Balanced BSTs

Abstract Data Type	Operations	Data Structures
Dictionary		

Abstract Data Type	Operations	Data Structures
Dictionary	**Search, Insert, Delete**	

Abstract Data Type	Operations	Data Structures
Dictionary	**Search, Insert, Delete**	BSTs

Recall the problem with BSTs…

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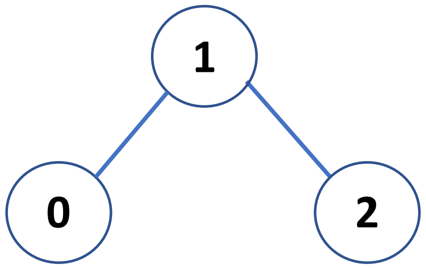
Suppose we start with
a “balanced” BST



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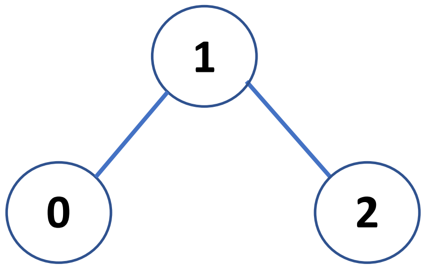
Suppose we start with a “balanced” BST

Insert keys 3, 4, 5, ..., n



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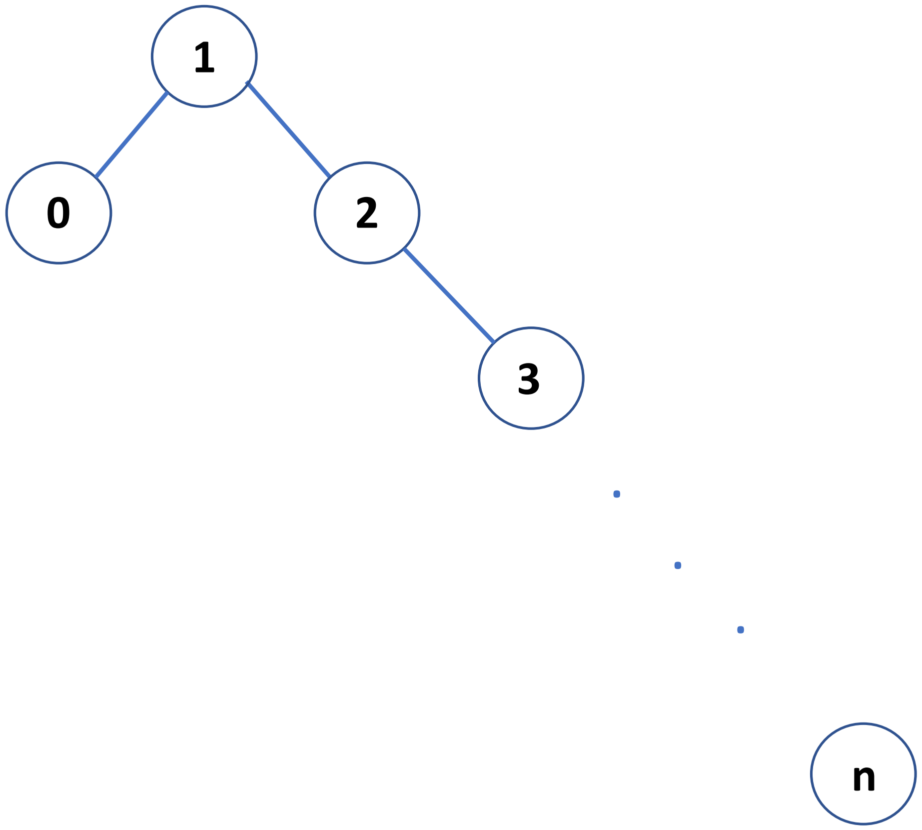

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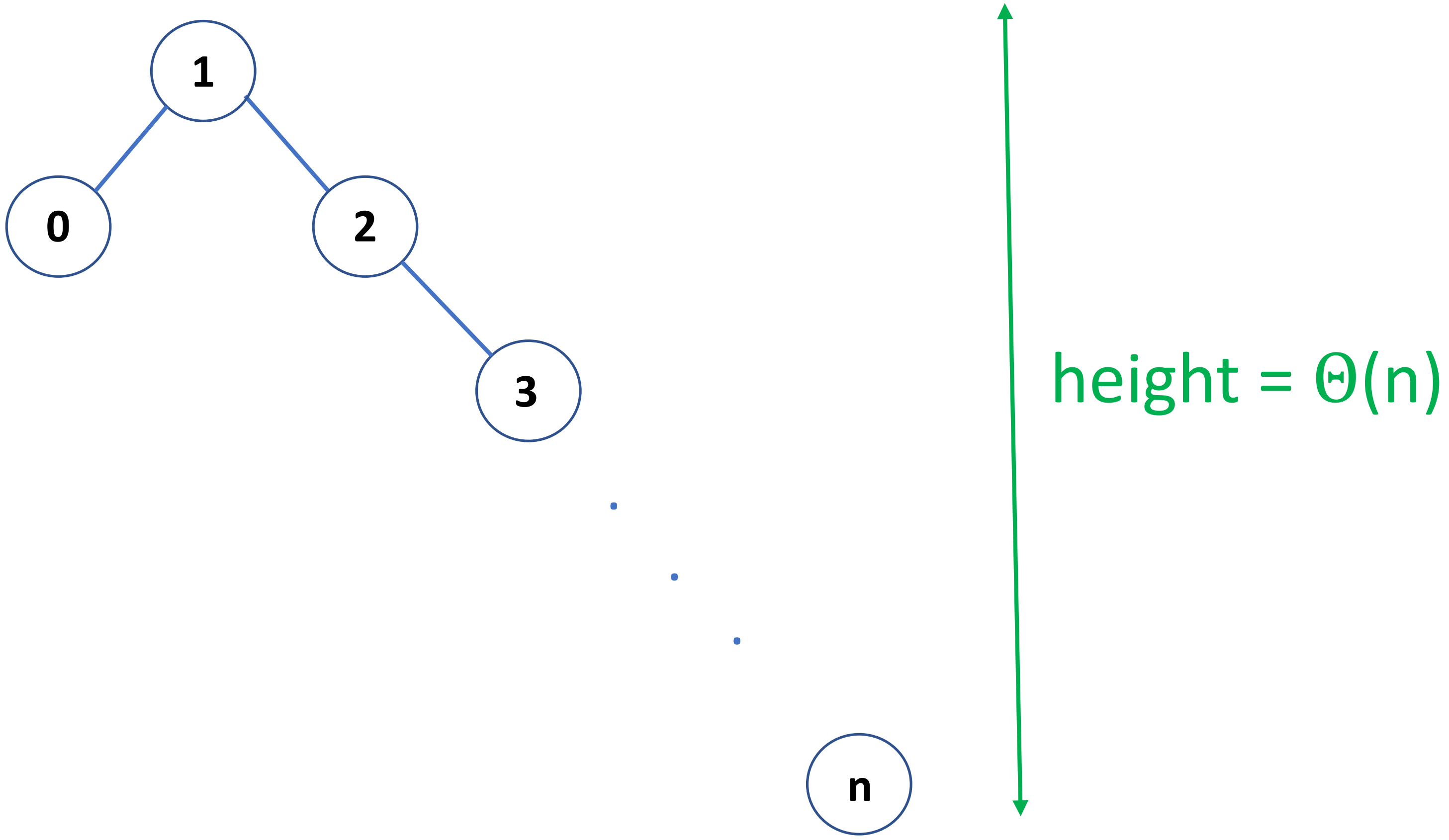

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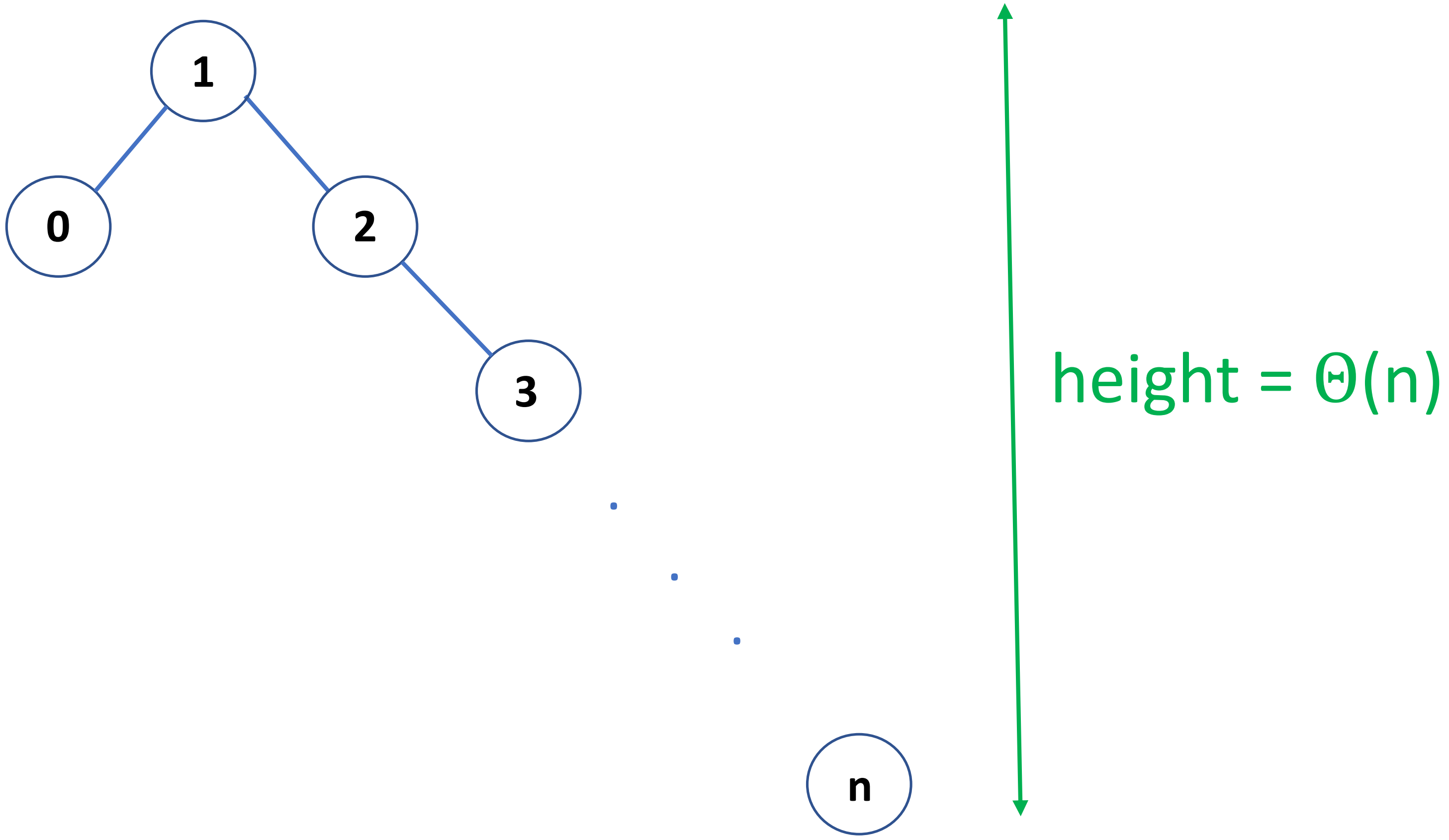
Search takes Θ(n) !



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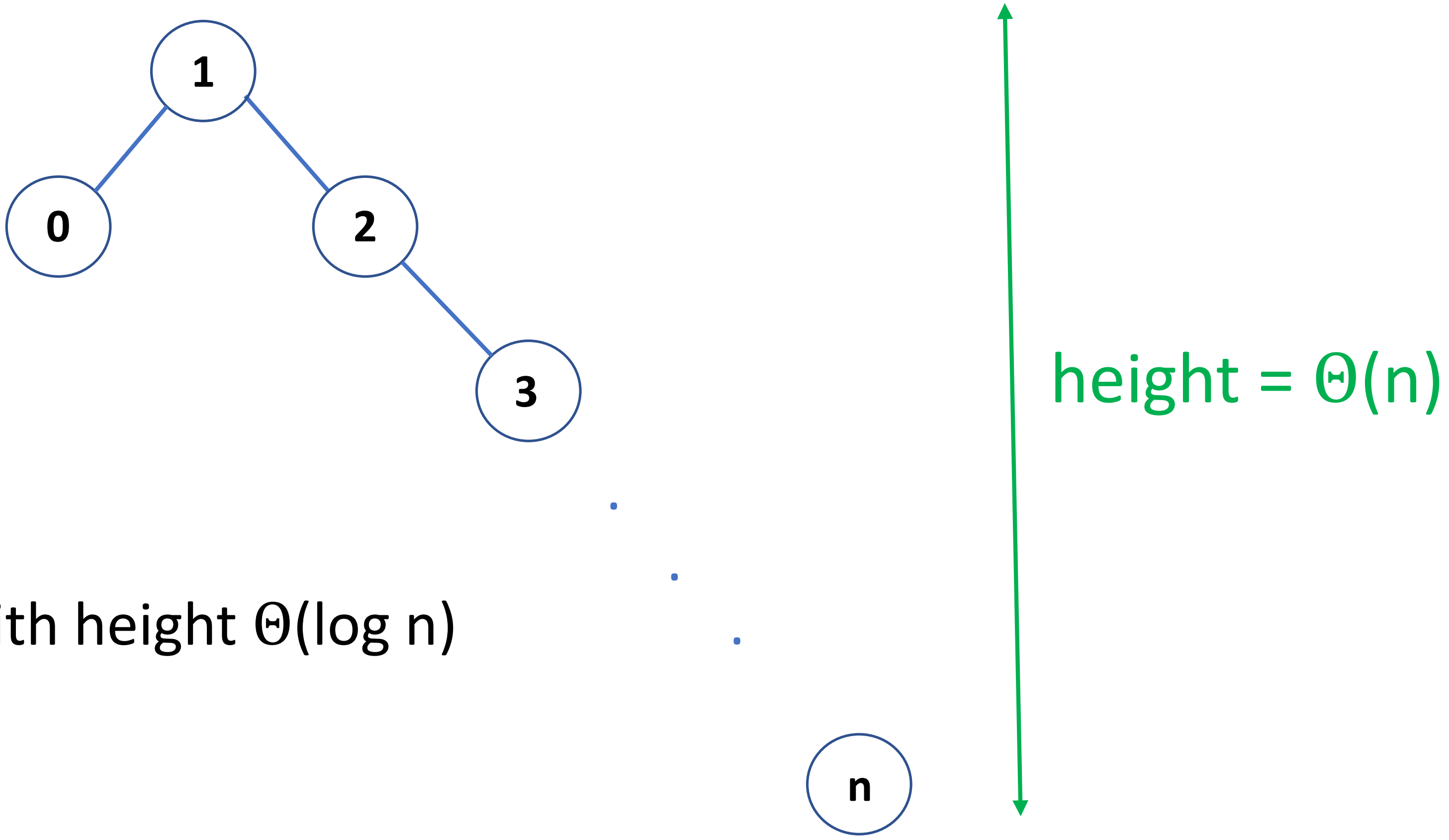
Intuitively, we want BST trees with height Θ(log n)



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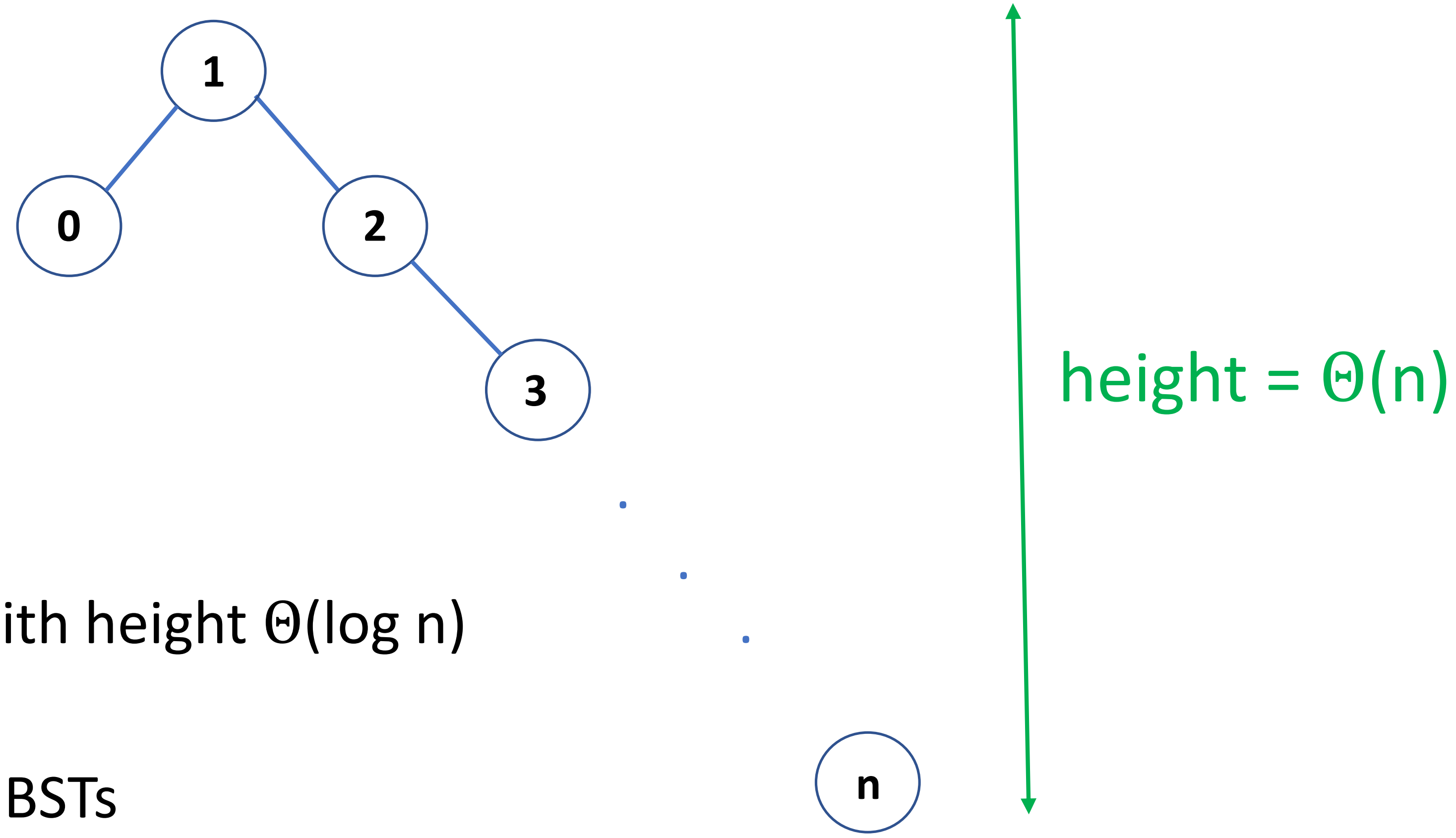
We achieve this using balanced BSTs



Abstract Data Type	Operations	Data Structures
Dictionary	**Search, Insert, Delete**	BSTs Balanced BSTs

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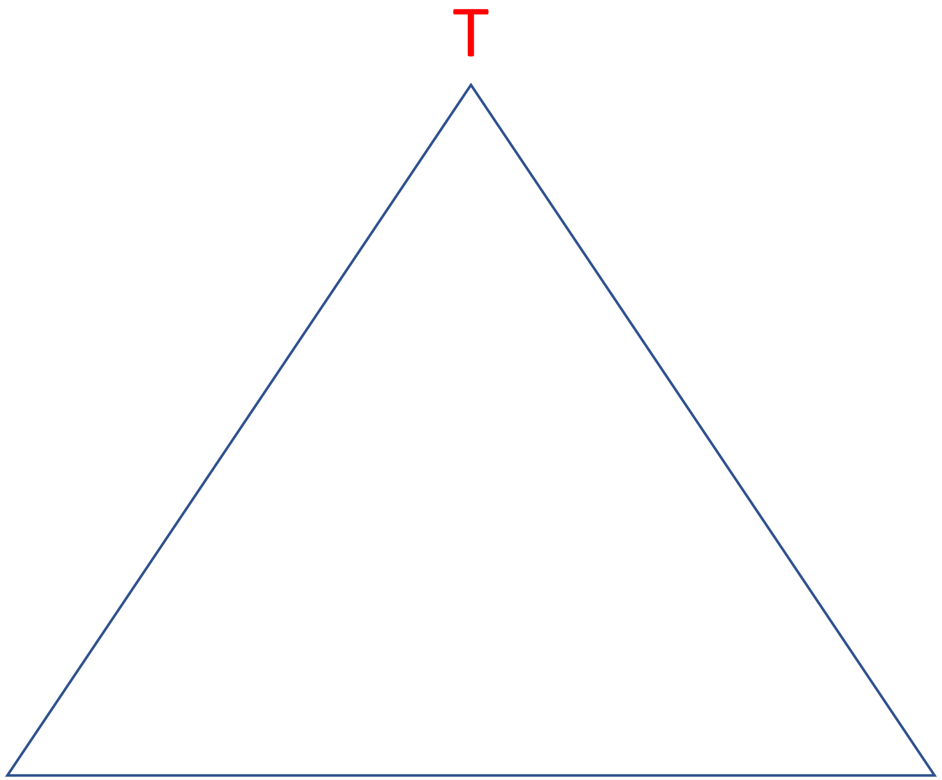
AVL Trees

Reminder : Height of a Node



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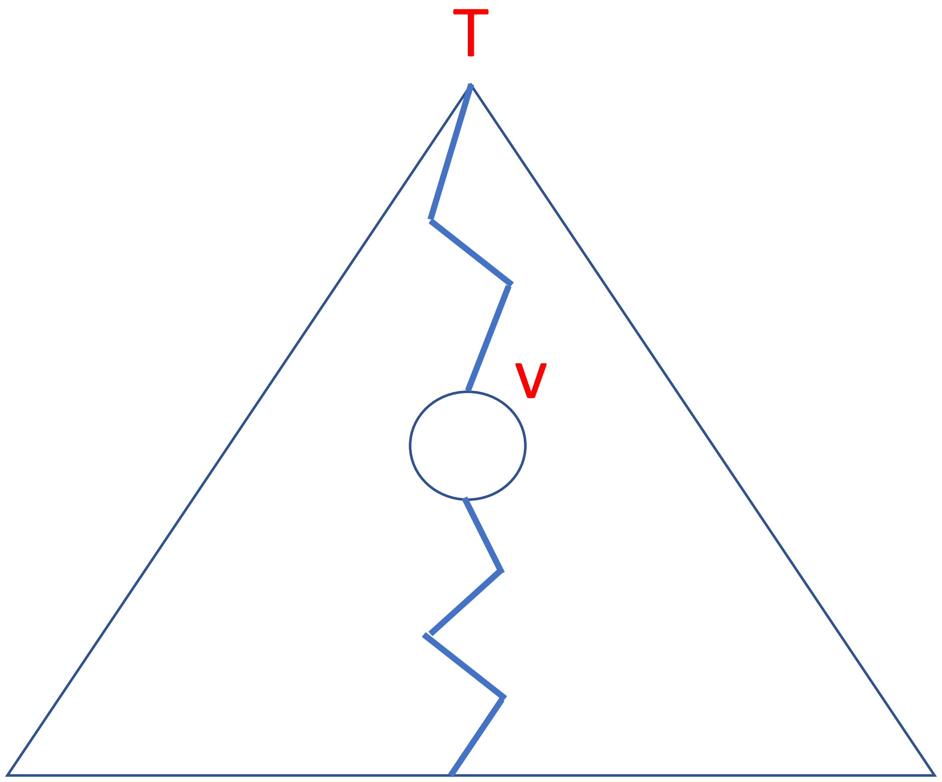
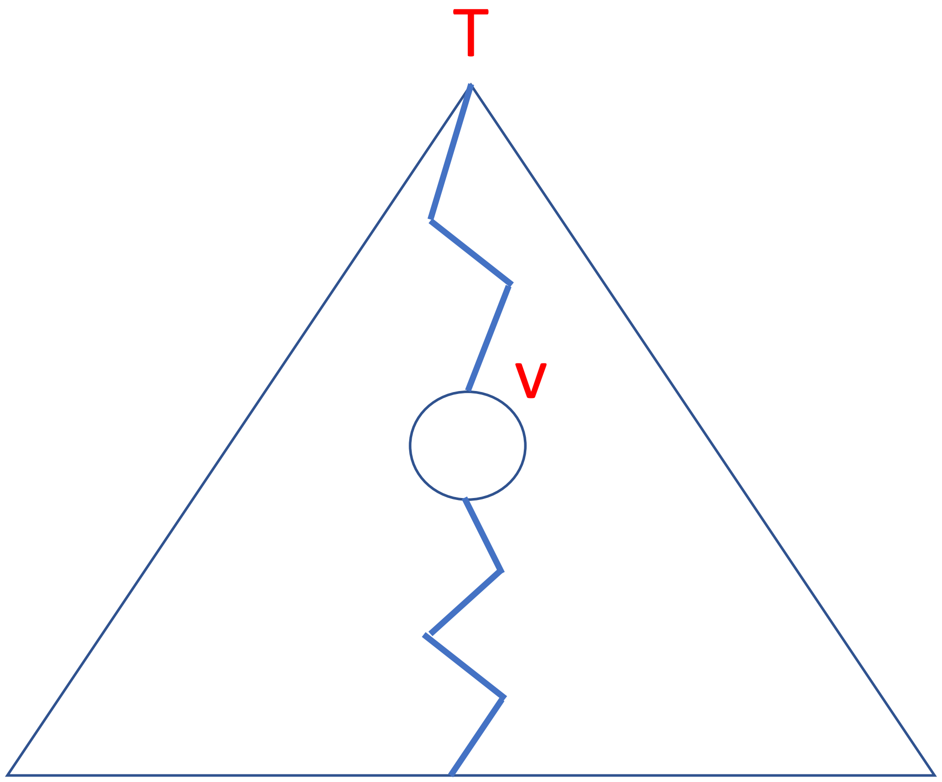


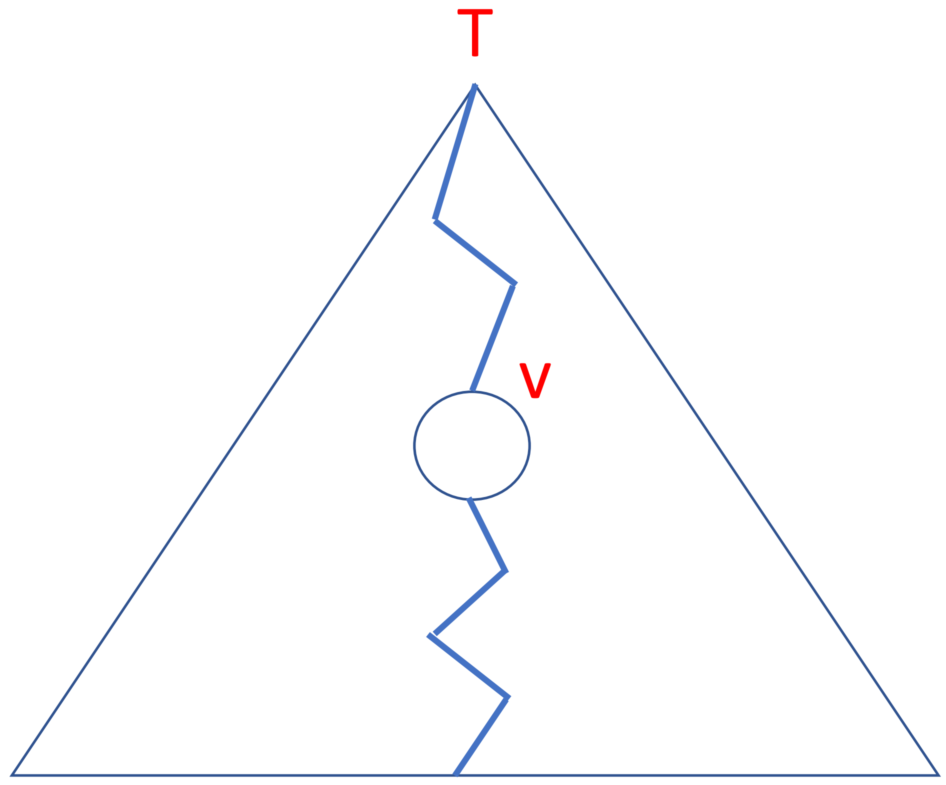
Reminder : Height of a Node

- height(v) :



Reminder : Height of a Node

- height(v) : Number of edges in the longest path from v to a leaf



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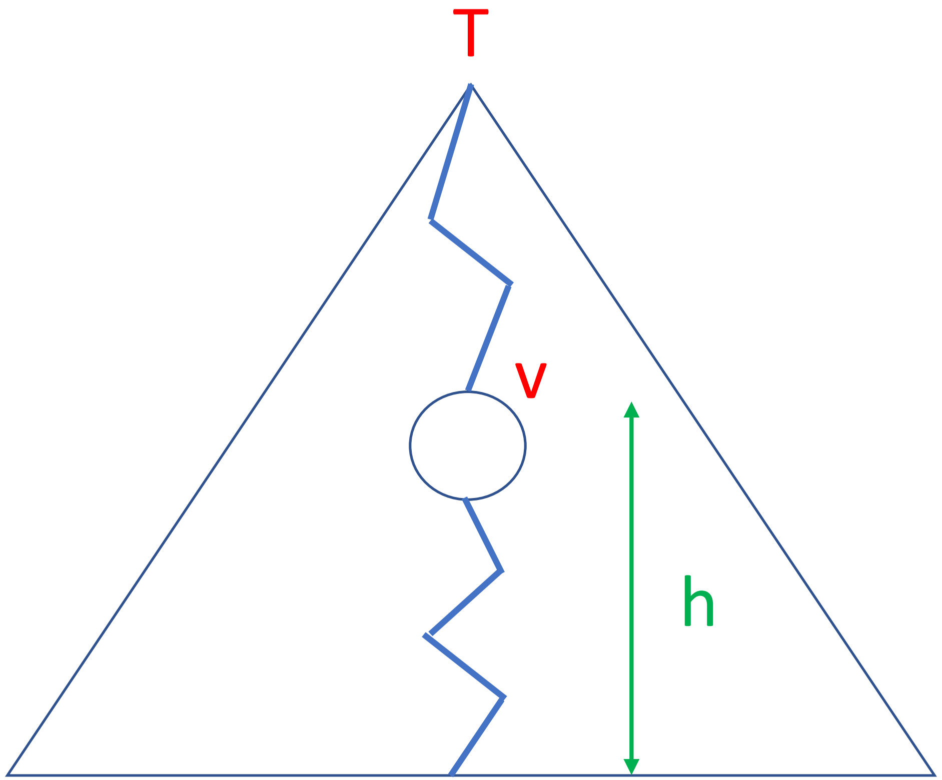

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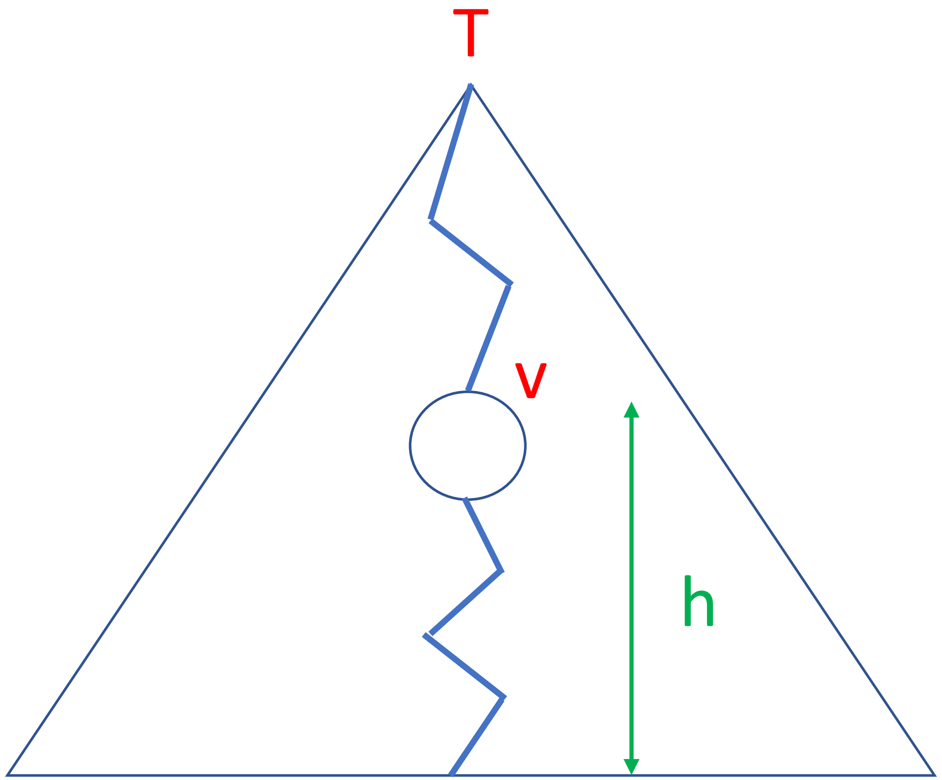
- height(v) : Number of edges in the longest path from v to a leaf

- height(T): height of the root node of T



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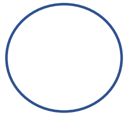
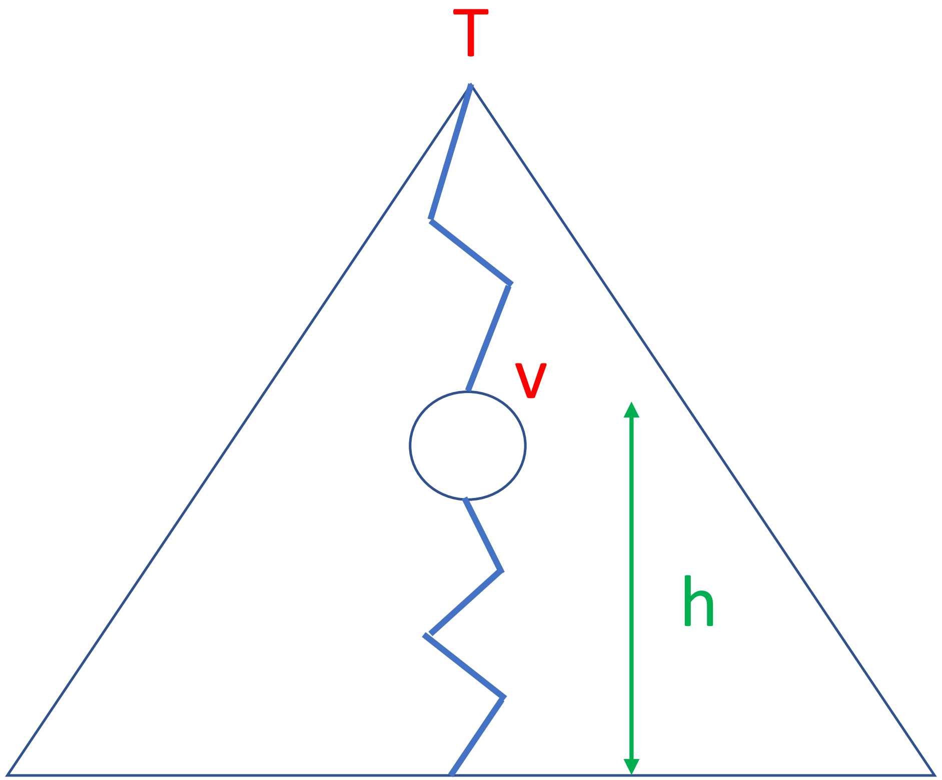
height() = 0



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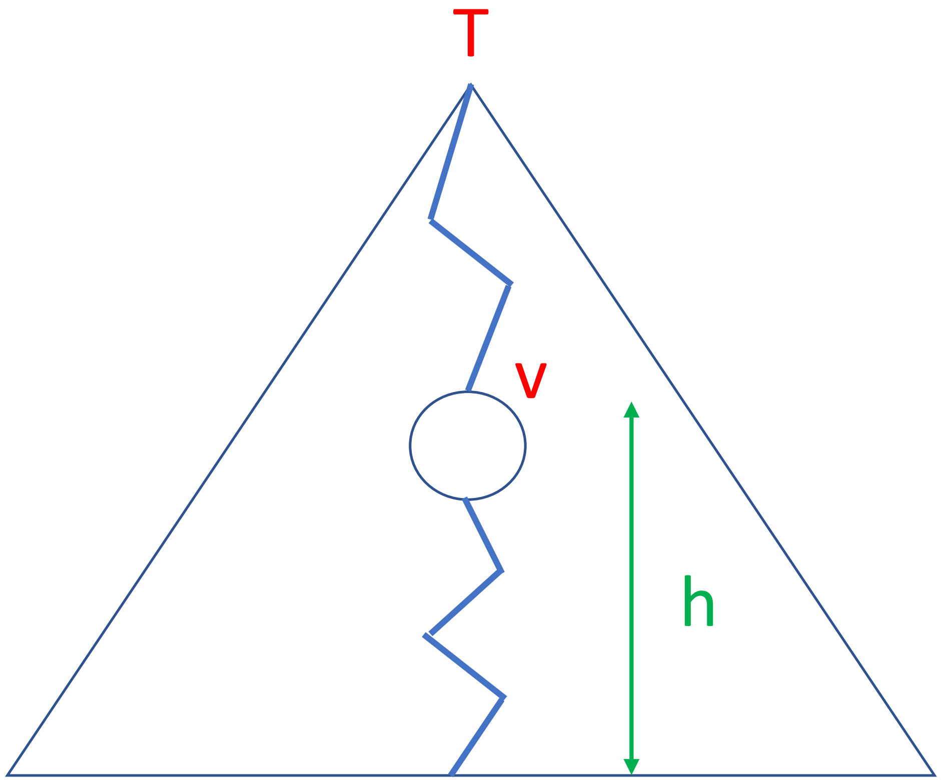
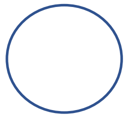
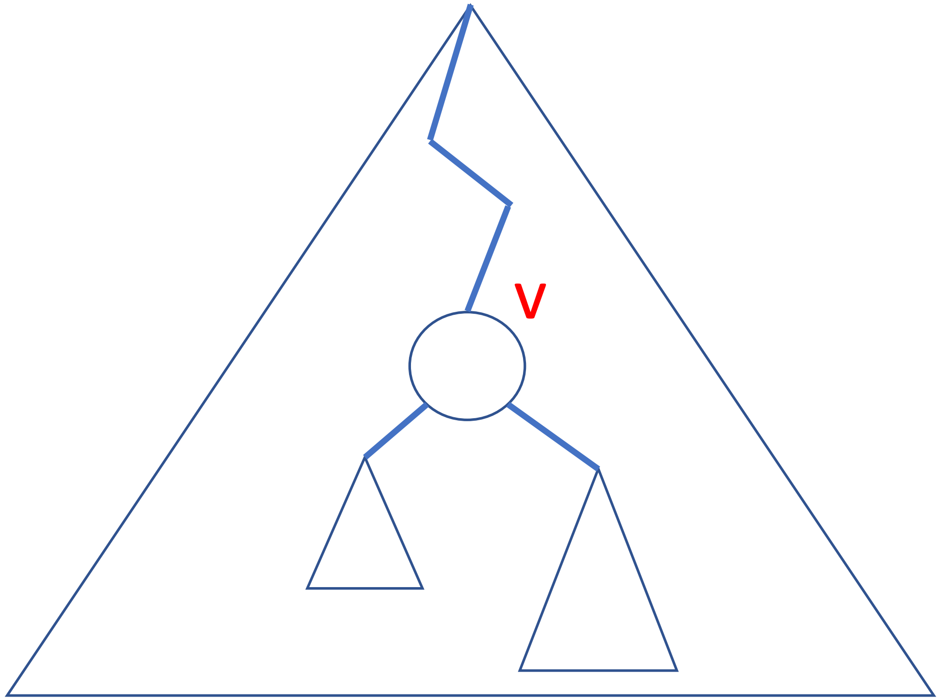


height() = 0

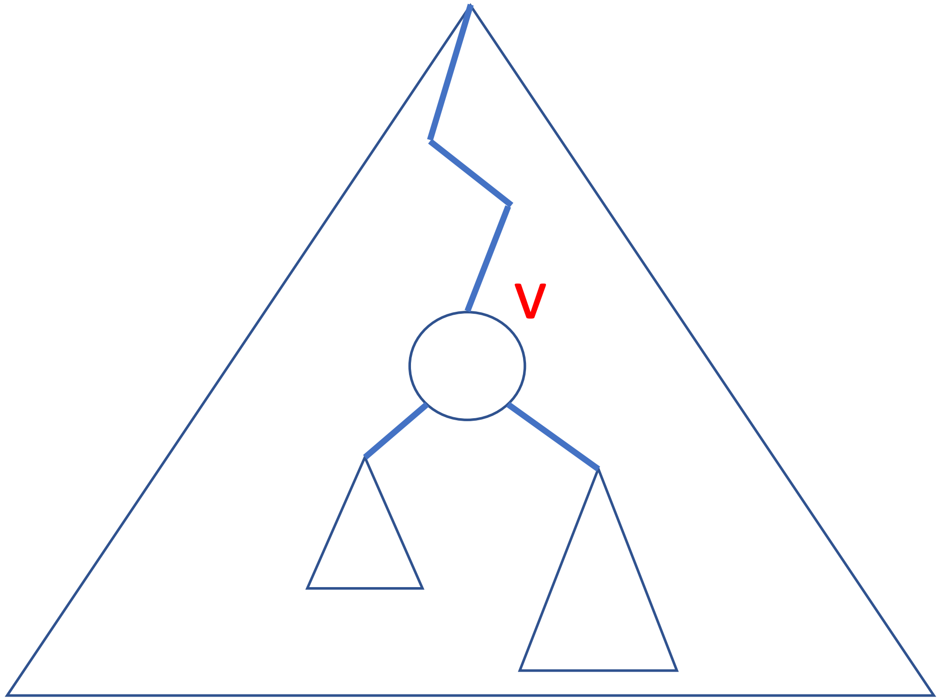
height(empty tree) = -1

Balance Factor (BF) of a node v



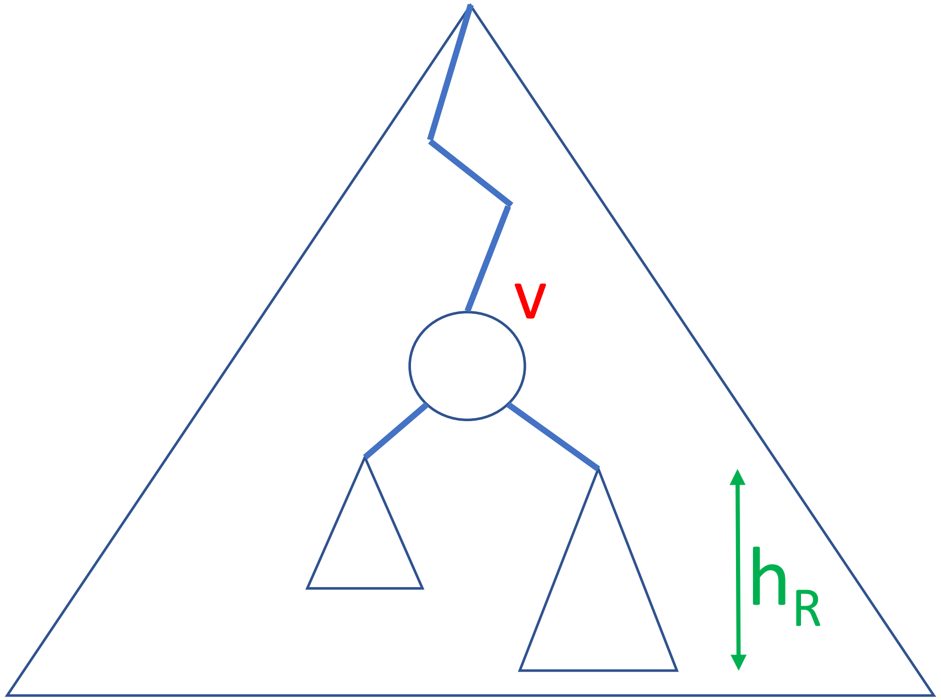
Balance Factor (BF) of a node v

BF(v) =



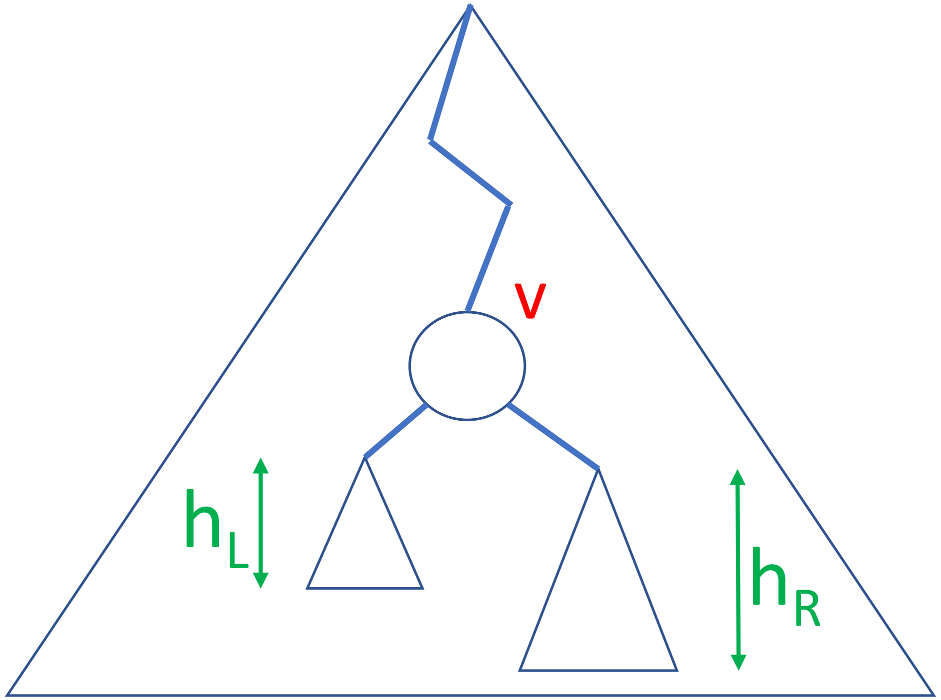
Balance Factor (BF) of a node v

BF(v) = height(right subtree of v)



Balance Factor (BF) of a node v

BF(v) = height(right subtree of v) – height(left subtree of v)



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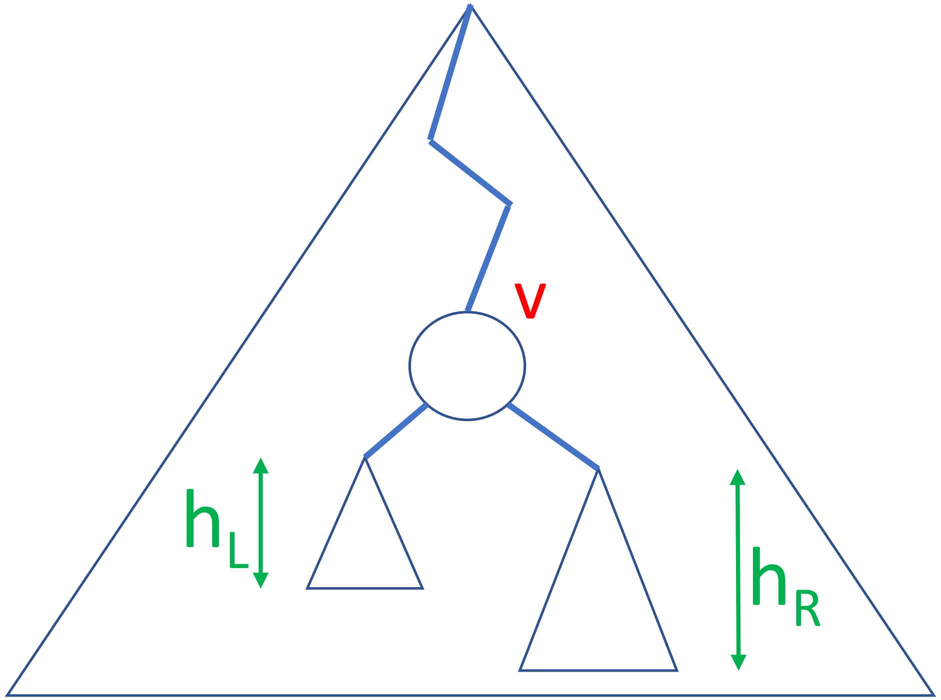
BF(v) = $h_R - h_L$

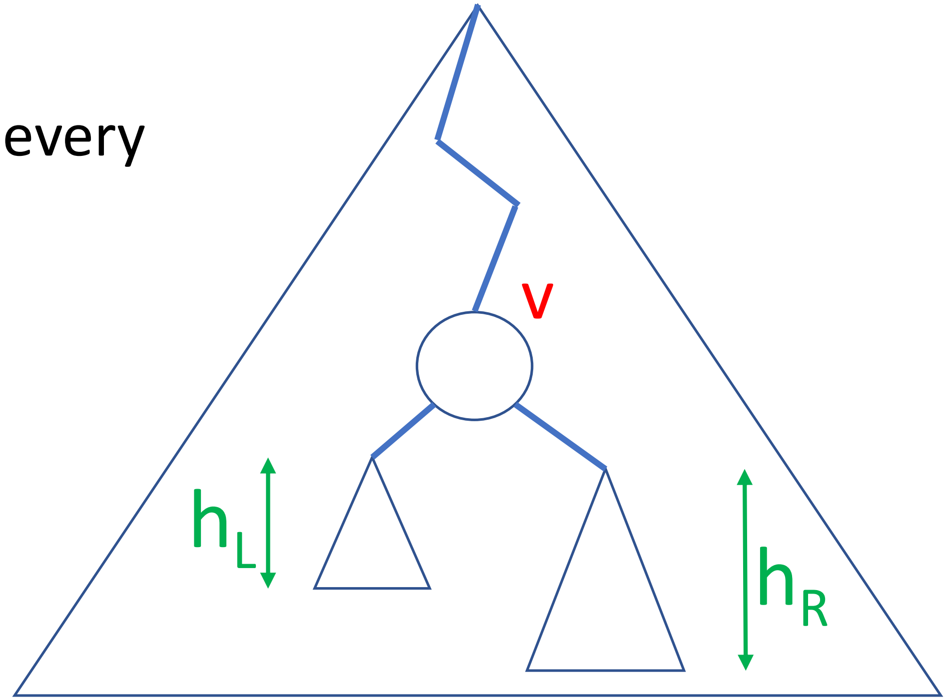


Balance Factor (BF) of a node v

BF(v) = height(right subtree of v) – height(left subtree of v)

$BF(v) = h_R - h_L$

We would like the BF of every node to be close to 0



Adelson-Velski-Landis Trees

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An AVL tree T is a BST where for every node v ∈ T :

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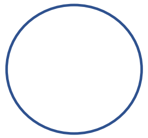
Is this an AVL tree?



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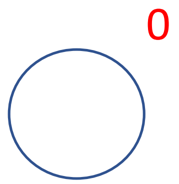

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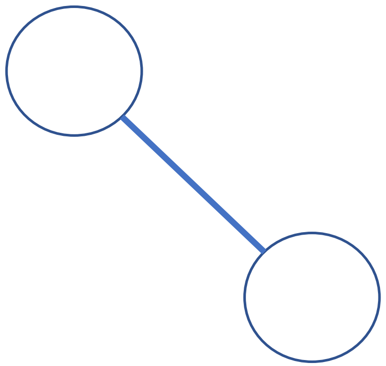

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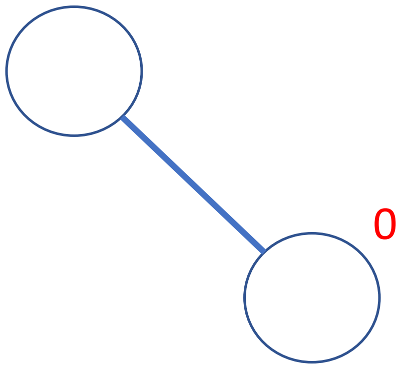

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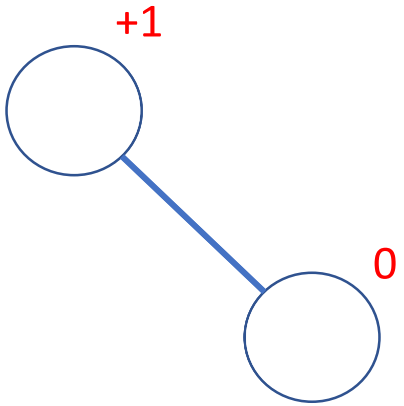

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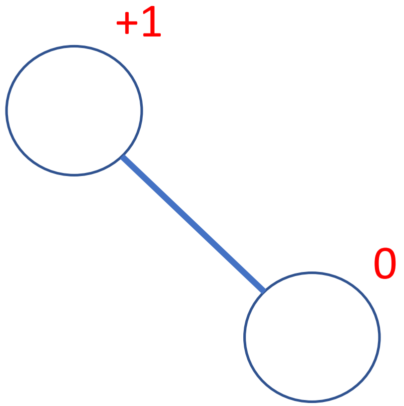

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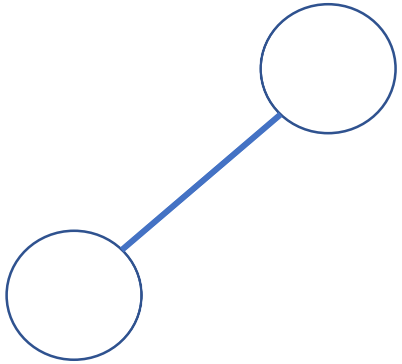

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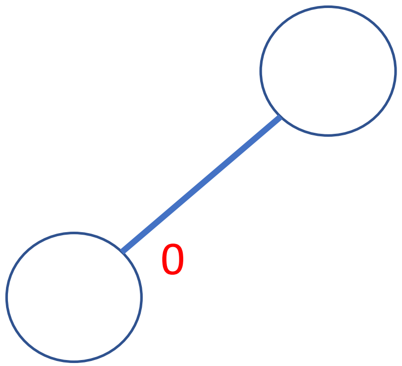

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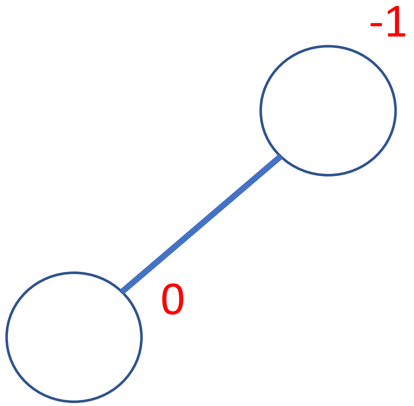

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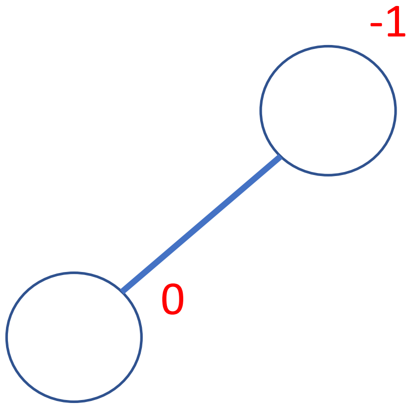

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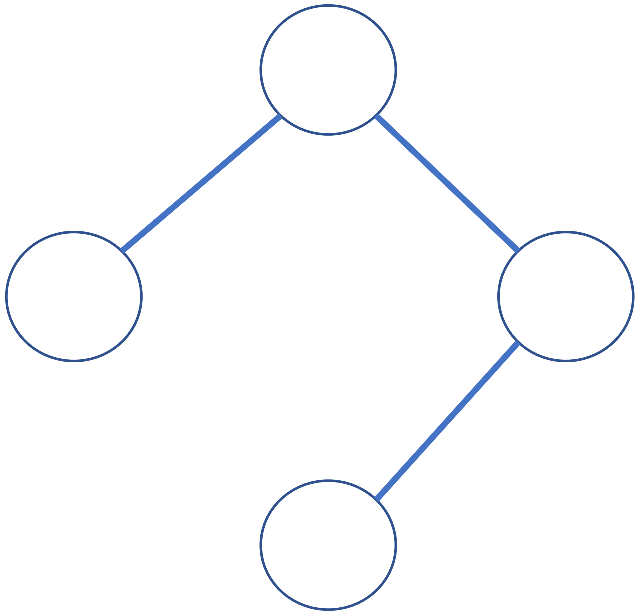

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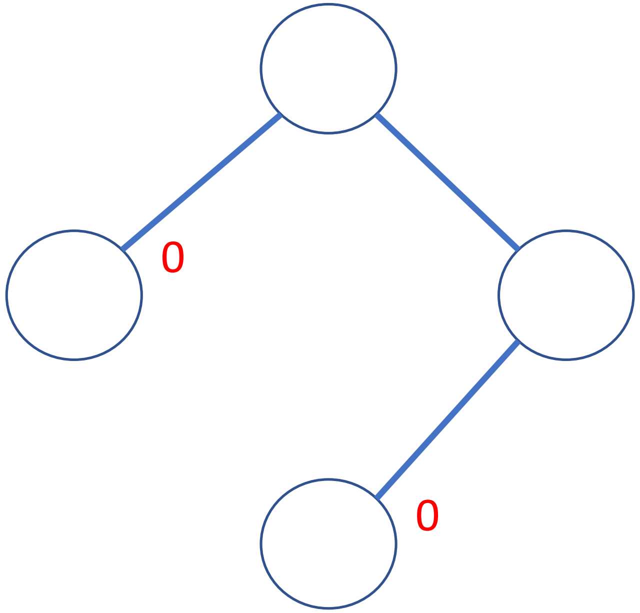

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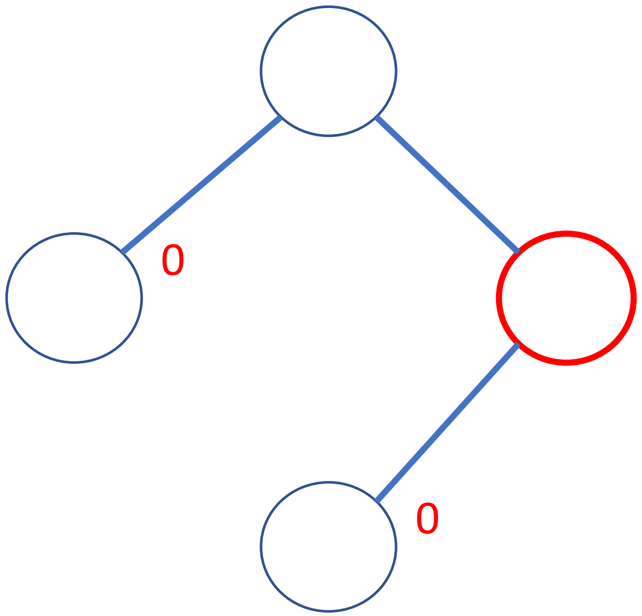

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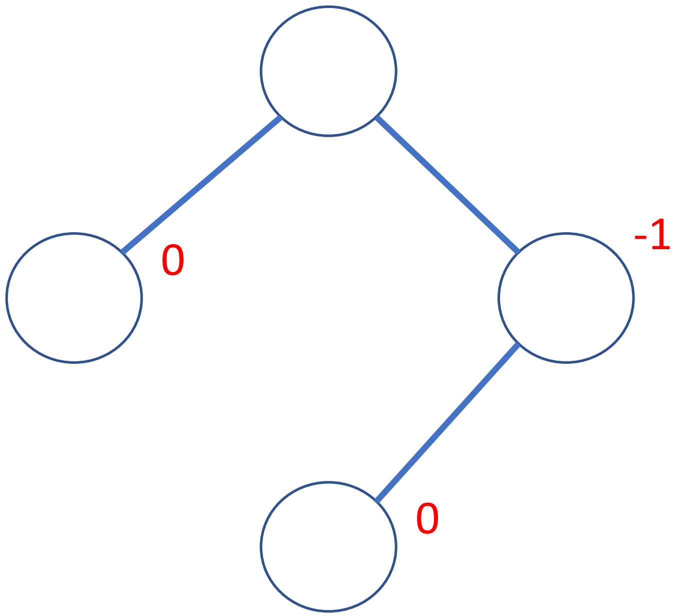

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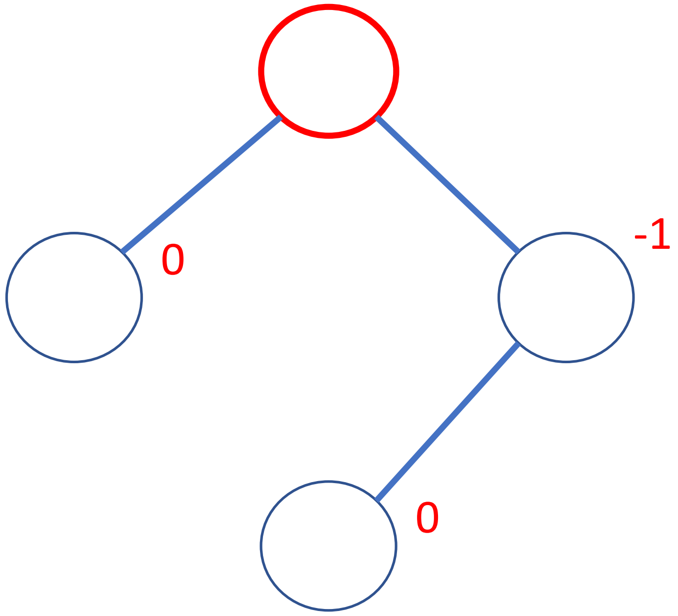

