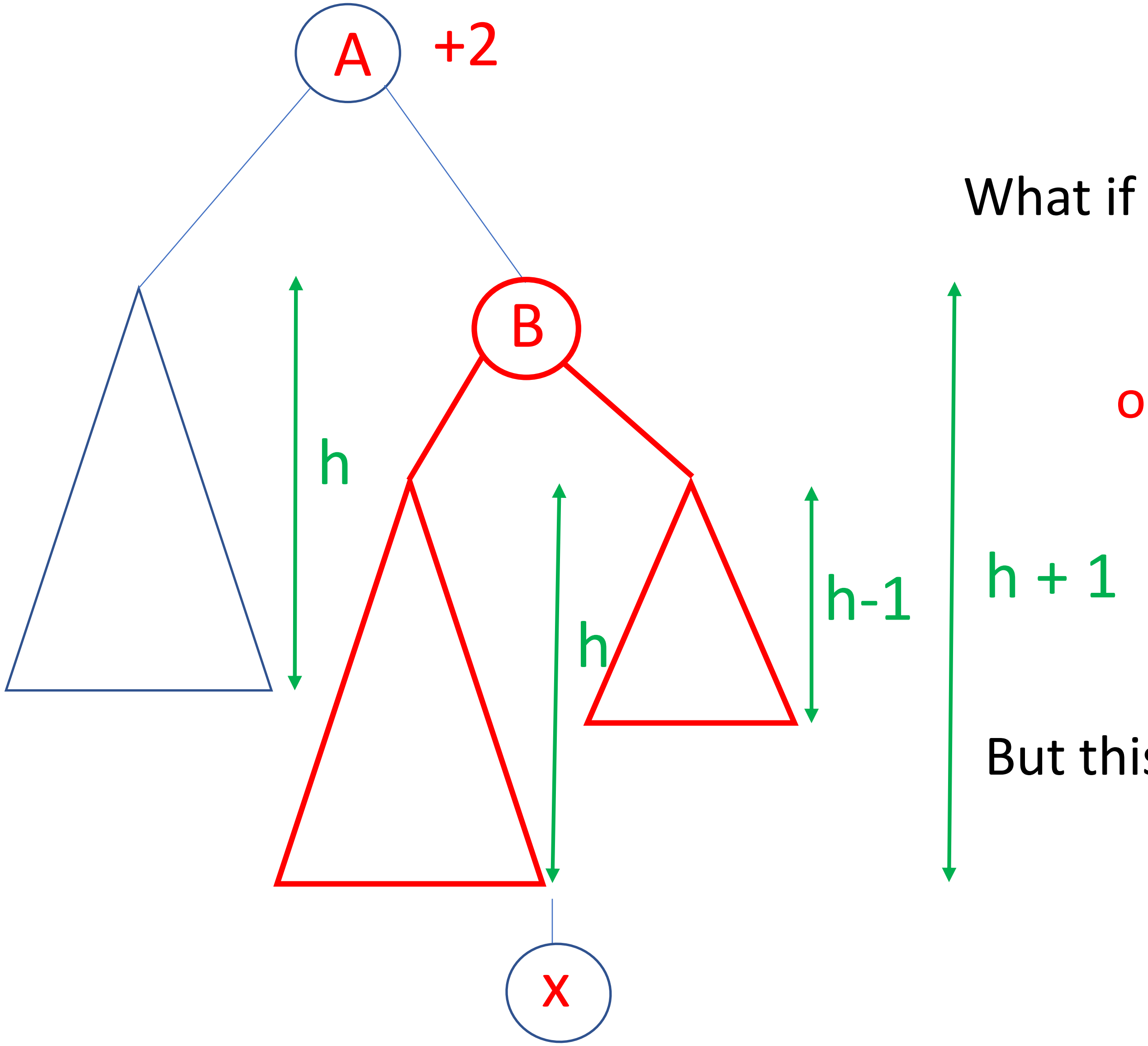
Case 1



What if only one of them has height h ?

or this

Case 1



What if only one of them has height h ?

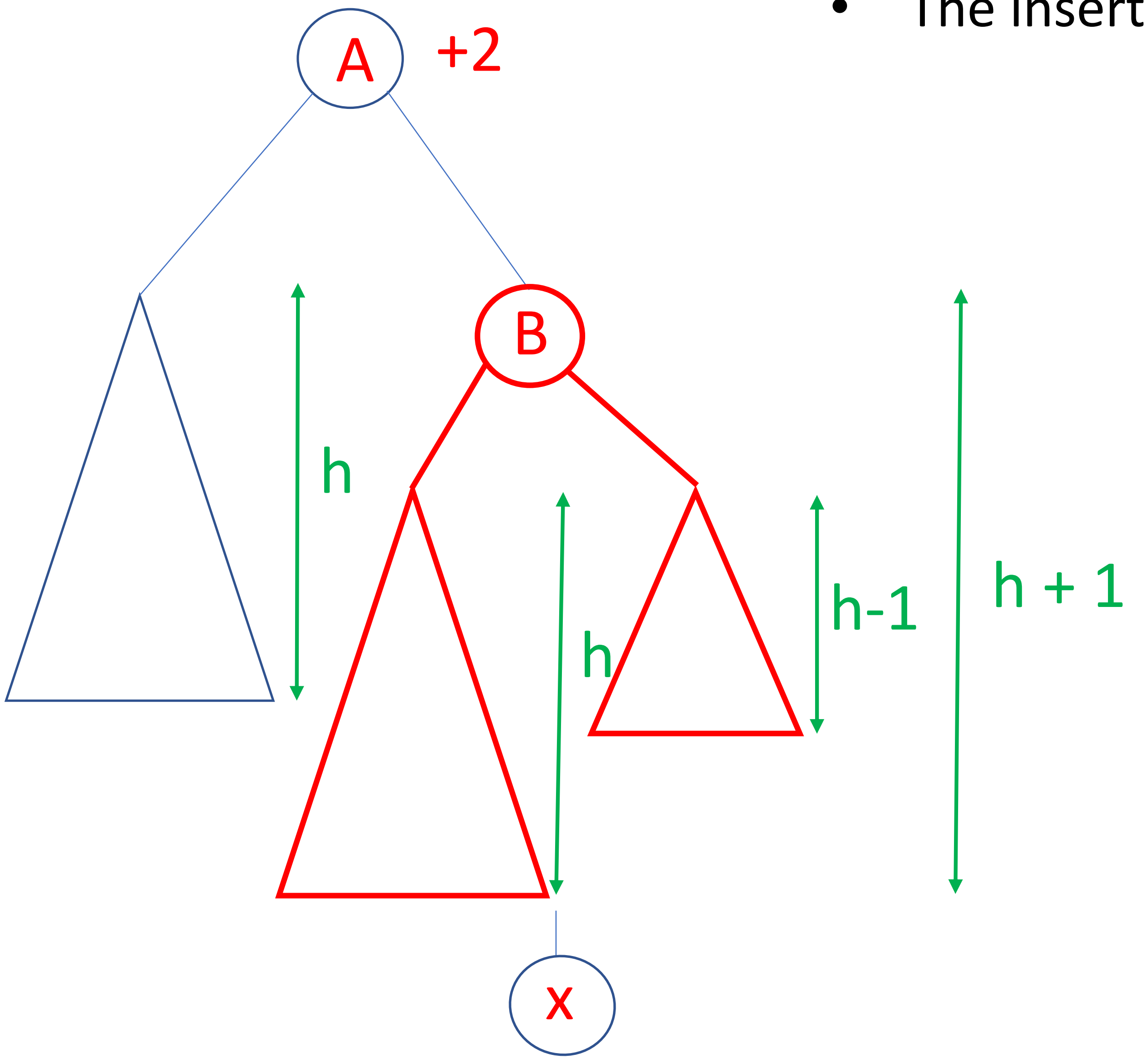
or this

But this is impossible!

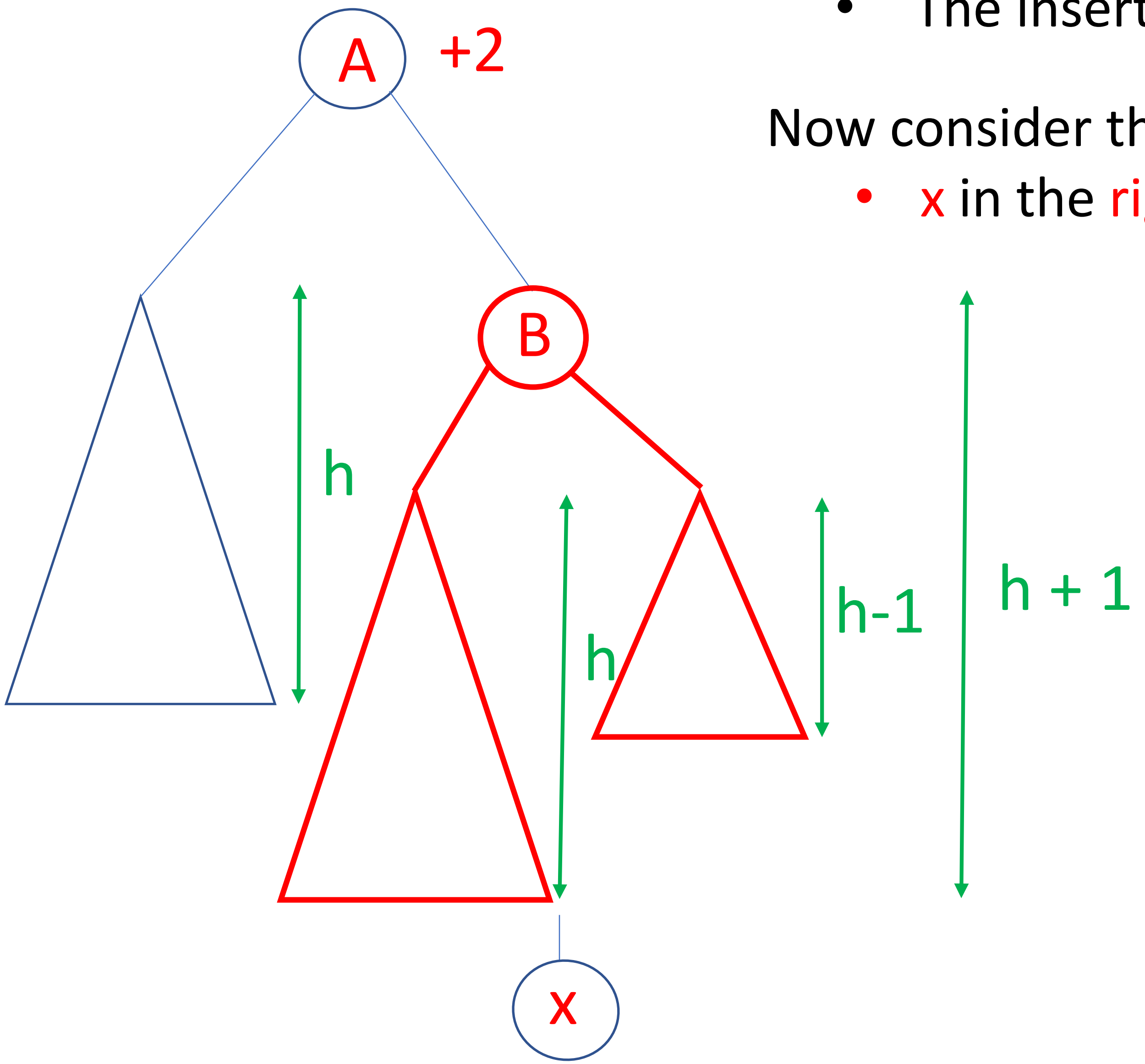
Case 1

To see why it is impossible, note that:

- The insertion of x must have increased the height of B



Case 1



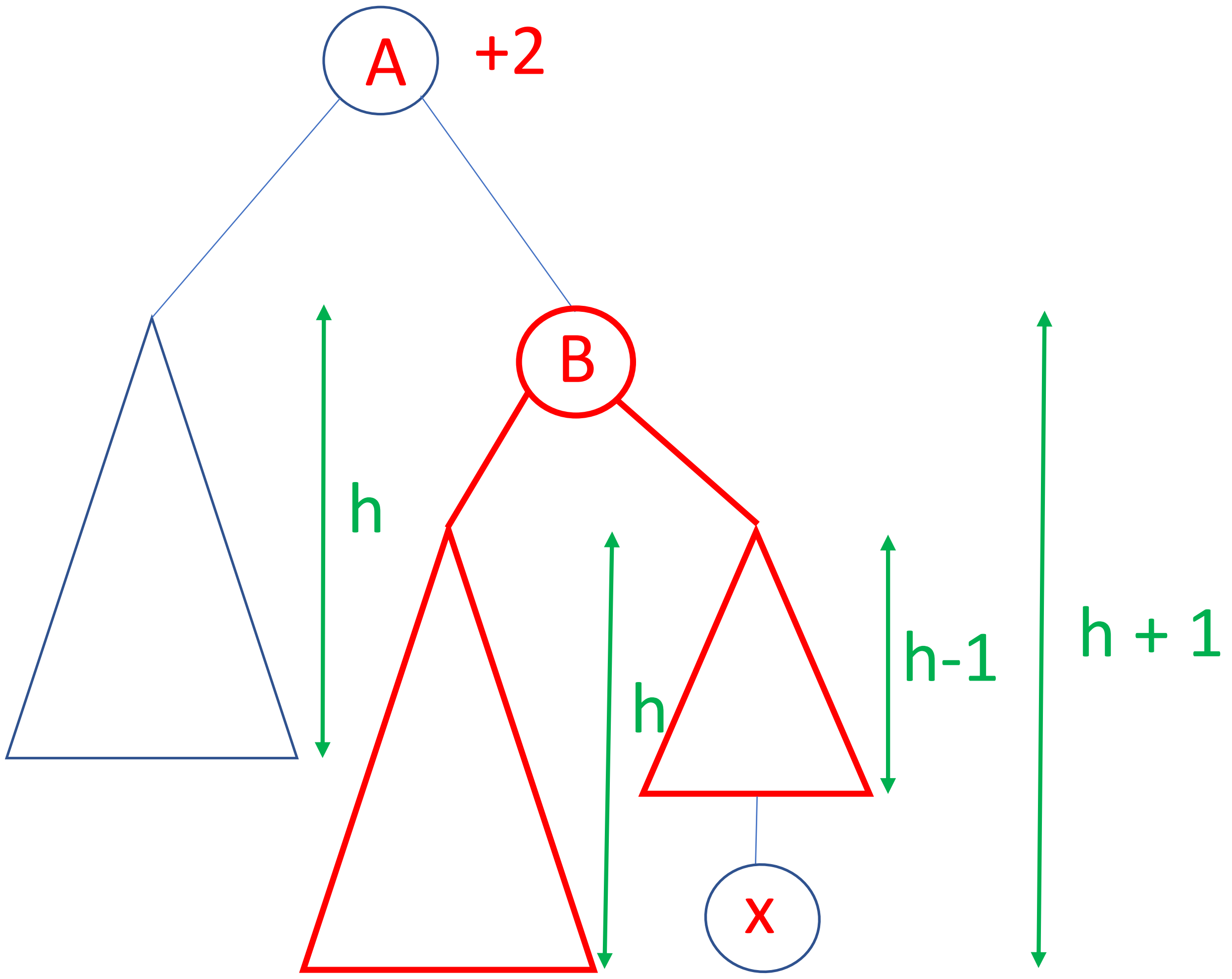
To see why it is impossible, note that:

- The insertion of x must have increased the height of B

Now consider the two possible cases:

- x in the right or left subtree of B

Case 1

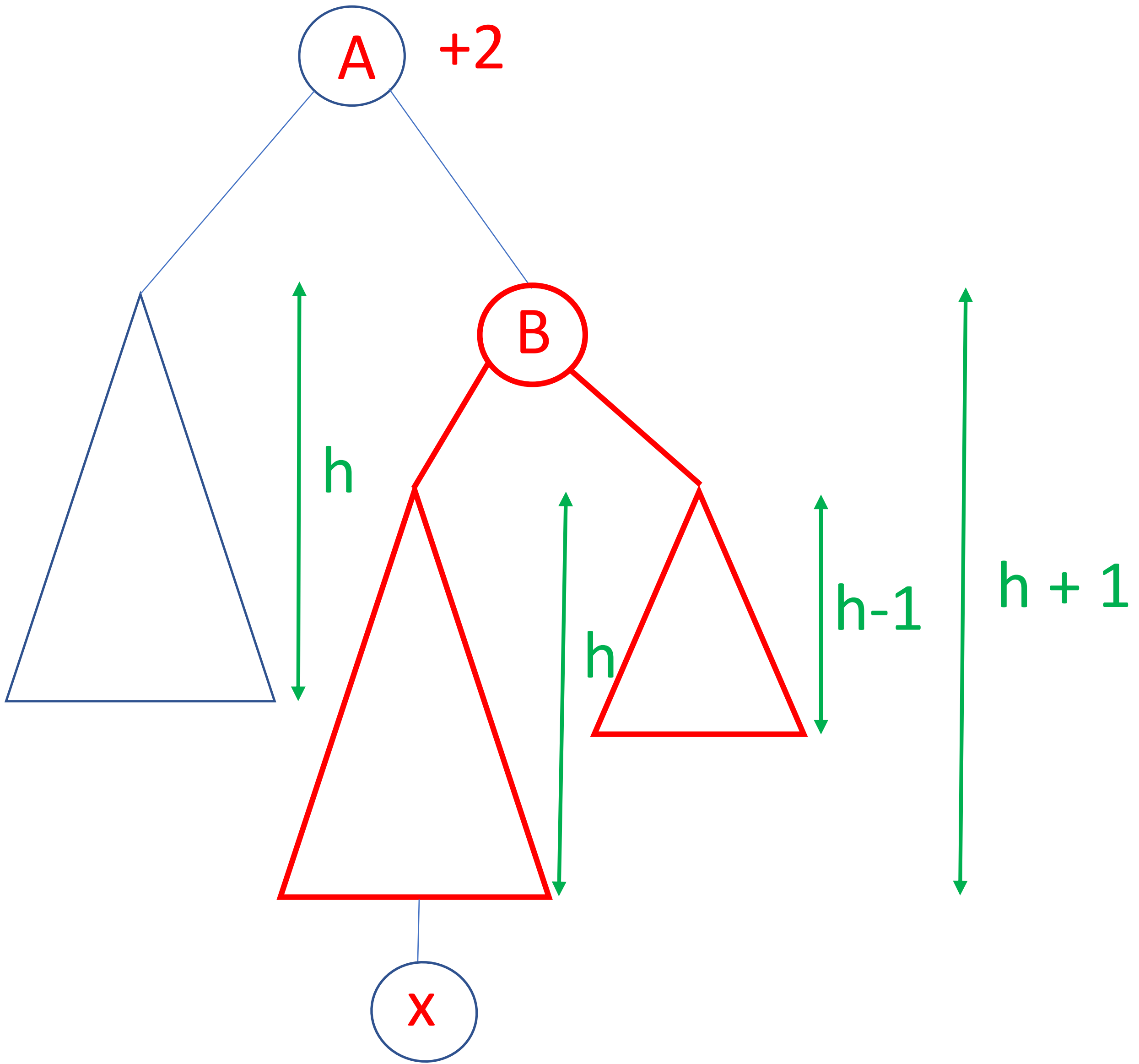


x in the right subtree of B

But this is impossible!

Do you see why?

Case 1

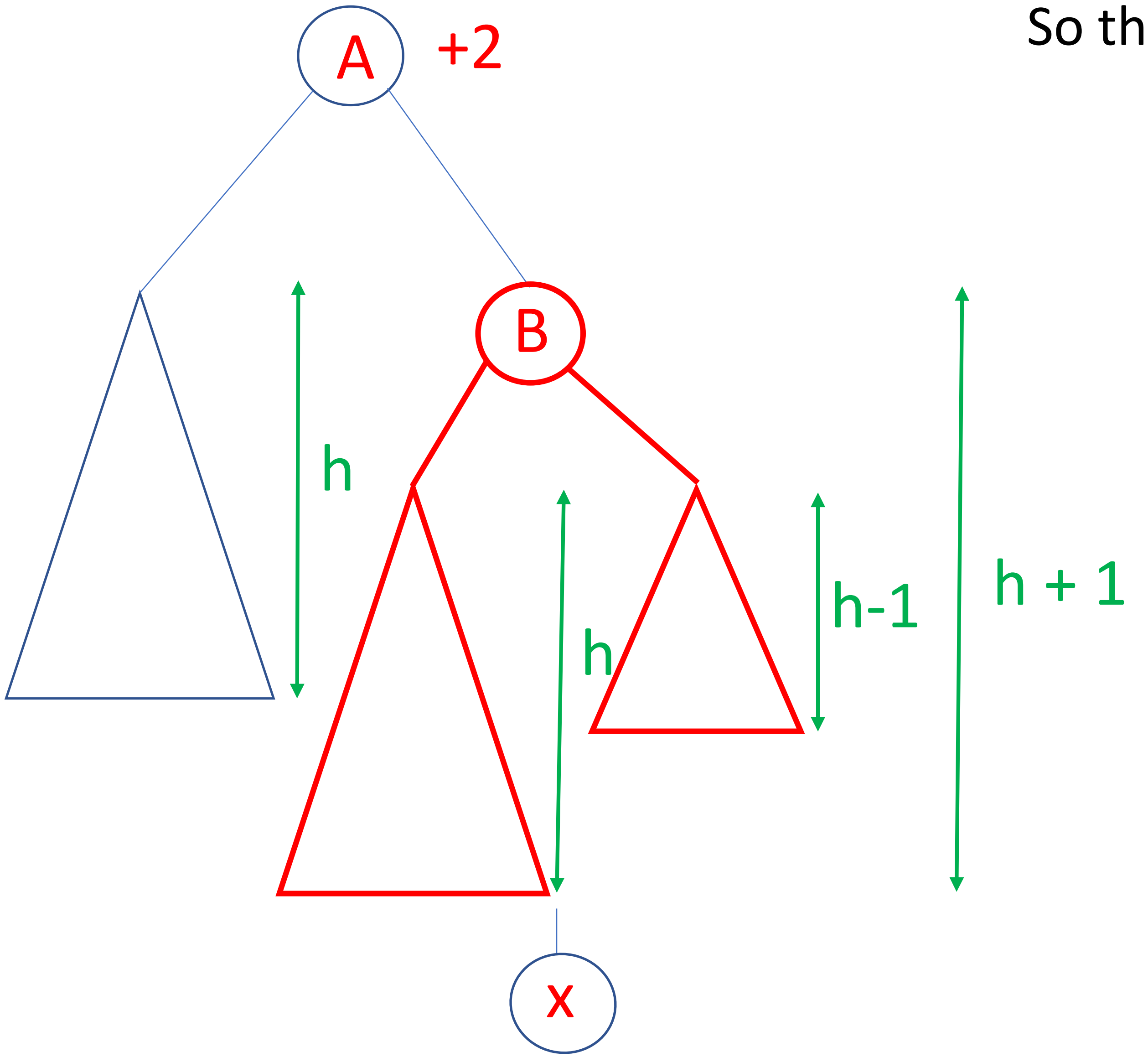


x in the left subtree of B

But this is also impossible!

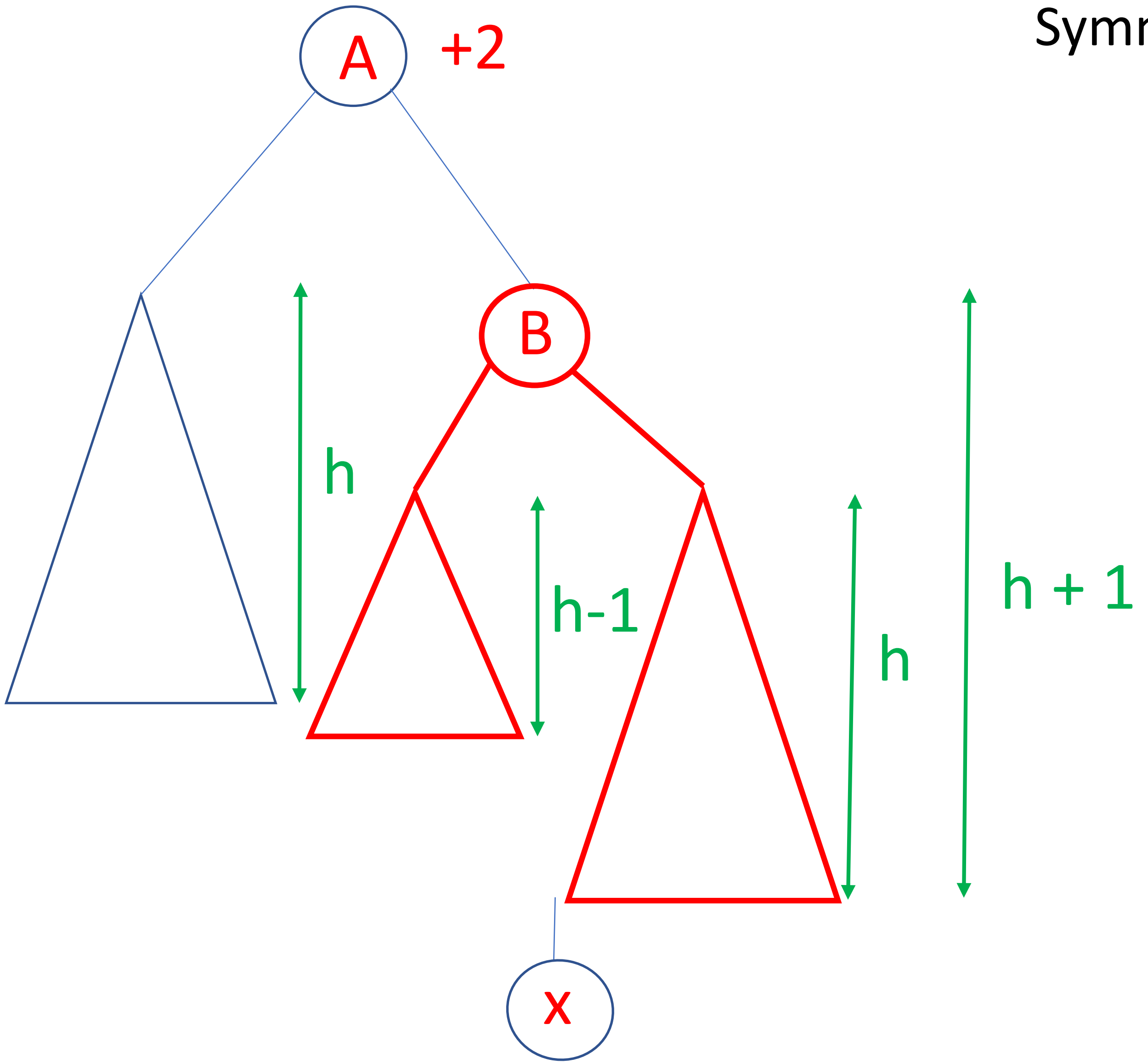
Do you see why?

Case 1



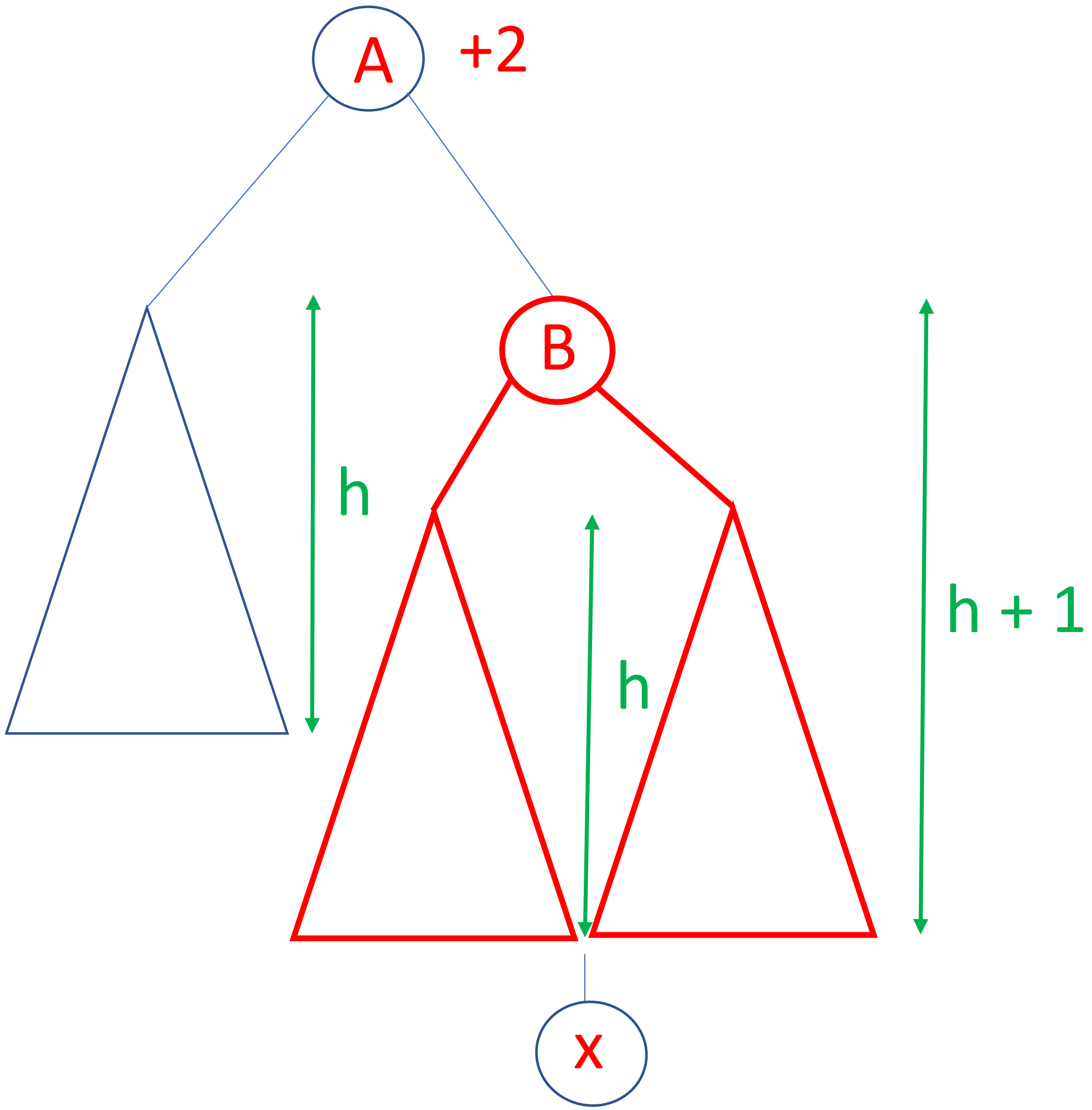
So this case is impossible!

Case 1



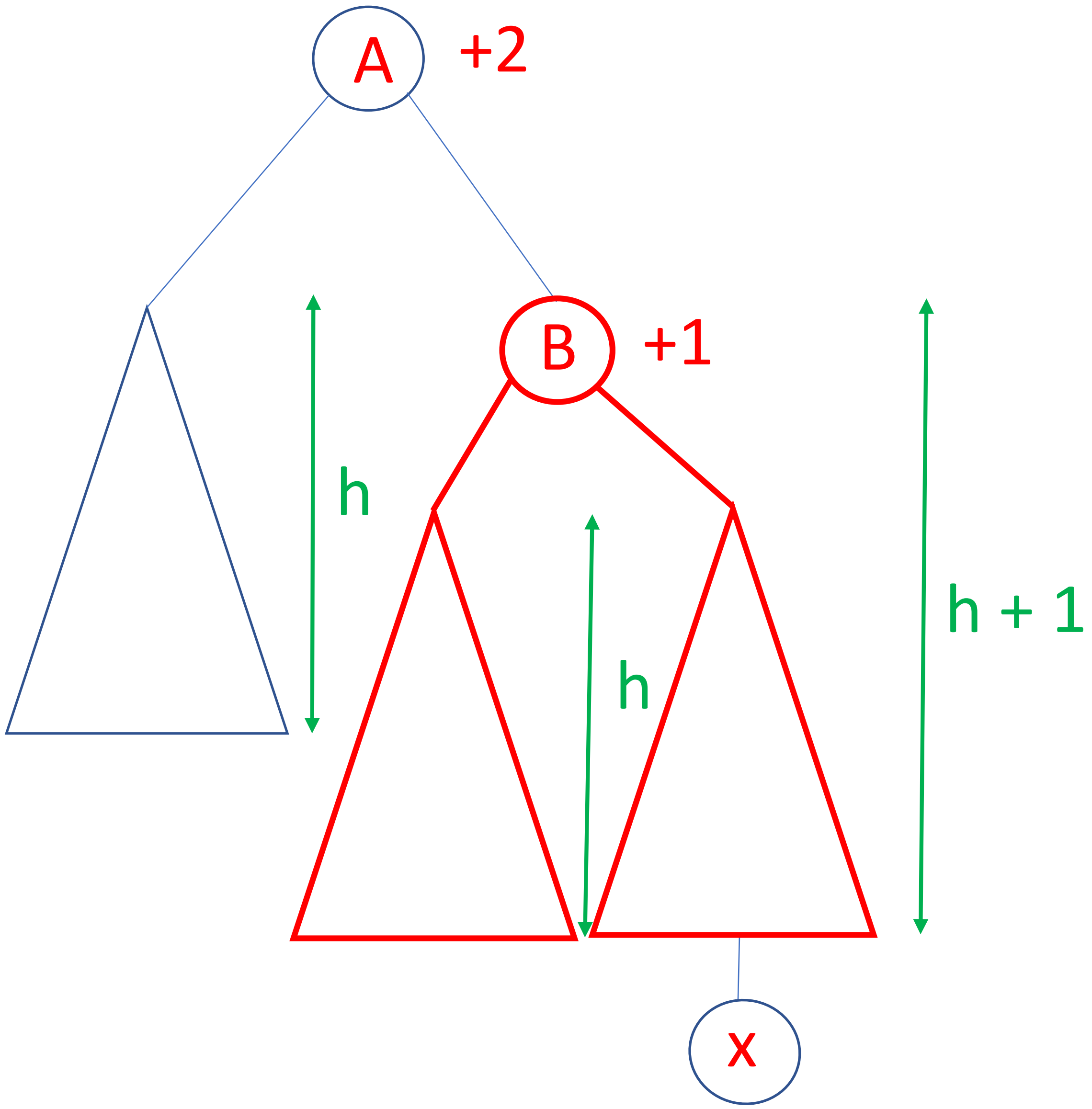
Symmetrically: this case is also impossible!

Case 1

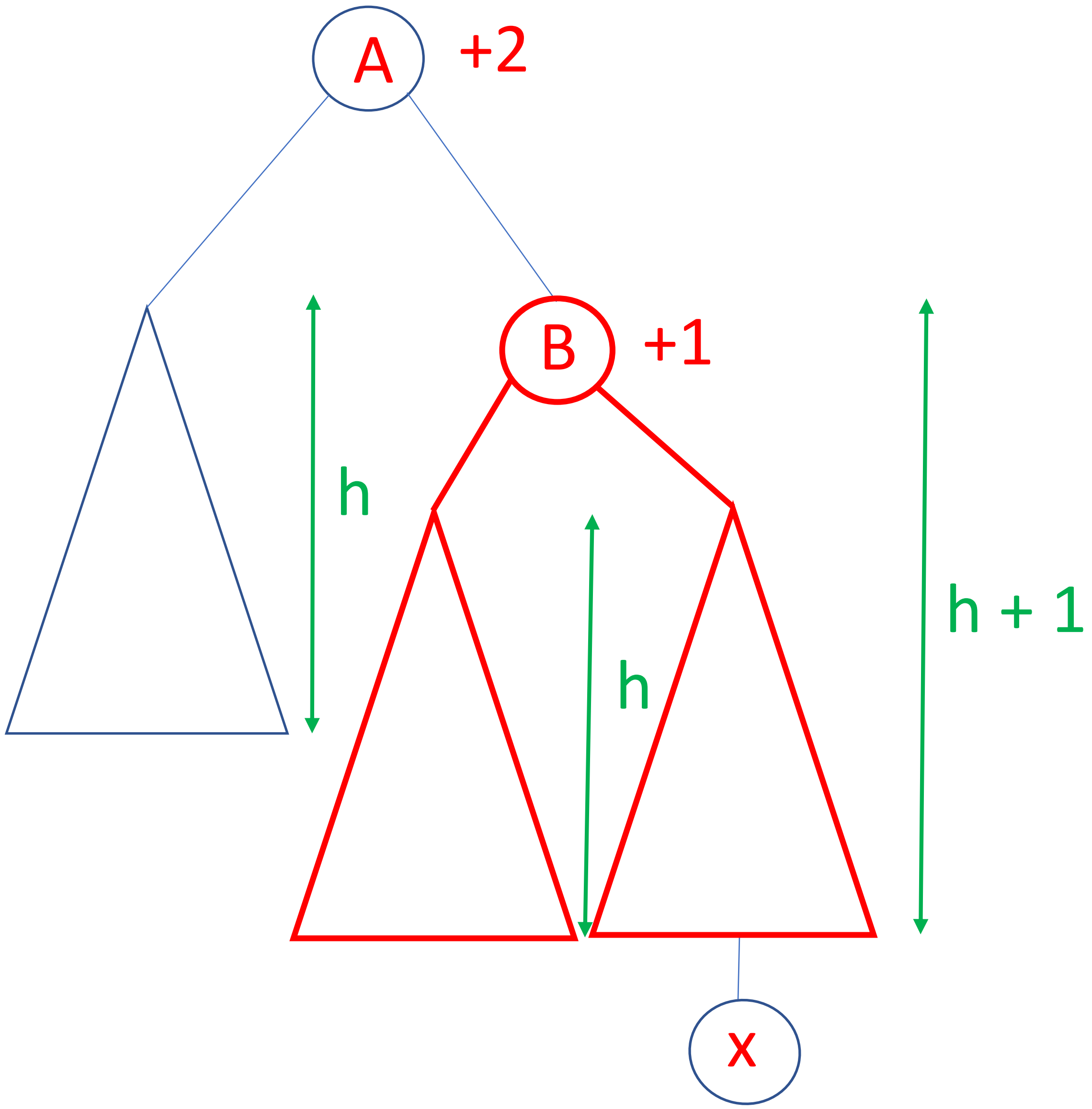


So this is indeed the only possible case 1

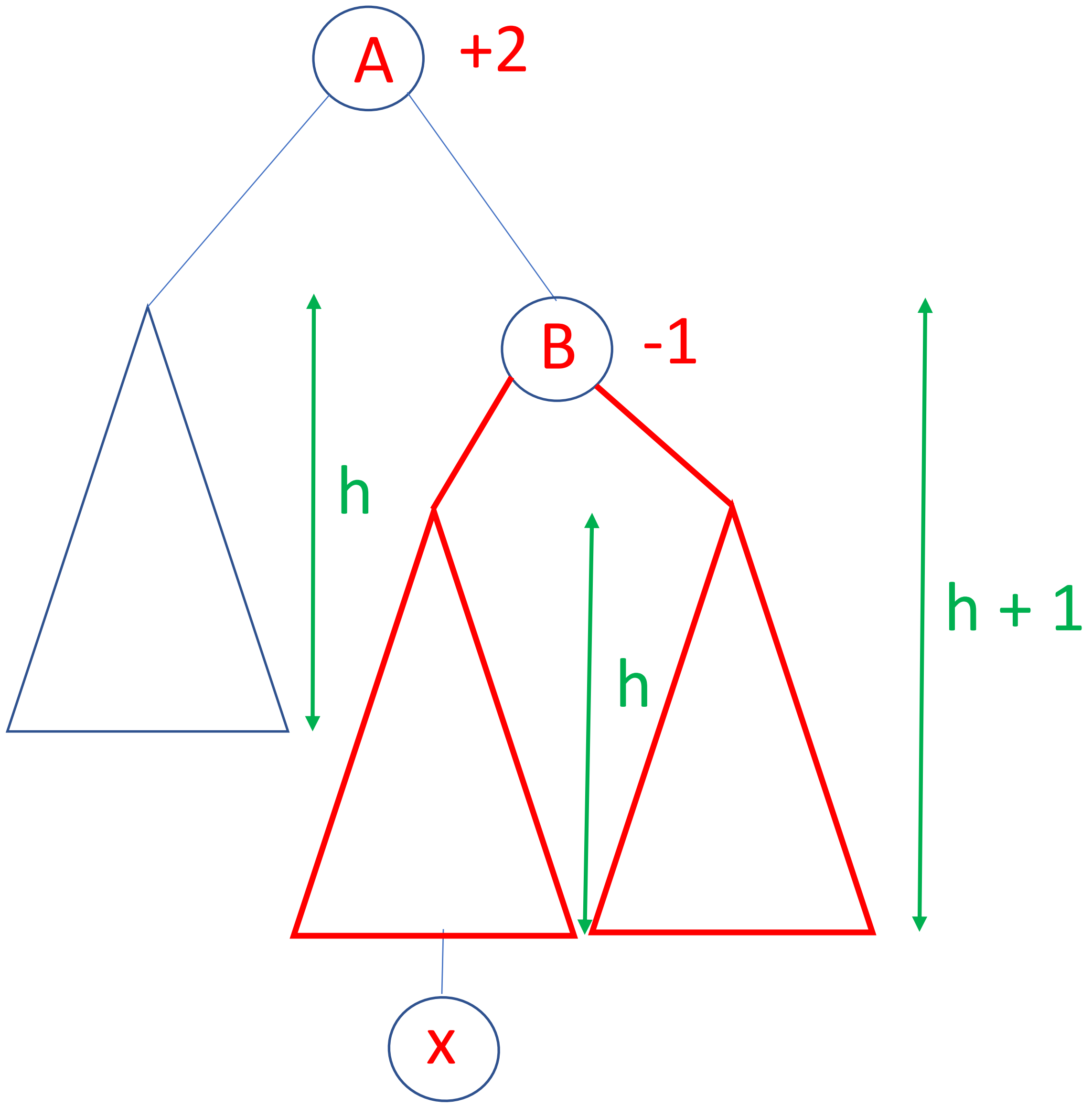
Case 1 (a)



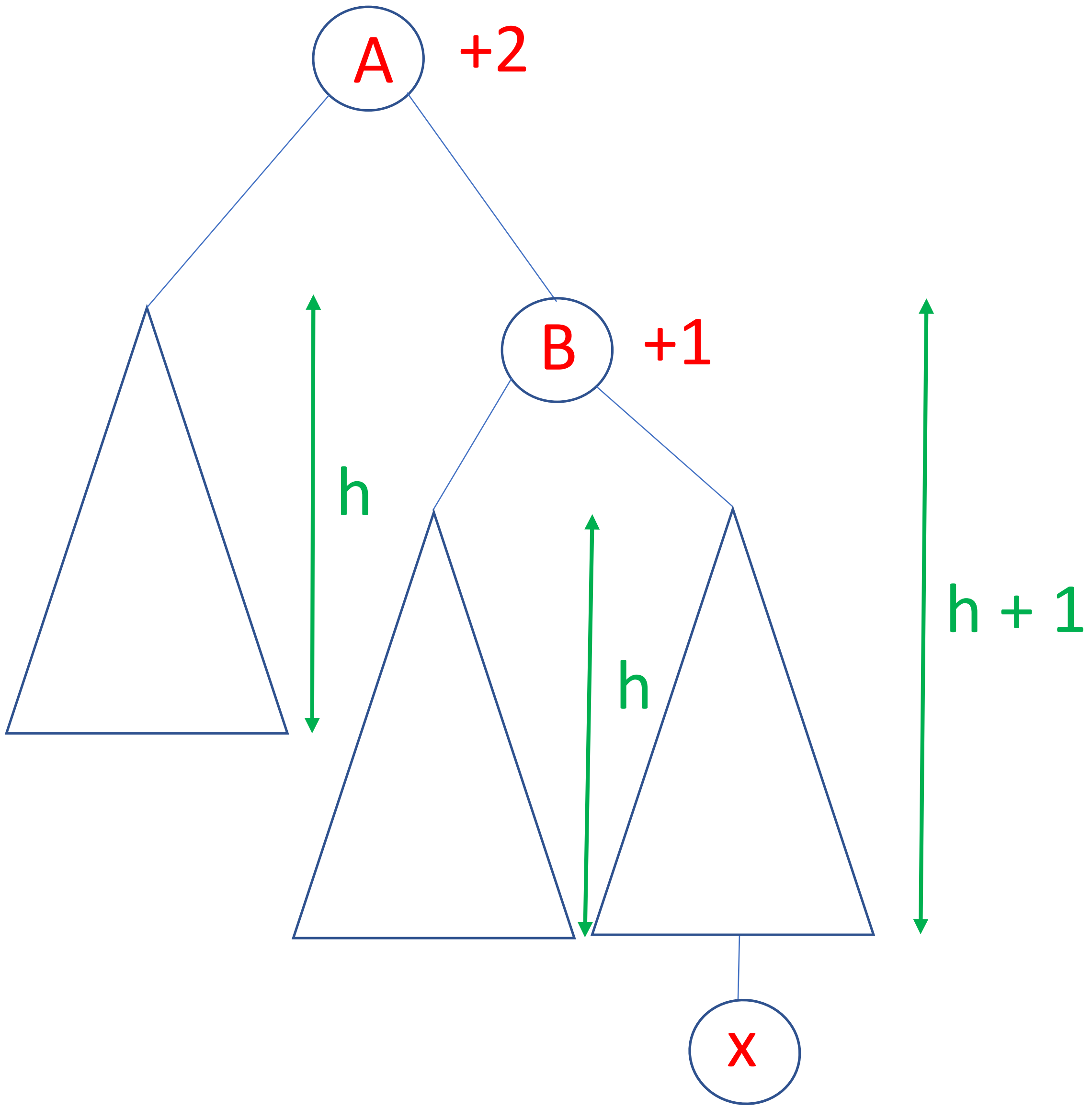
Case 1 (a)