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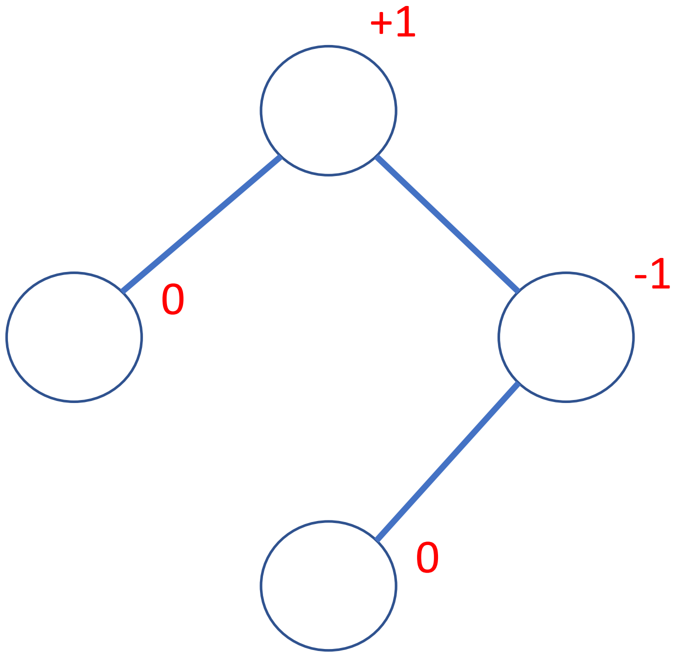

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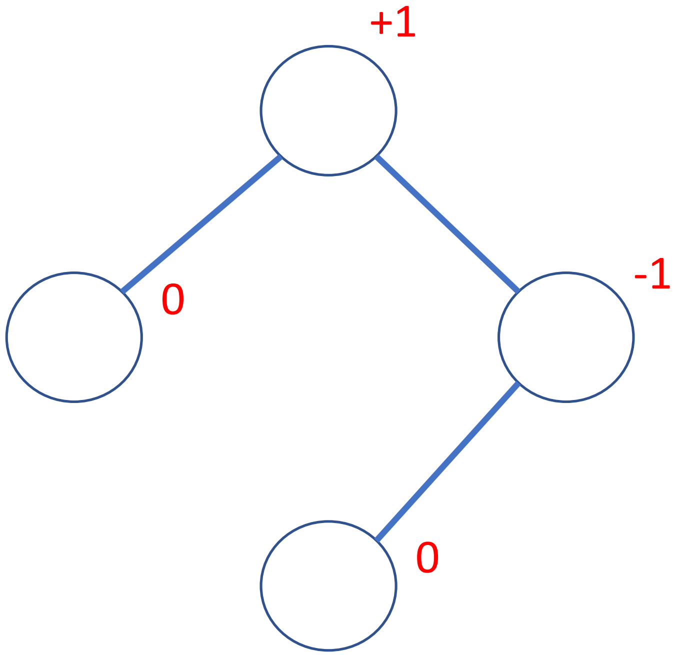

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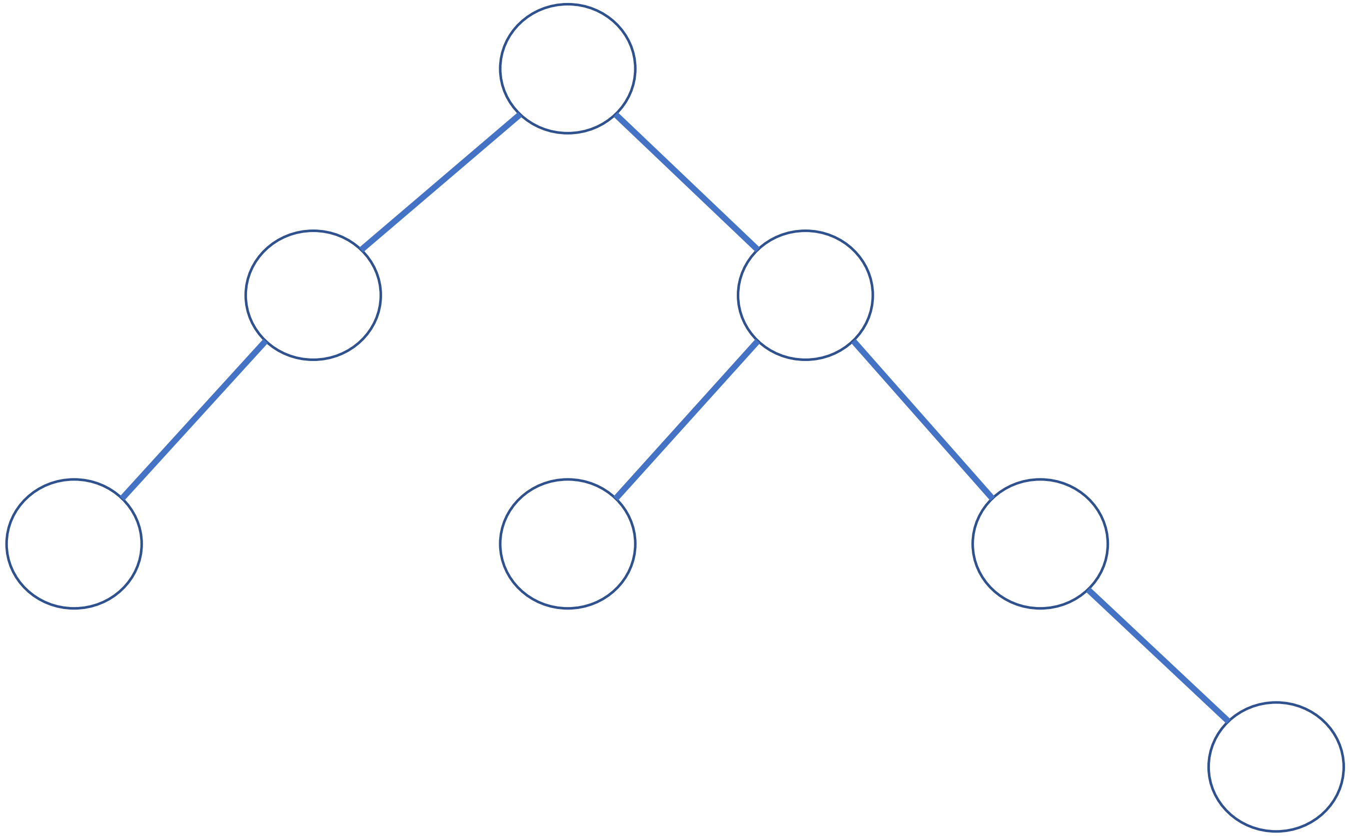

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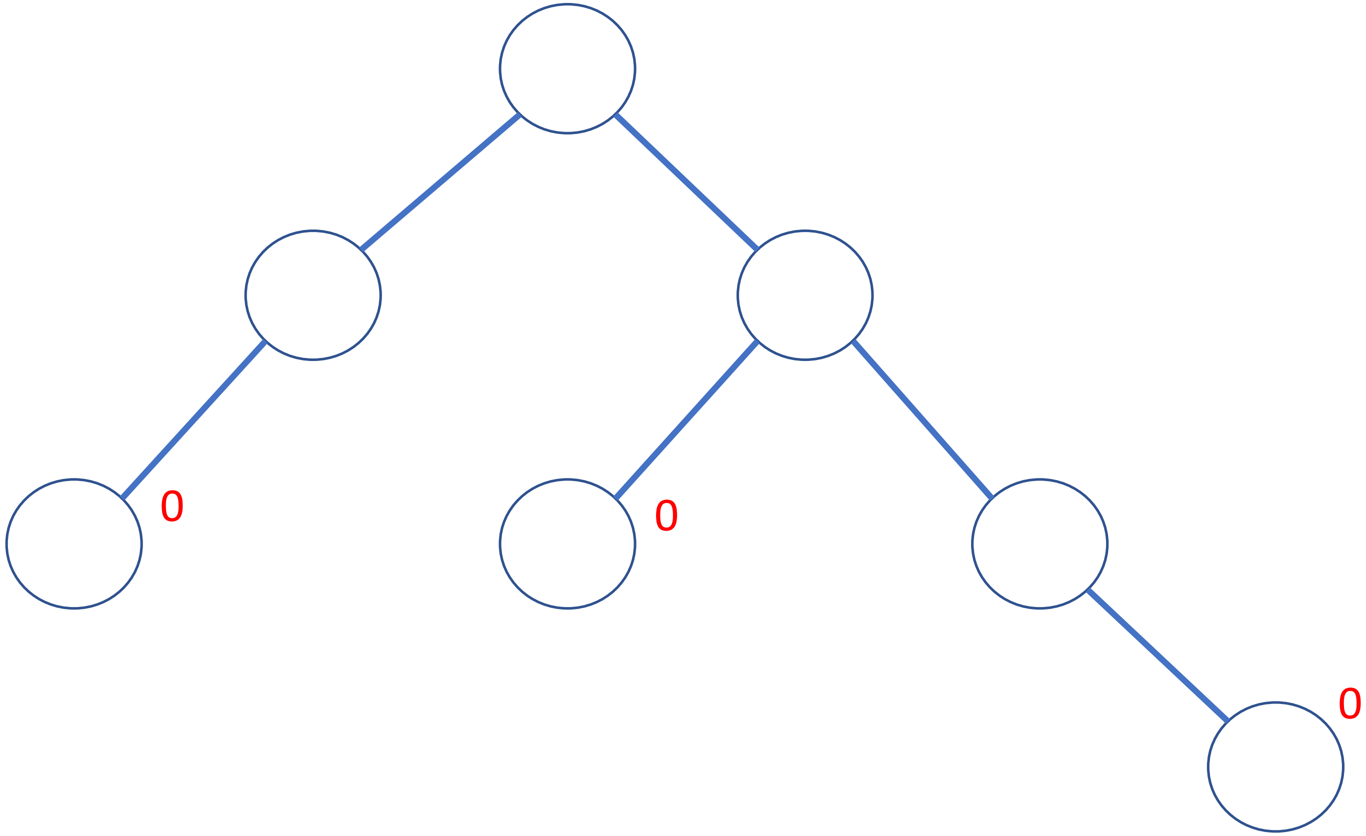

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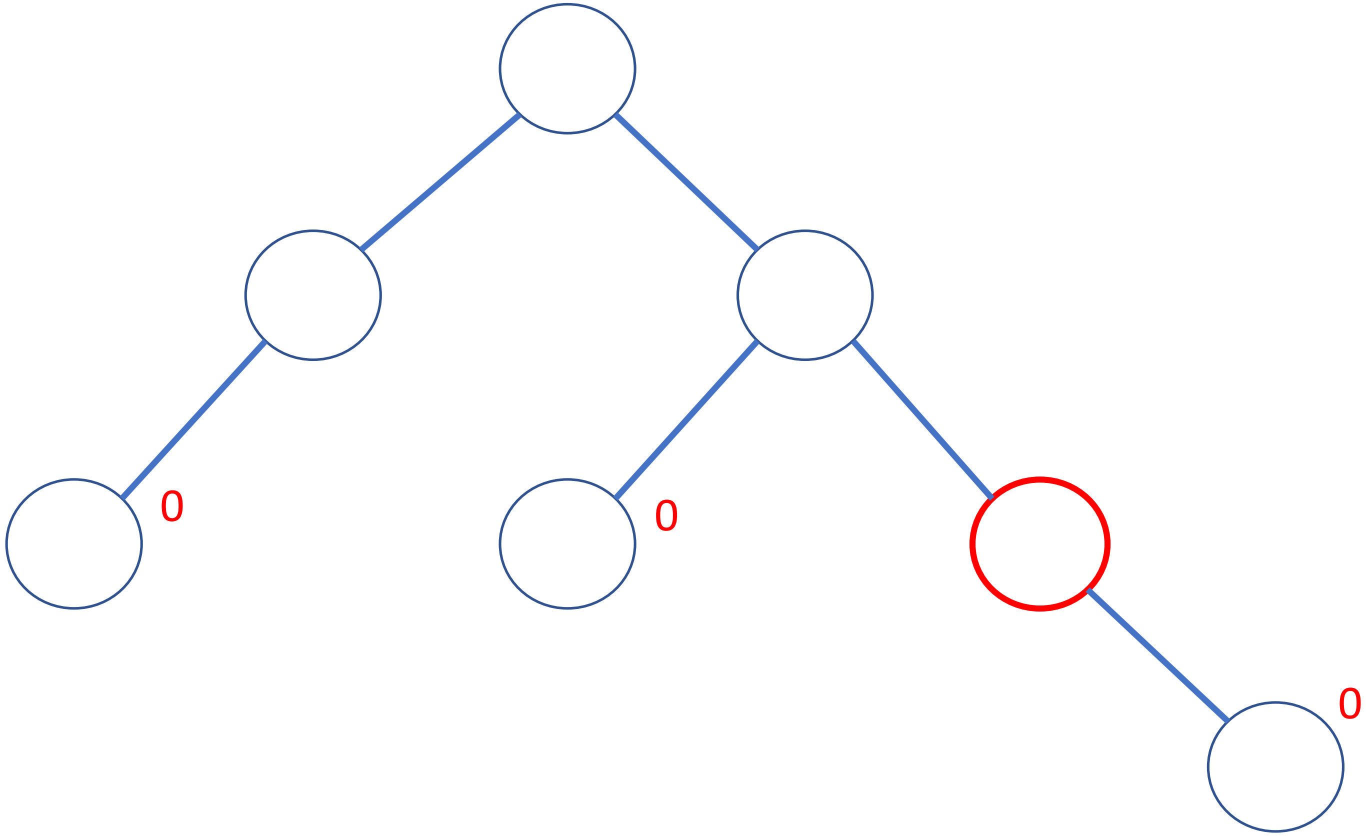

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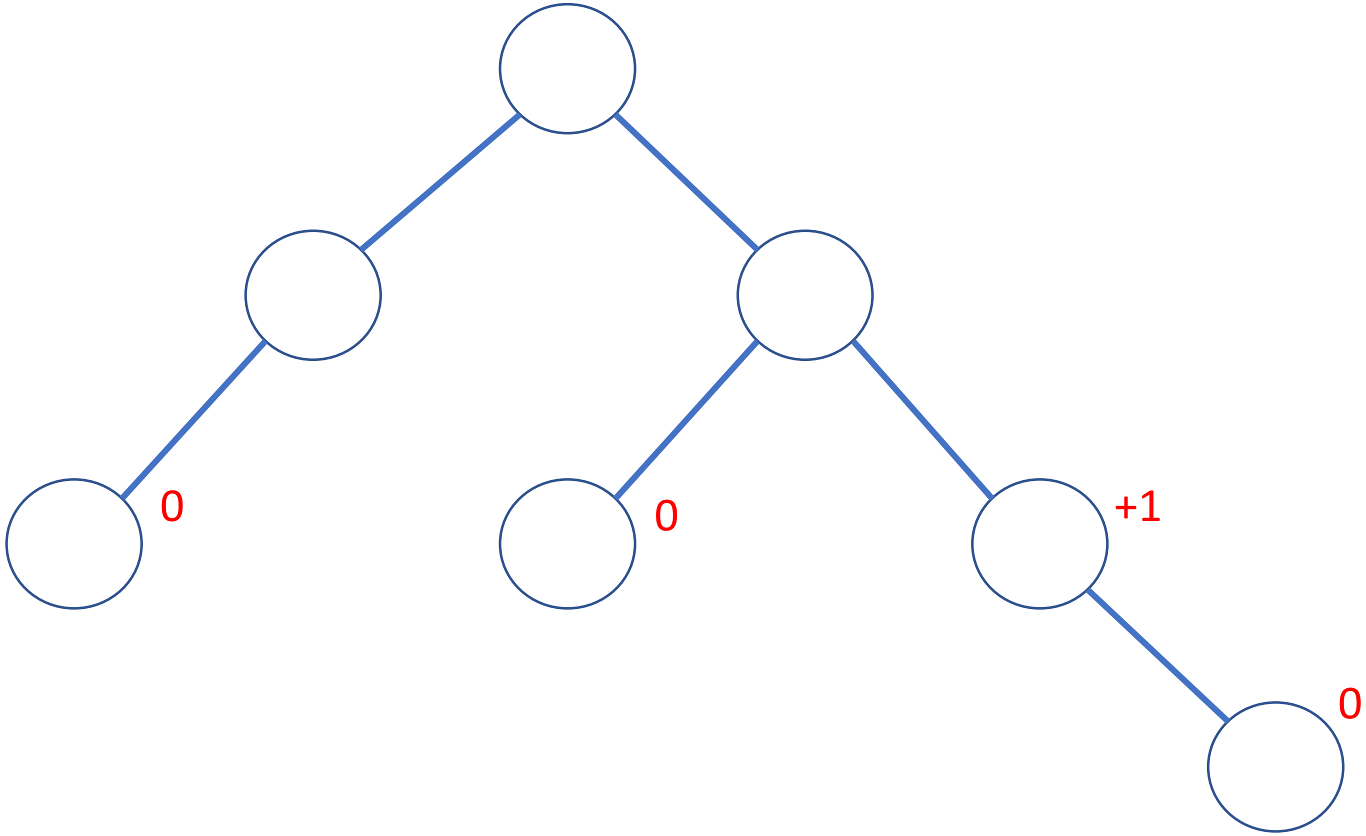

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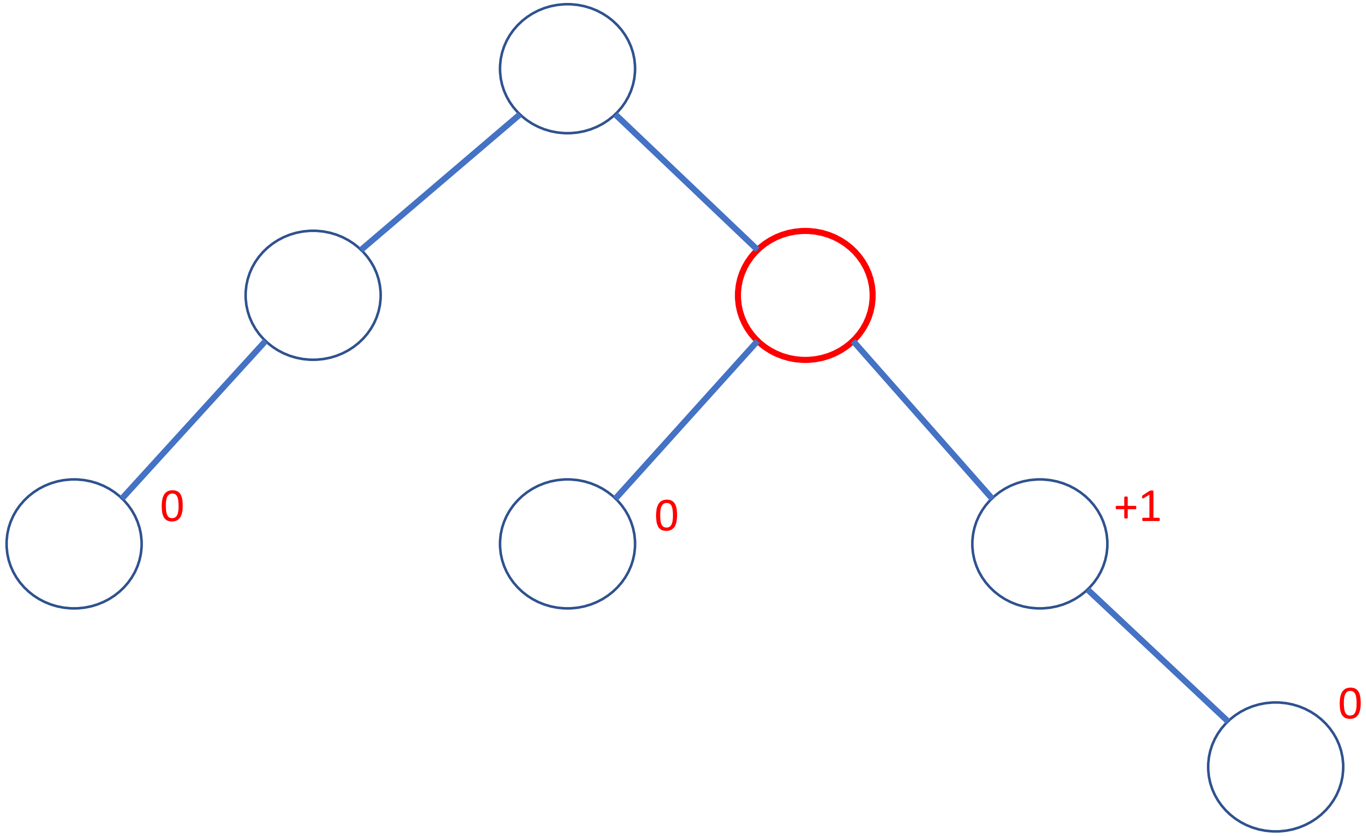

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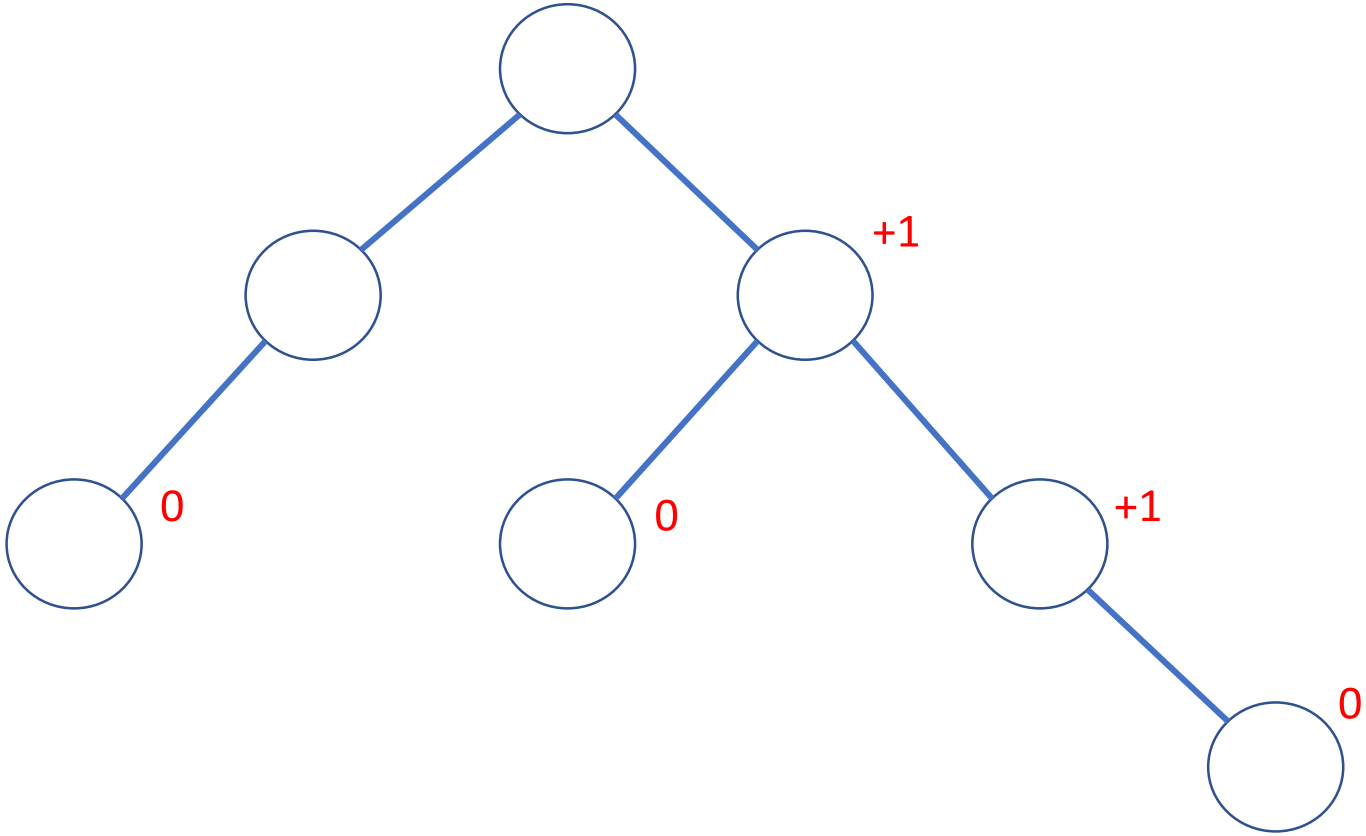

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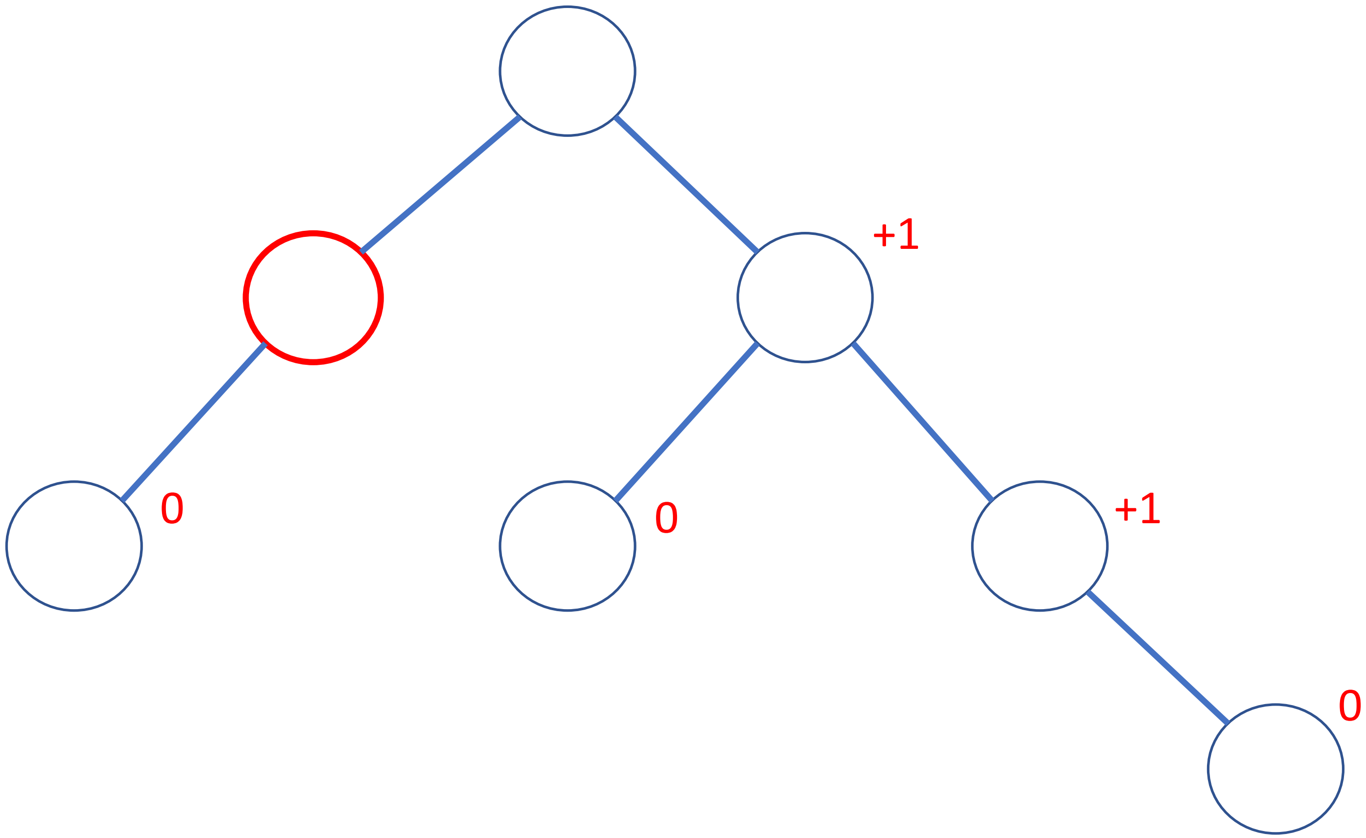

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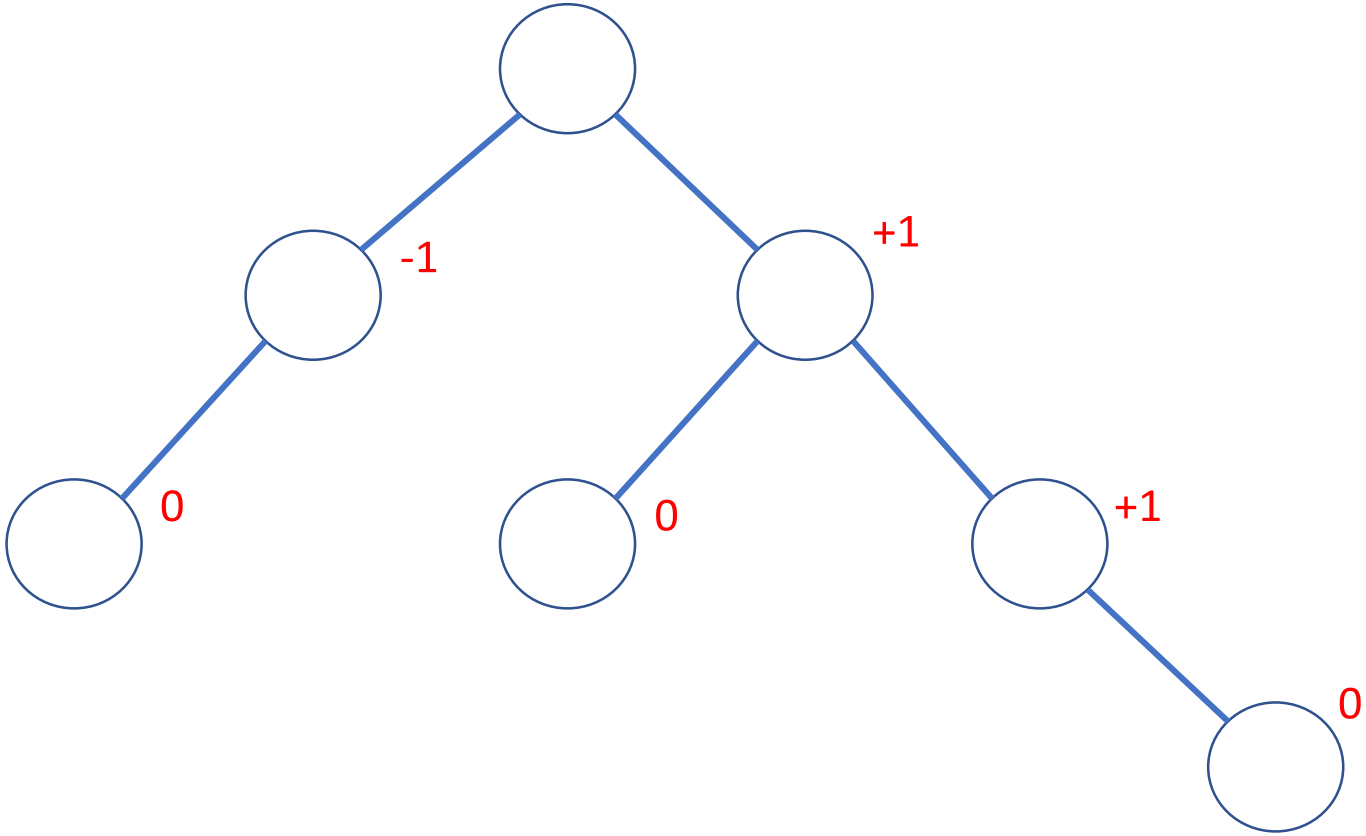

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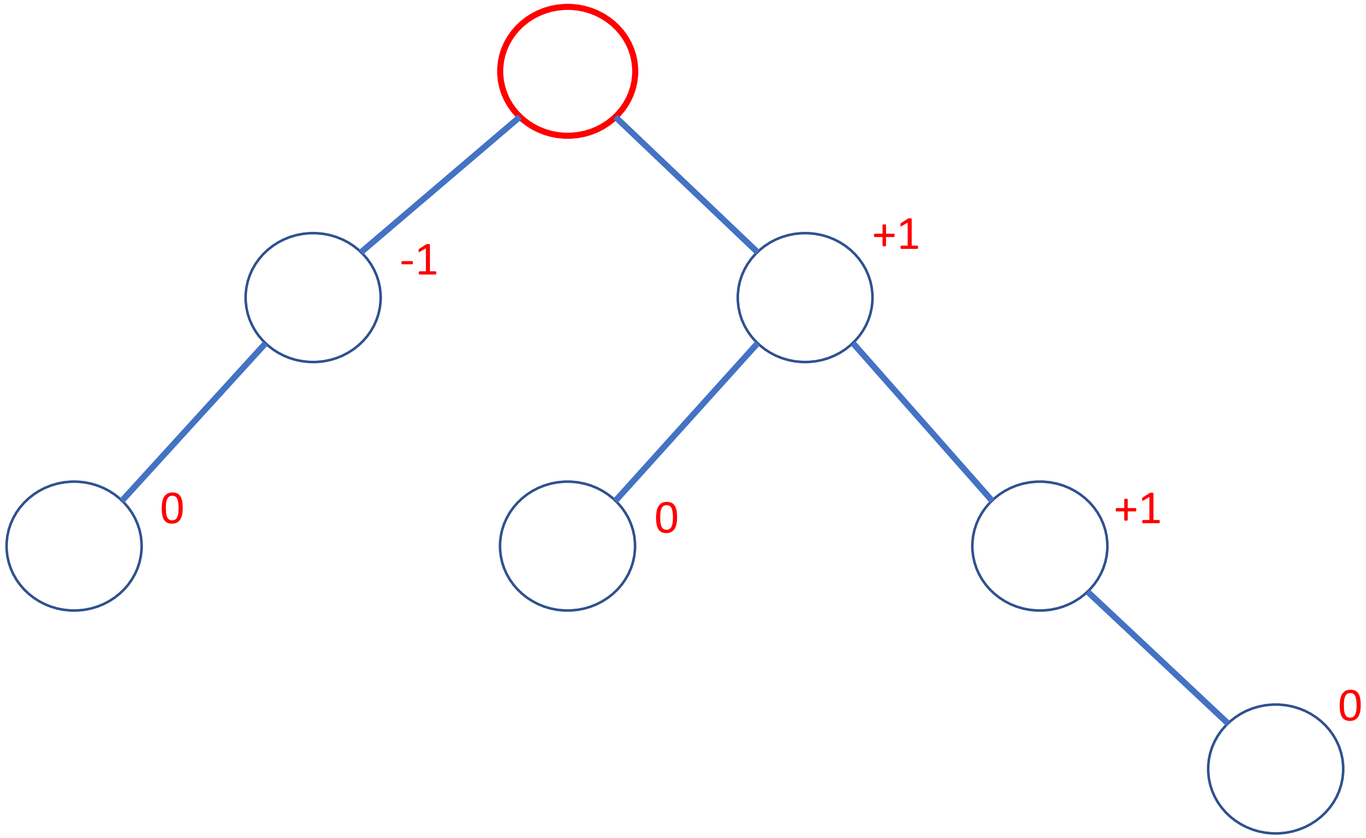

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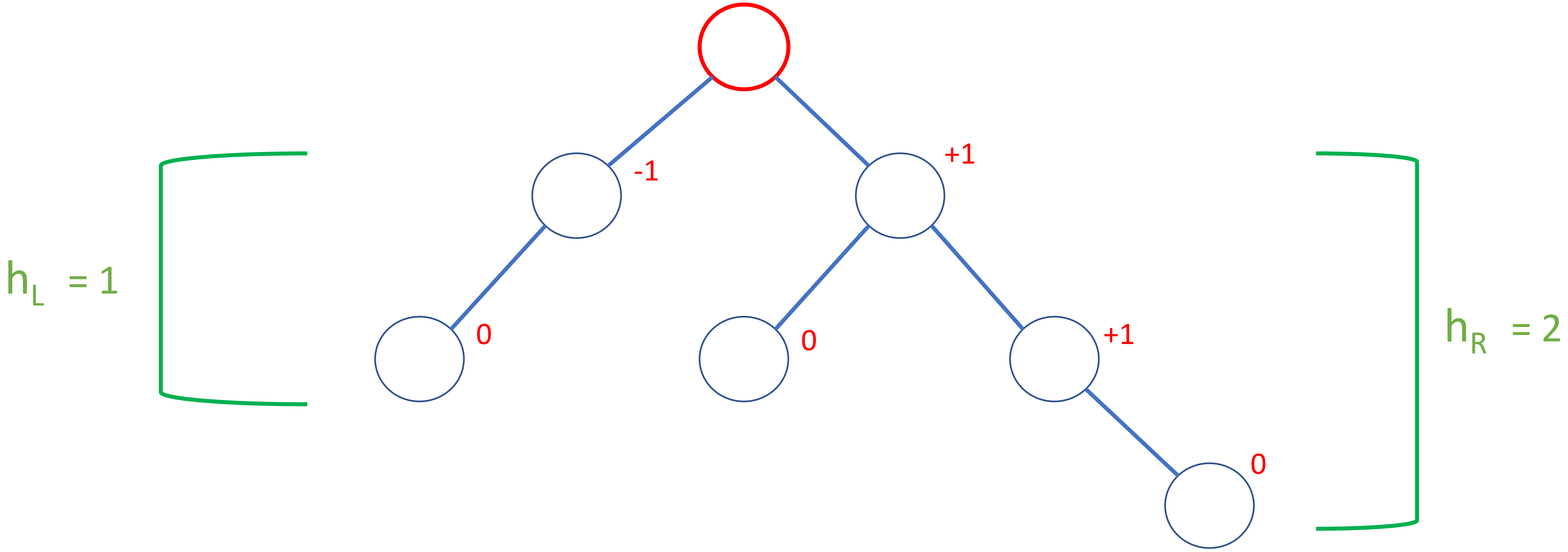

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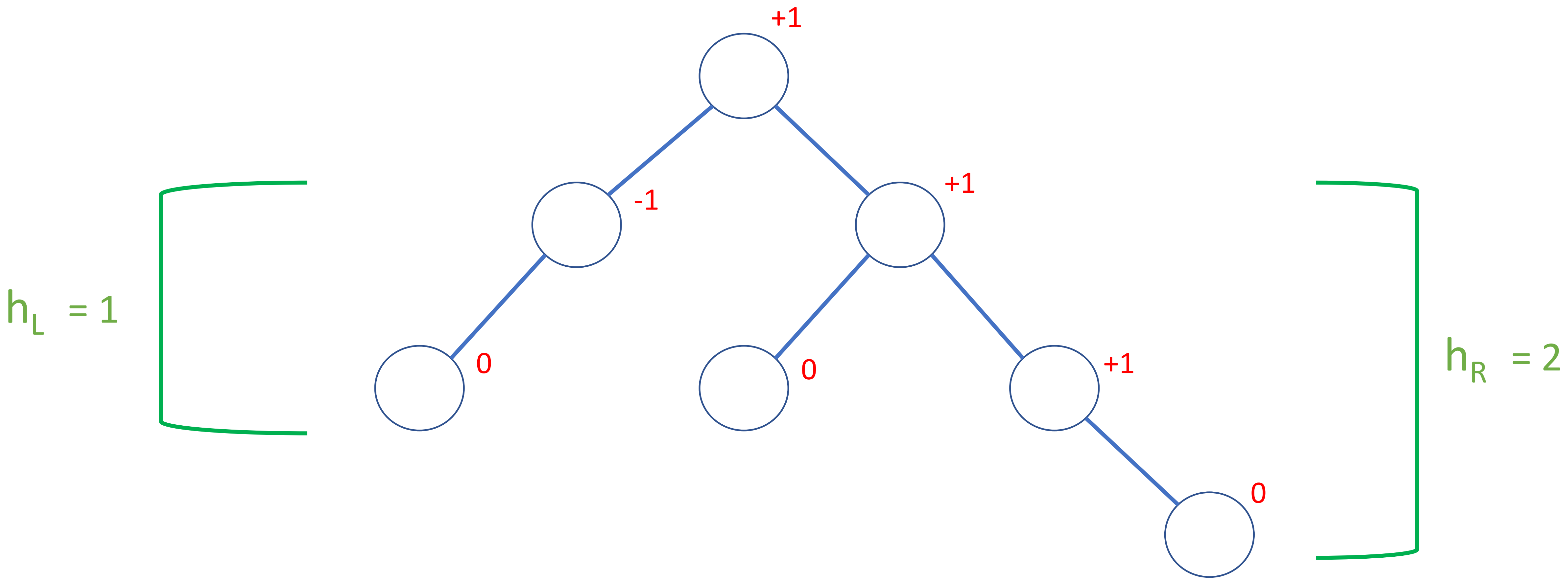

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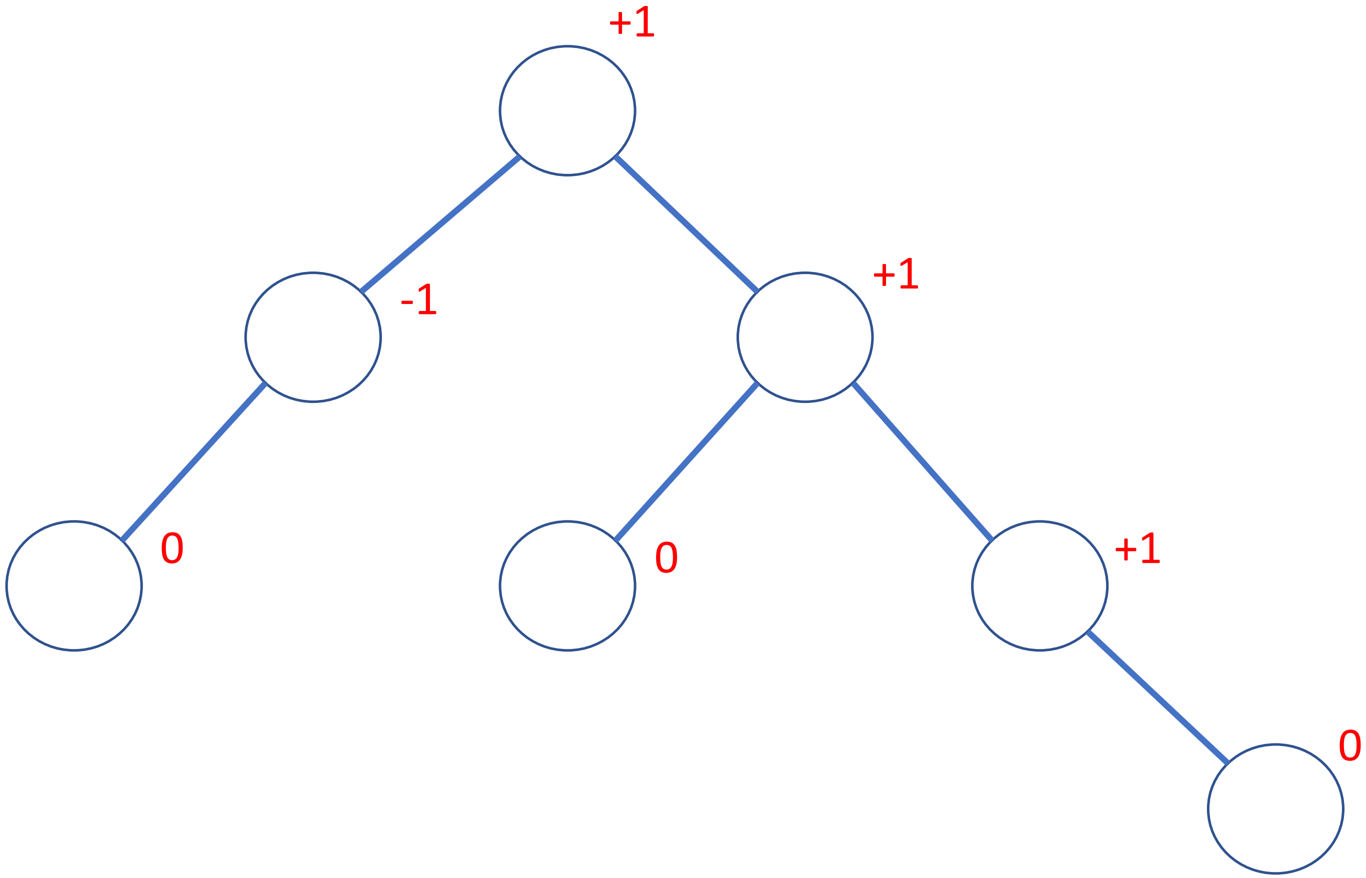

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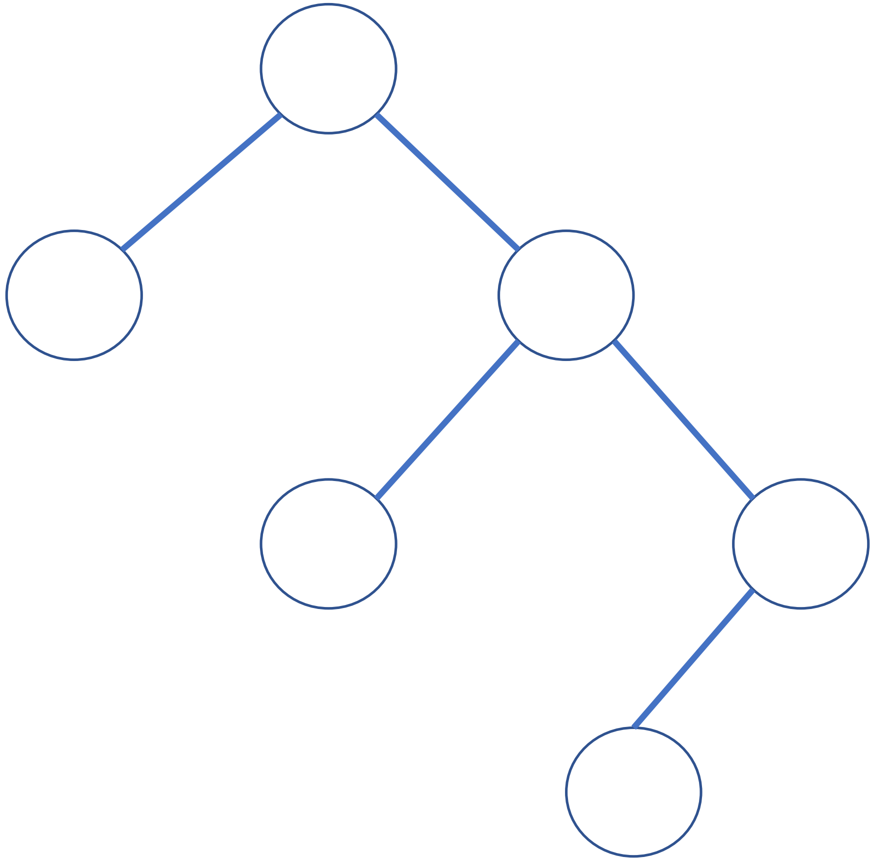

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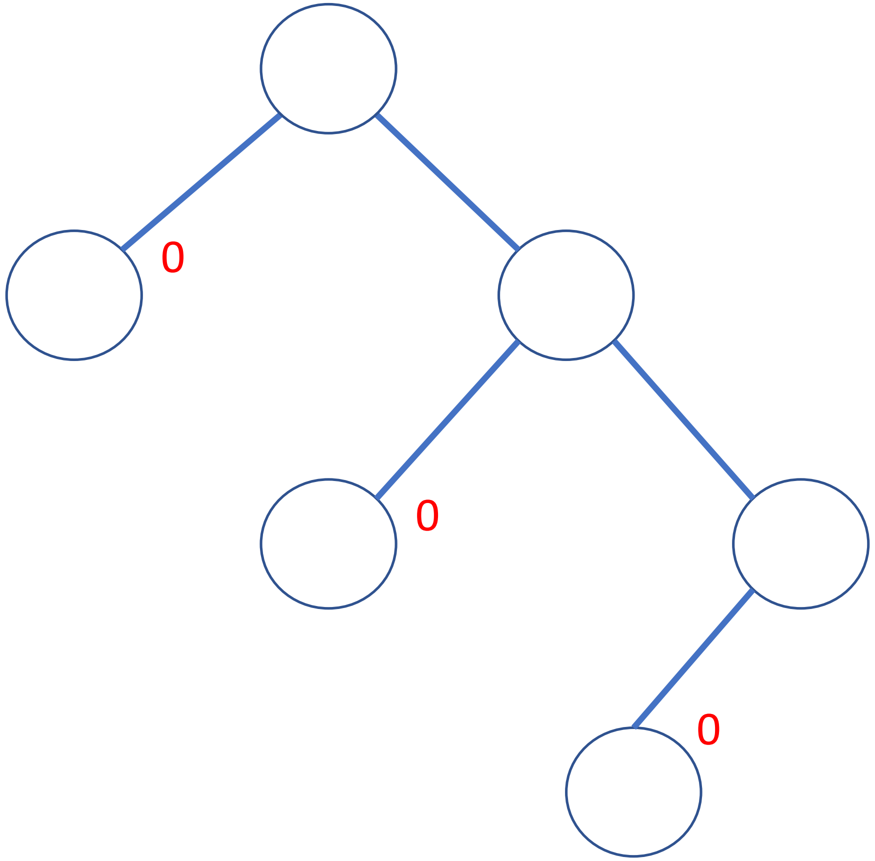

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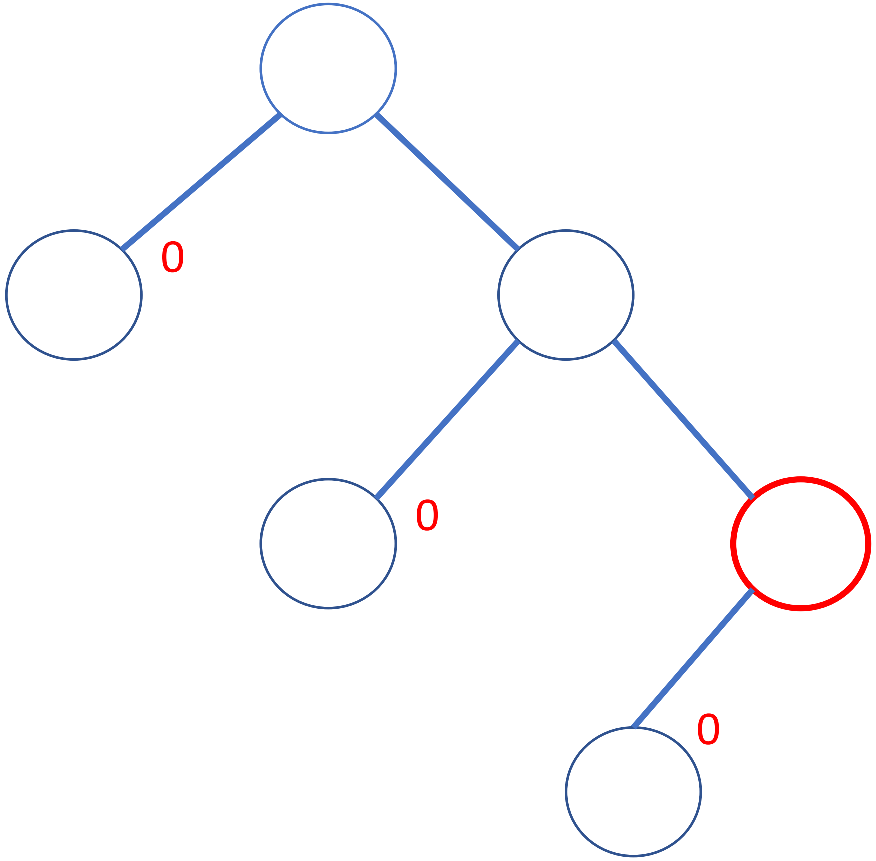

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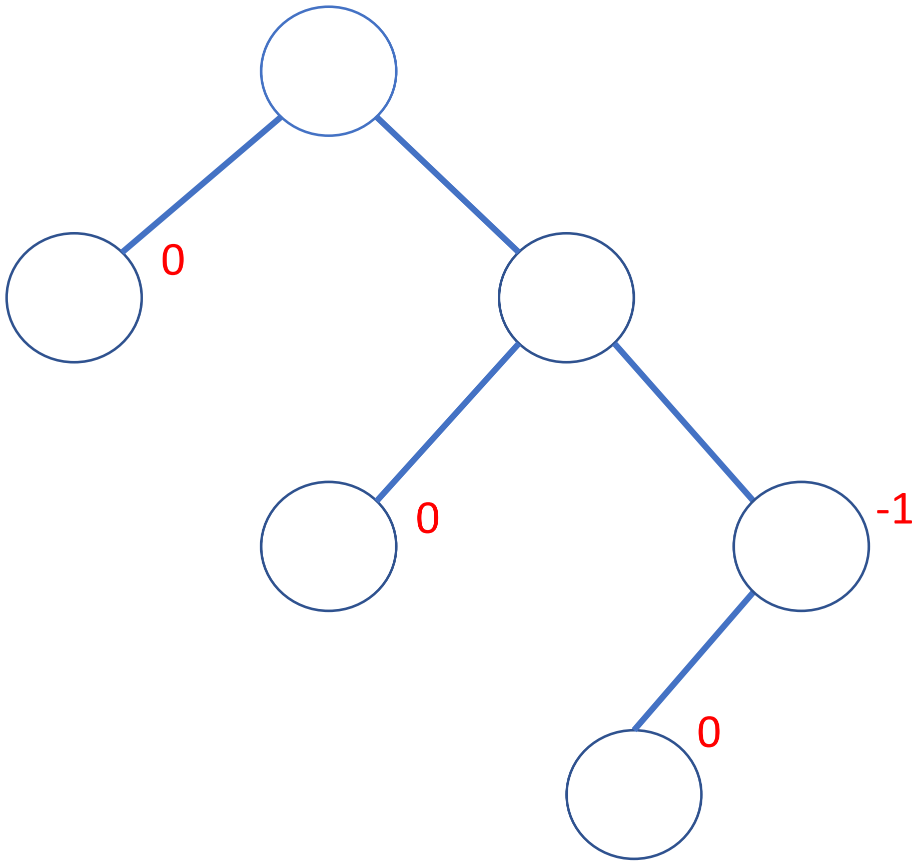

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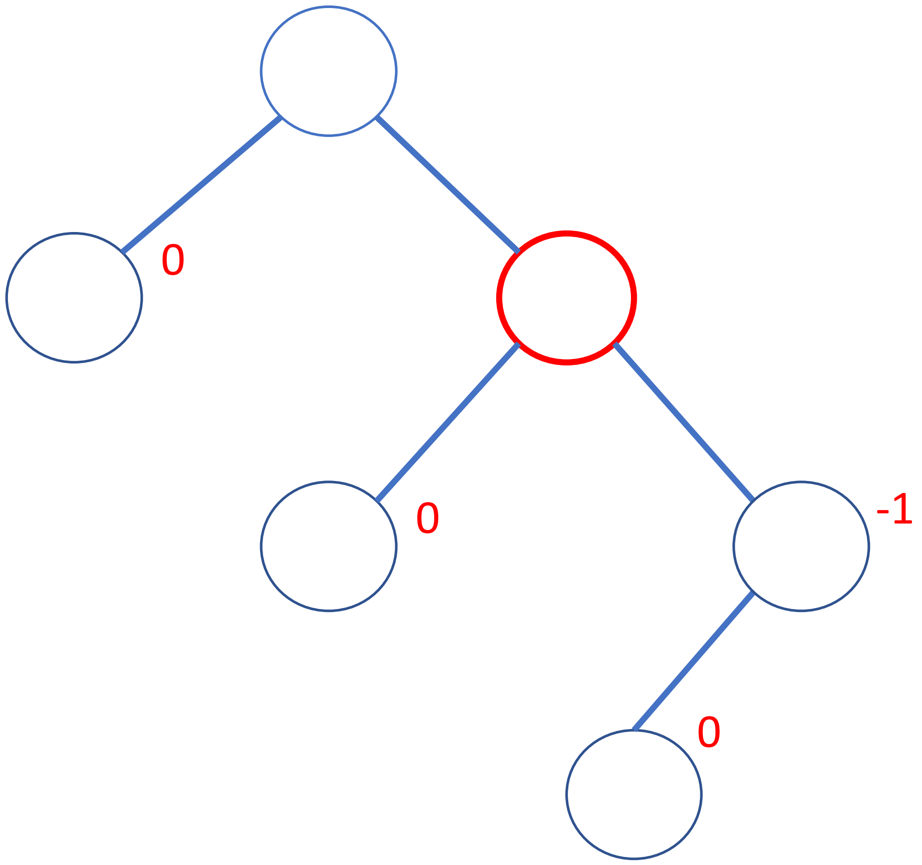

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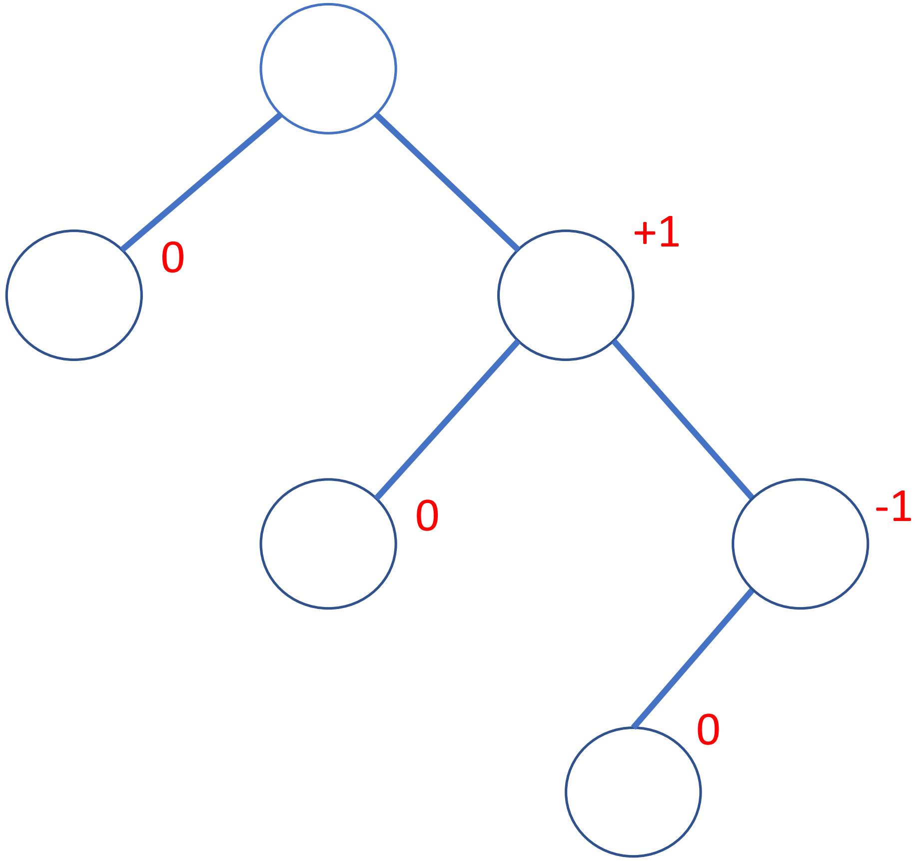

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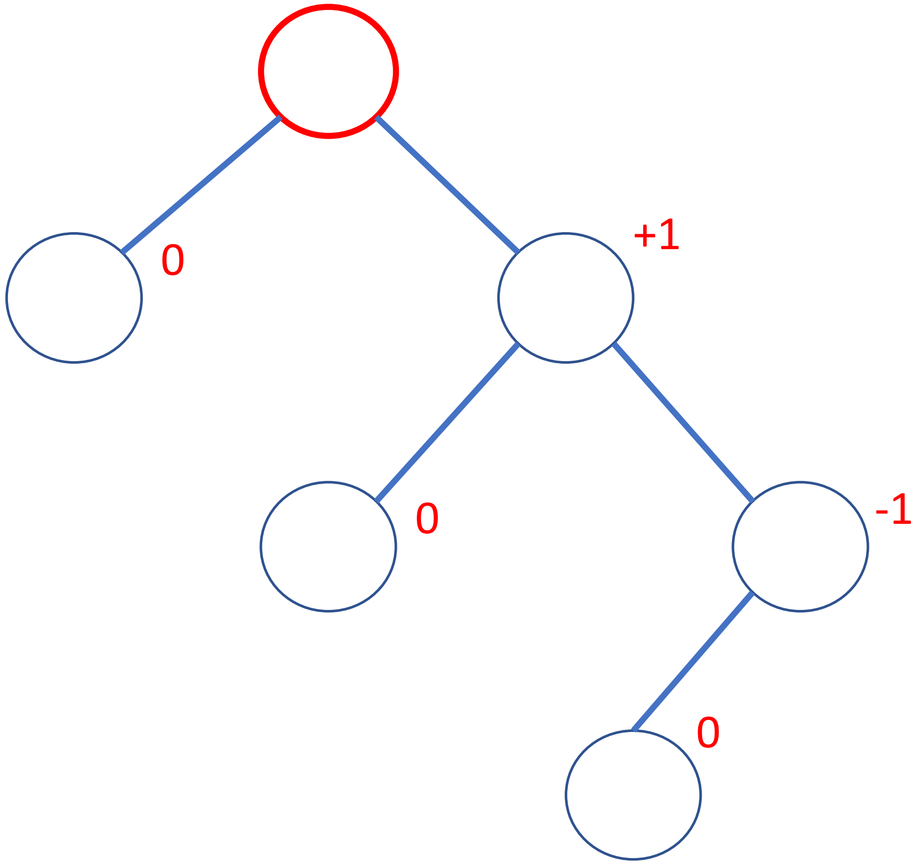

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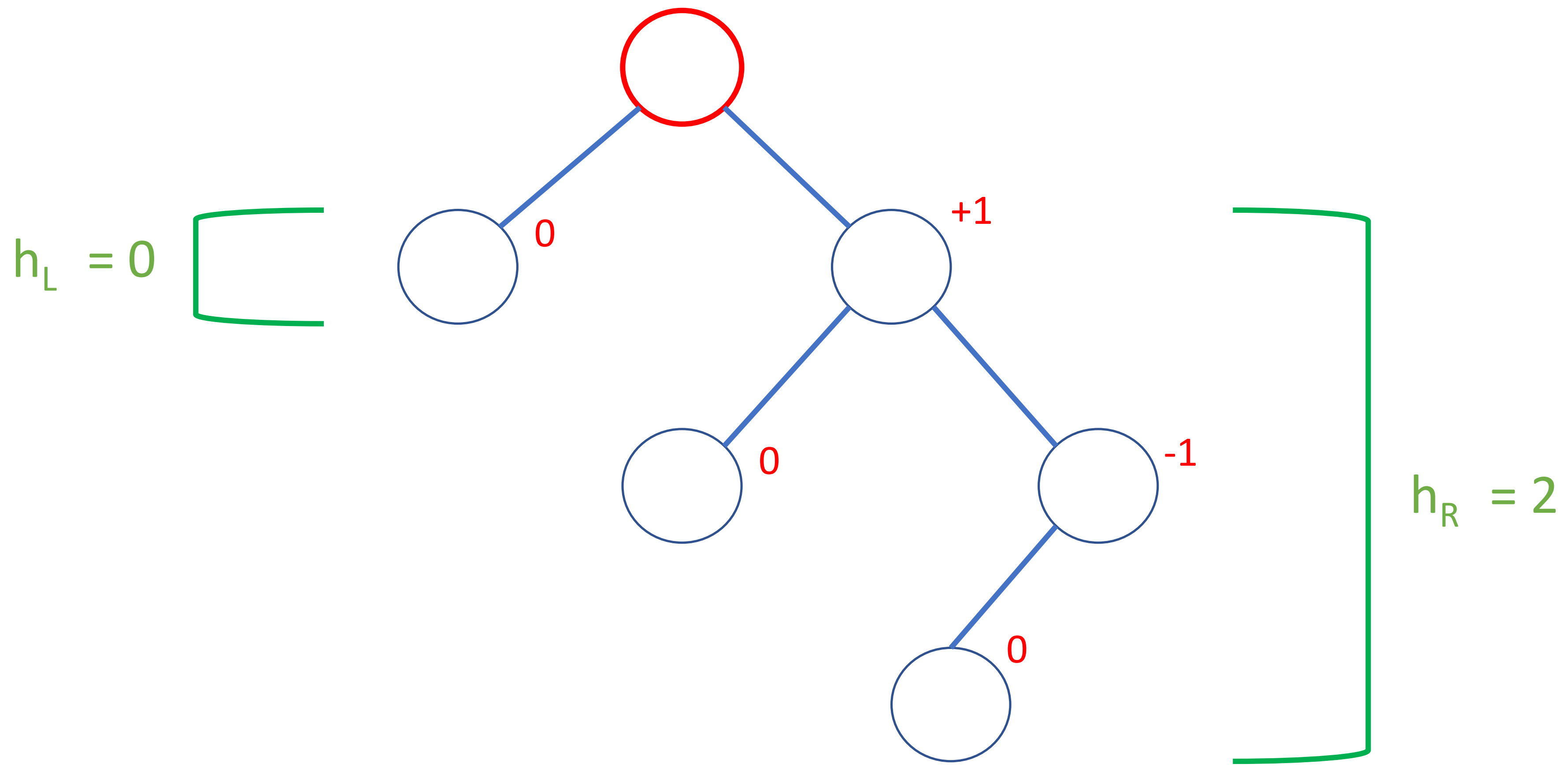

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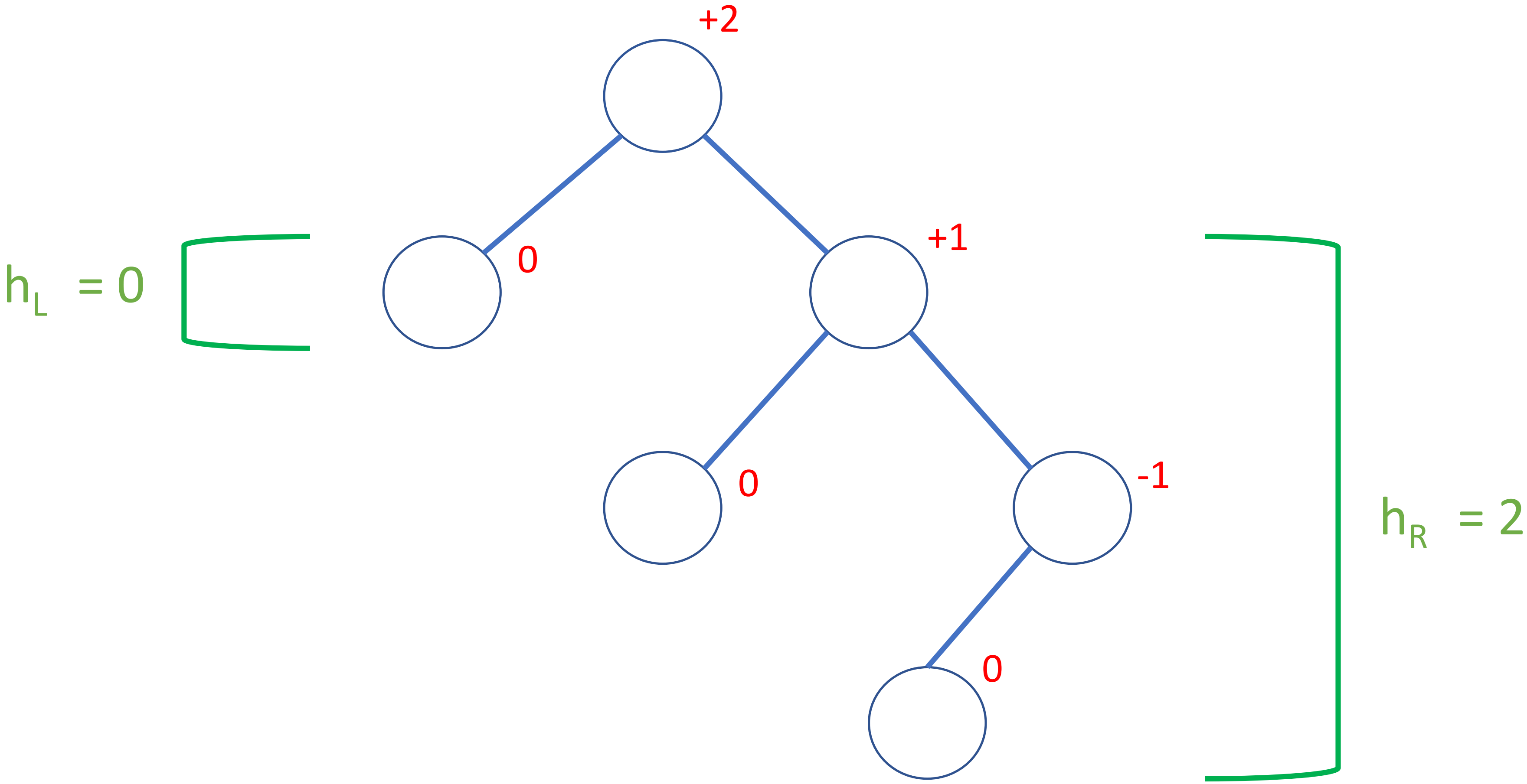
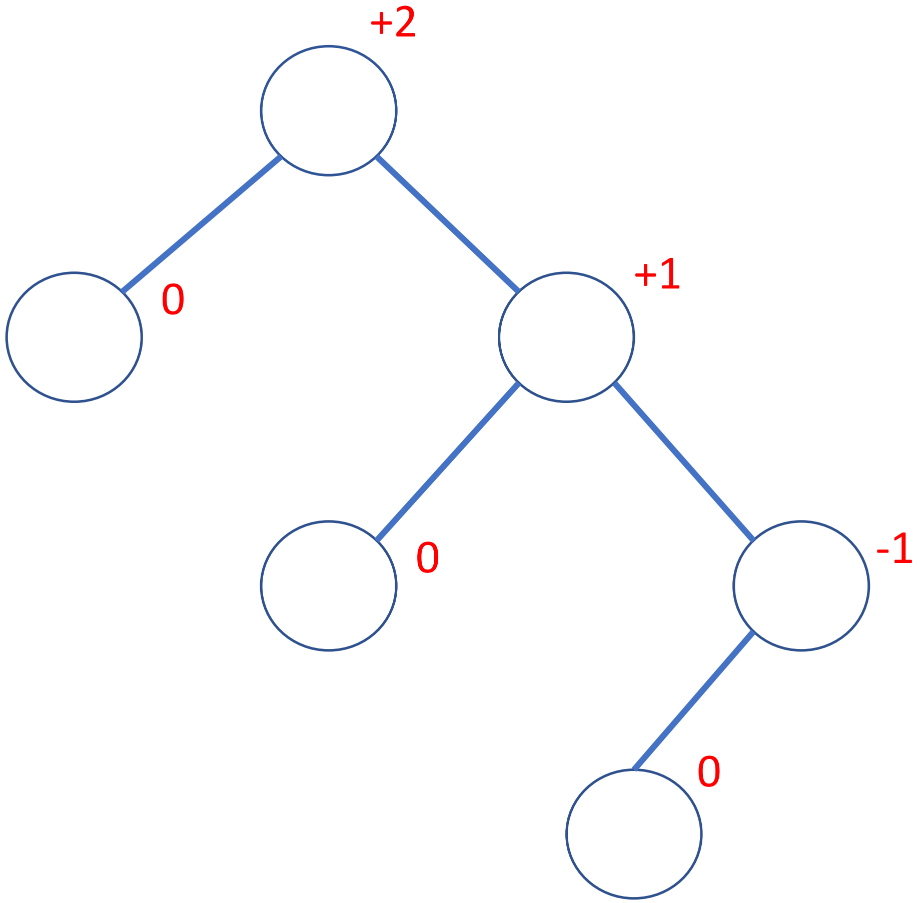


Is this an AVL tree? **No!**



Properties of AVL trees

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- AVL trees of n nodes has height $\Theta(\log n)$ [height $\leq$ $1.44 \log_2(n+2)$]

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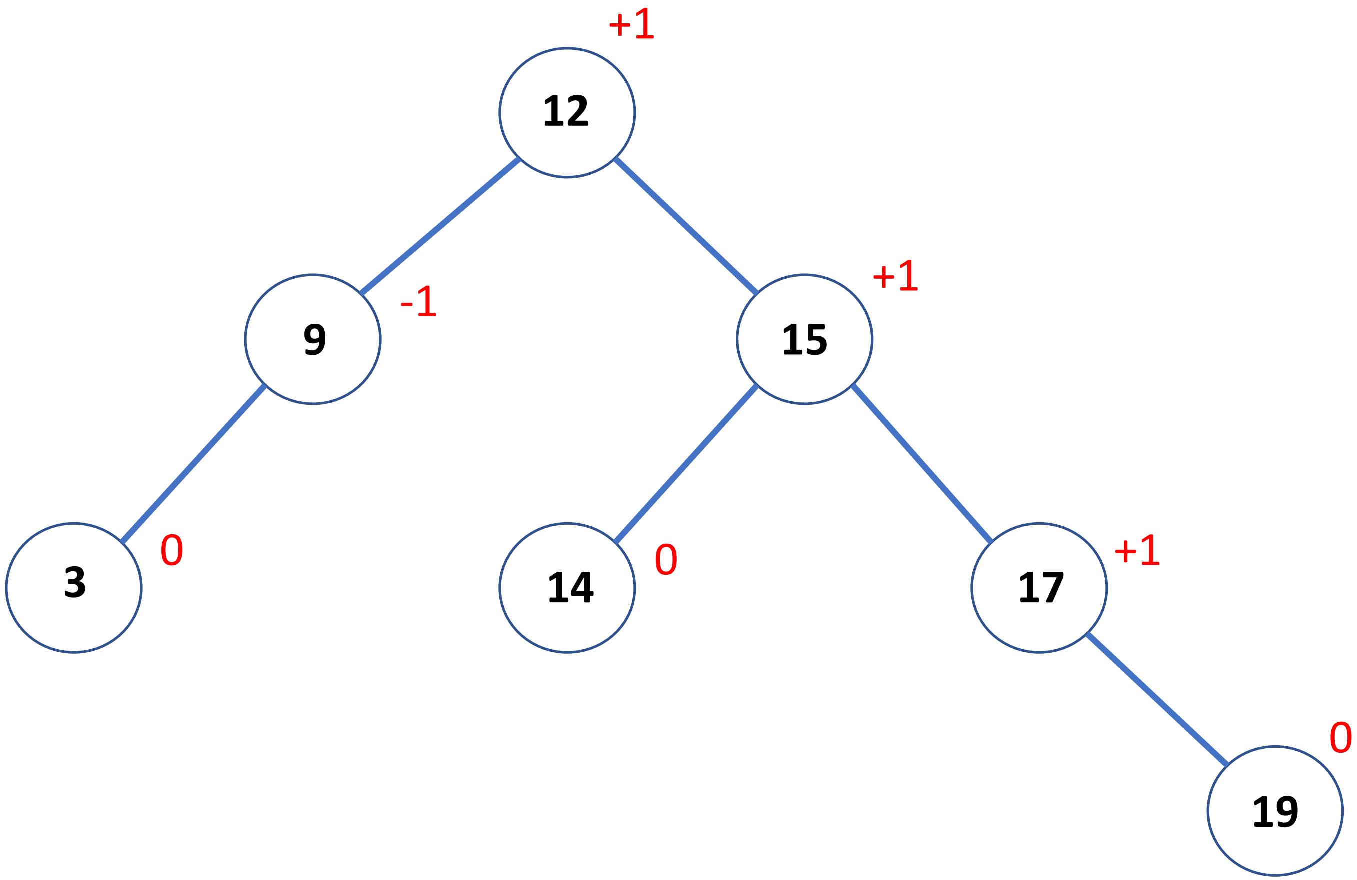
- Can do inserts, deletes while maintaining the tree balance in $\Theta(\log n)$ time

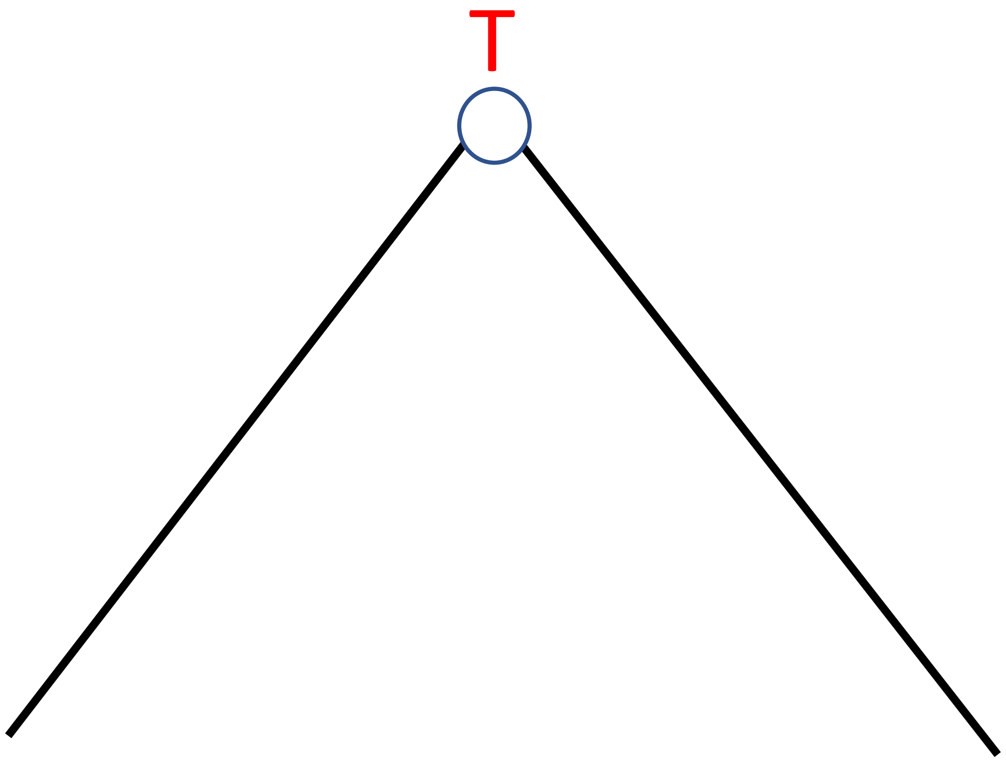
Elegant, relatively clean and simple, works well in practice

AVL tree operations

- **Search**(T, x)
- **Insert**(T, x)
- **Delete**(T, x)



Insert(T, x) : General idea



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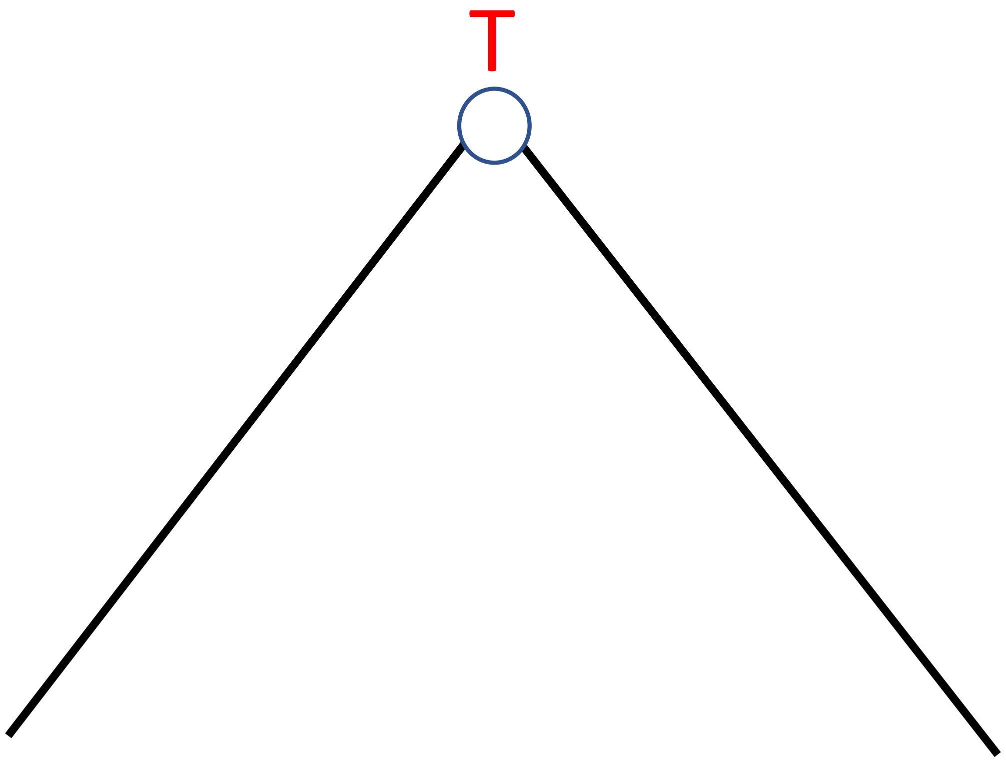
- Insert x into T as in any BST



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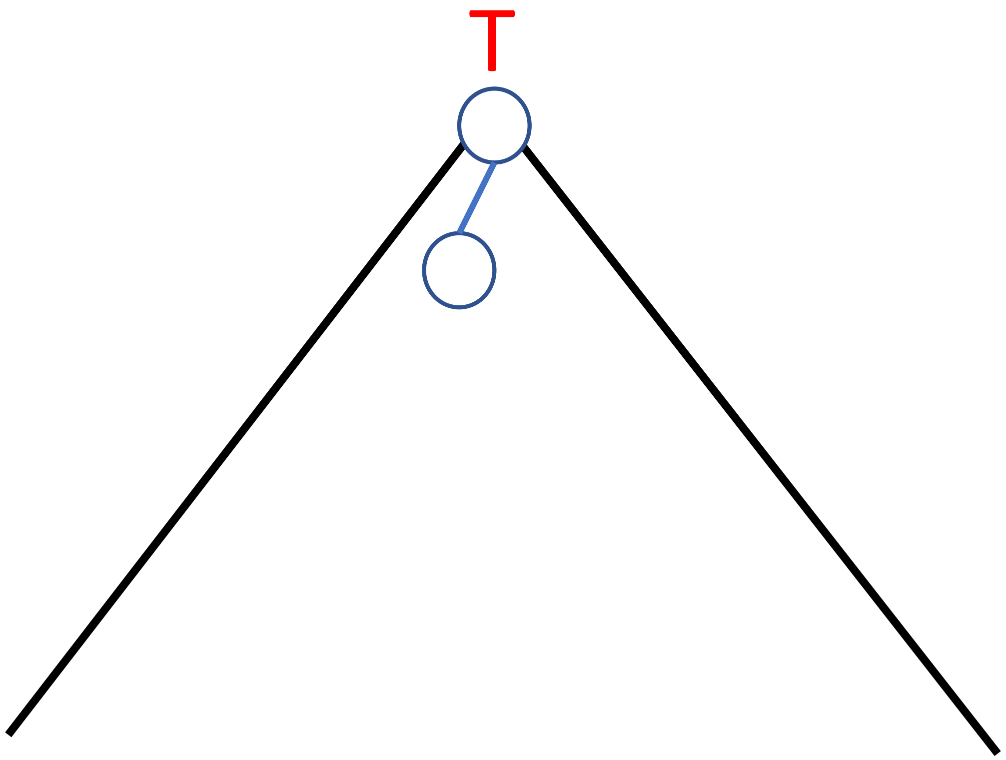

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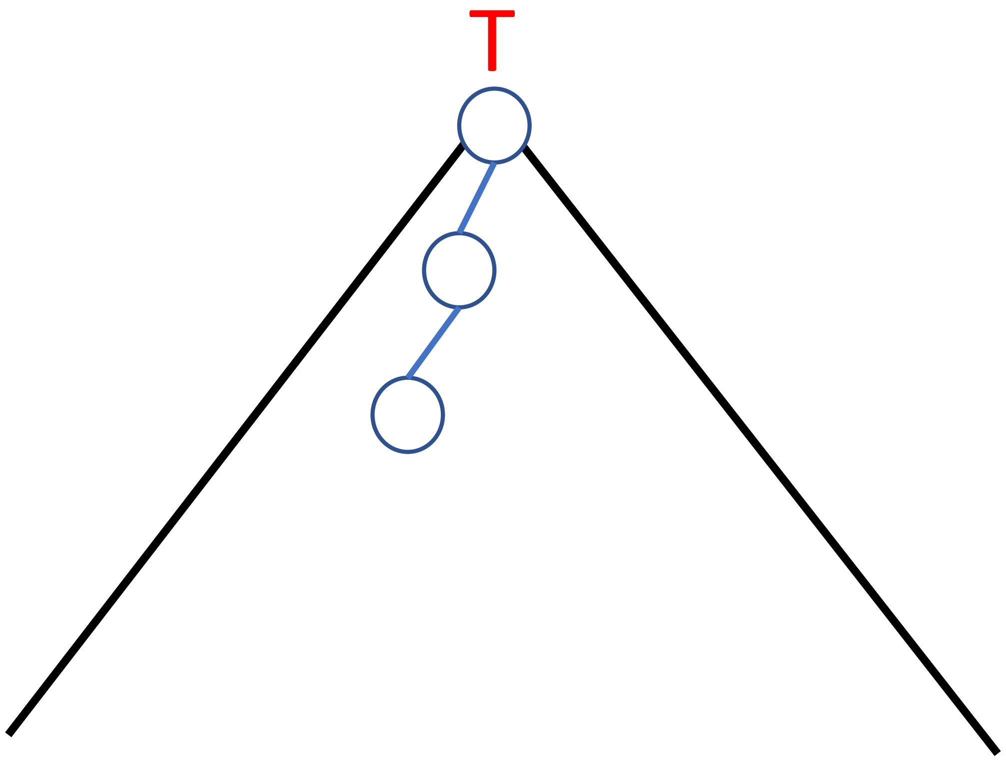

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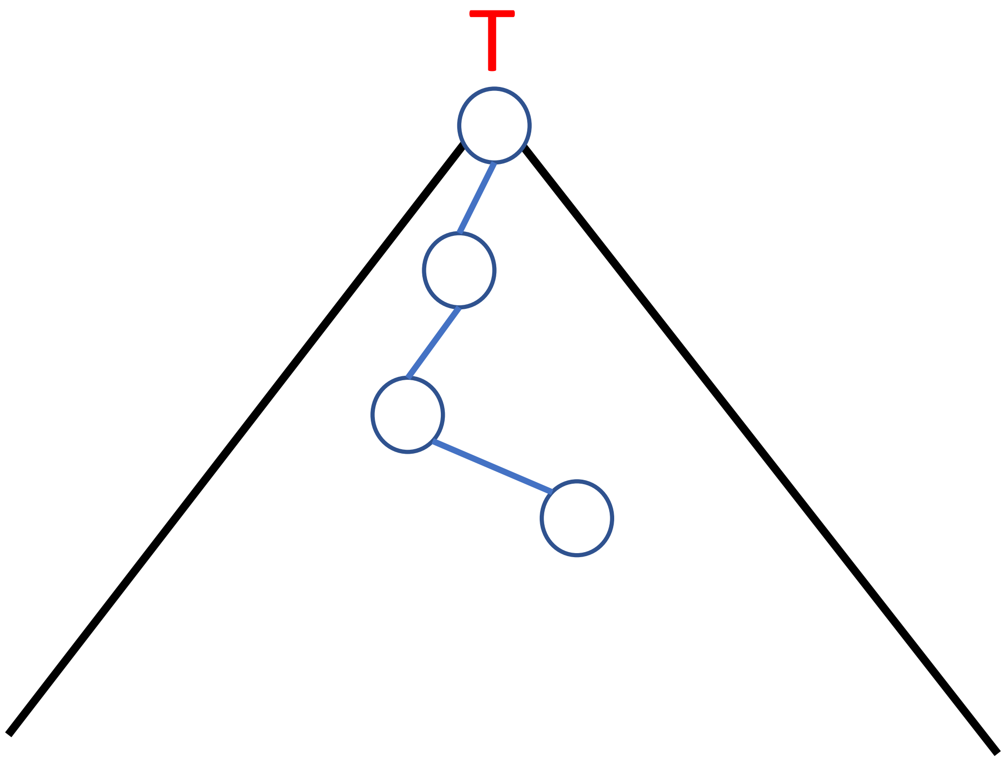

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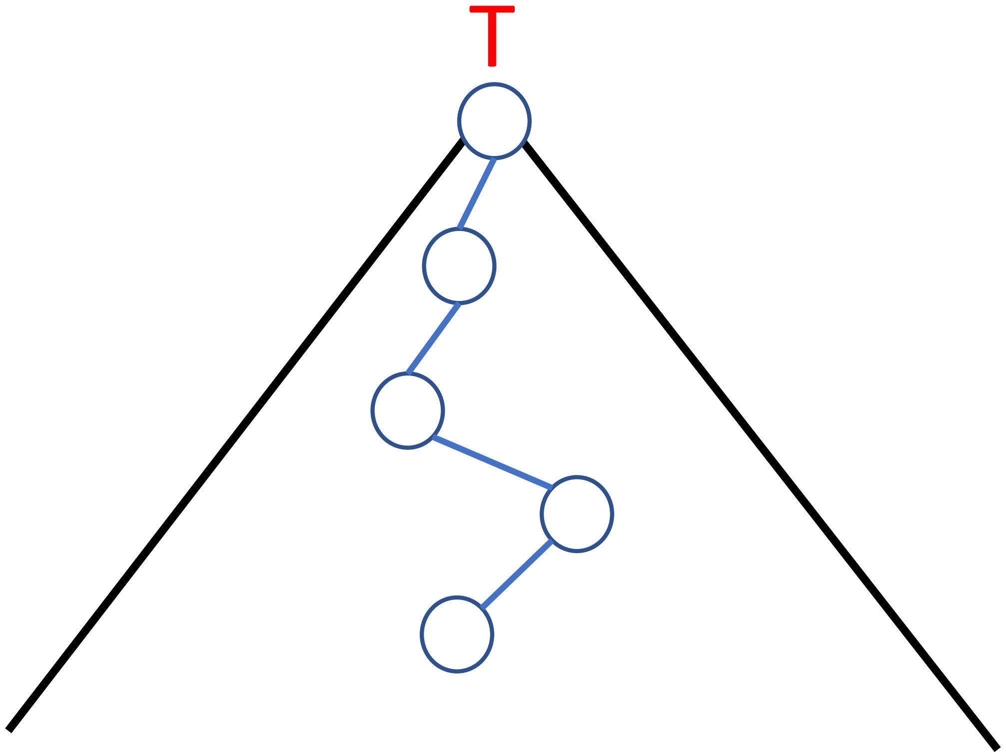

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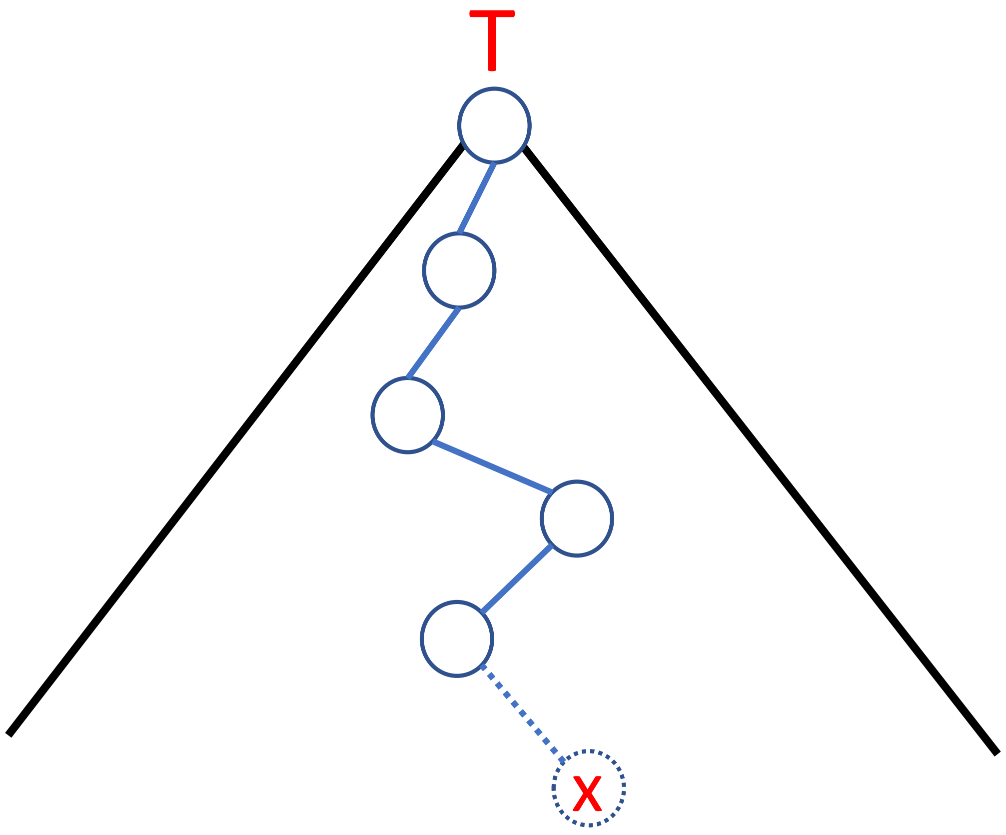
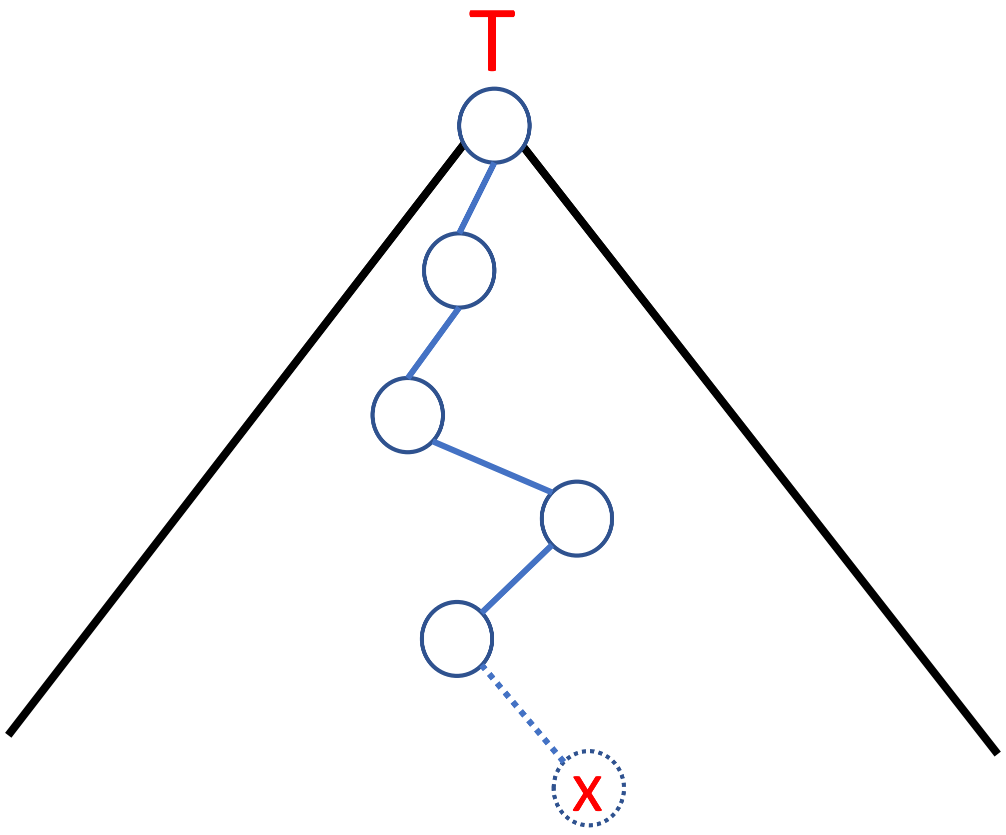


Insert(T, x) : General idea

- Insert x into T as in any BST :
  - x is now a leaf



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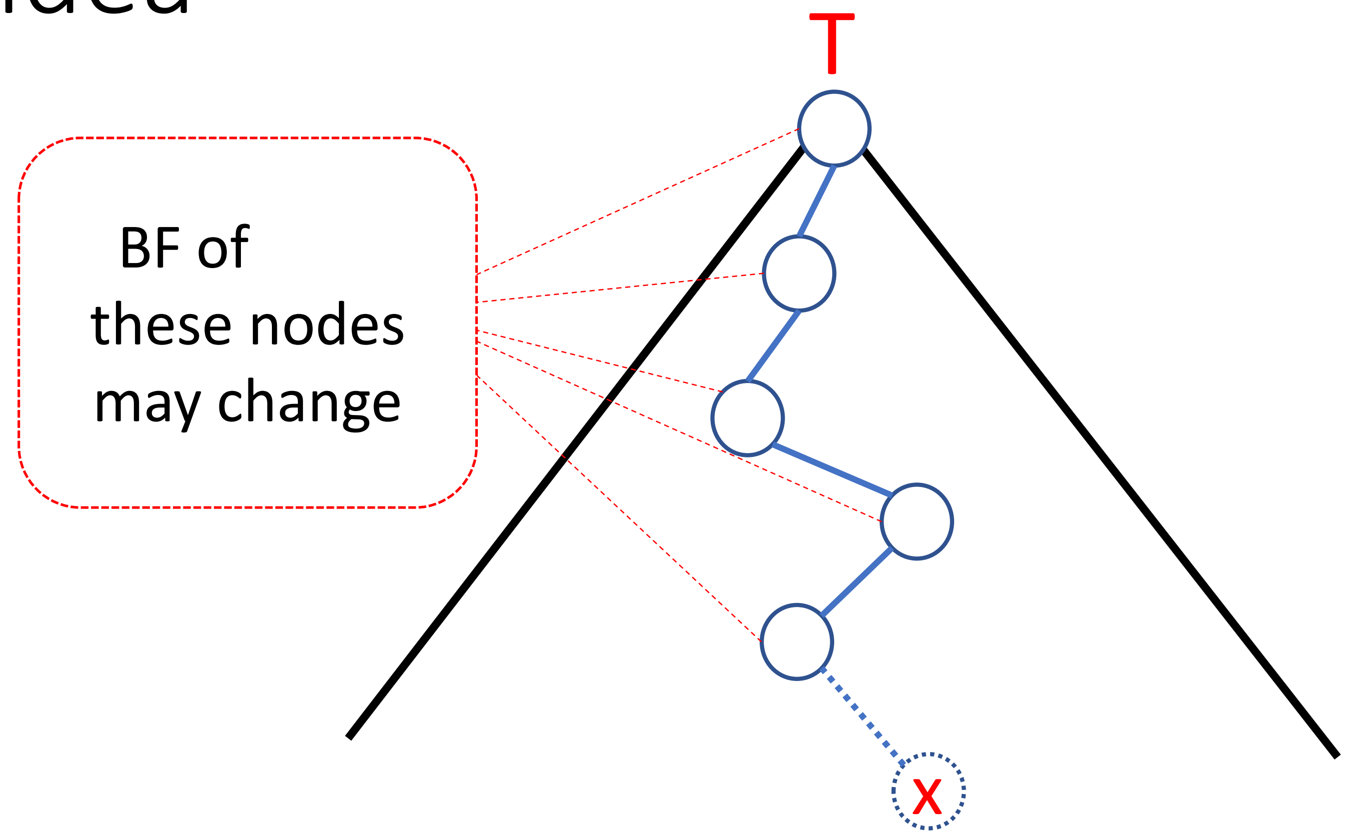

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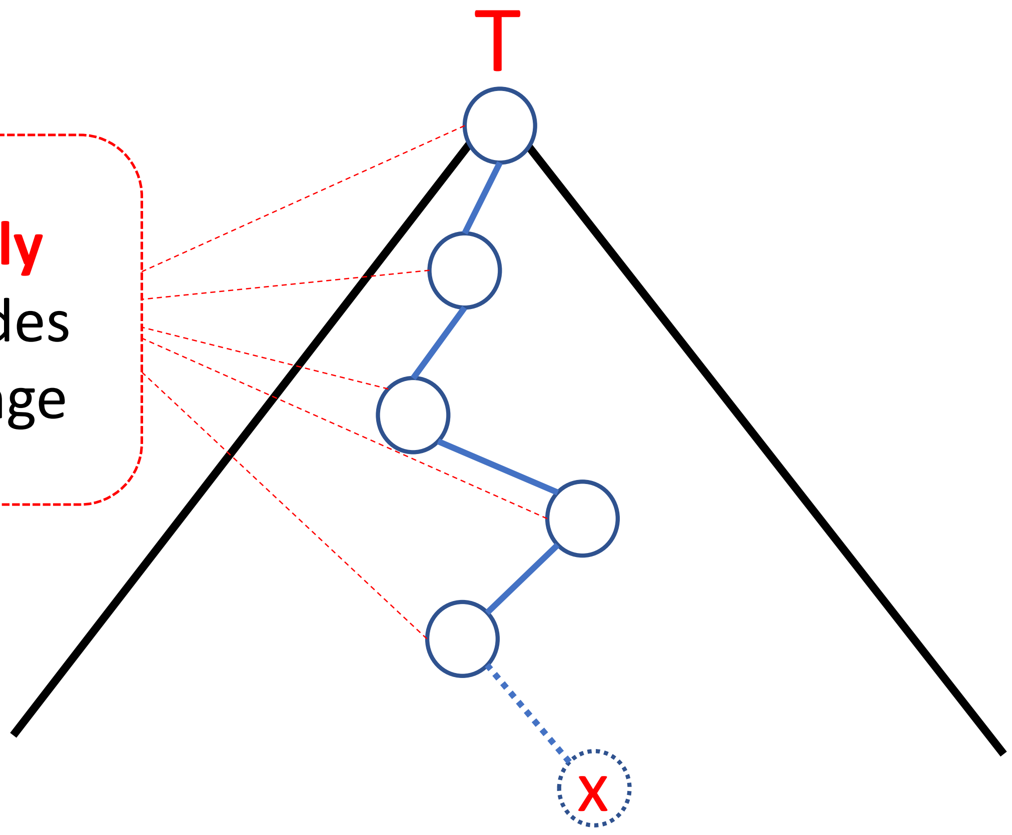
BF of **only** these nodes may change



Insert(T, x) : General idea

- Insert x into T as in any BST :
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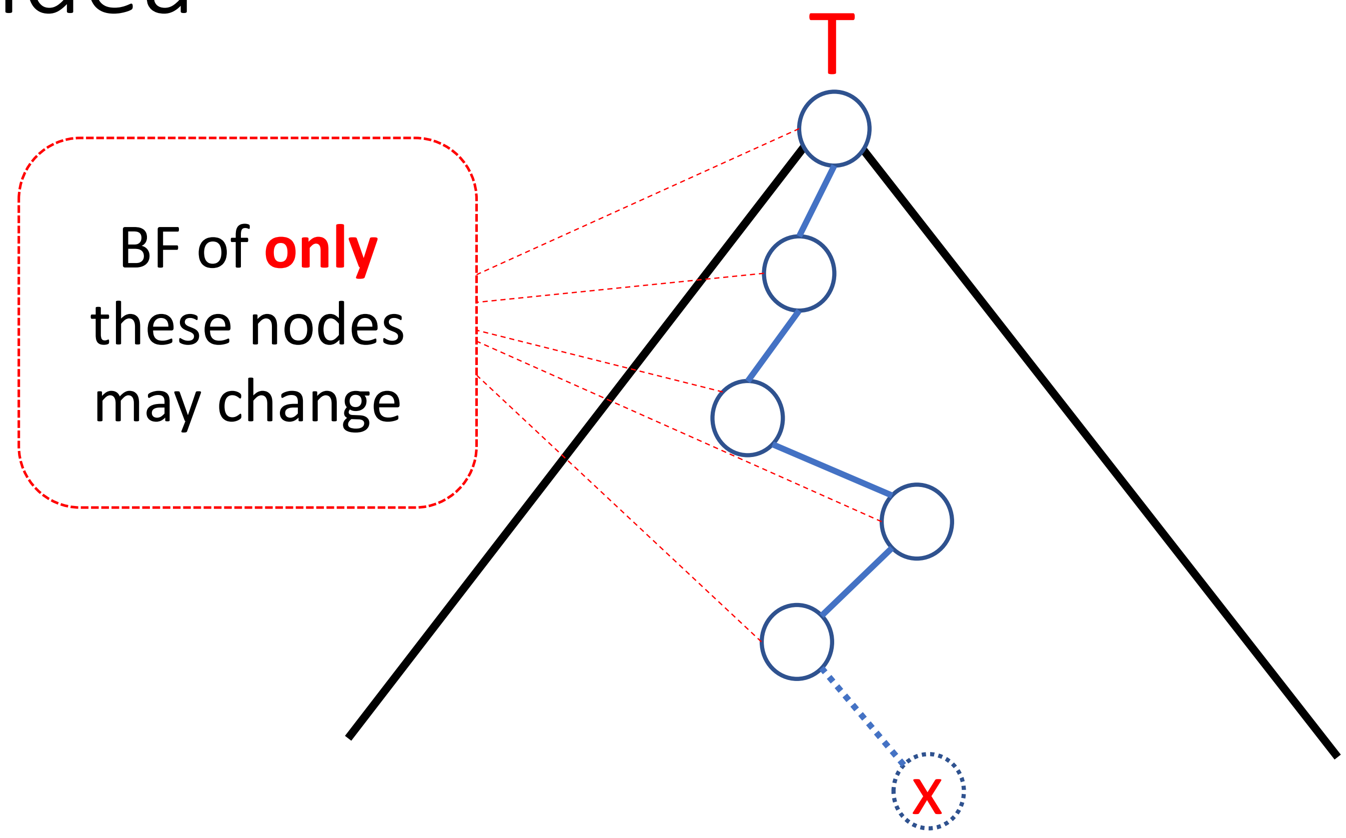
- Go up from x to the root



Insert(T, x) : General idea

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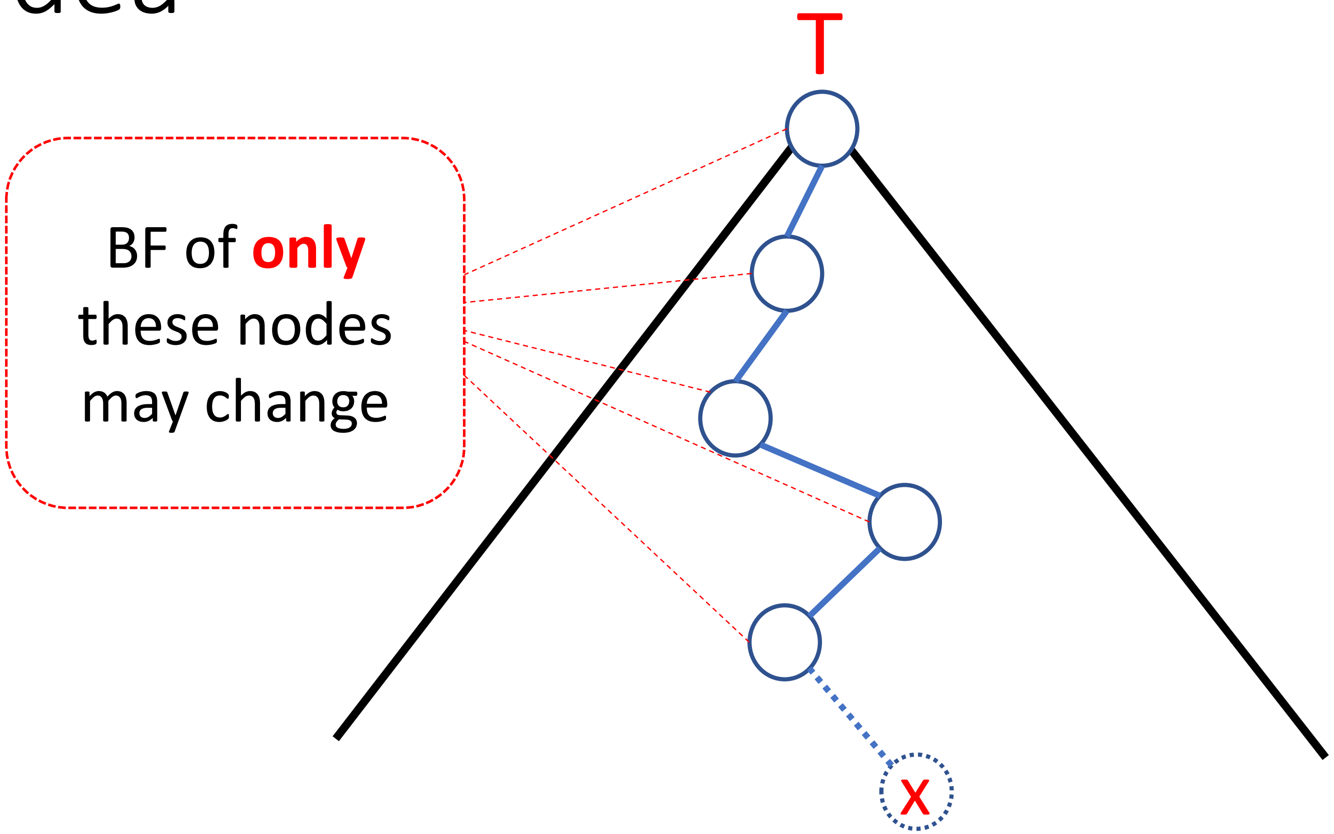
  - x is now a leaf

- Go up from x to the root and for each node :



Insert(T, x) : General idea

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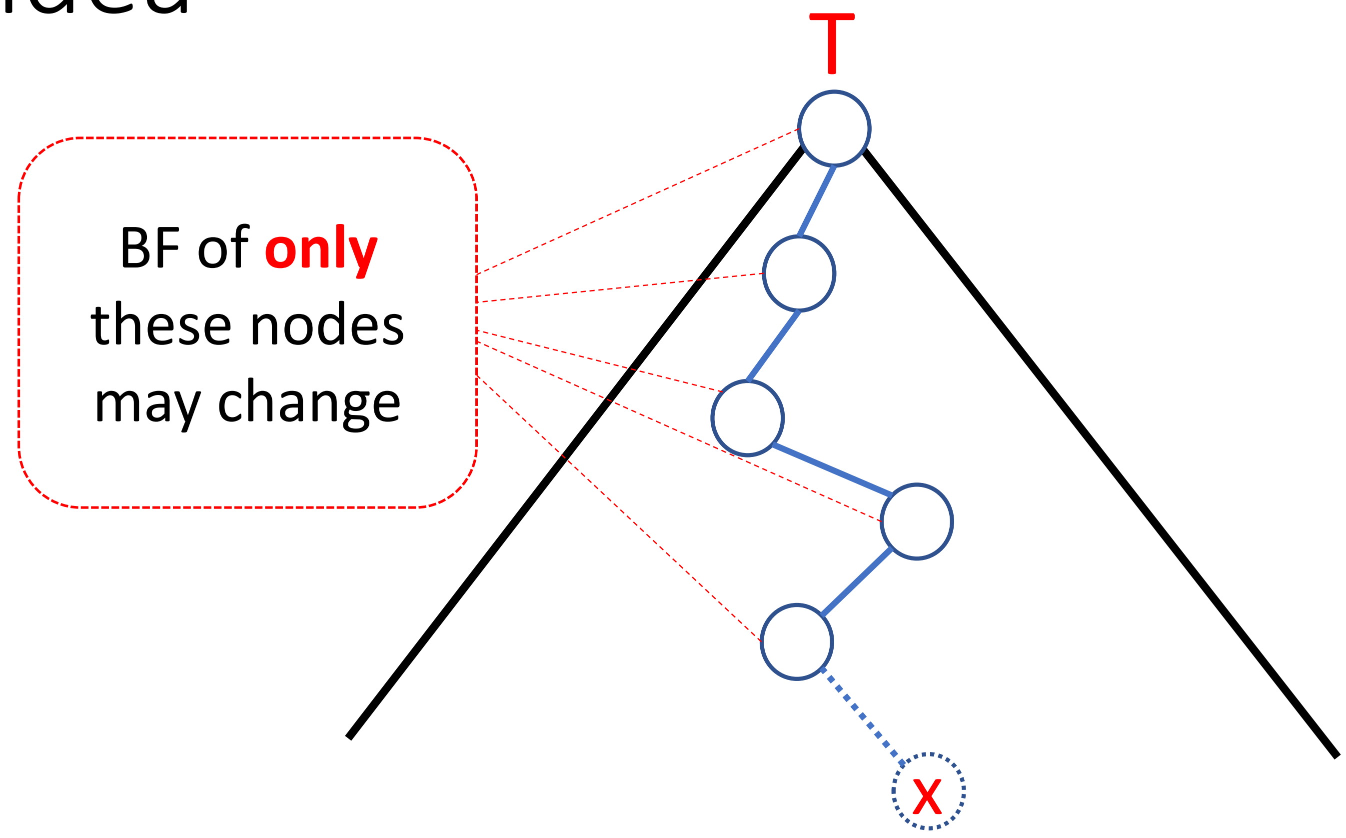
- Go up from x to the root and for each node :
  - Adjust the BF

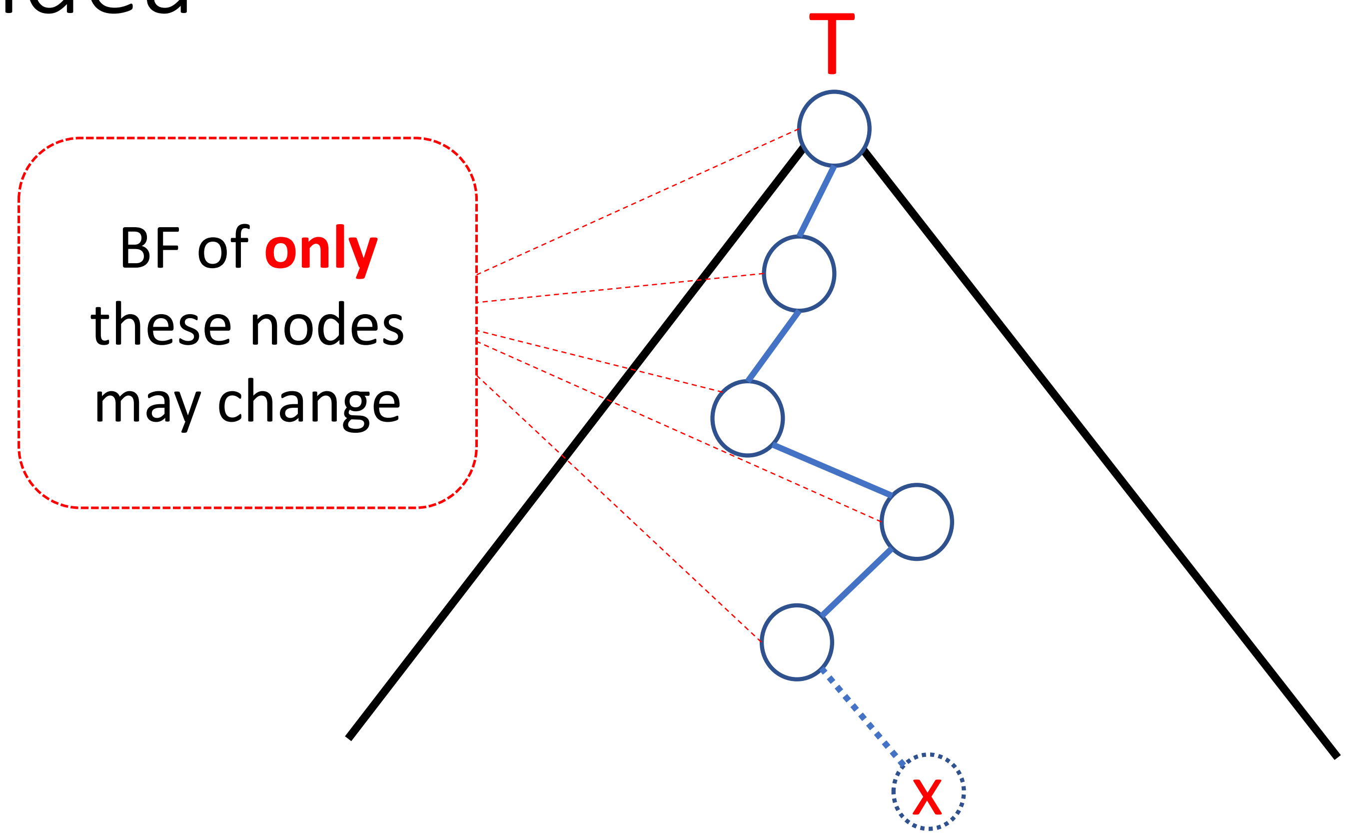


Insert(T, x) : General idea

- Insert x into T as in any BST :
  - x is now a leaf

- Go up from x to the root and for each node :
  - Adjust the BF
  - “Rebalance” if necessary

    i.e. if BF > 1 or BF < -1



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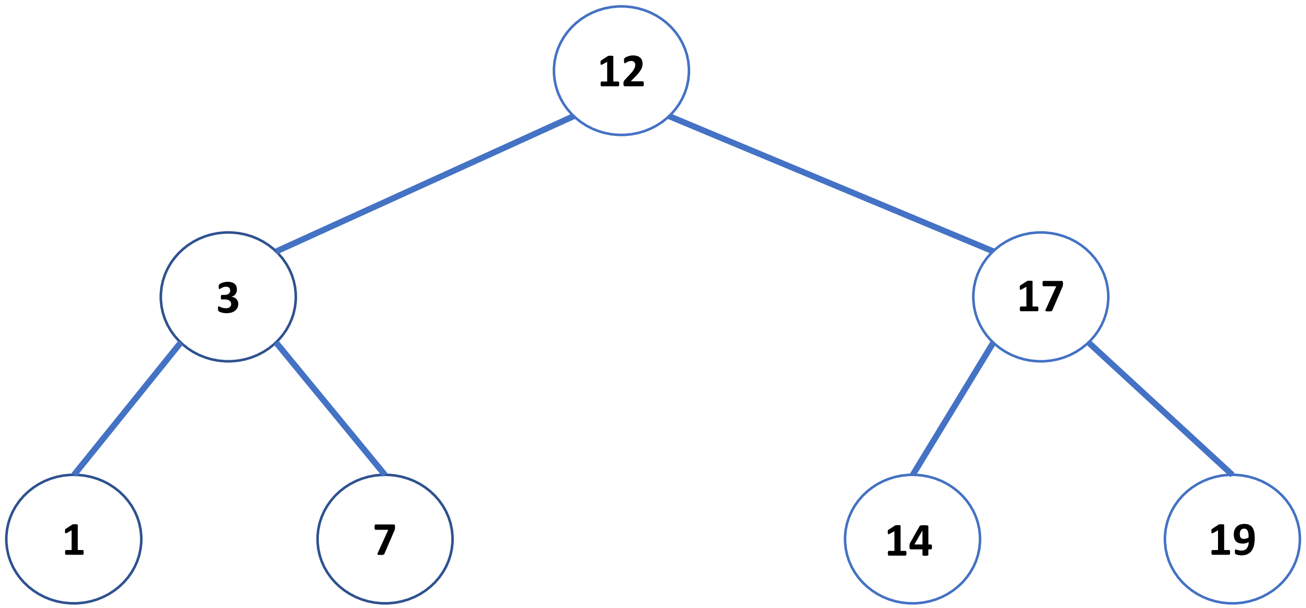

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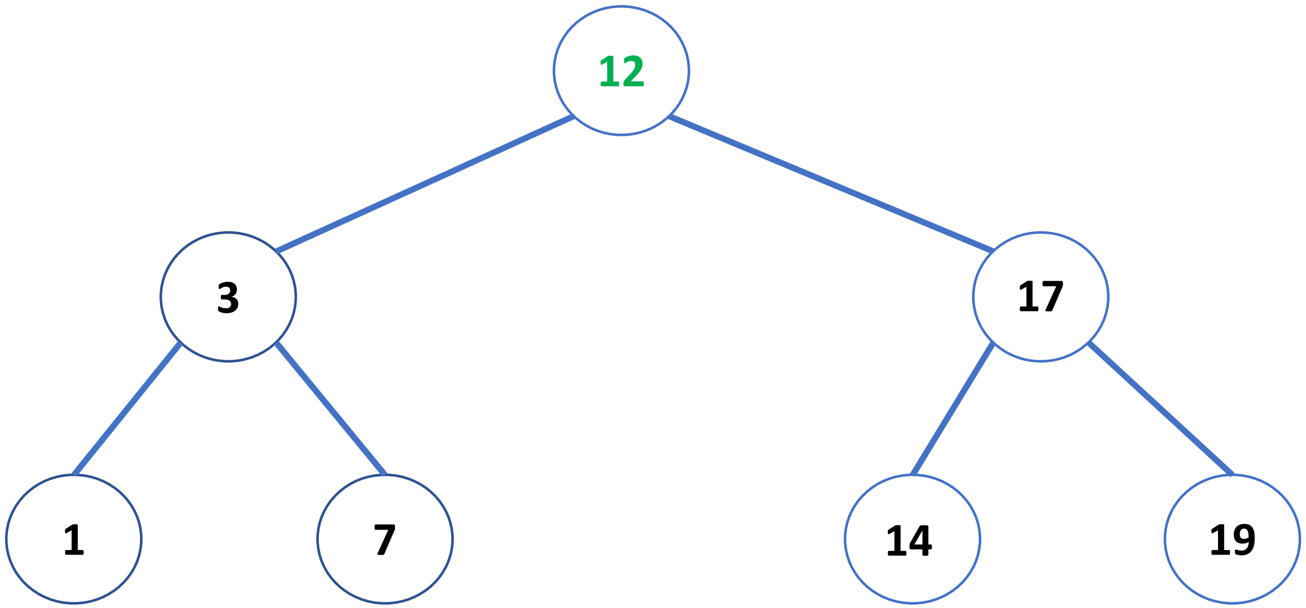