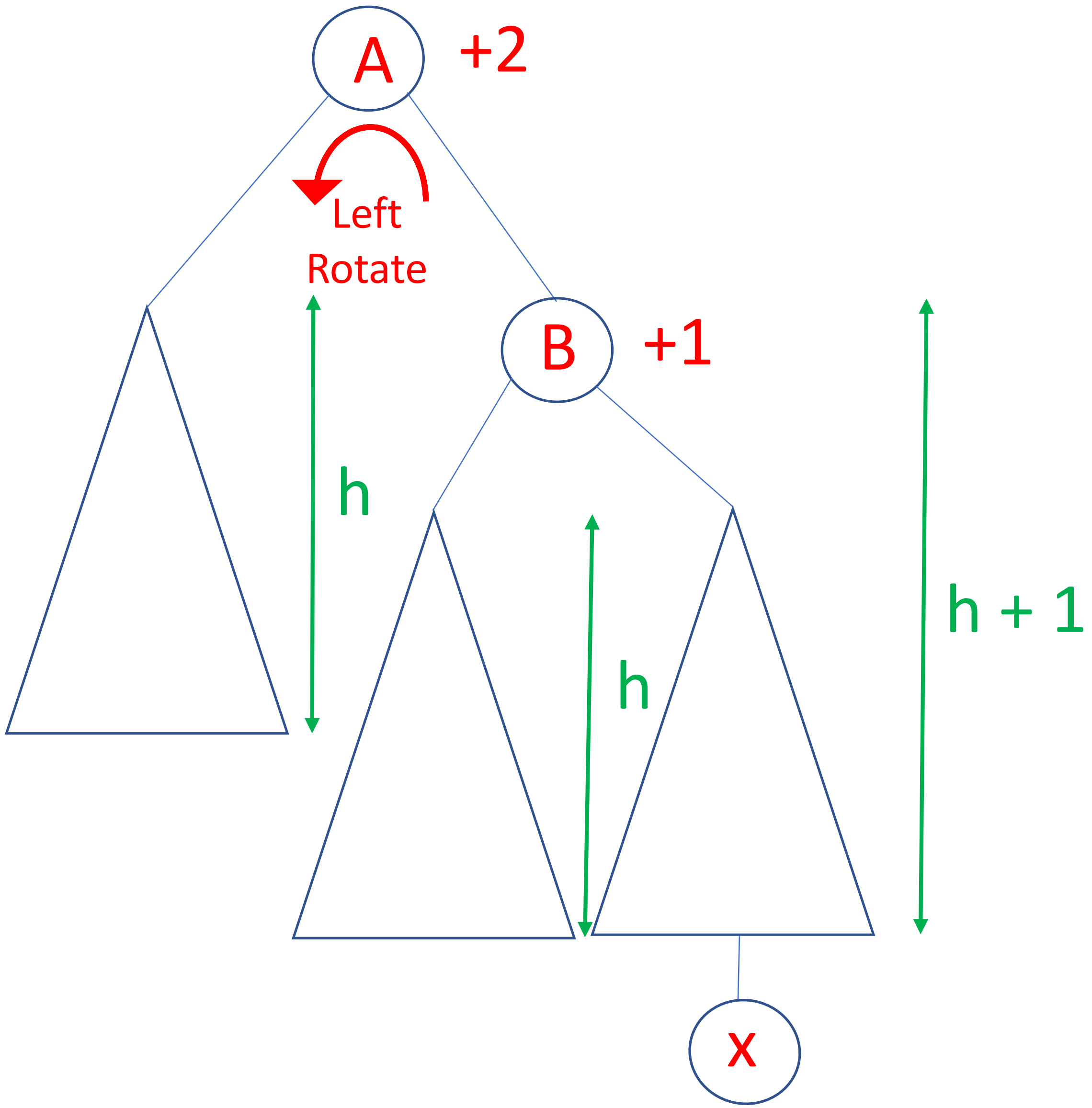
Case 1 (b)



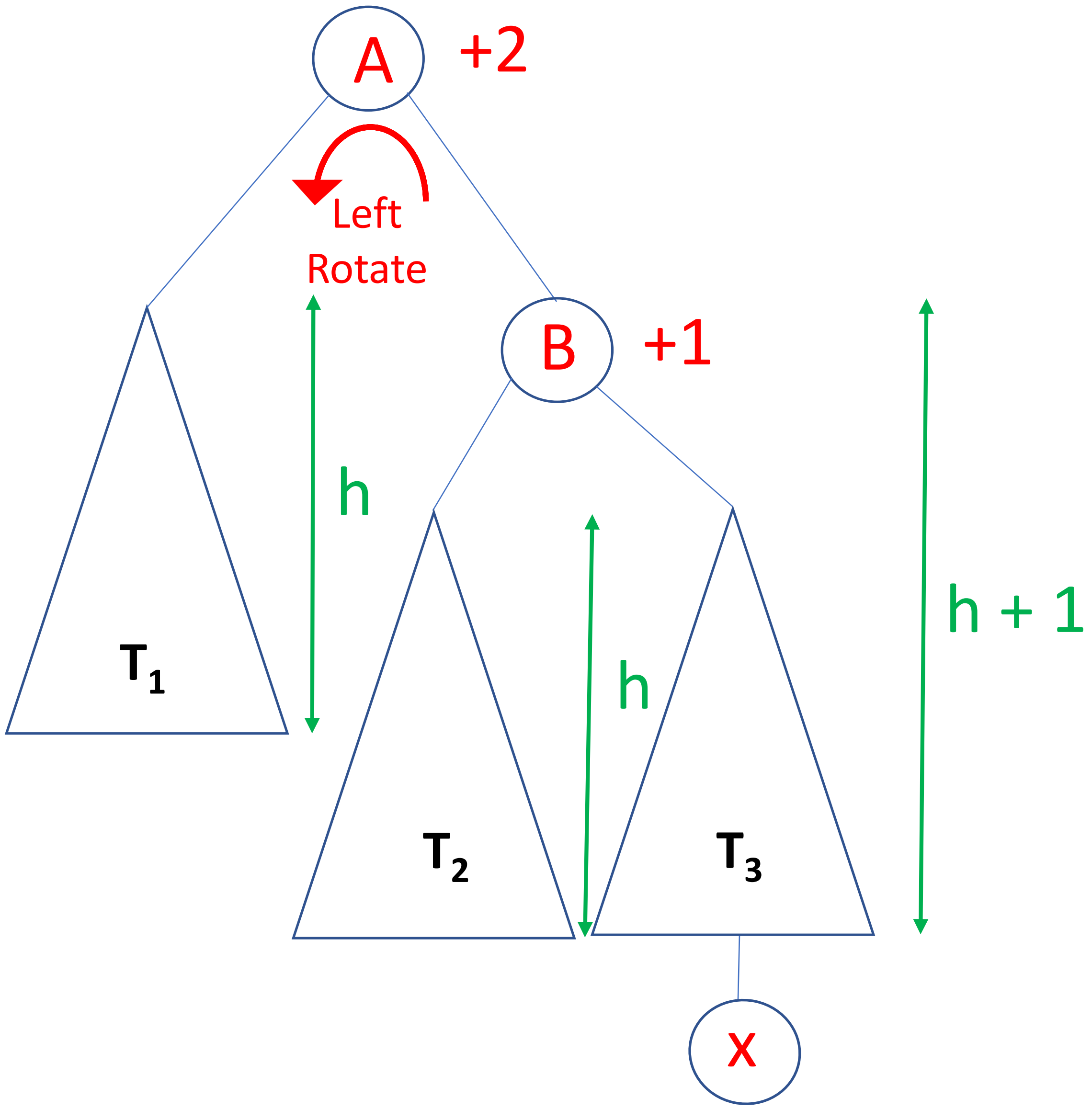
Case 1 (a)



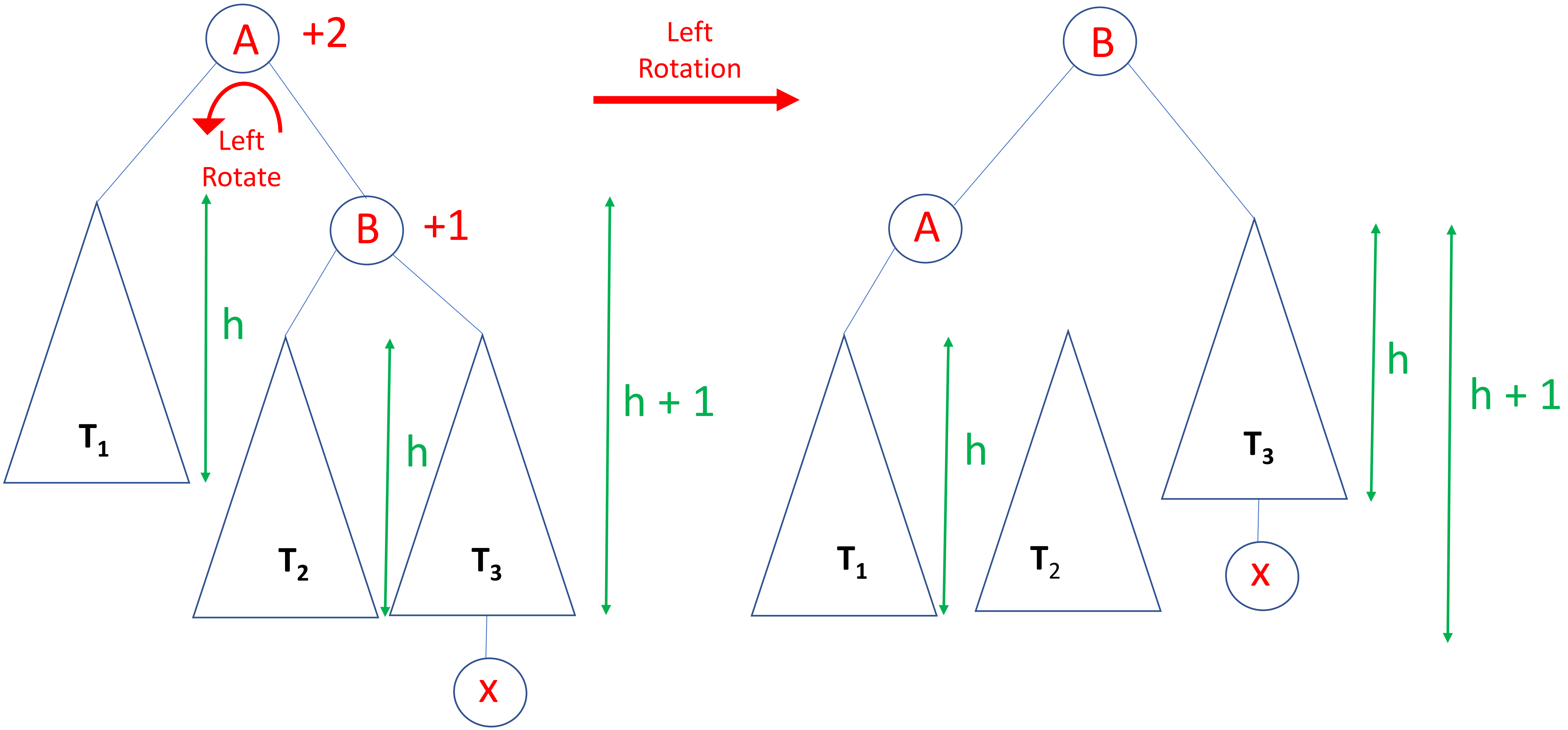
Case 1 (a)



Case 1 (a)

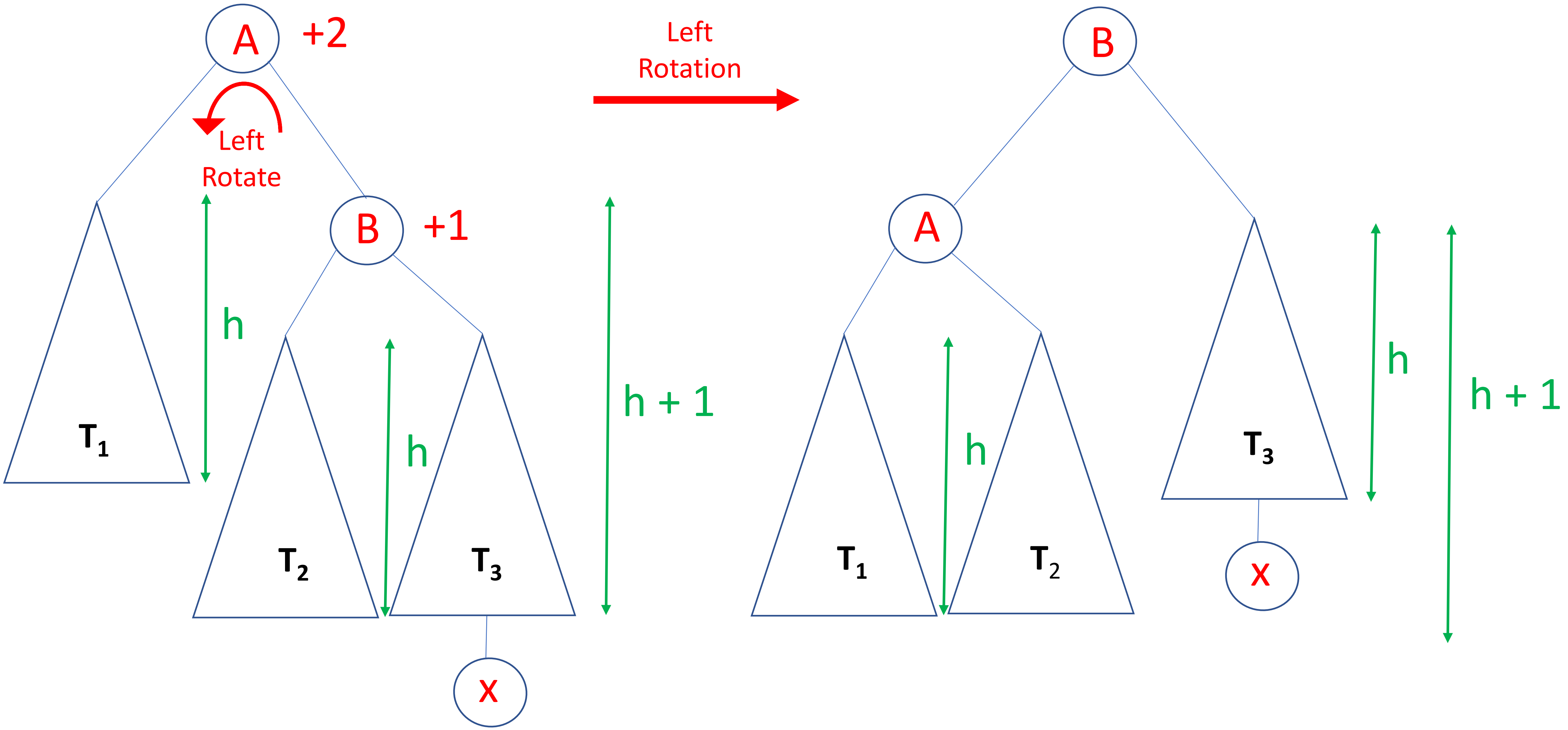


Case 1 (a)

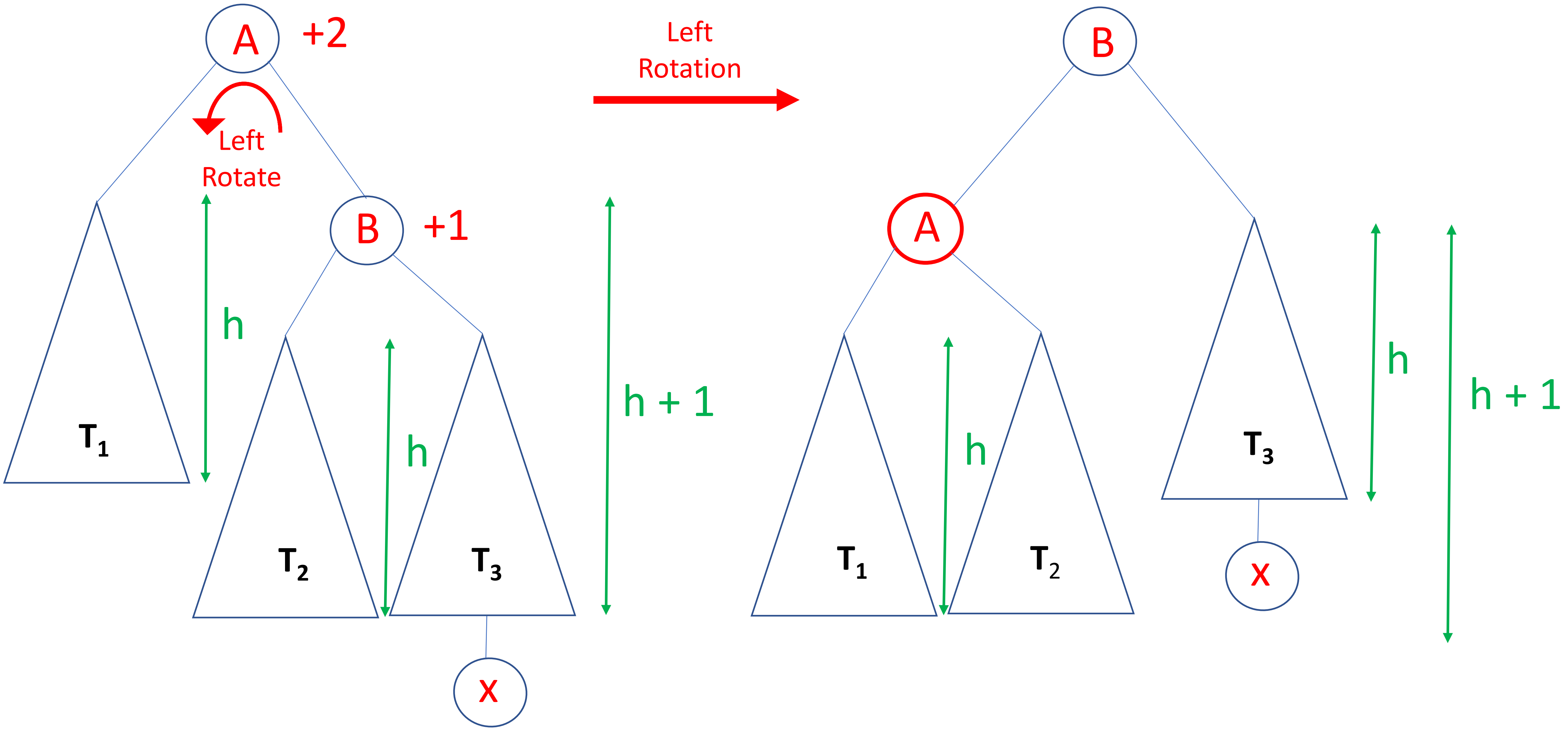


A
+2
Left
Rotate
B
+1
h
T1
T2
h
T3
X
h + 1
Left
Rotation
B
A
T1
h
T2
T3
h
X
h + 1

Case 1 (a)

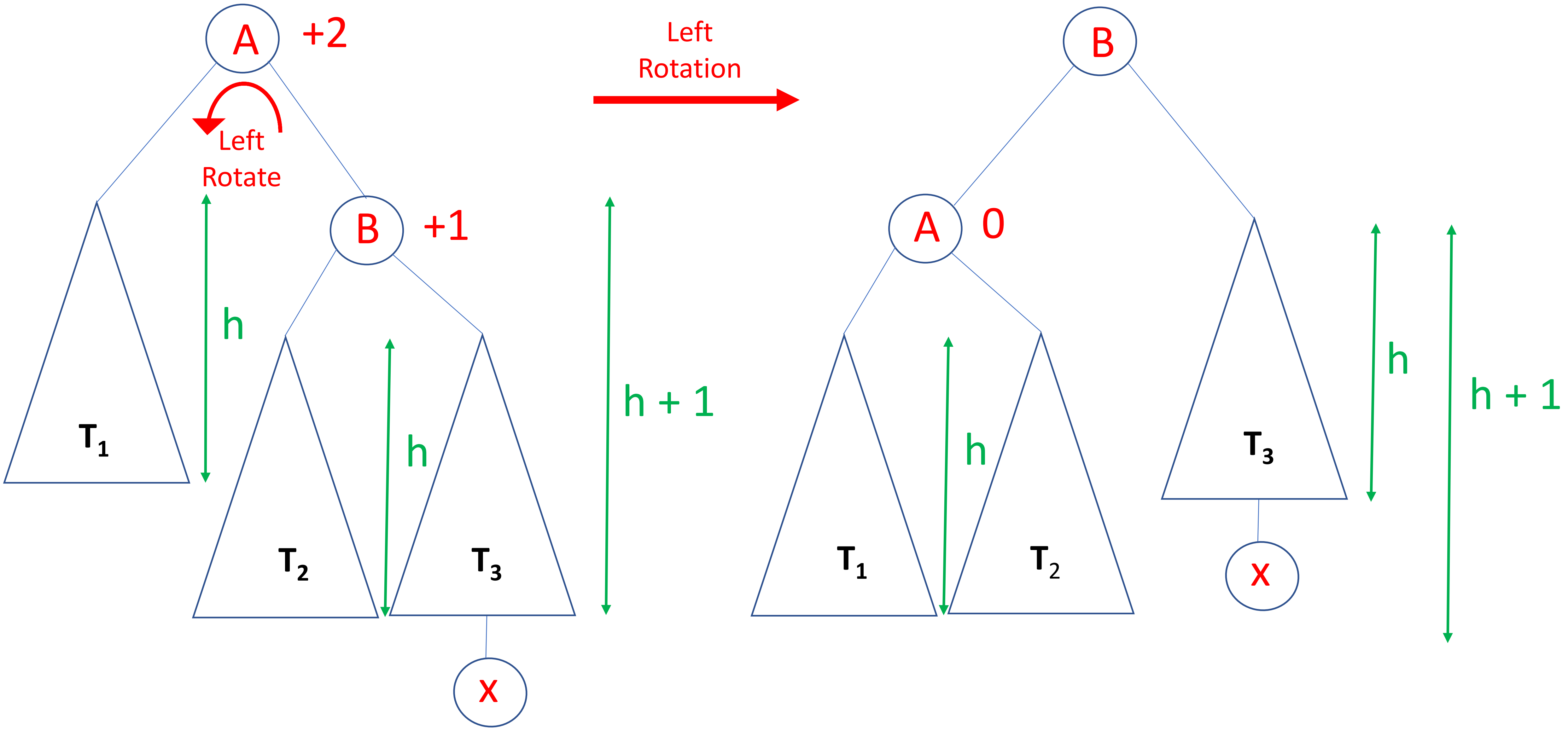

A
+2
Left
Rotate
B
+1
h
h
h + 1
T1
T2
T3
X
Left
Rotation
B
A
T1
T2
T3
X
h
h
h + 1

Case 1 (a)

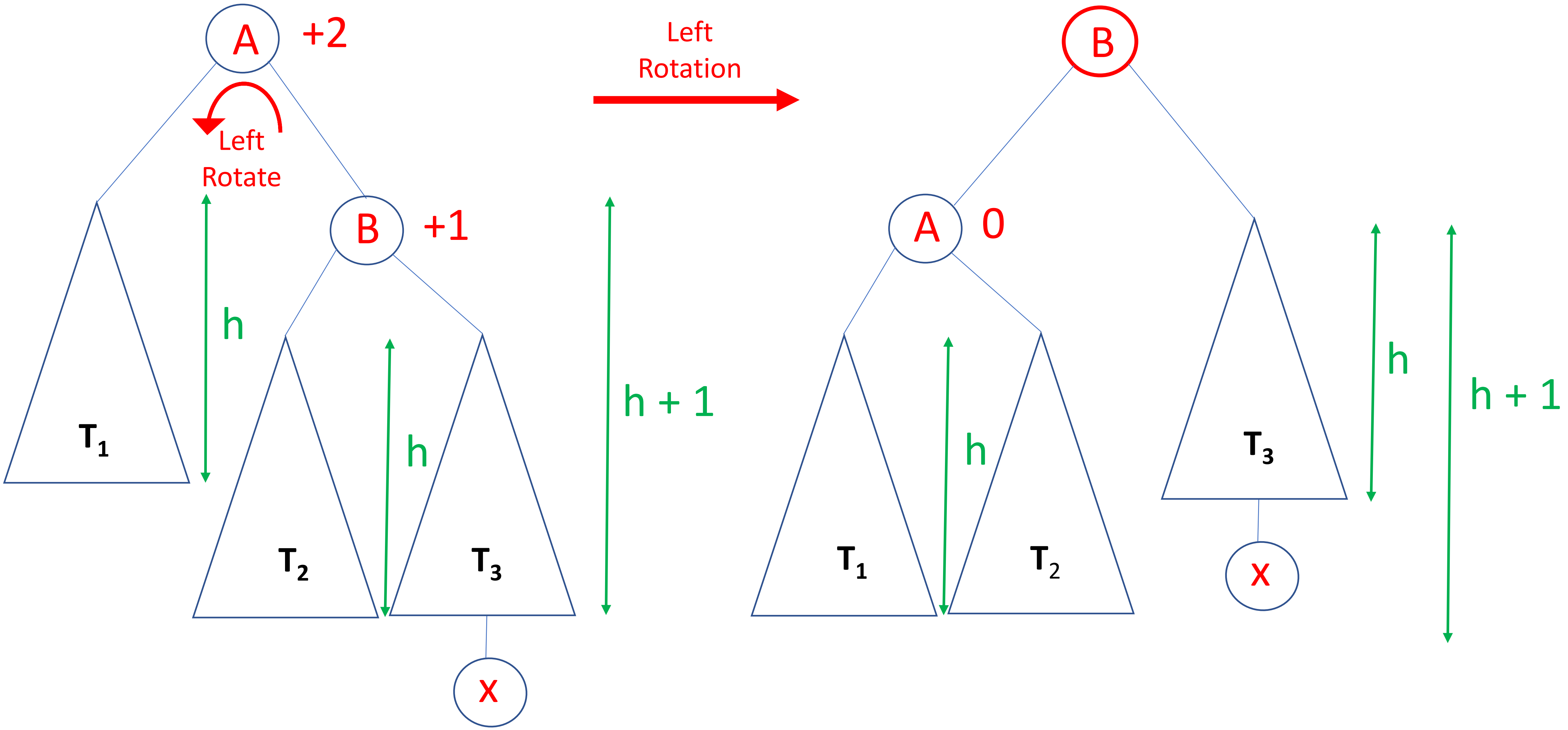


A
+2
Left
Rotate
B
+1
h
T1
T2
h
T3
X
h + 1
Left
Rotation
B
A
T1
h
T2
T3
h
h + 1
X

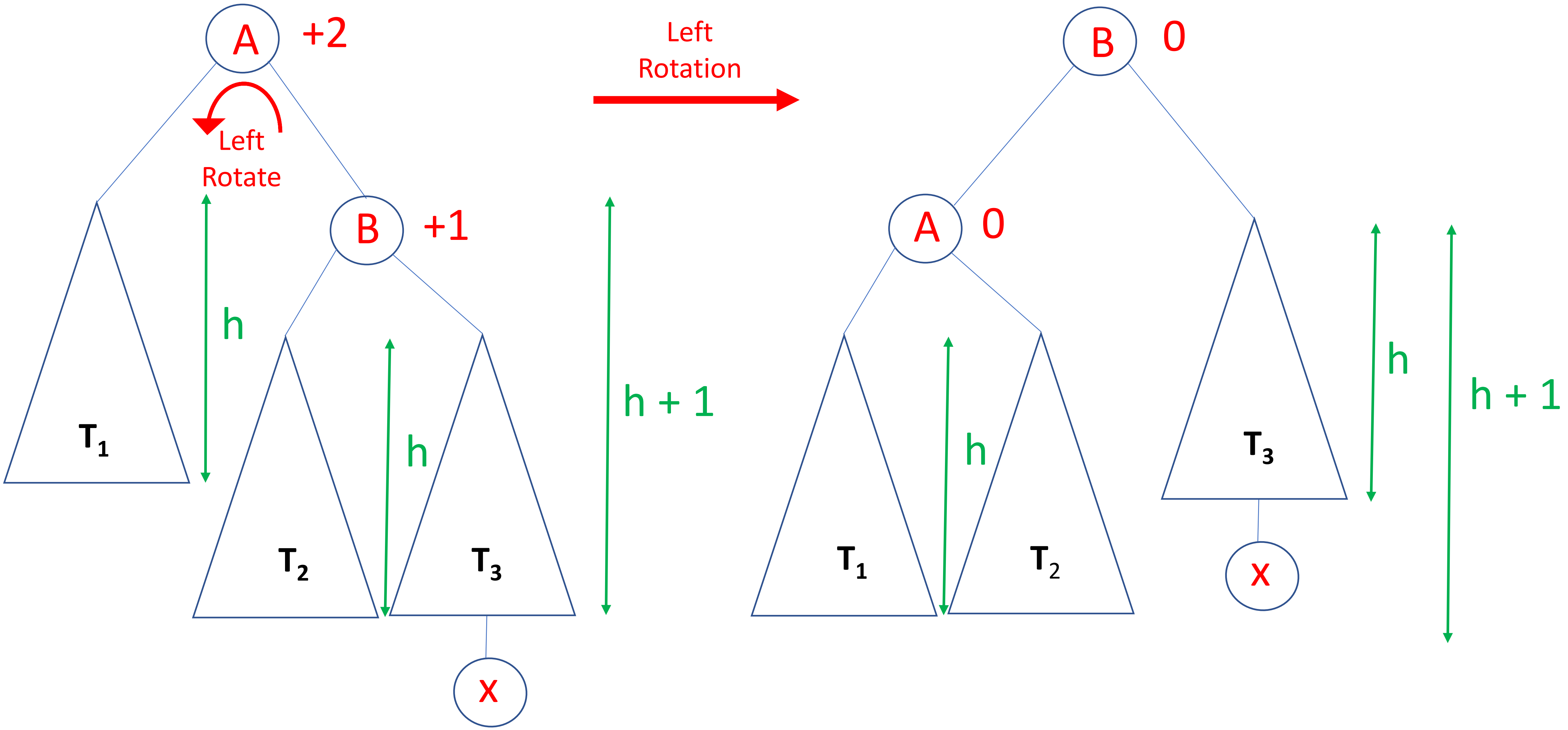
Case 1 (a)


A
+2
Left
Rotate
B
+1
h
h
h + 1
T1
T2
T3
X
Left
Rotation
B
A
0
h
h
h + 1
T1
T2
T3
X

Case 1 (a)

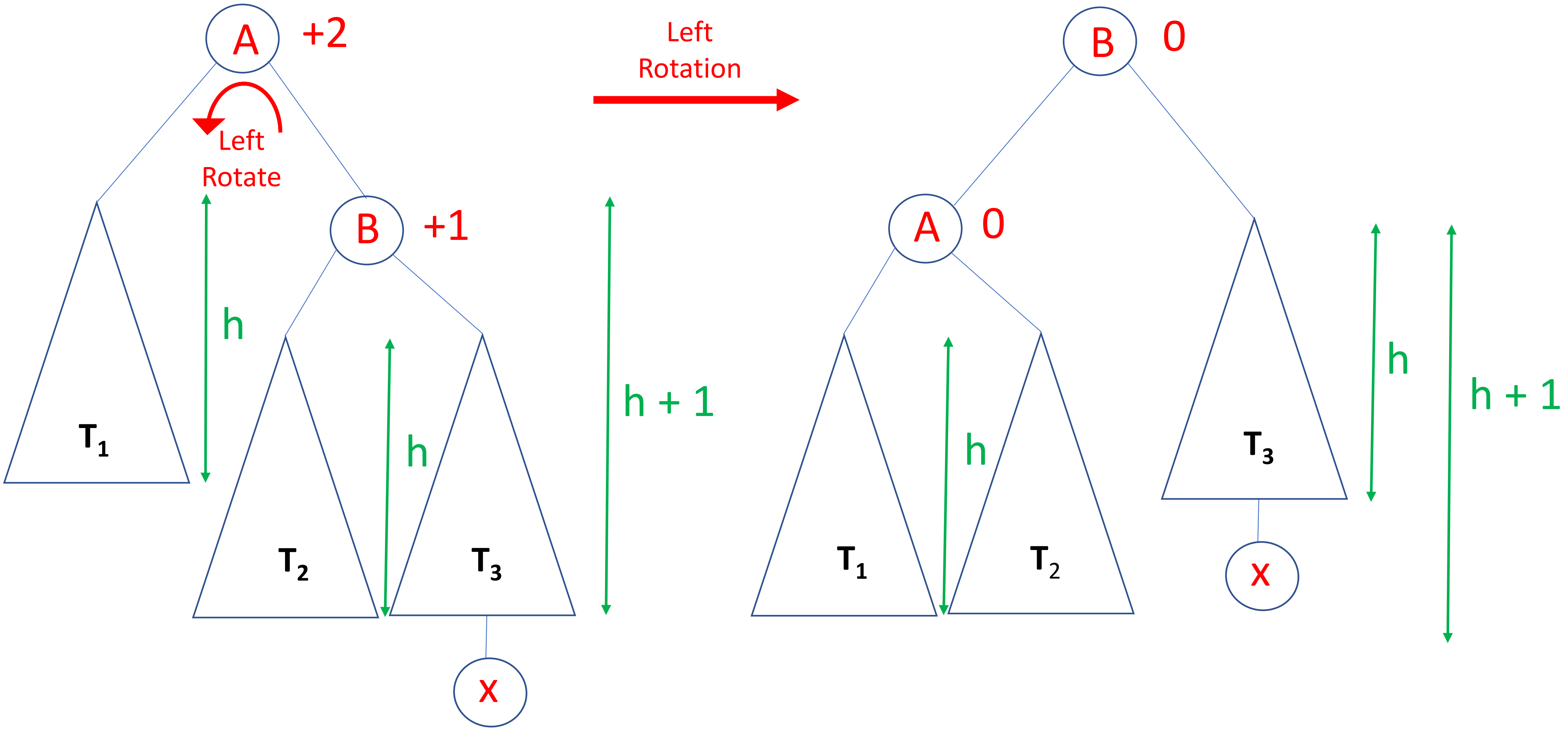

A
+2
Left
Rotate
B
+1
h
h
h + 1
T1
T2
T3
X
Left
Rotation
B
A
0
h
h
h + 1
T1
T2
T3
X

Case 1 (a)


A
+2
Left
Rotate
B
+1
h
h
h + 1
T1
T2
T3
X
Left
Rotation
B
0
A
0
h
h
h + 1
T1
T2
T3
X

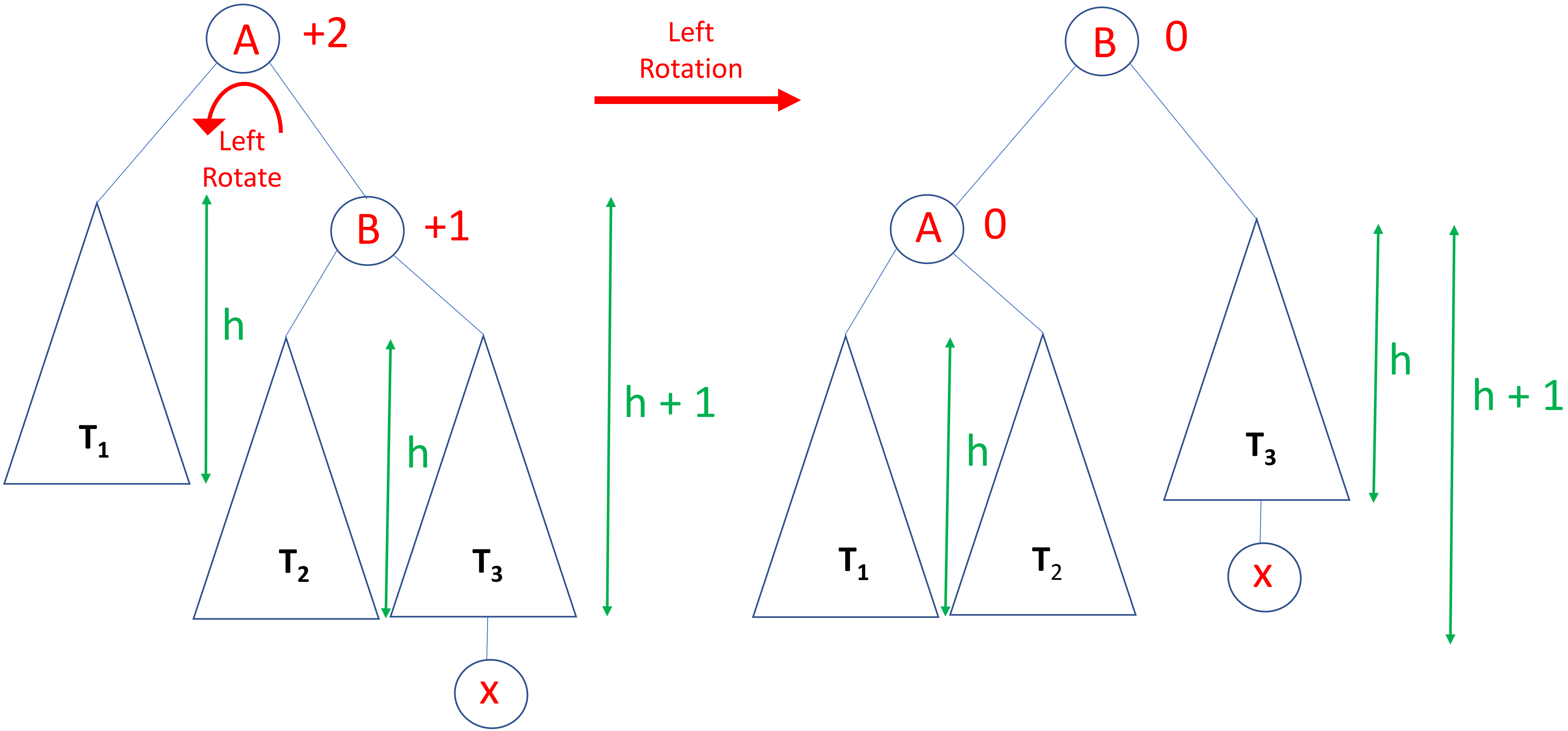
Rotation: (1) Rebalances the subtree

Case 1 (a)



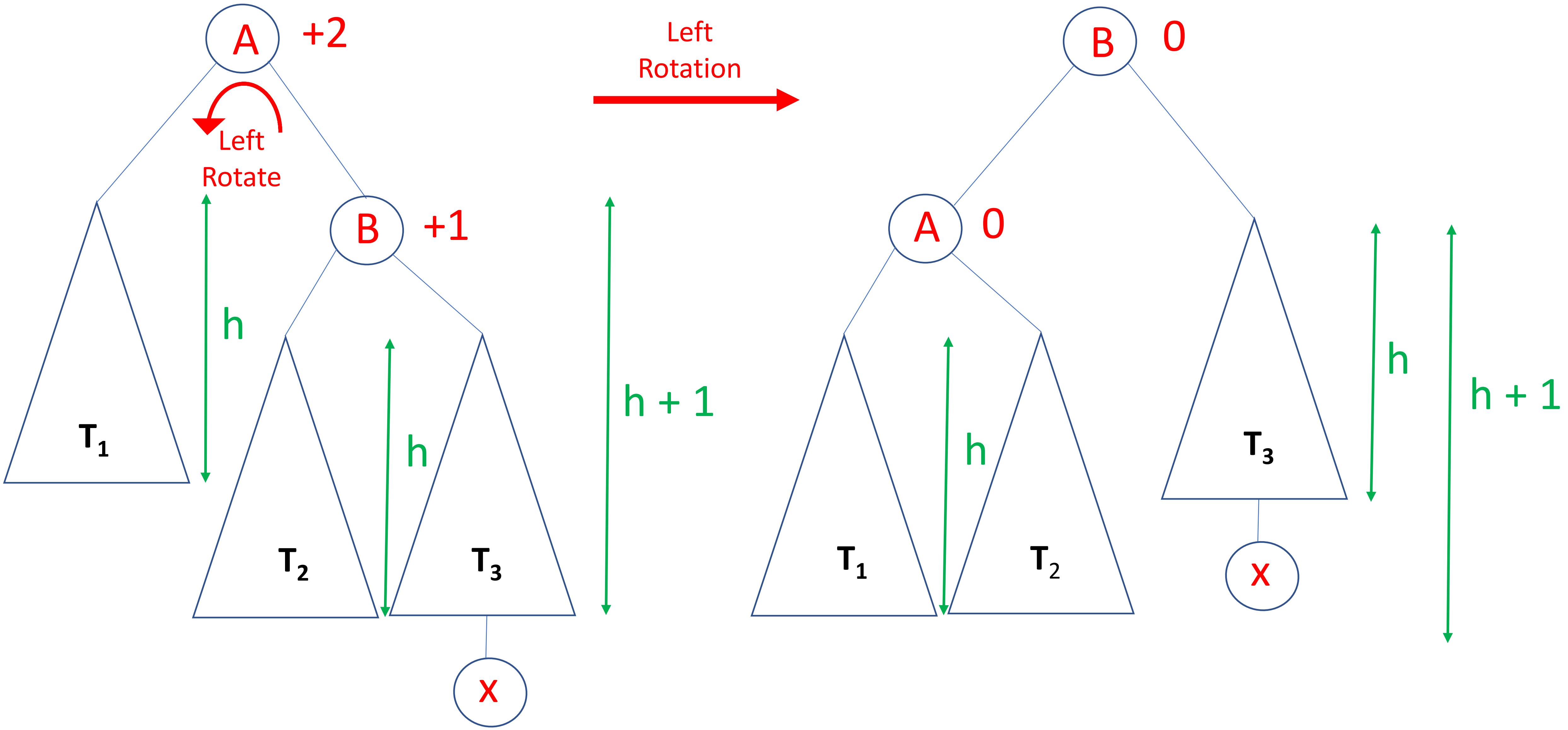
Rotation: (2) Preserves BST property

Case 1 (a)



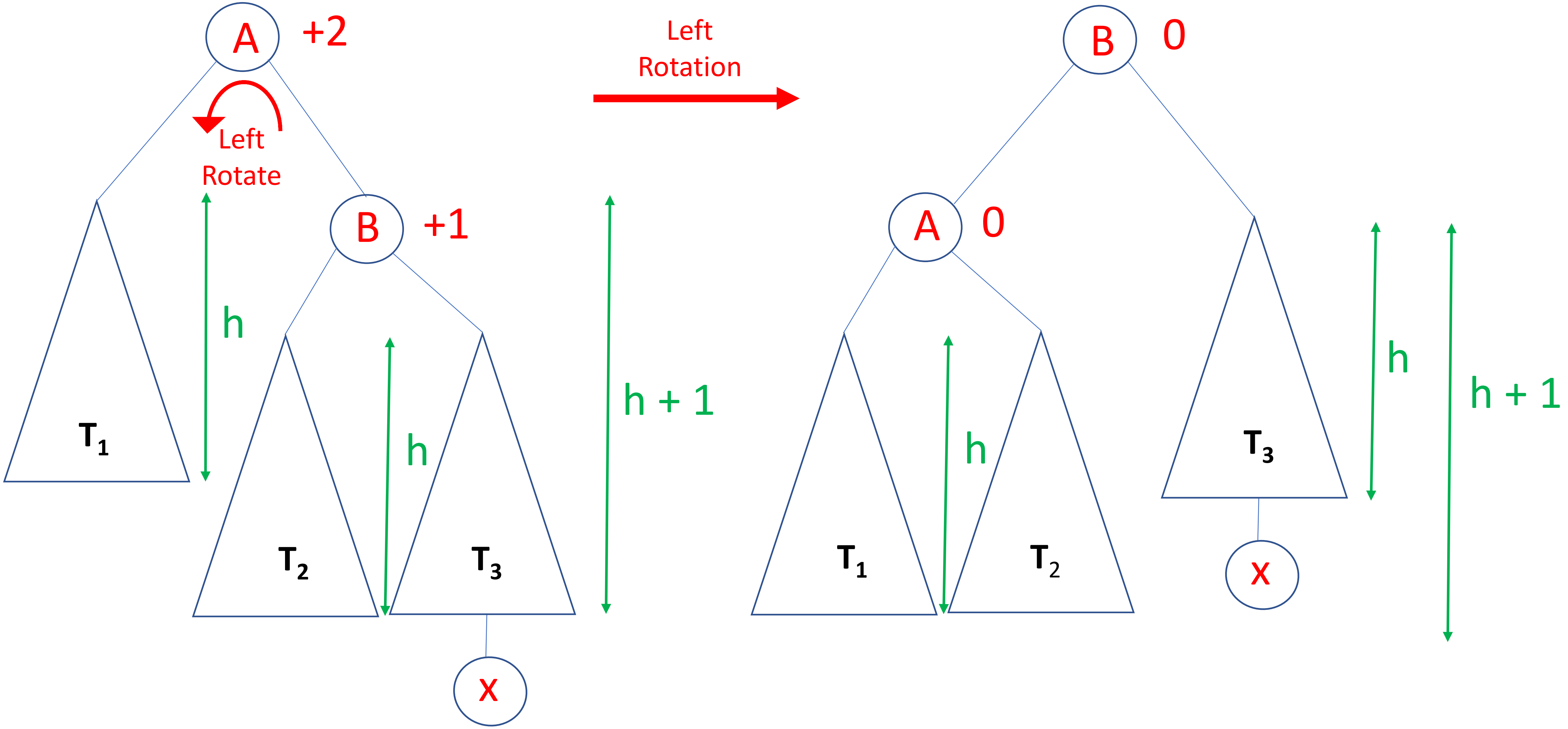
Rotation: (2) Preserves BST property (order:)

Case 1 (a)