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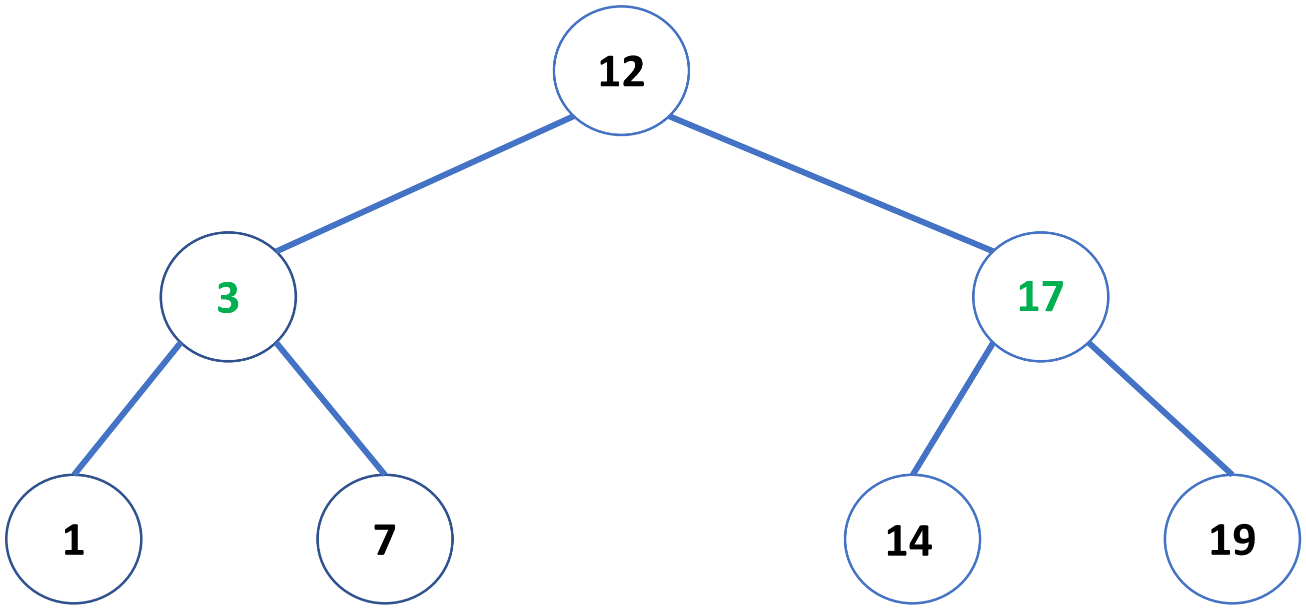

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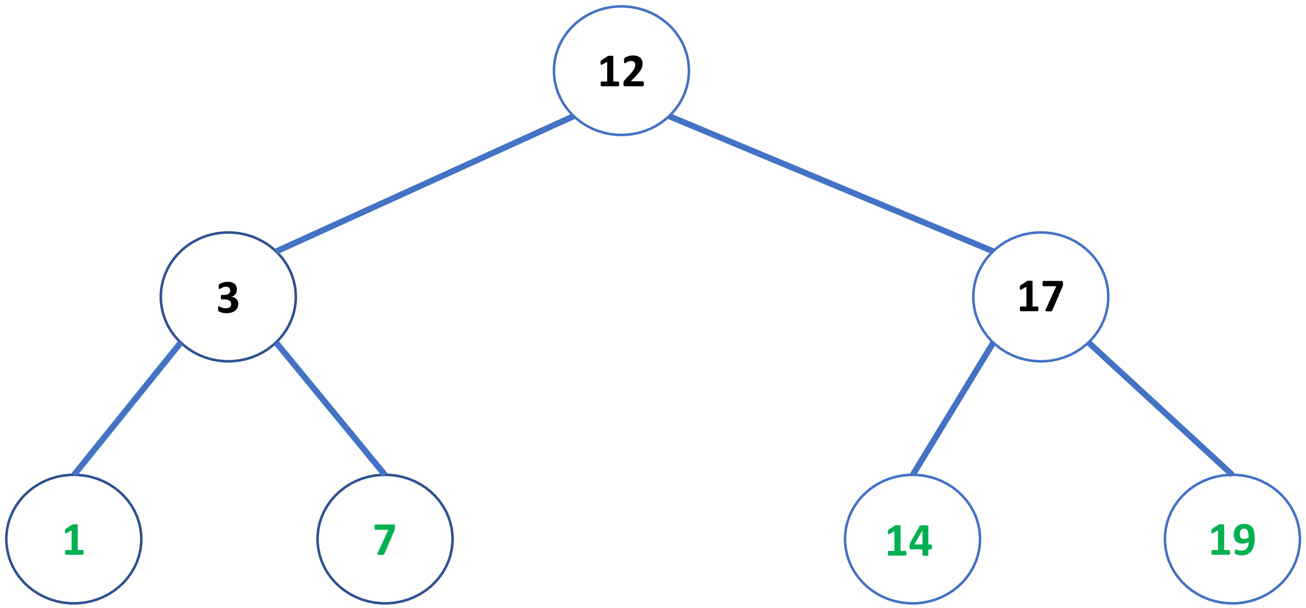

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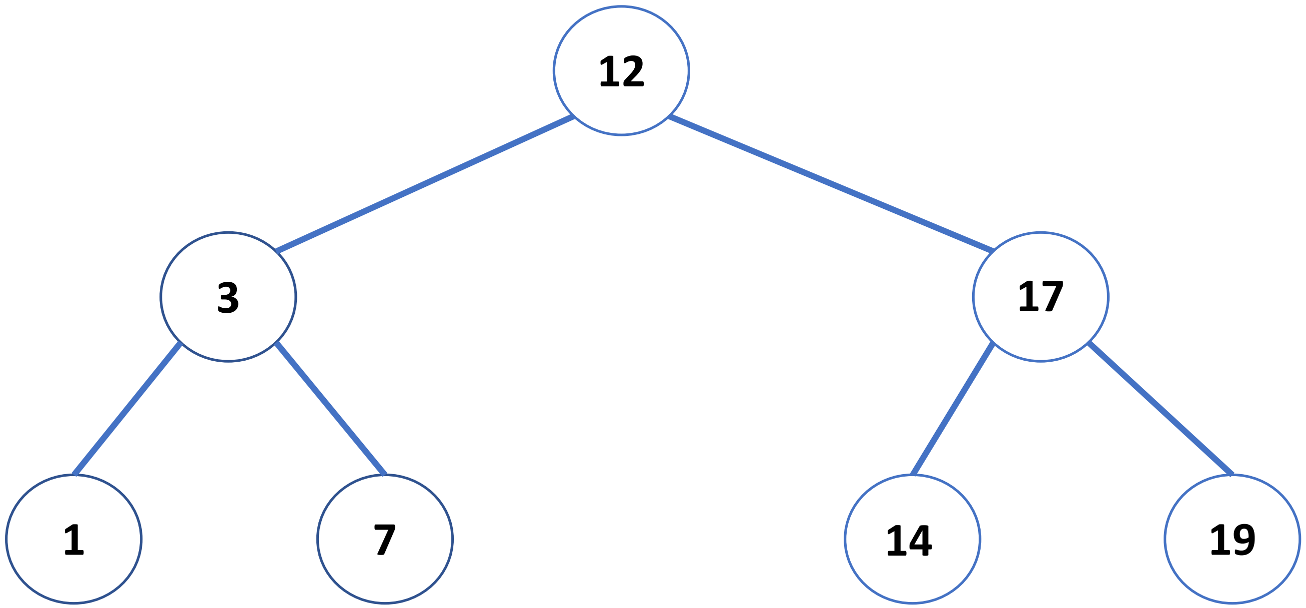

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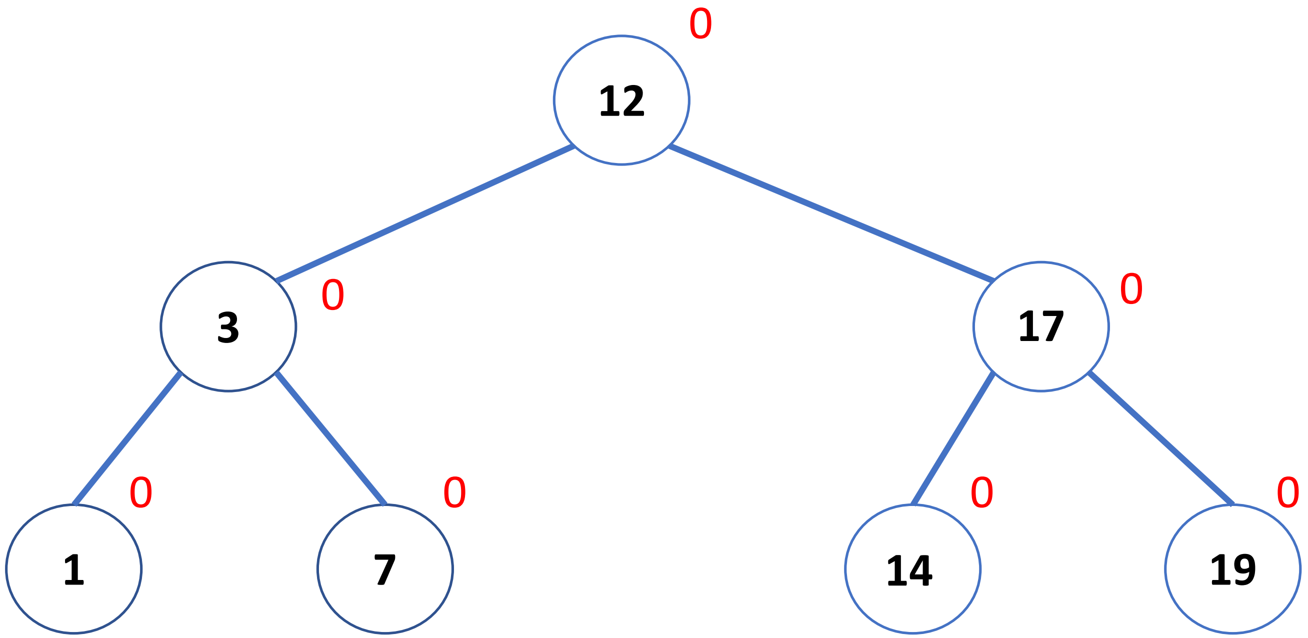

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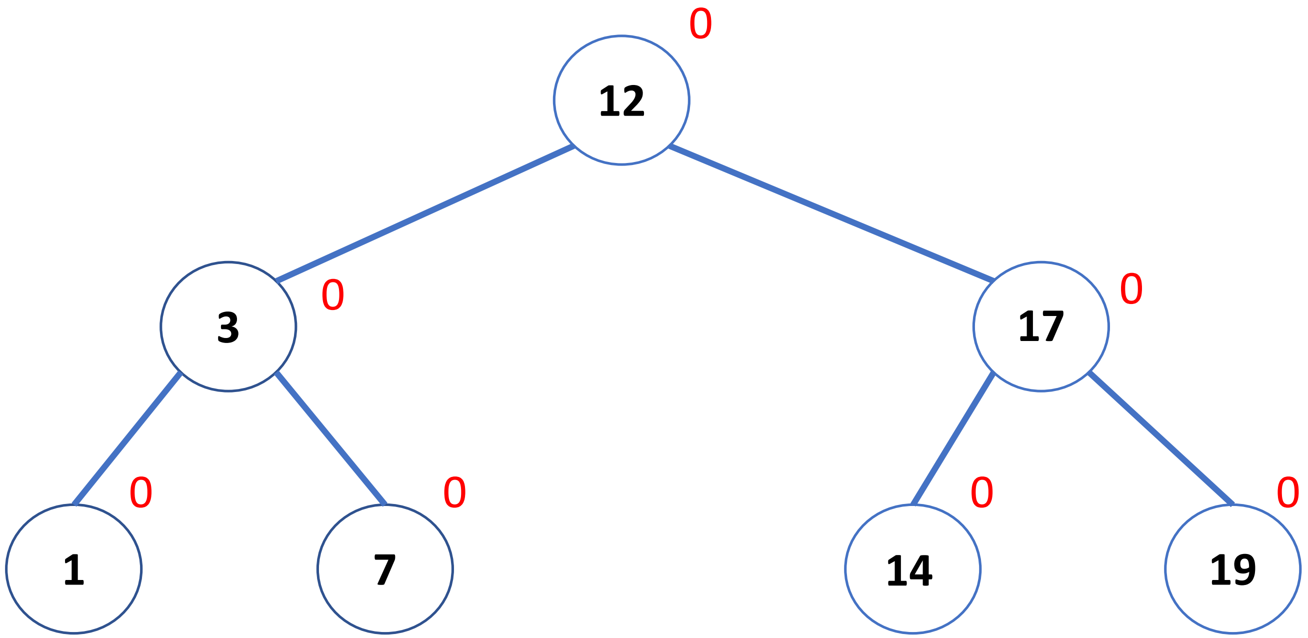

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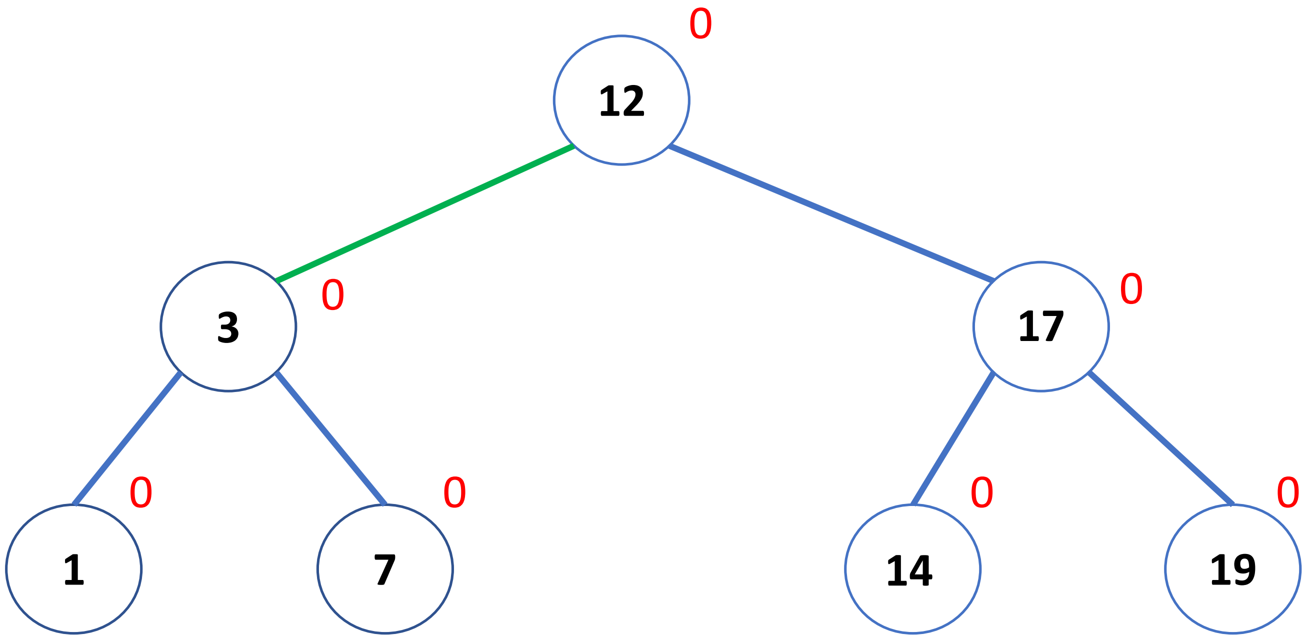

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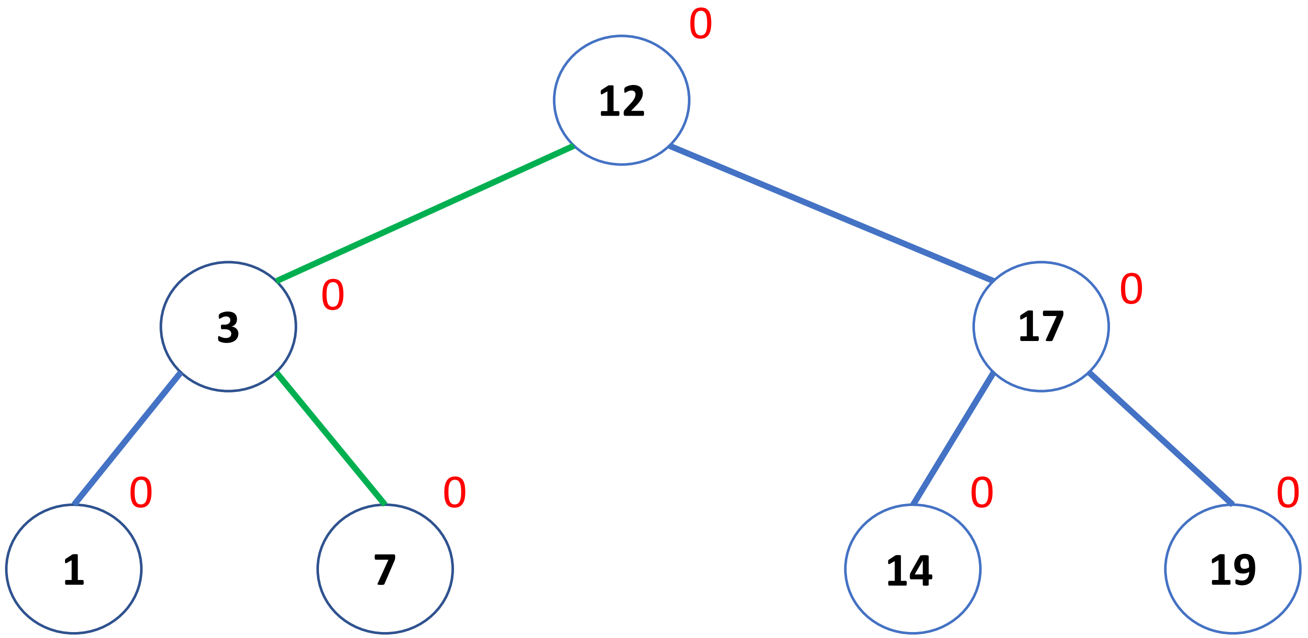

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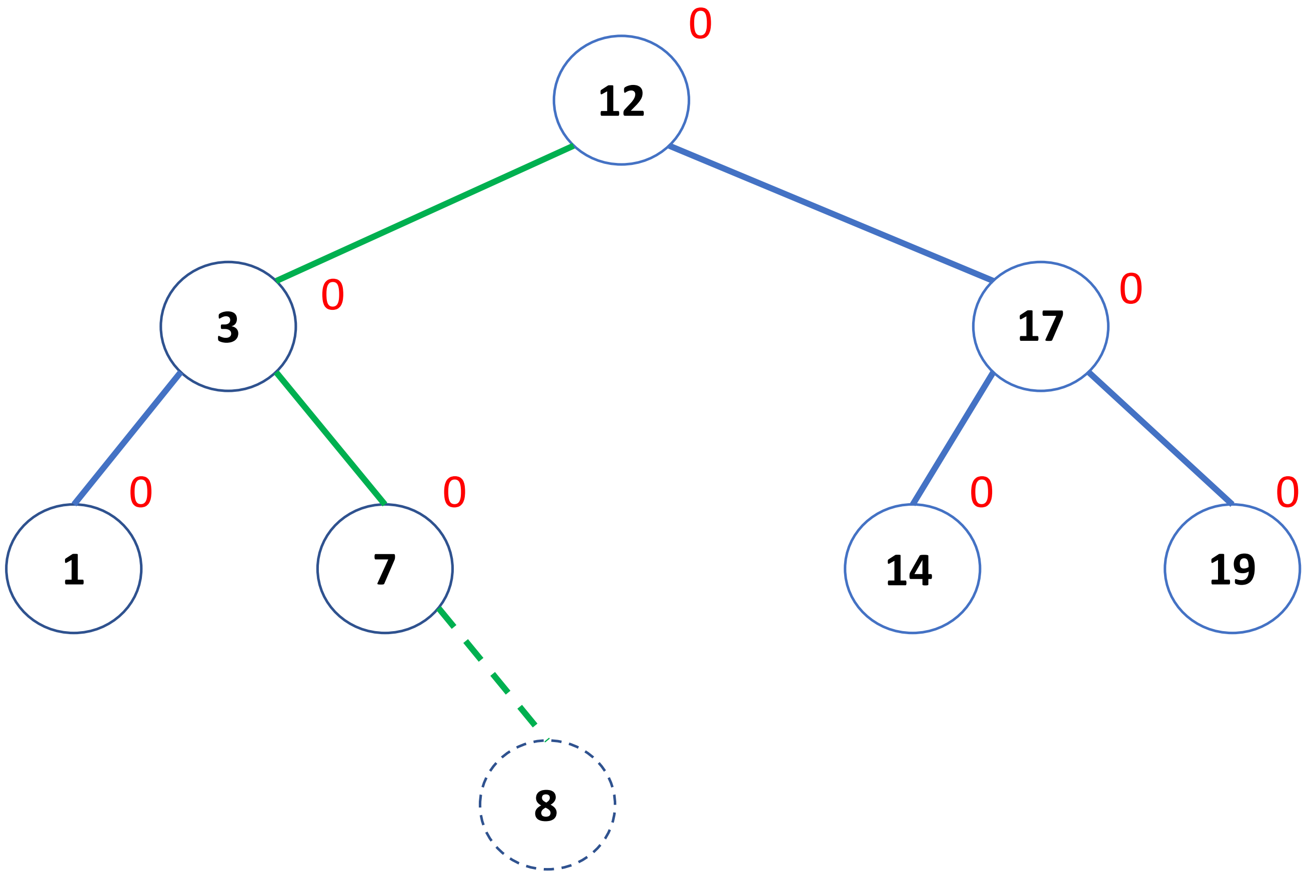

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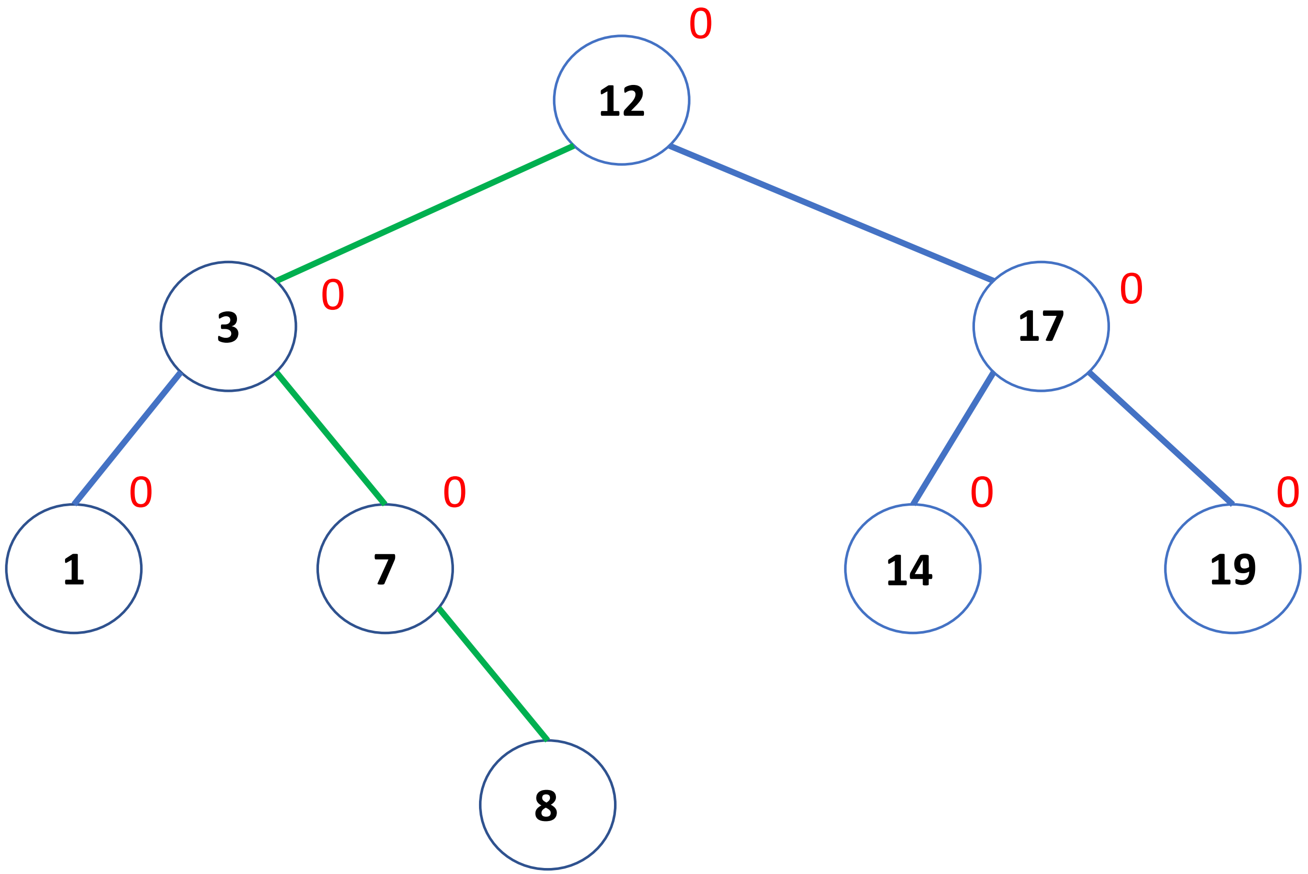

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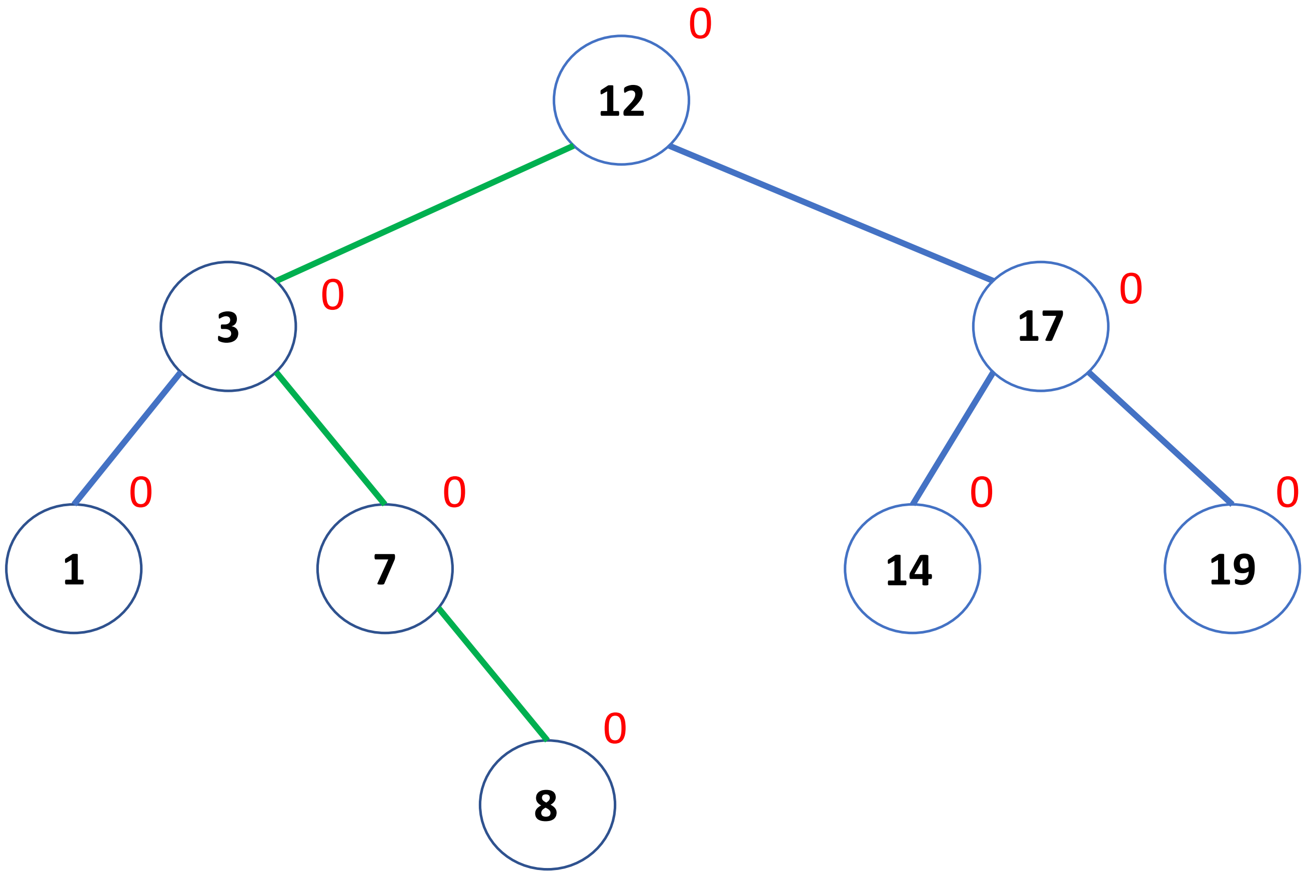

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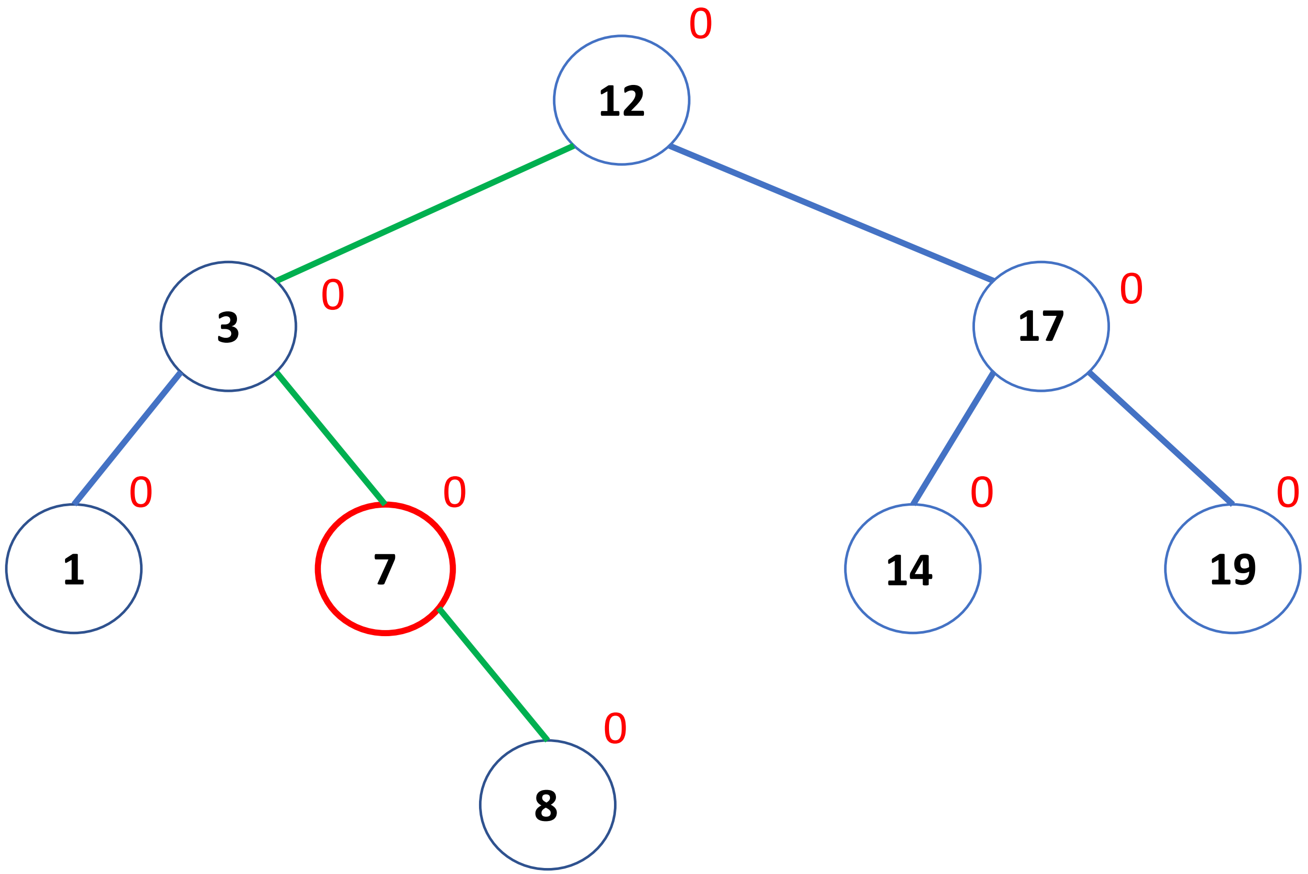

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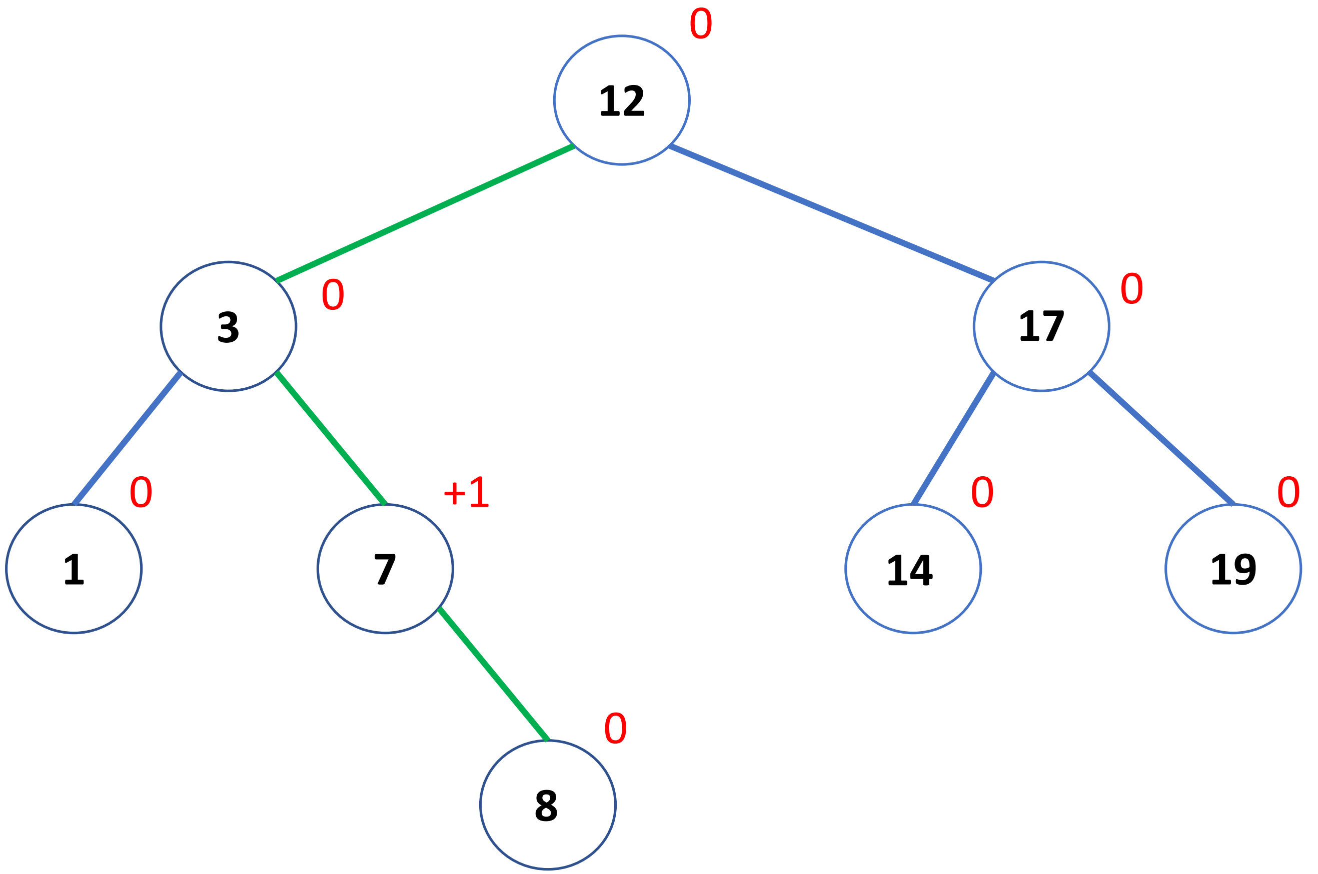

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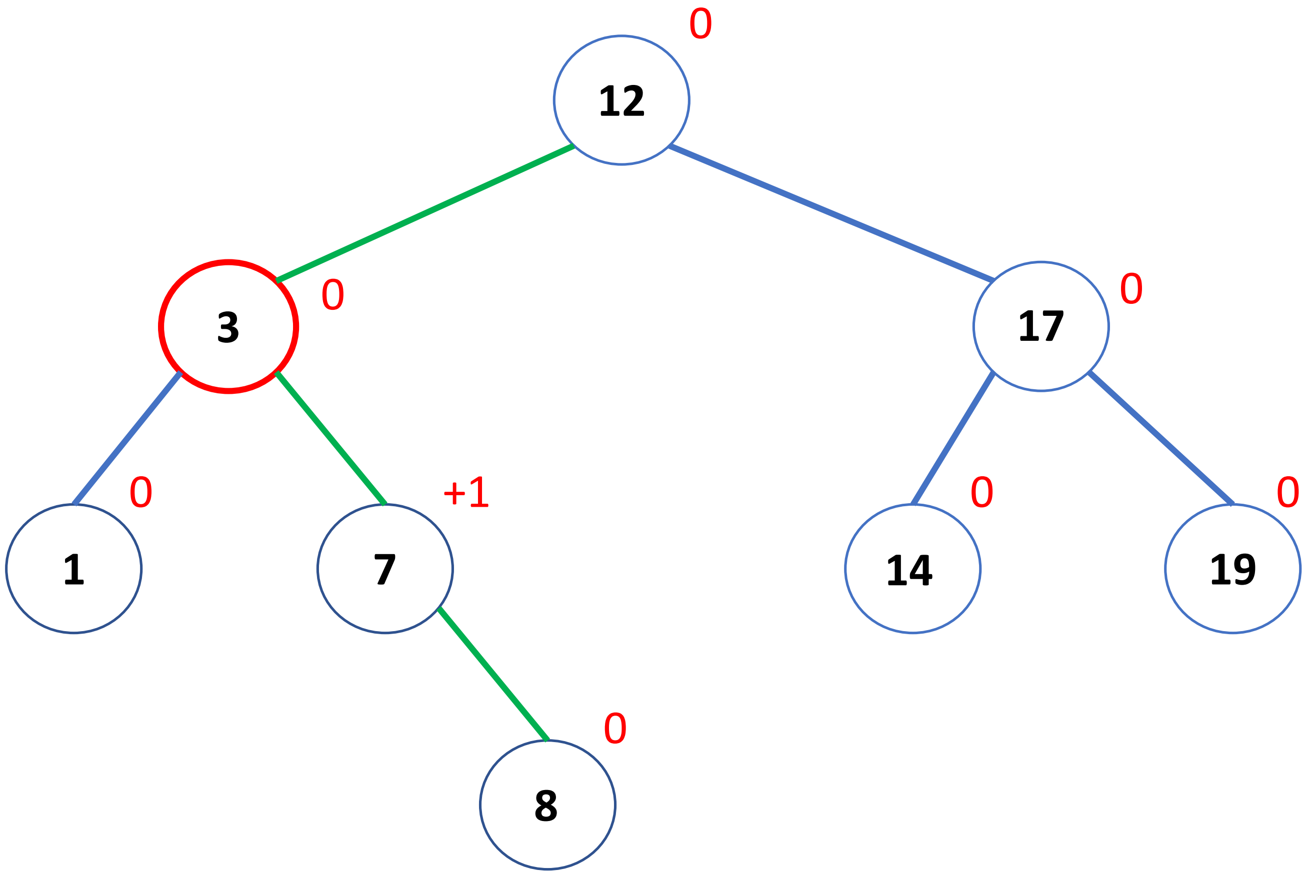

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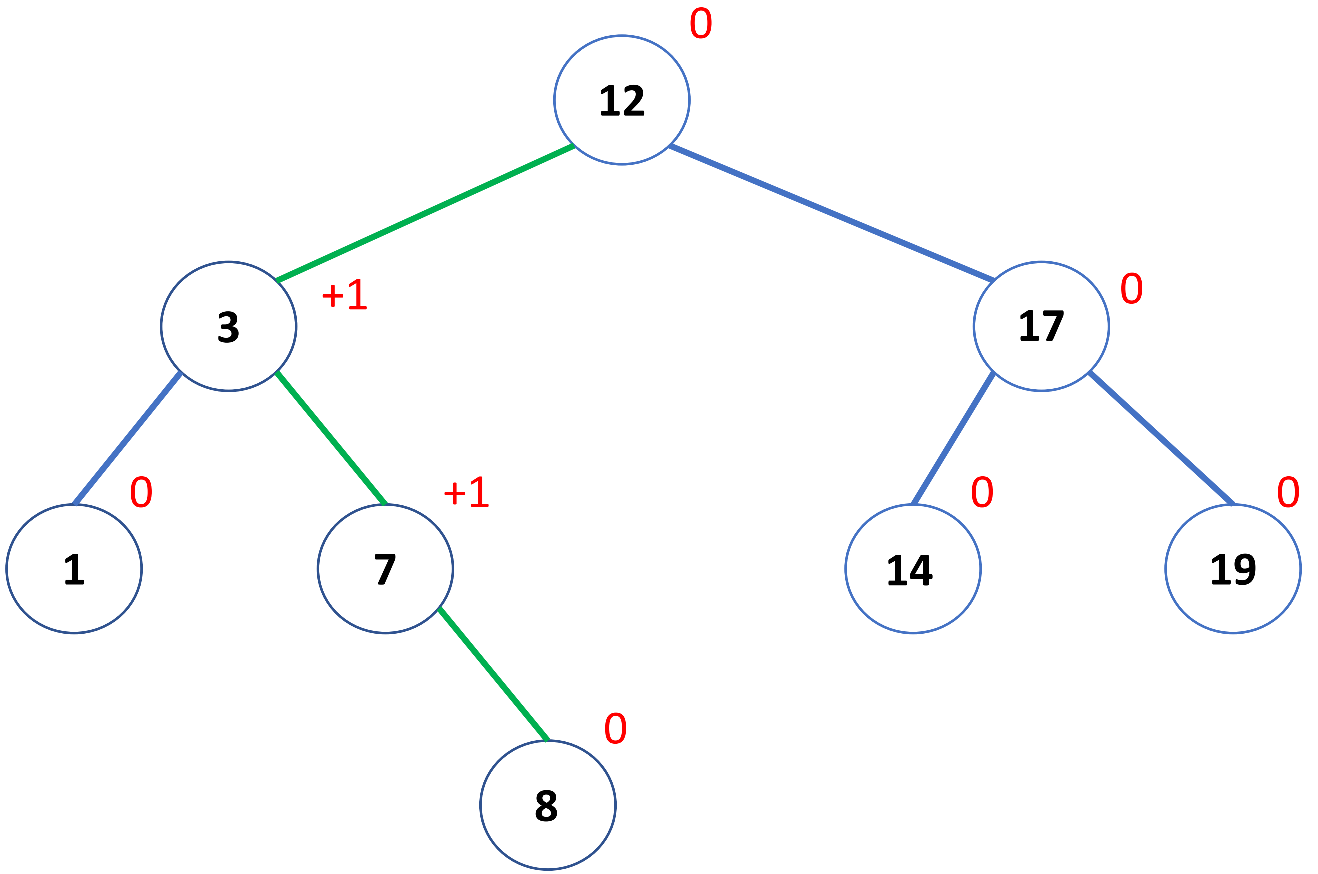

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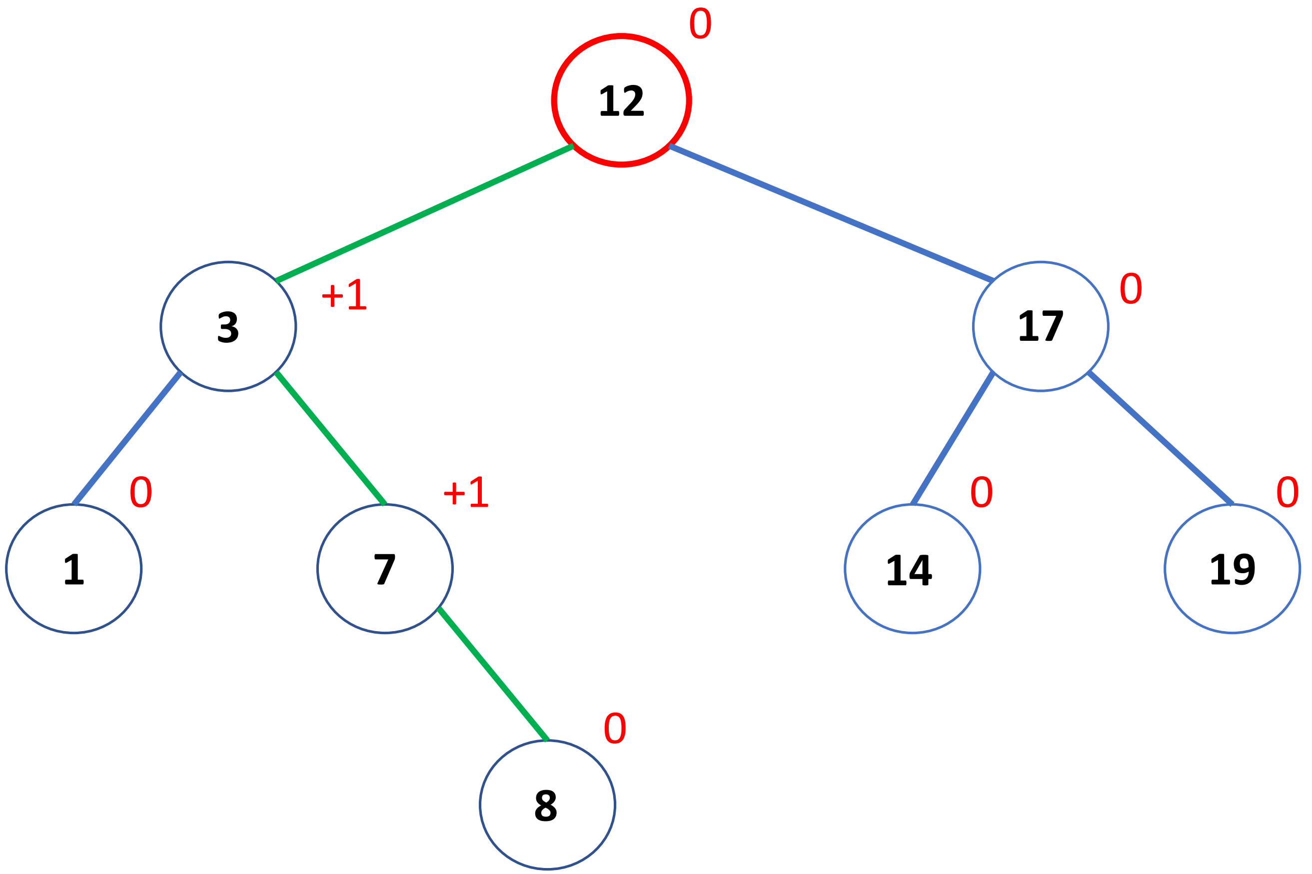

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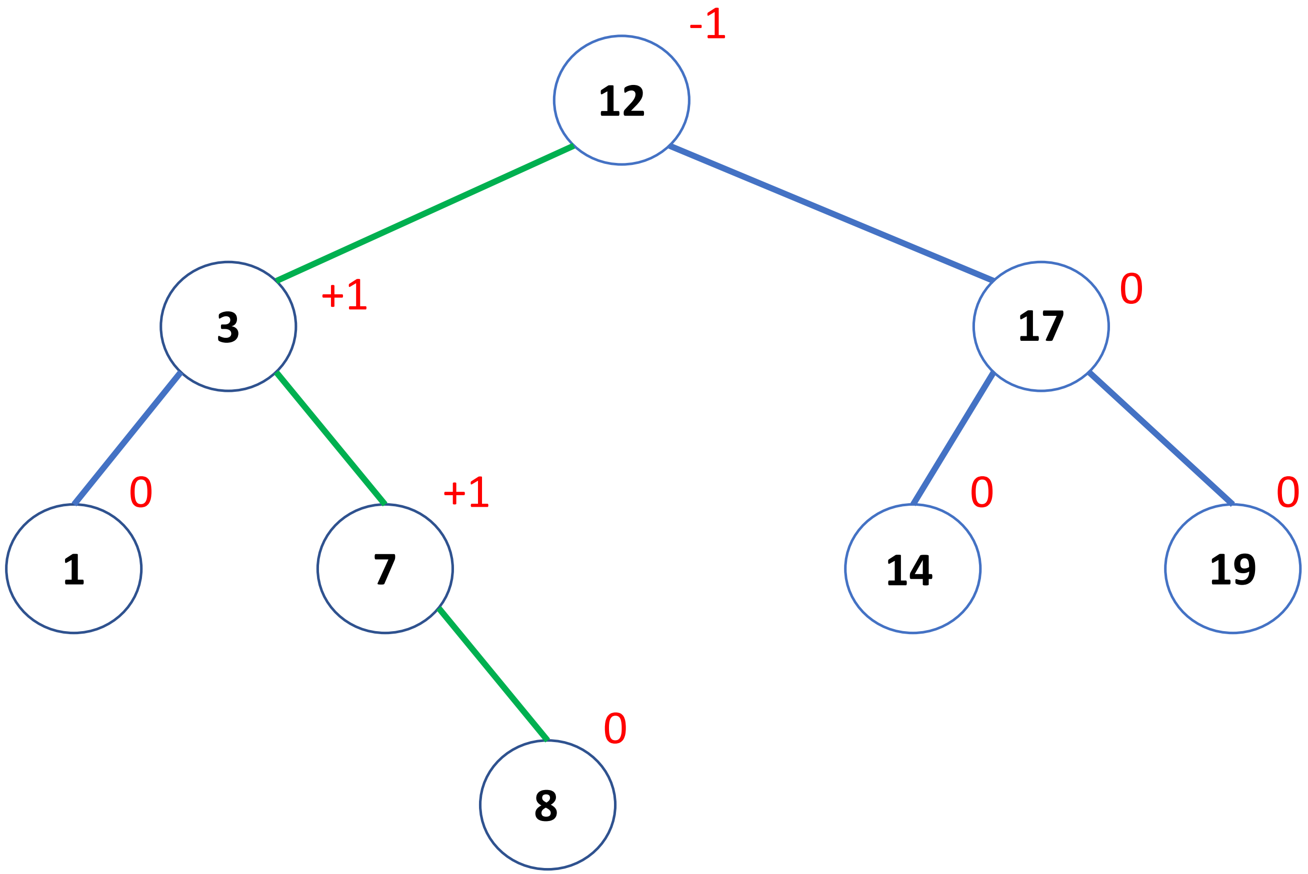

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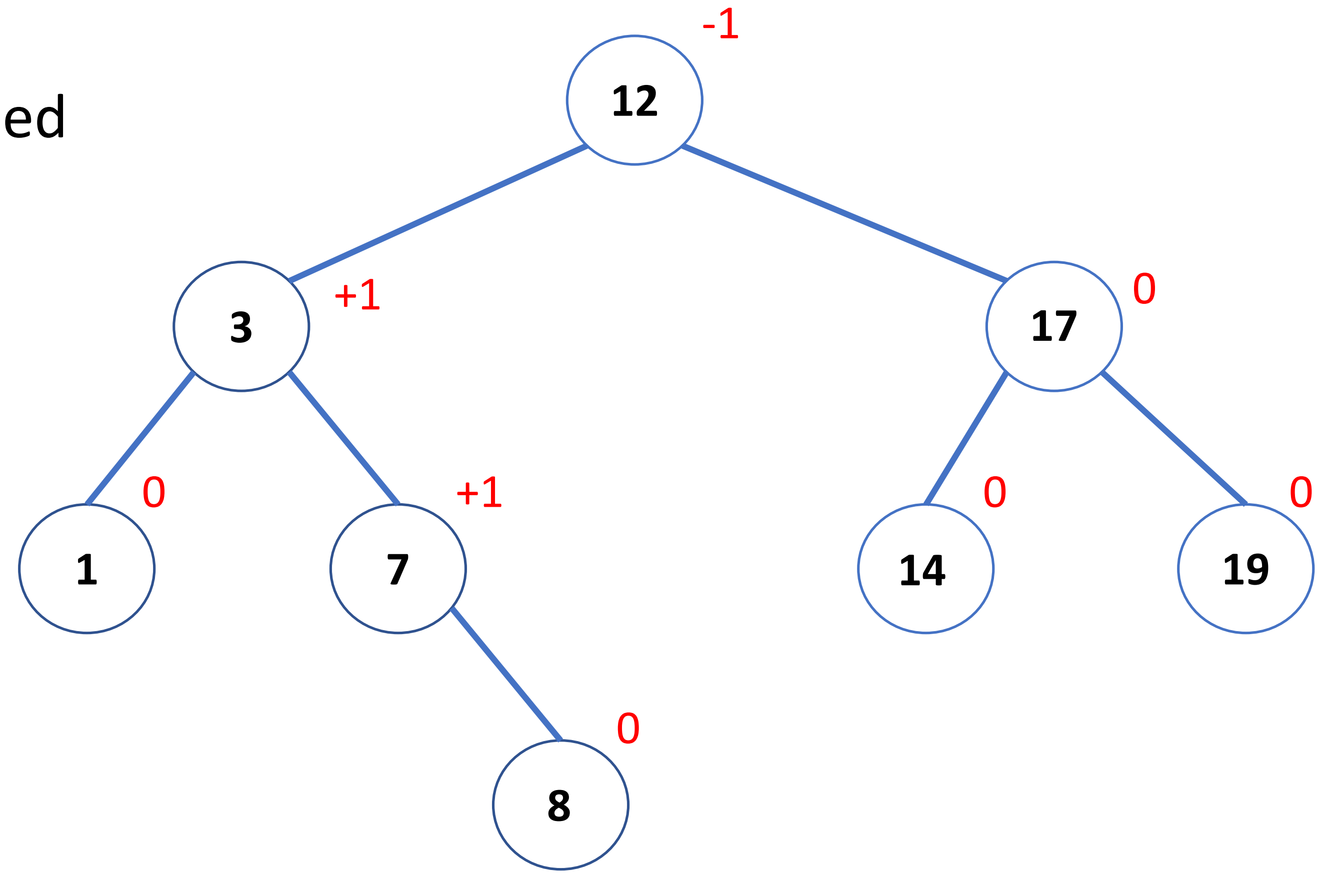
Insert(T, 8) :

No rebalancing needed



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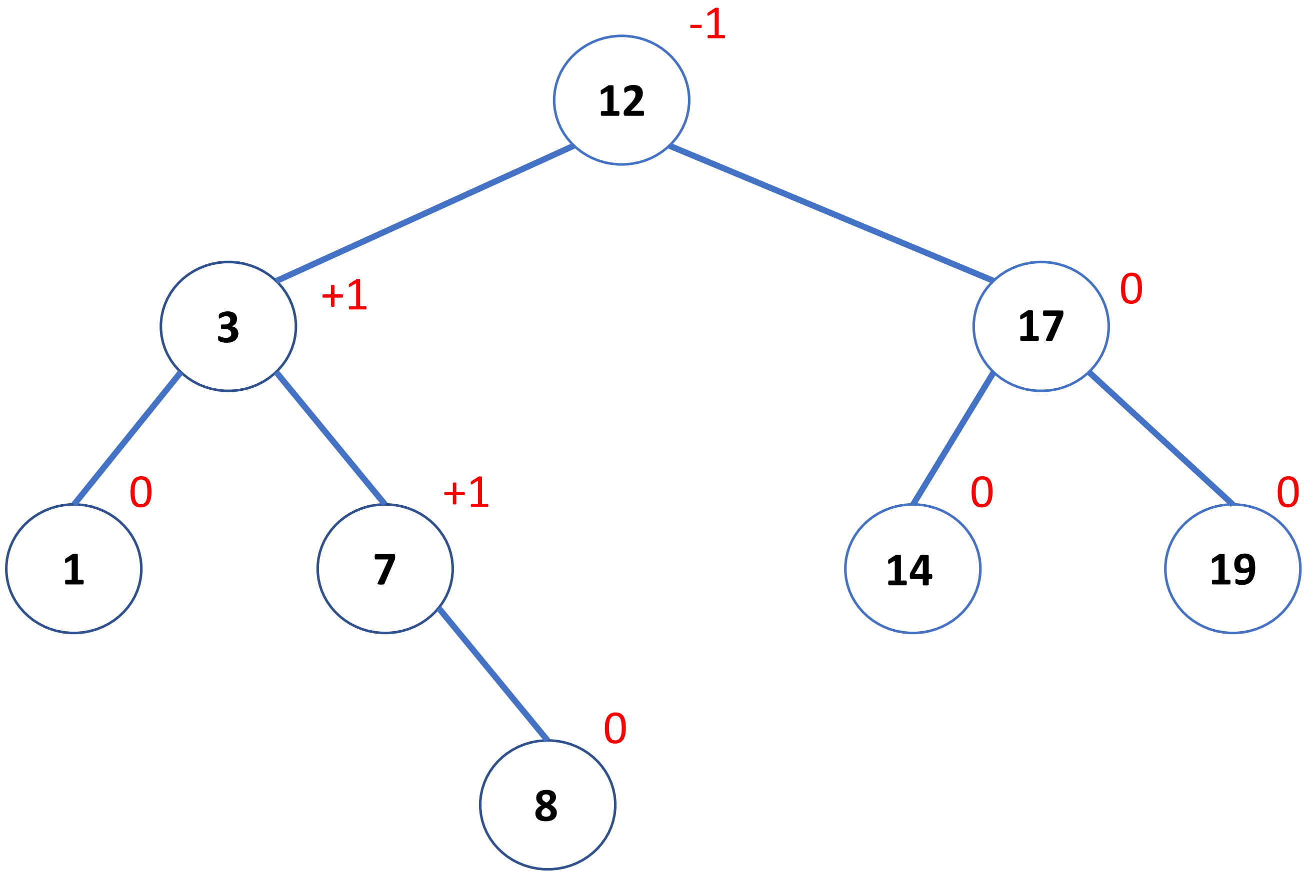
Insert(T, 5)



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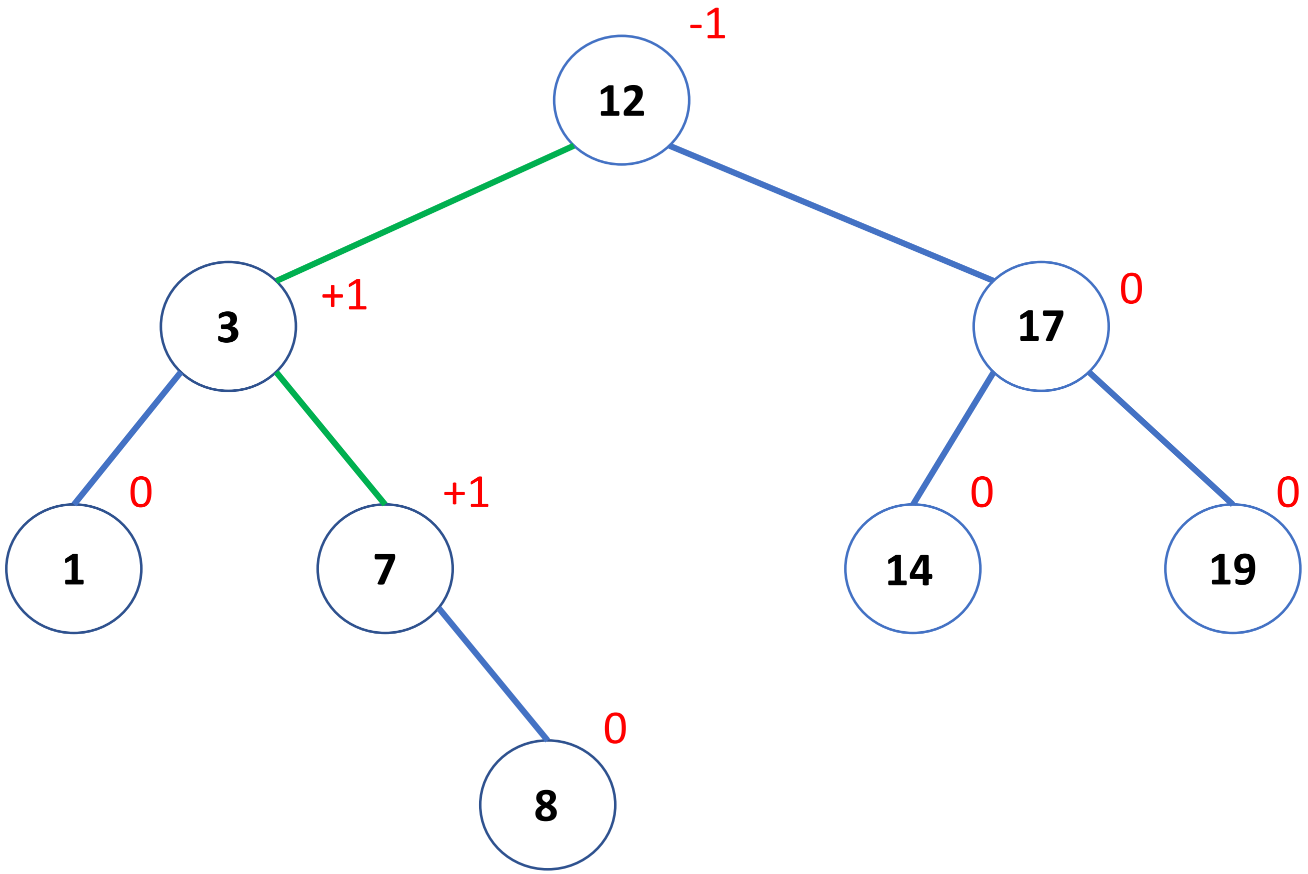