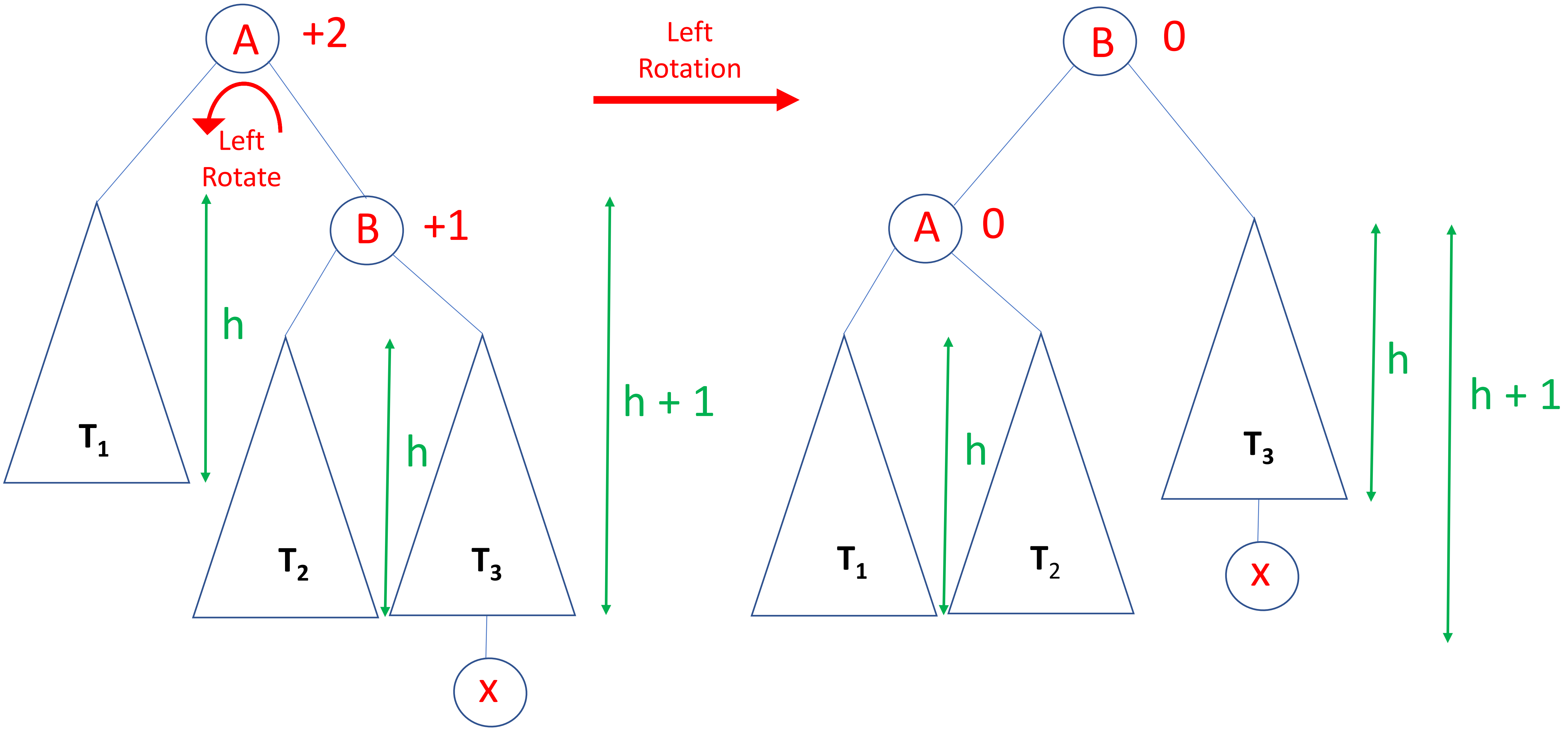
Rotation: (2) Preserves BST property (order: T_1)

Case 1 (a)



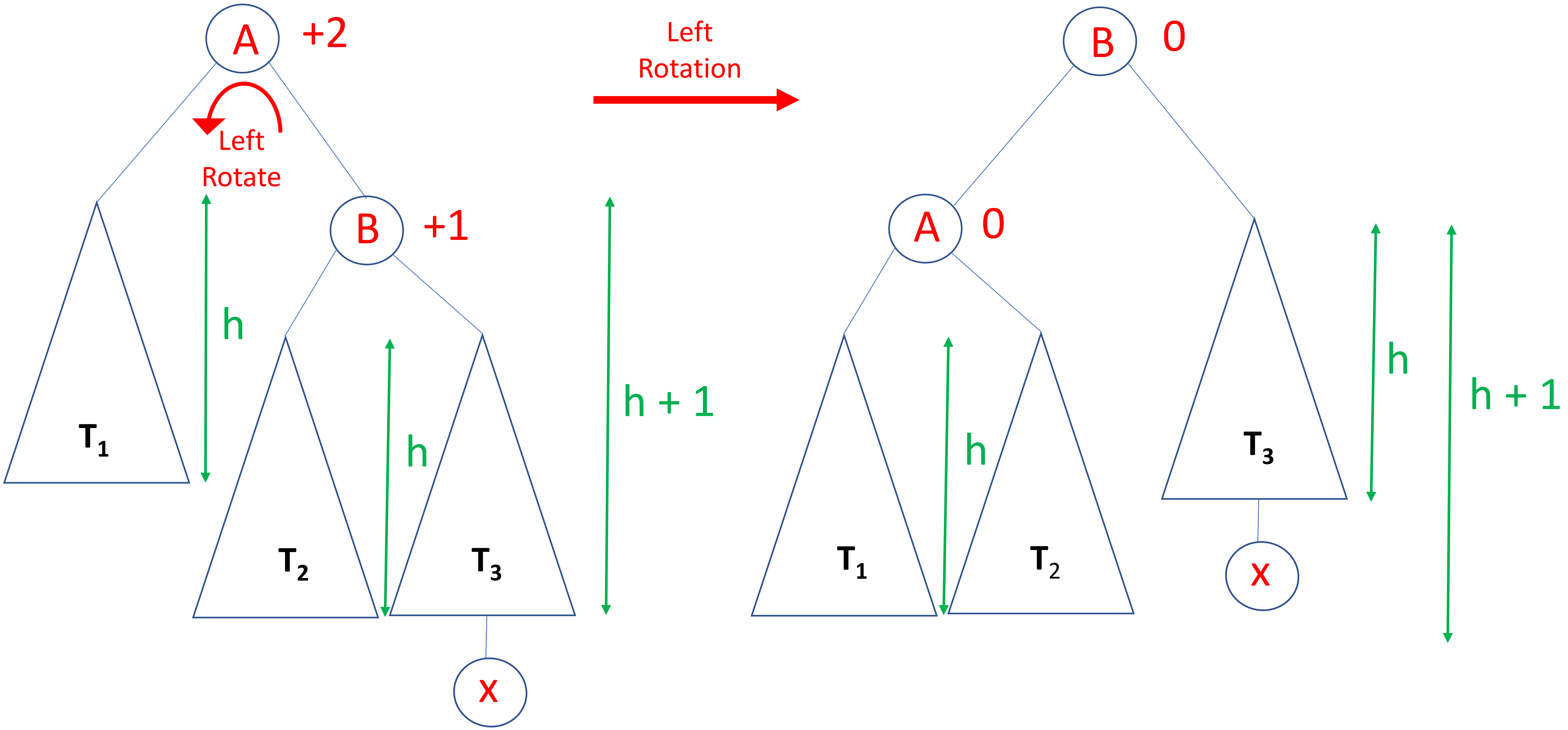
Rotation: (2) Preserves BST property (order: T_1 A)

Case 1 (a)



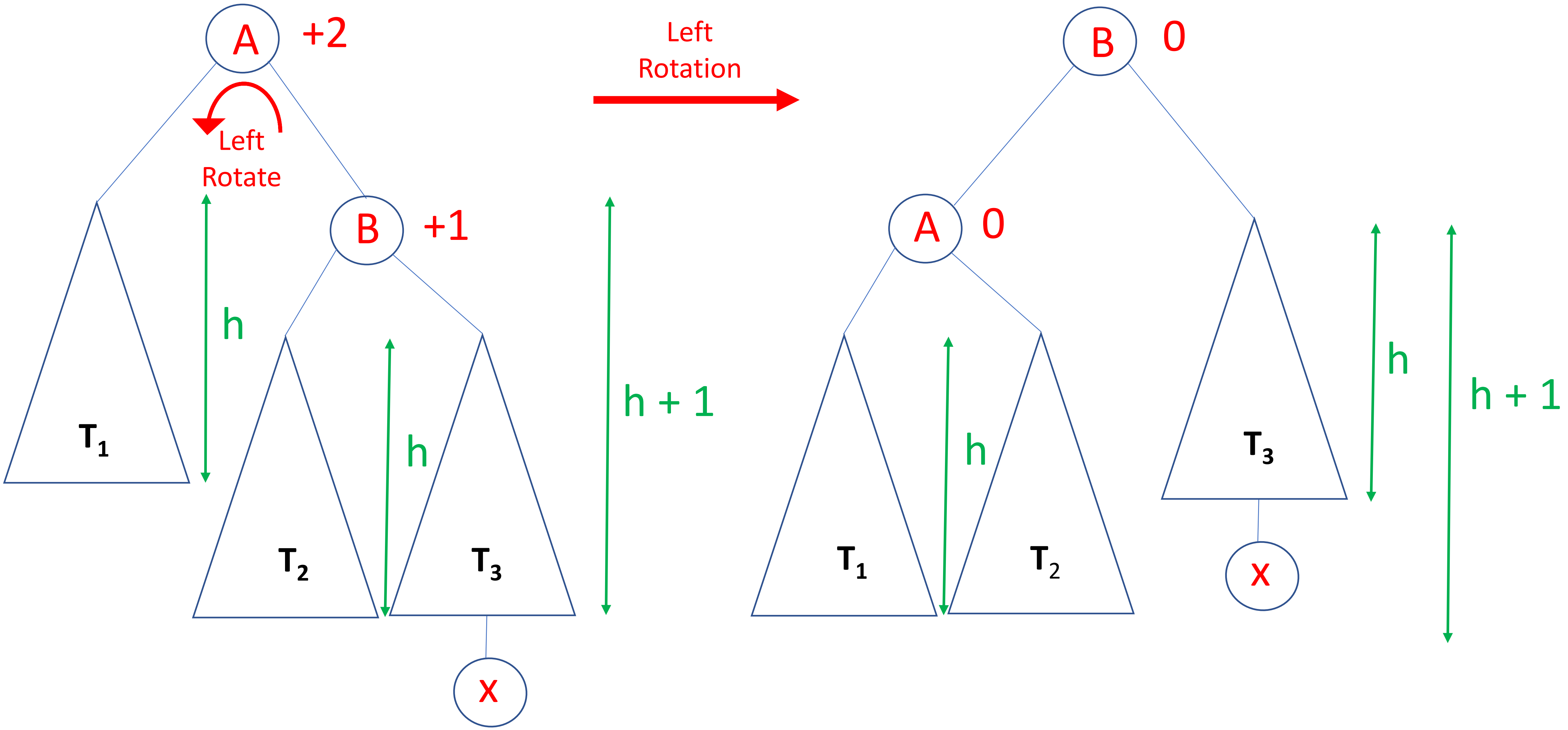
Rotation: (2) Preserves BST property (order: T_1 A T_2)

Case 1 (a)



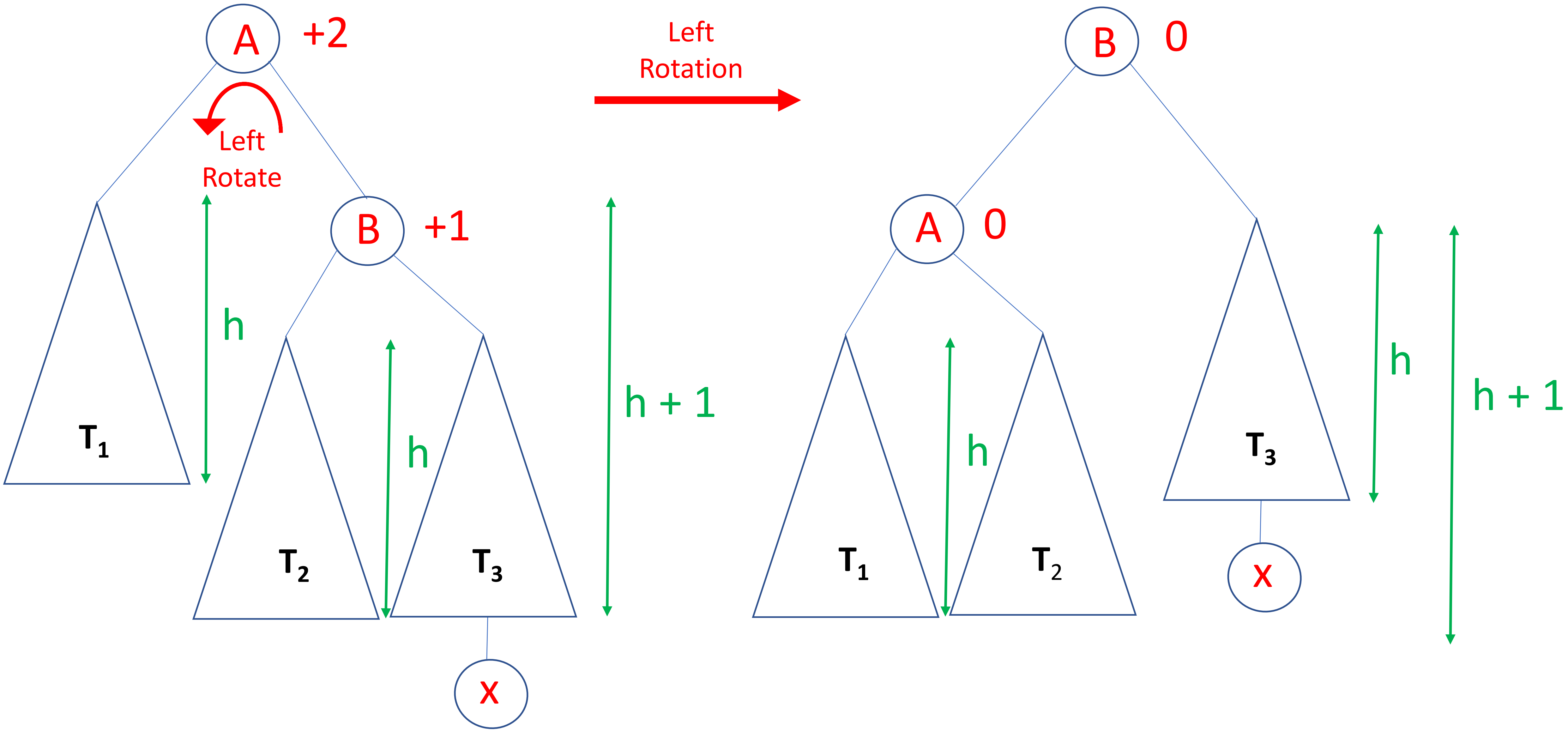
Rotation: (2) Preserves BST property (order: T_1 A T_2 B)

Case 1 (a)

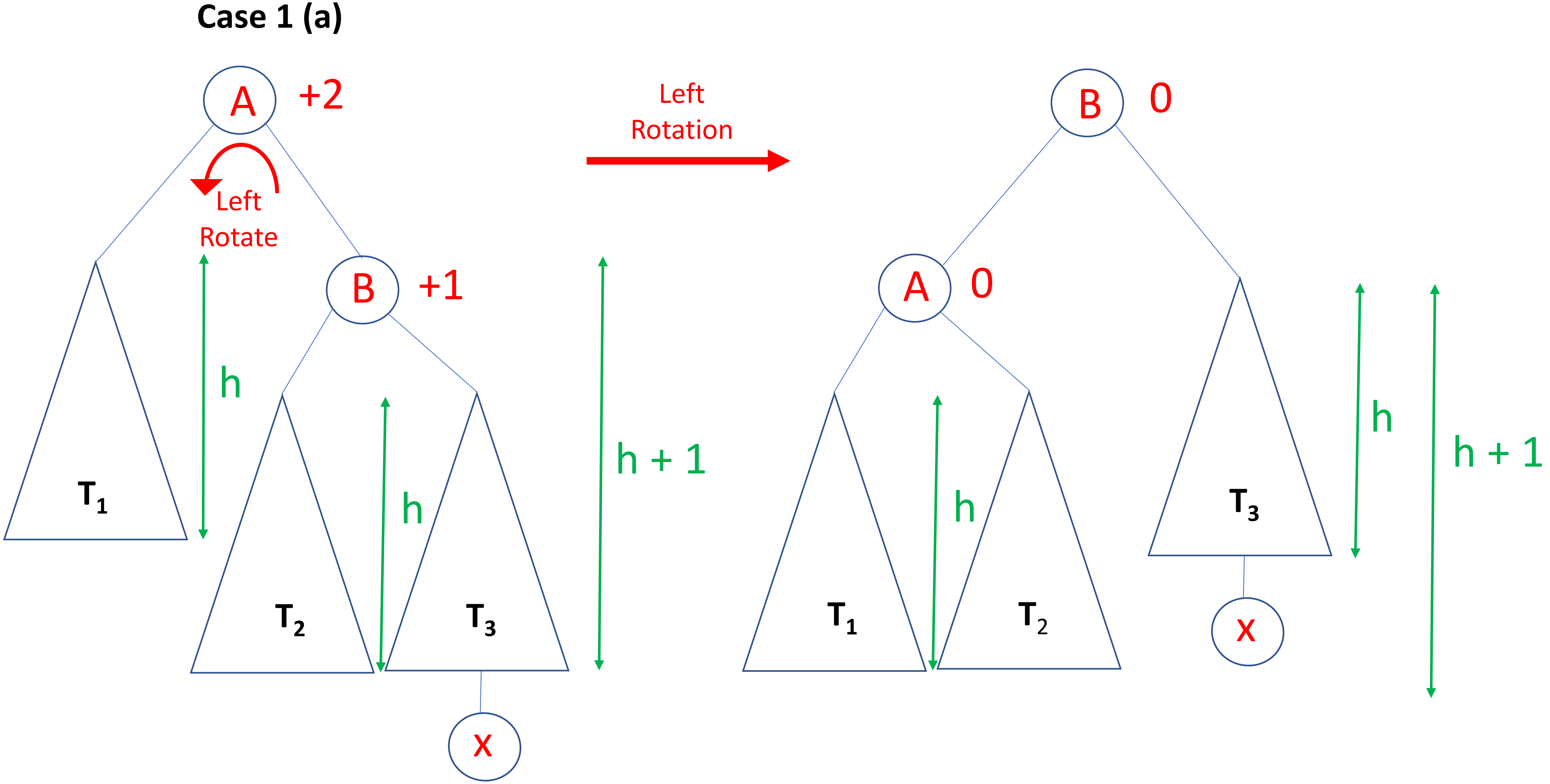


Rotation: (2) Preserves BST property (order: T_1 A T_2 B T_3)

Case 1 (a)

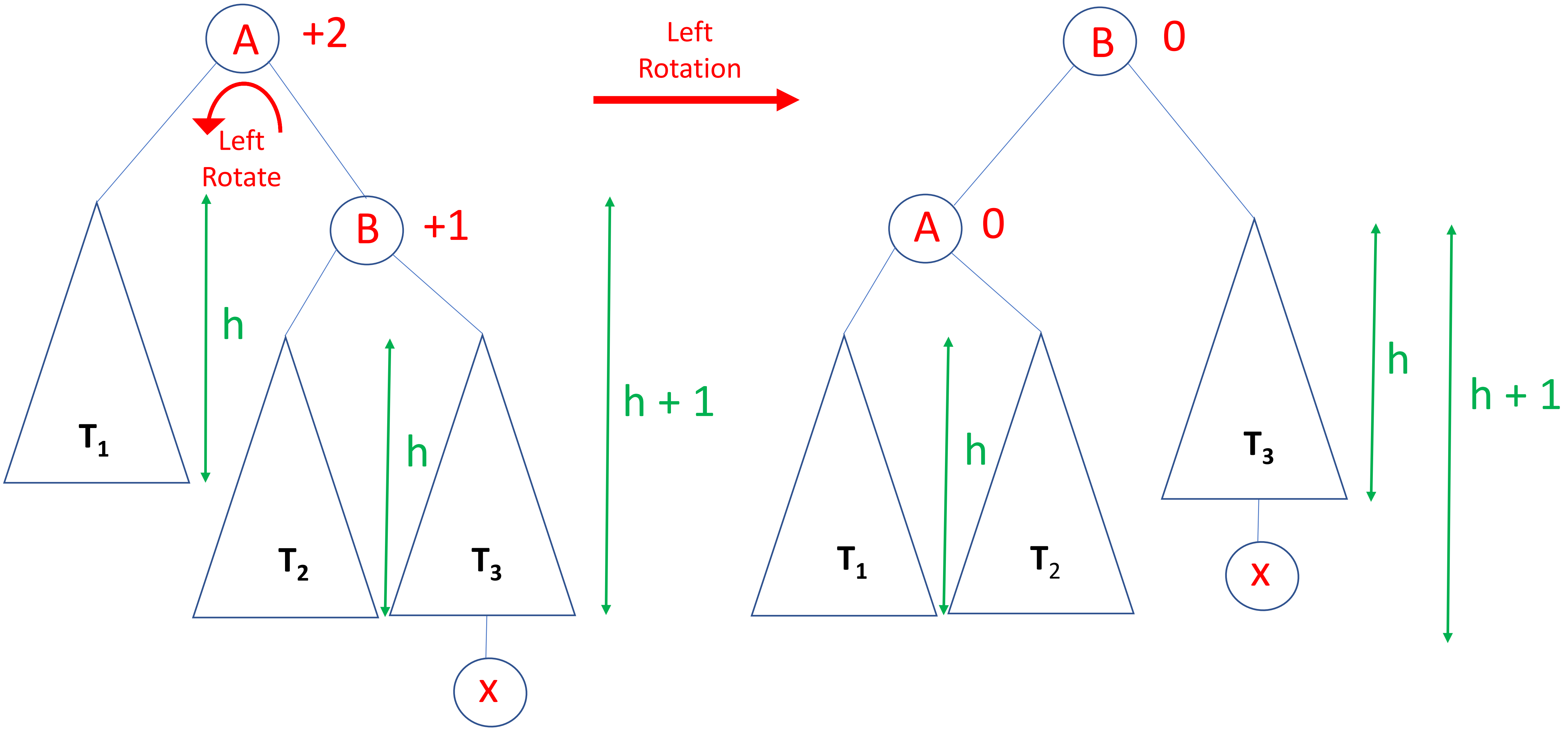


Rotation: (3) Bonus?

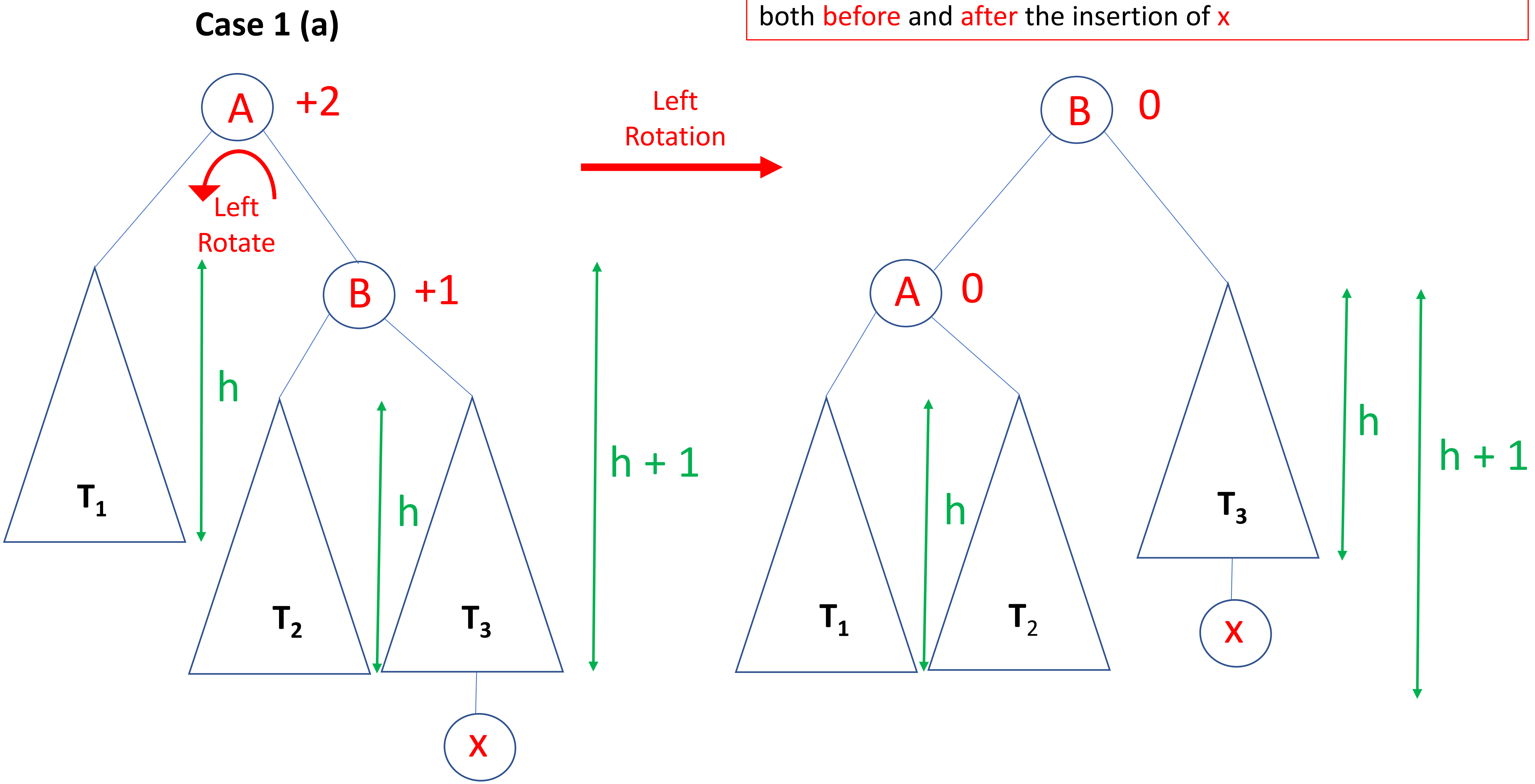


Rotation: (3) Bonus? YES!

Case 1 (a)

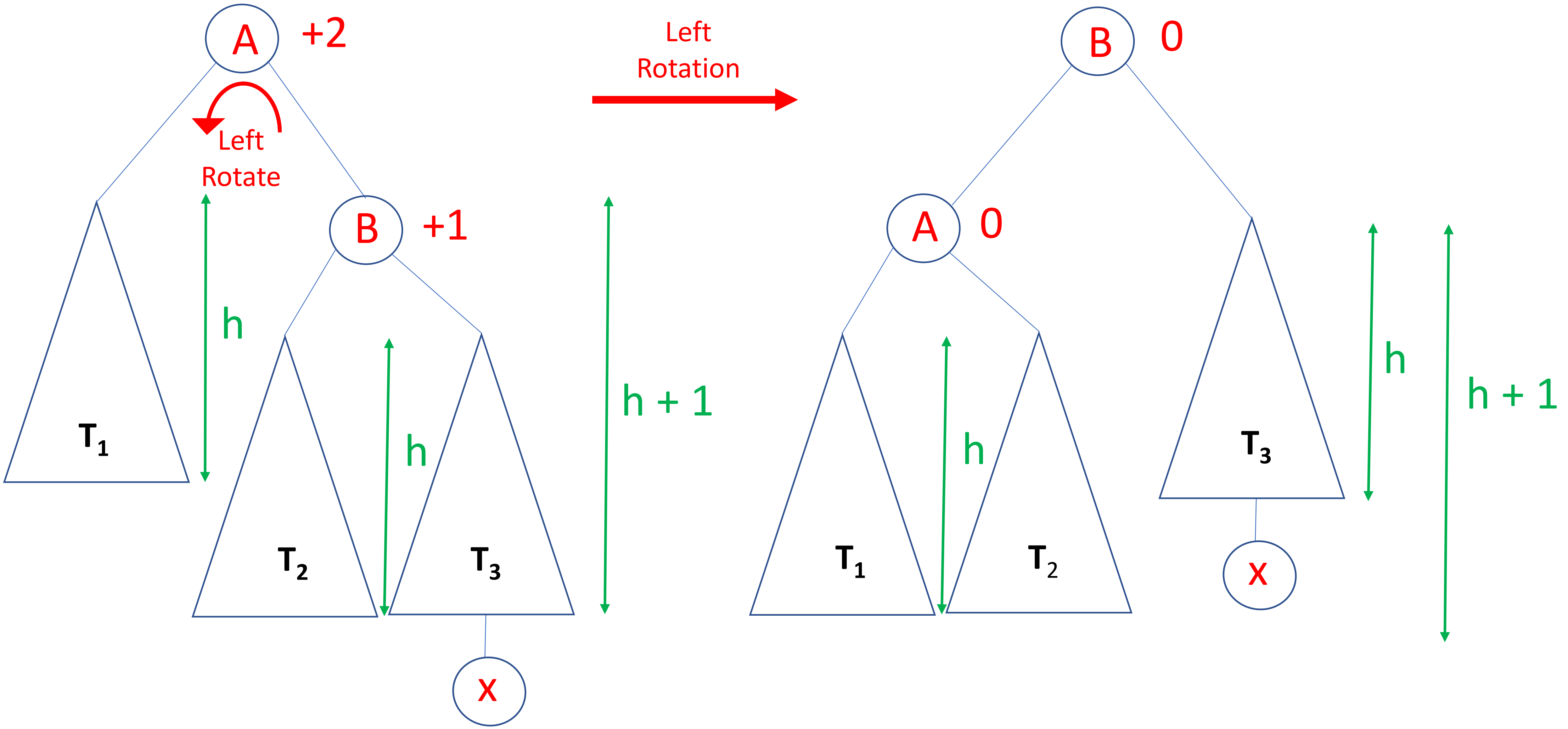


Rotation: (3) Bonus: height of this subtree is the same both before and after the insertion of x



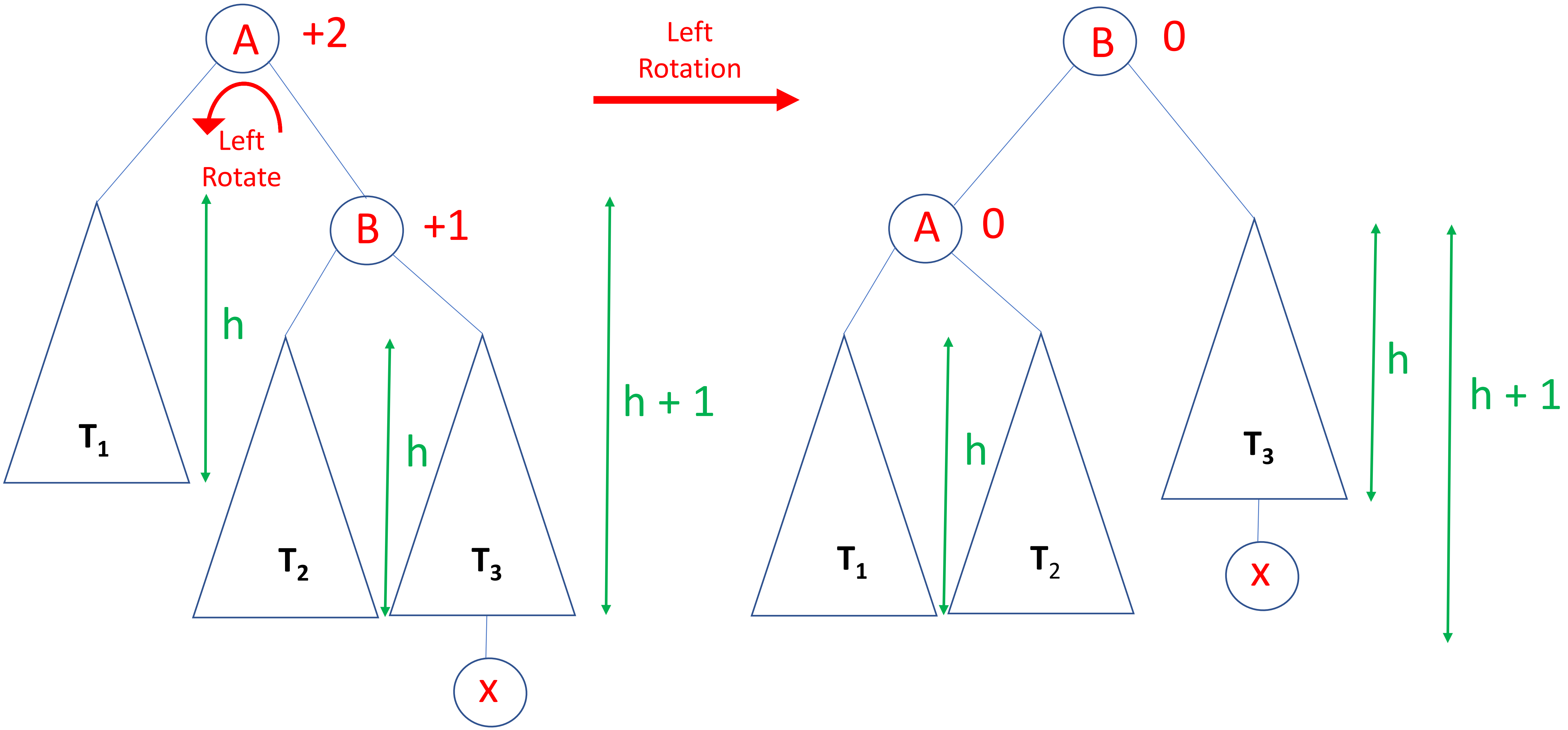
Rotation: (3) Bonus: height of this subtree is the same (h+2) both before and after the insertion of x

Case 1 (a)



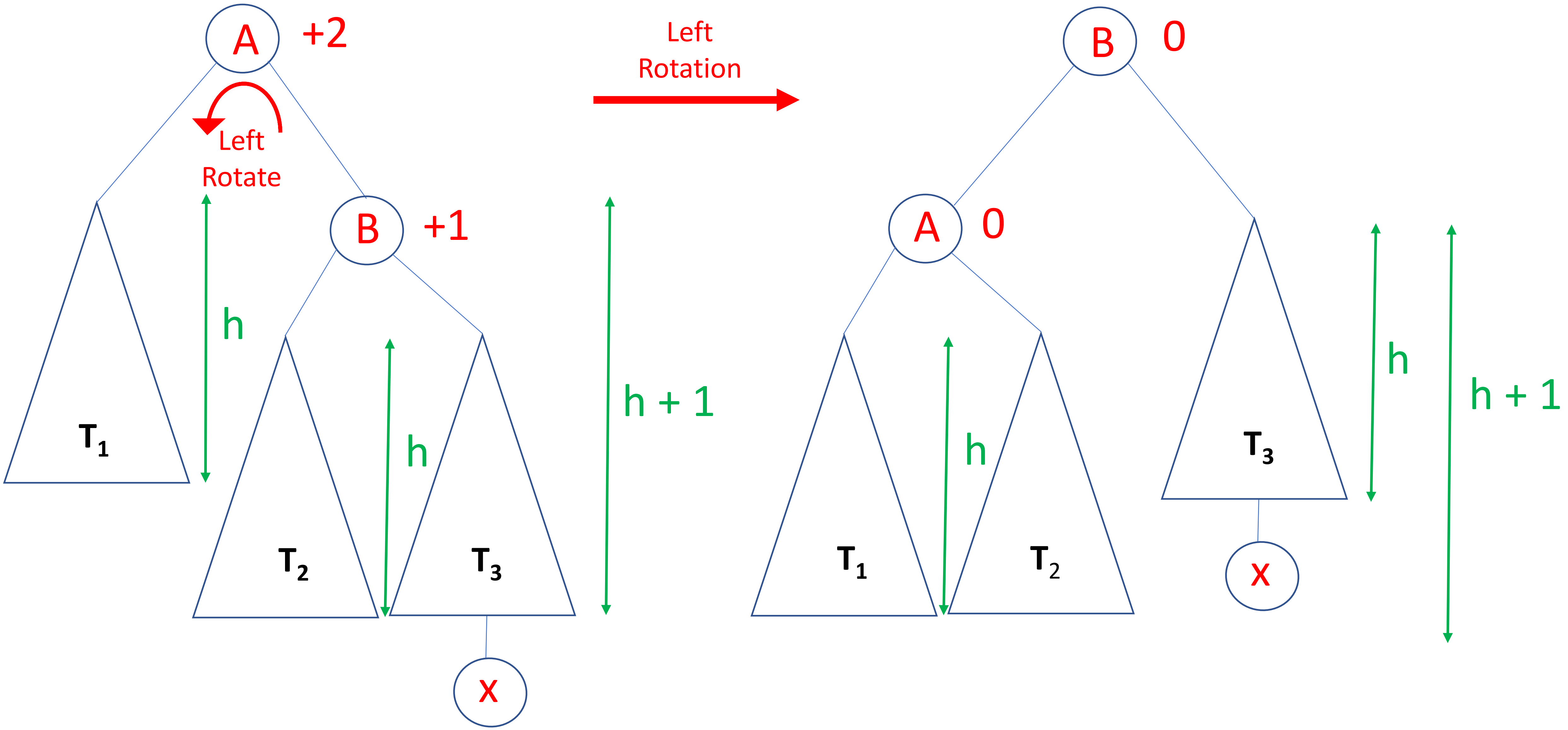
Rotation: (3) Bonus ➡ insertion algorithm is done

Case 1 (a)

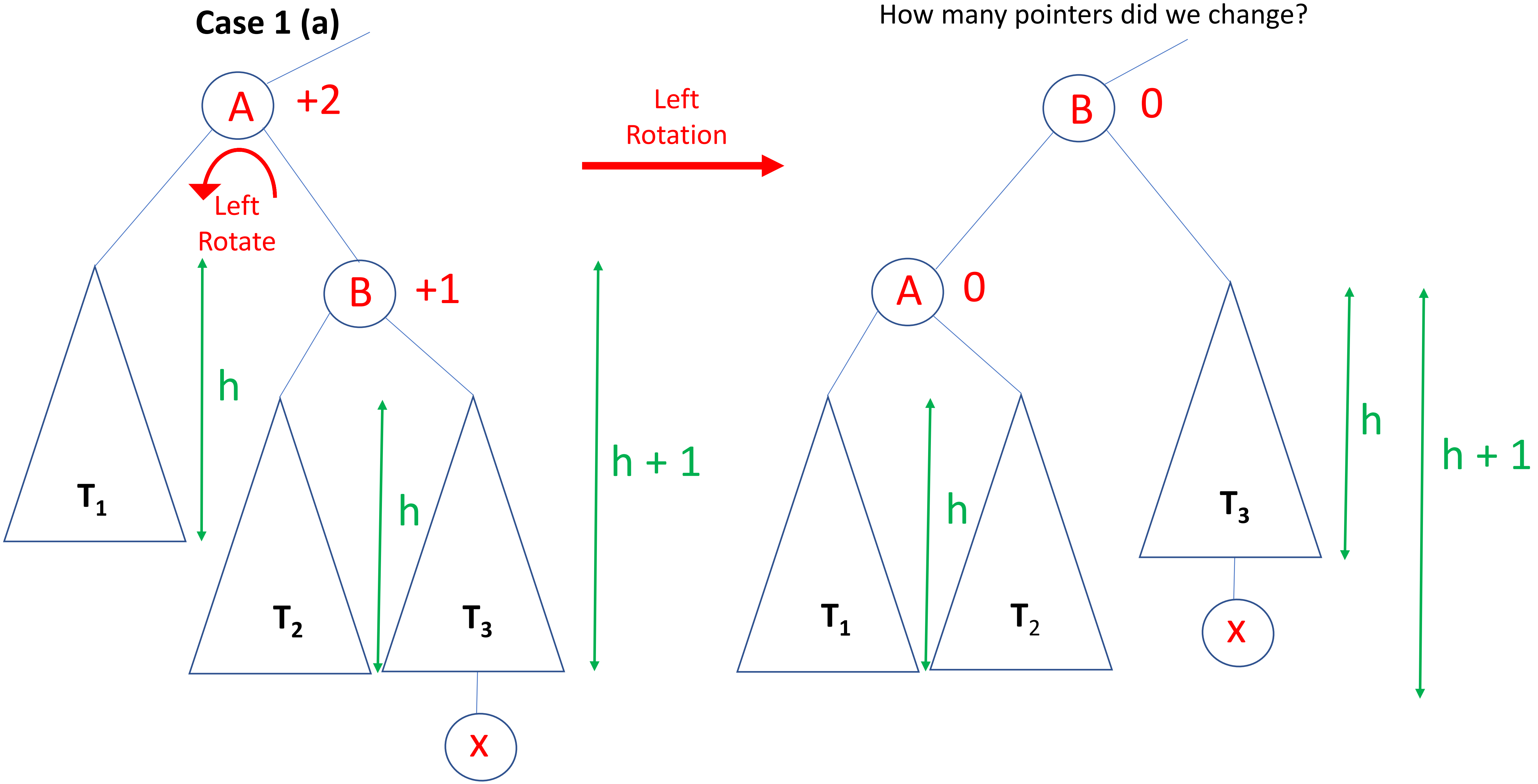


Rotation: What is the time complexity?

Case 1 (a)

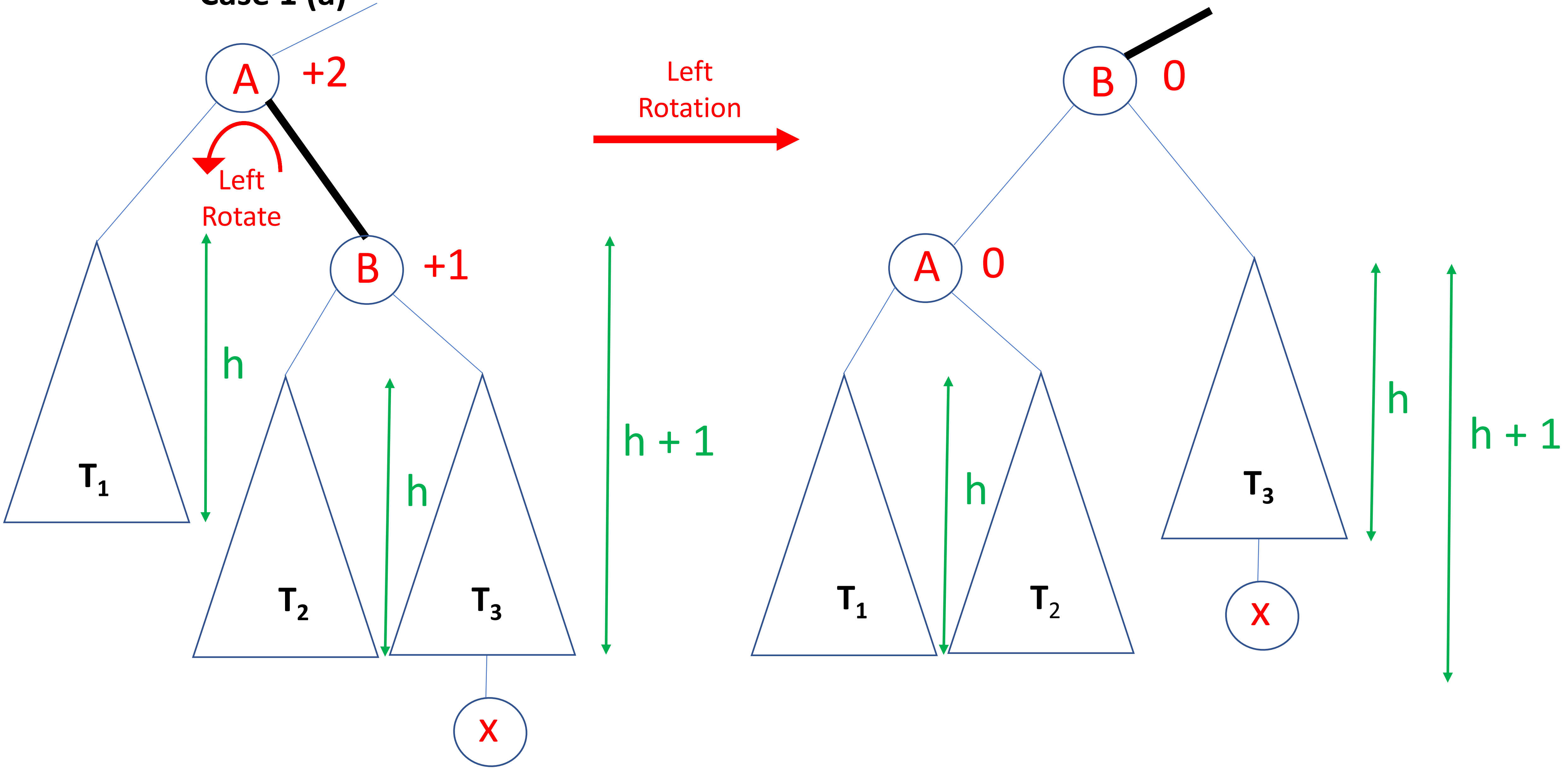


Rotation: What is the time complexity?
How many pointers did we change?



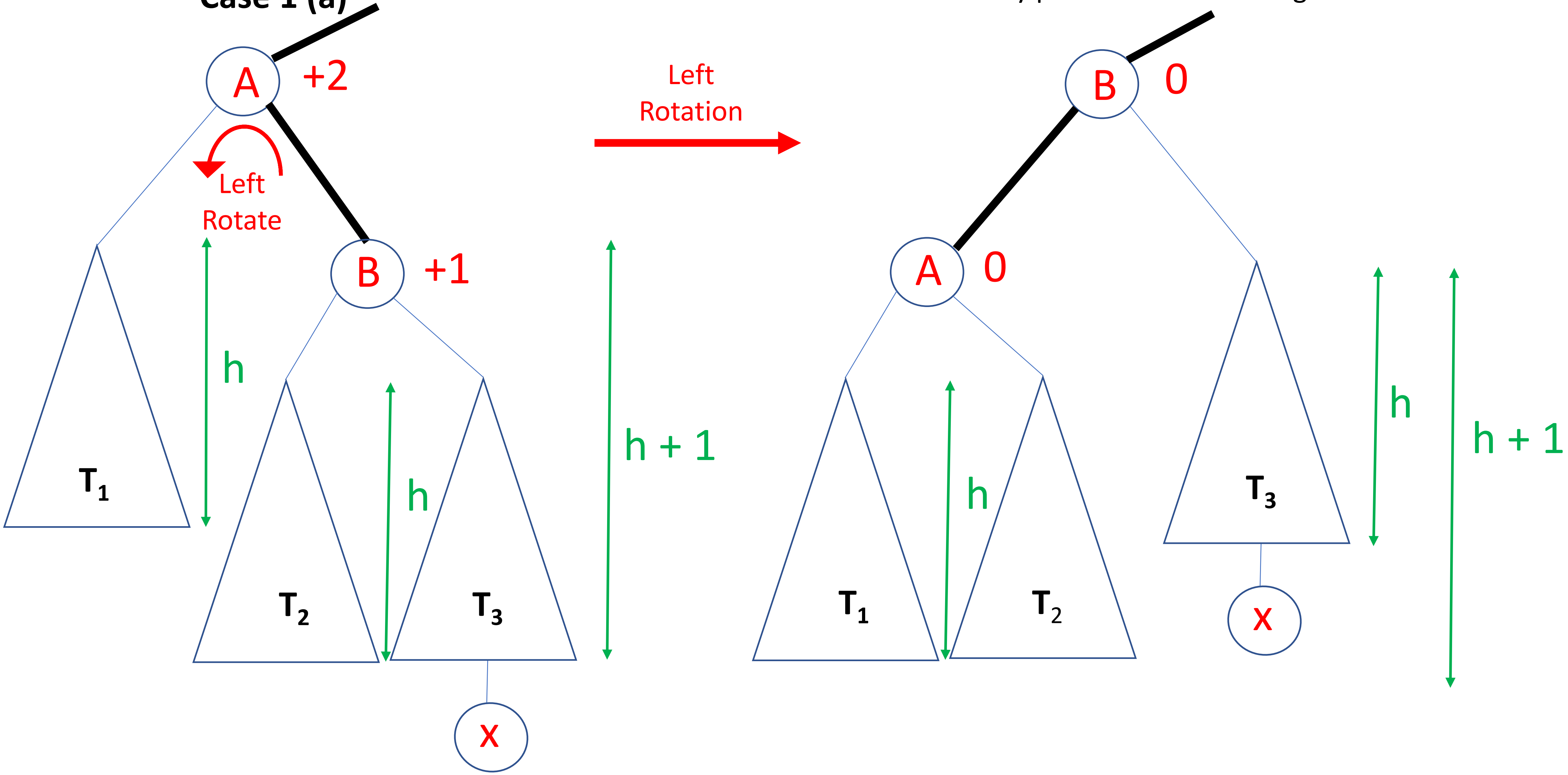
Rotation: What is the time complexity?
How many pointers did we change?

Case 1 (a)

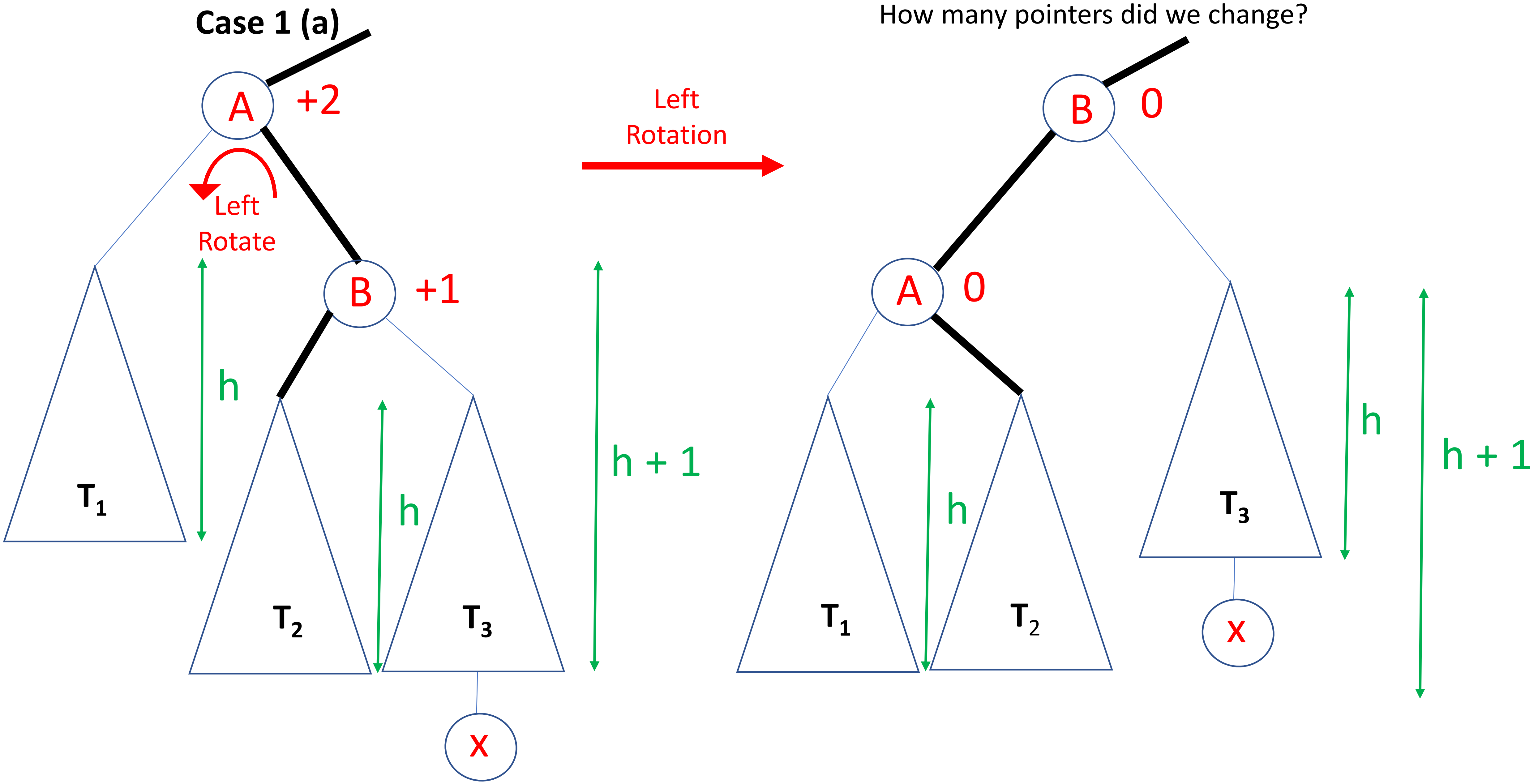


Rotation: What is the time complexity?
How many pointers did we change?

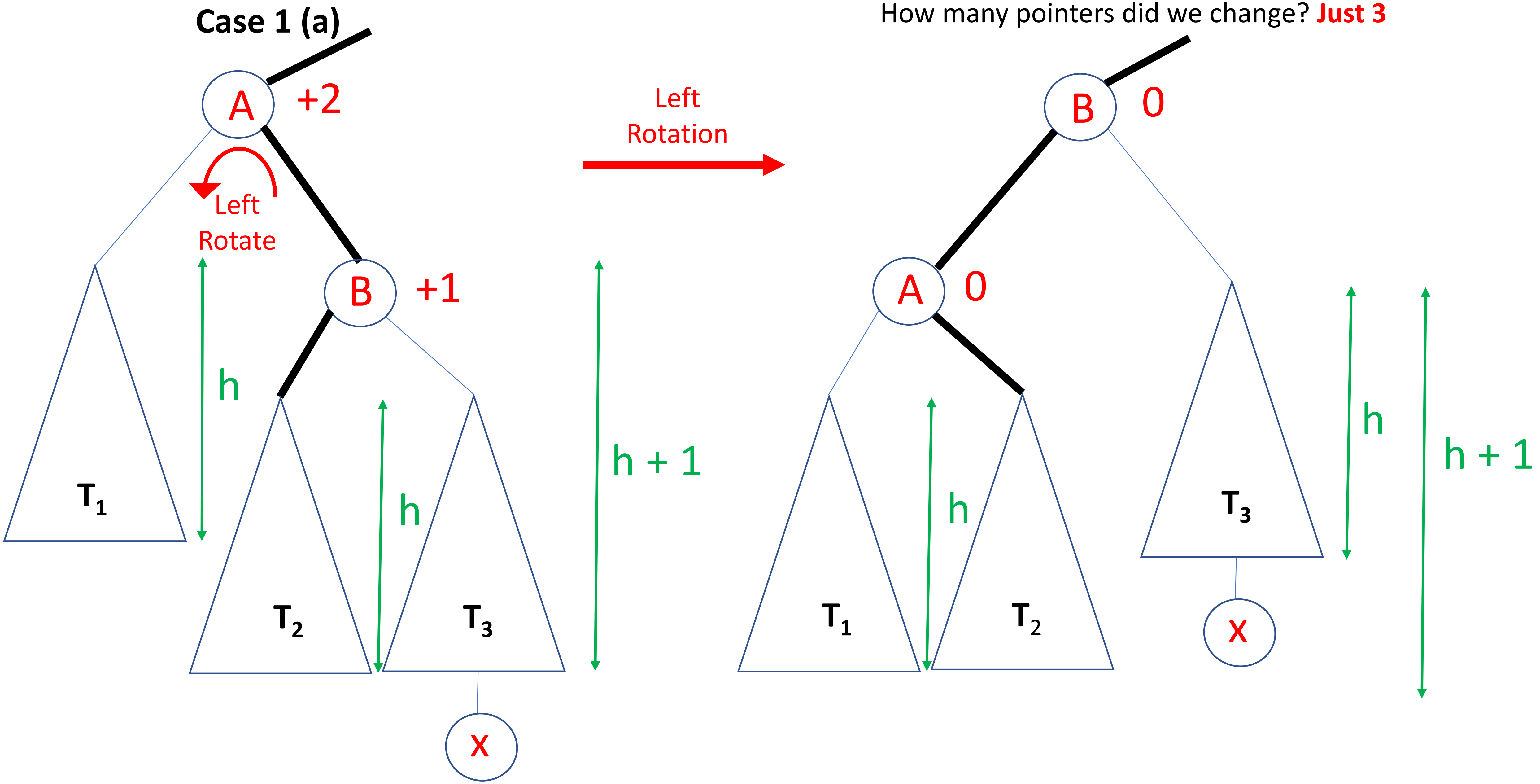
Case 1 (a)



Rotation: What is the time complexity?
How many pointers did we change?



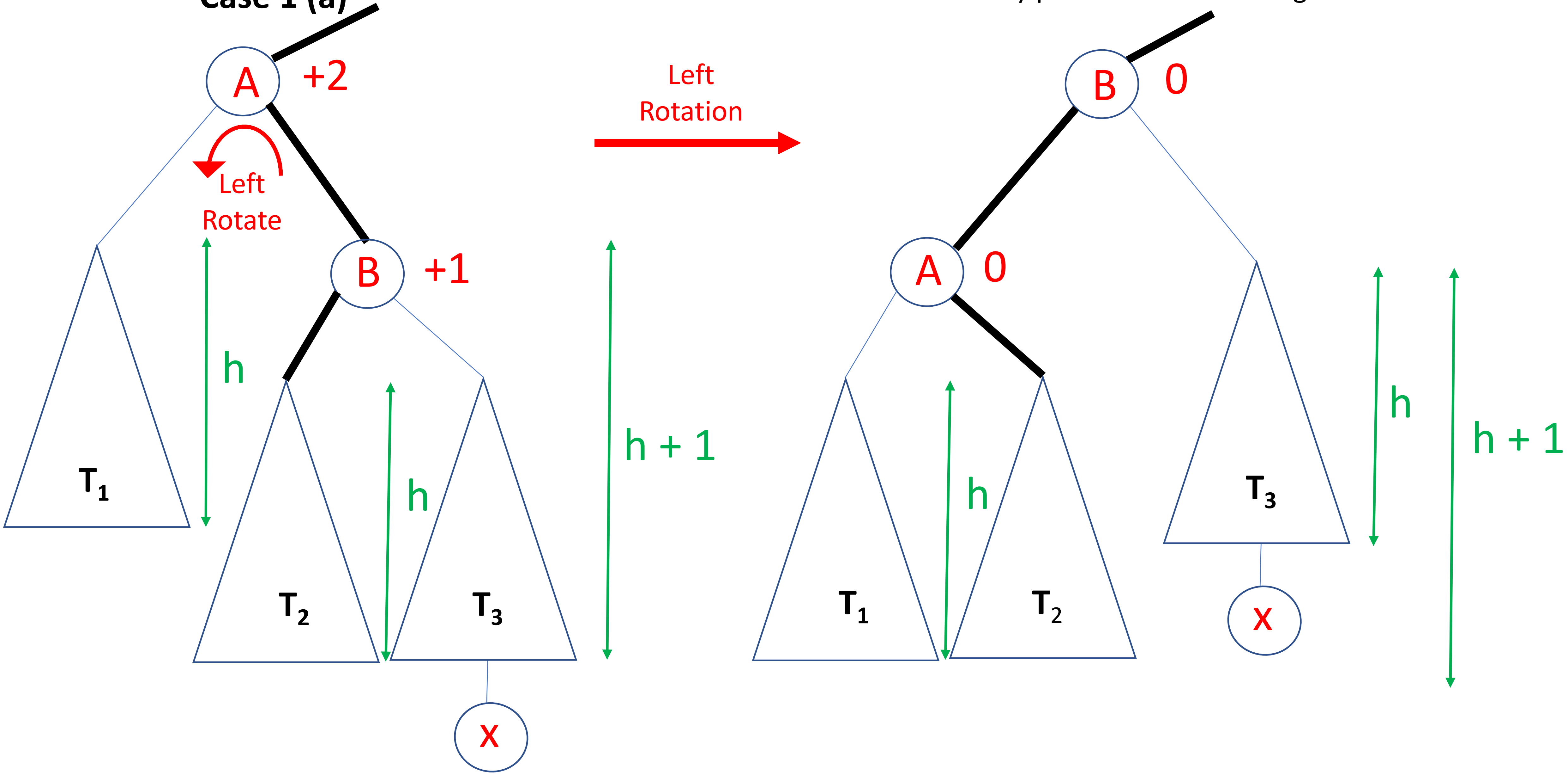
Rotation: What is the time complexity?
How many pointers did we change? **Just 3**



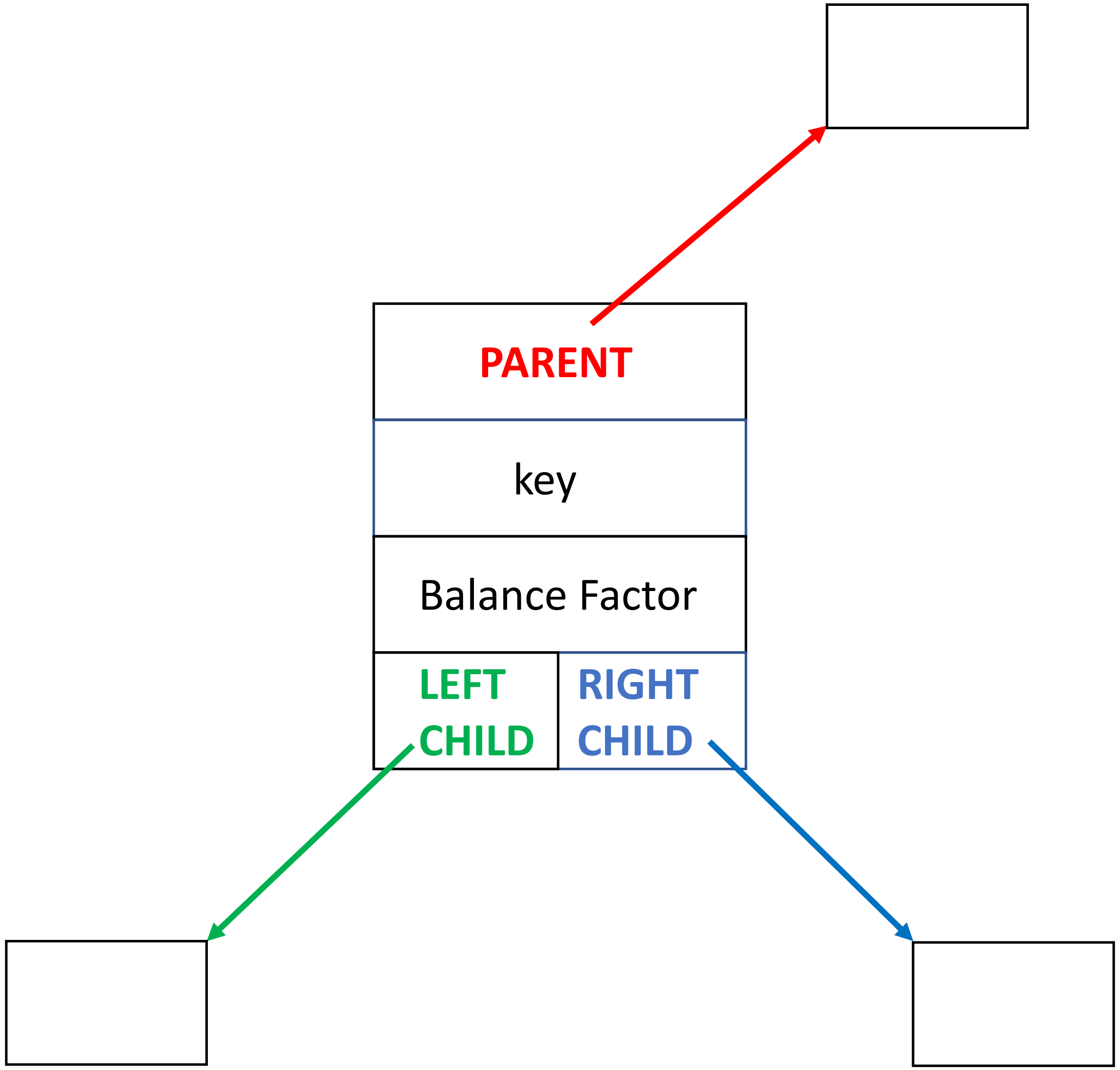
Rotation: What is the time complexity? **Θ(1)**

How many pointers did we change? **Just 3**

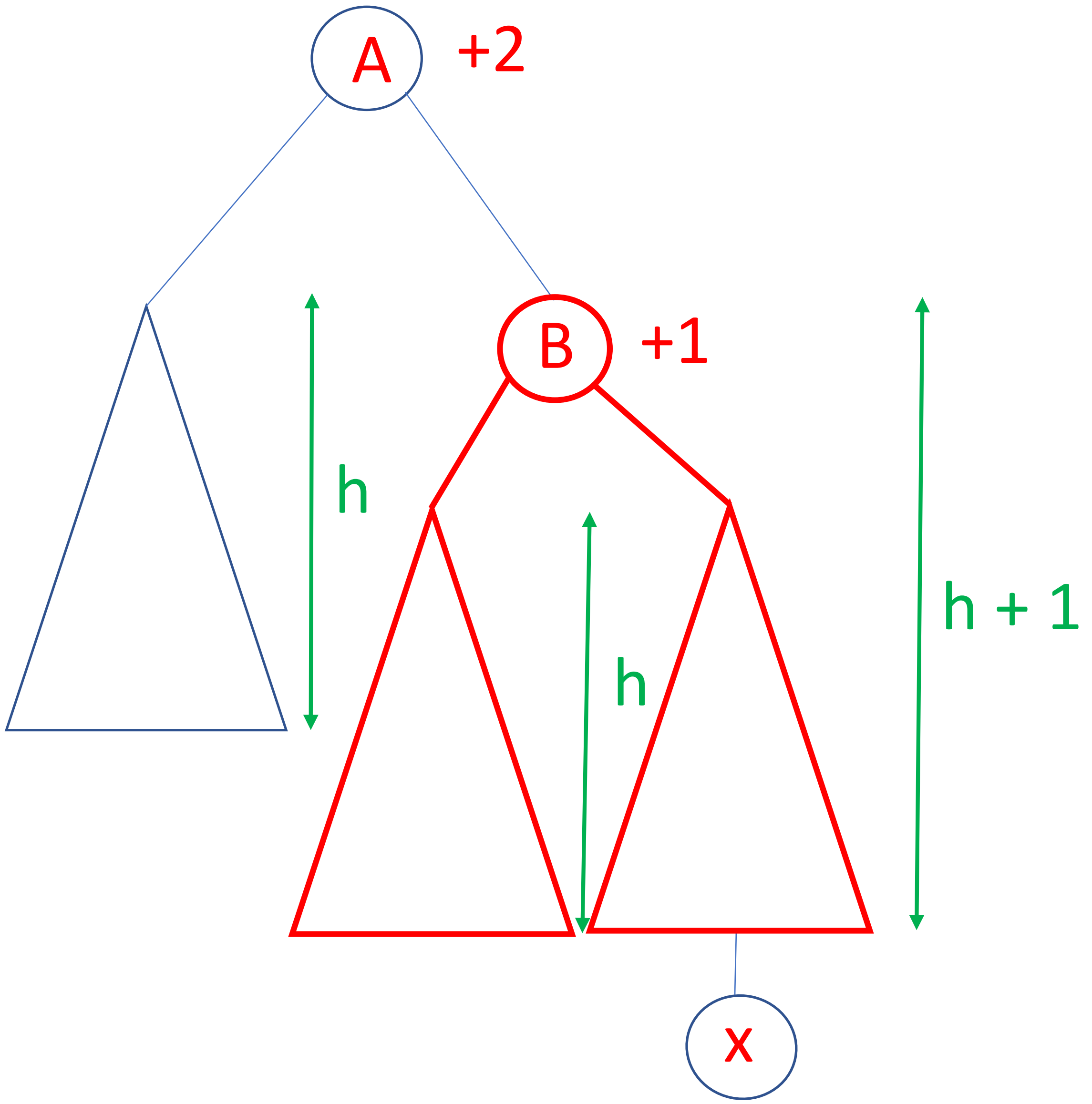
Case 1 (a)



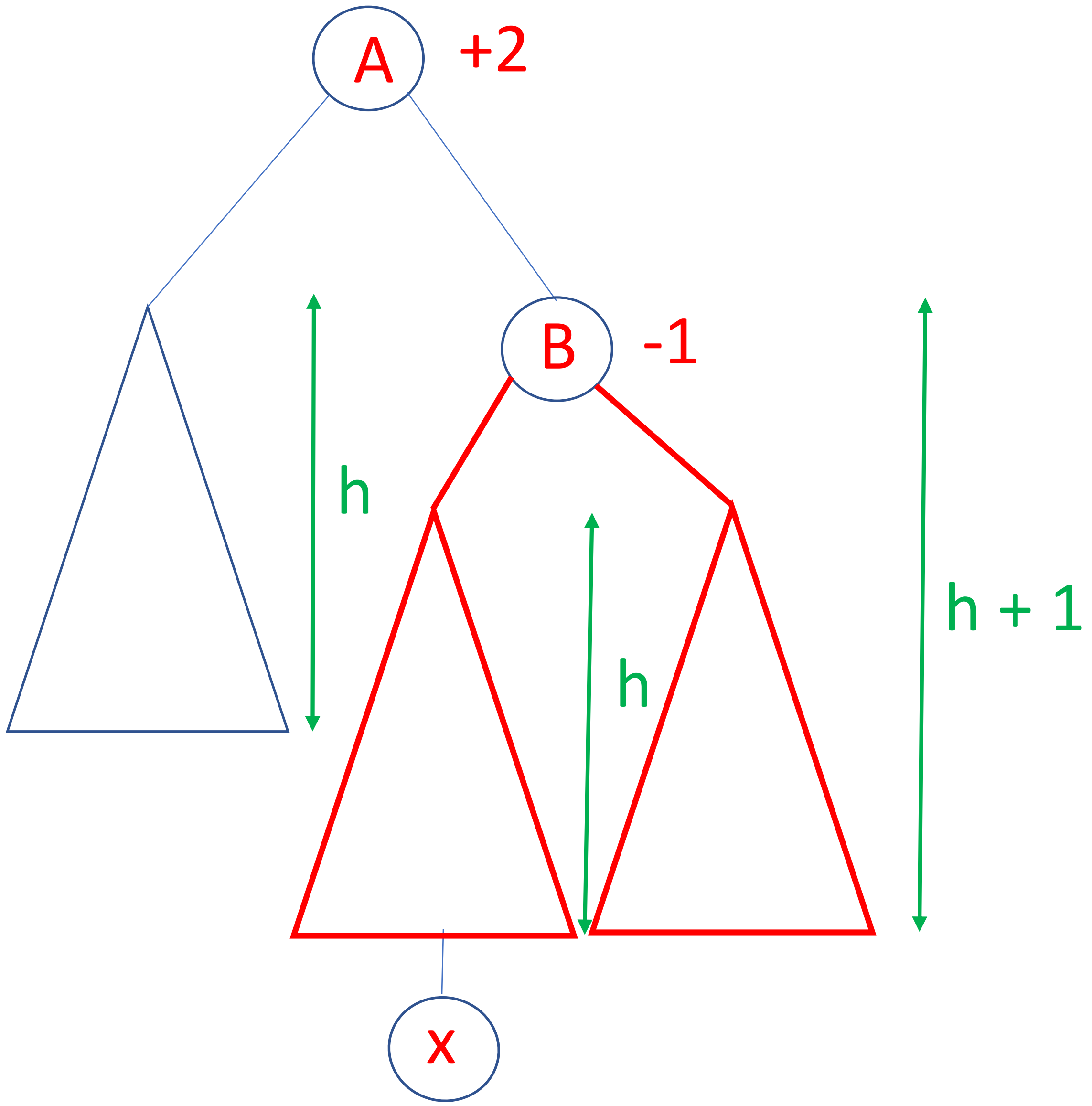
Contents of an AVL node



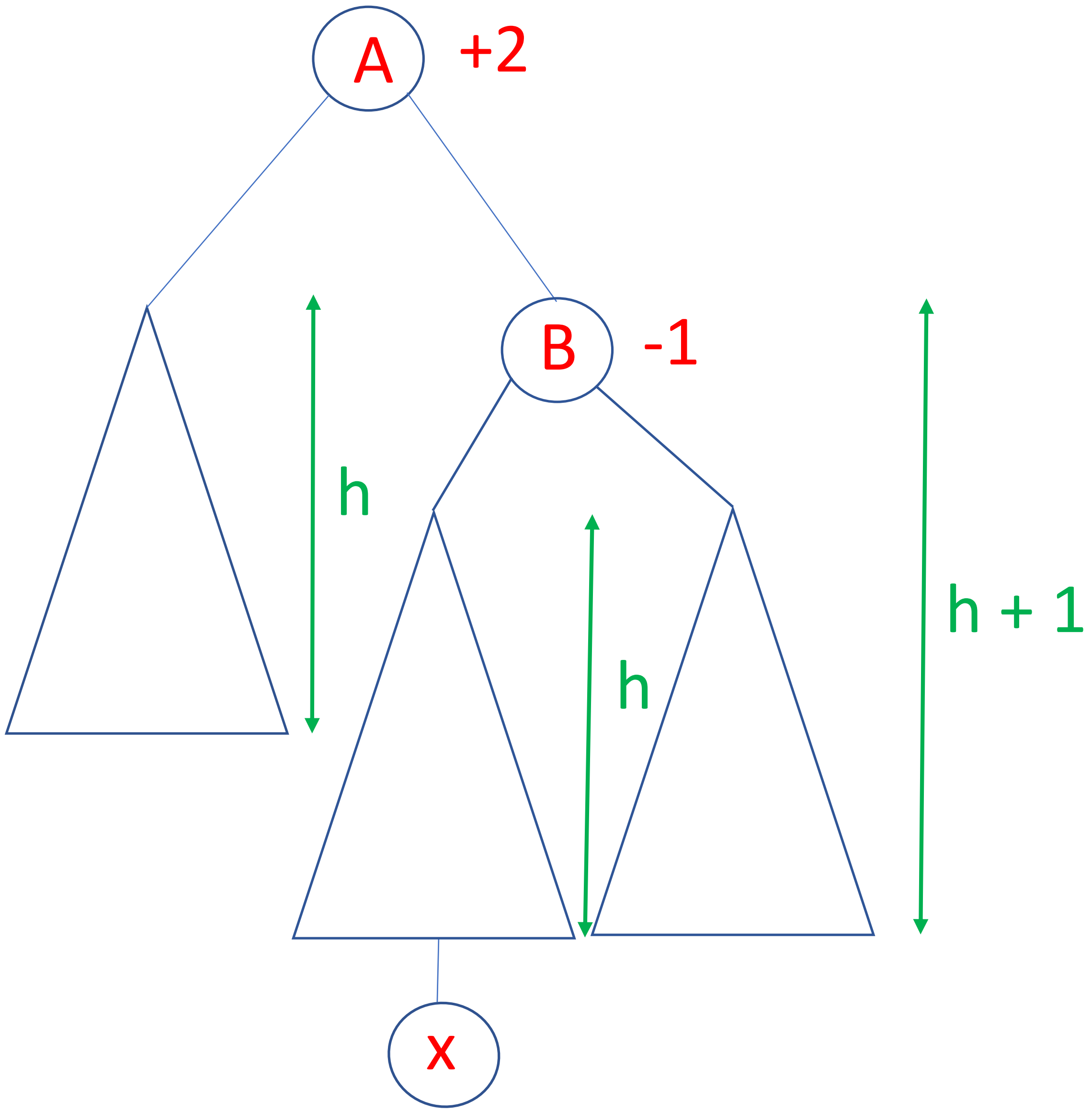
Case 1 (a) ✓



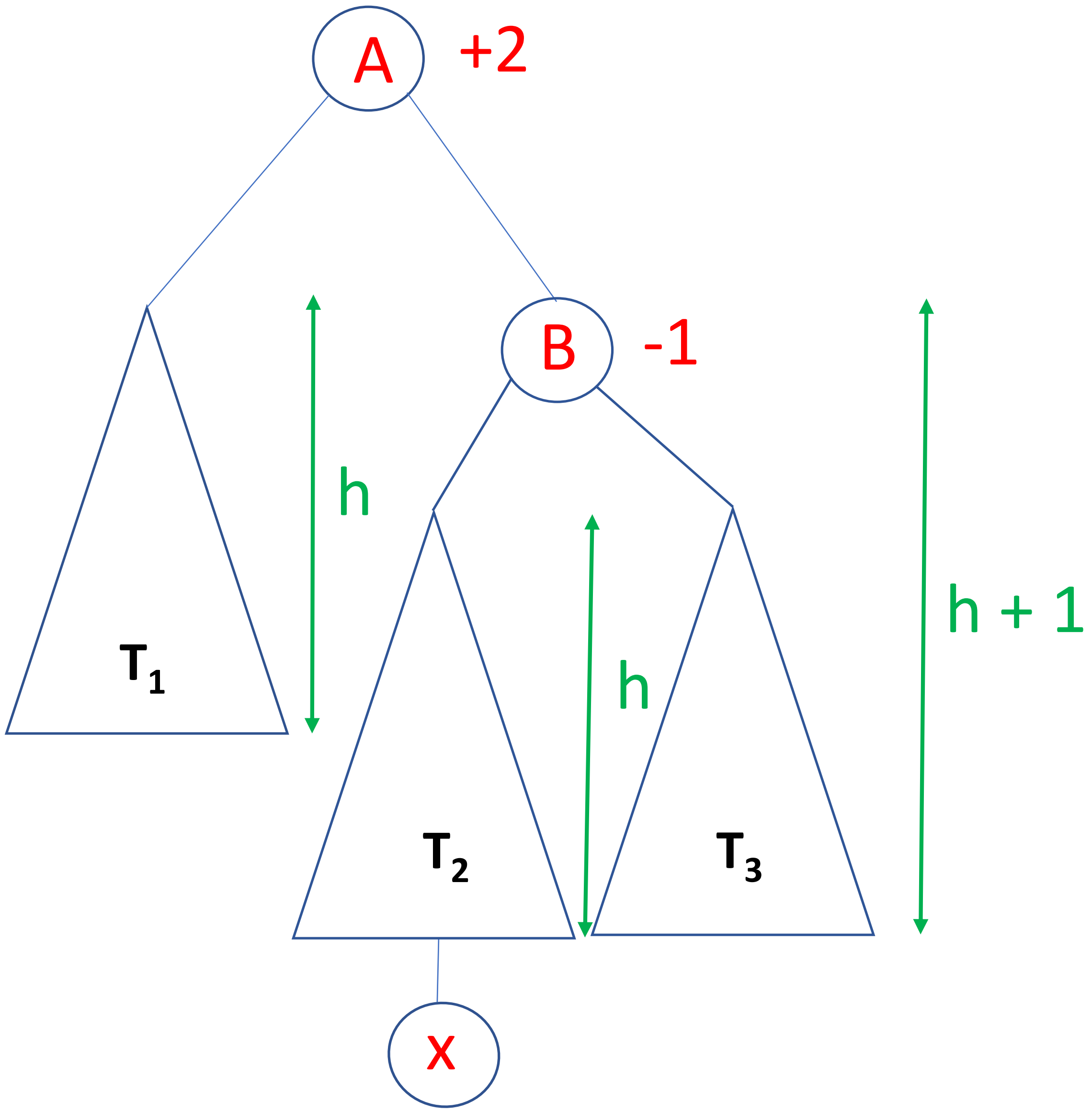
Case 1 (b)



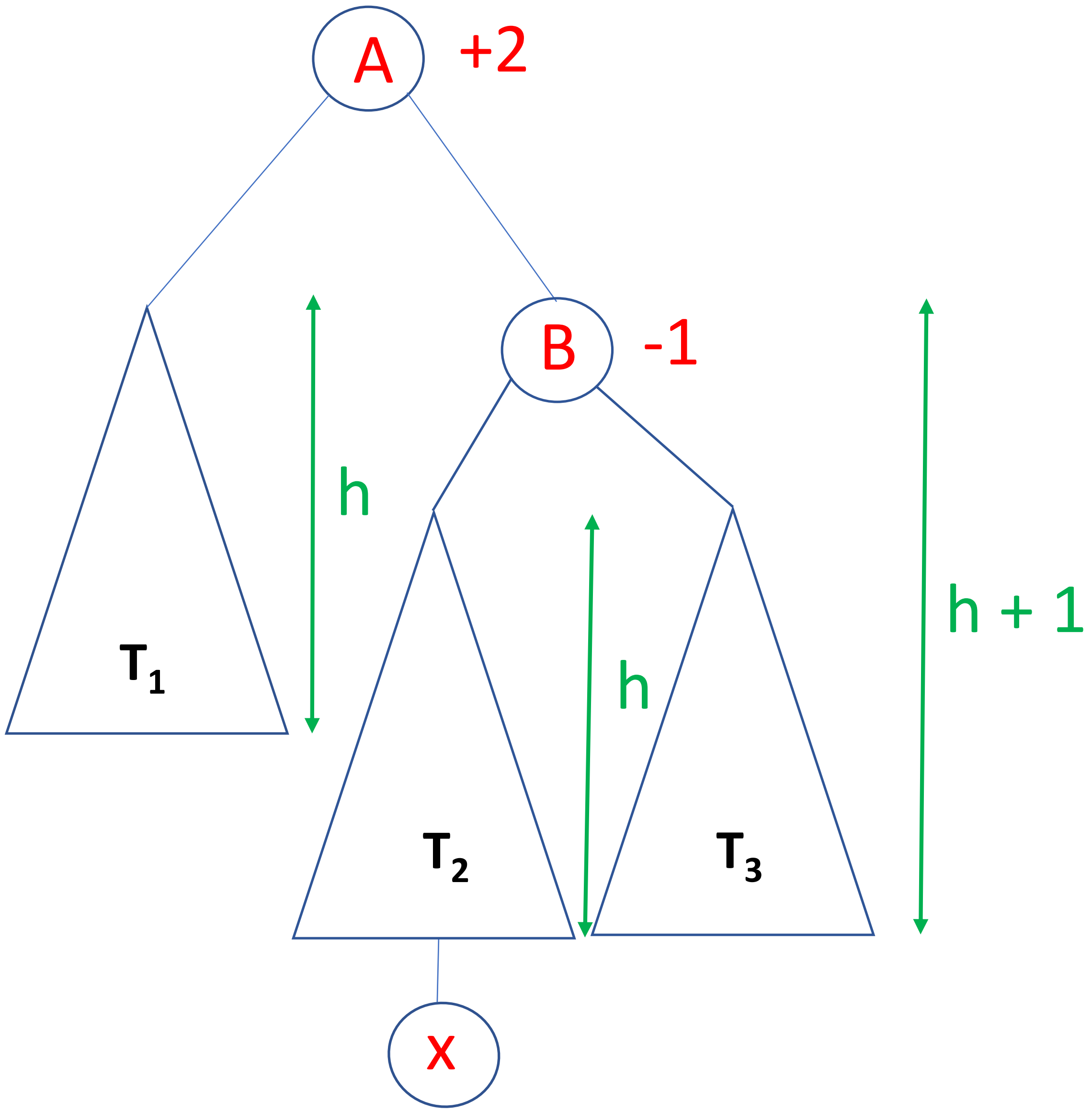
Case 1 (b)



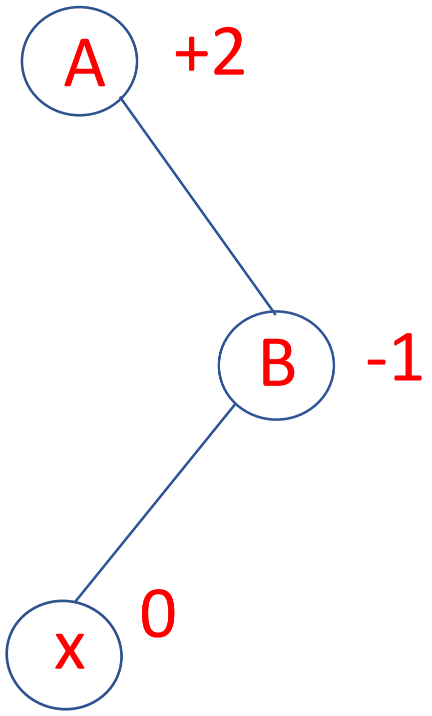
Case 1 (b)