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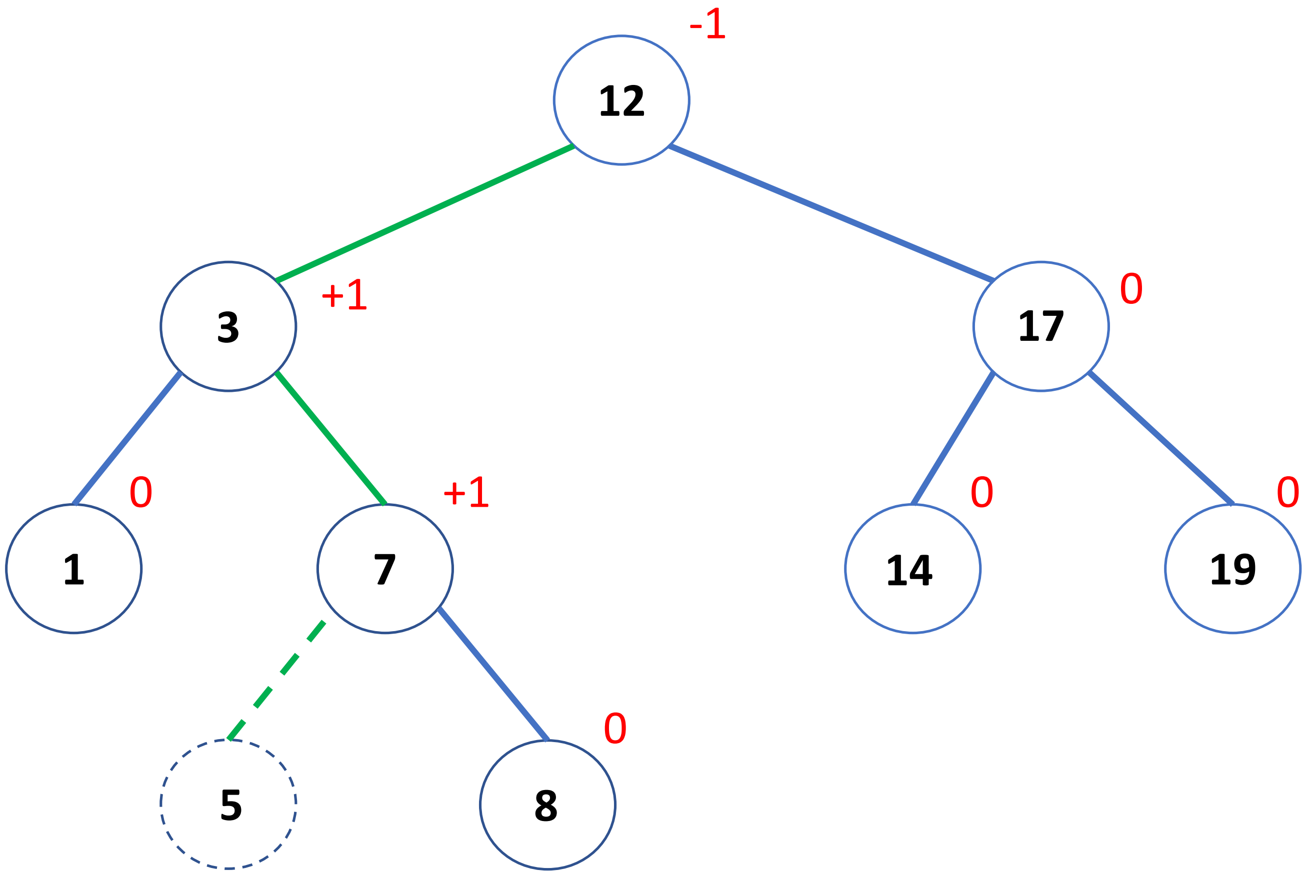

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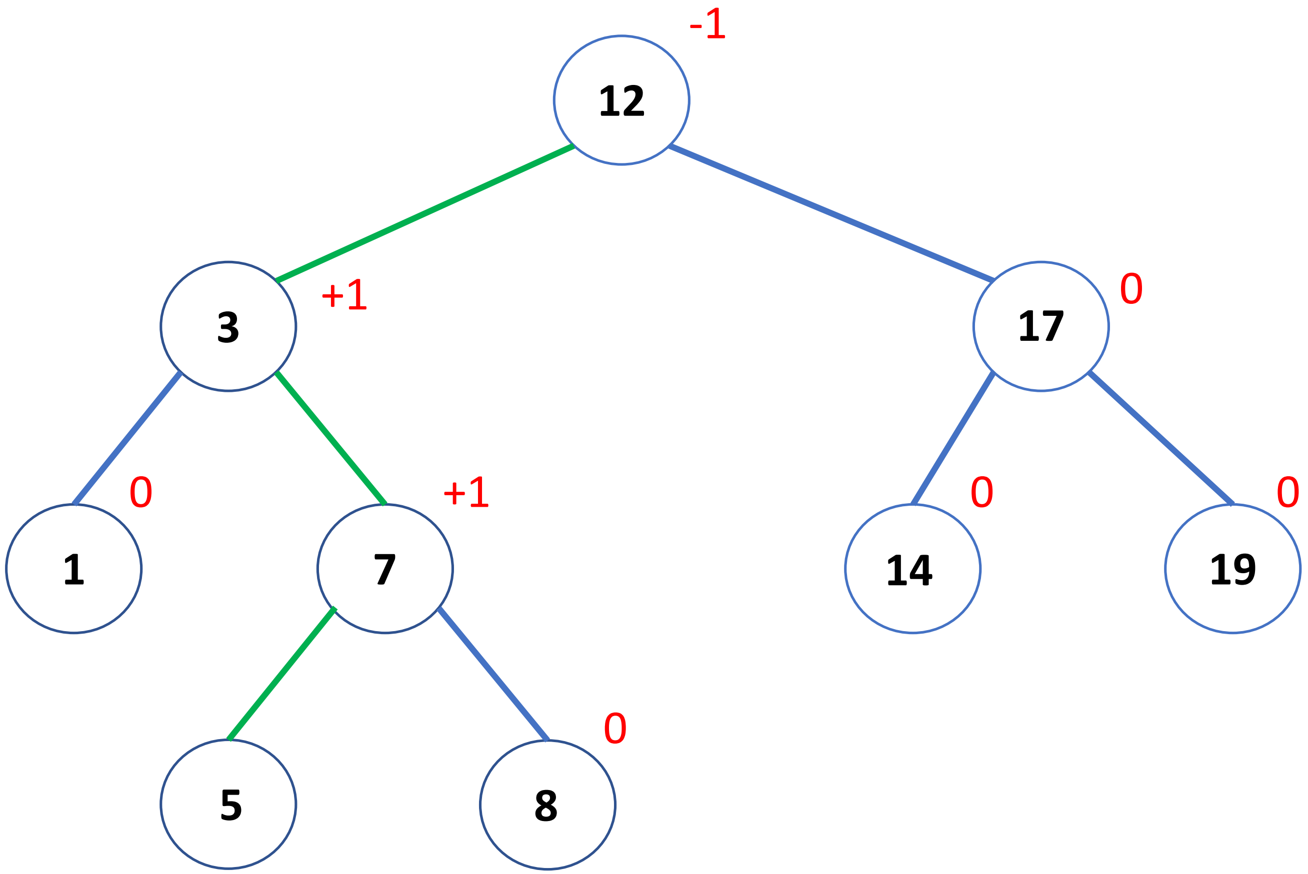

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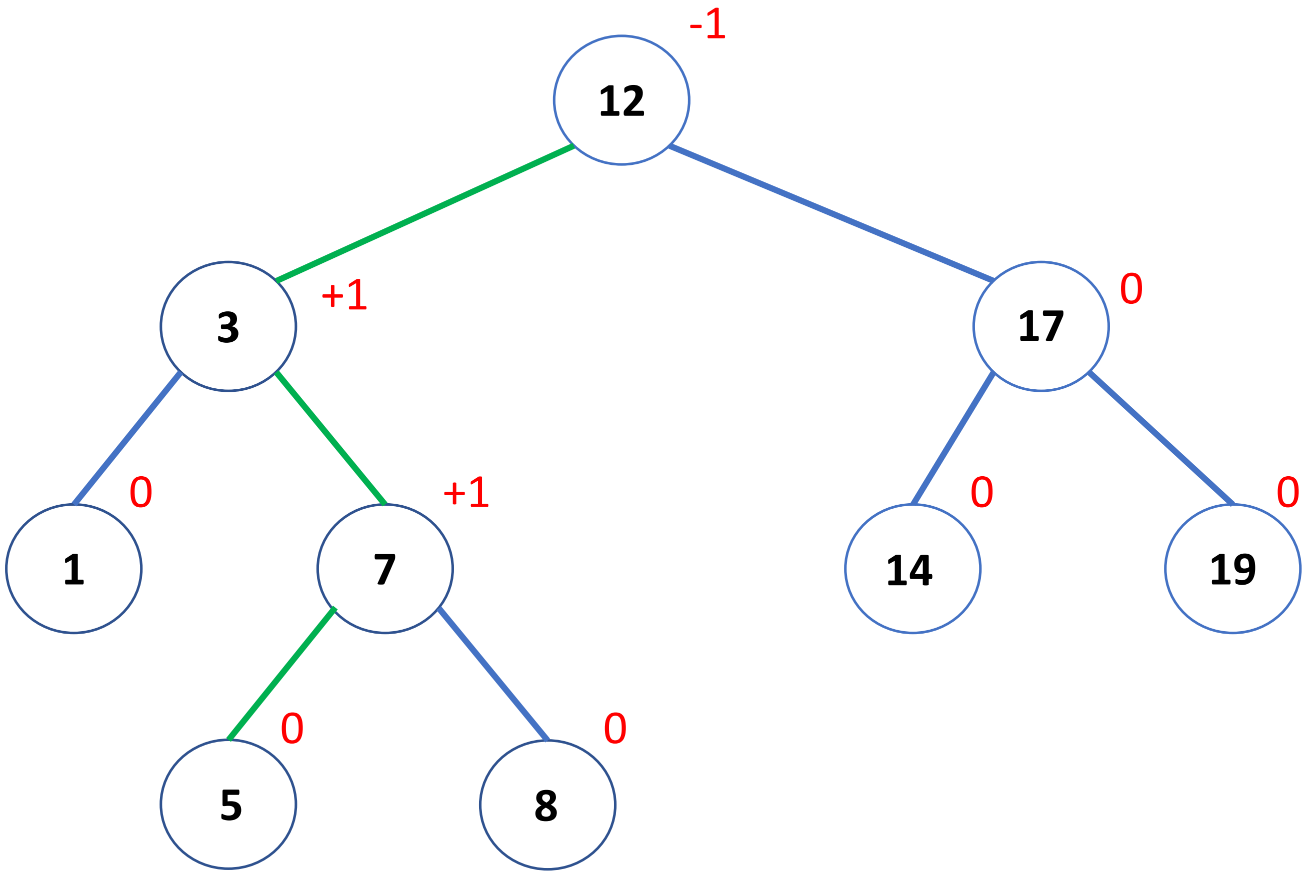

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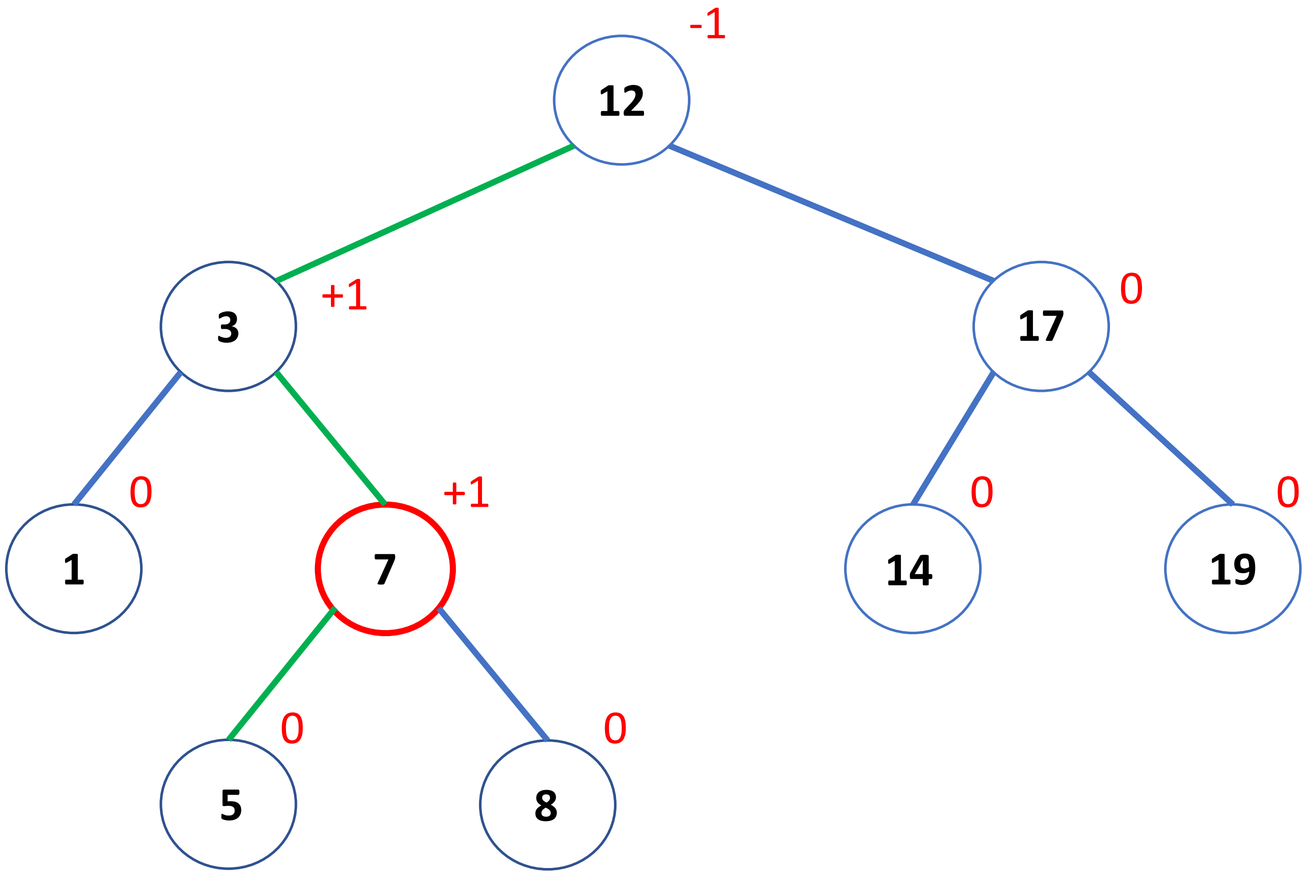

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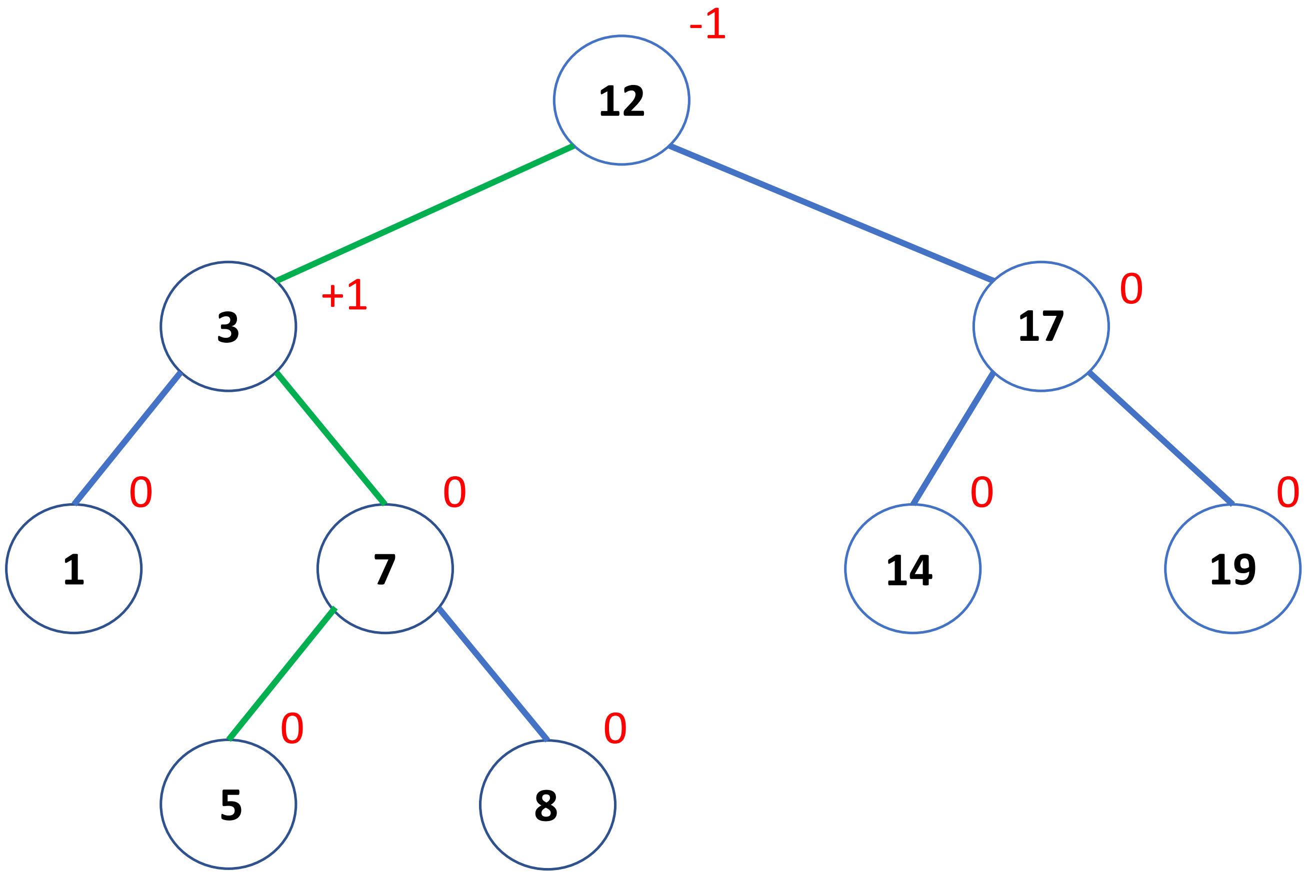
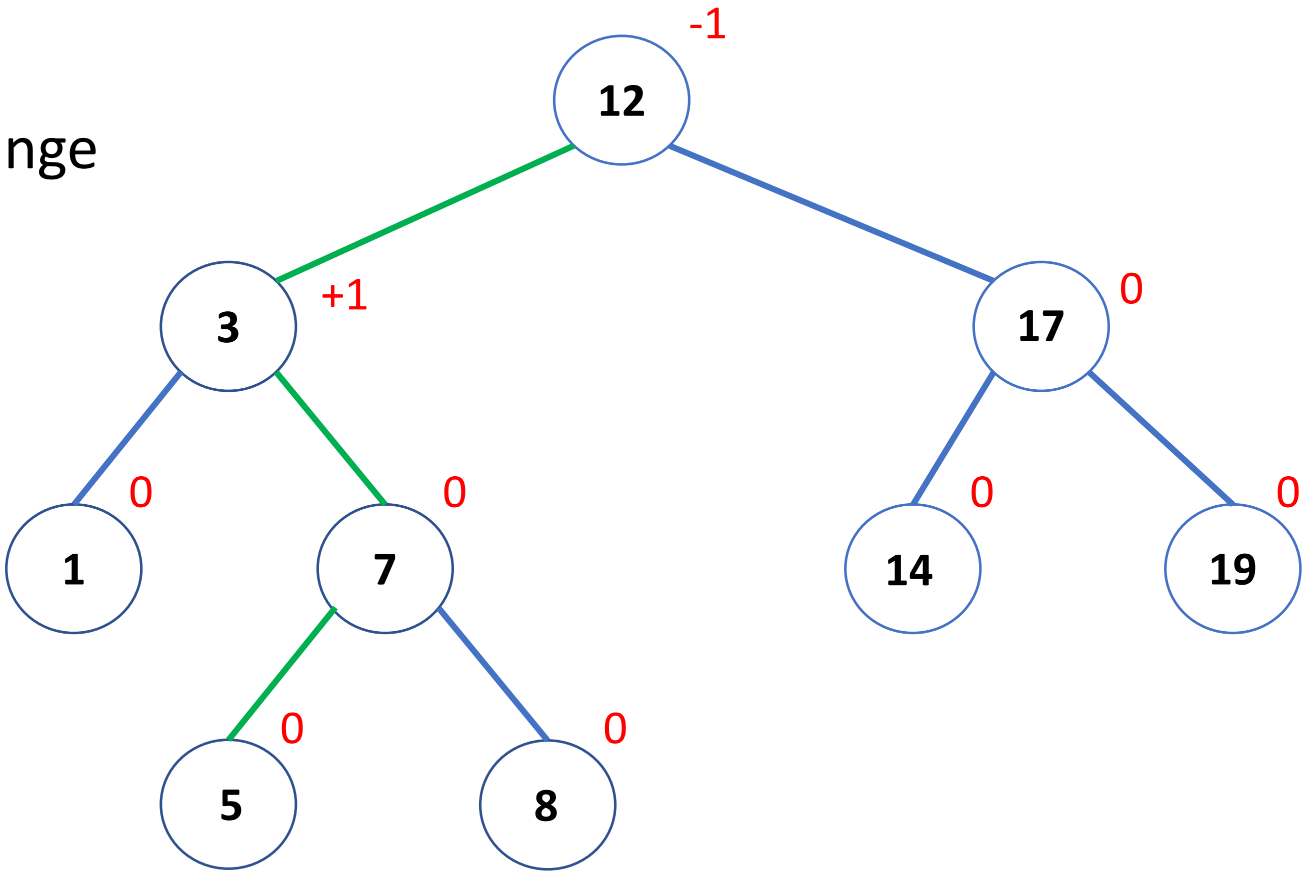


Example: AVL of {1, 3, 7, 8, 12, 14, 17, 19}

Insert(T, 5):

height(7) doesn't change



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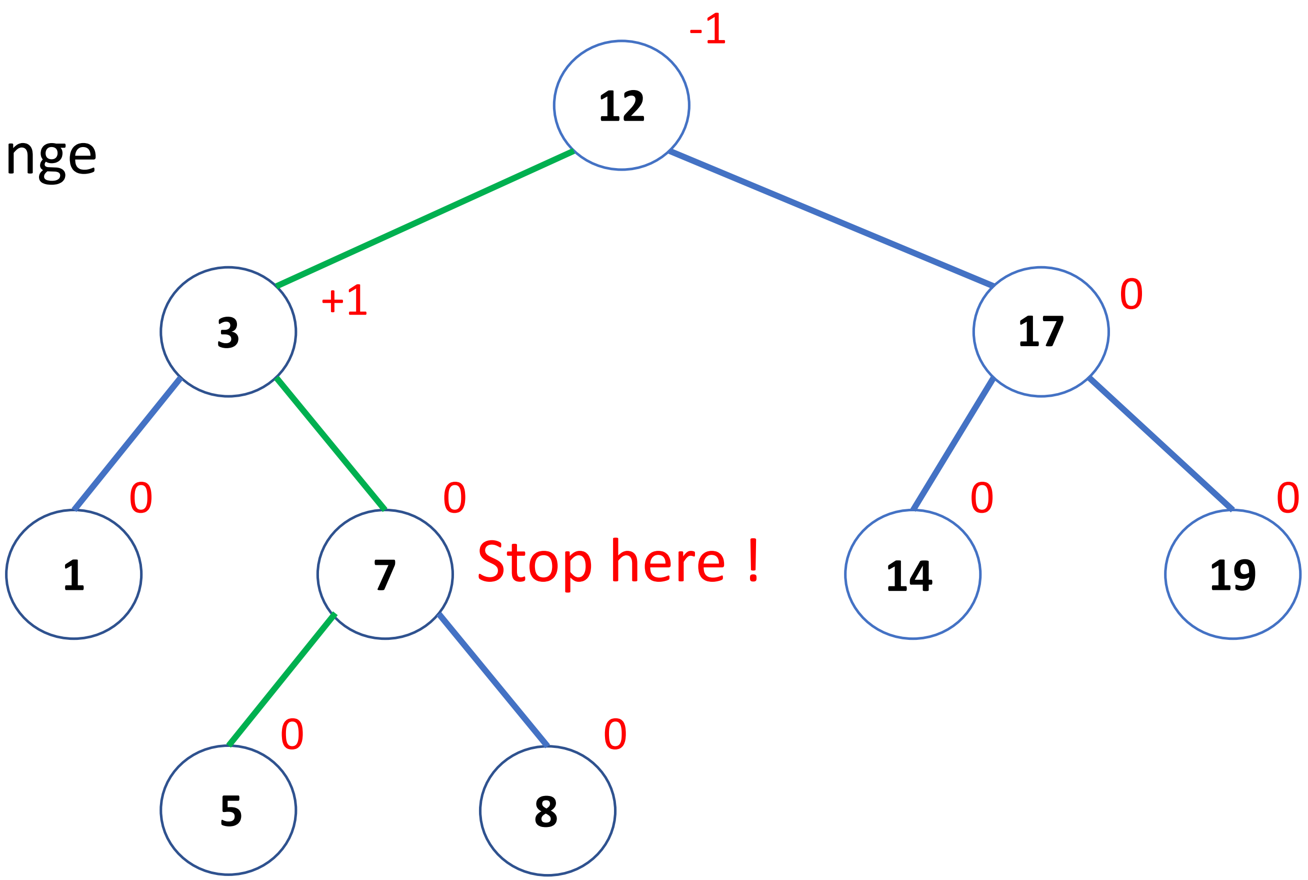
height(7) doesn't change
⇒ Stop at 7 !



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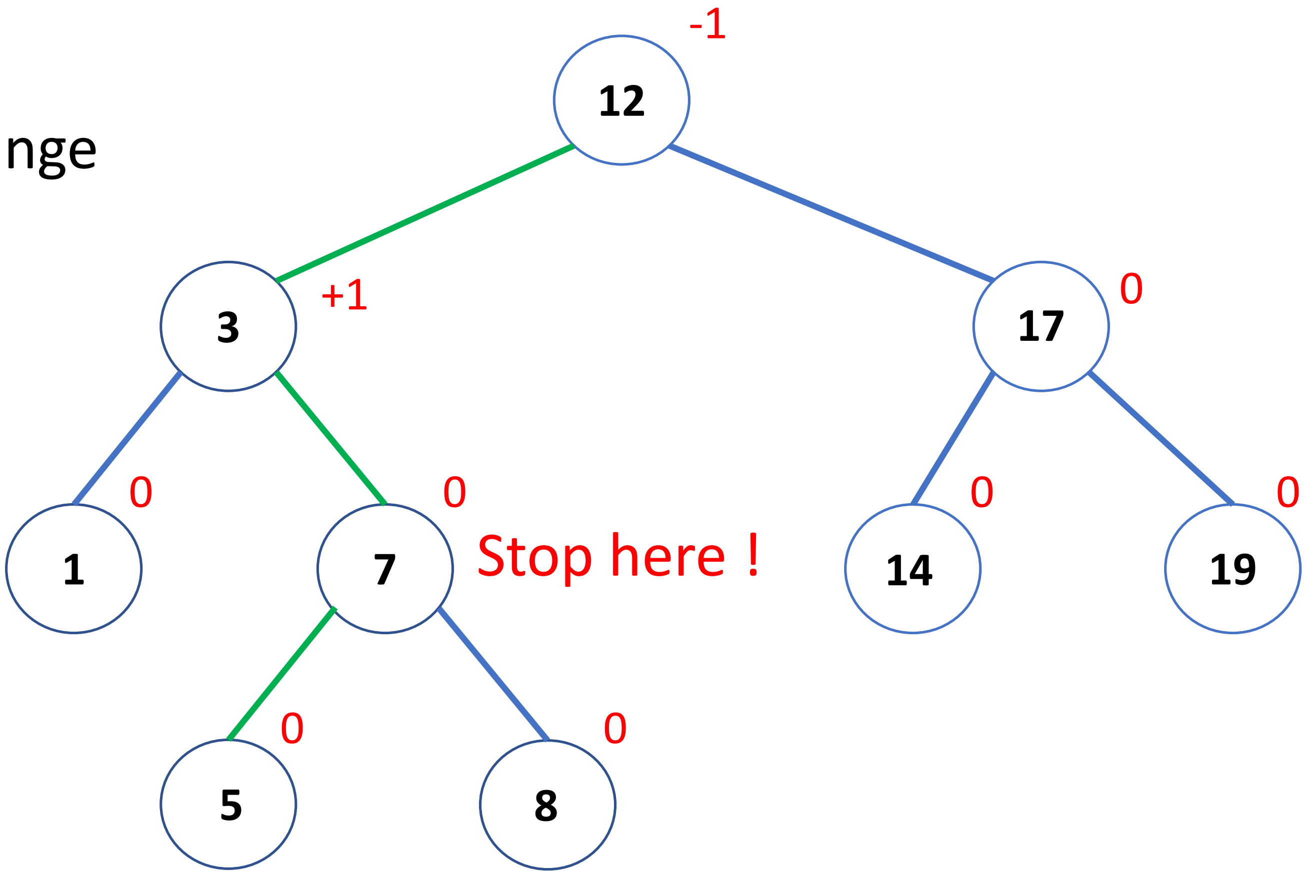
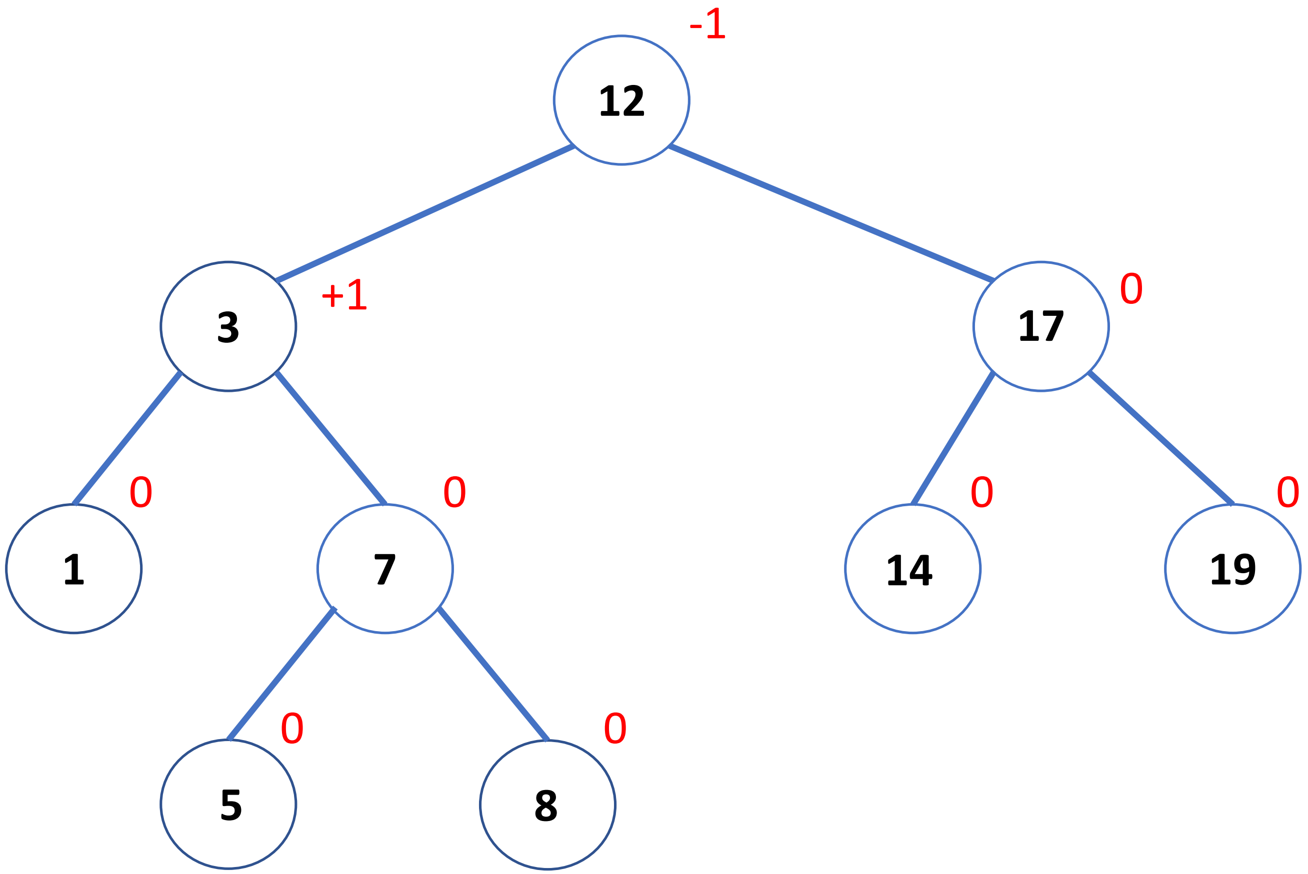


No rebalancing needed !

Example: AVL of {1, 3, 5, 7, 8, 12, 14, 17, 19}

Insert(T, 10)



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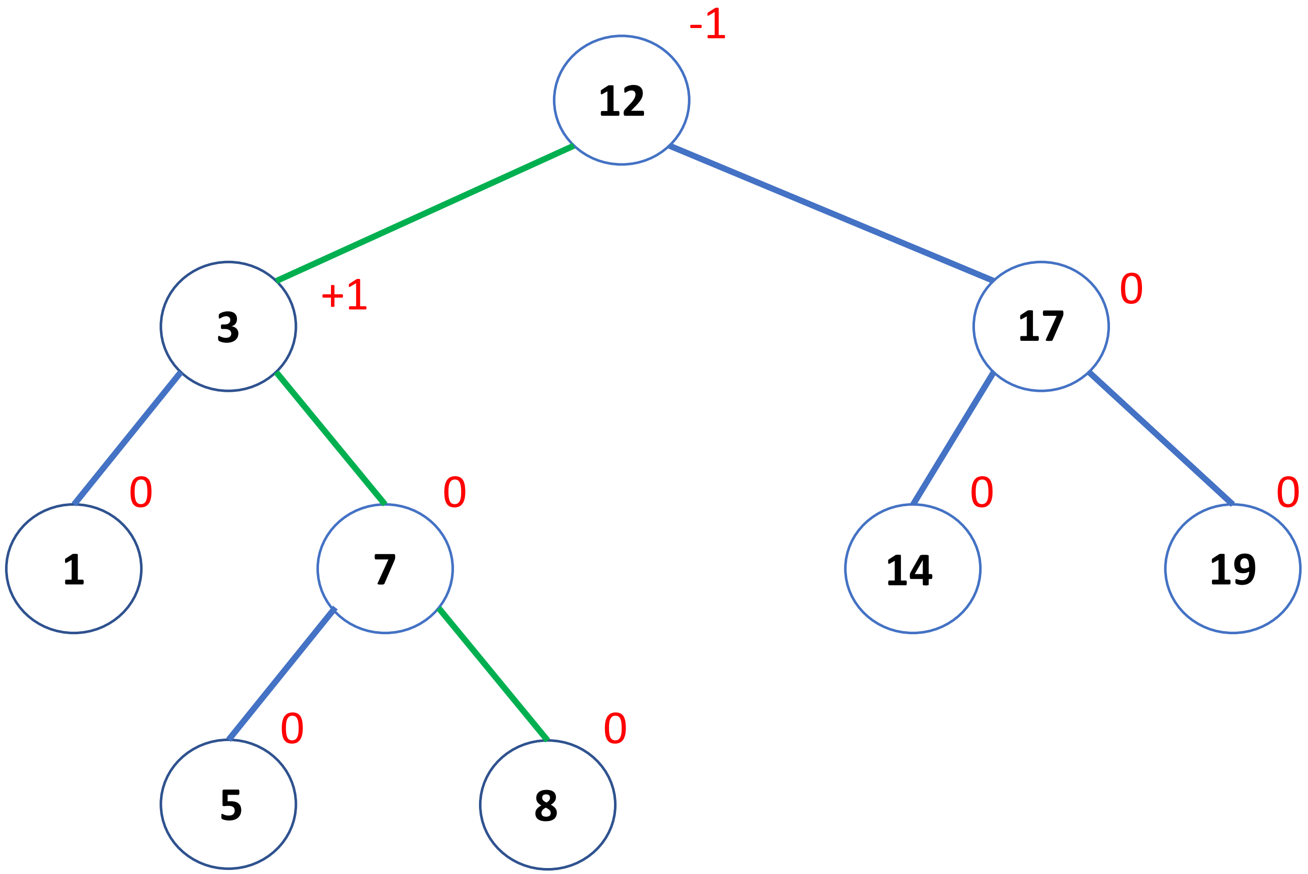

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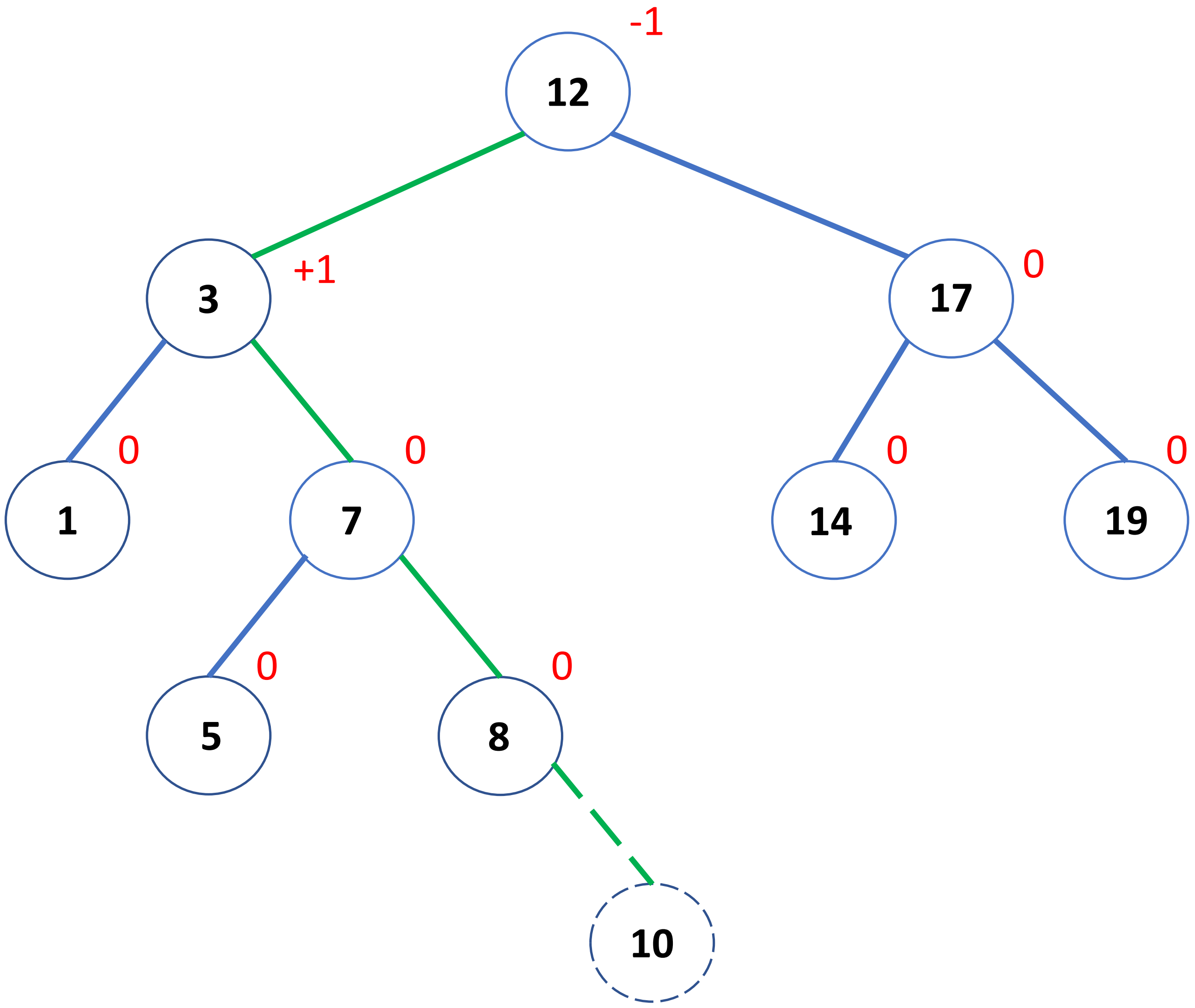

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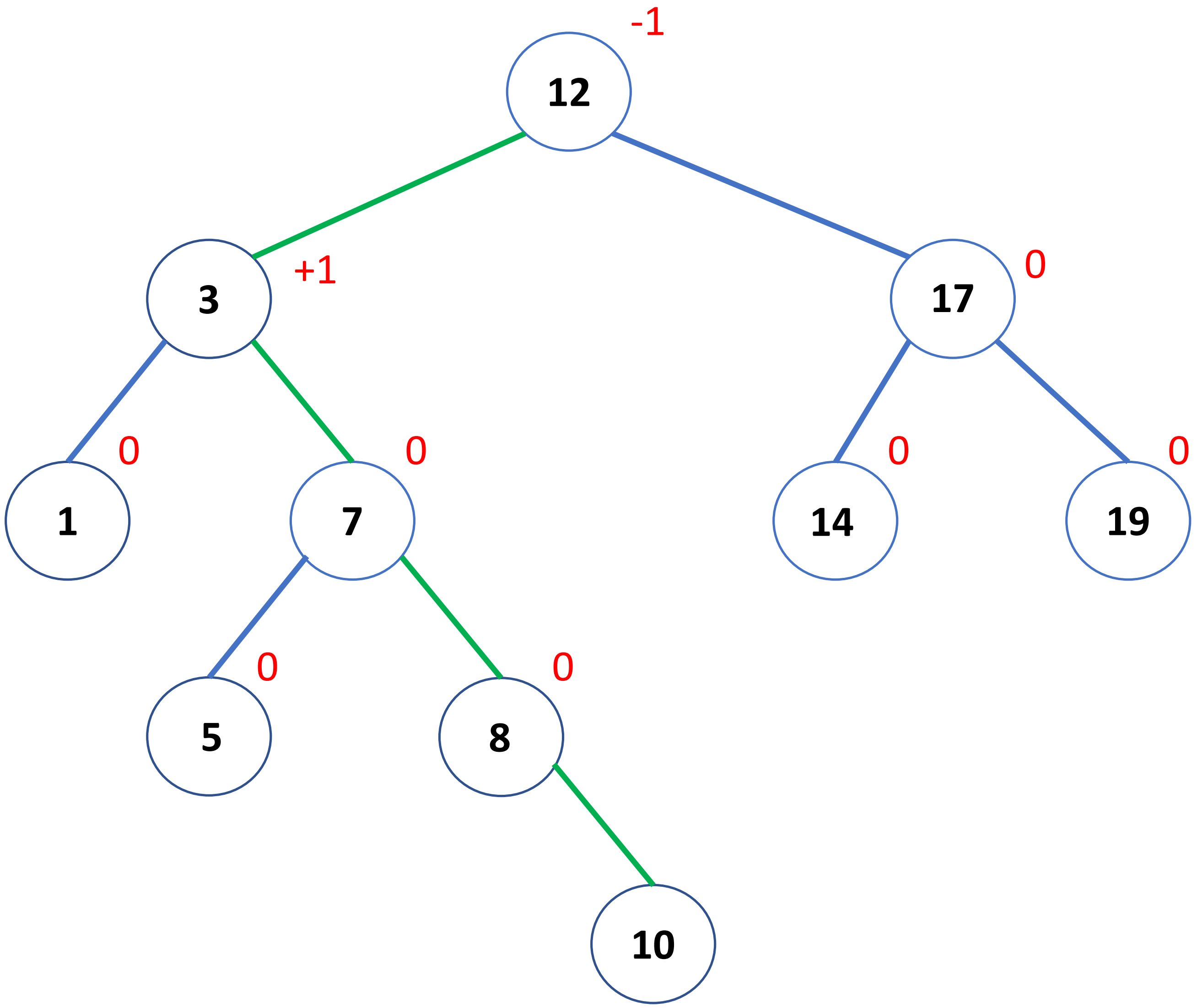

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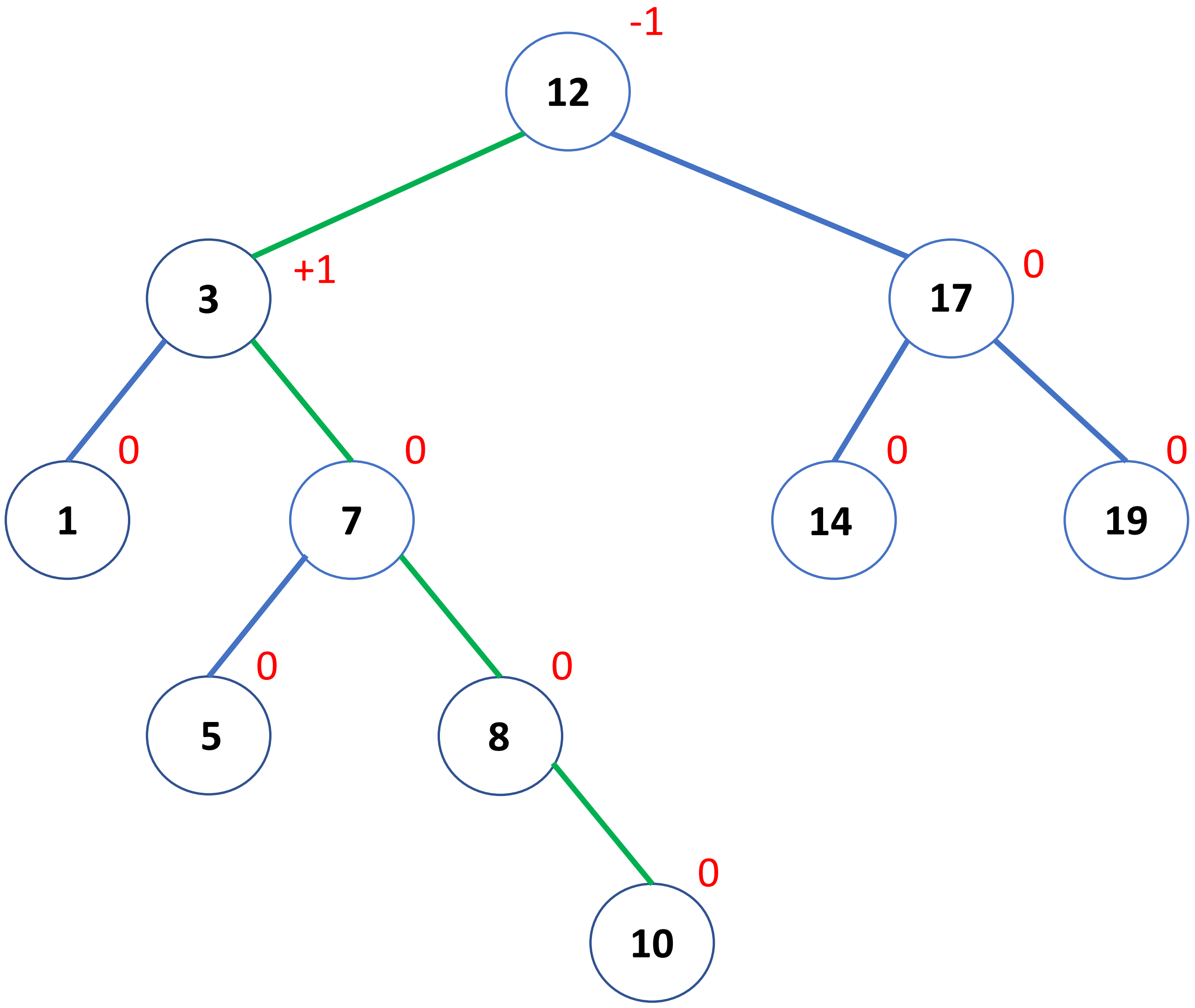

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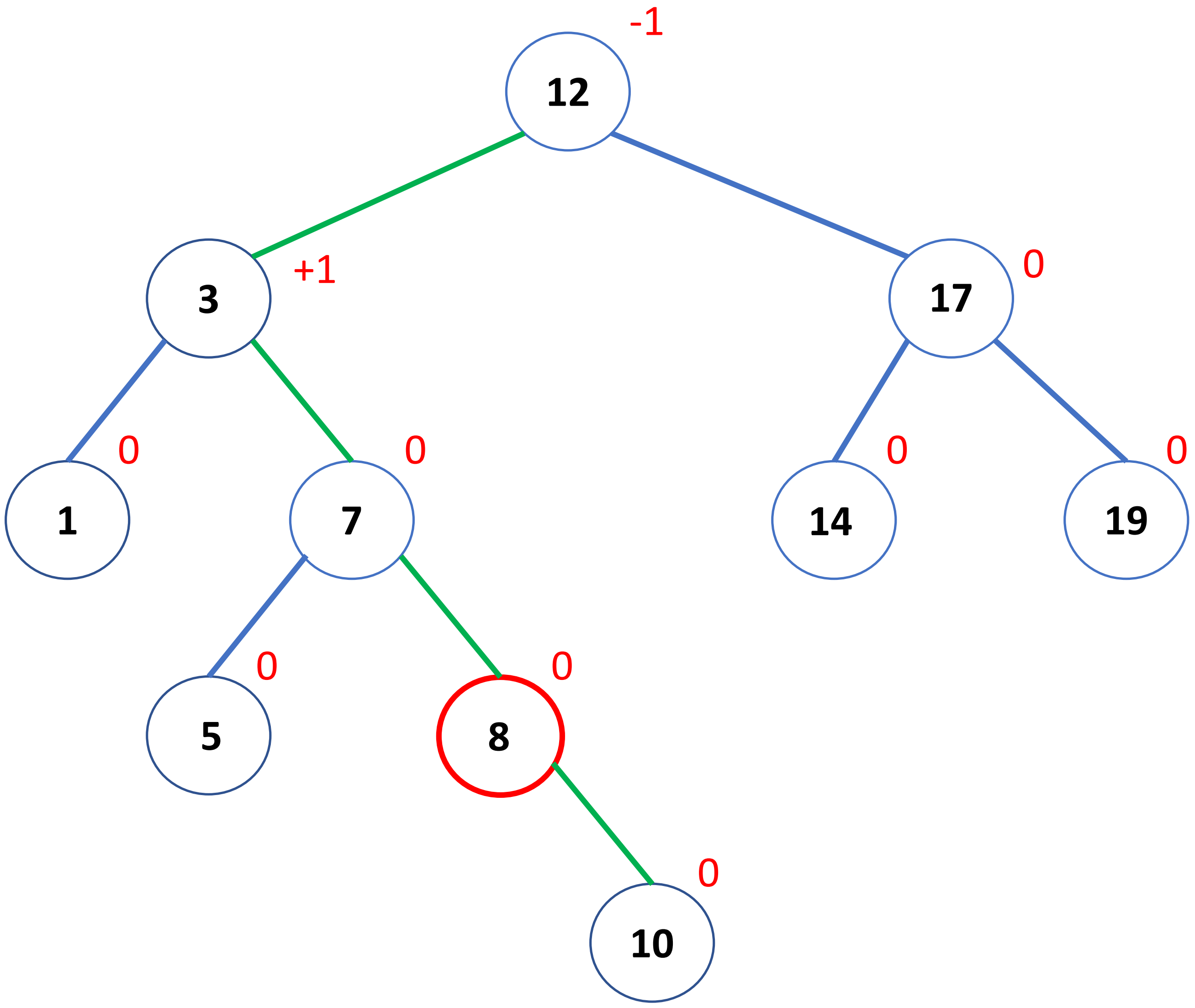

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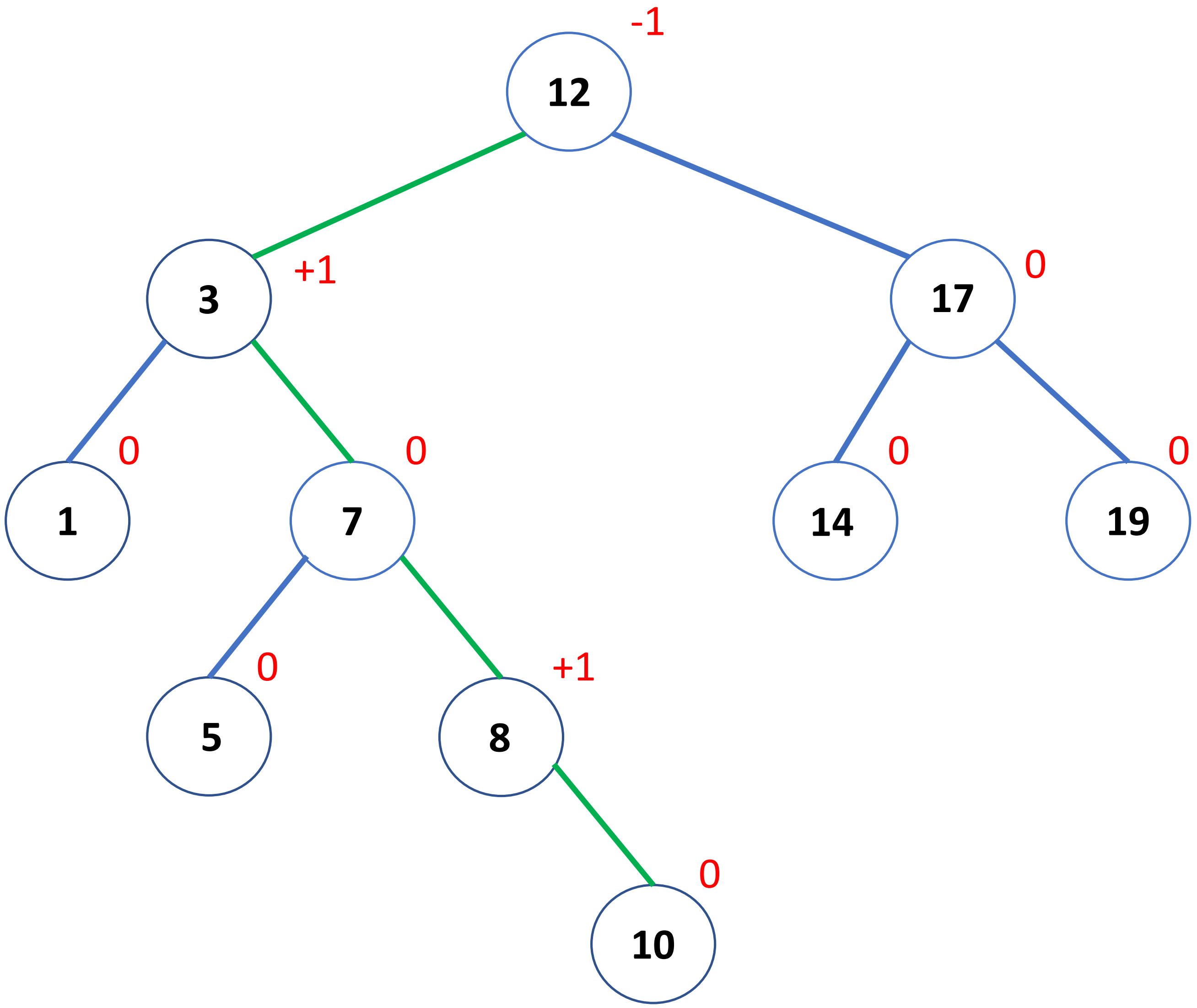

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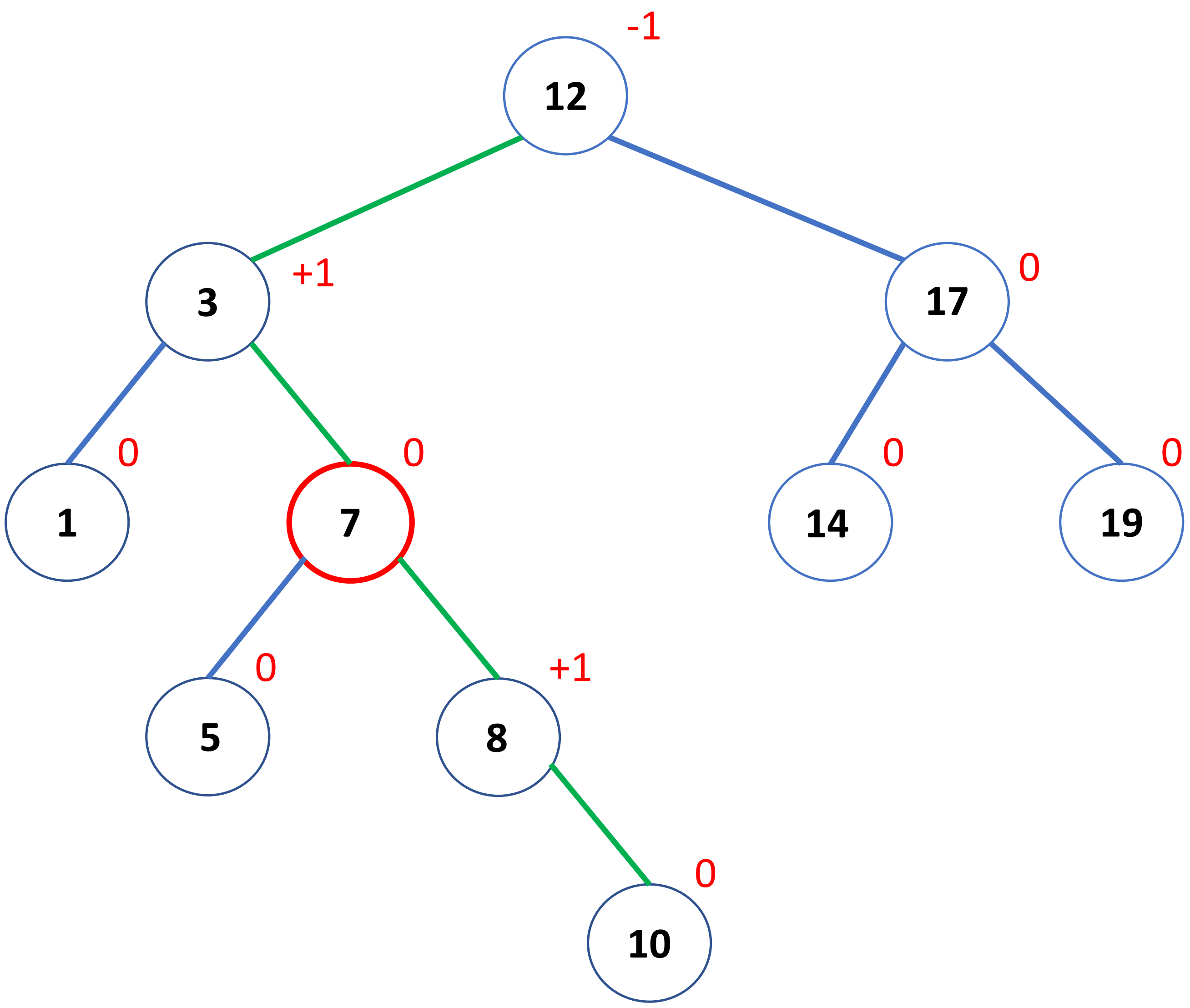

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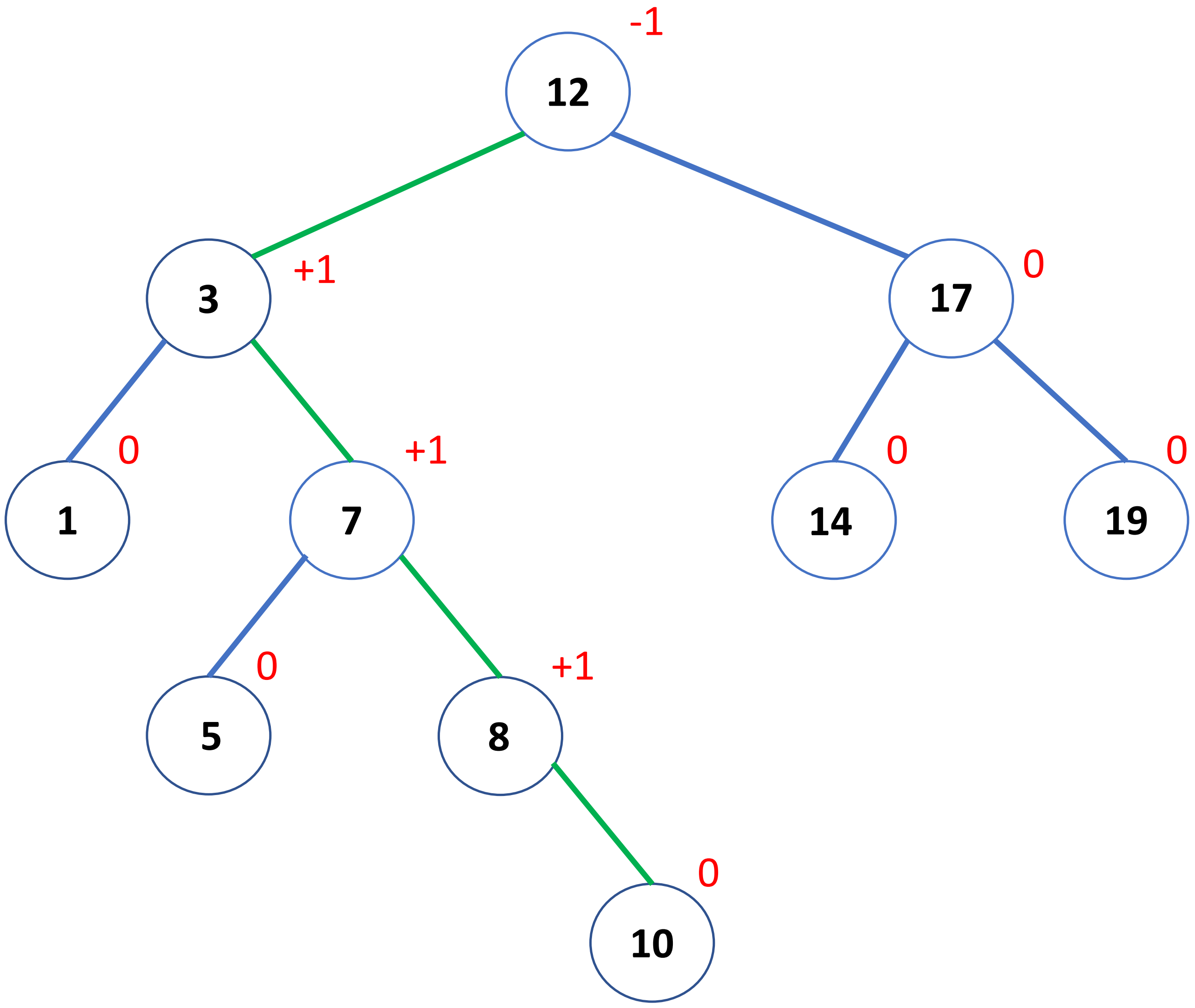