Case 1 (b) : special case h = -1

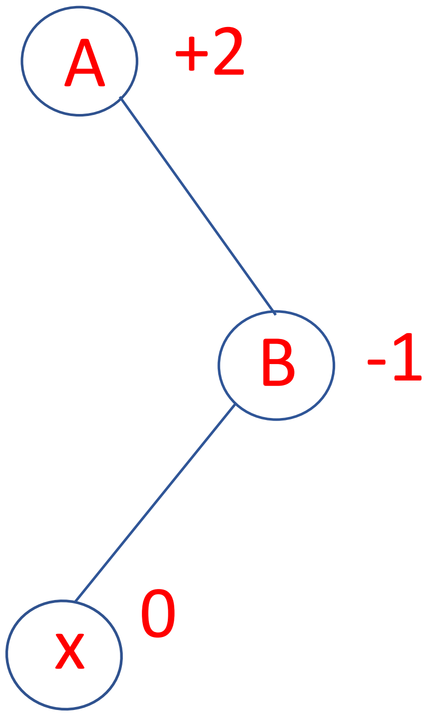


Case 1 (b) : special case h = -1



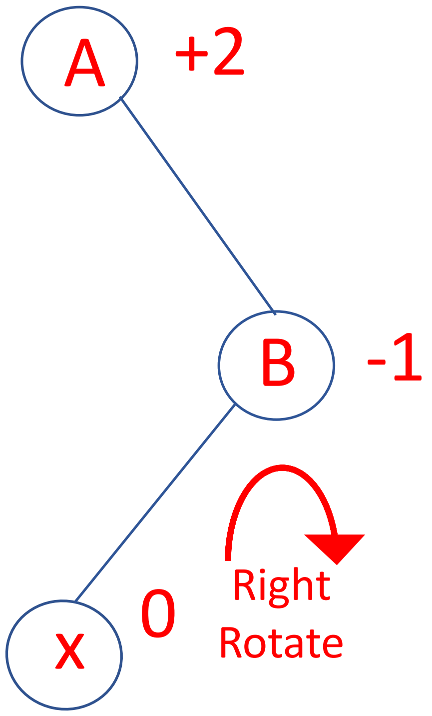
Order of keys: A x B

Case 1 (b) : special case h = -1



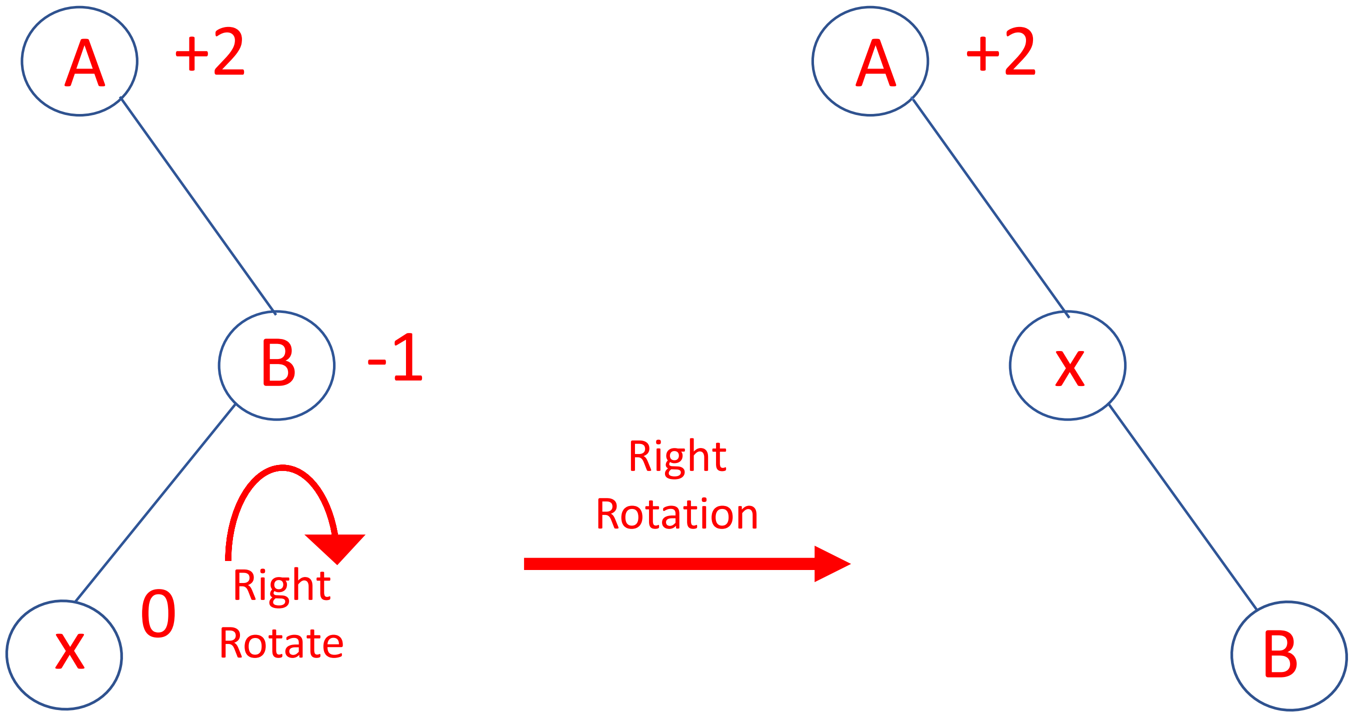
Order of keys: A x B

Case 1 (b) : special case h = -1



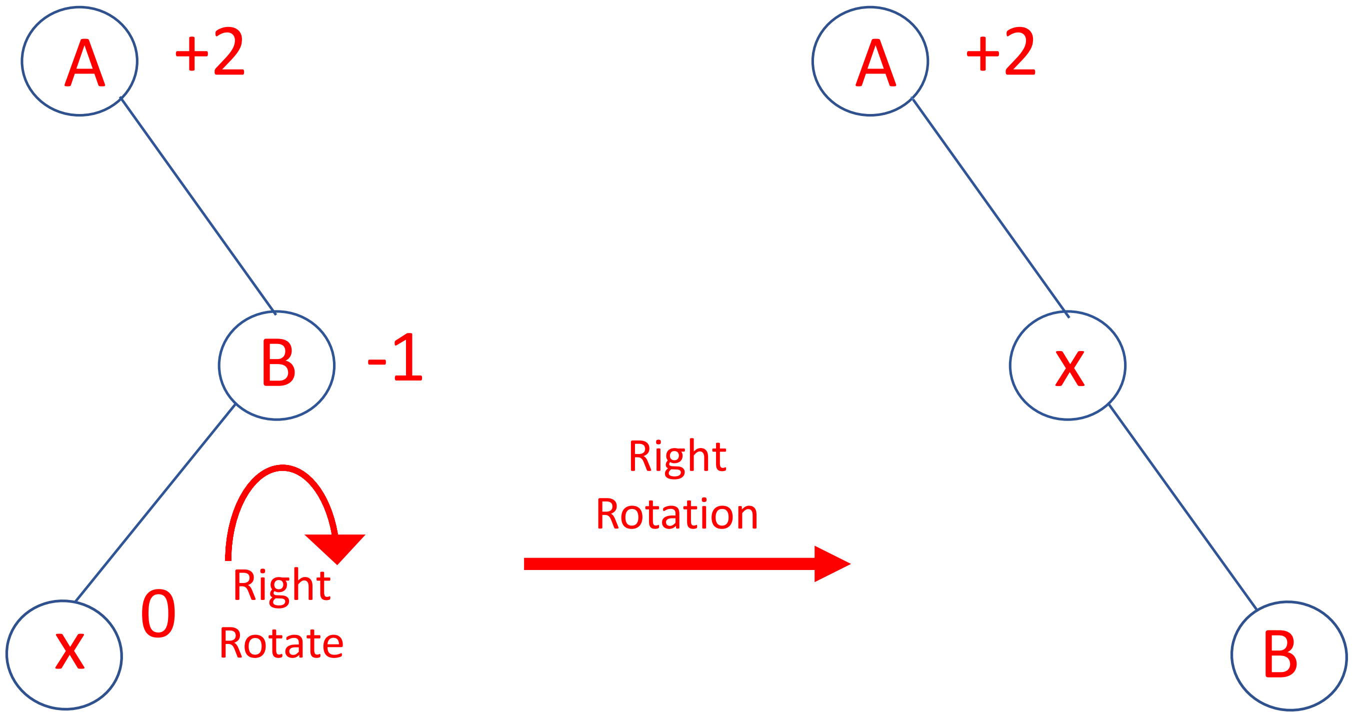
Order of keys: A x B

Case 1 (b) : special case h = -1



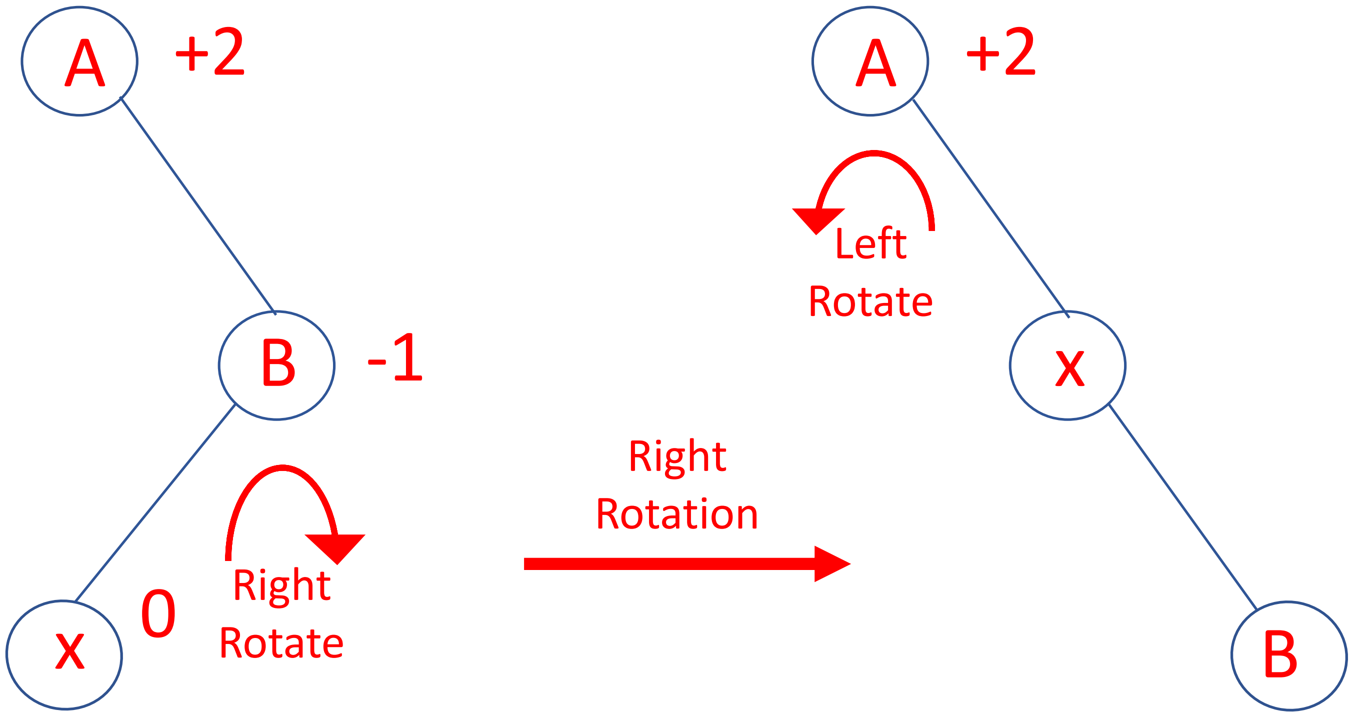
Order of keys: A x B

Case 1 (b) : special case h = -1



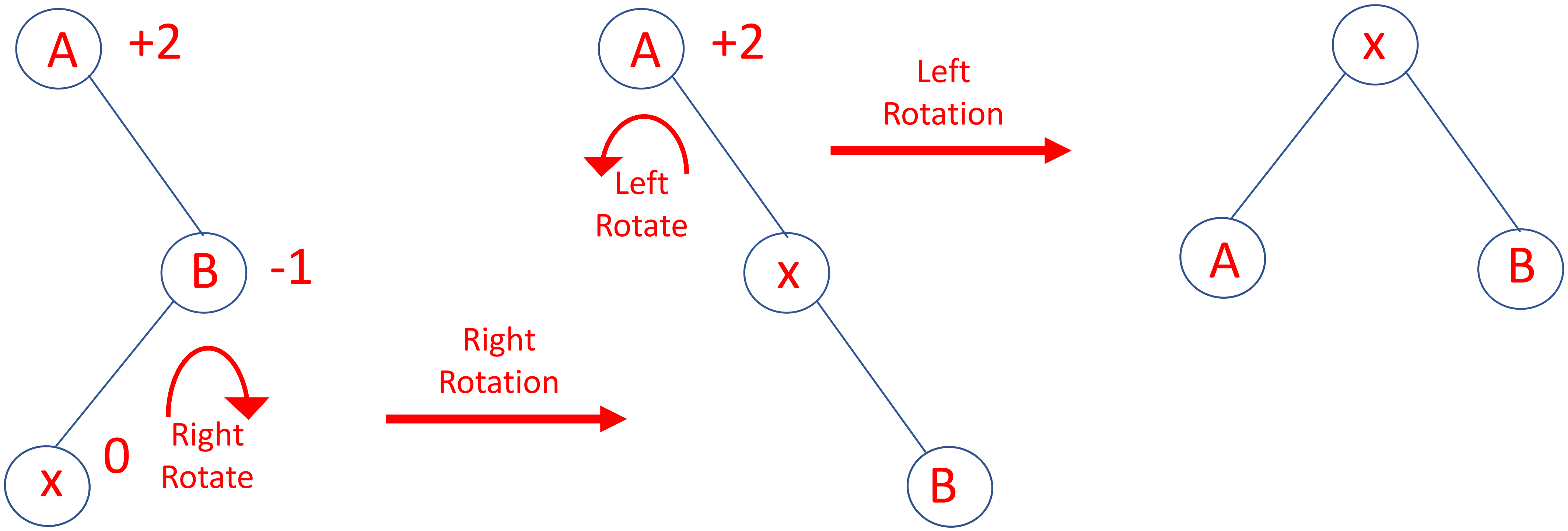
Order of keys: A x B

Case 1 (b) : special case h = -1



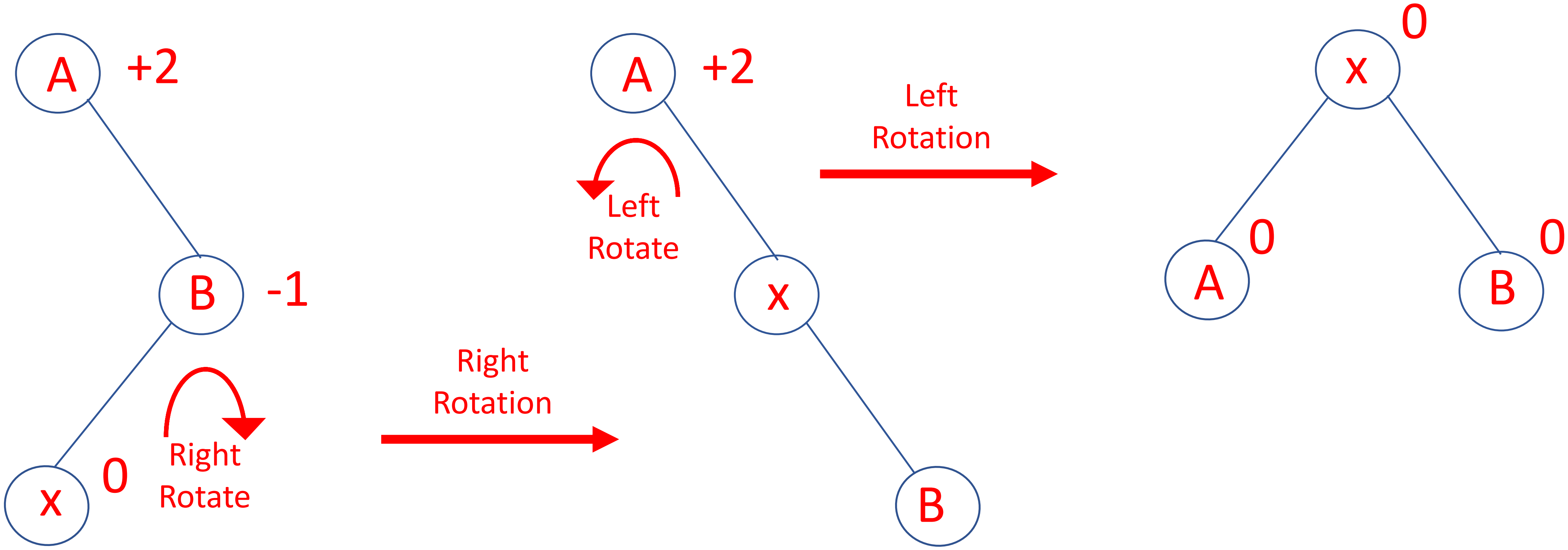
Order of keys: A x B

Case 1 (b) : special case h = -1



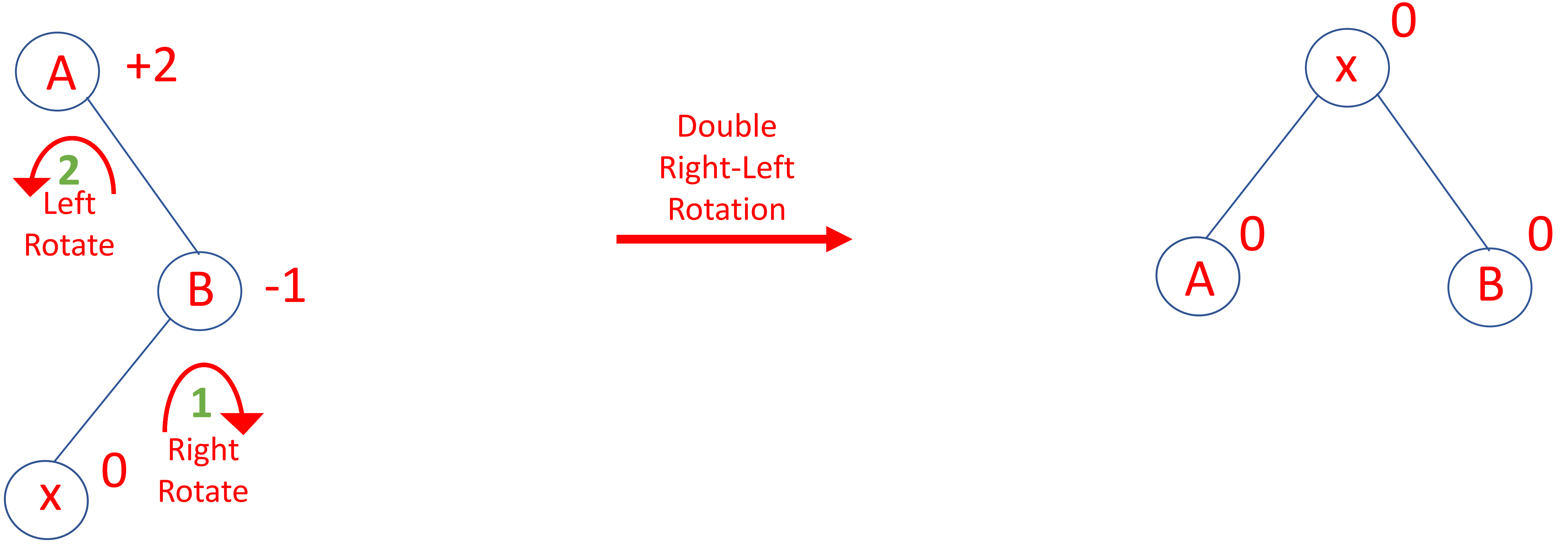
Order of keys: A x B

Case 1 (b) : special case h = -1



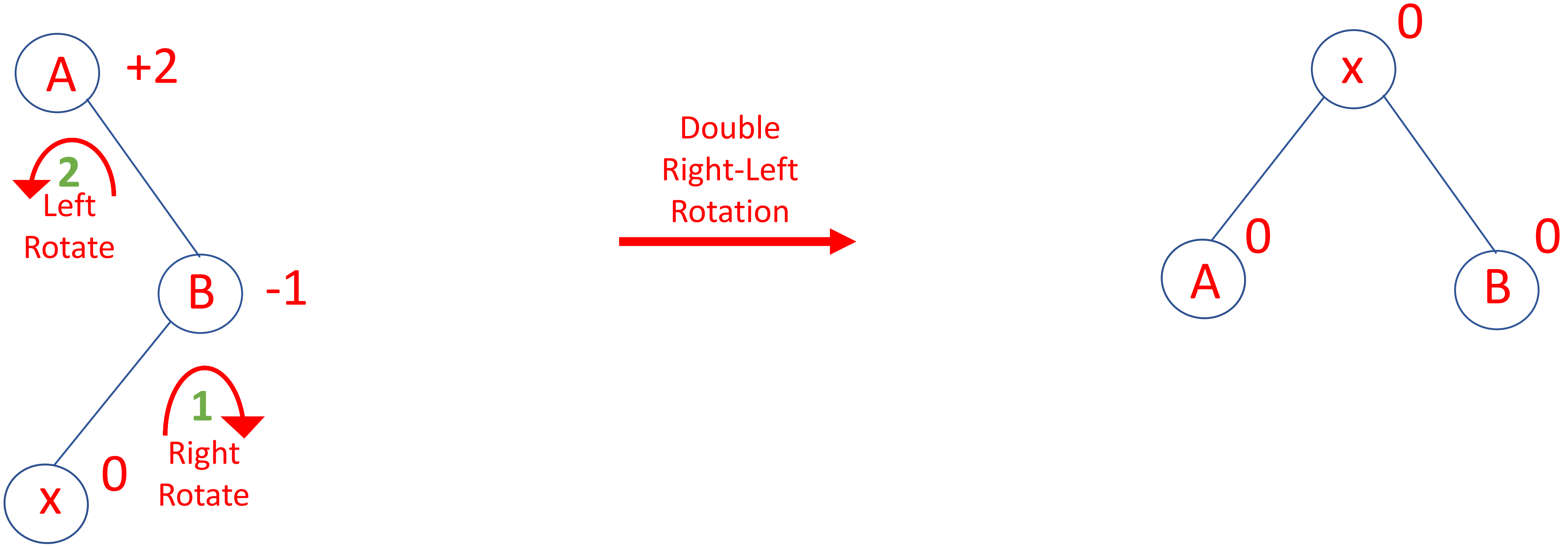
Order of keys: A x B

Case 1 (b) : special case h = -1



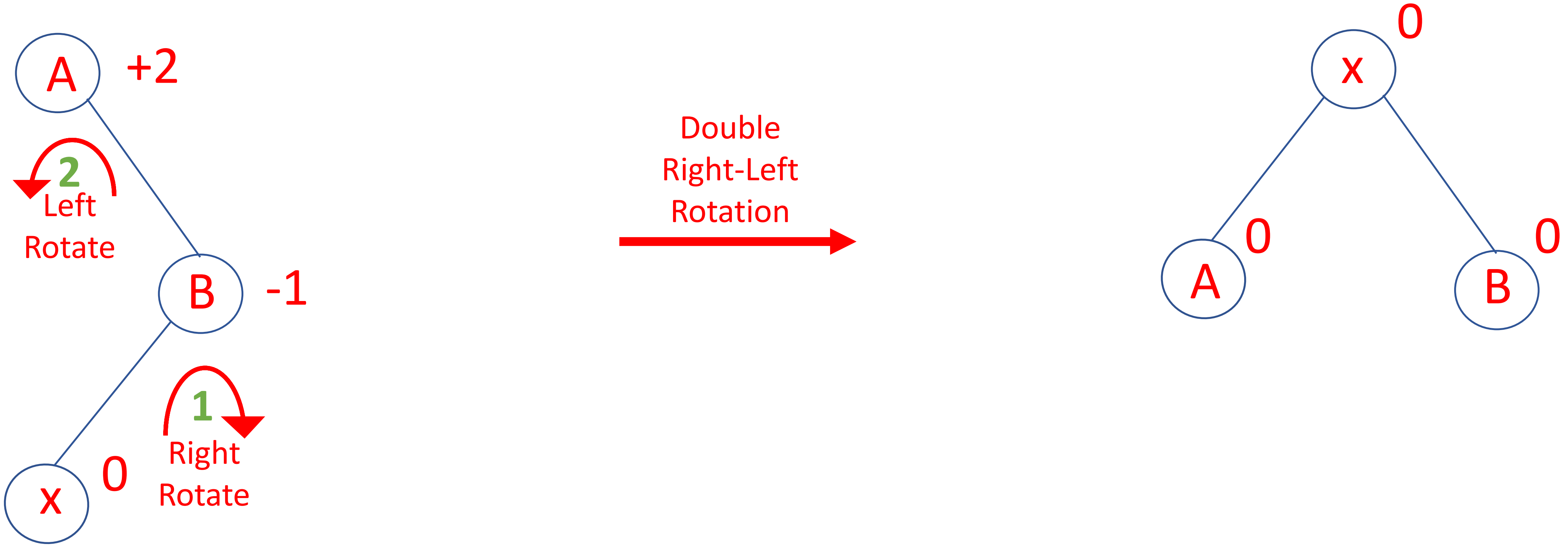
Rotation: (1) Rebalances the subtree

Case 1 (b) : special case h = -1



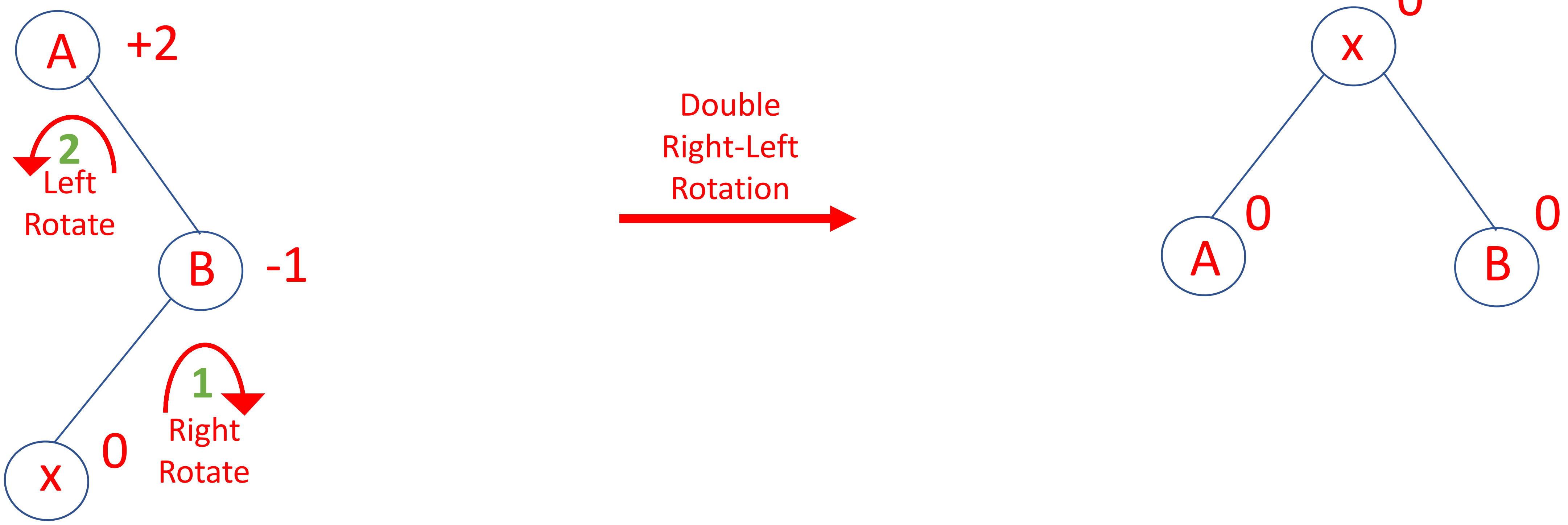
Rotation: (2) Preserves BST property (order: A x B)

Case 1 (b) : special case h = -1



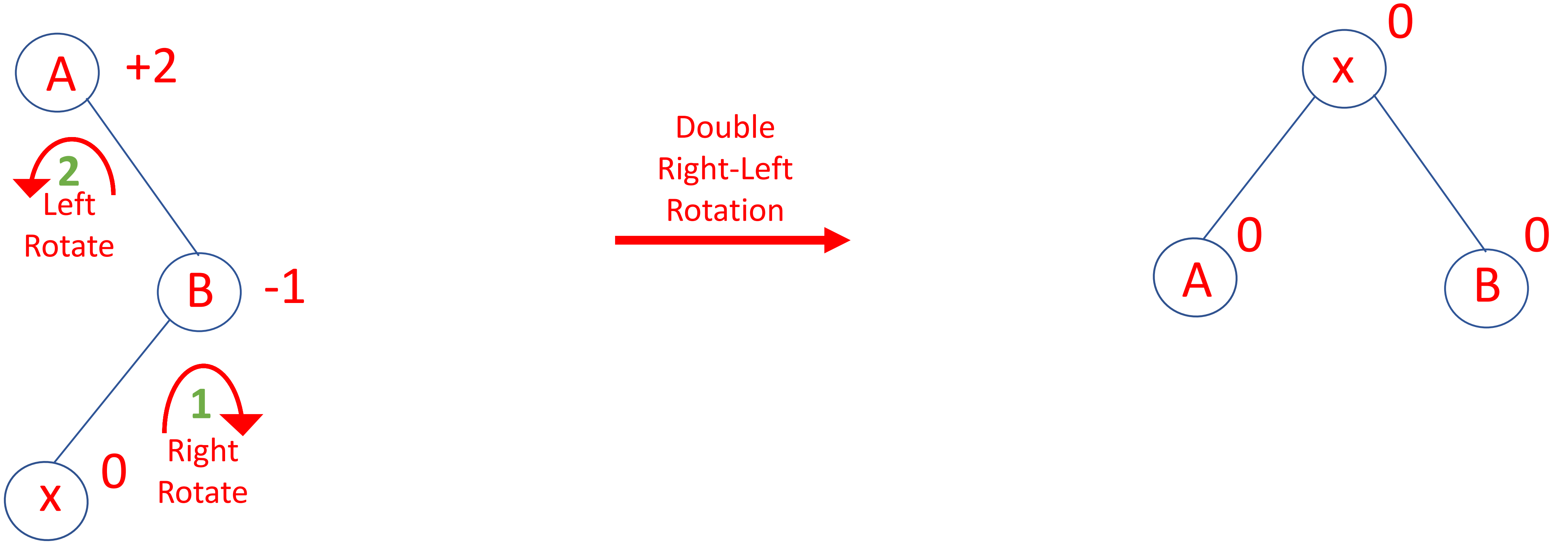
Rotation: (3) Bonus?

Case 1 (b) : special case h = -1



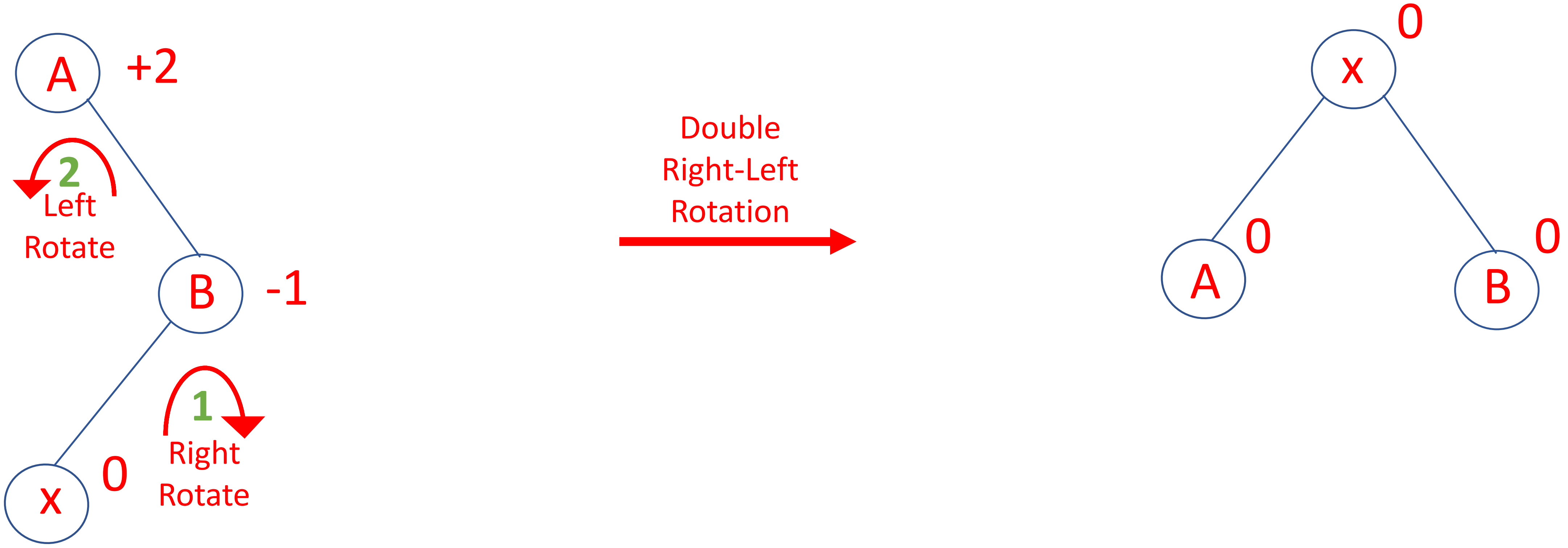
Rotation: (3) Bonus? YES!

Case 1 (b) : special case h = -1



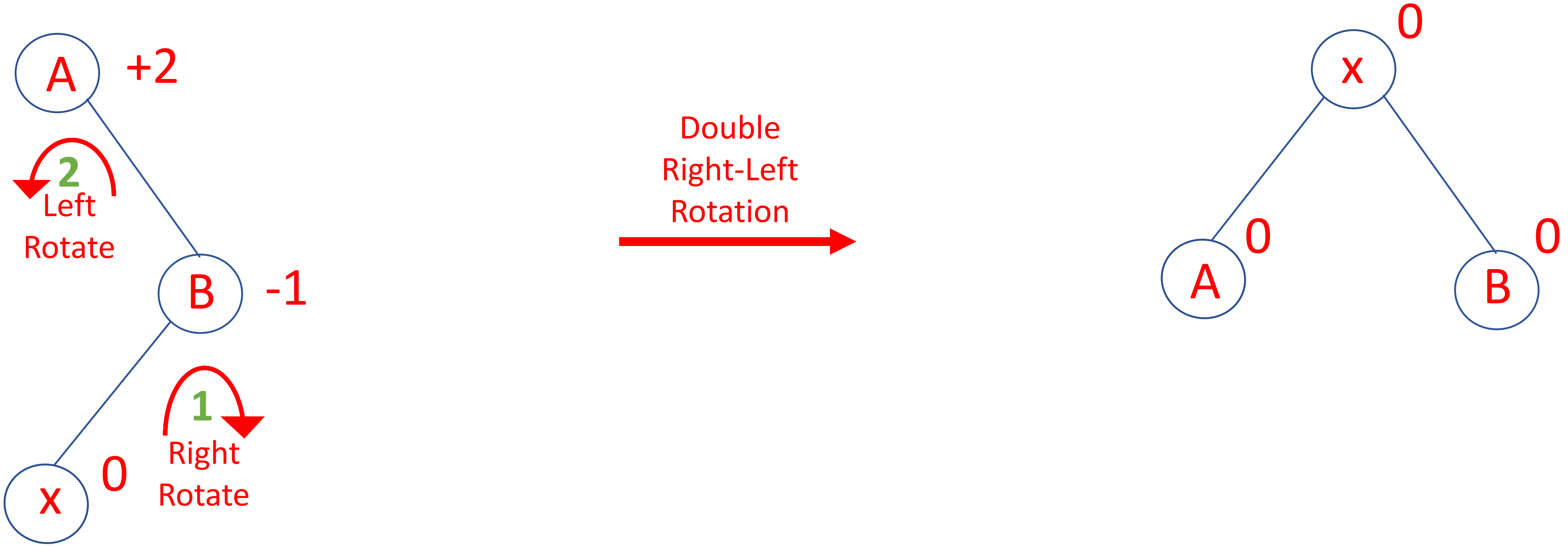
Case 1 (b) : special case h = -1

Rotation: (3) Bonus: height of this subtree is the same both before and after the insertion of x

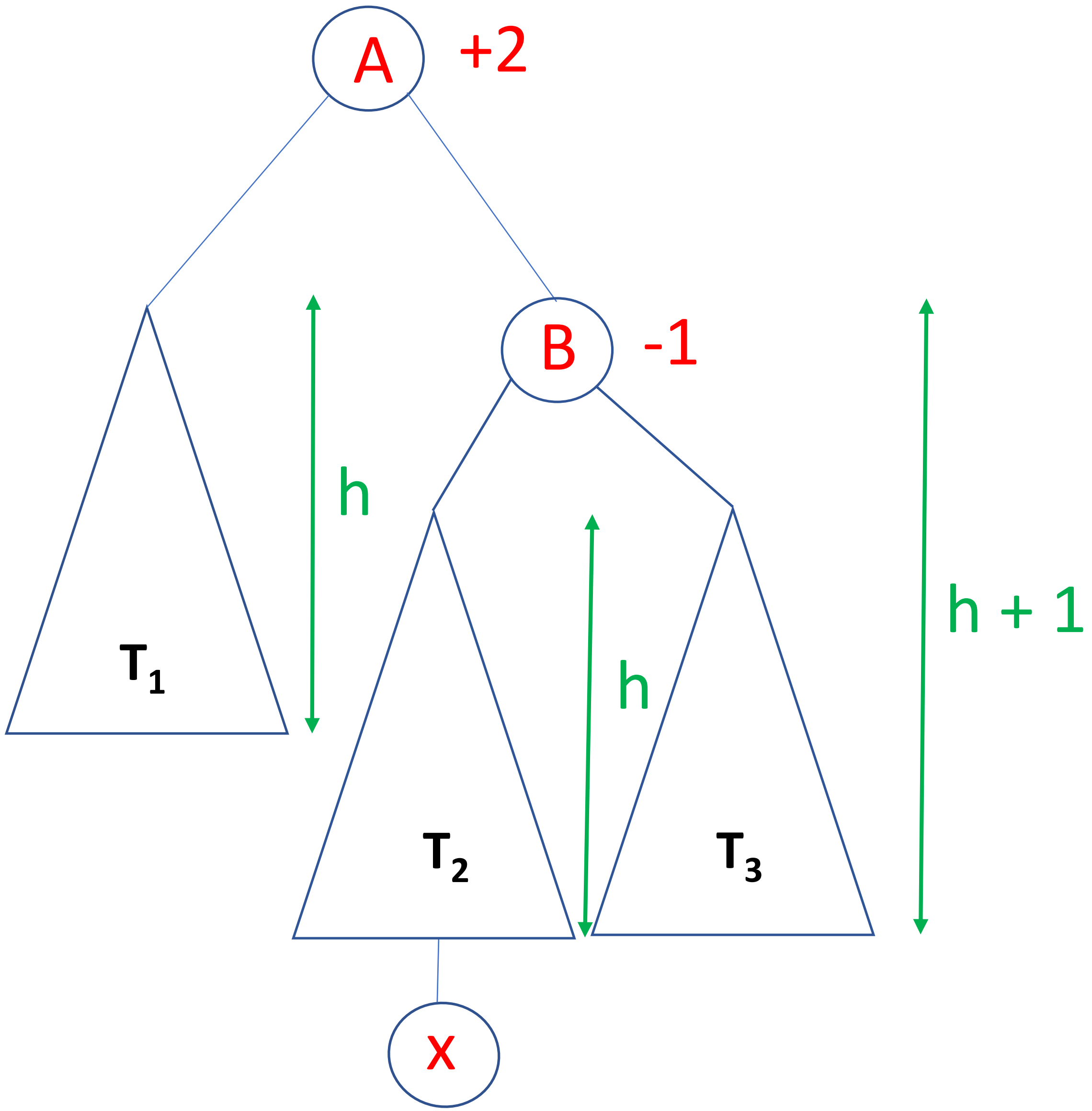


Rotation: (3) Bonus → insertion algorithm is done

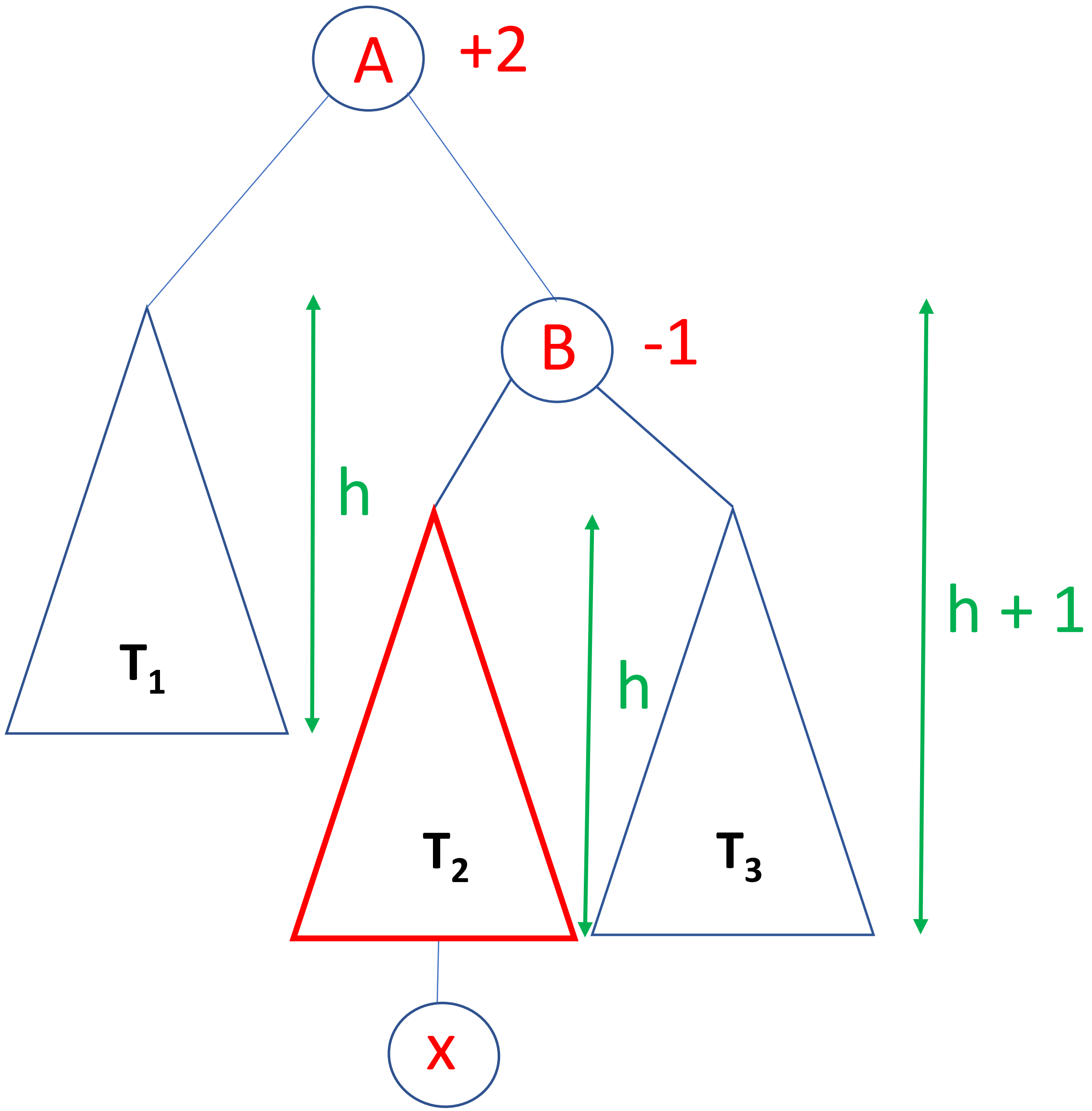
Case 1 (b) : special case h = -1



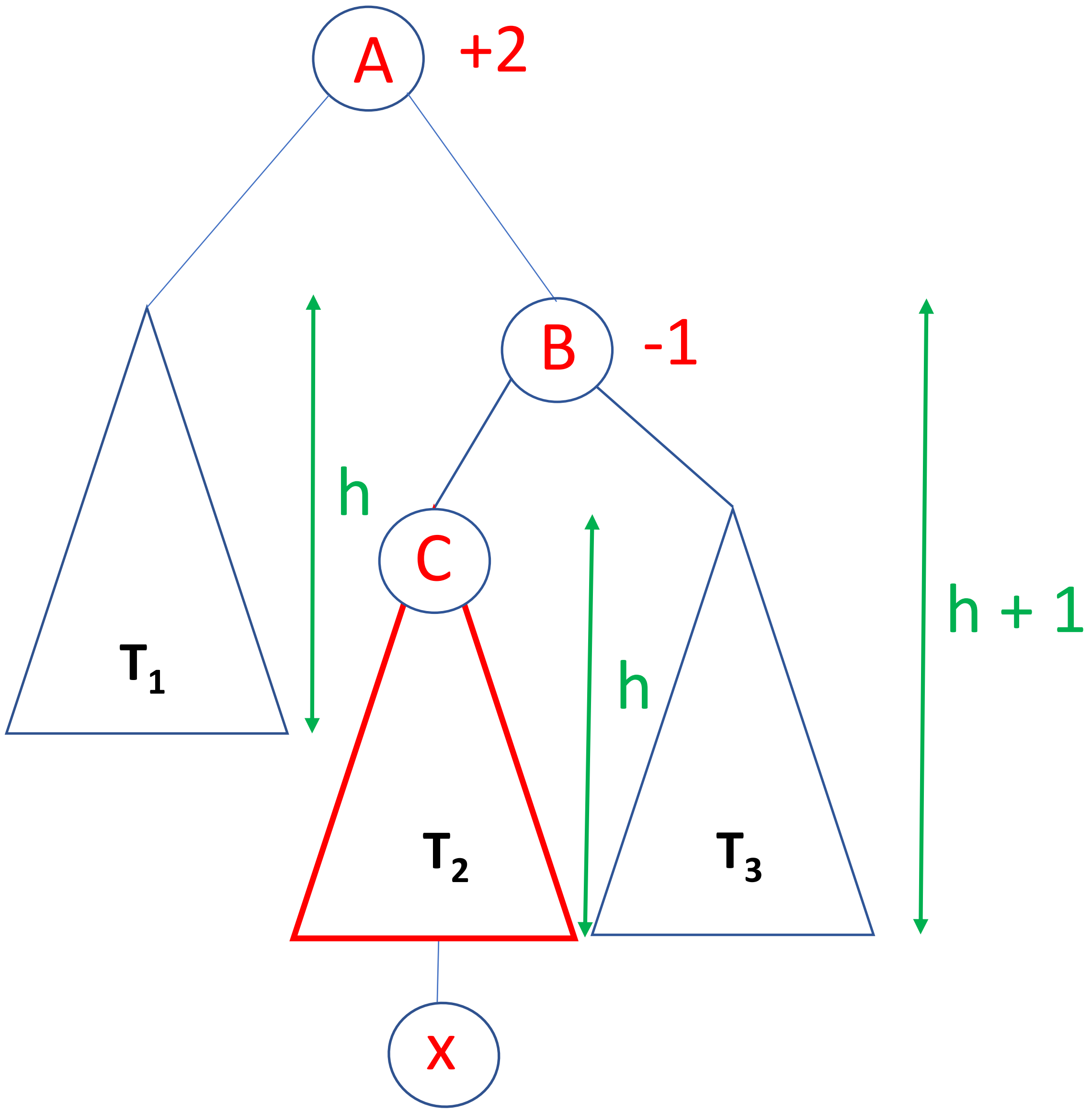
Case 1 (b) : h > -1



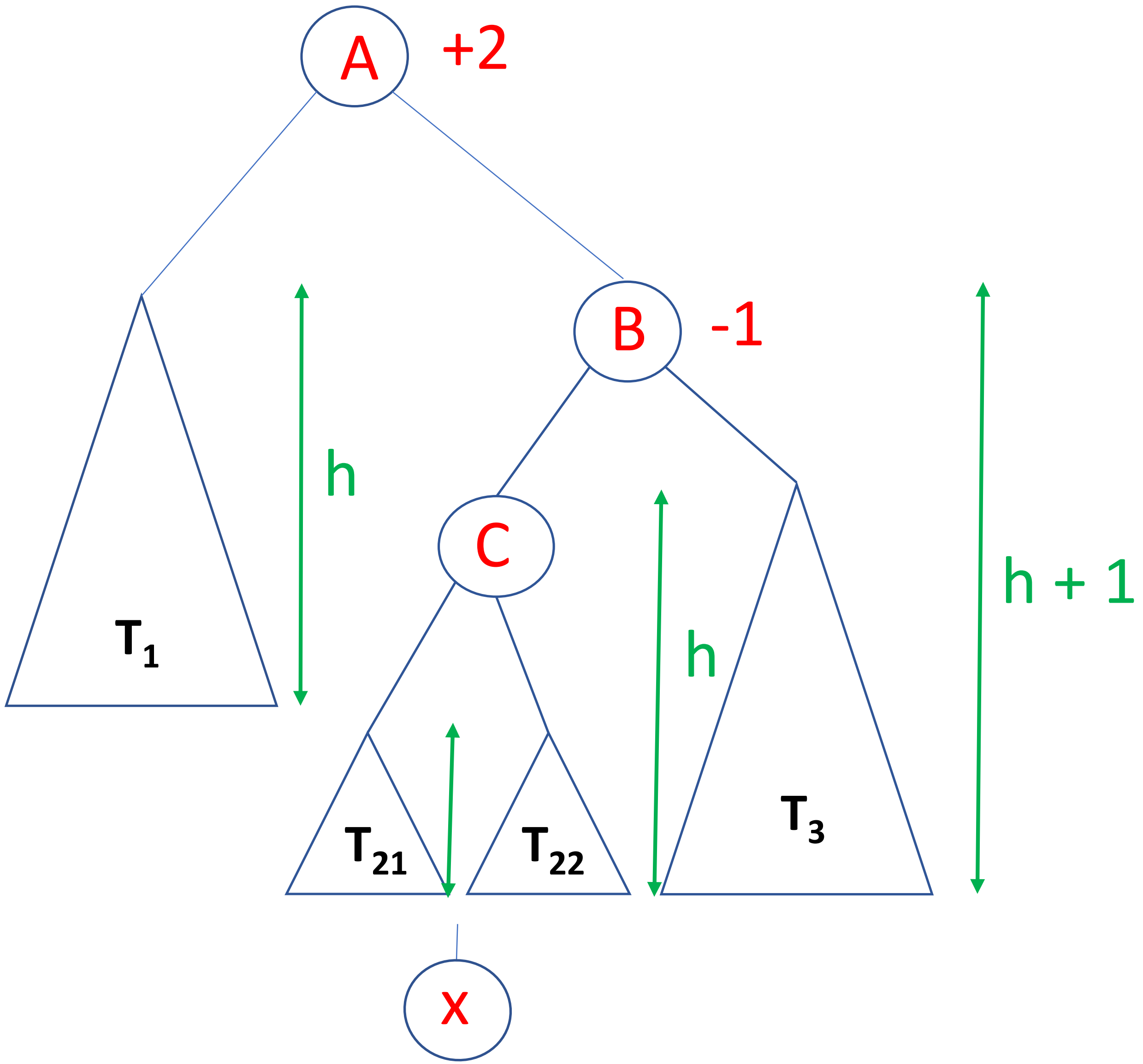
Case 1 (b) : h > -1



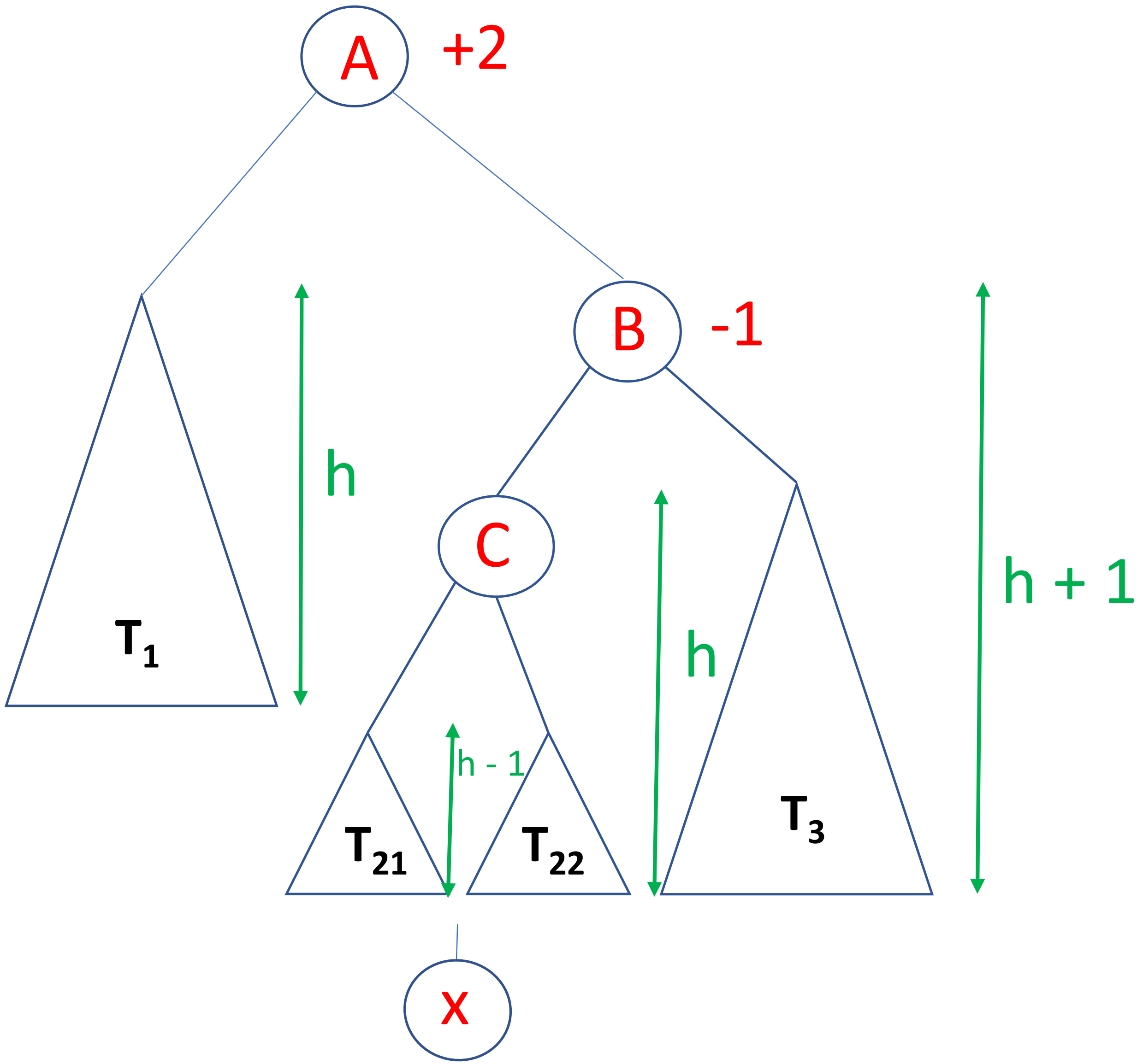
Case 1 (b) : h > -1



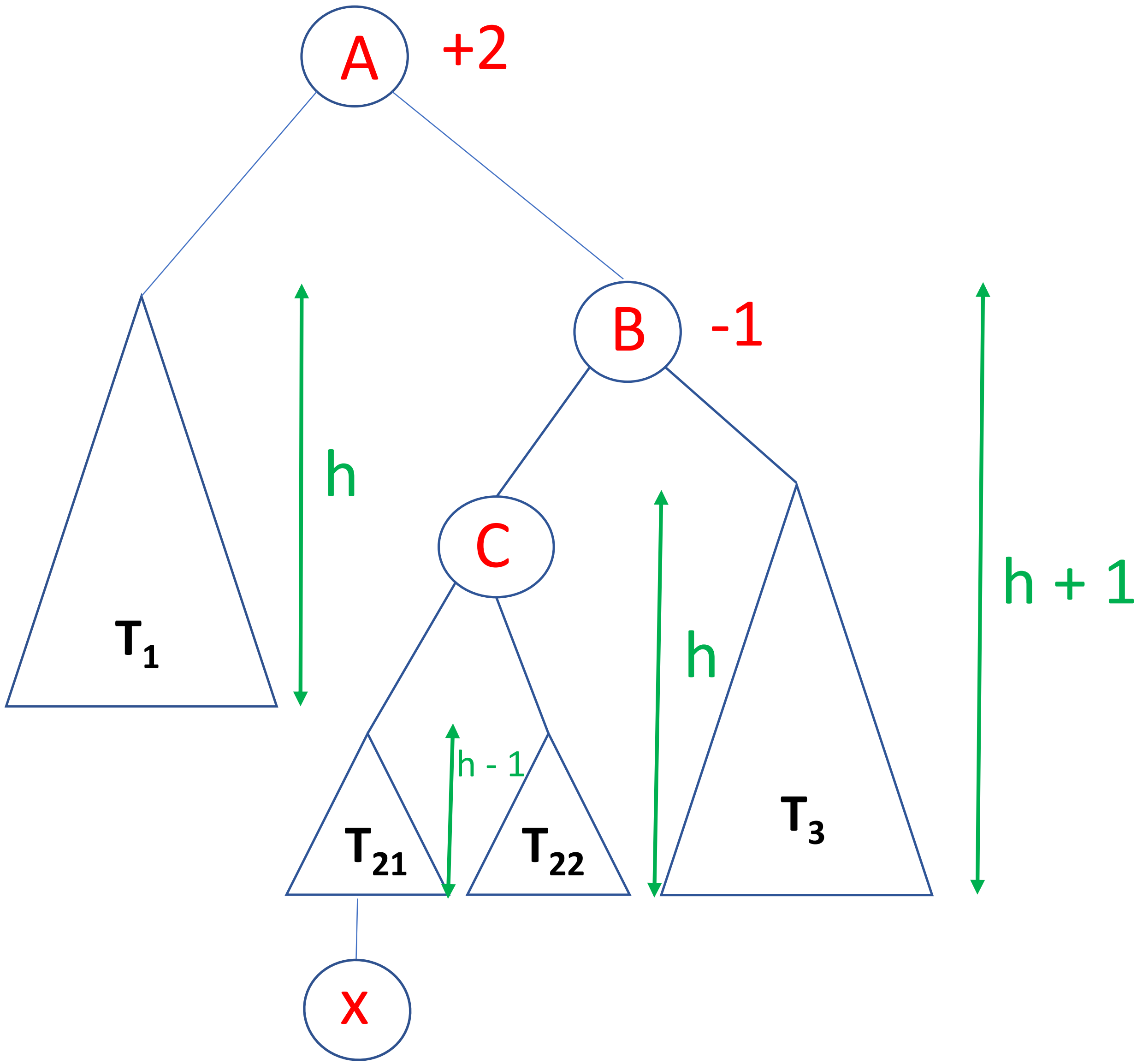
Case 1 (b) : h > -1



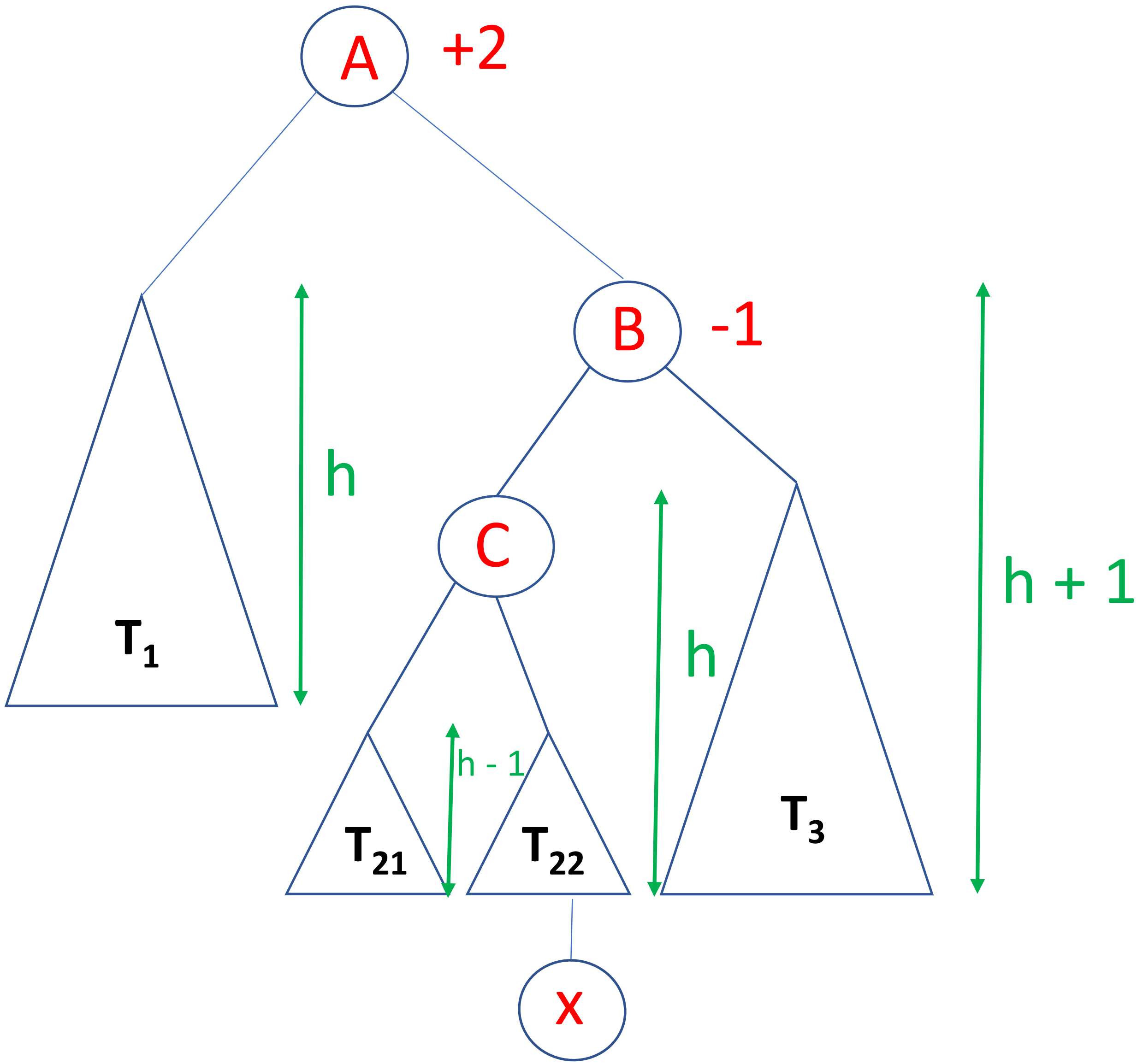
Case 1 (b) : h > -1



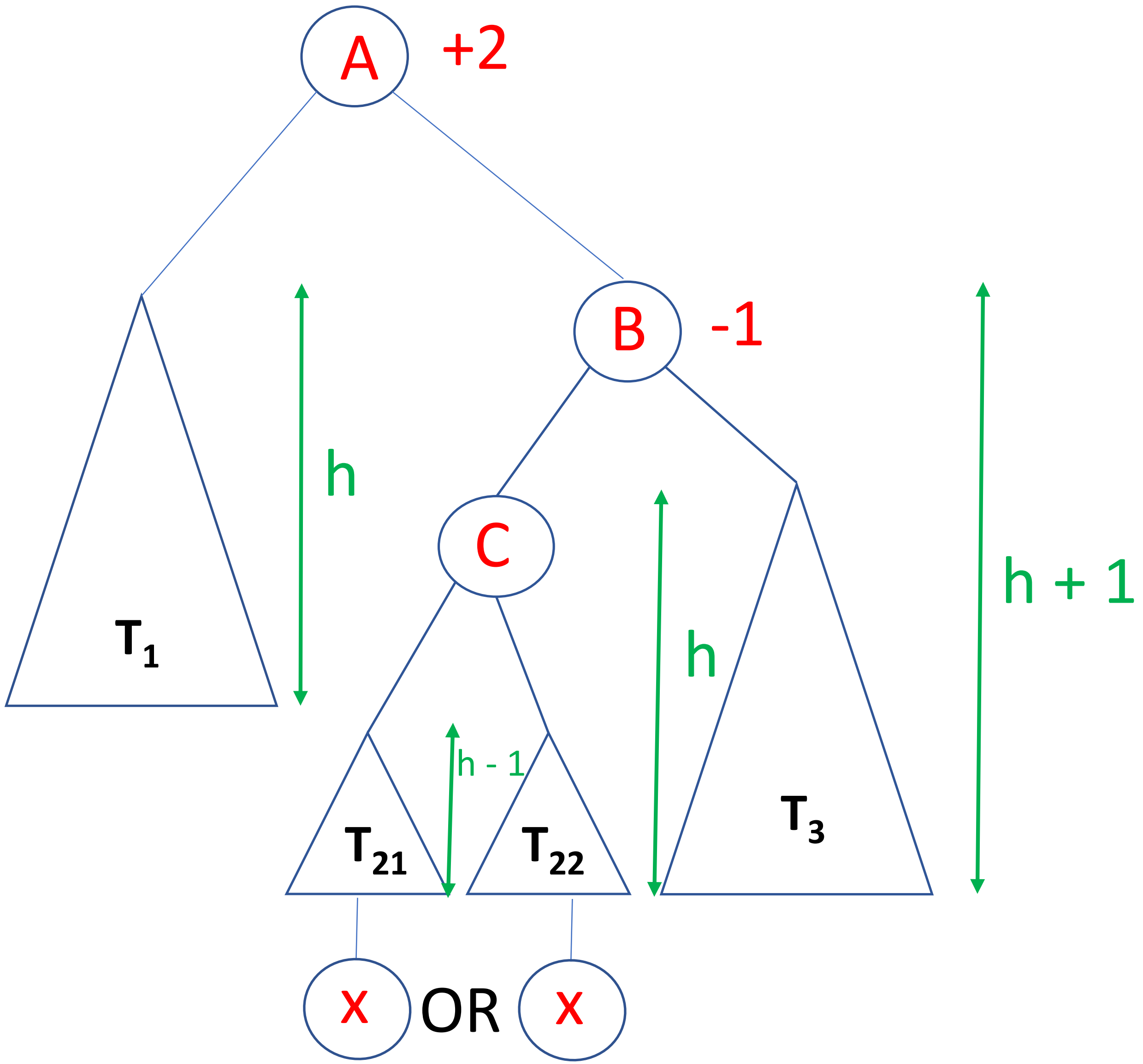
Case 1 (b) : h > -1



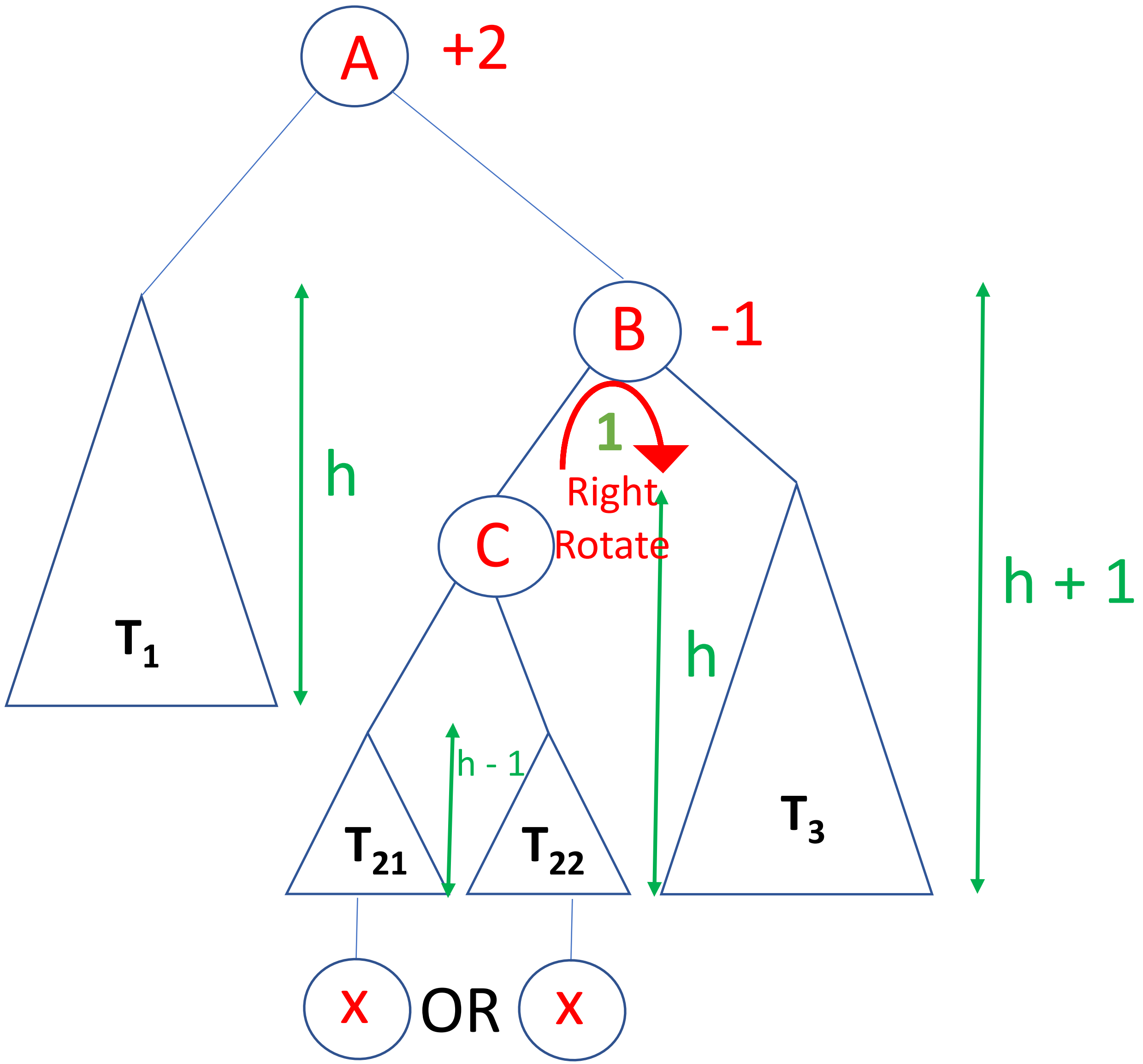
Case 1 (b) : h > -1



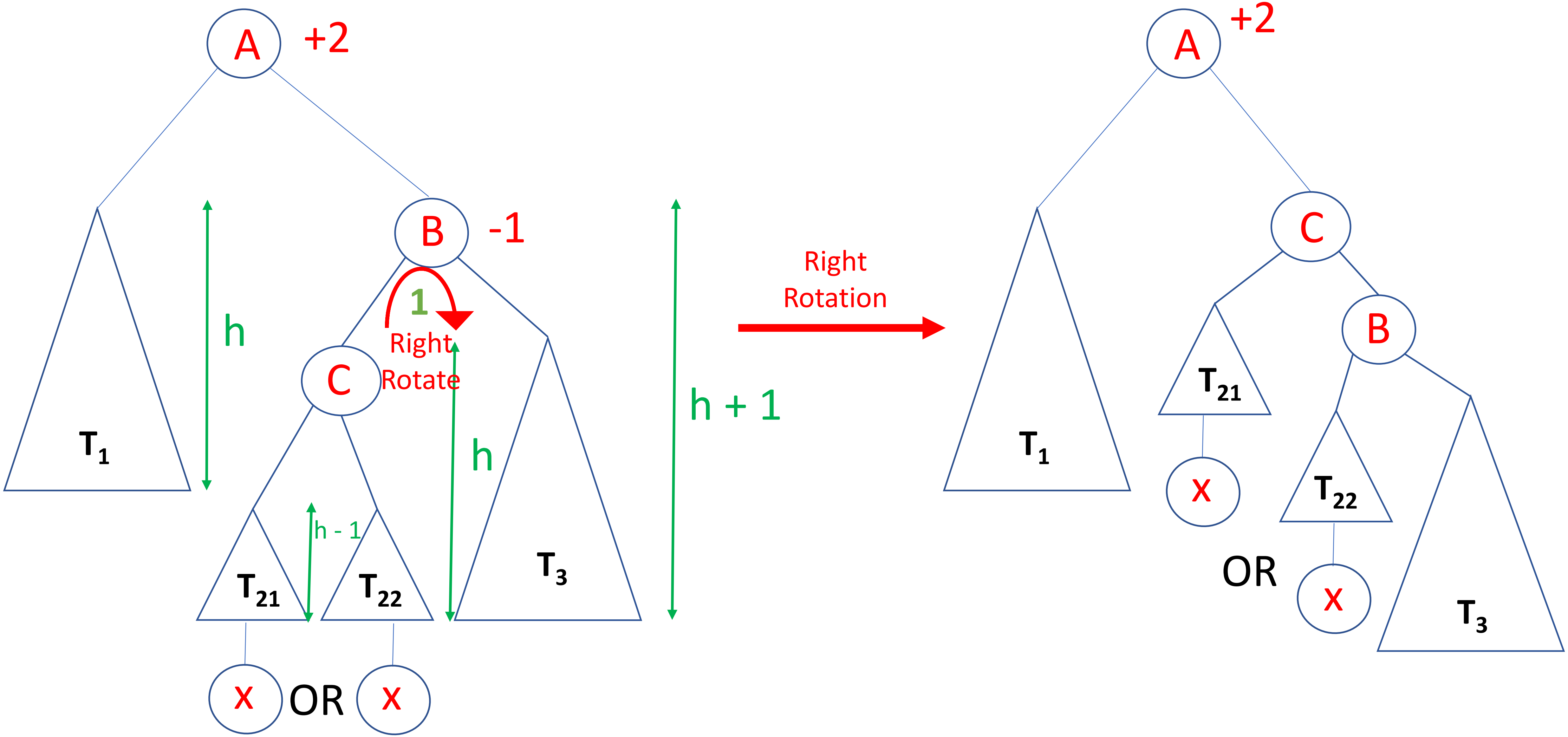
Case 1 (b) : h > -1



Case 1 (b) : h > -1

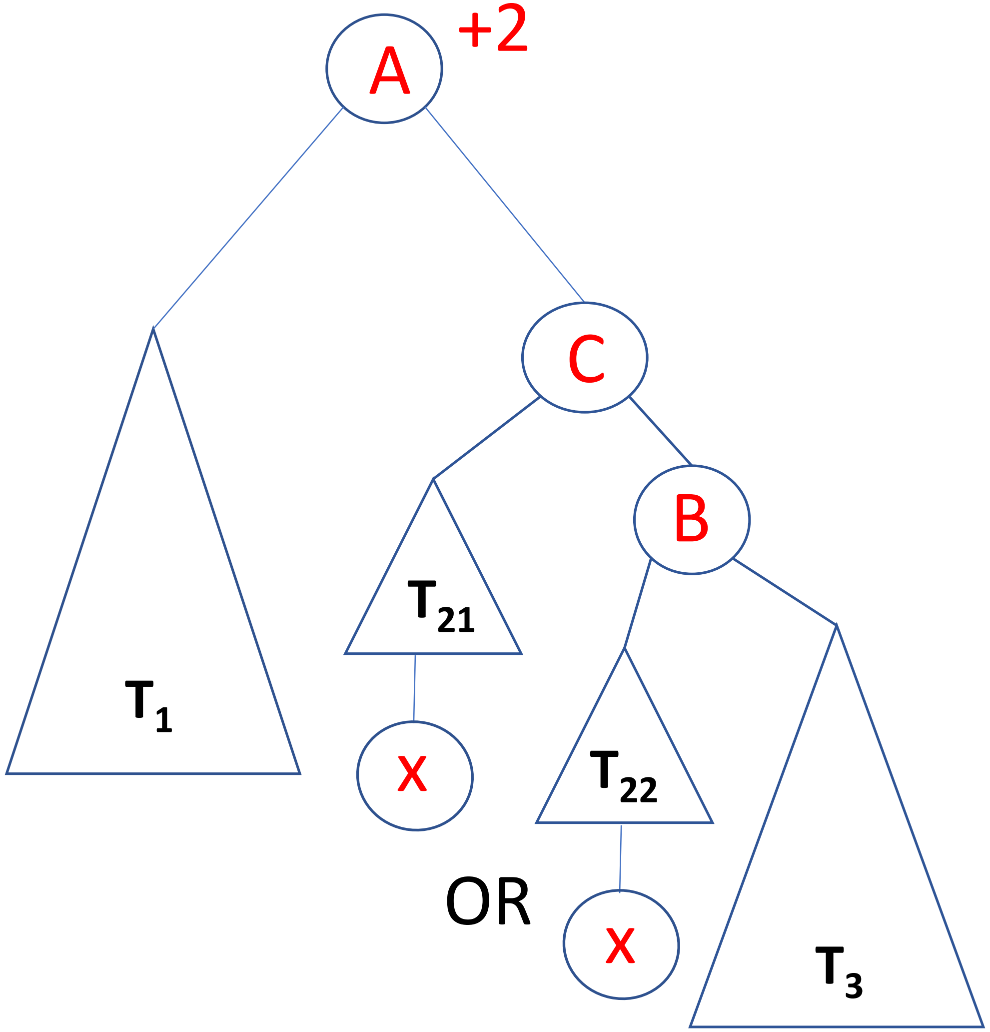


Case 1 (b) : h > -1

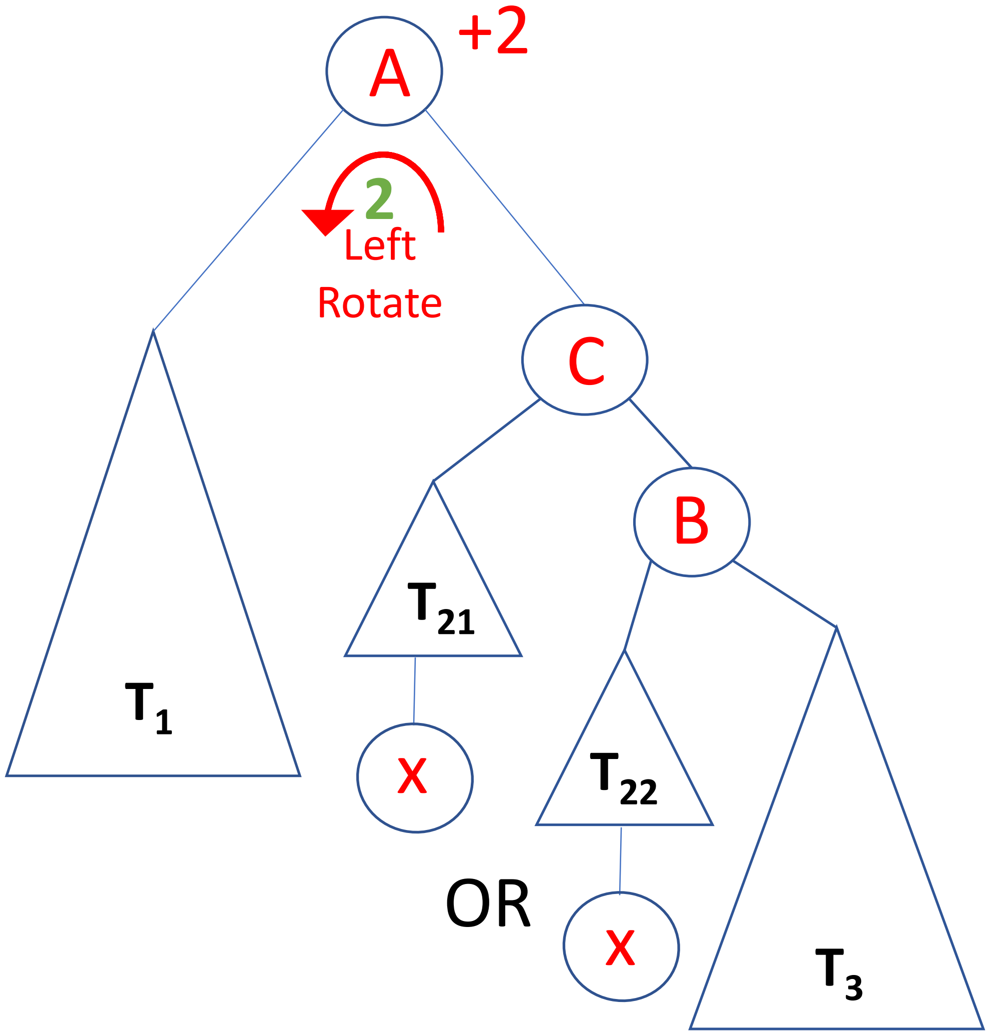


A
+2
B
-1
C
1
Right
Rotate
h
h
h - 1
h + 1
T_1
T_{21}
T_{22}
T_3
X
OR
X
Right
Rotation
A
+2
C
B
T_1
T_{21}
X
T_{22}
OR
X
T_3

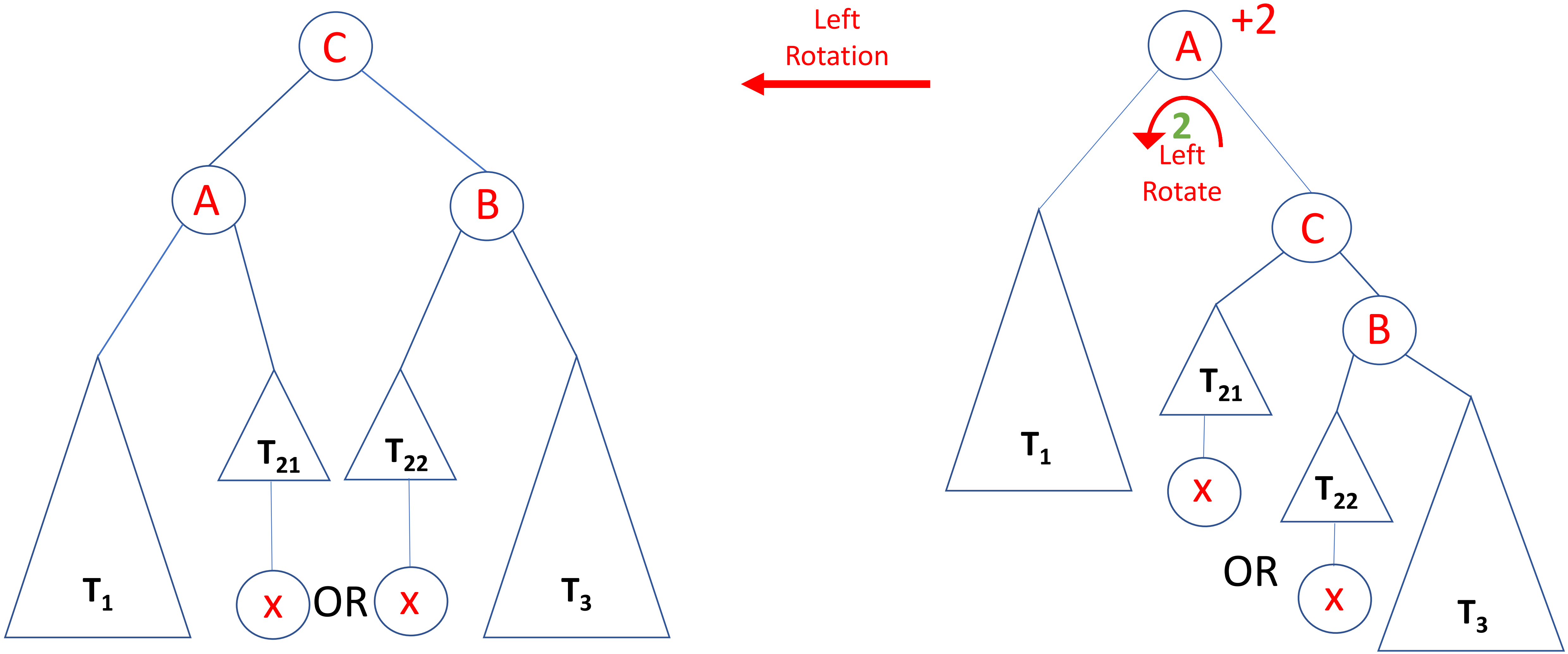
Case 1 (b) : h > -1



Case 1 (b) : h > -1

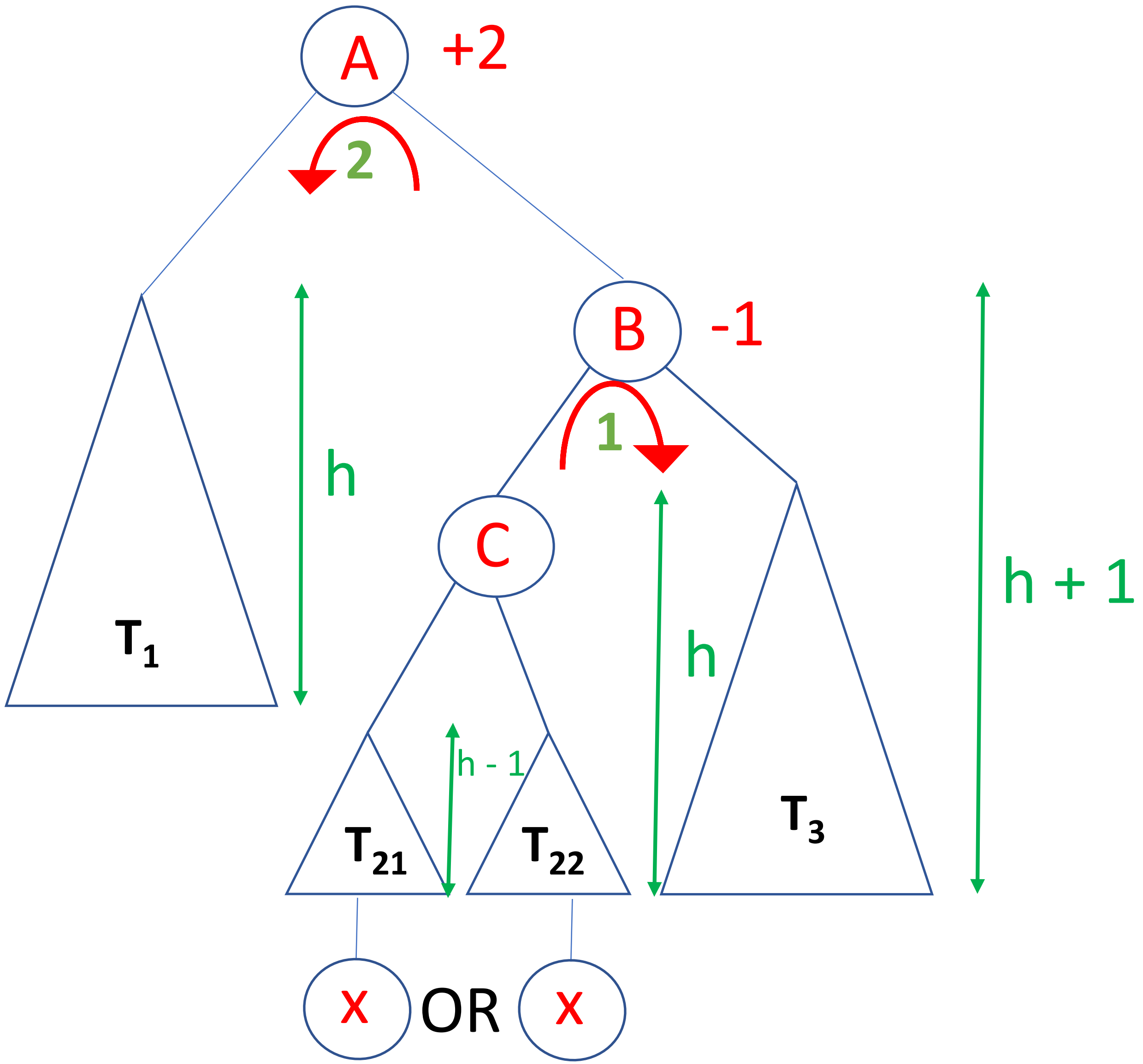


Case 1 (b) : h > -1

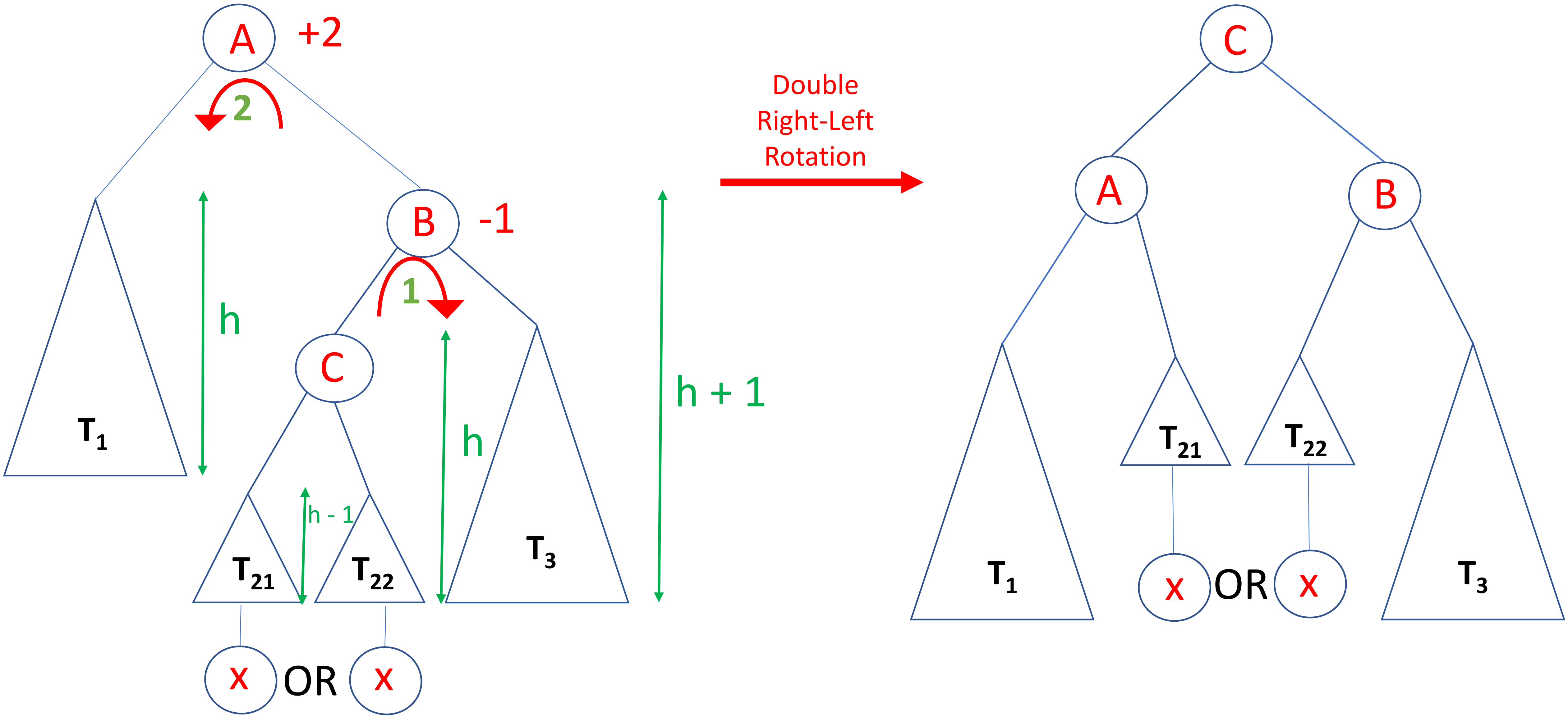


C
A
B
T_1
T_{21}
T_{22}
T_3
X
OR
X
Left
Rotation
A
+2
2
Left
Rotate
C
B
T_1
T_{21}
X
T_{22}
OR
X
T_3

Case 1 (b) : h > -1

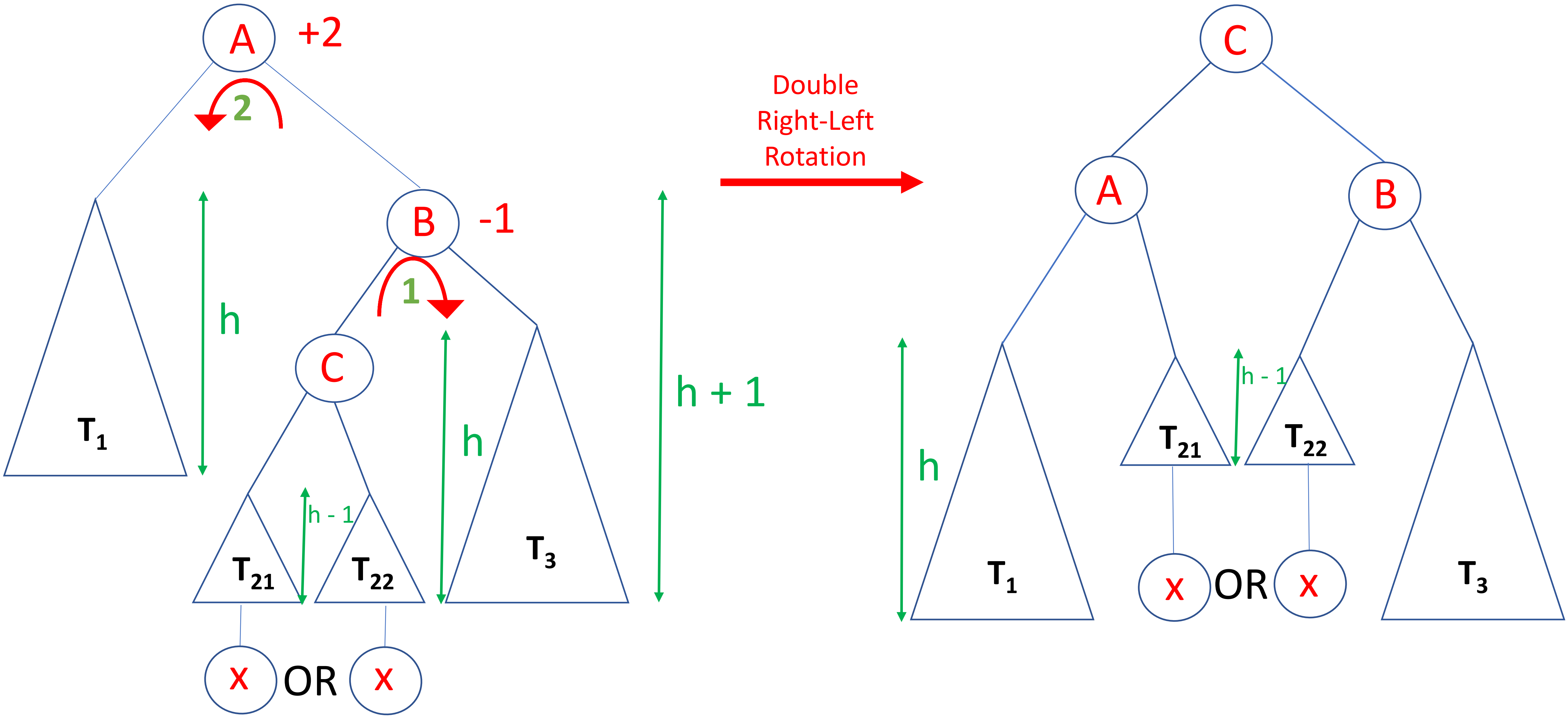


Case 1 (b) : h > -1

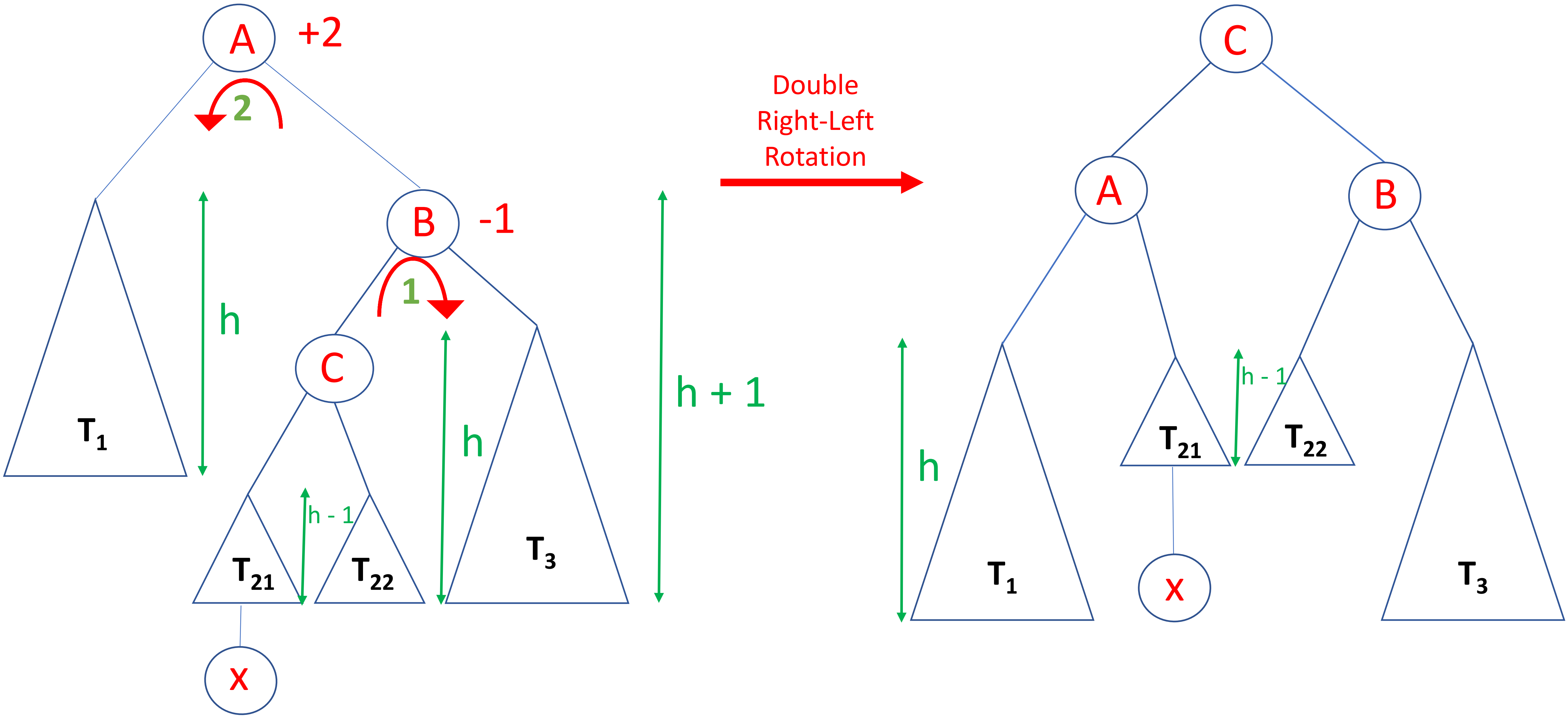


A
+2
2
B
-1
1
C
h
h
h - 1
h + 1
T1
T21
T22
T3
X
OR
X
Double
Right-Left
Rotation
C
A
B
T1
T21
T22
T3
X
OR
X

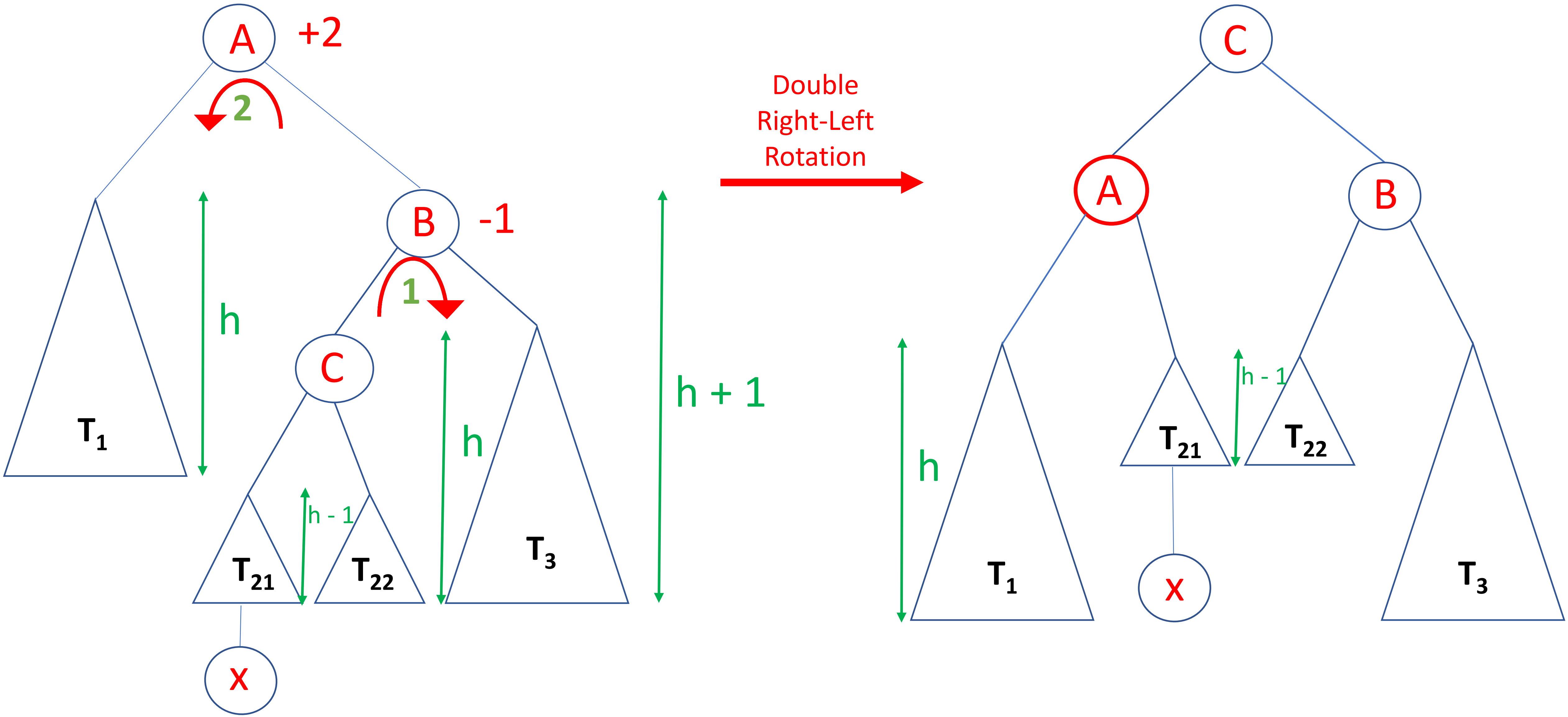
Case 1 (b) : h > -1


A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T1
T21
T22
T3
X
OR
X
Double
Right-Left
Rotation
C
A
B
h
h - 1
T1
T21
T22
T3
X
OR
X