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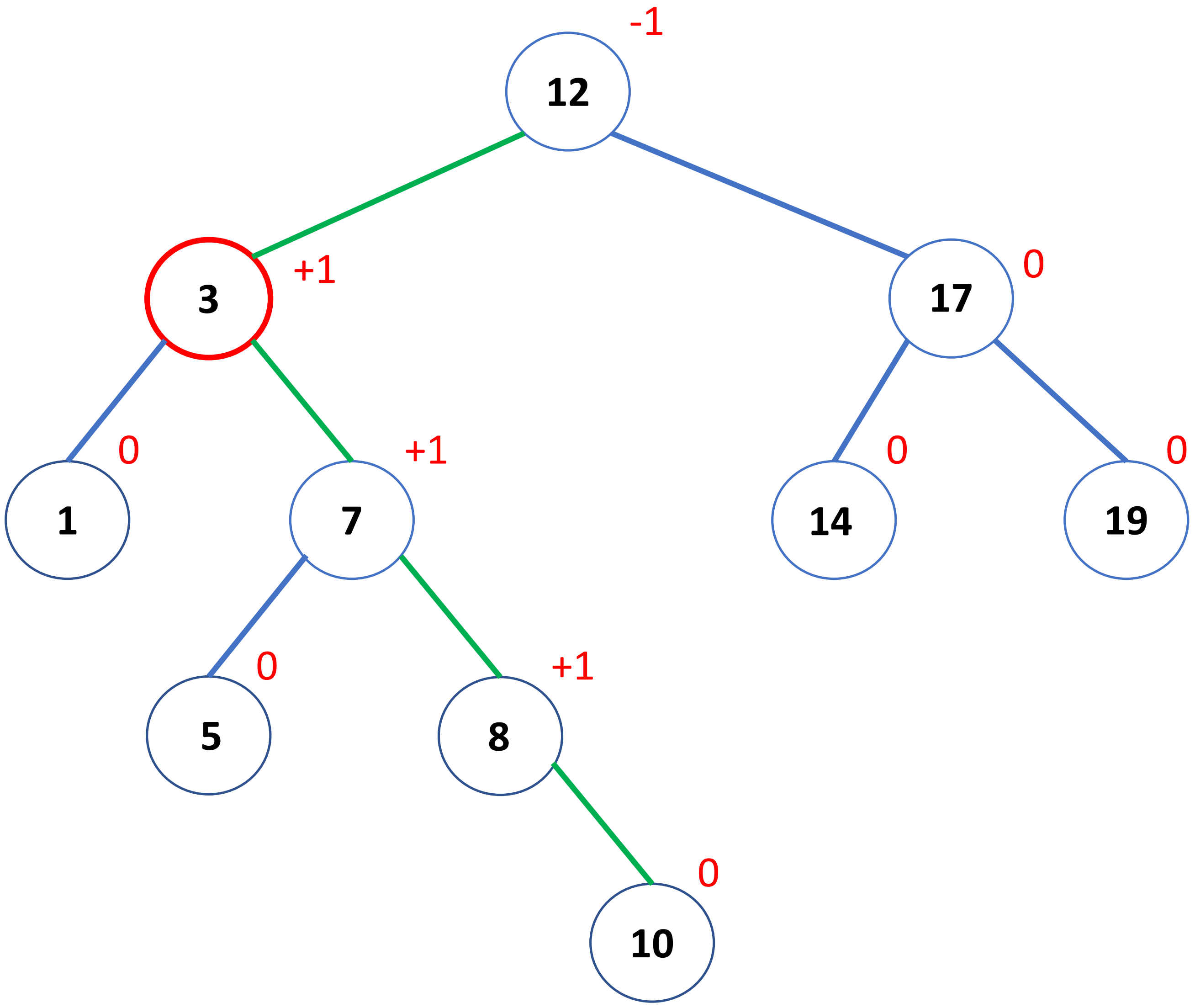

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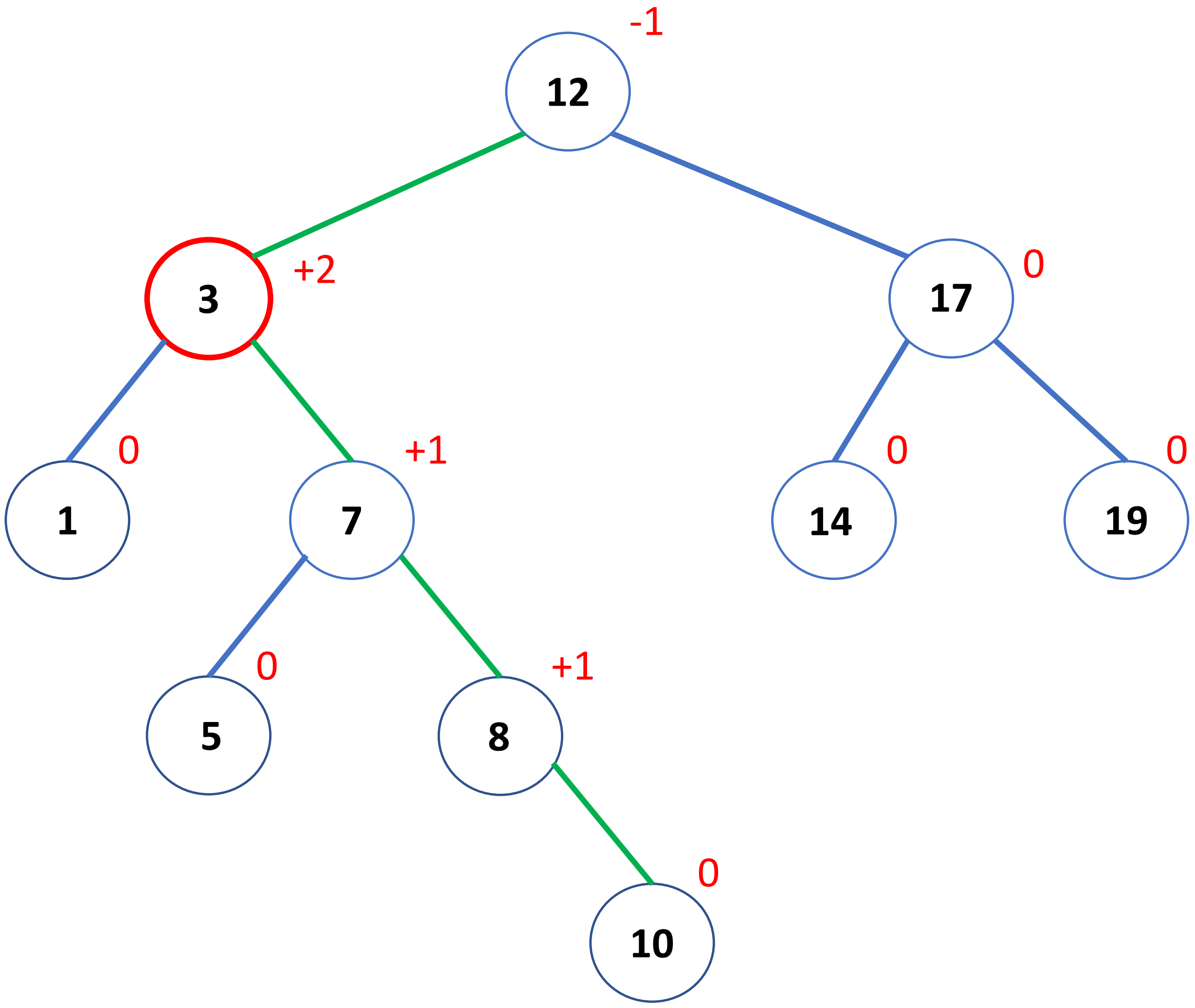
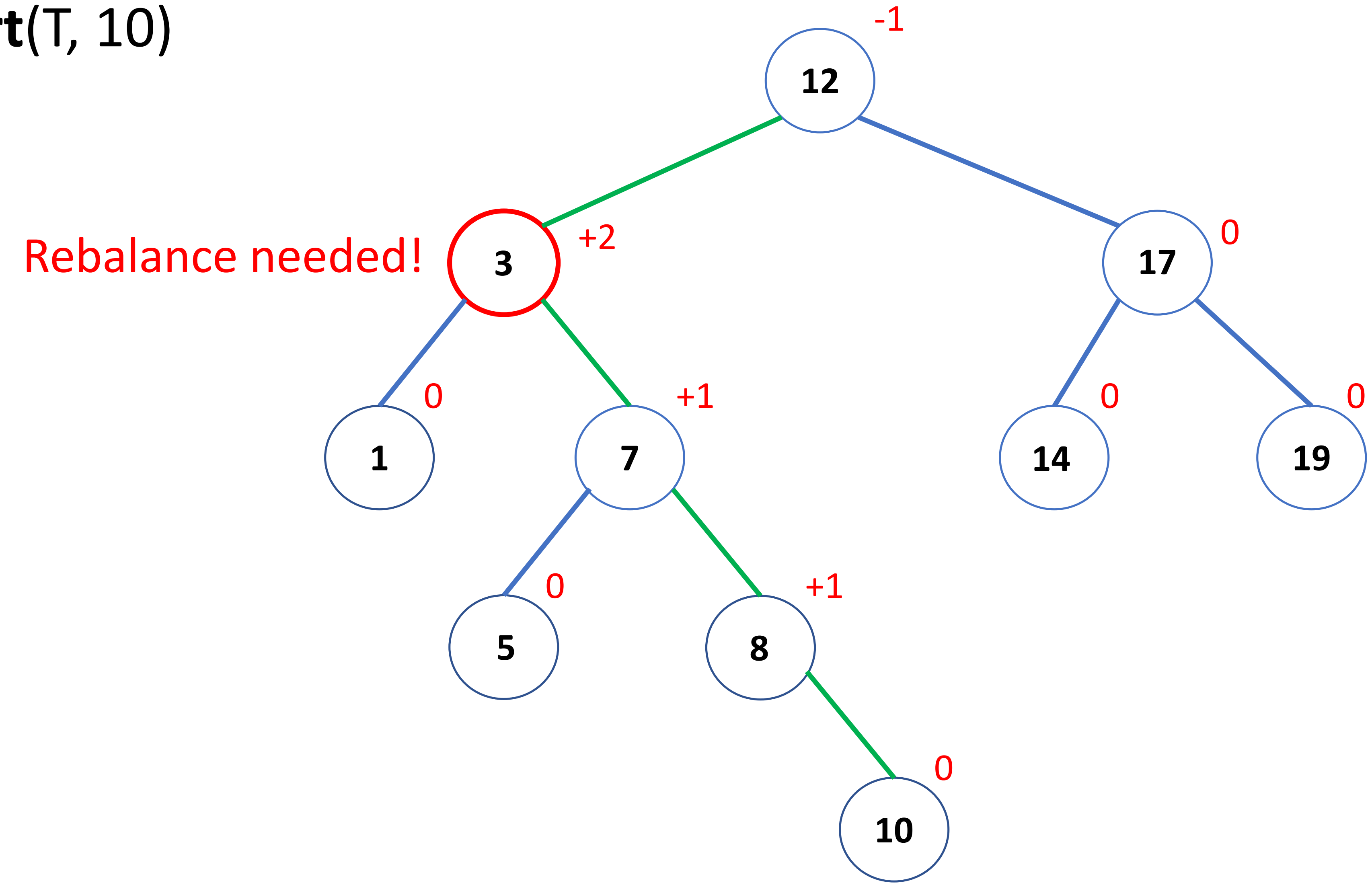
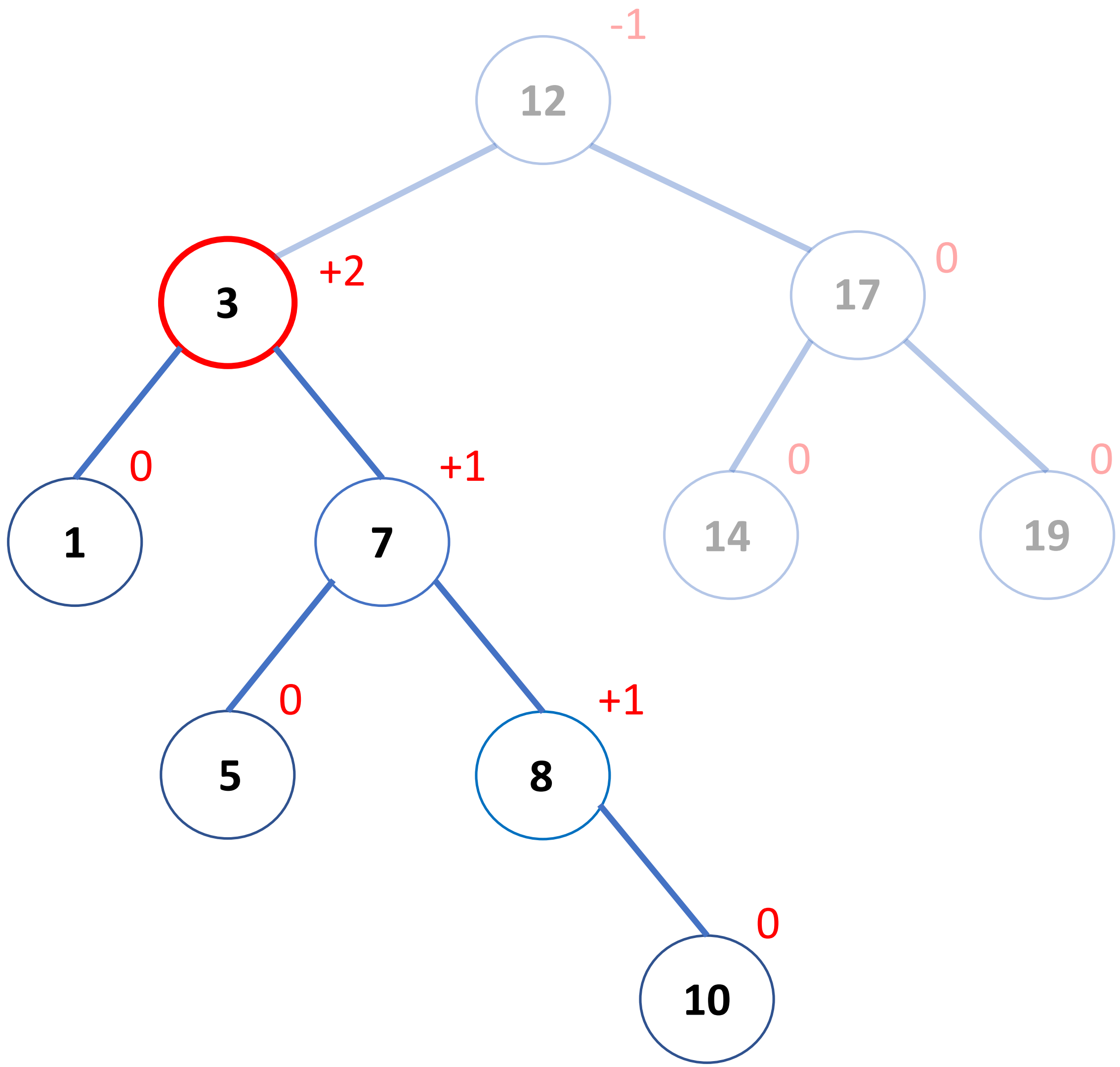


Example: AVL of {1, 3, 5, 7, 8, 12, 14, 17, 19}

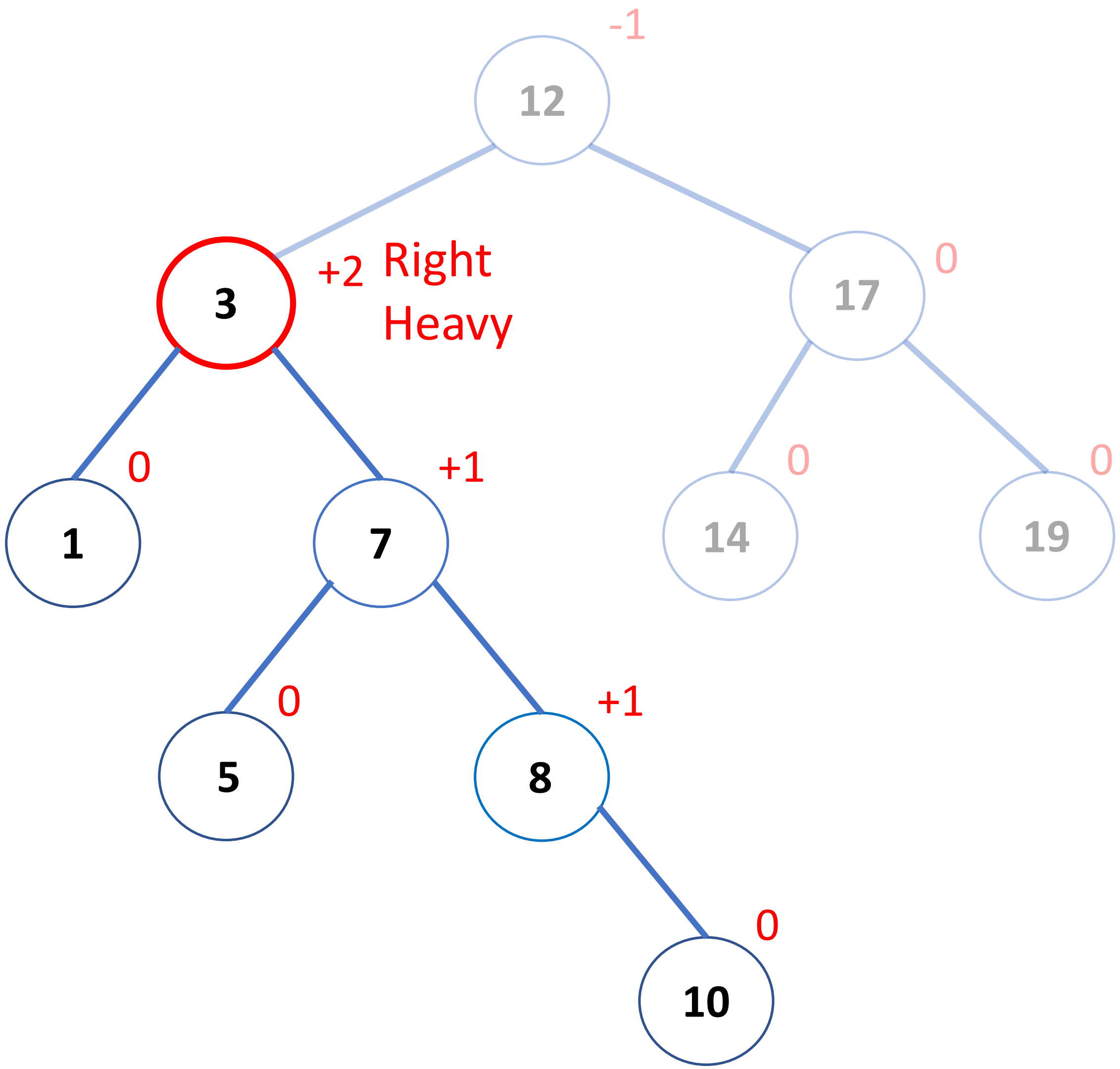
Insert(T, 10)



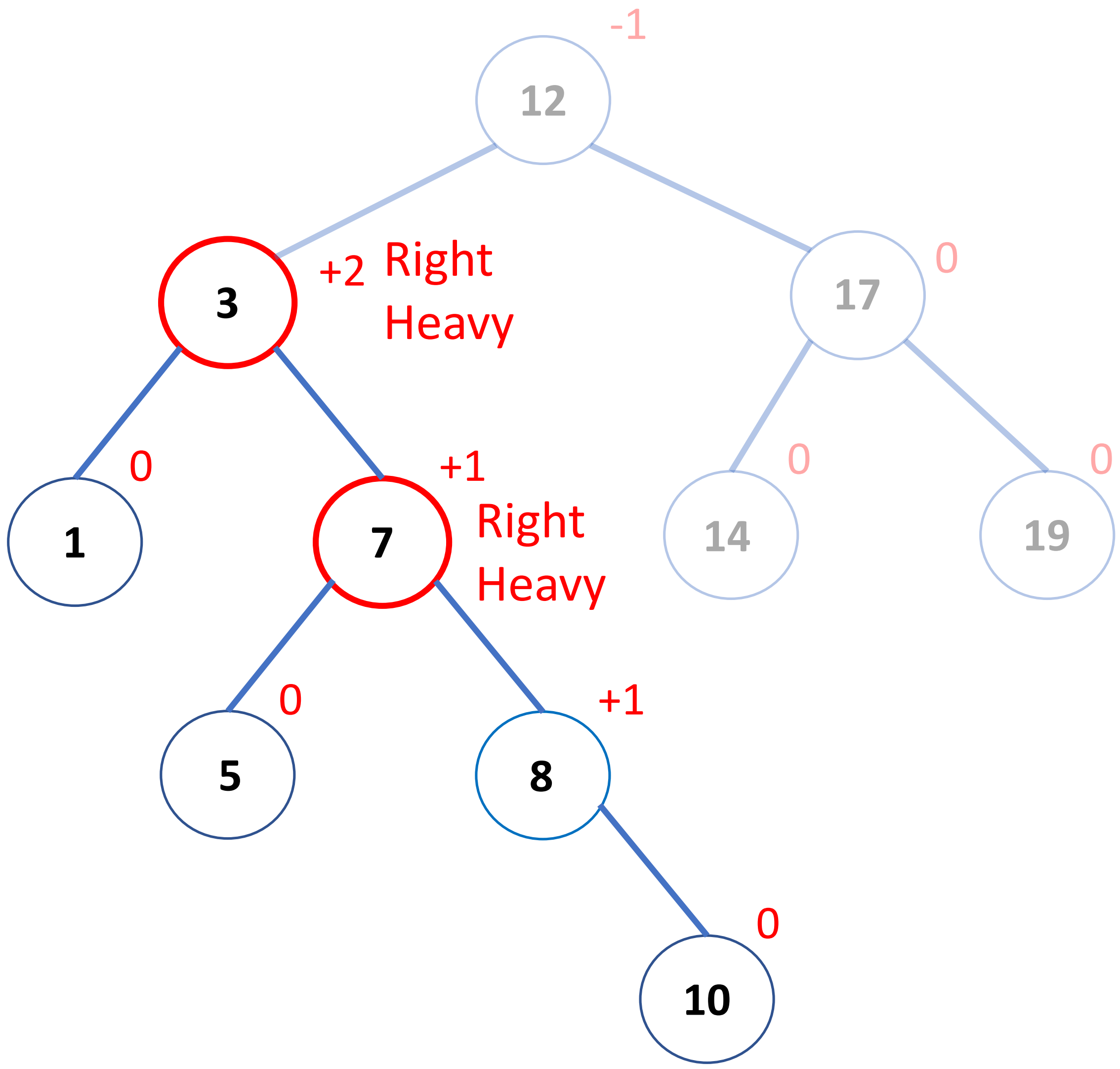
Rebalancing left subtree of 12:



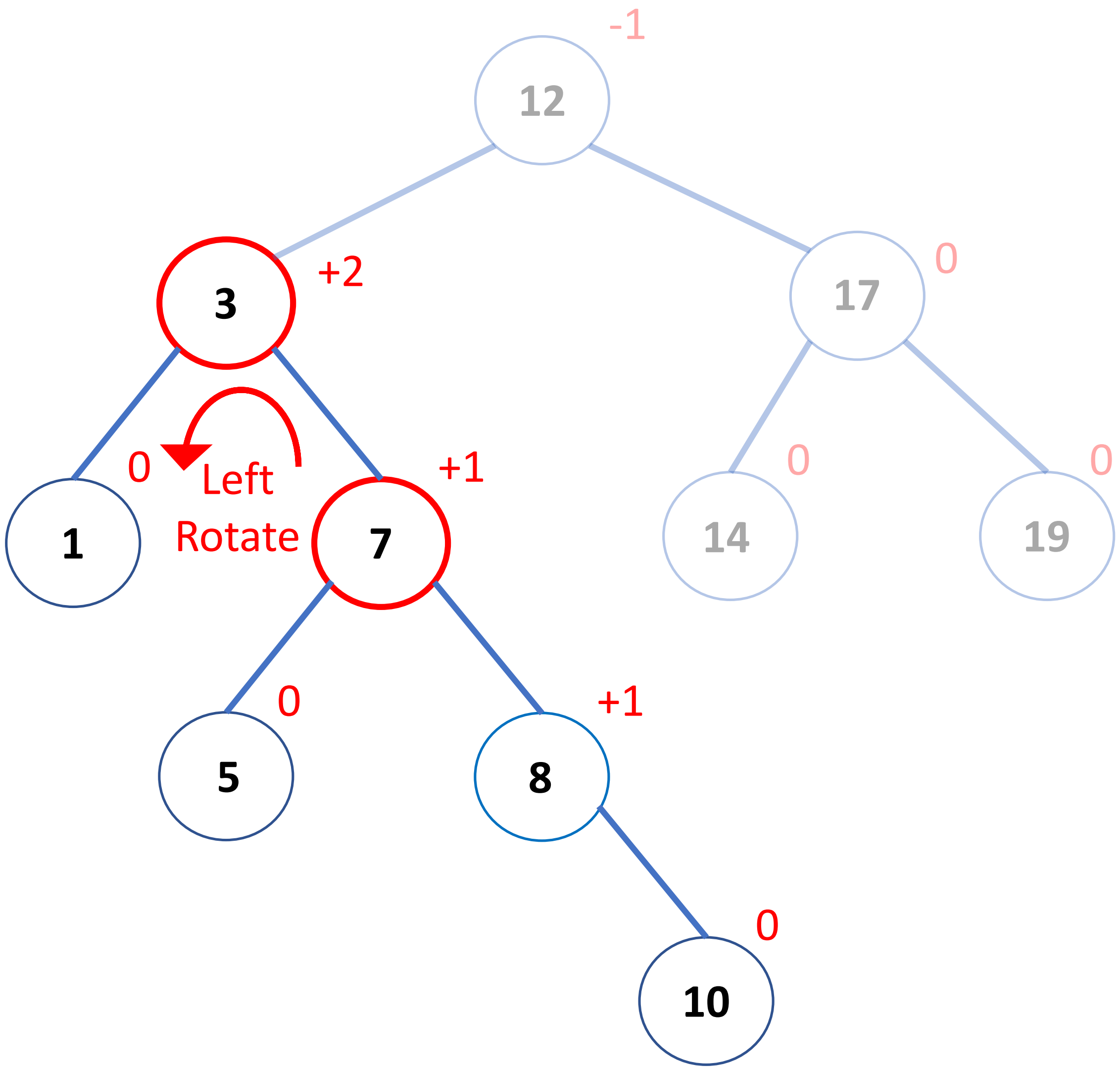
Rebalancing left subtree of 12:



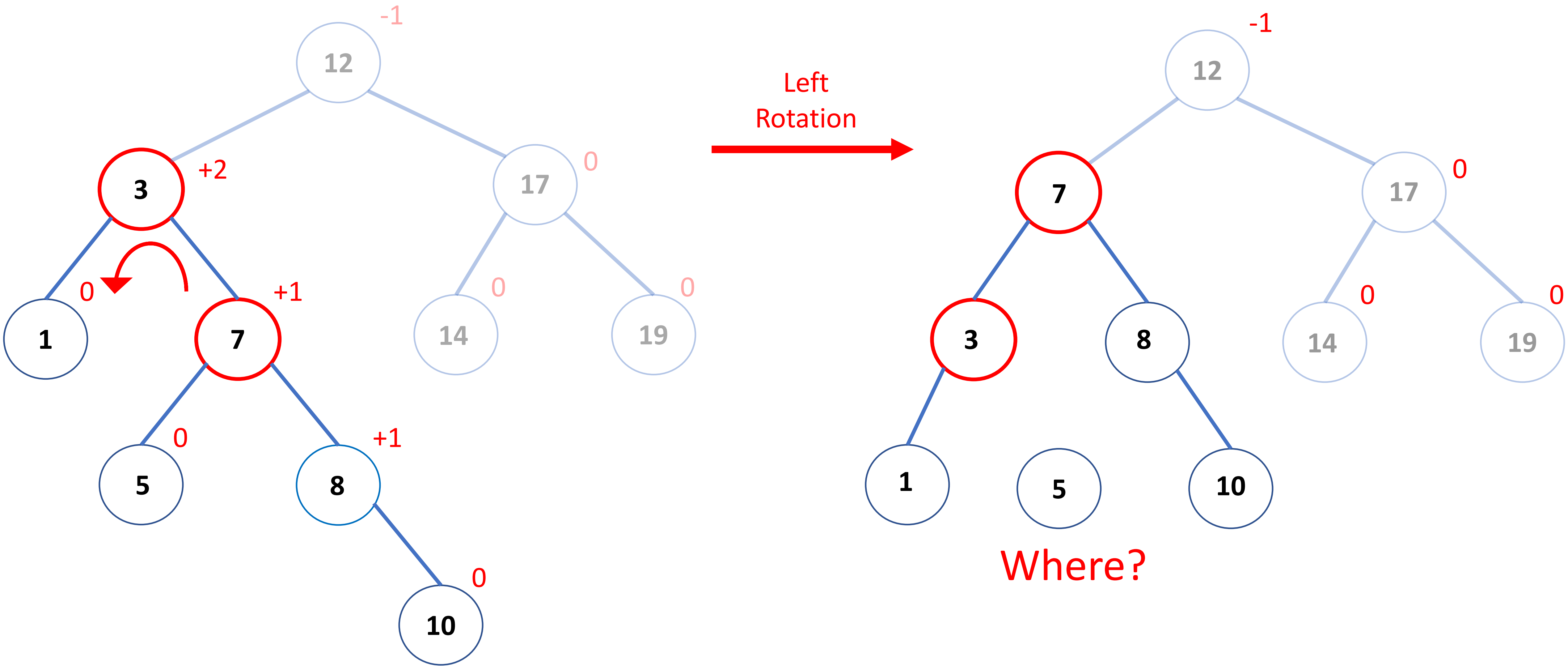
Rebalancing left subtree of 12:



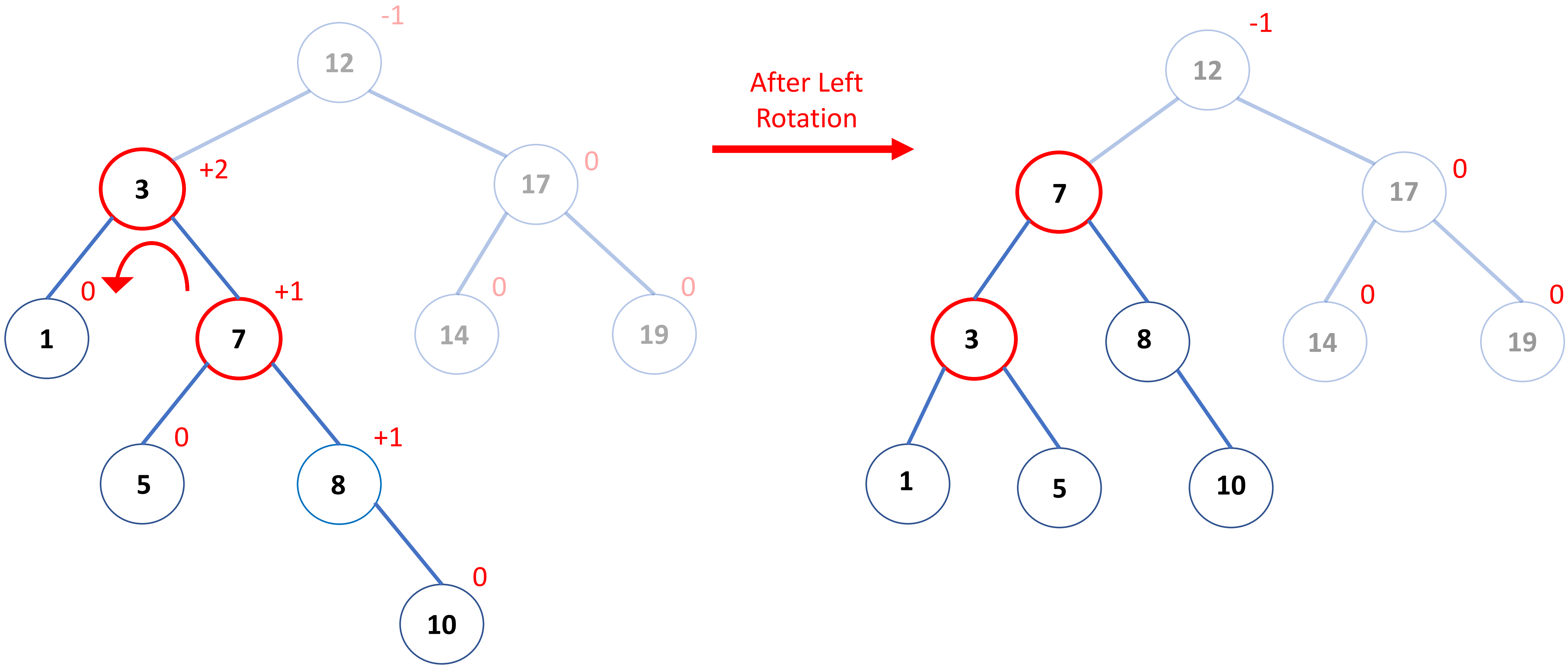
Rebalancing left subtree of 12:



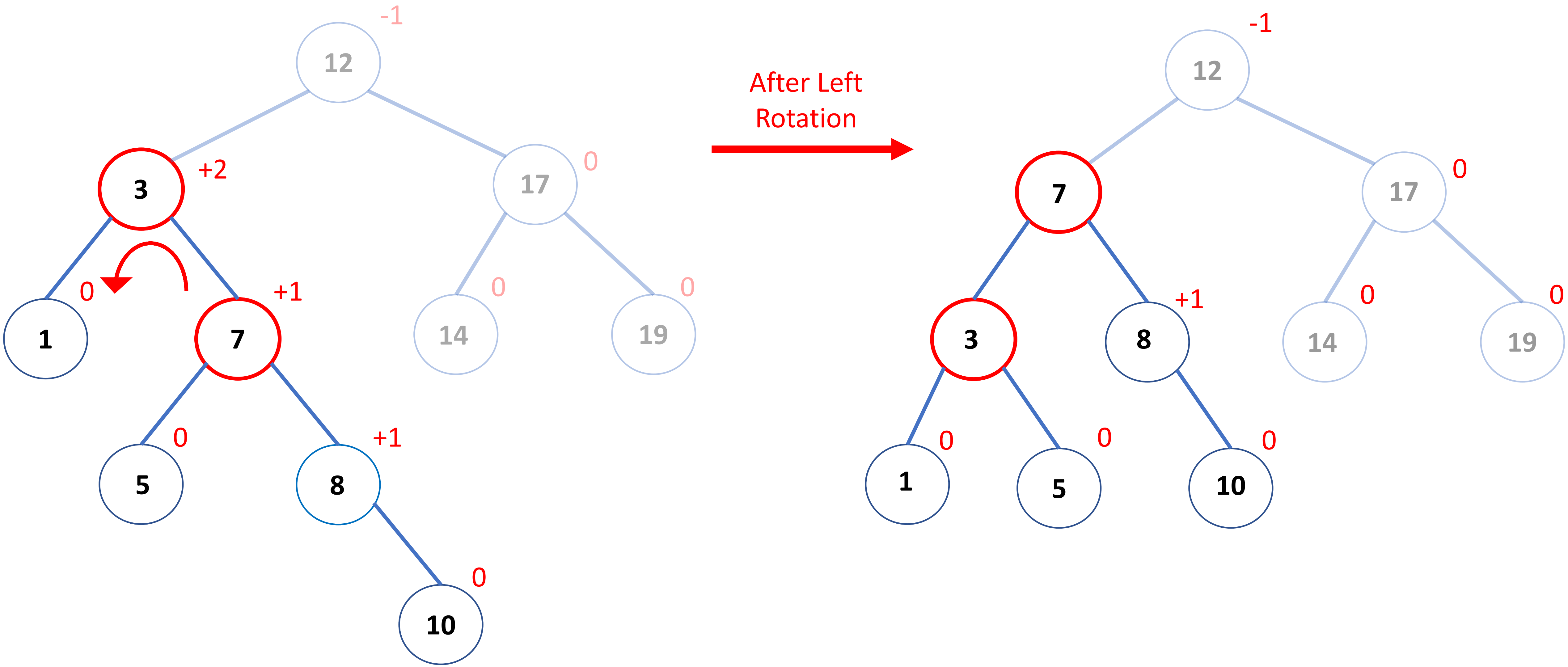
Rebalancing left subtree of 12:



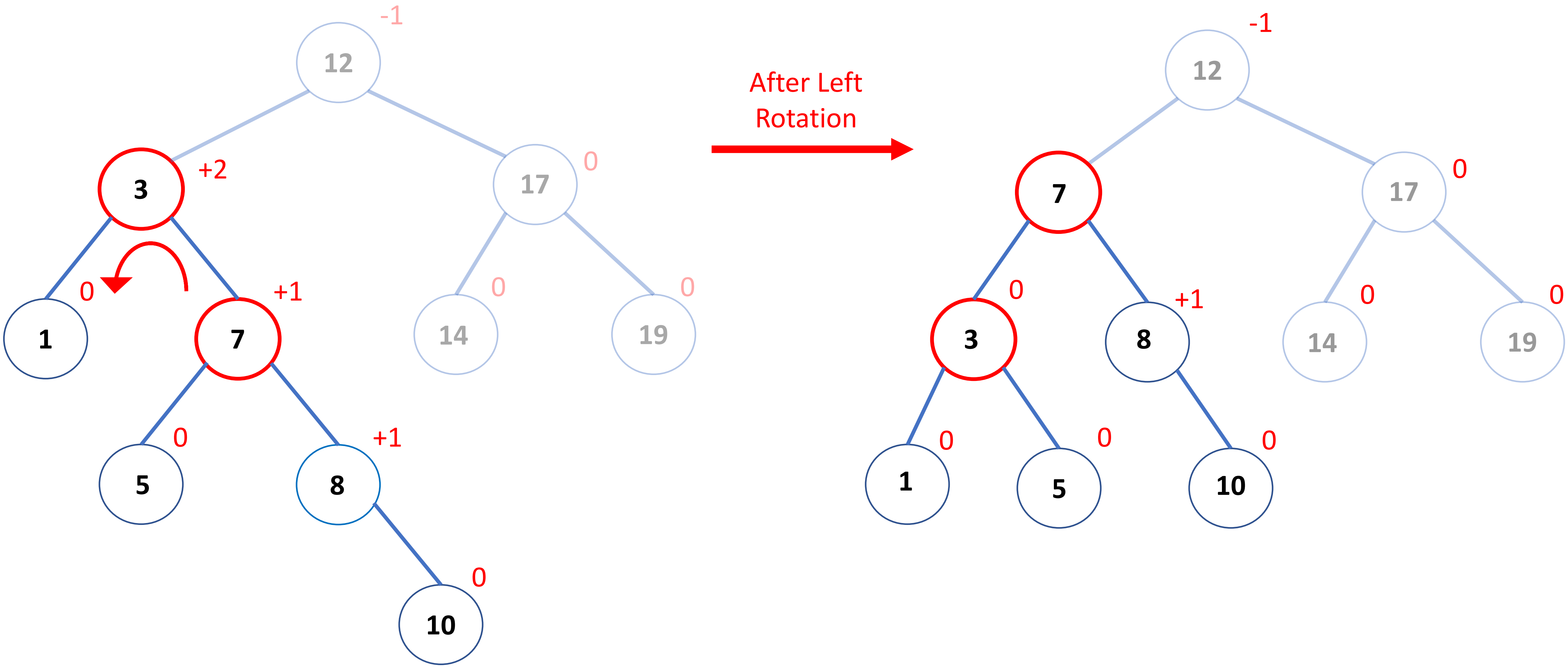
Rebalancing left subtree of 12:



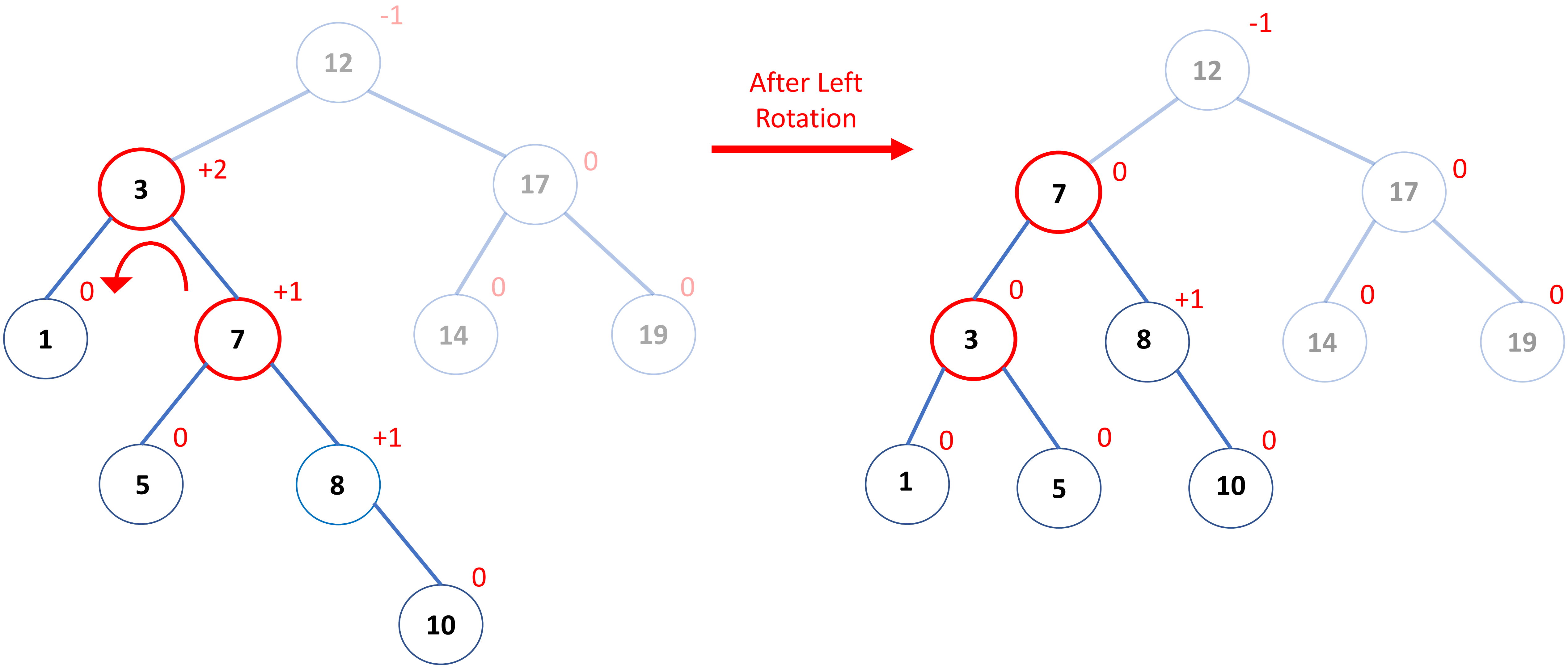
Rebalancing left subtree of 12:



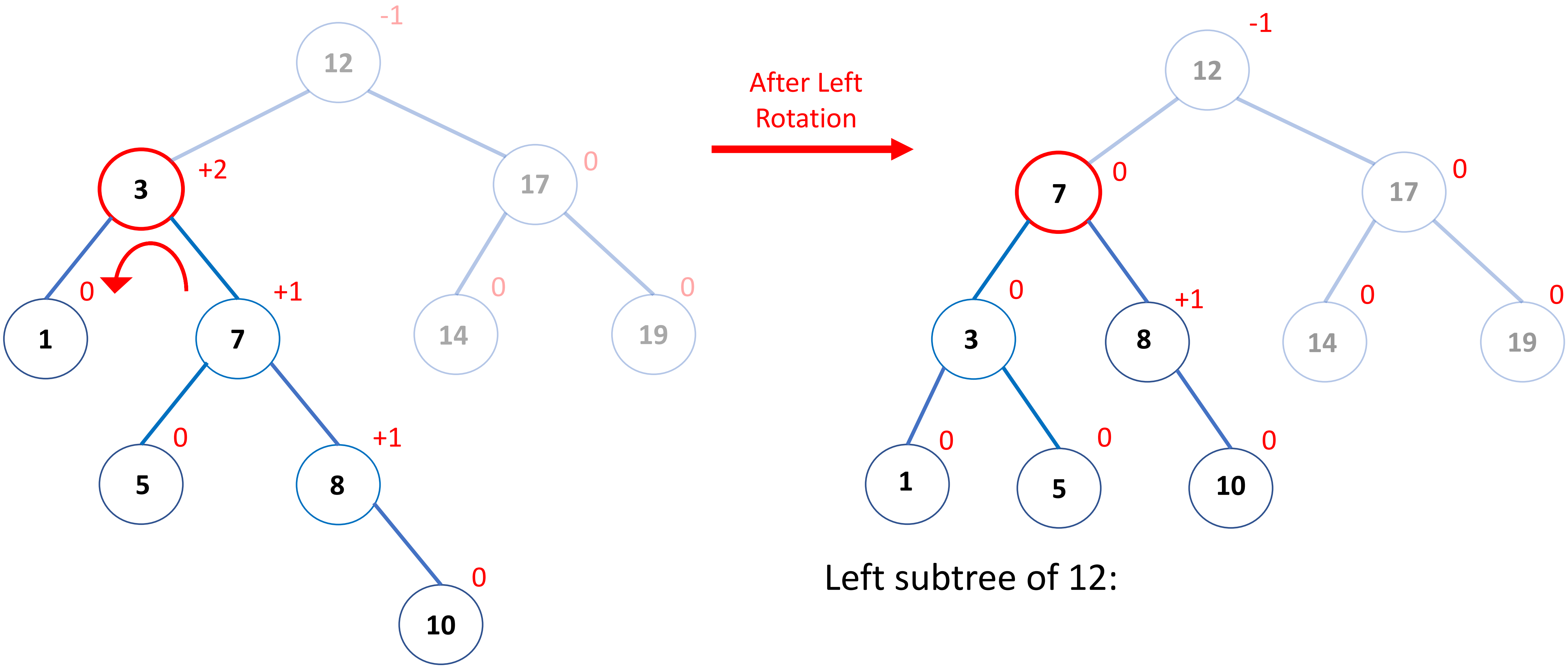
Rebalancing left subtree of 12:



Rebalancing left subtree of 12:

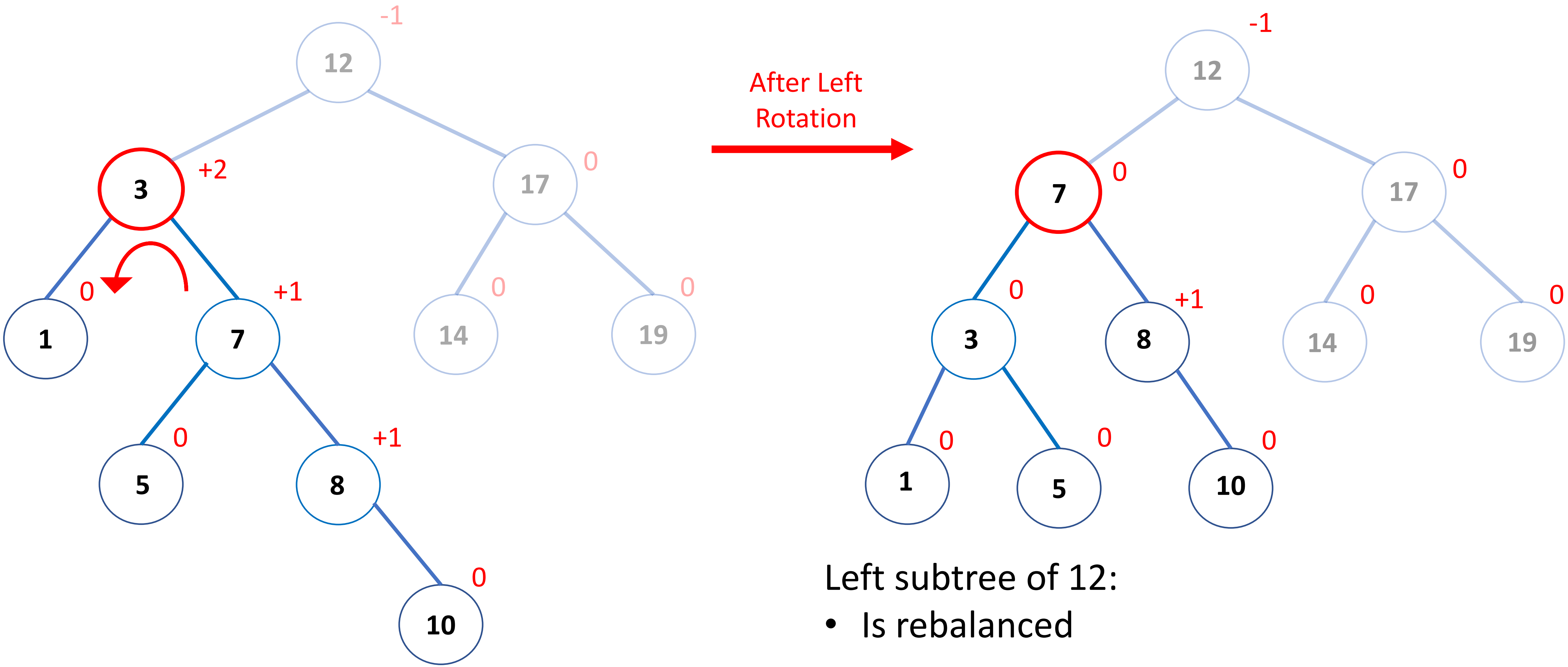


Rebalancing left subtree of 12:



Left subtree of 12:

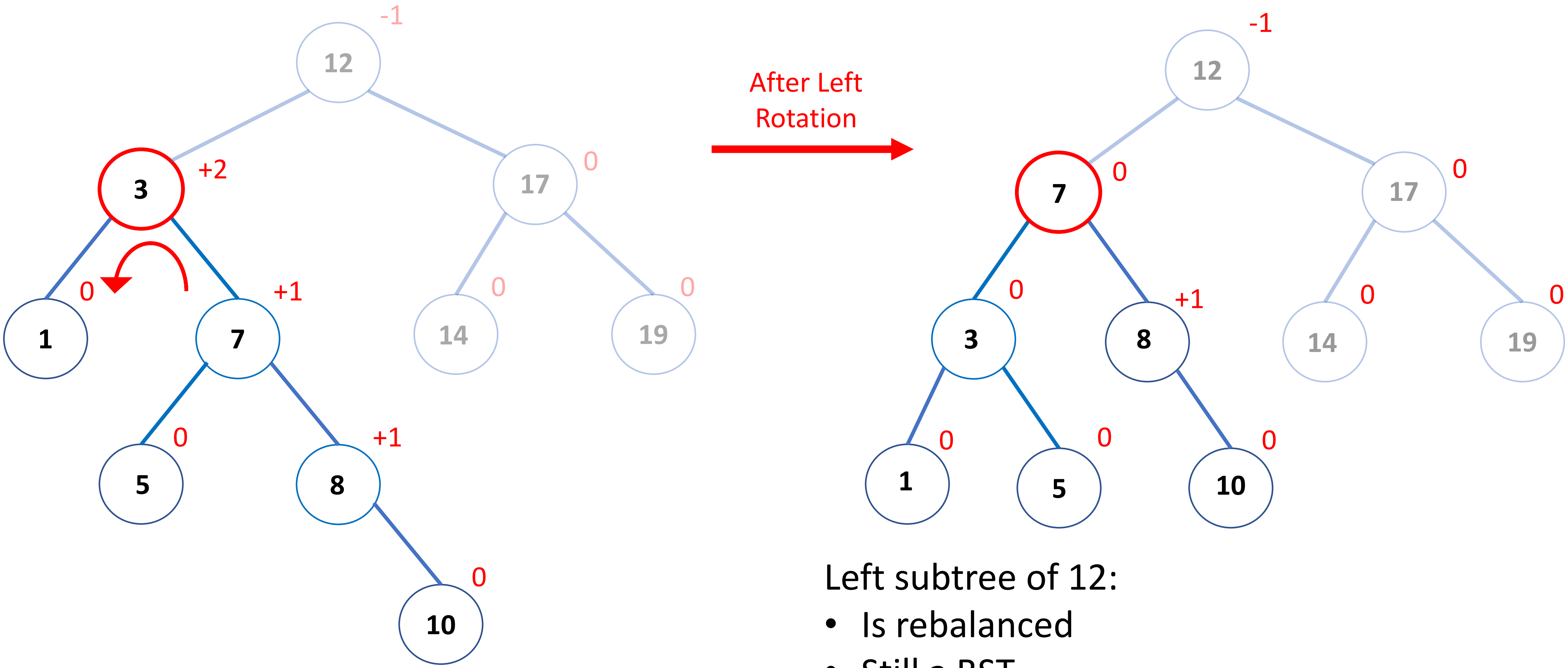
Rebalancing left subtree of 12:



Left subtree of 12:

- Is rebalanced

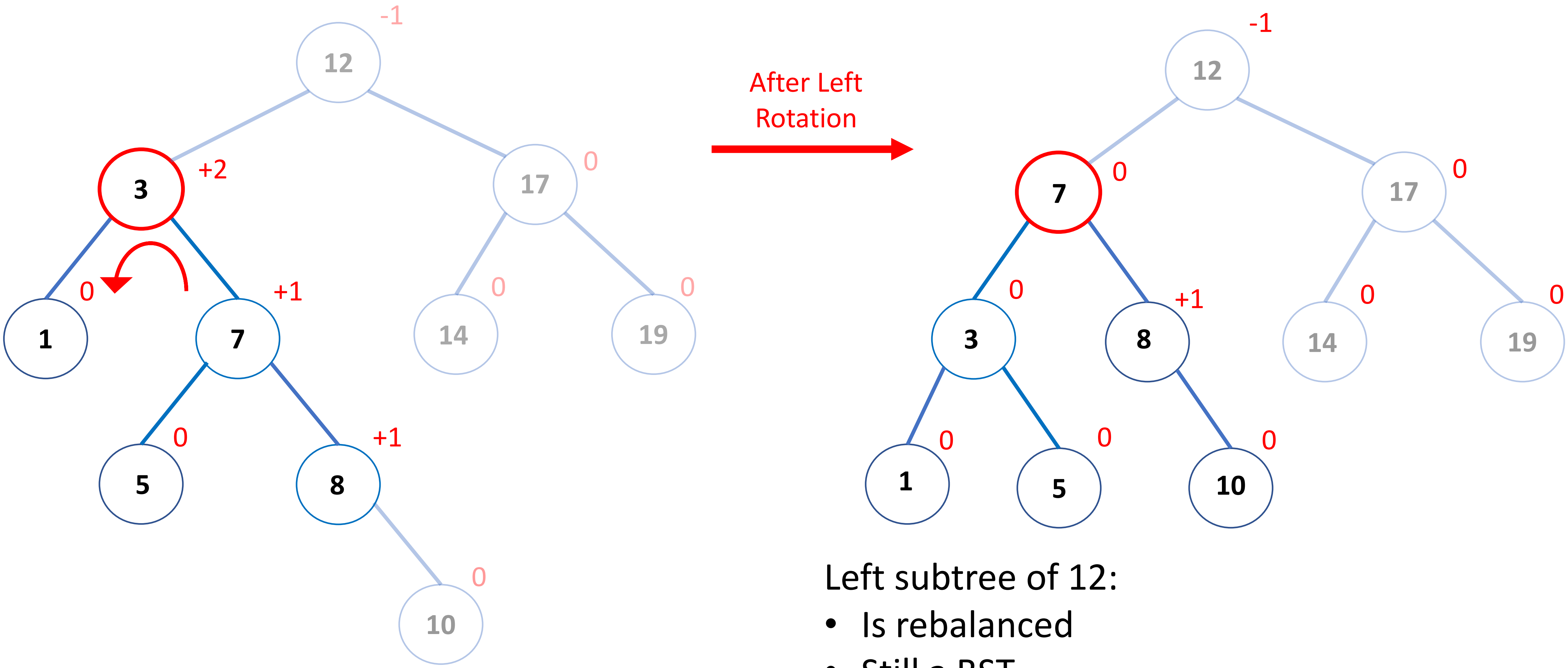
Rebalancing left subtree of 12:



Left subtree of 12:

- Is rebalanced
- Still a BST

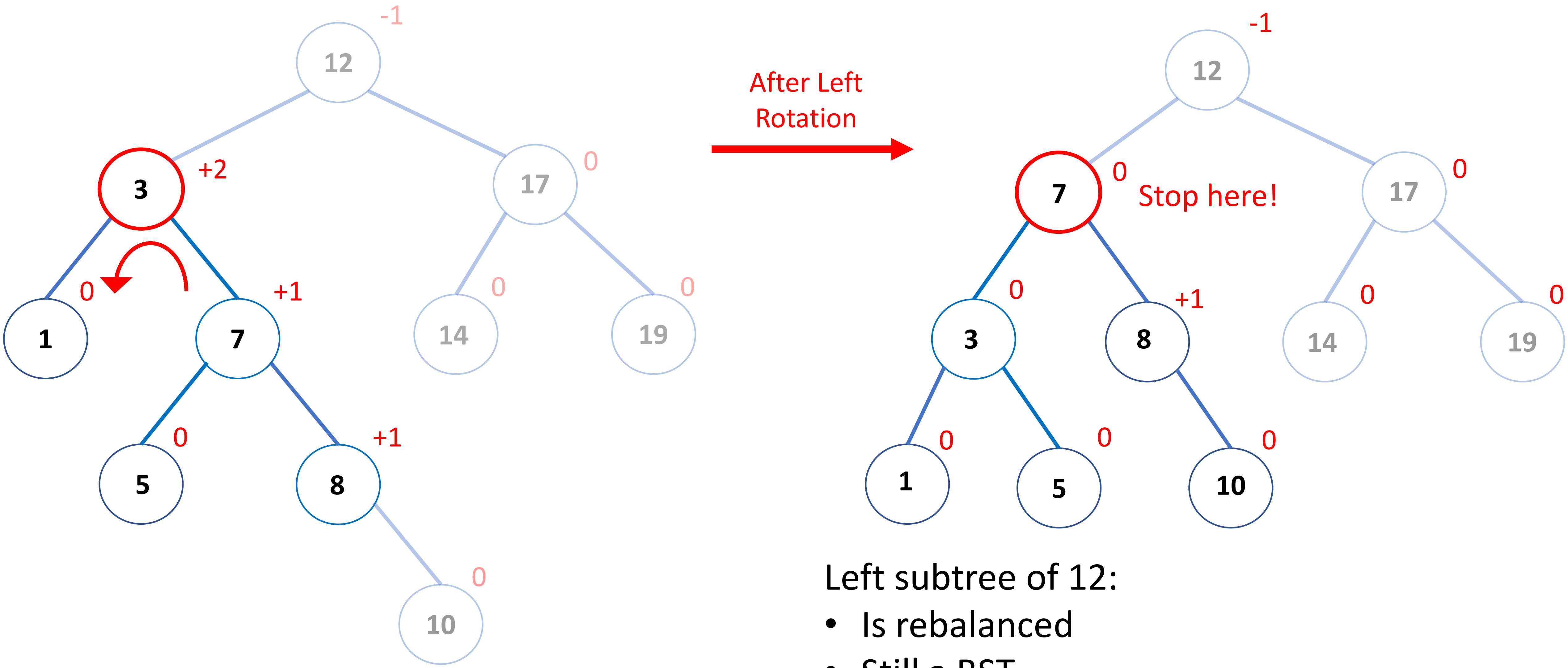
Rebalancing left subtree of 12:



Left subtree of 12:

- Is rebalanced
- Still a BST
- **Bonus**: height same as before inserting 10

Rebalancing left subtree of 12:

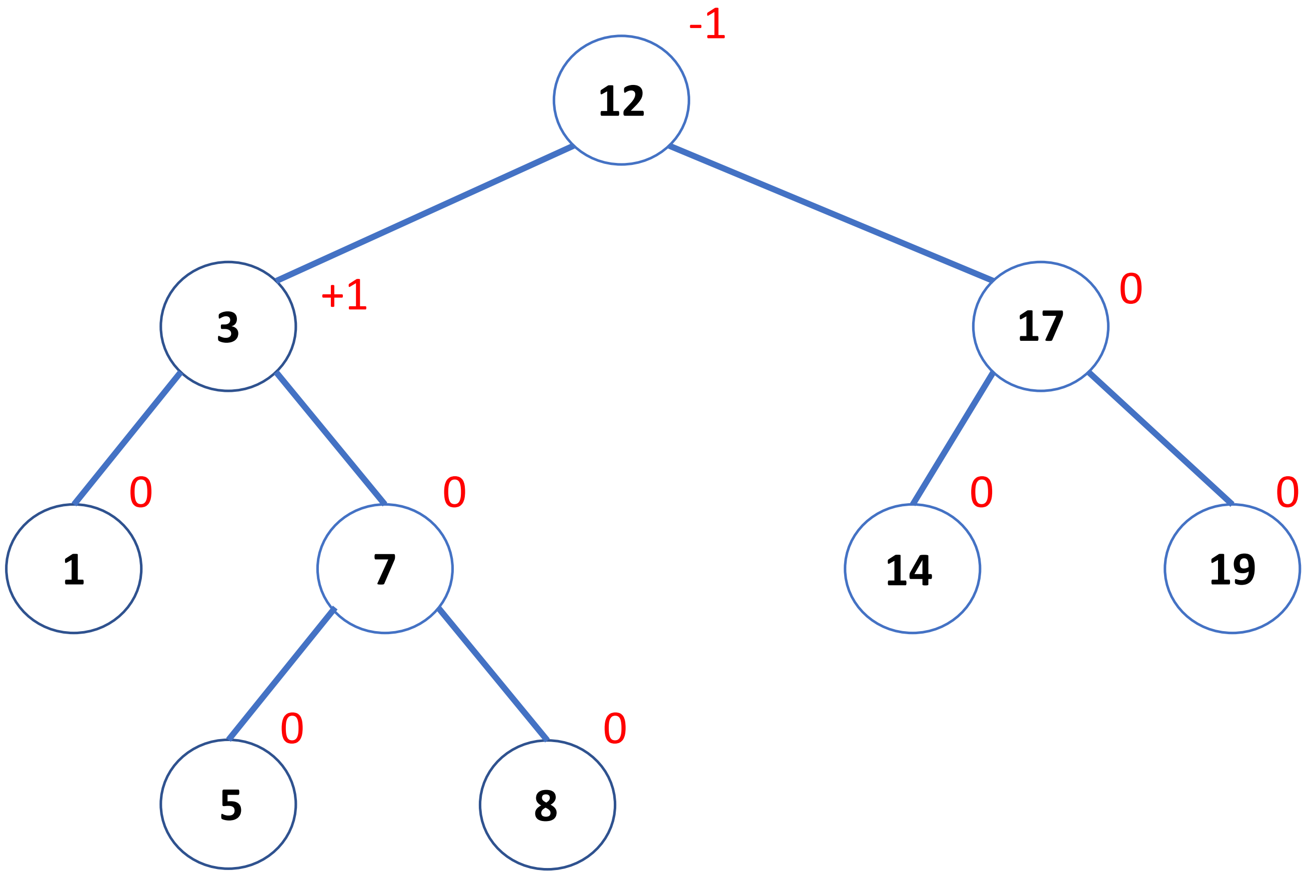


Left subtree of 12:

- Is rebalanced
- Still a BST
- **Bonus**: height same as before inserting 10

Example: AVL of {1, 3, 5, 7, 8, 12, 14, 17, 19}

Insert(T, 6)



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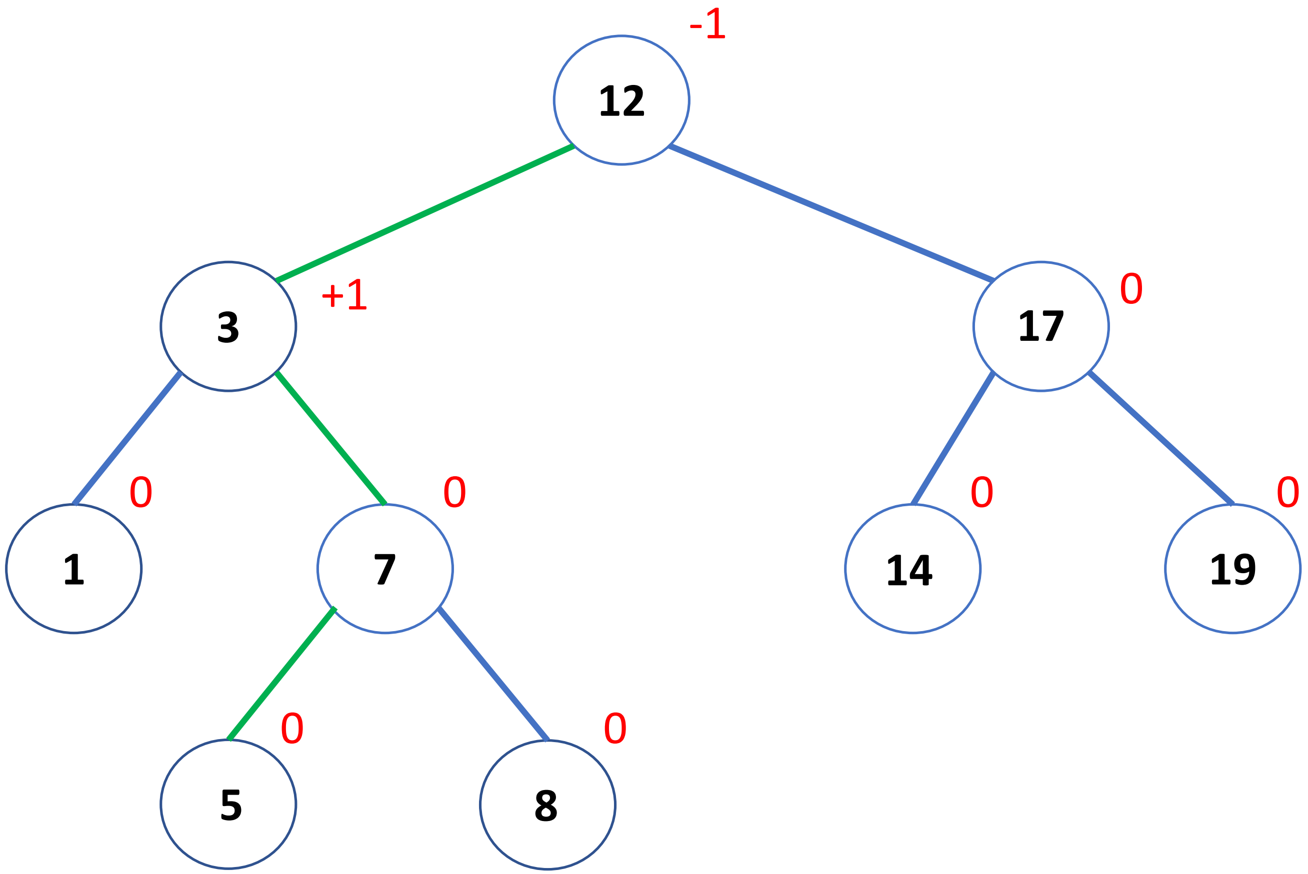

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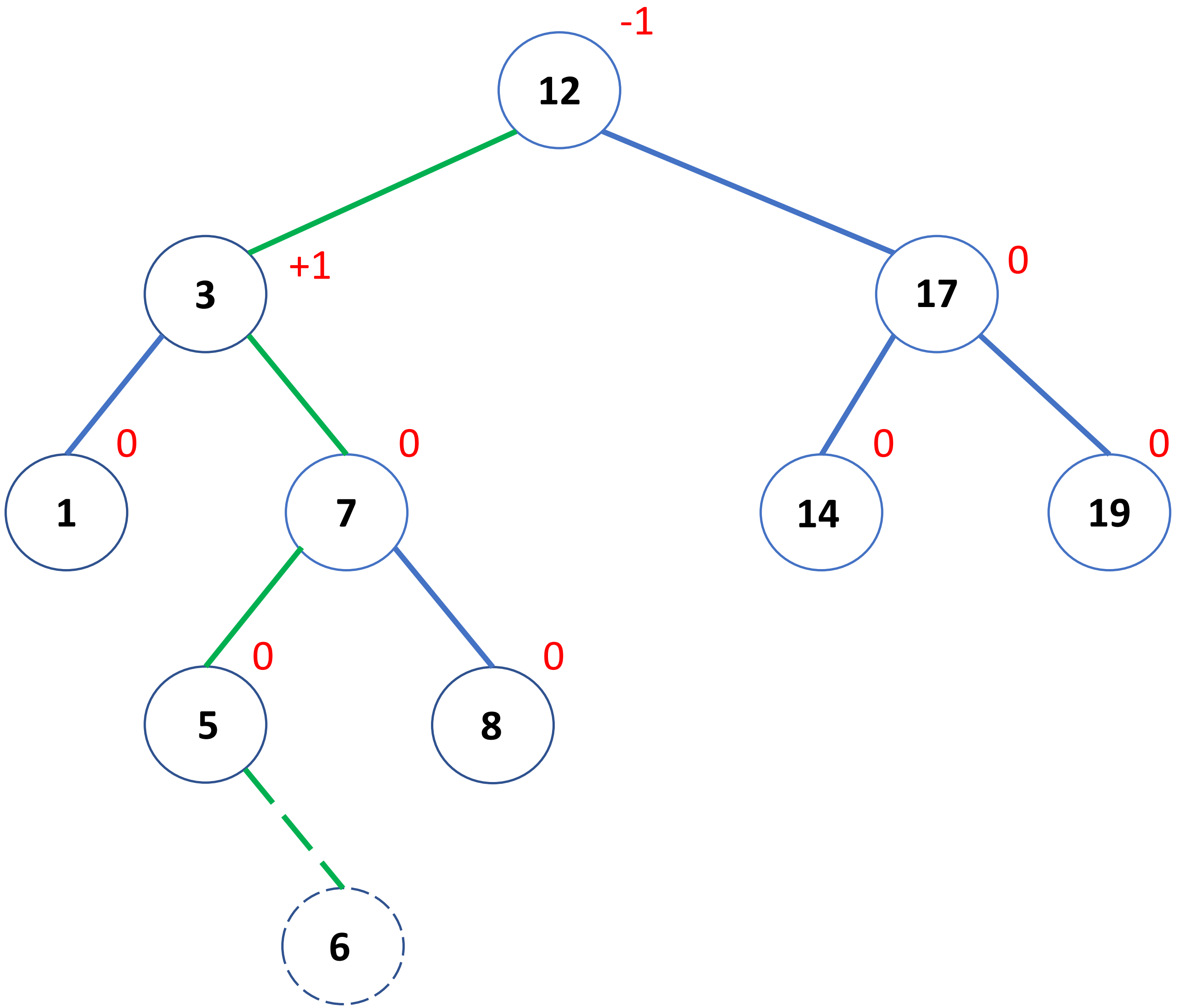

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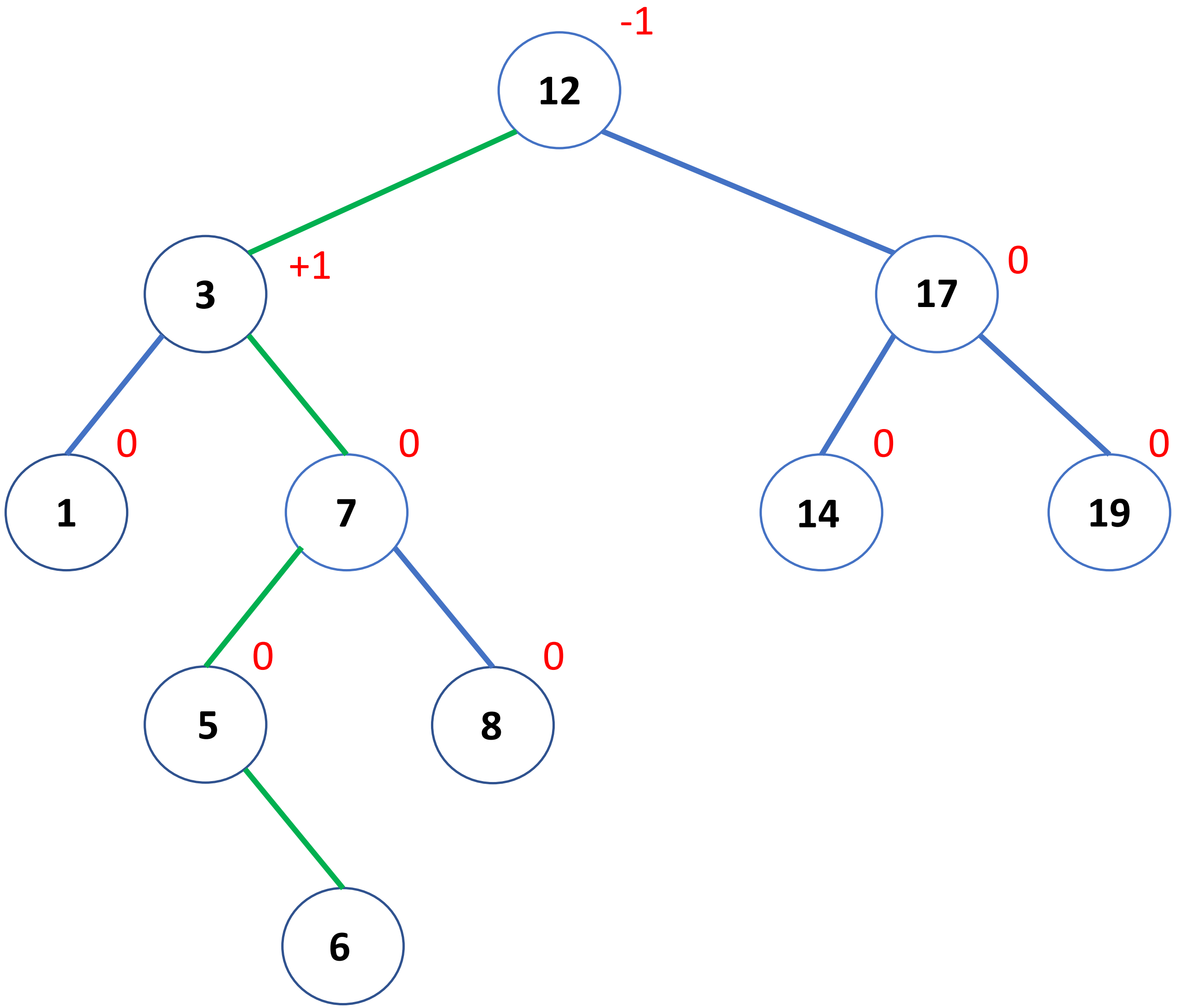

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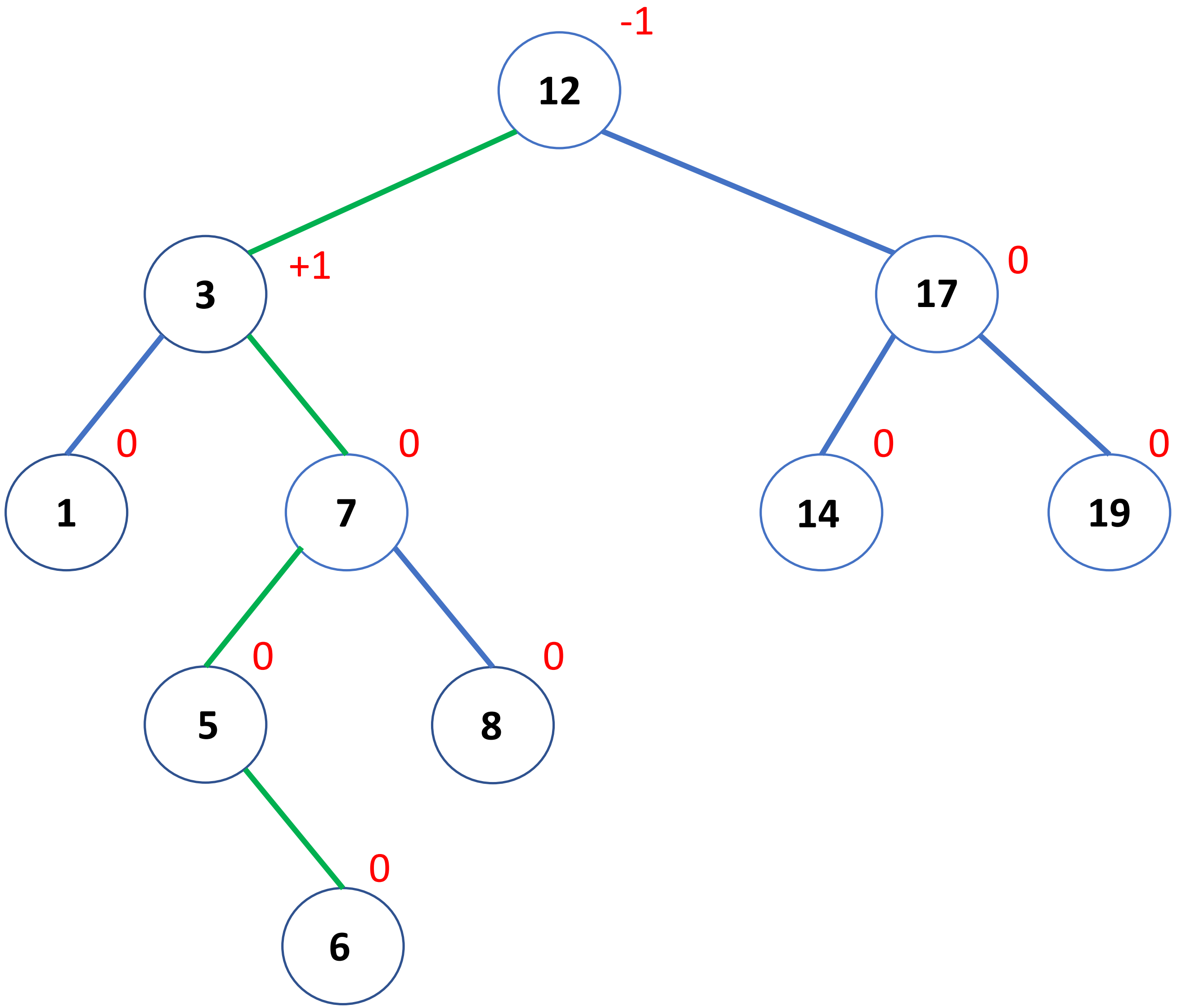

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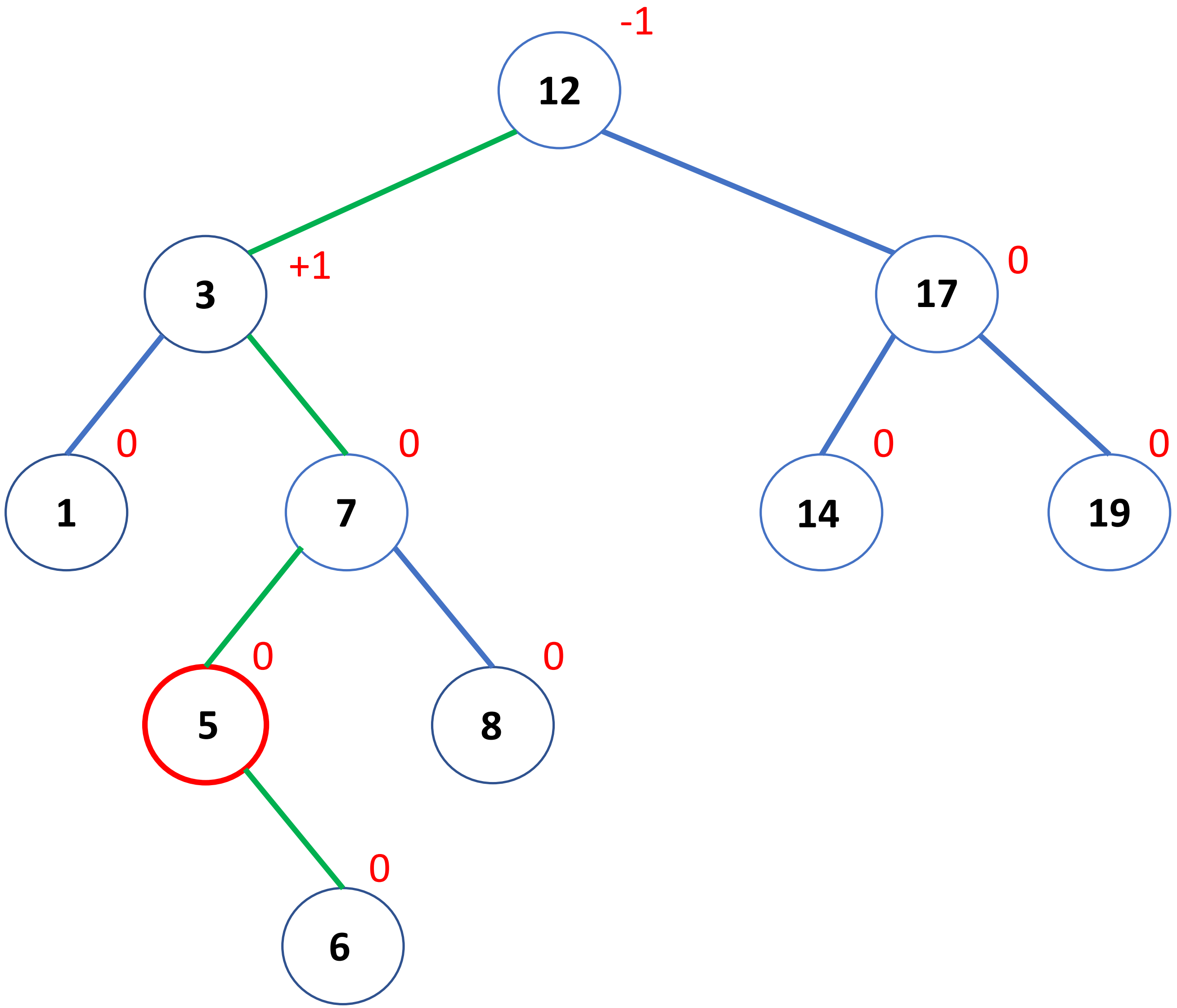

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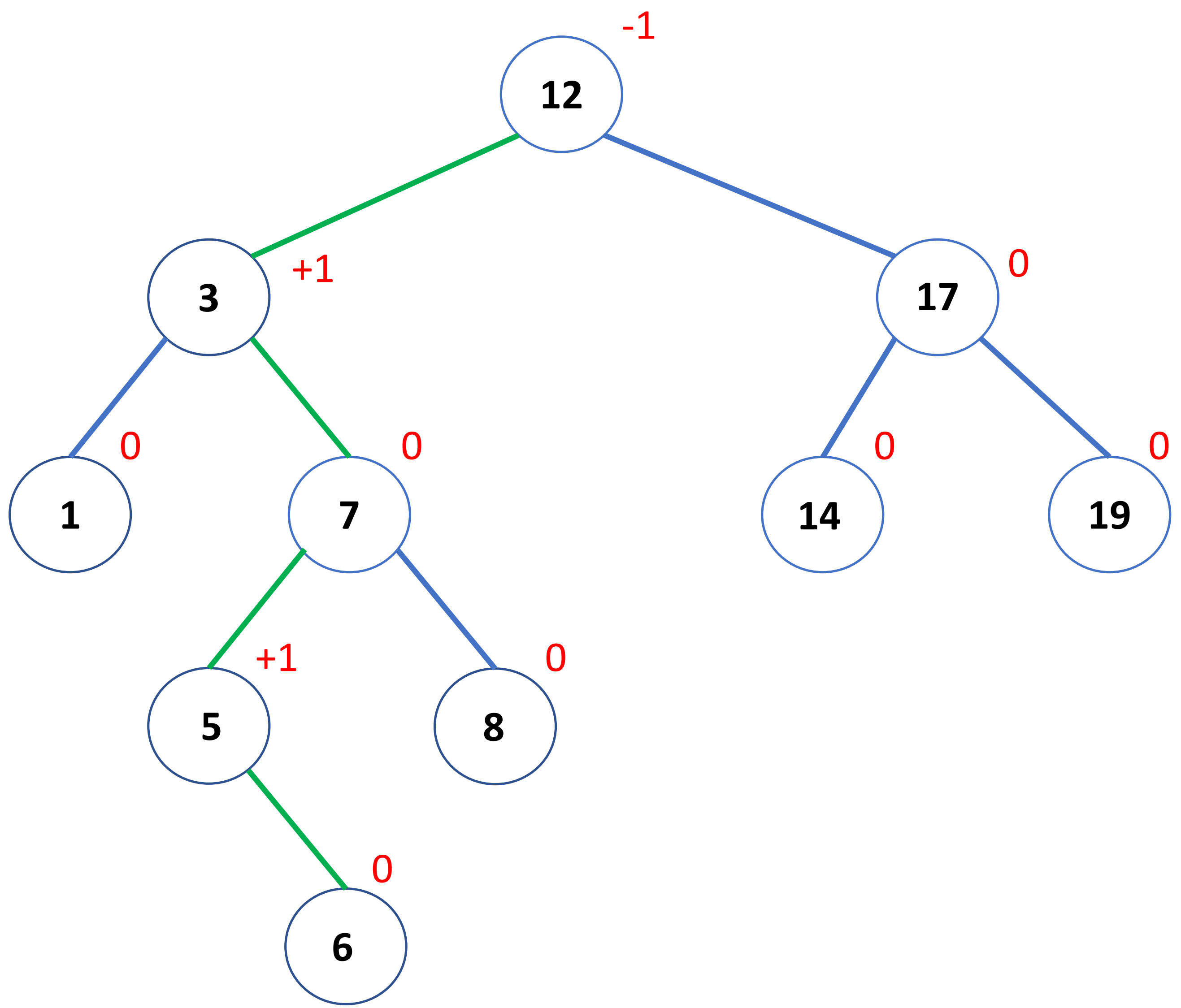

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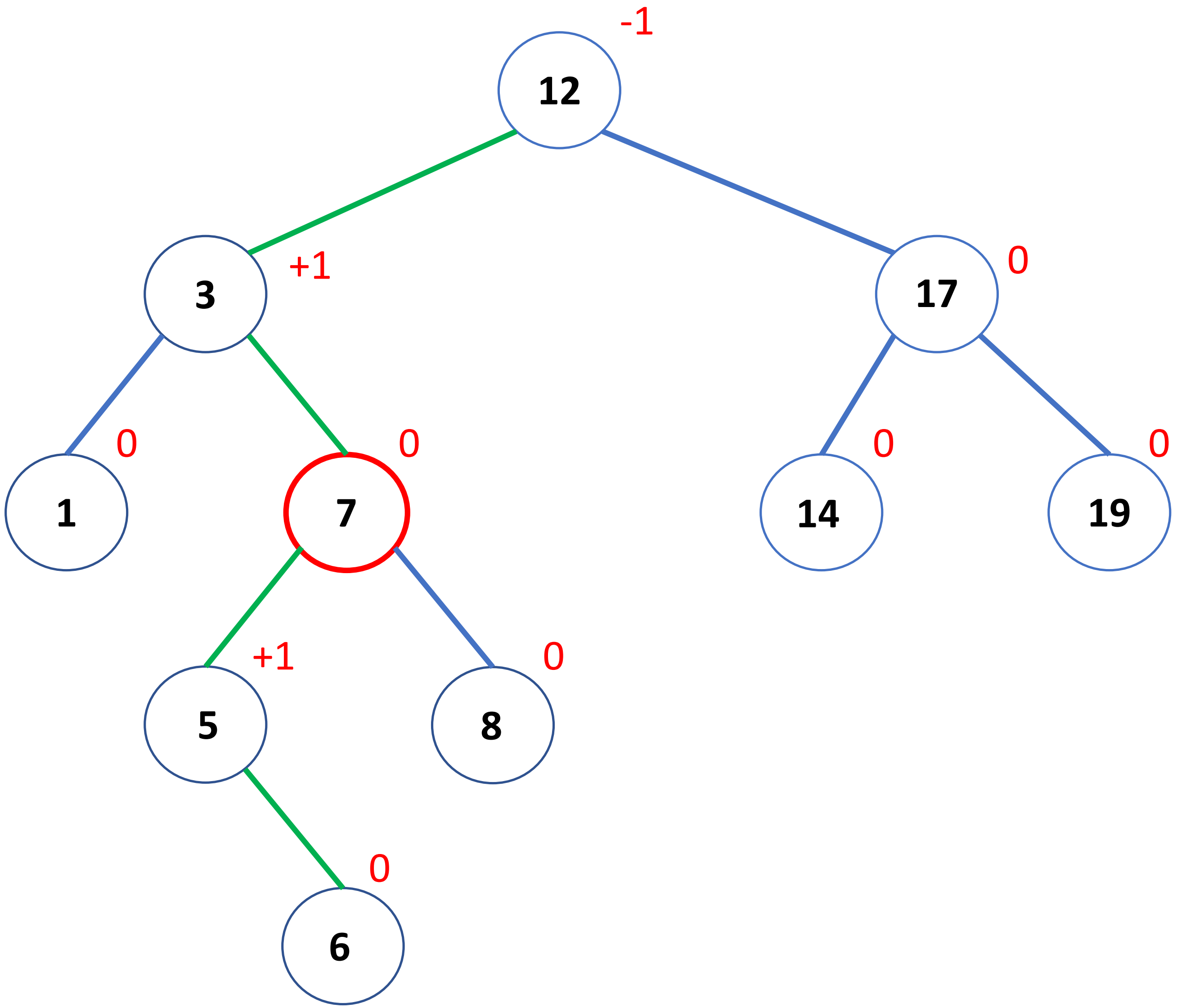
Insert(T, 6)



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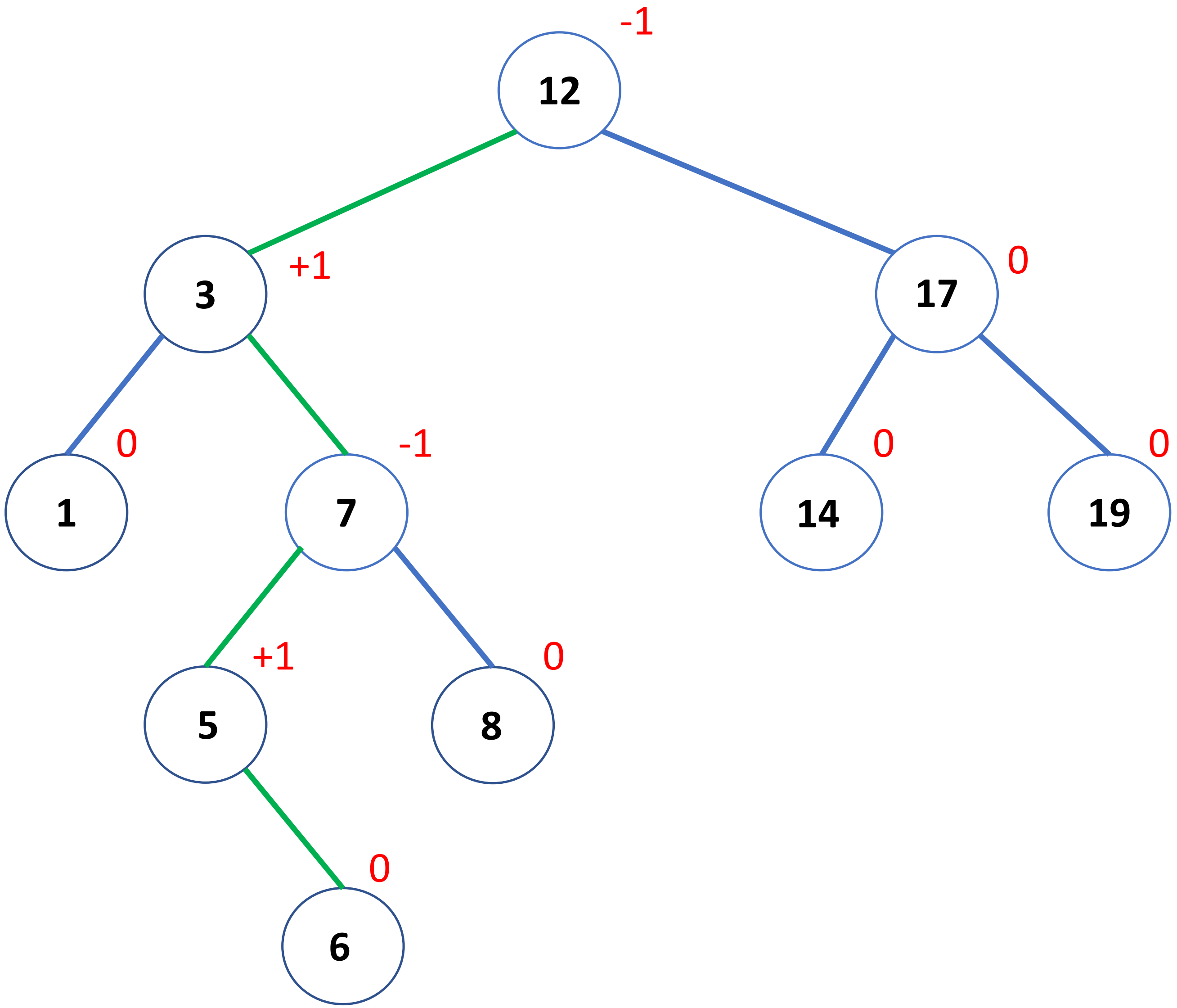

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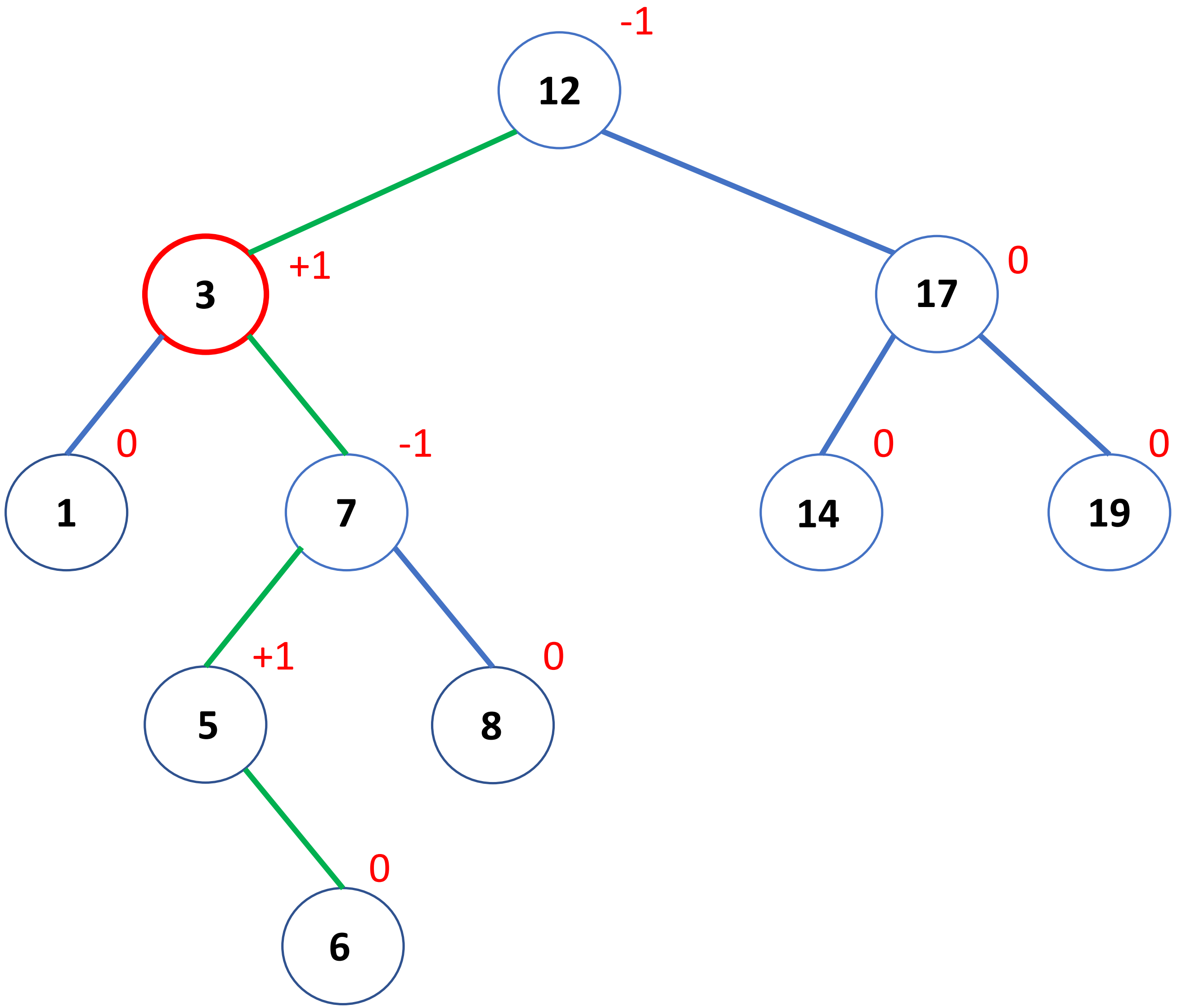

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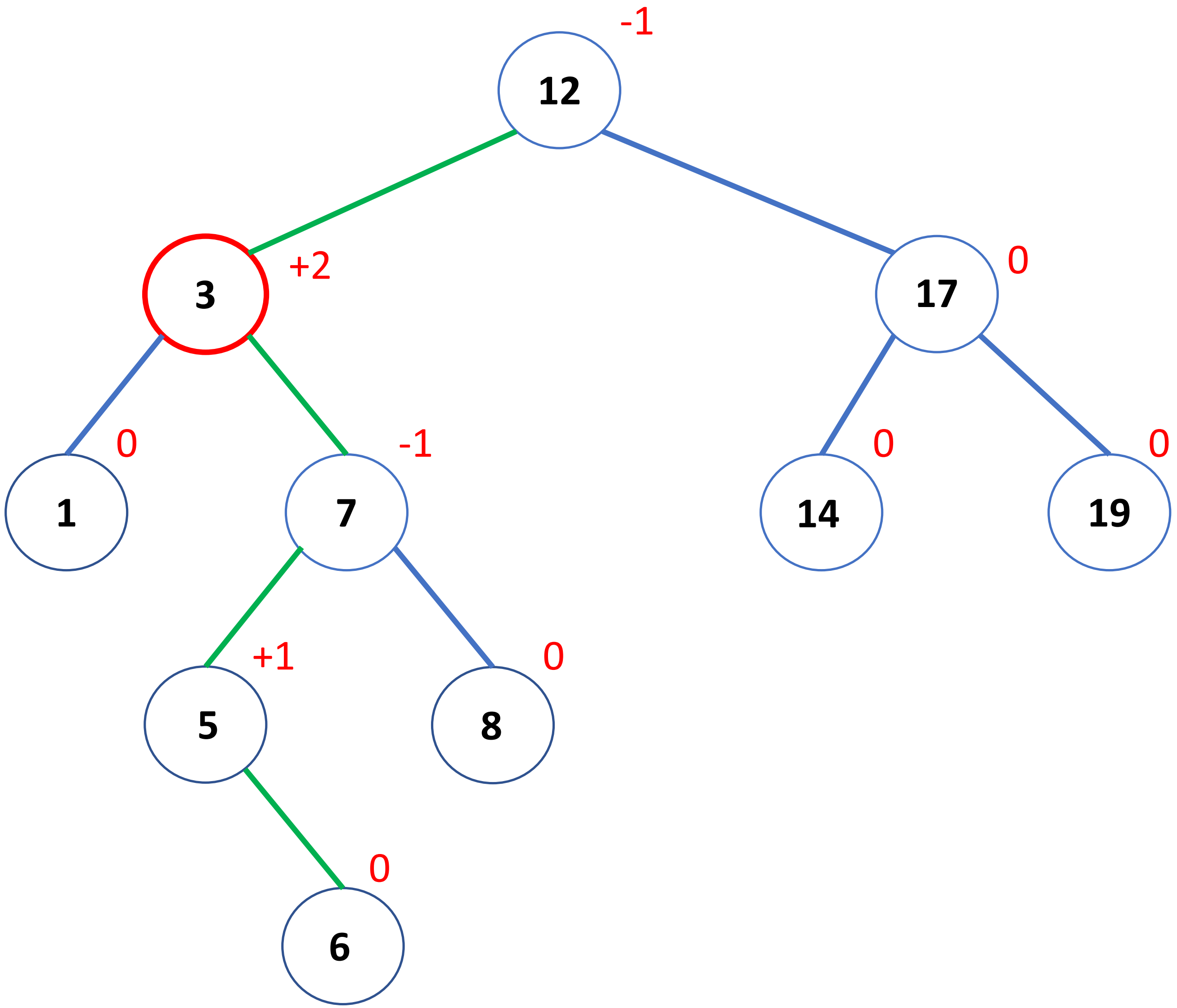

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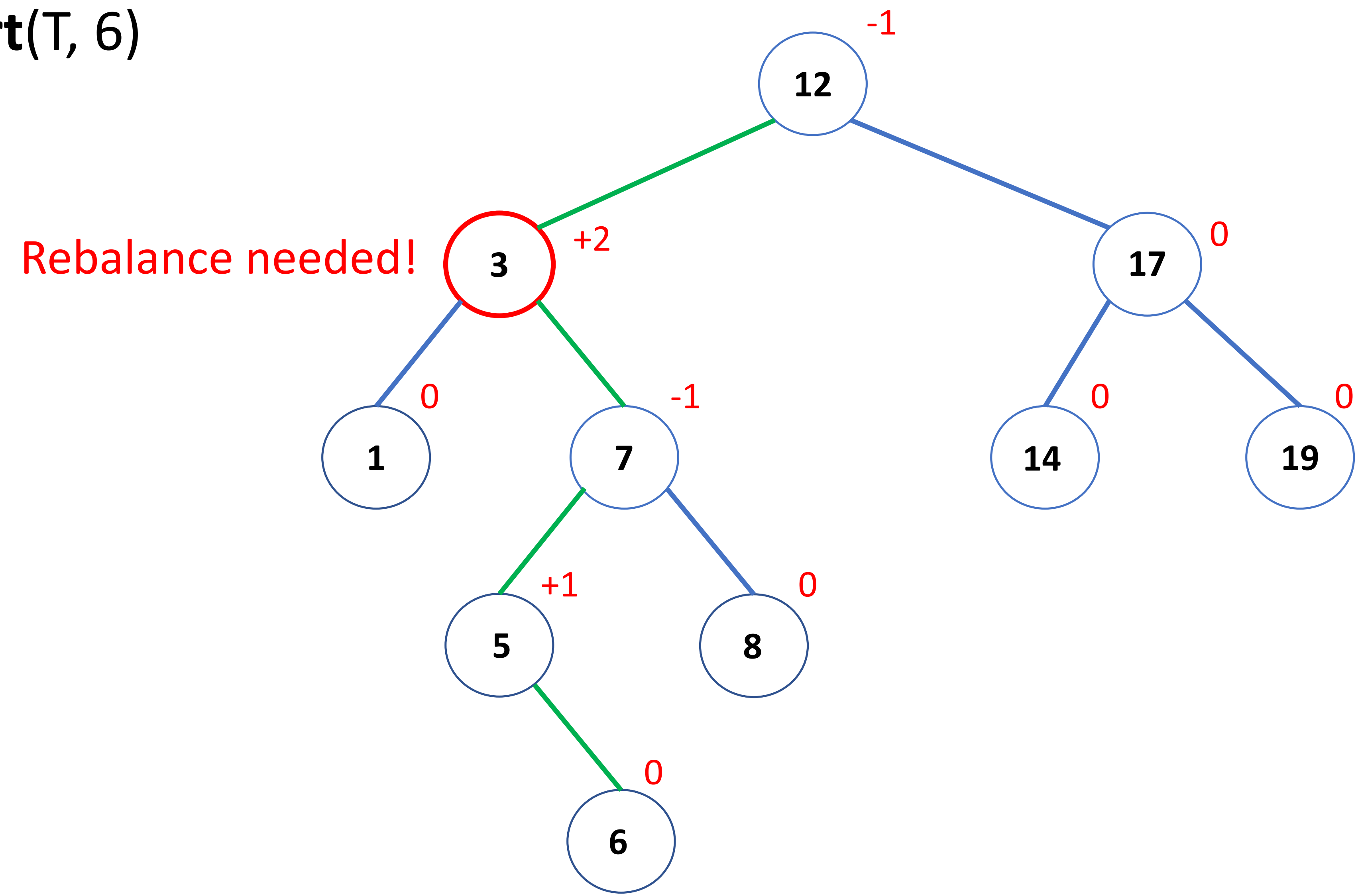
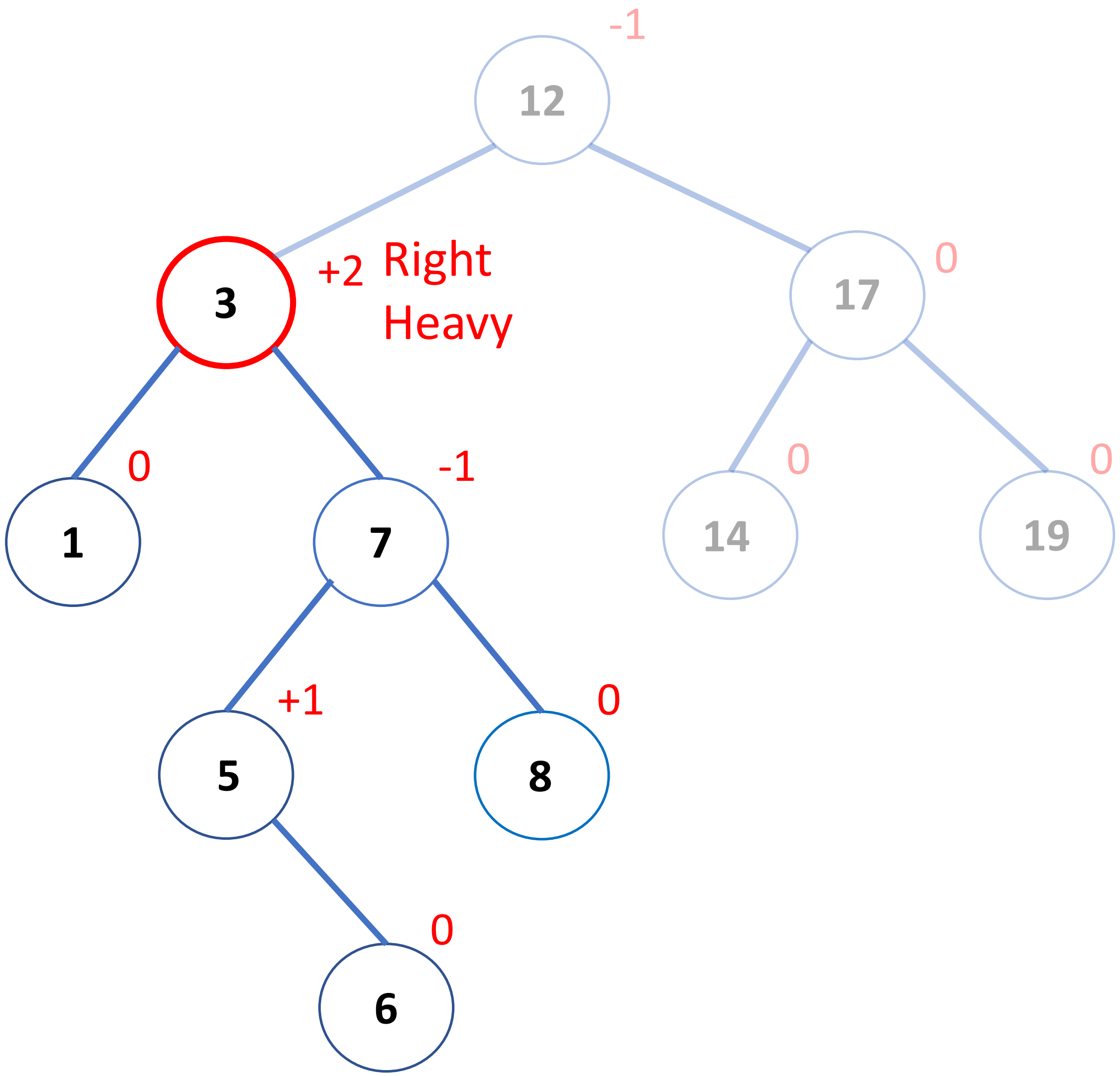
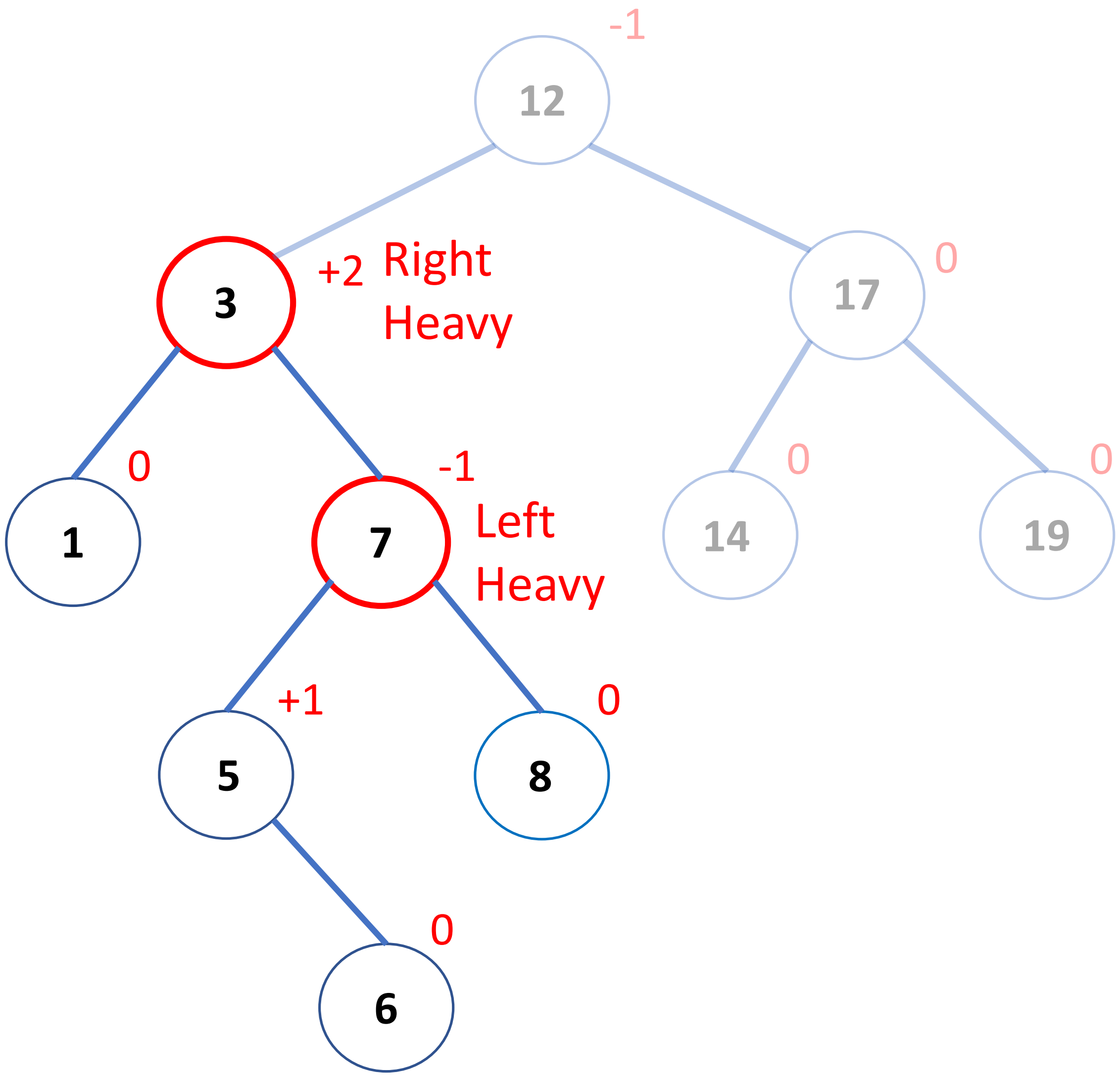


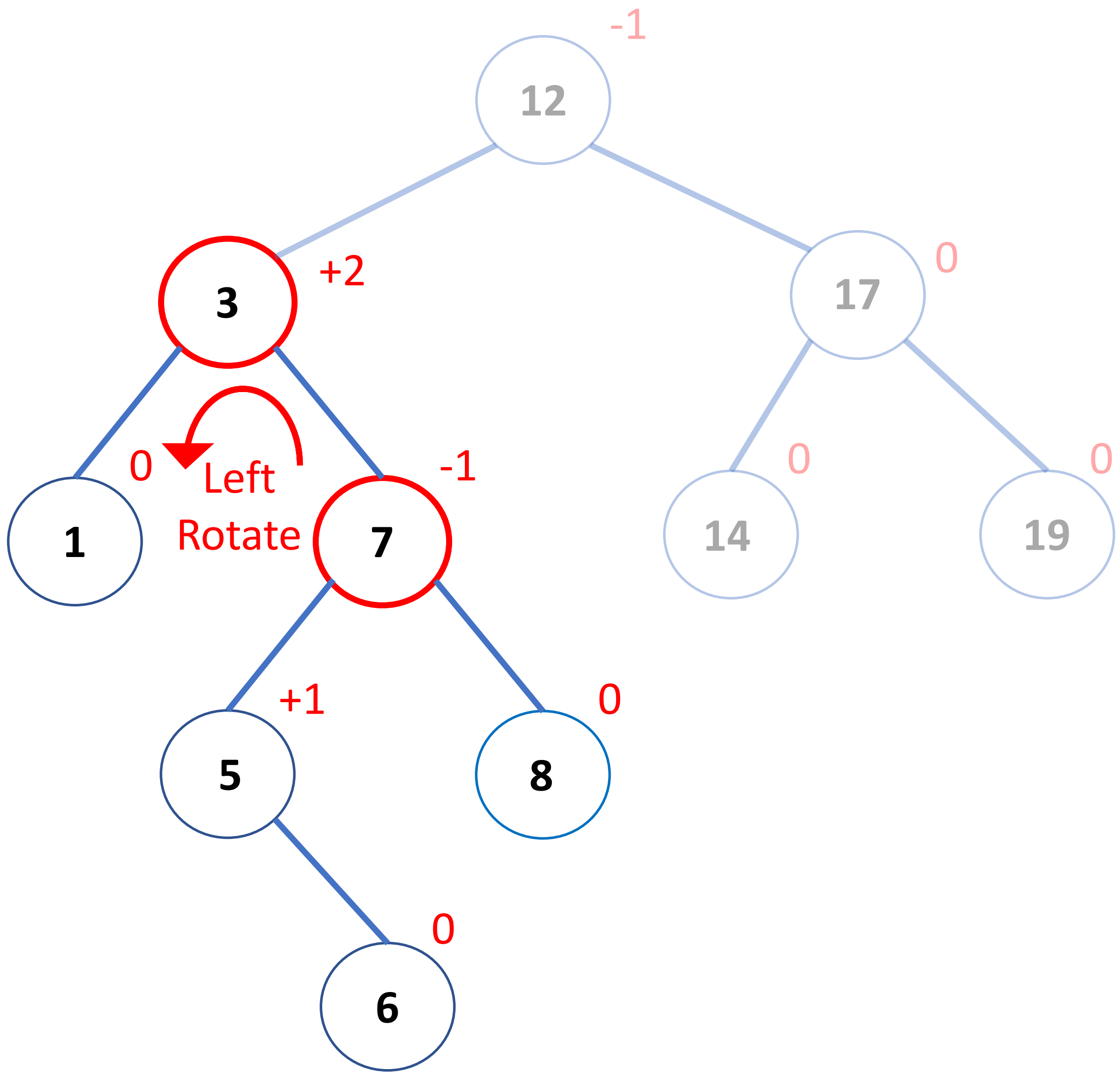
Rebalancing left subtree of 12:



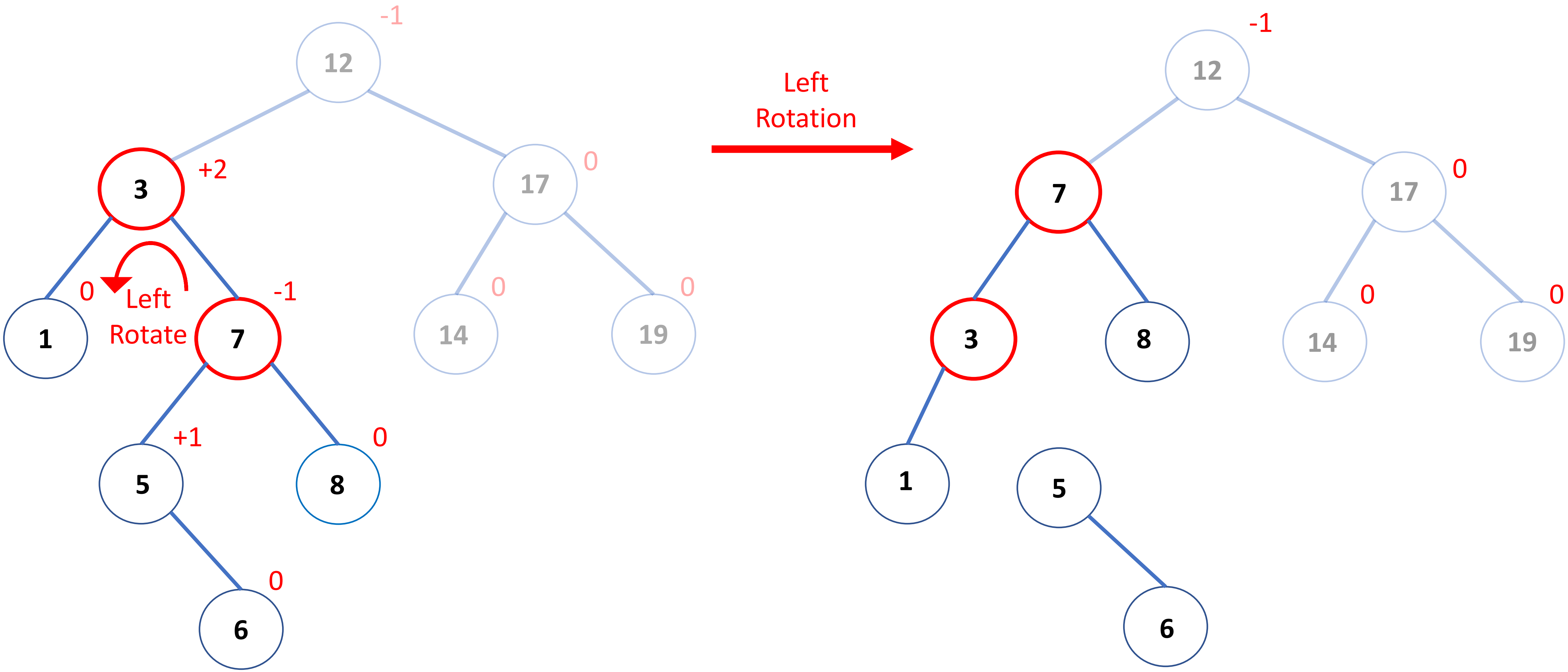
Rebalancing left subtree of 12:



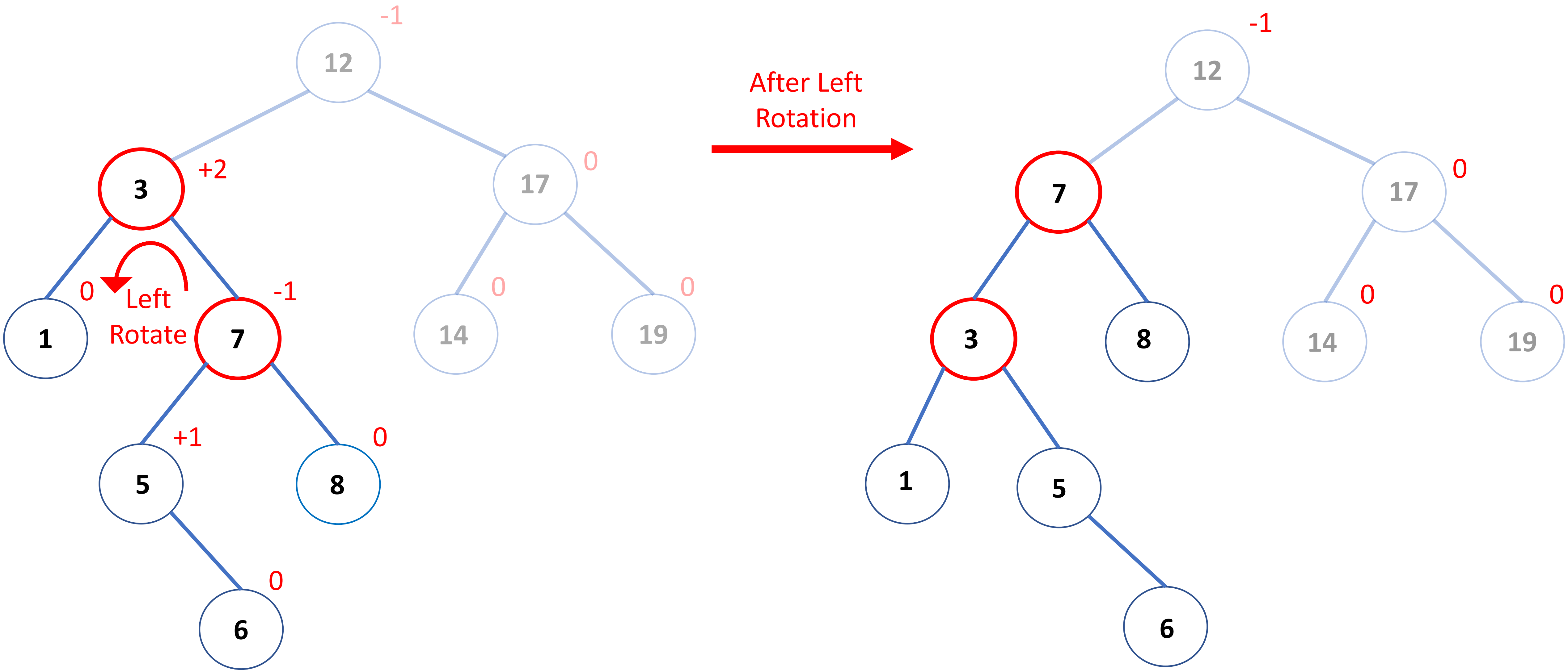
Rebalancing left subtree of 12: Try Left Rotation !