Case 1 (b) : h > -1

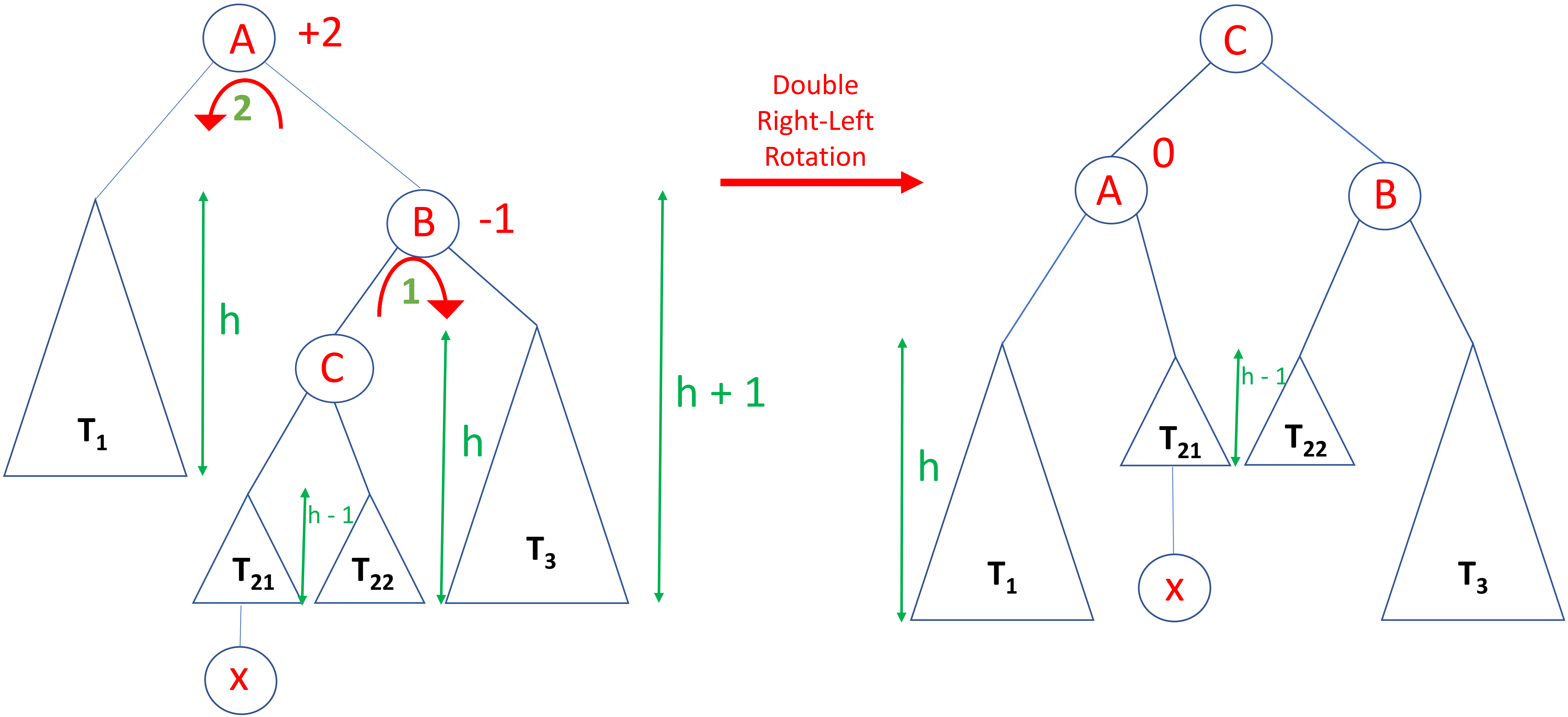

A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T_1
T_{21}
T_{22}
T_3
X
Double
Right-Left
Rotation
C
A
B
h
h - 1
T_1
T_{21}
T_{22}
T_3
X

Case 1 (b) : h > -1

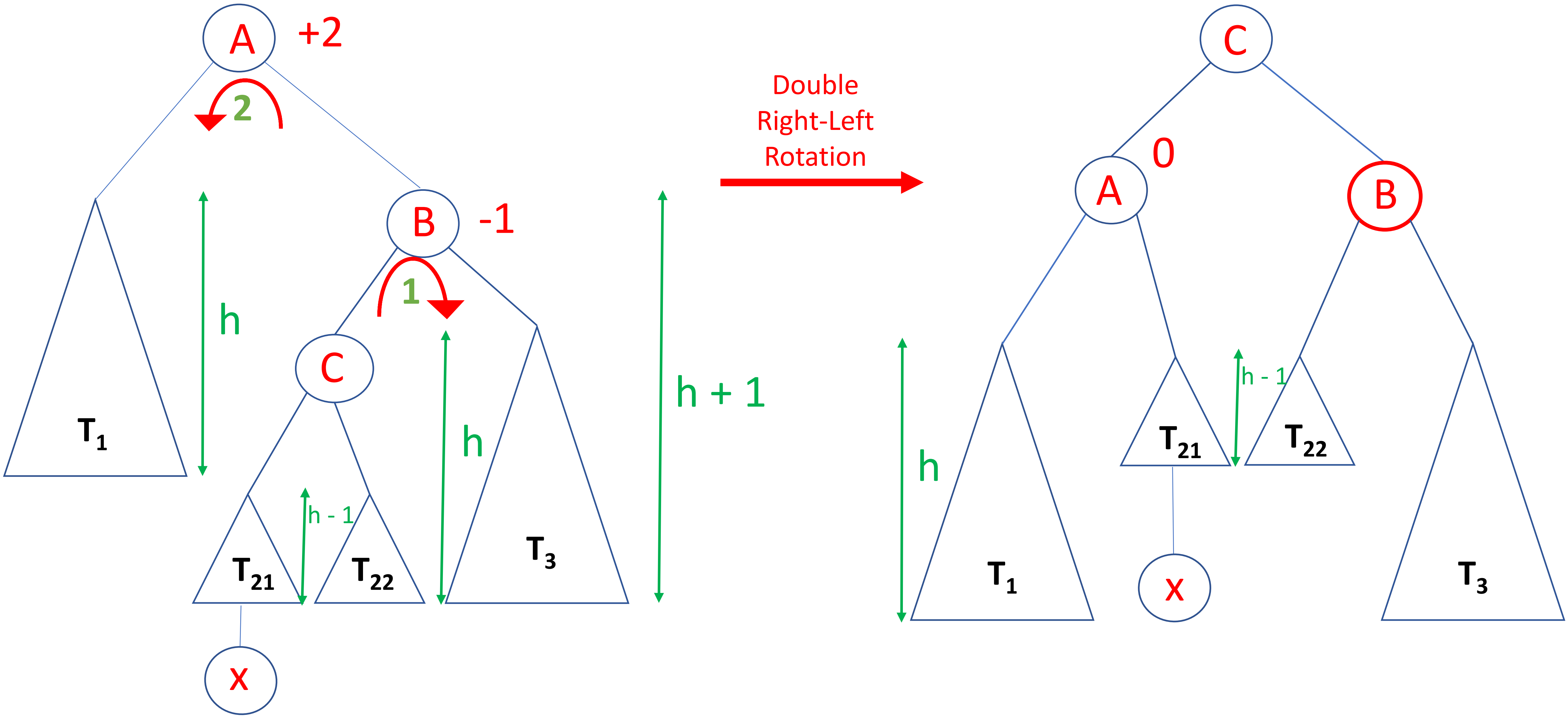


A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T1
T21
T22
T3
X
Double
Right-Left
Rotation
C
A
B
h
h - 1
T1
T21
T22
T3
X

Case 1 (b) : h > -1

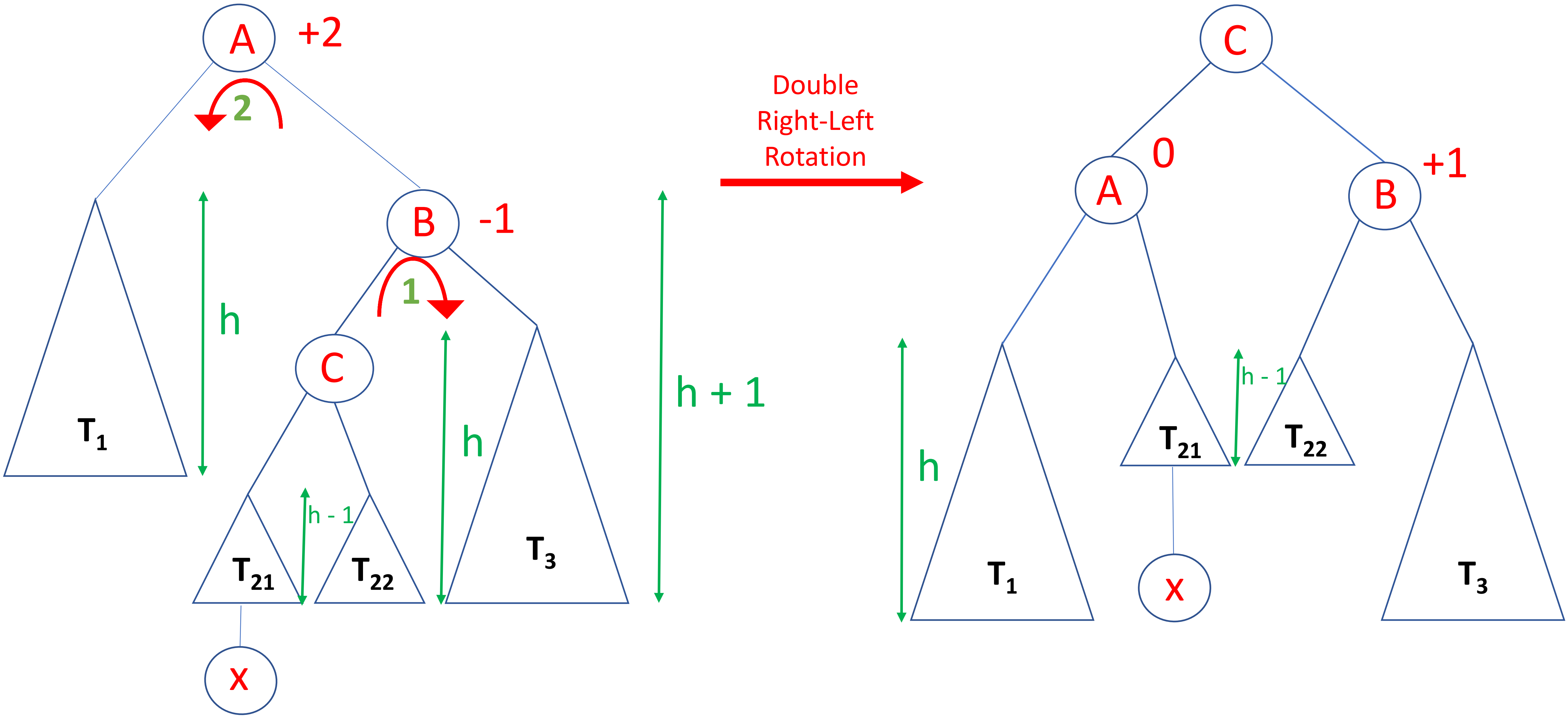

A
+2
2
B
-1
1
C
h
h
h + 1
T1
T21
T22
T3
h - 1
X
Double
Right-Left
Rotation
C
A
0
B
h
T1
T21
h - 1
T22
X
T3

Case 1 (b) : h > -1



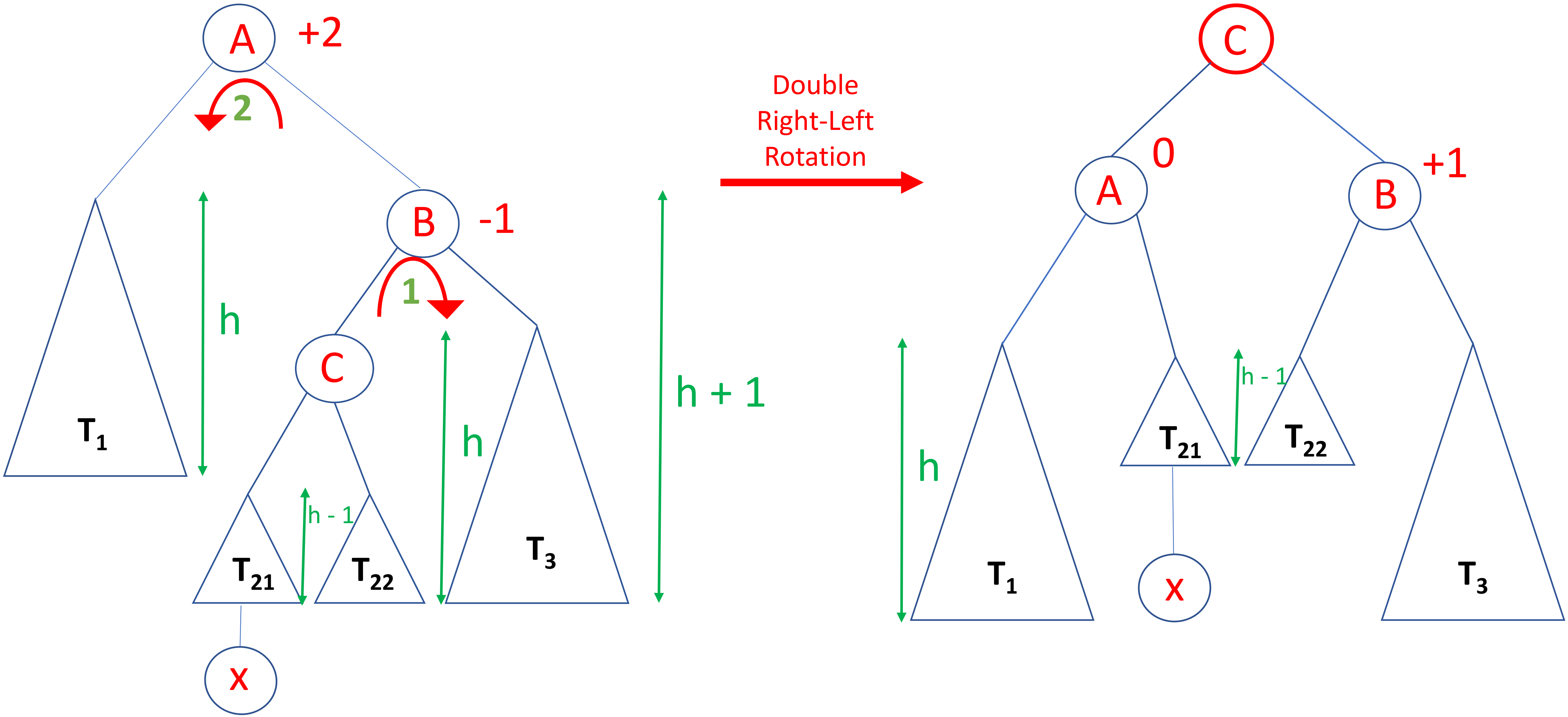
A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T1
T21
T22
T3
X
Double
Right-Left
Rotation
C
A
0
B
h
h - 1
T1
T21
T22
T3
X

Case 1 (b) : h > -1



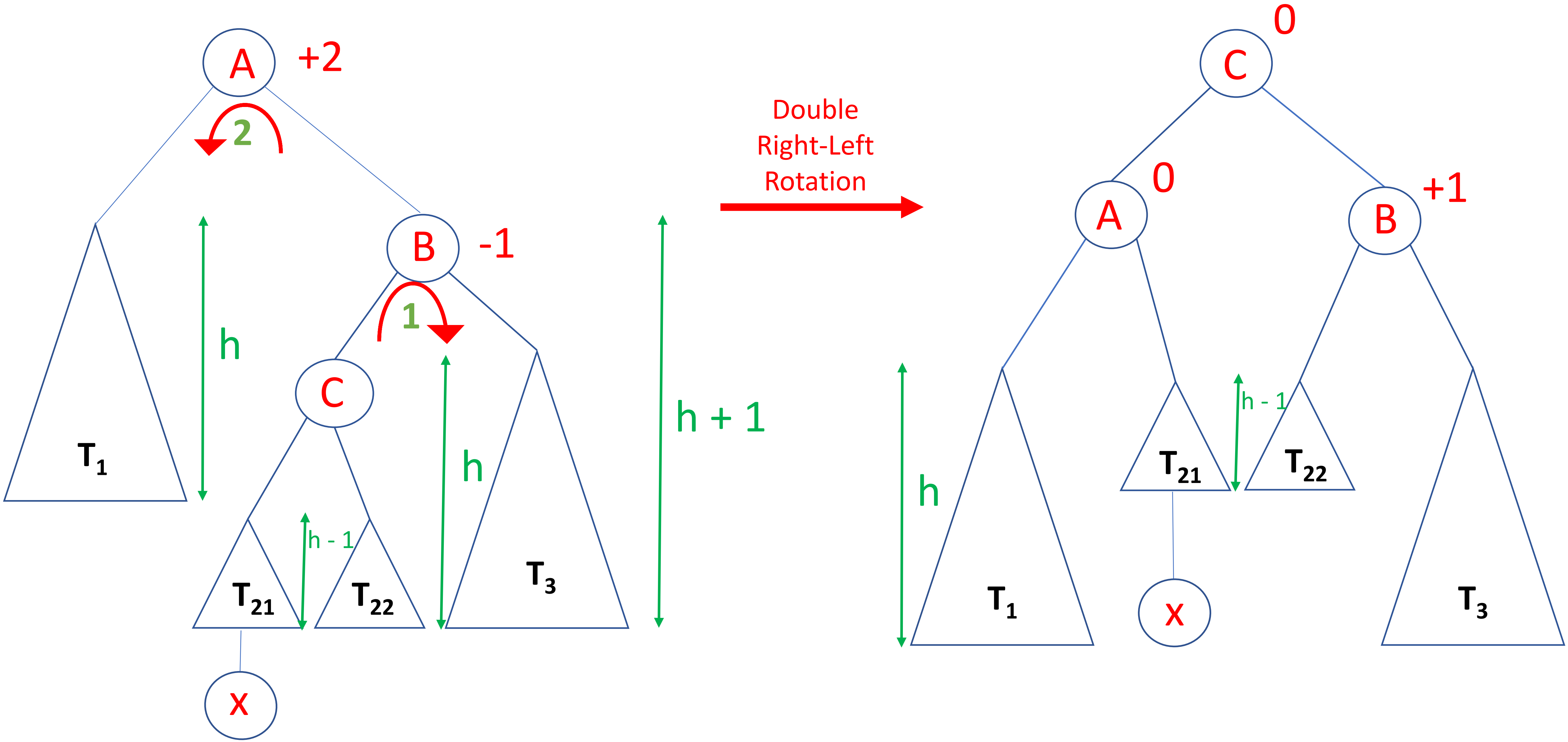
A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T_1
T_{21}
T_{22}
T_3
X
Double
Right-Left
Rotation
C
A
0
B
+1
h
h - 1
T_1
T_{21}
T_{22}
T_3
X

Case 1 (b) : h > -1

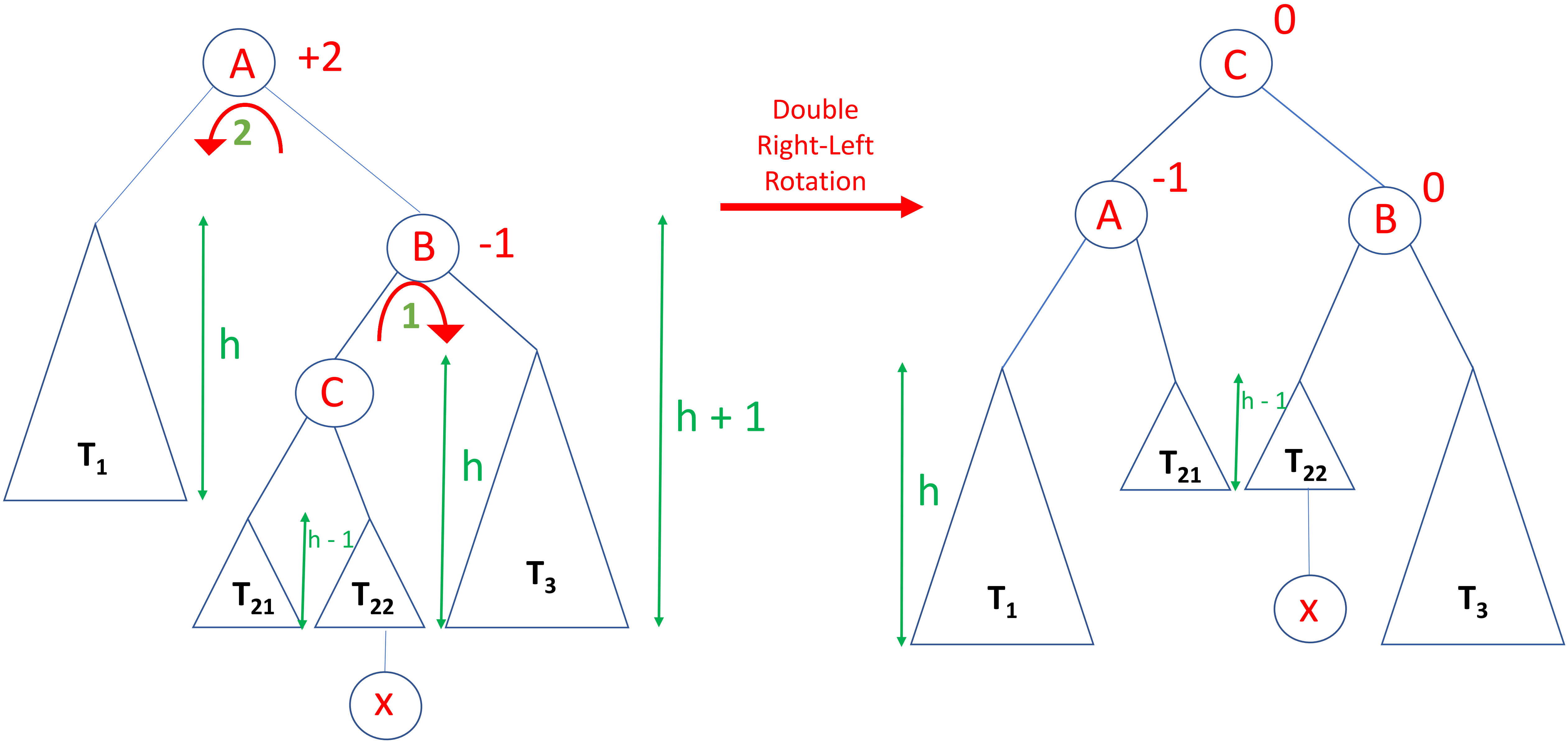


A
+2
2
B
-1
1
C
h
h
h + 1
T1
T21
T22
T3
h - 1
X
Double
Right-Left
Rotation
C
A
0
B
+1
h
h - 1
T1
T21
T22
T3
X

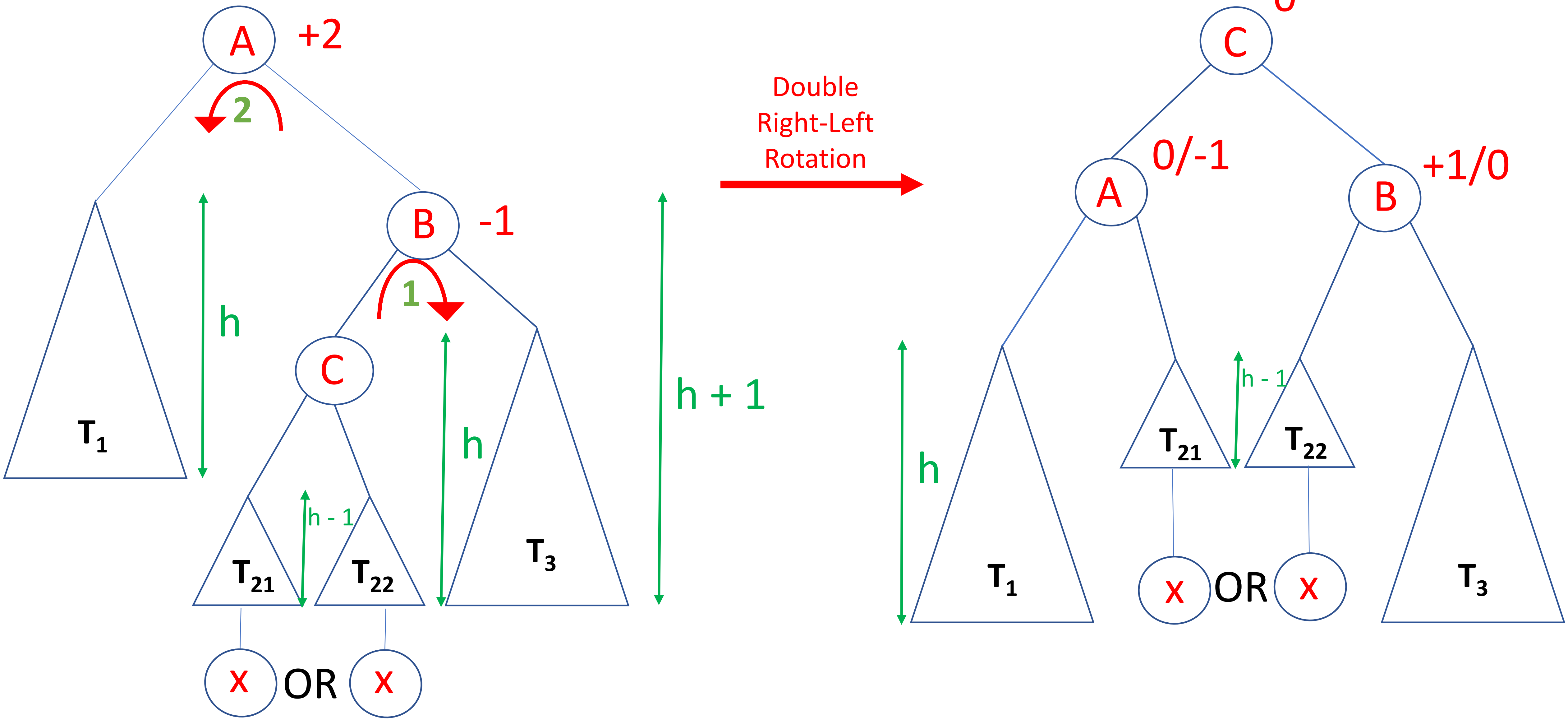
Case 1 (b) : h > -1


A
+2
2
B
-1
1
C
h
h
h + 1
T_1
T_{21}
T_{22}
T_3
h - 1
X
Double
Right-Left
Rotation
C
0
A
0
B
+1
h
h - 1
T_1
T_{21}
T_{22}
T_3
X

Case 1 (b) : h > -1

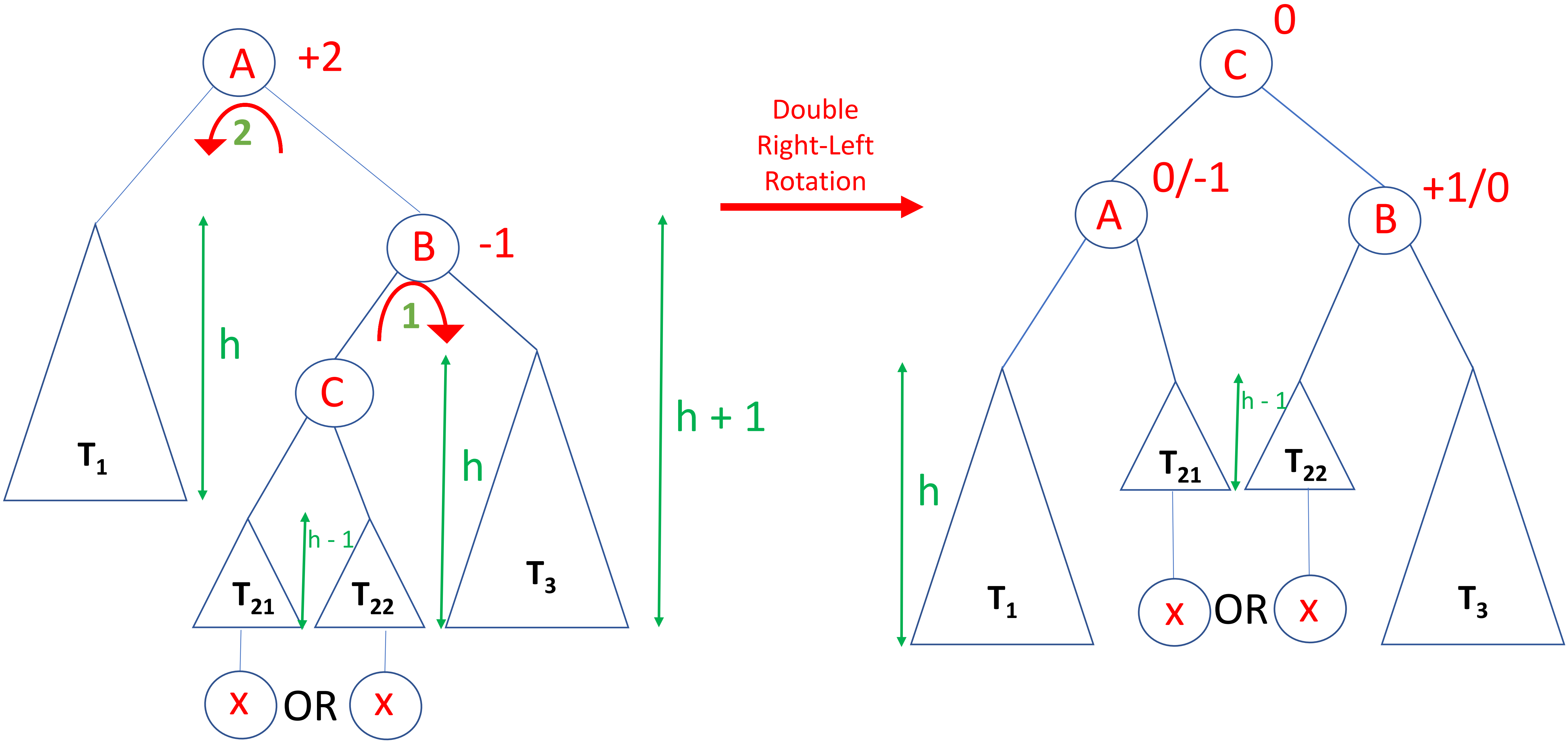

A
+2
2
B
-1
1
C
h
h
h + 1
h - 1
T_1
T_{21}
T_{22}
T_3
X
Double
Right-Left
Rotation
C
0
A
-1
B
0
h
h - 1
T_1
T_{21}
T_{22}
X
T_3

Case 1 (b) : h > -1


A
+2
2
B
-1
1
C
h
h
h + 1
T_1
T_{21}
T_{22}
T_3
h - 1
X
OR
X
Double
Right-Left
Rotation
0
C
0/-1
A
+1/0
B
h
h - 1
T_1
T_{21}
T_{22}
T_3
X
OR
X

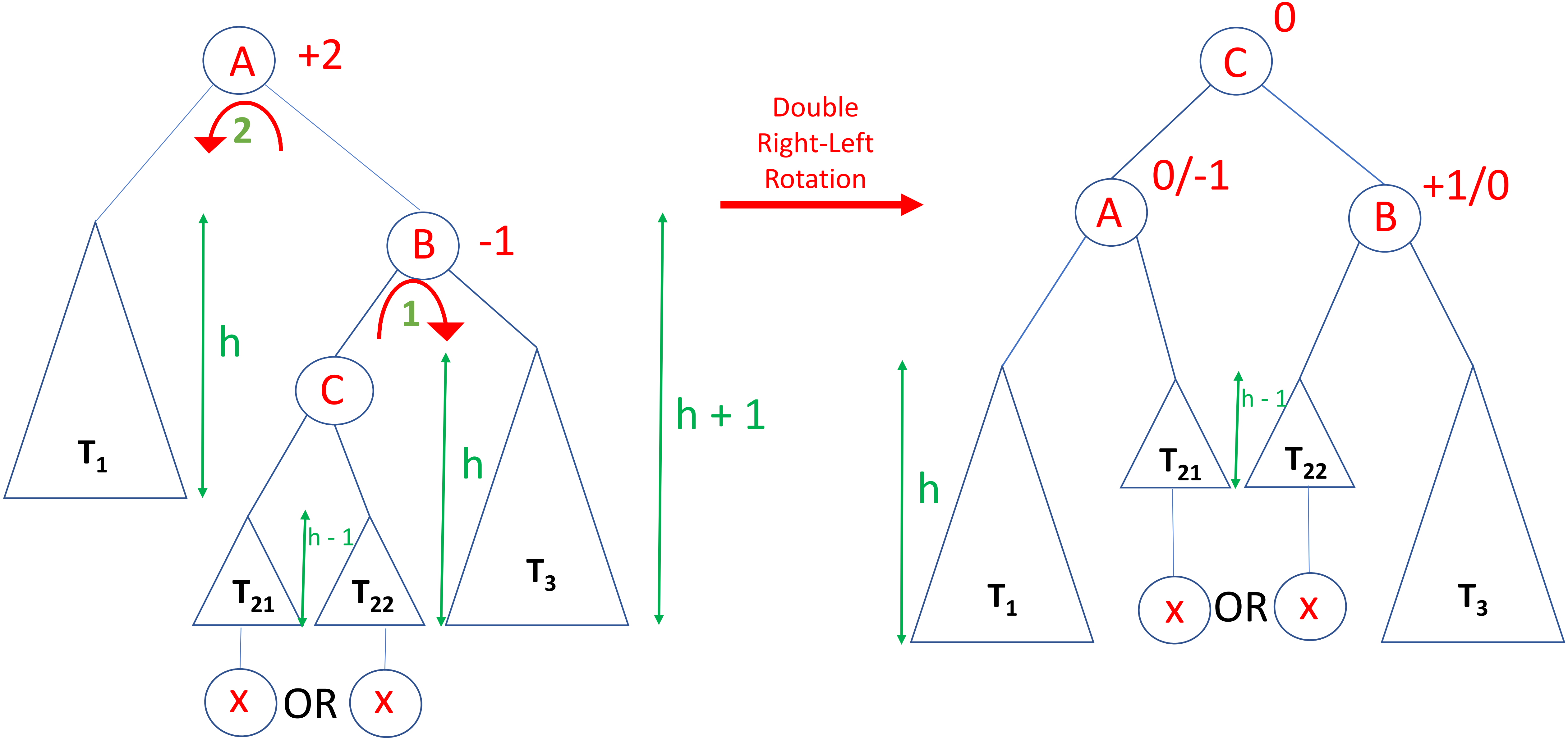
Rotation: (1) Rebalances the subtree

Case 1 (b) : h > -1



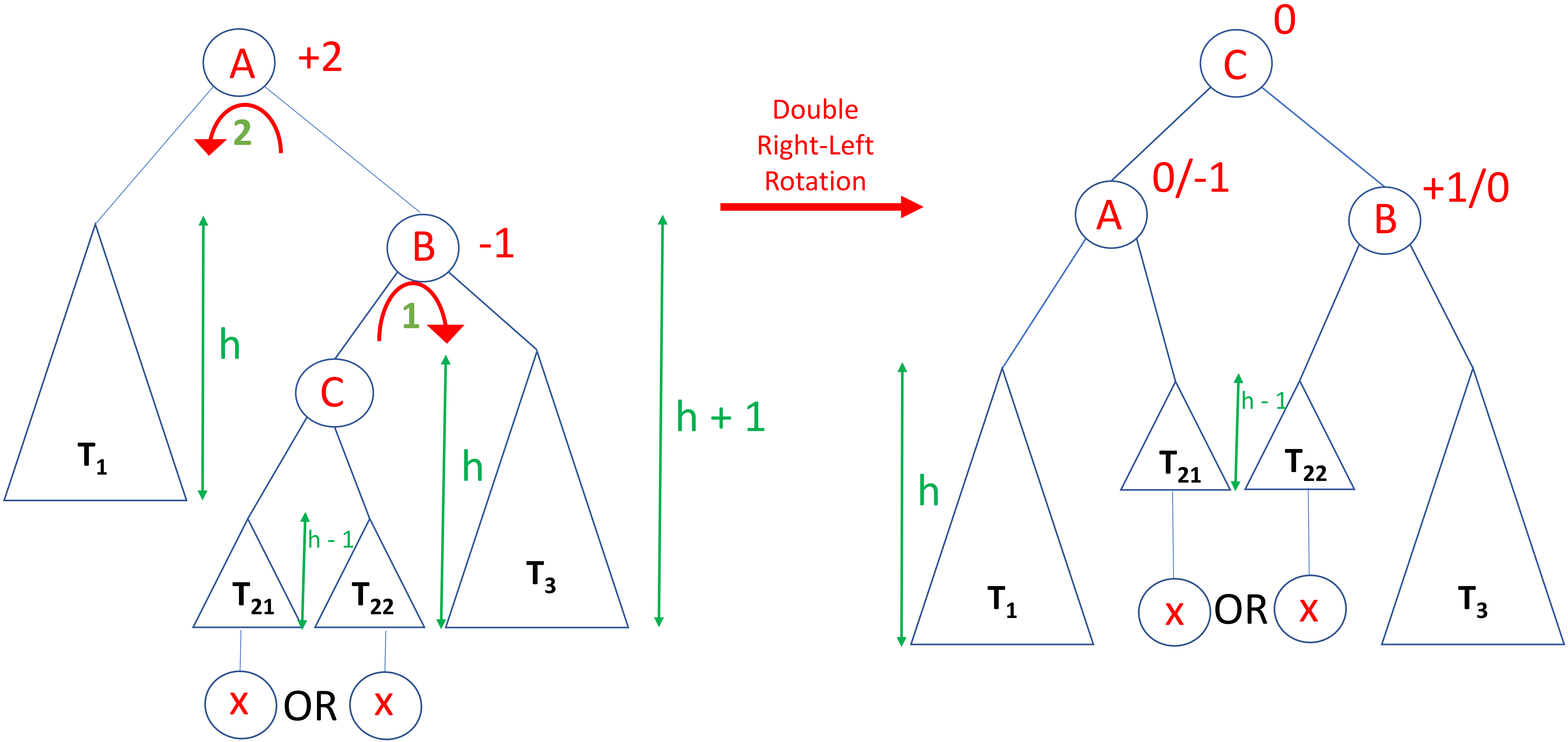
Rotation: (2) Preserves BST property

Case 1 (b) : h > -1



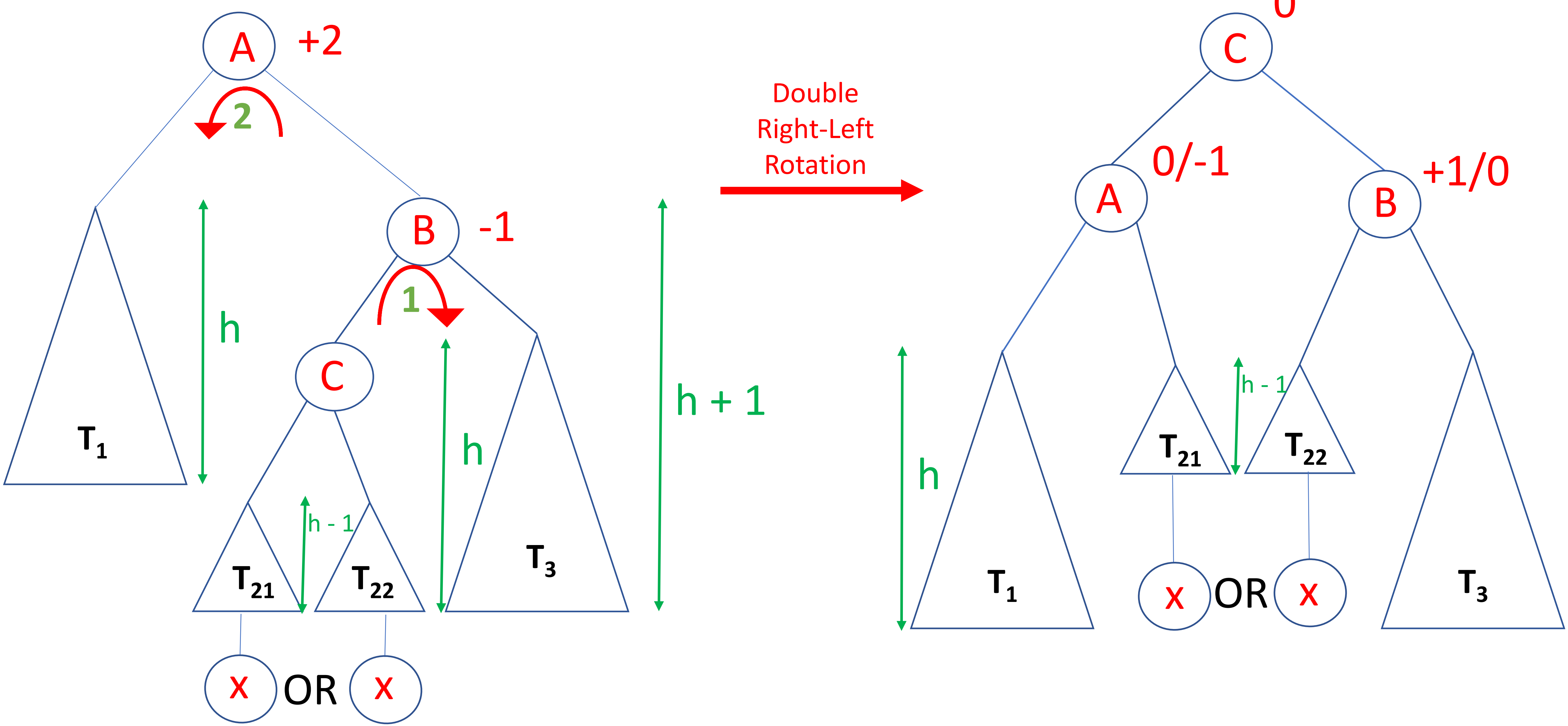
Rotation: (2) Preserves BST property (order:)

Case 1 (b) : h > -1



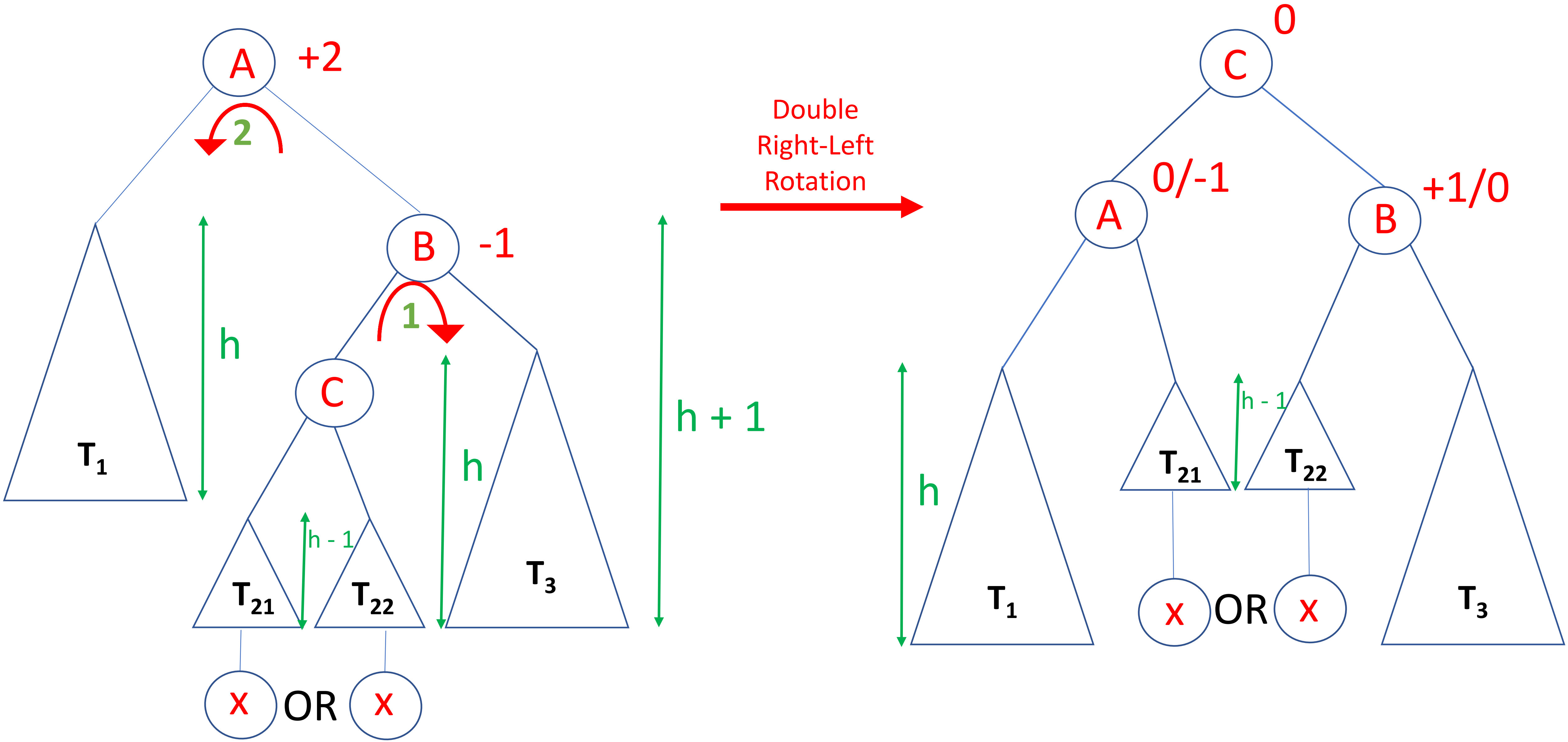
Rotation: (2) Preserves BST property (order: T_1)