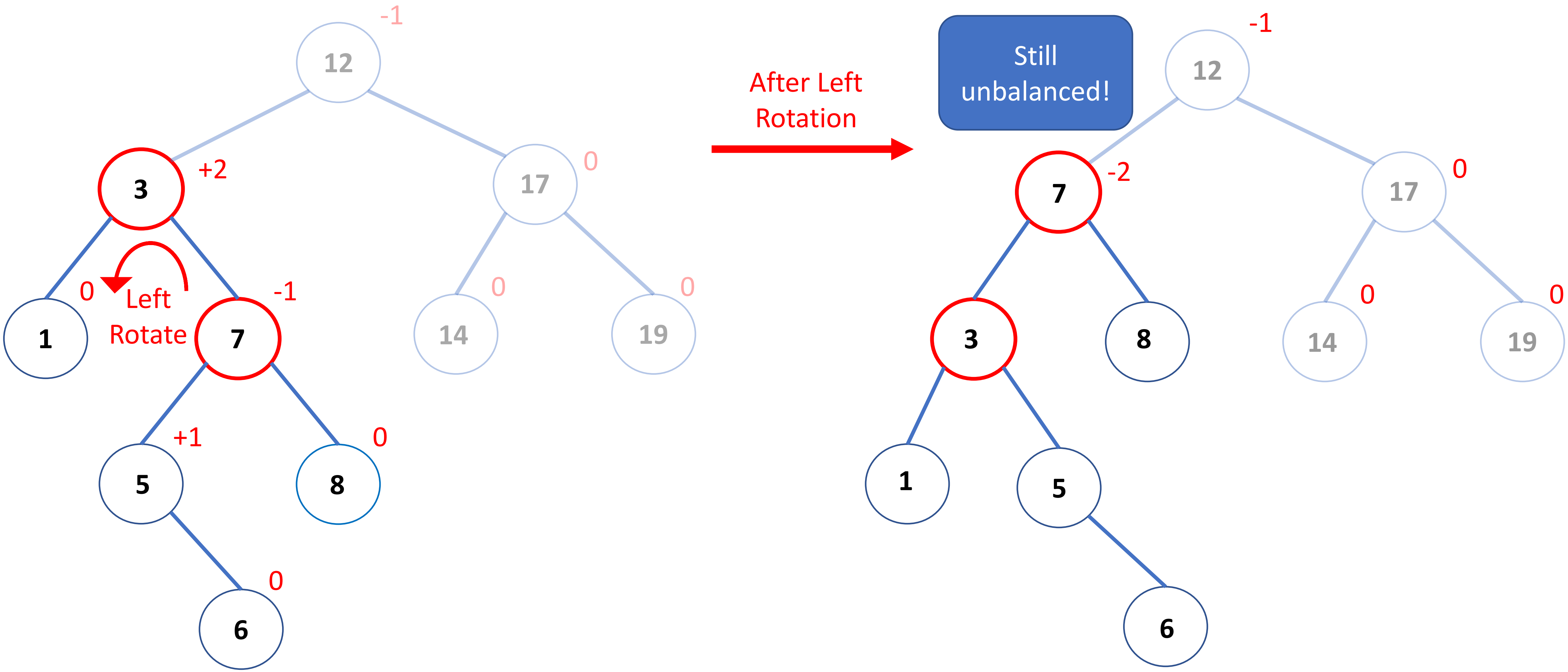
Rebalancing left subtree of 12: Try Left Rotation !



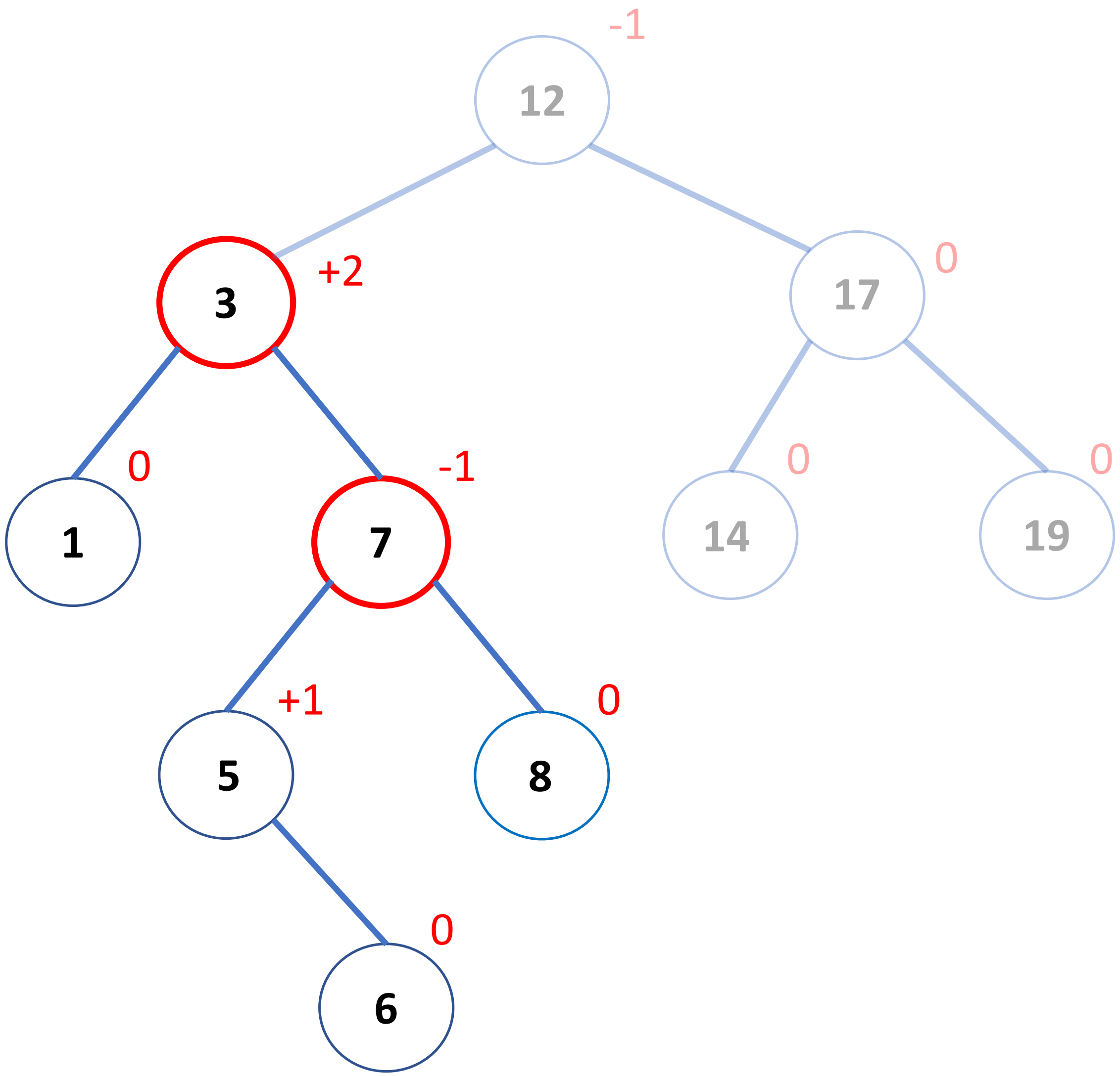
Rebalancing left subtree of 12: Try Left Rotation !



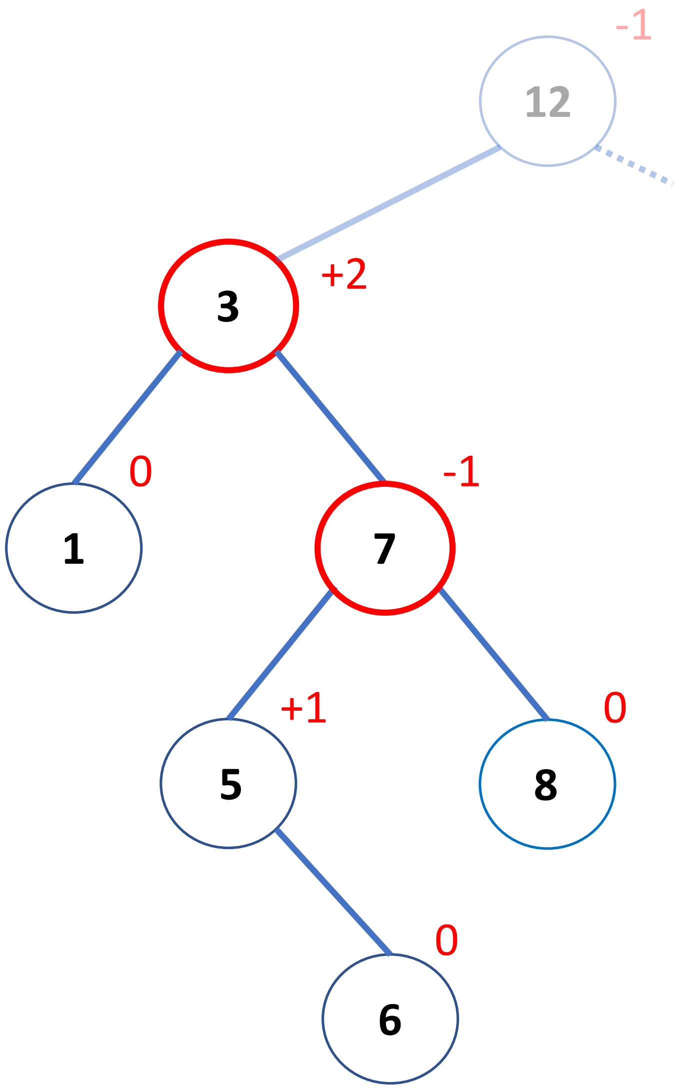
Rebalancing left subtree of 12: Try Left Rotation !



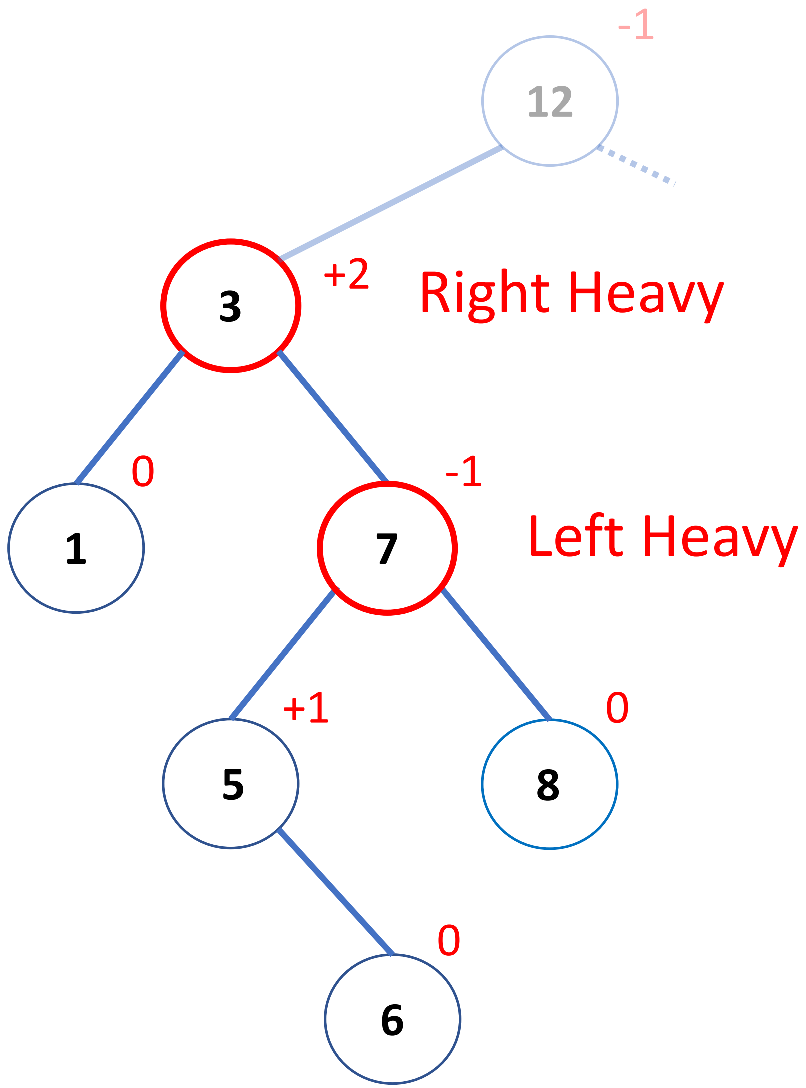
Rebalancing left subtree of 12: Attempt two!



Rebalancing left subtree of 12: Attempt two!

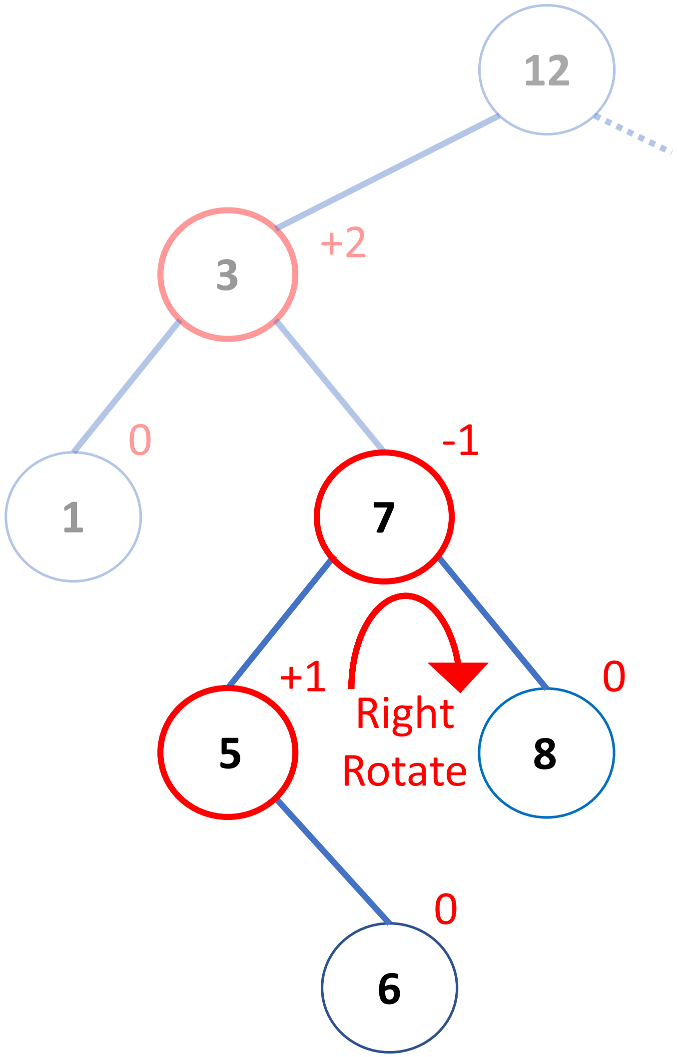


Rebalancing left subtree of 12: Attempt two!



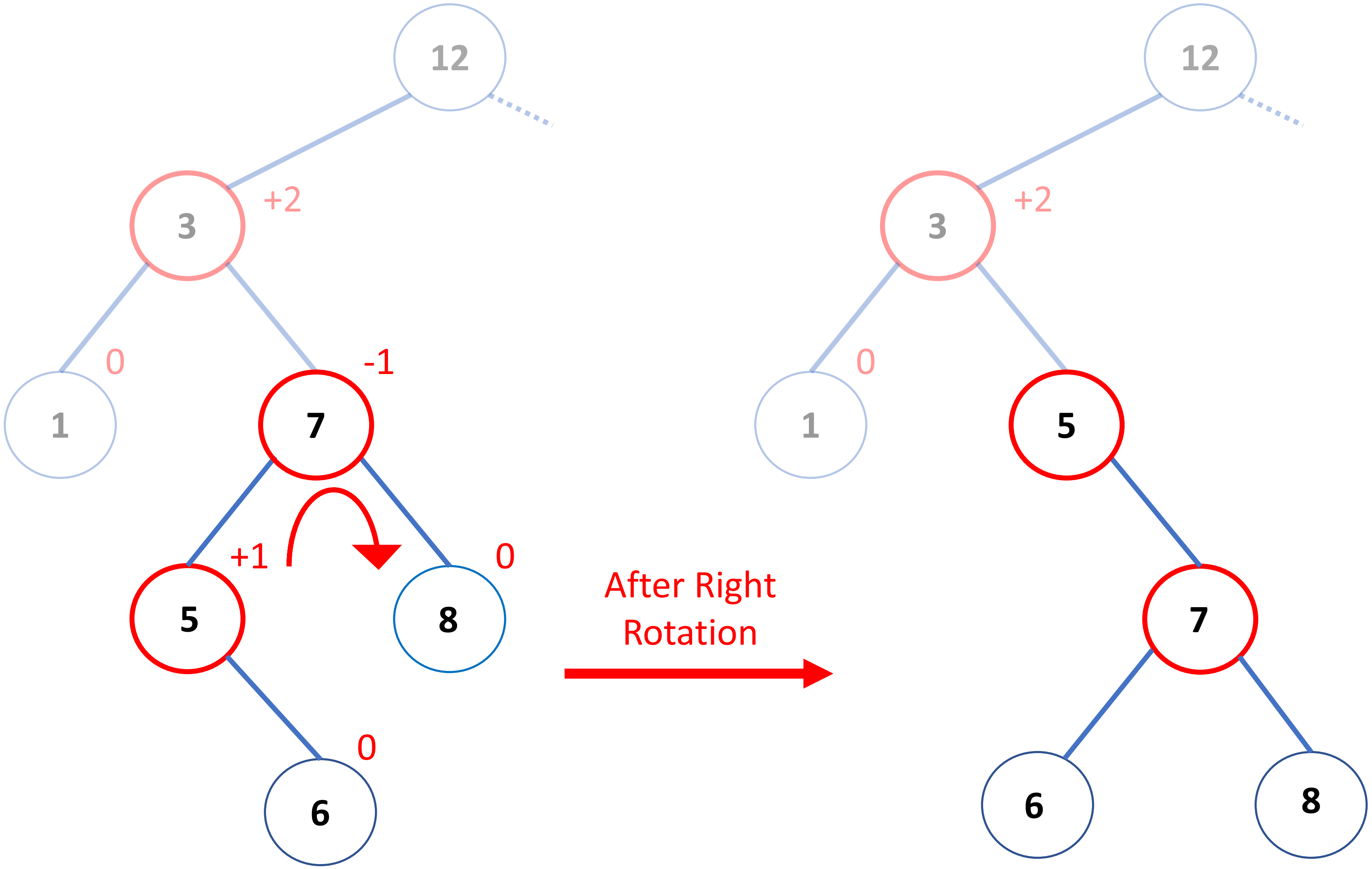
Rebalancing left subtree of 12: Attempt two!

Step 1. Right rotate right subtree of 3



Rebalancing left subtree of 12: Attempt two!

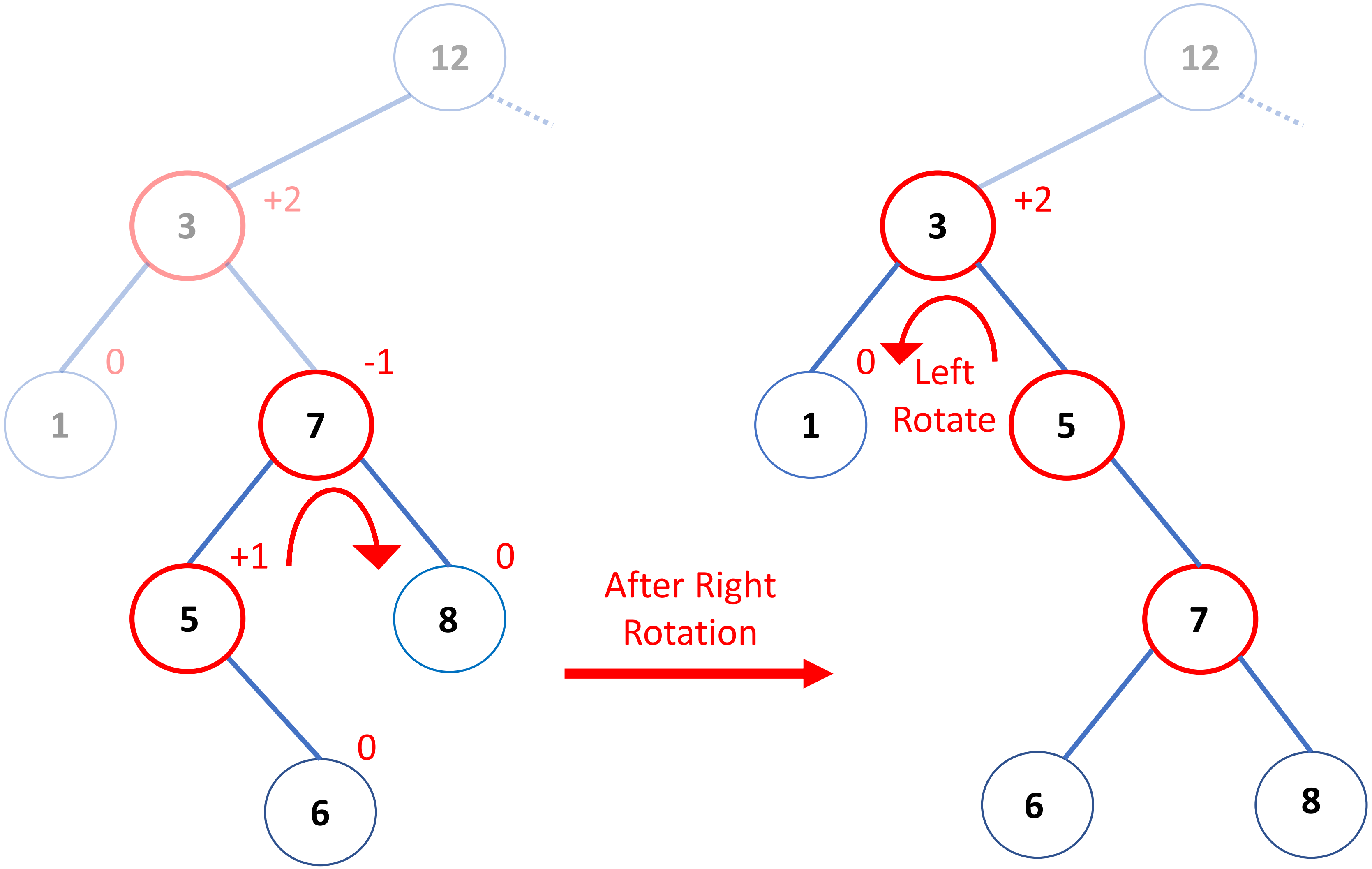
Step 1. Right rotate right subtree of 3



Rebalancing left subtree of 12: Attempt two!

Step 1. Right rotate right subtree of 3

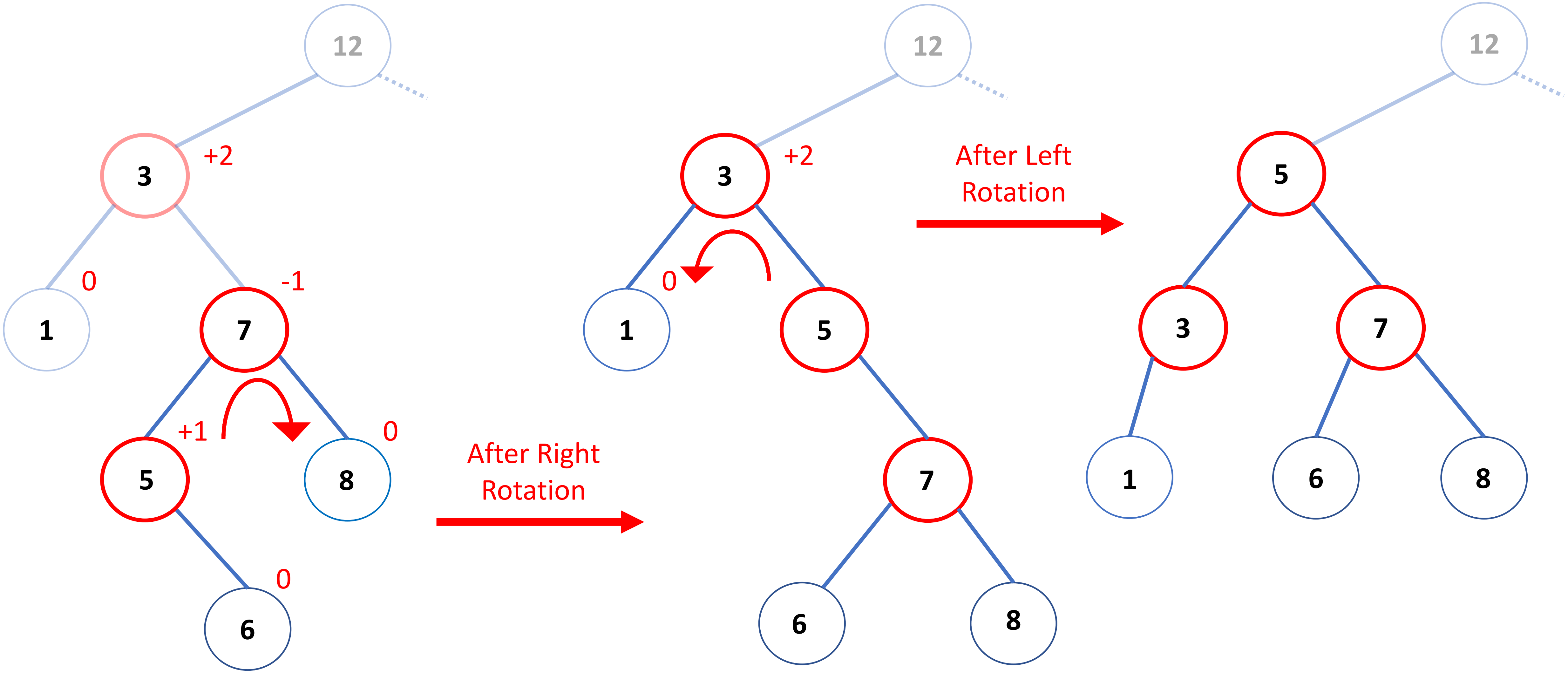
Step 2. Left rotate left subtree of 12



Rebalancing left subtree of 12: Attempt two!

Step 1. Right rotate right subtree of 3

Step 2. Left rotate left subtree of 12



Rebalancing left subtree of 12: Double Right-Left Rotation

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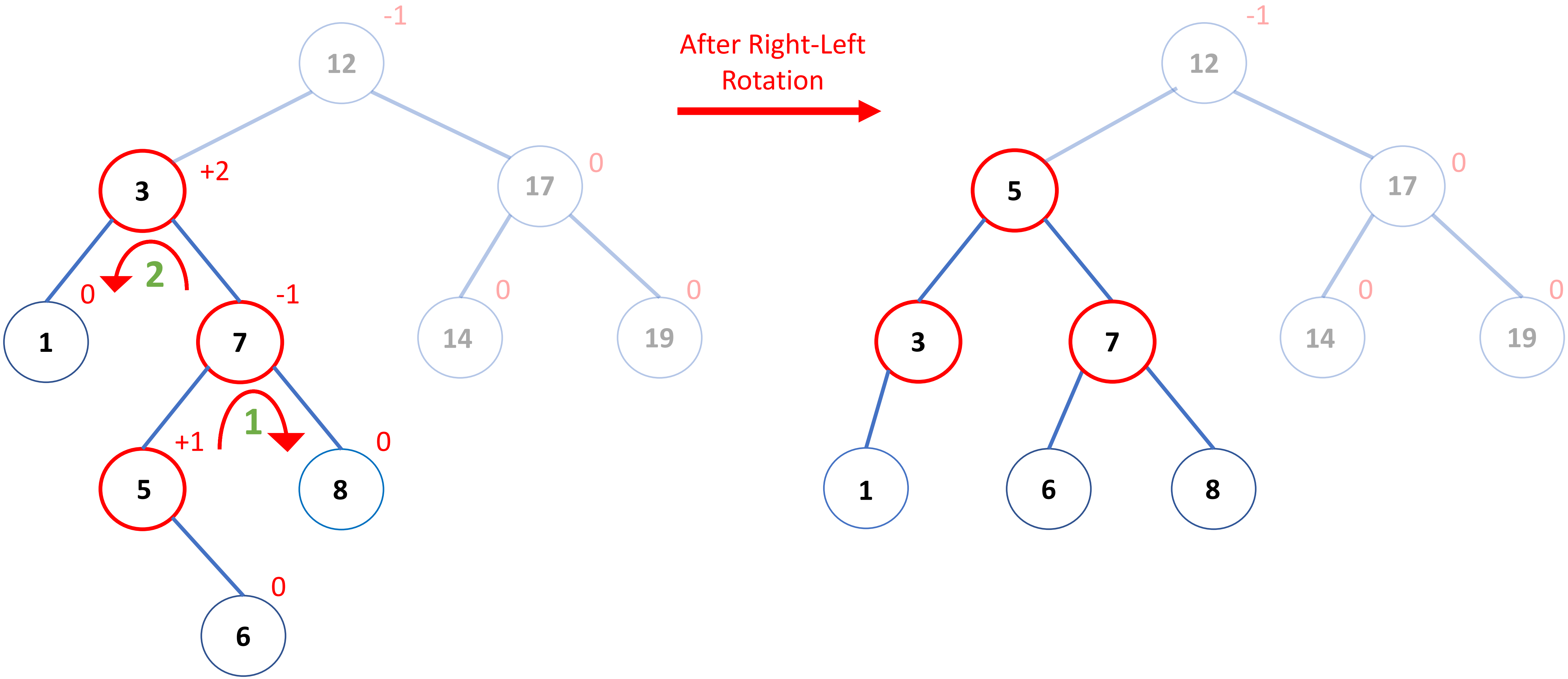

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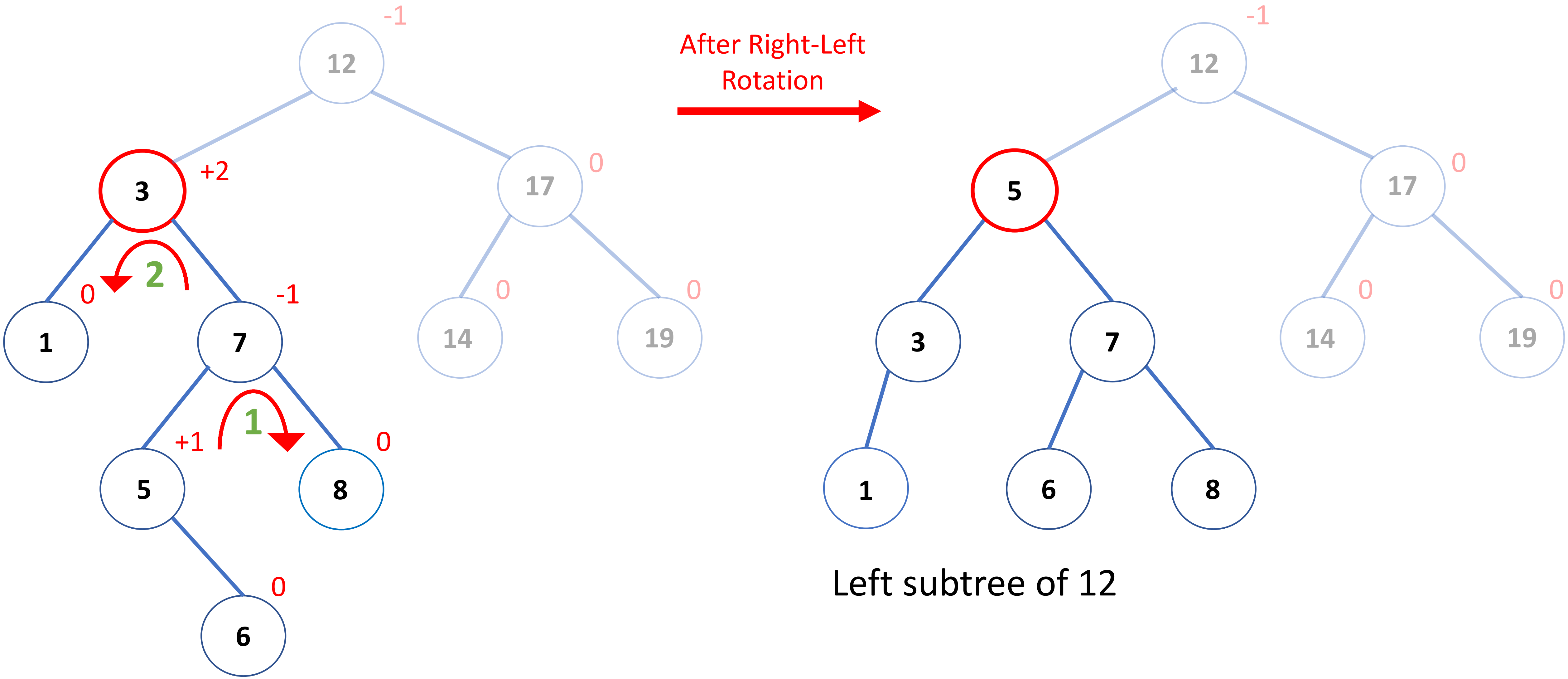
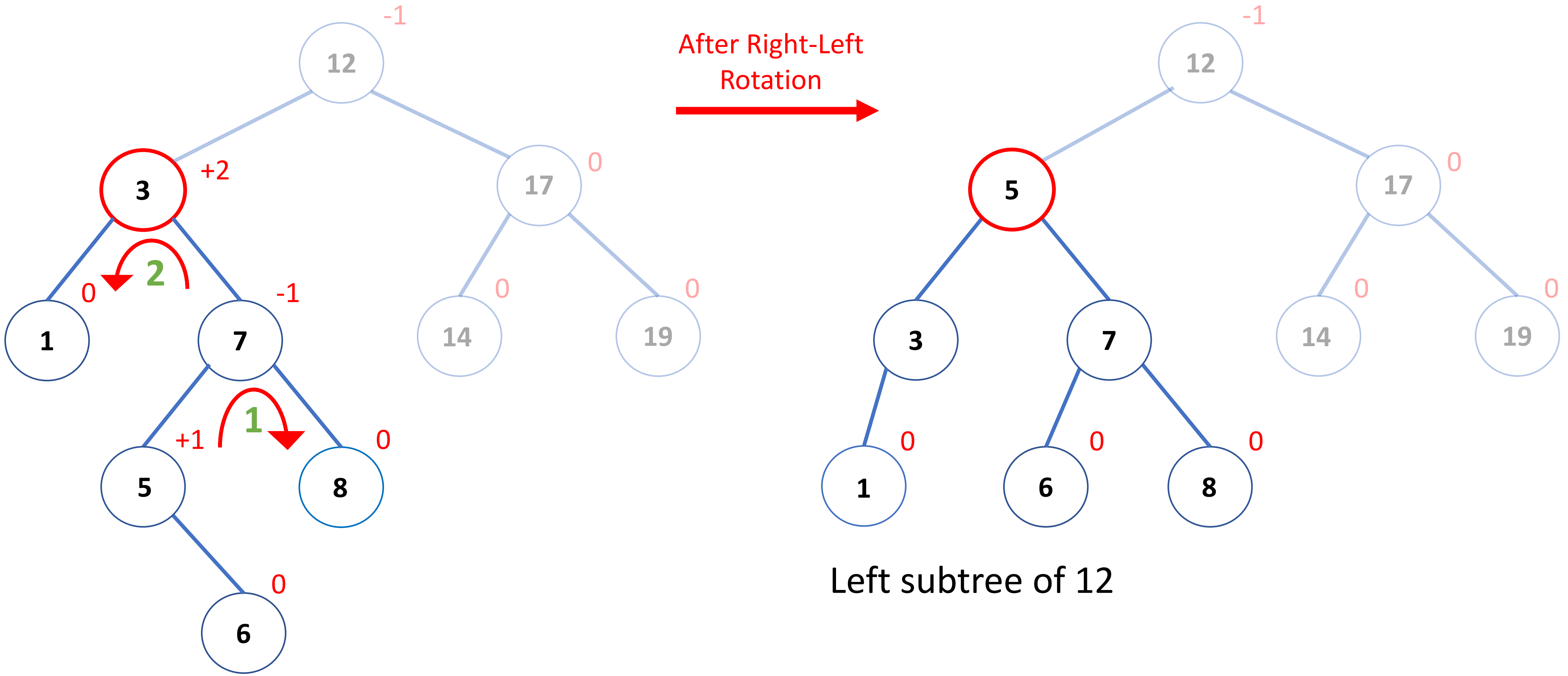
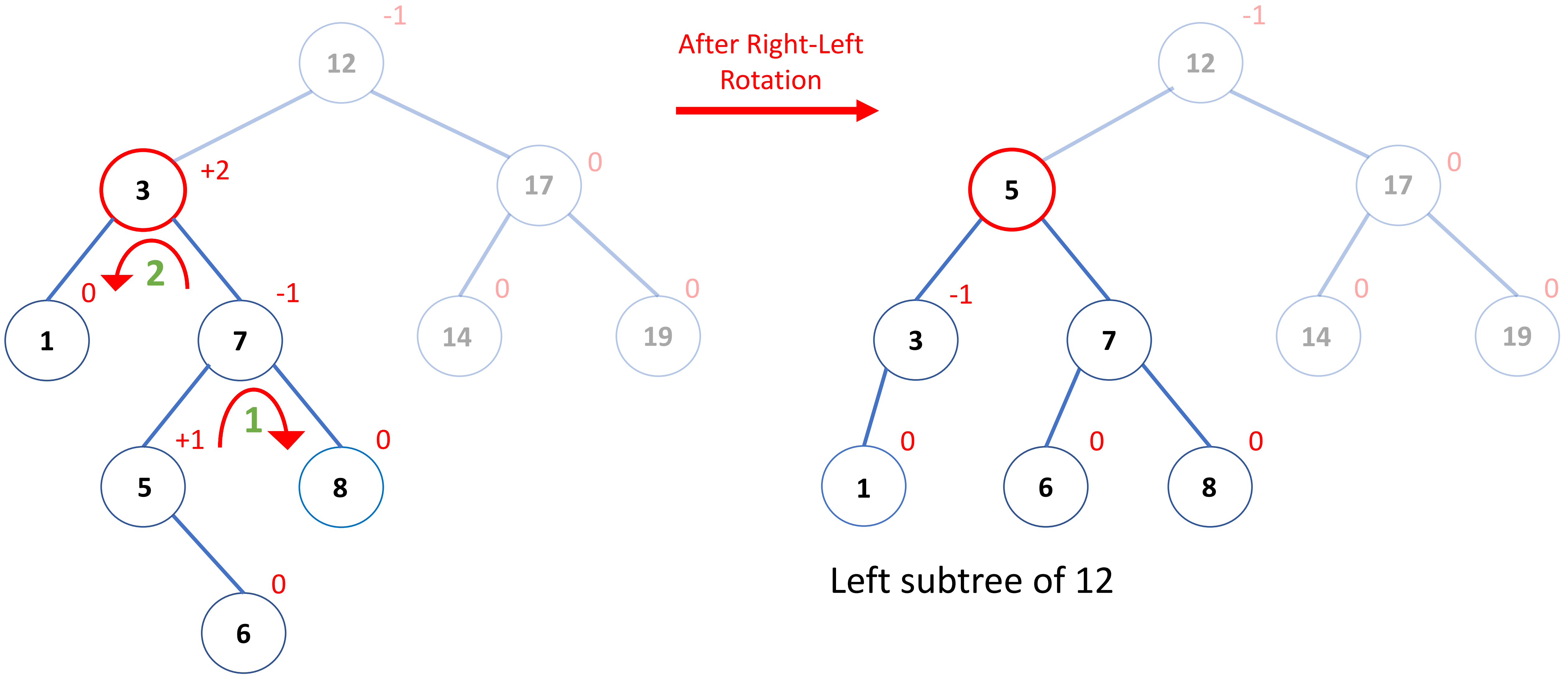


Rebalancing left subtree of 12: Double Right-Left Rotation



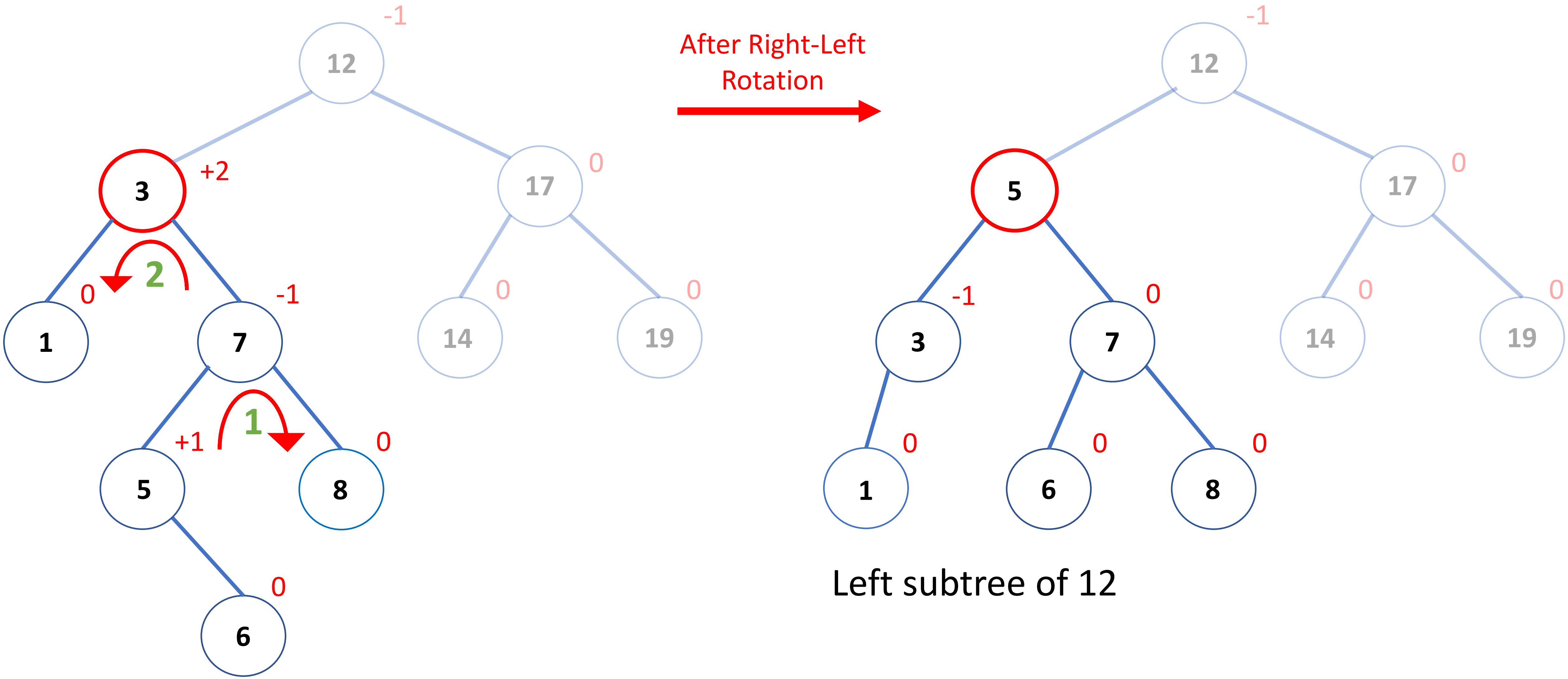
Left subtree of 12

Rebalancing left subtree of 12: Double Right-Left Rotation



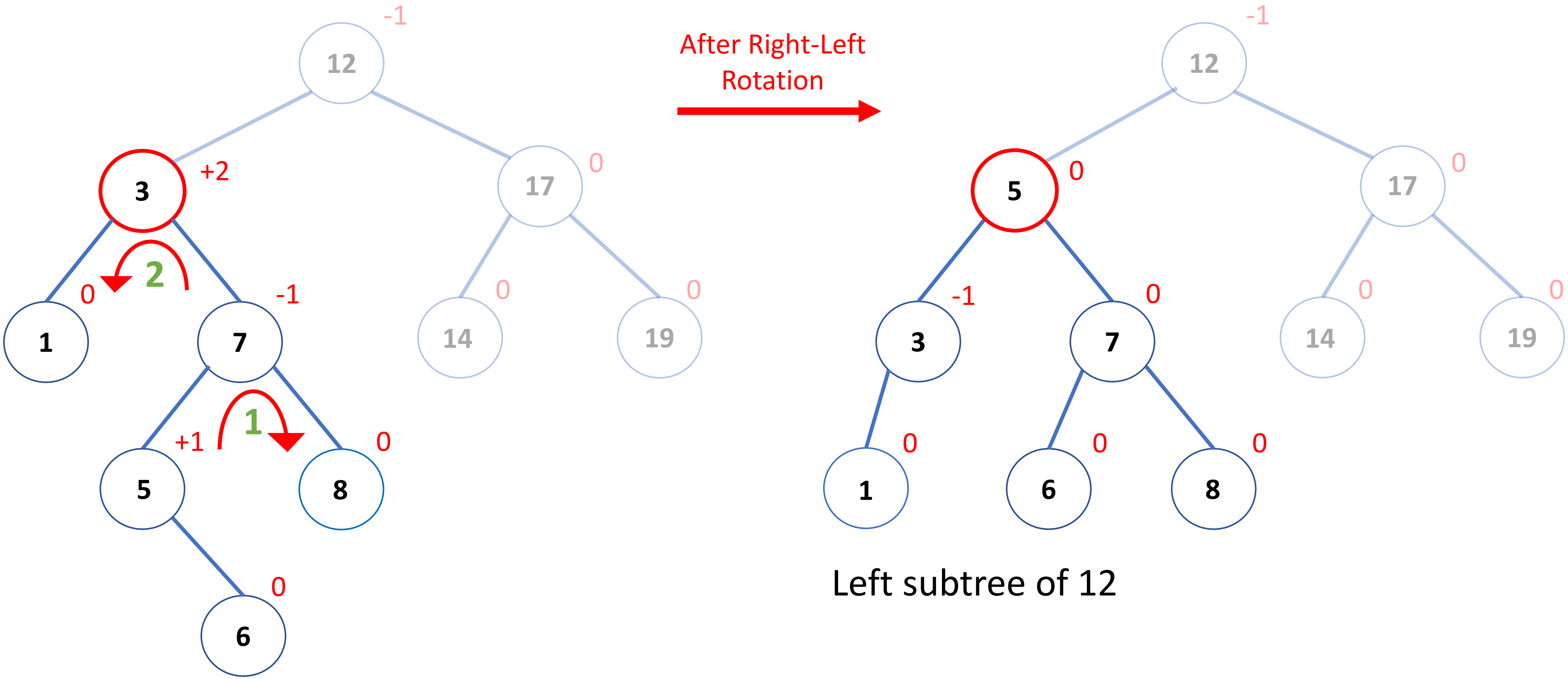
Left subtree of 12

Rebalancing left subtree of 12: Double Right-Left Rotation



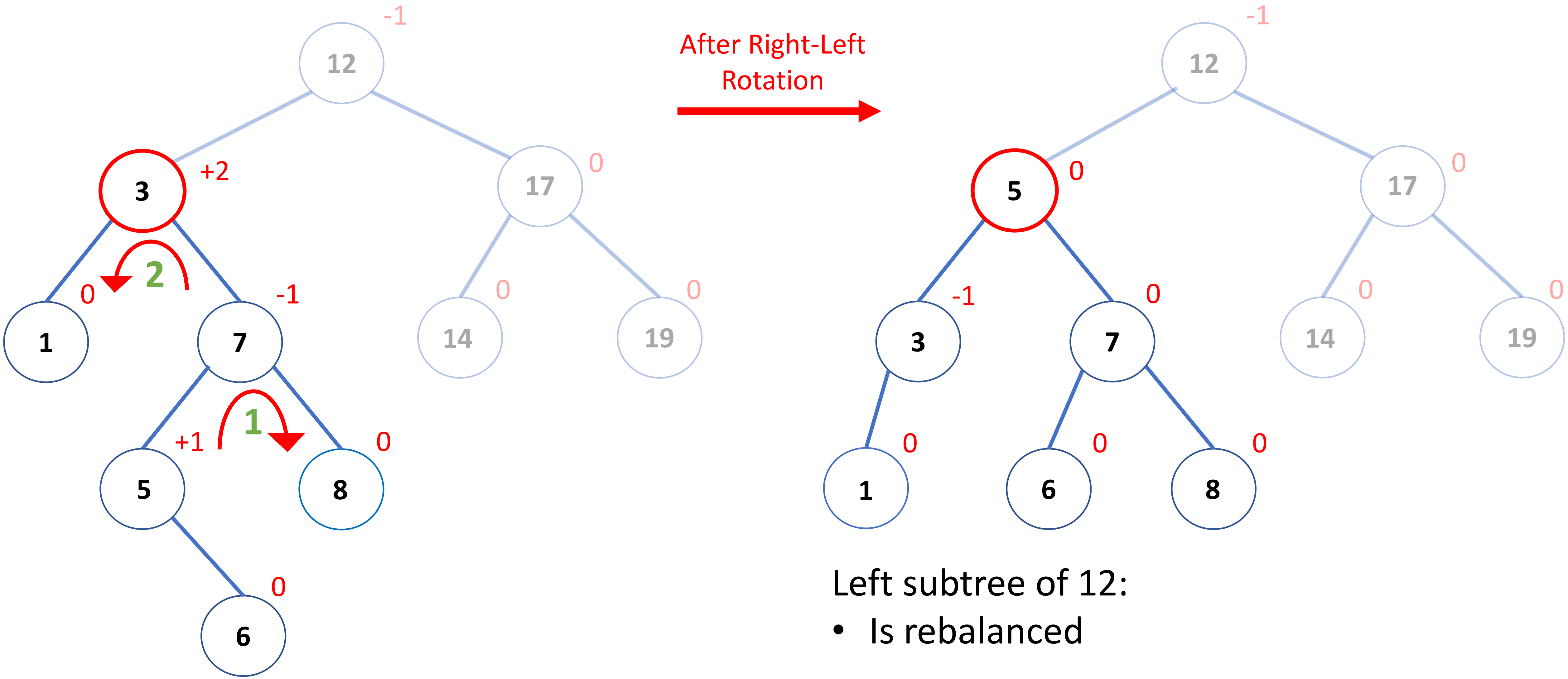
Left subtree of 12

Rebalancing left subtree of 12: Double Right-Left Rotation



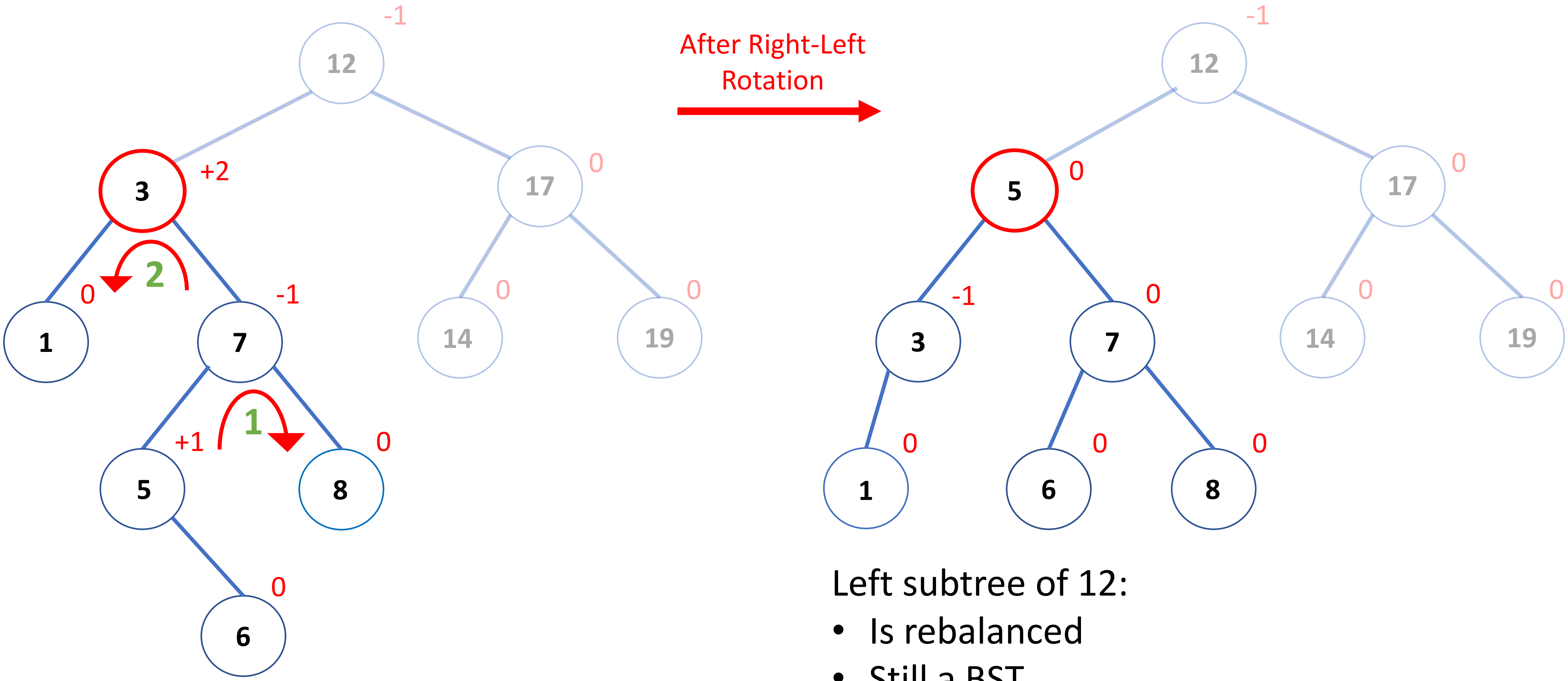
Left subtree of 12

Rebalancing left subtree of 12: Double Right-Left Rotation



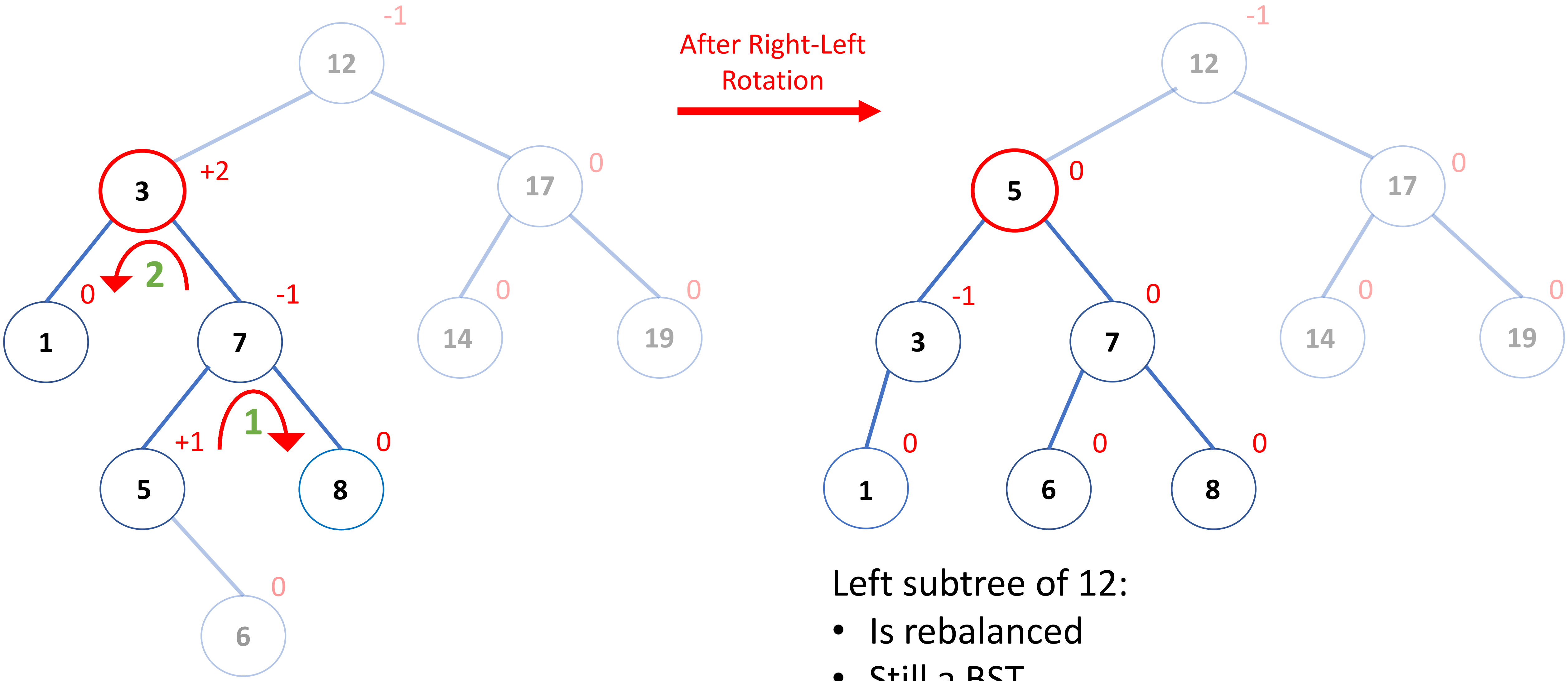
Left subtree of 12:
- Is rebalanced

Rebalancing left subtree of 12: Double Right-Left Rotation



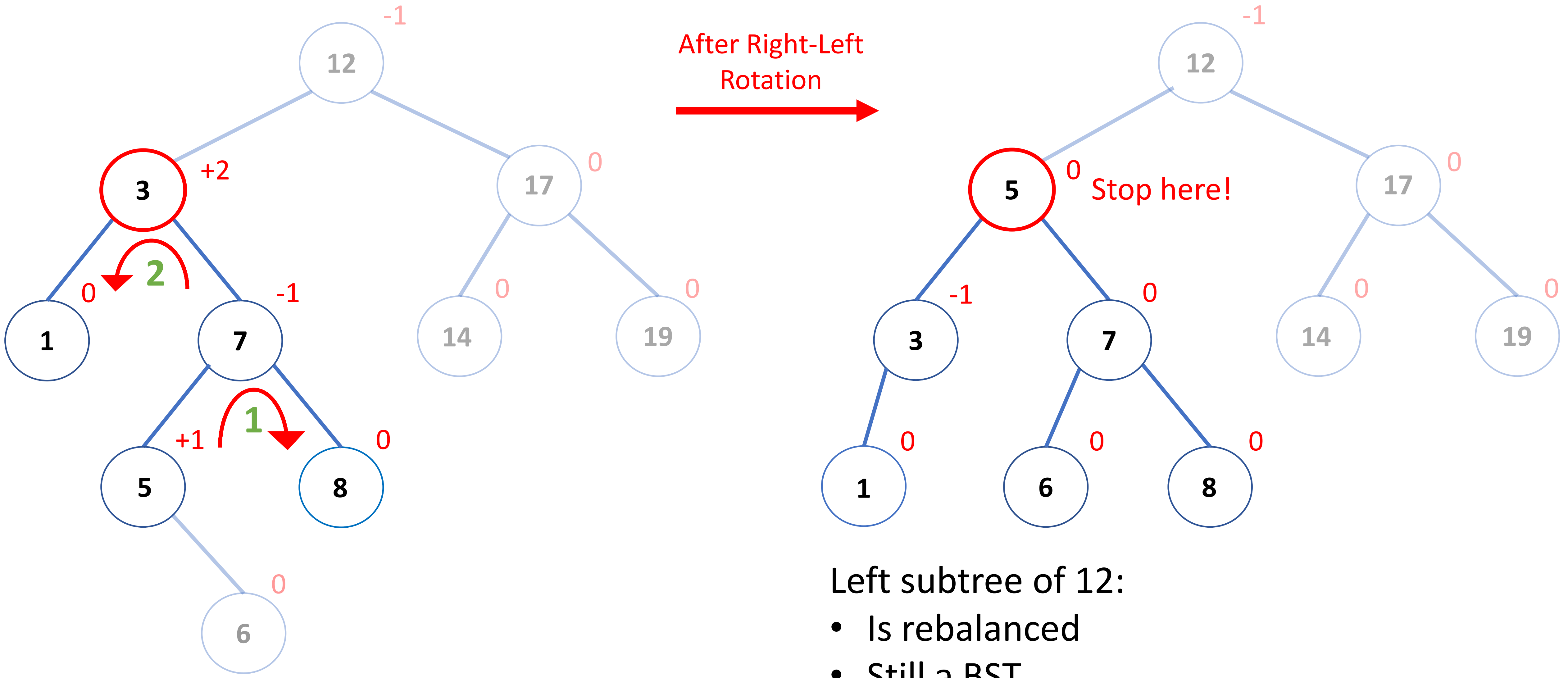
Left subtree of 12:
- Is rebalanced
- Still a BST

Rebalancing left subtree of 12: Double Right-Left Rotation



Left subtree of 12:
- Is rebalanced
- Still a BST
- **Bonus**: height same as before inserting 6

Rebalancing left subtree of 12: Double Right-Left Rotation



Left subtree of 12:

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