Case 1 (b) : h > -1



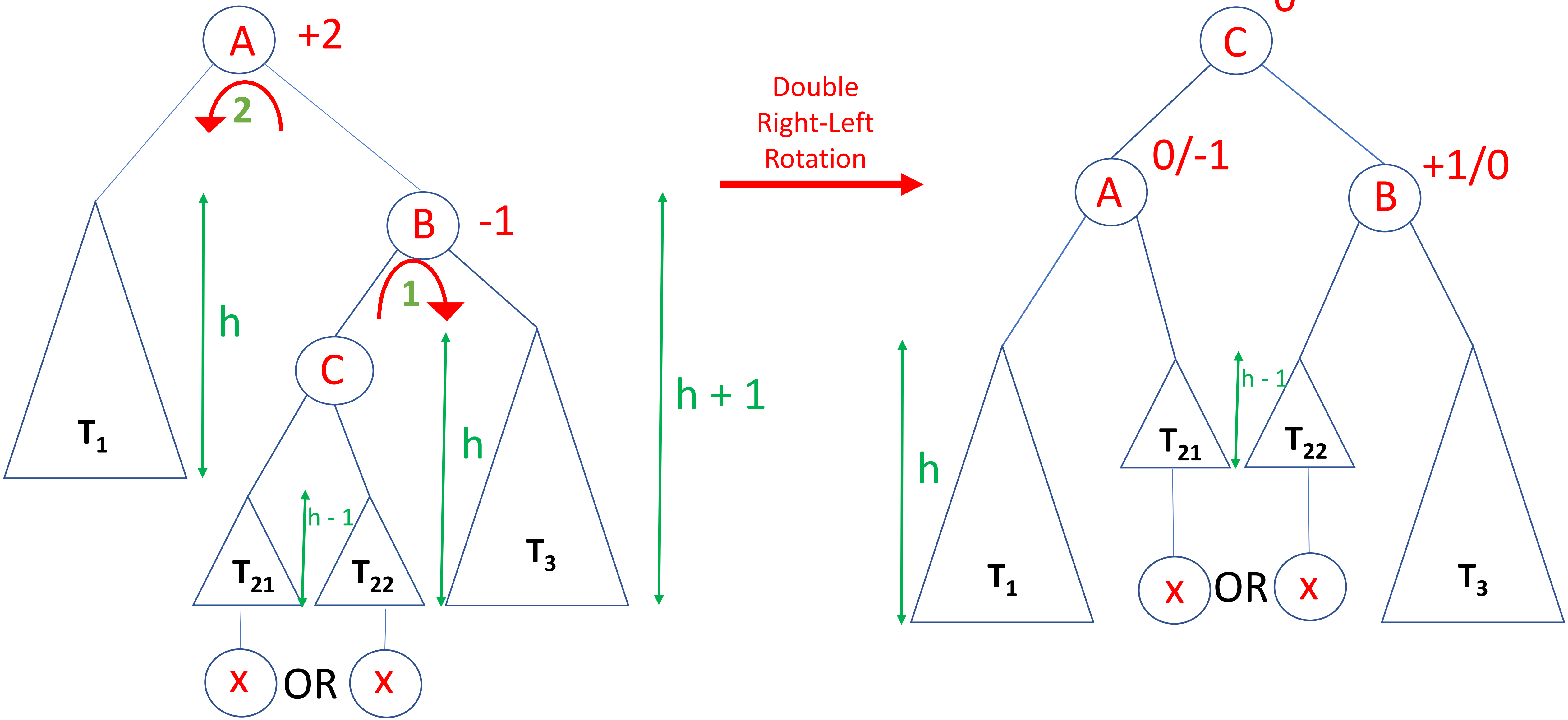
Rotation: (2) Preserves BST property (order: T_1 A)

Case 1 (b) : h > -1



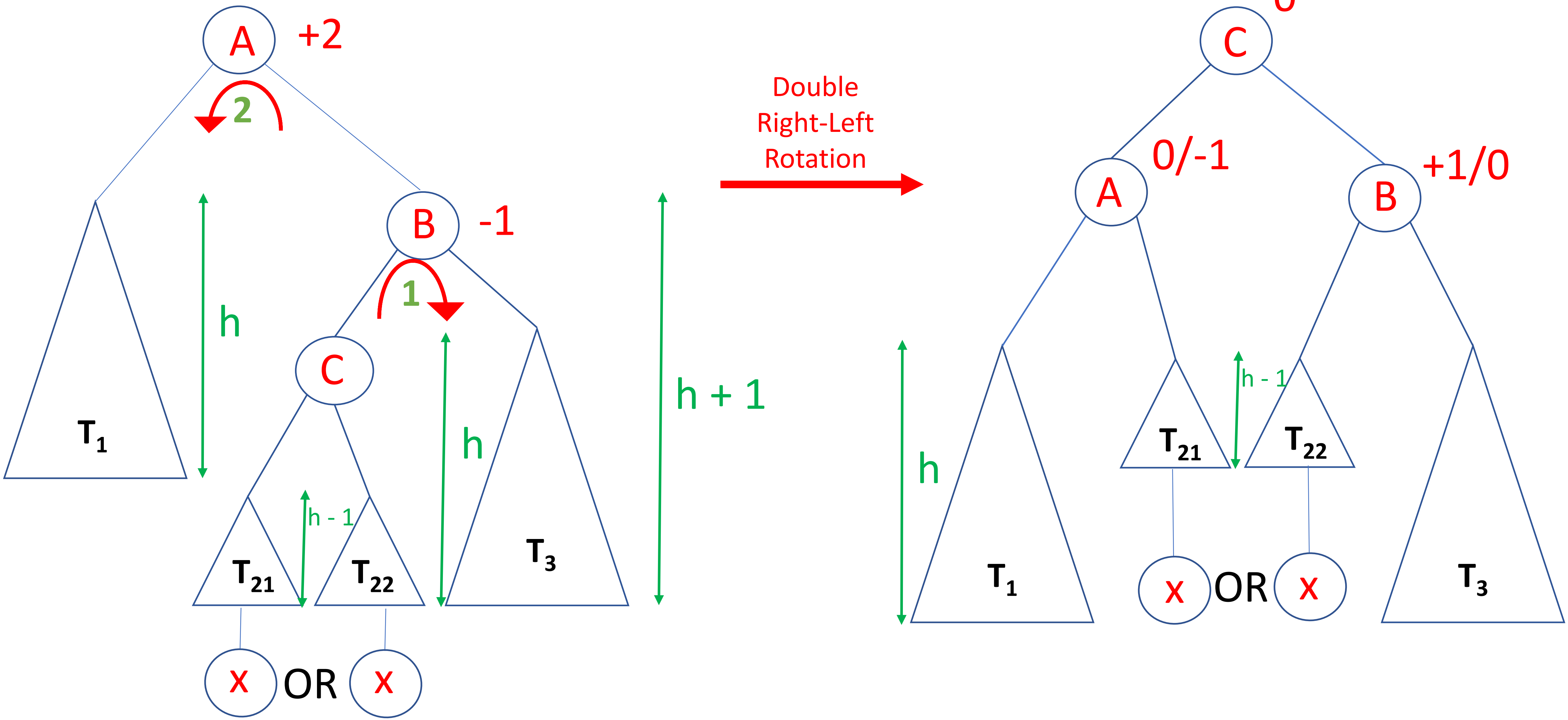
Rotation: (2) Preserves BST property (order: T_1 A T_{21})

Case 1 (b) : h > -1



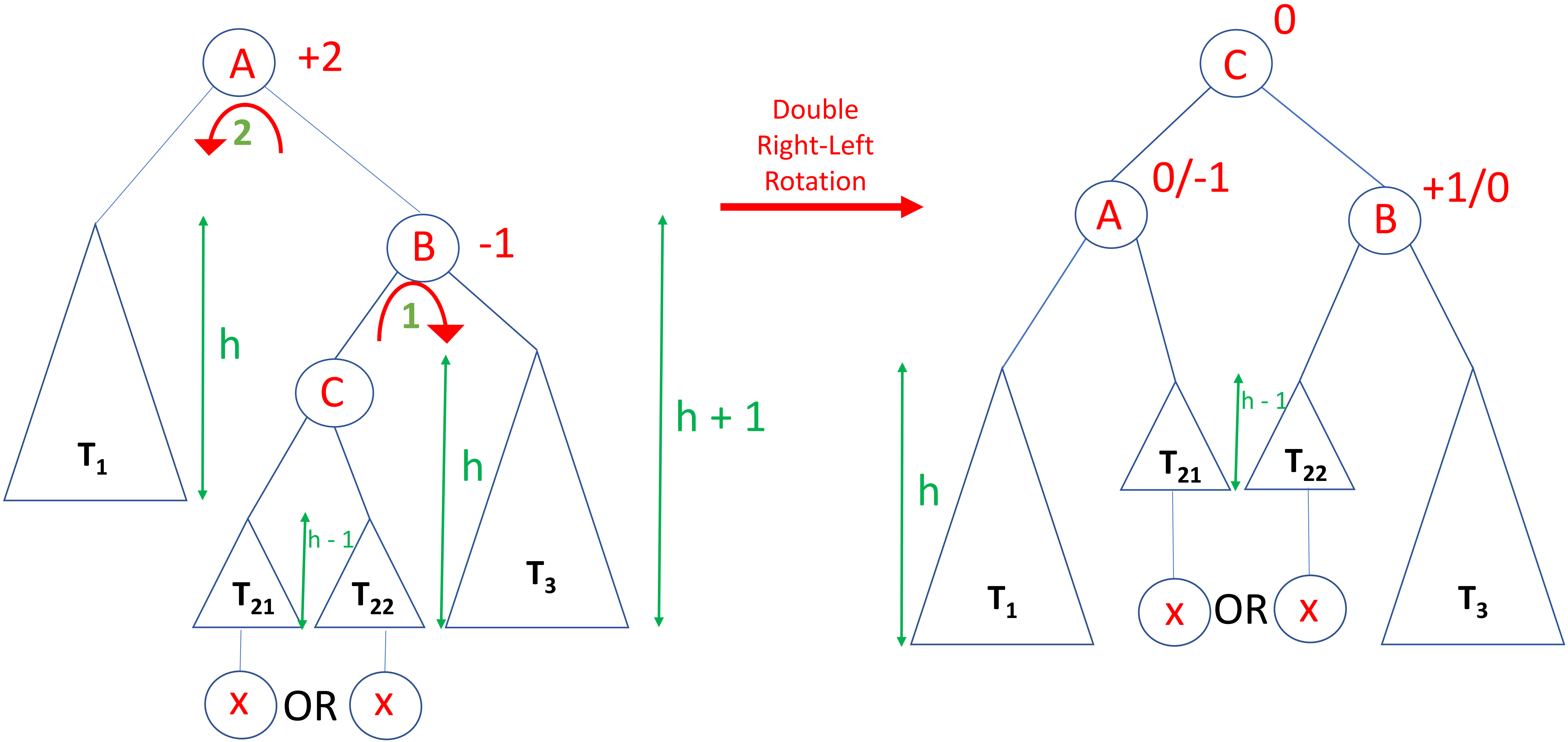
Rotation: (2) Preserves BST property (order: T_1 A T_{21} C)

Case 1 (b) : h > -1



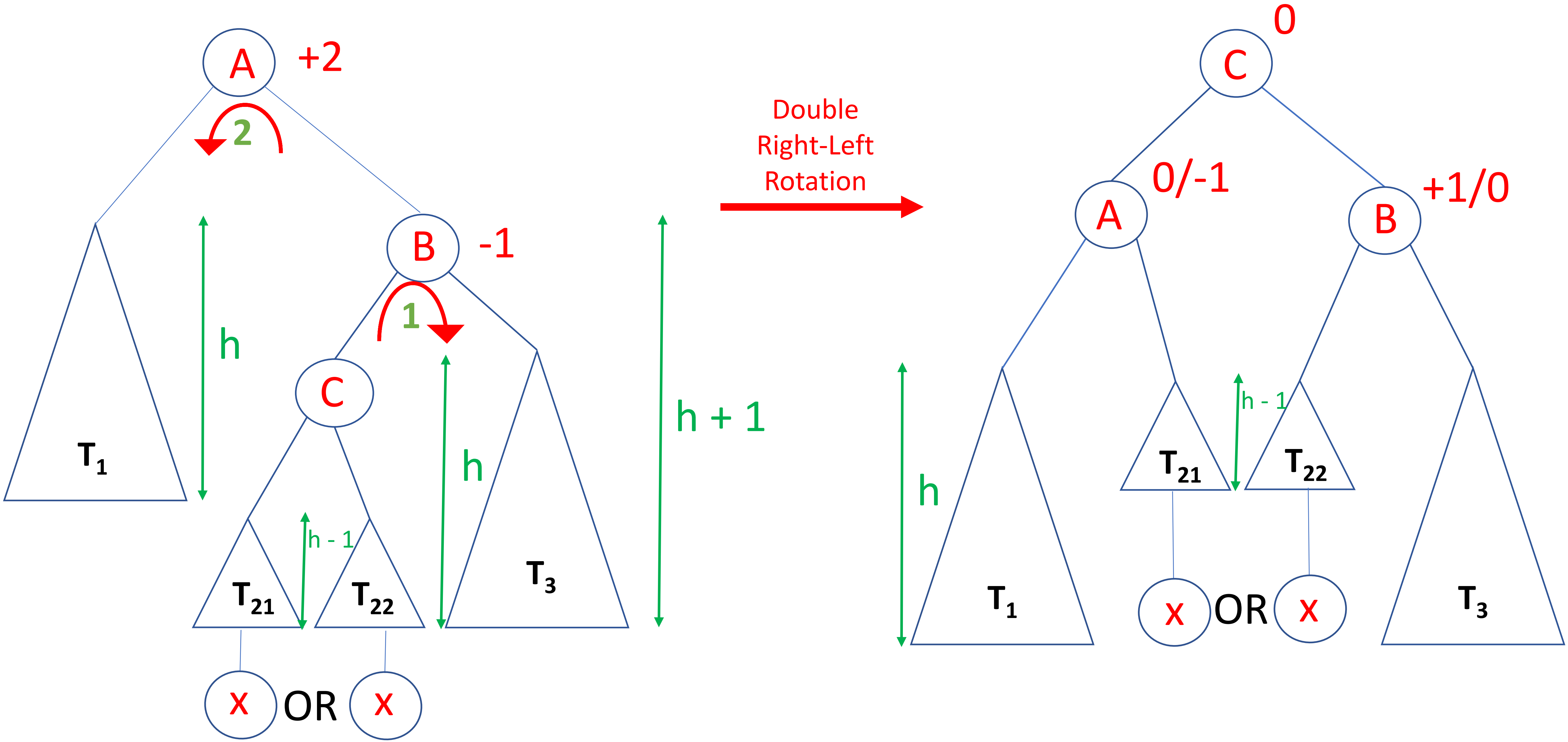
Rotation: (2) Preserves BST property (order: T_1 A T_{21} C T_{22})

Case 1 (b) : h > -1



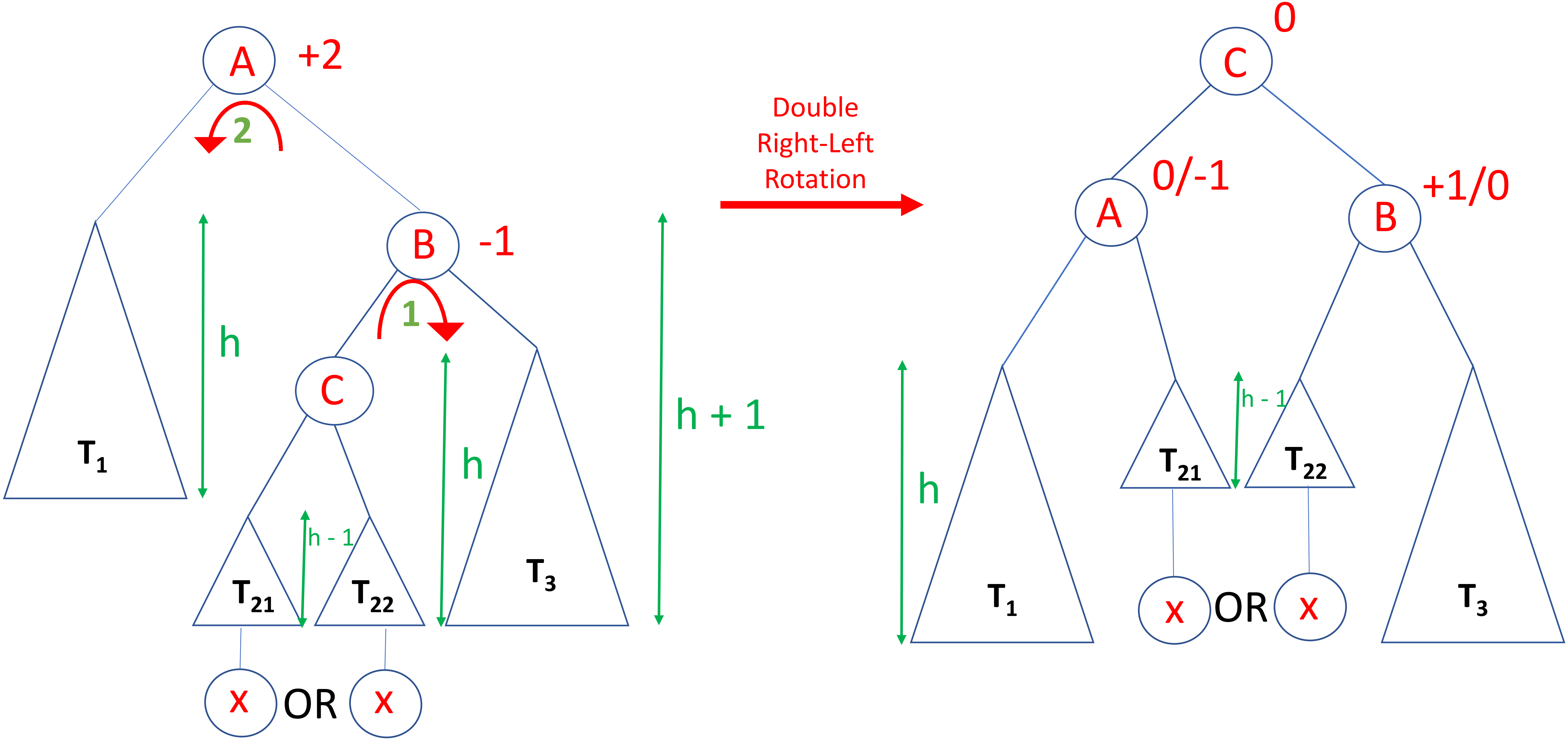
Rotation: (2) Preserves BST property (order: T_1 A T_{21} C T_{22} B)

Case 1 (b) : h > -1



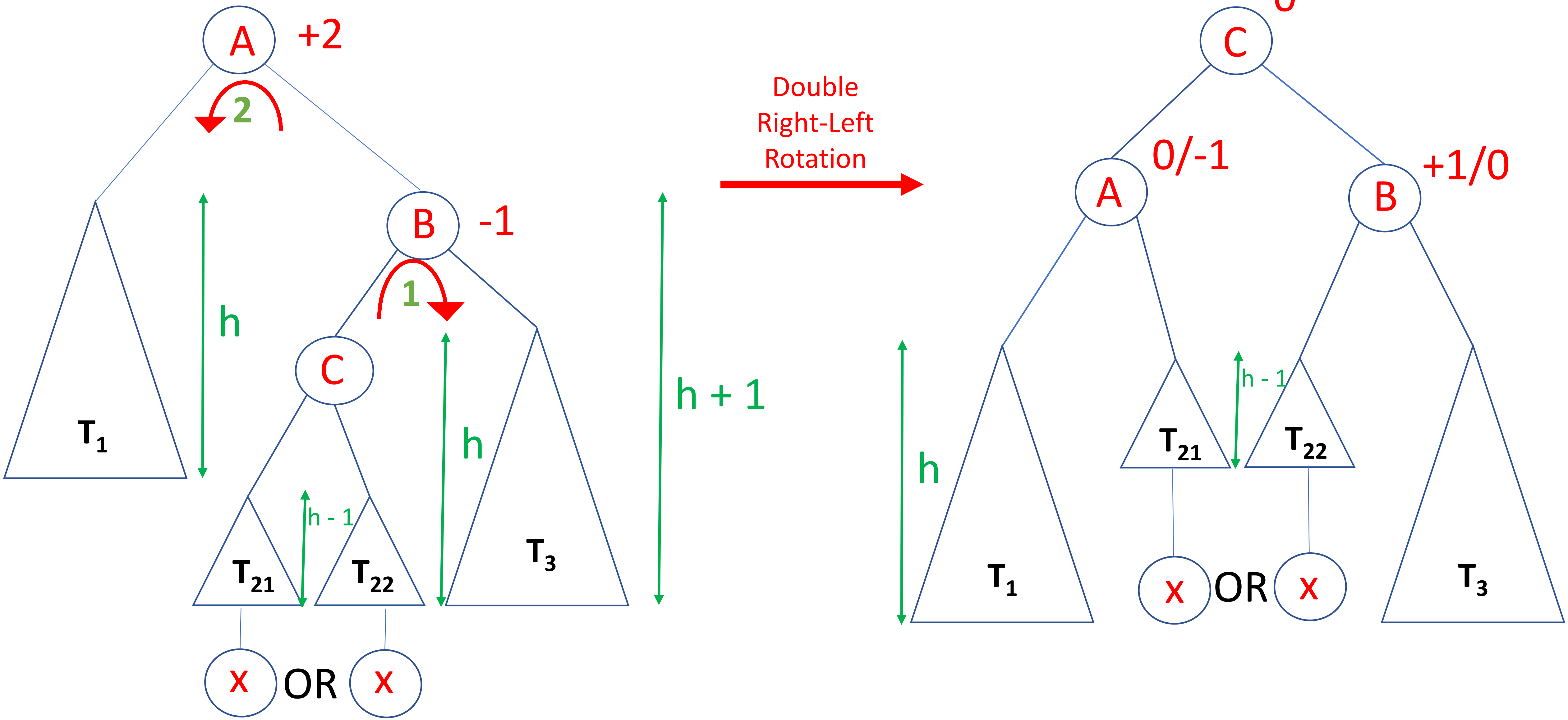
Rotation: (2) Preserves BST property (order: T_1 A T_{21} C T_{22} B T_3)

Case 1 (b) : h > -1



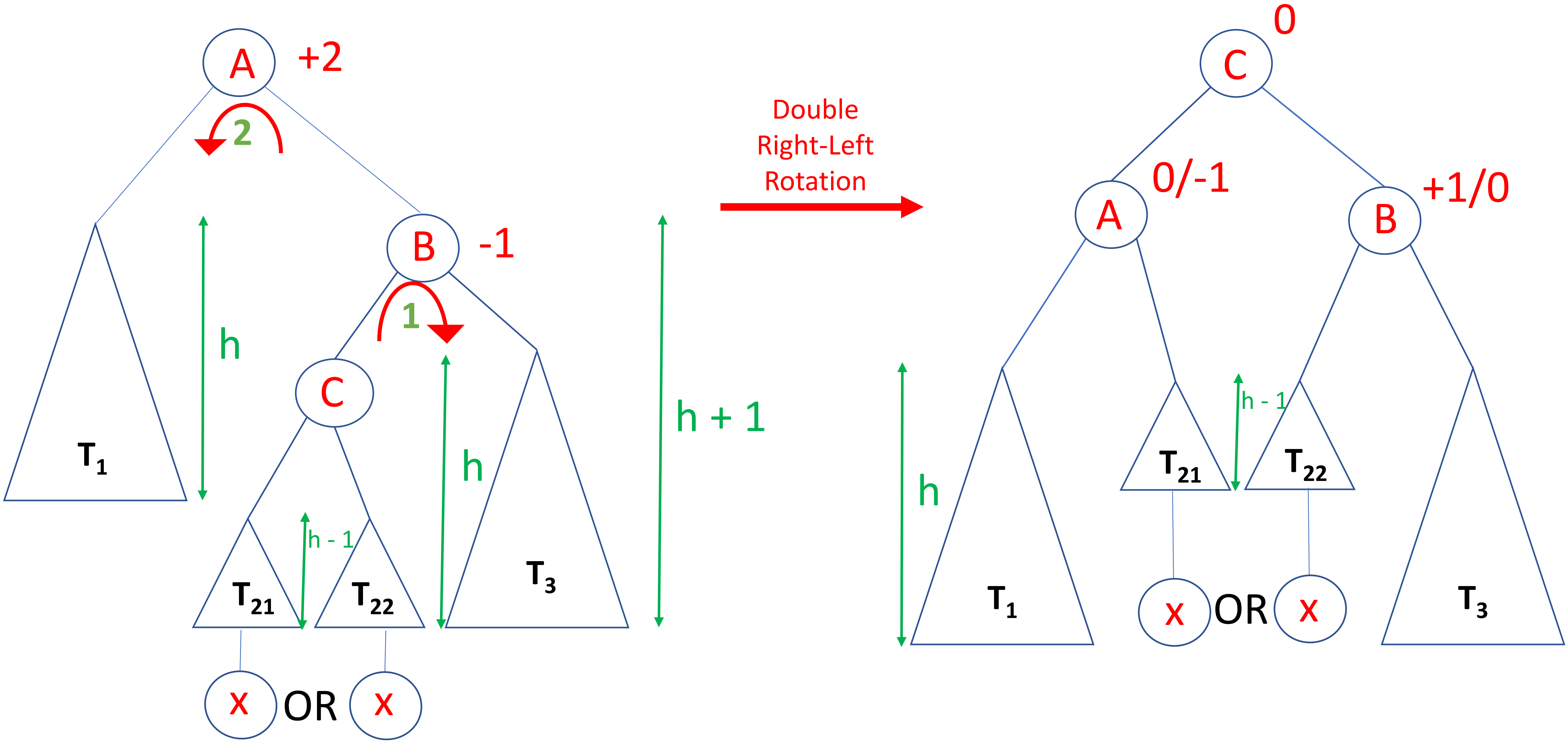
Rotation: (3) Bonus?

Case 1 (b) : **h > -1**



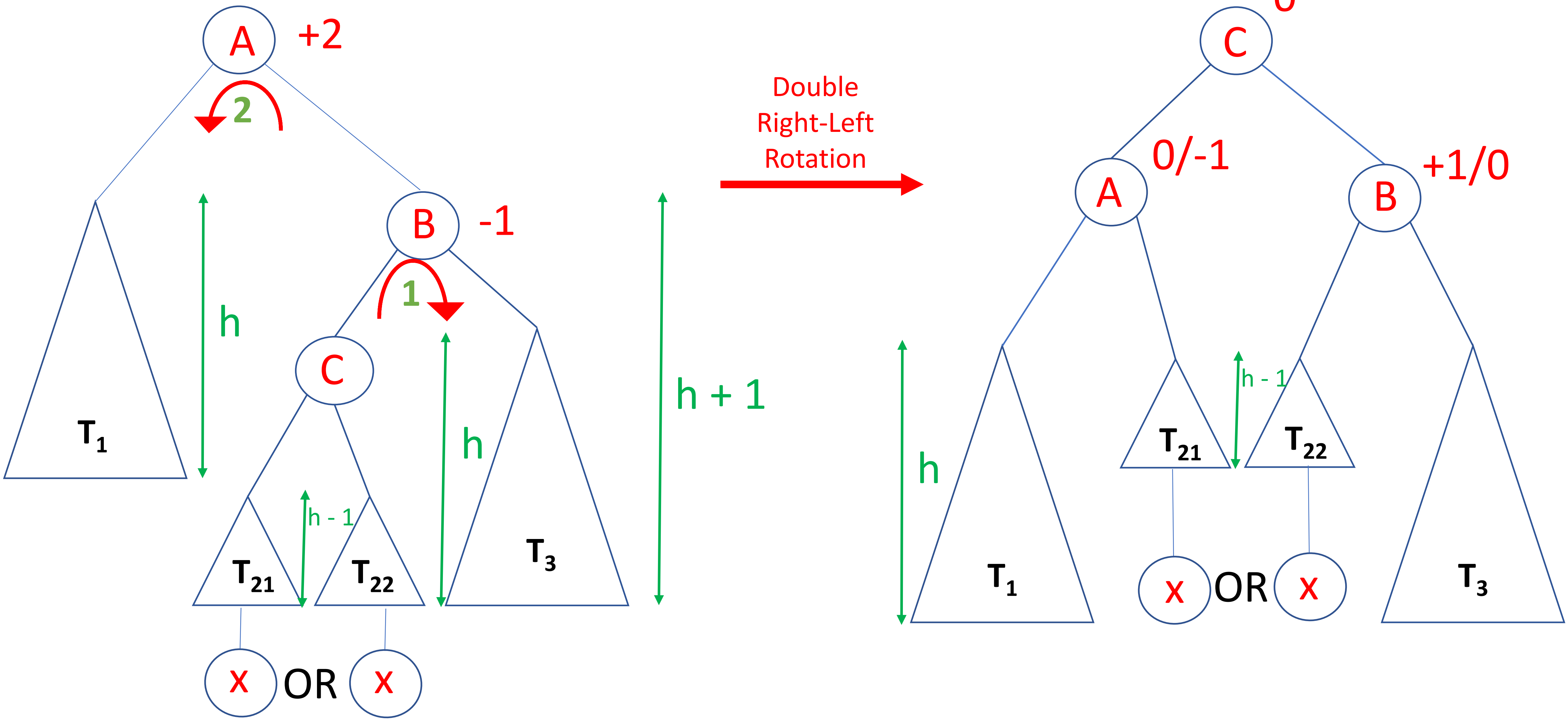
Rotation: (3) Bonus: height of this subtree is the same both before and after the insertion of x

Case 1 (b) : h > -1



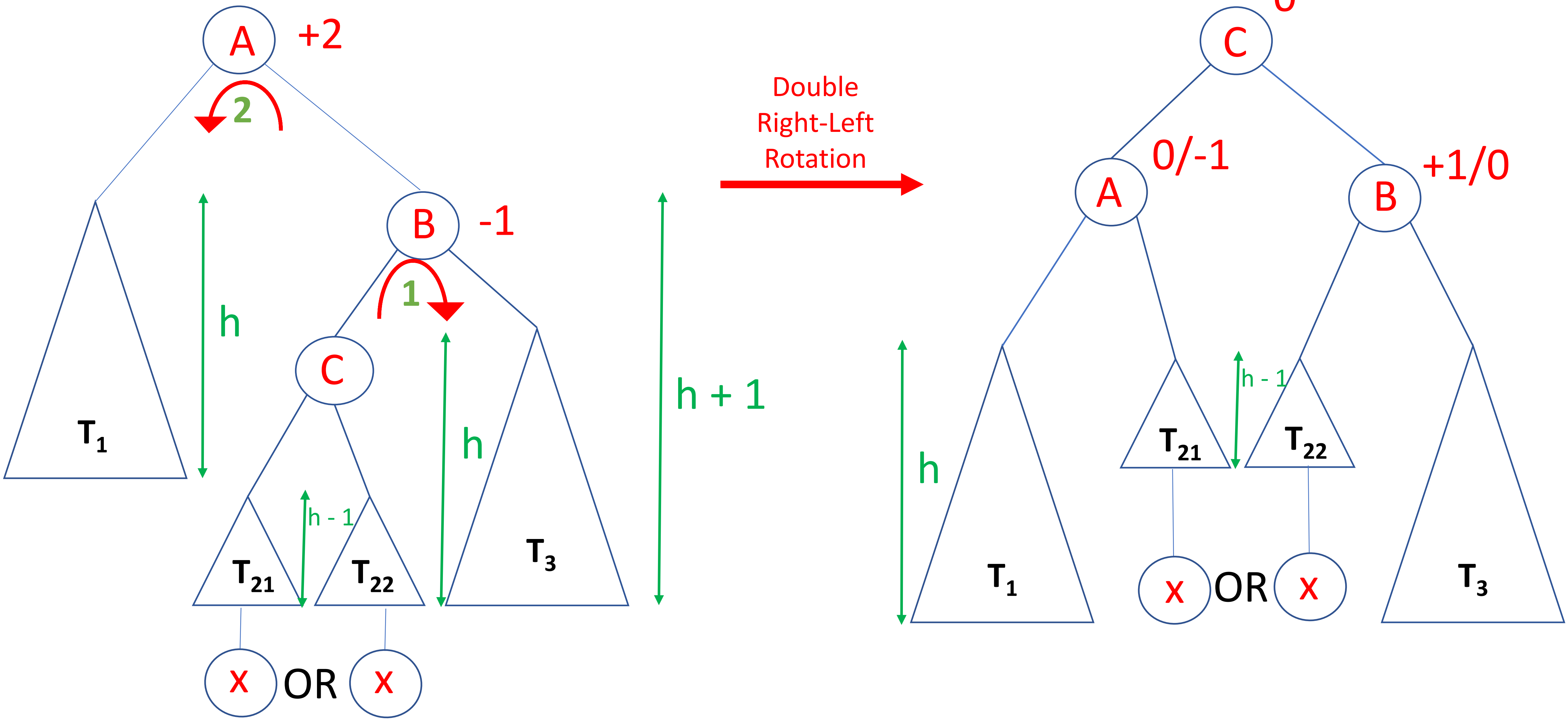
Rotation: (3) Bonus: height of this subtree is the same ($h + 2$) both before and after the insertion of x

Case 1 (b) : h > -1

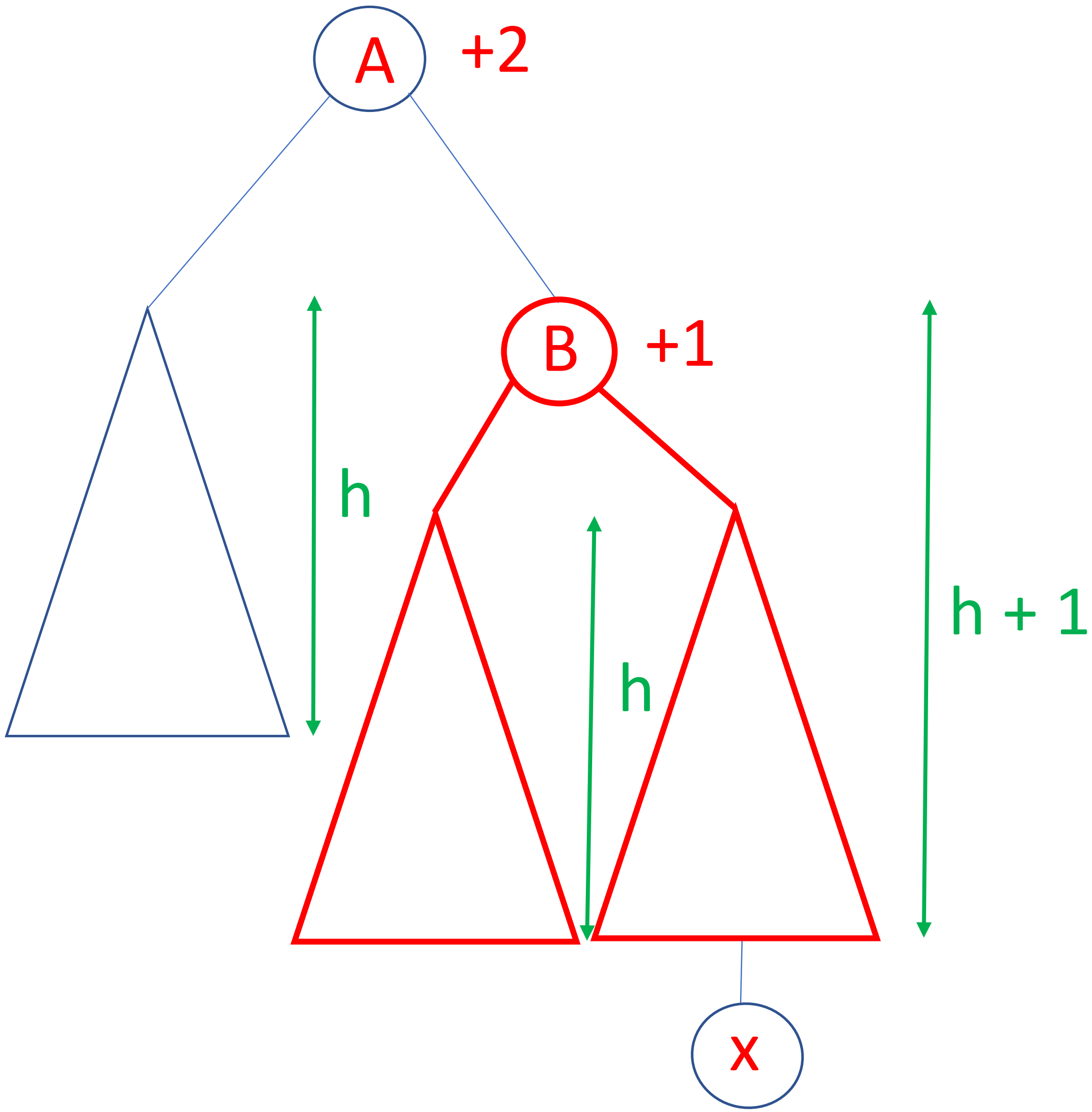


Rotation: (3) Bonus ➡ insertion algorithm is done

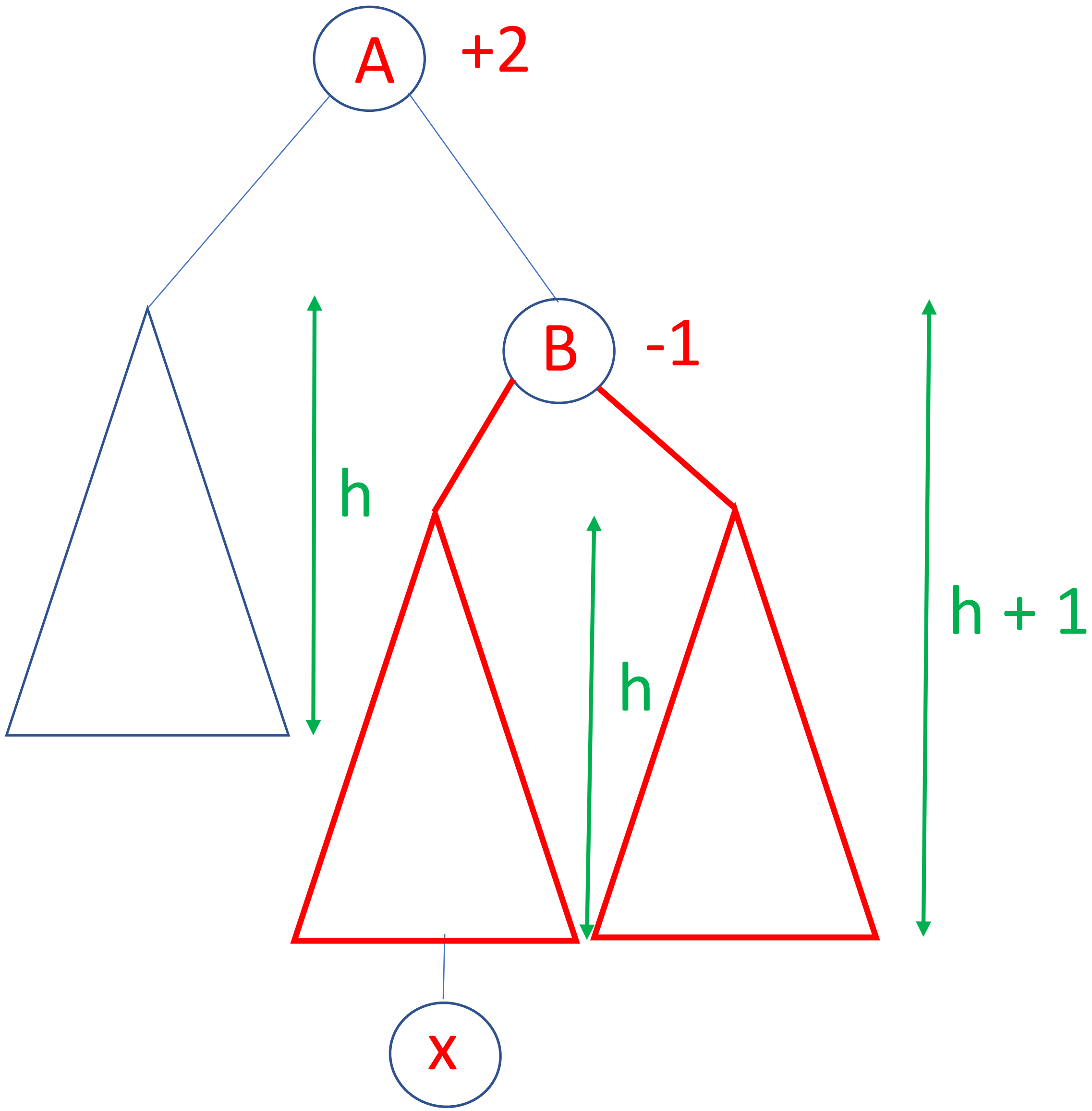
Case 1 (b) : h > -1



Case 1 (a) ✓



Case 1 (b) ✓



Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
- Go up from x to the root and for each node v in this path
  - Adjust the BF:
  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
    - x is now a leaf
    - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
    - Adjust the BF:
    - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v :

  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:
    if x is in right subtree of v : Increment BF(v)

  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v :

  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
    - x is now a leaf
    - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
    - Adjust the BF:

      if x is in right subtree of v : Increment BF(v)

      if x is in left subtree of v : Decrement BF(v)

    - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 :
  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

            Do Left Rotation, update BFs of rotated nodes, and stop

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

            Do Left Rotation, update BFs of rotated nodes, and stop

        if BF(v.right) = -1

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

      if BF(v.right) = +1

        Do Left Rotation, update BFs of rotated nodes, and stop

      if BF(v.right) = -1

        Do Right-Left Rotation, update BFs of rotated nodes, and stop

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

            Do Left Rotation, update BFs of rotated nodes, and stop

        if BF(v.right) = -1

            Do Right-Left Rotation, update BFs of rotated nodes, and stop

    if BF(v) = -2

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

            Do Left Rotation, update BFs of rotated nodes, and stop

        if BF(v.right) = -1

            Do Right-Left Rotation, update BFs of rotated nodes, and stop

    if BF(v) = -2

        Symmetric to above case

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0

O(log n)

- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)
    if x is in left subtree of v : Decrement BF(v)
    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2
        if BF(v.right) = +1
            Do Left Rotation, update BFs of rotated nodes, and stop
        if BF(v.right) = -1
            Do Right-Left Rotation, update BFs of rotated nodes, and stop
    if BF(v) = -2
        Symmetric to above case

Insert(T, x)

- Insert x into T as in any BST: $\quad$ O(log n)
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF: $\quad$ O(1)

    if x is in right subtree of v : Increment BF(v)

    if x is in left subtree of v : Decrement BF(v)

    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2

        if BF(v.right) = +1

            Do Left Rotation, update BFs of rotated nodes, and stop

        if BF(v.right) = -1

            Do Right-Left Rotation, update BFs of rotated nodes, and stop

    if BF(v) = -2

        Symmetric to above case

Insert(T, x)

- Insert x into T as in any BST:
  - x is now a leaf
  - Set BF(x) to 0

O(log n)

- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)
    if x is in left subtree of v : Decrement BF(v)
    if BF(v) = 0 : Stop

    O(1)
  - Rebalance if necessary:

    if BF(v) = + 2
        if BF(v.right) = +1
            Do Left Rotation, update BFs of rotated nodes, and stop
        if BF(v.right) = -1
            Do Right-Left Rotation, update BFs of rotated nodes, and stop
    if BF(v) = -2
        Symmetric to above case

    O(1)

Insert(T, x)

- Insert x into T as in any BST: O(log n)
  - x is now a leaf
  - Set BF(x) to 0
- Go up from x to the root and for each node v in this path
  - Adjust the BF:

    if x is in right subtree of v : Increment BF(v)
    if x is in left subtree of v : Decrement BF(v) O(1)
    if BF(v) = 0 : Stop
  - Rebalance if necessary:

    if BF(v) = + 2
        if BF(v.right) = +1 O(log n)
            Do Left Rotation, update BFs of rotated nodes, and stop
        if BF(v.right) = -1 O(1)
            Do Right-Left Rotation, update BFs of rotated nodes, and stop
    if BF(v) = -2
        Symmetric to above case

Delete(T, x)

- Same general idea as **Insert**(T, x):
    - Delete x, go up to adjust BFs and rebalance (with single or double rotations) if necessary

Delete(T, x)

- Same general idea as **Insert**(T, x):
  - Delete x, go up to adjust BFs and rebalance (with single or double rotations) if necessary
- Main difference: No bonus!
  - After a rebalancing, height of the rebalanced subtree might be smaller than before

  $\Rightarrow$ Might have to do $\Theta(\log n)$ rotations. This is still OK because each one takes constant time!

Delete(T, x)

- Same general idea as **Insert**(T, x):
  - Delete x, go up to adjust BFs and rebalance (with single or double rotations) if necessary
- Main difference: No bonus!
  - After a rebalancing, height of the rebalanced subtree might be smaller than before

  $\Rightarrow$ Might have to do $\Theta(\log n)$ rotations. This is still OK because each one takes constant time!
- To see how **Delete**(T, x) works:
  - Read the AVL notes posted on the course website
  - Go to the tutorial this week!