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It is not an automatic process: needs creativity!

Dynamic Order Statistics

Maintain a dynamic set S of elements with distinct keys, supporting the following operations:

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Rank of x : Position of x in the sorted order of S

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5. **Rank**(x) : Given a pointer to x, find the rank of x

Rank of x : Position of x in the sorted order of S

Example

$S = \{5, 15, 27, 30, 56\}$

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Select(4) =

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Rank(15) =

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S = {5, 15, 27, 30, 56}

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Example

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Select(4) = 30

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Example

S = {5, 15, 27, 30, 56}

Select(4) = 30

Select(1) = 5

Rank(15) = 2

Rank(30) = 4

Example

S = {5, 15, 27, 30, 56}

Select(4) = 30

Select(1) = 5

Rank(15) = 2

Rank(30) = 4 (Note: Exactly 3 elements < 30)

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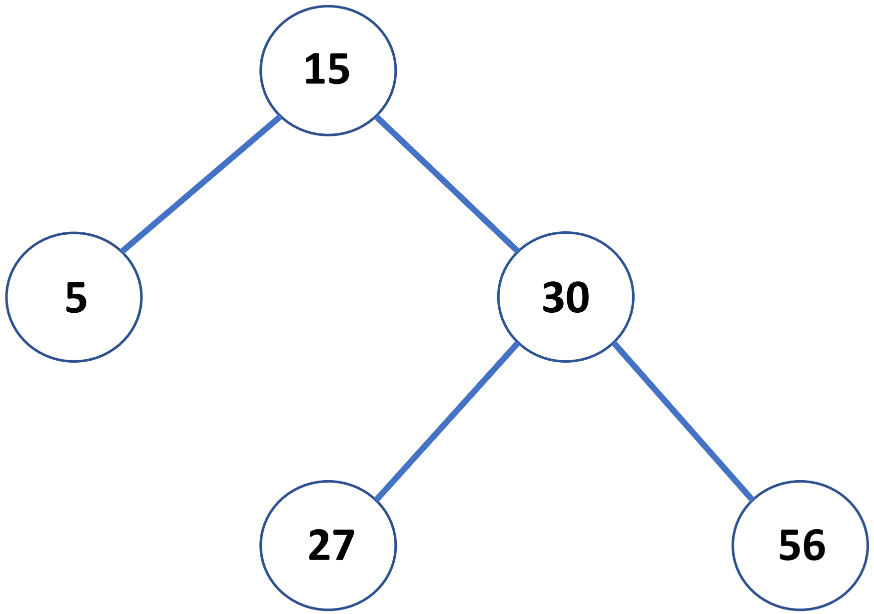
  keep S in an AVL tree



What data structure do we use?

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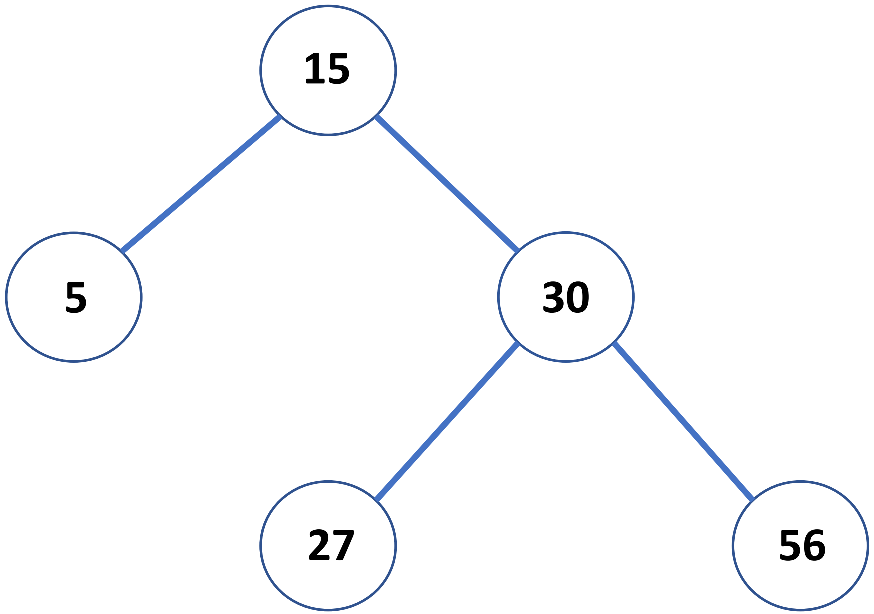

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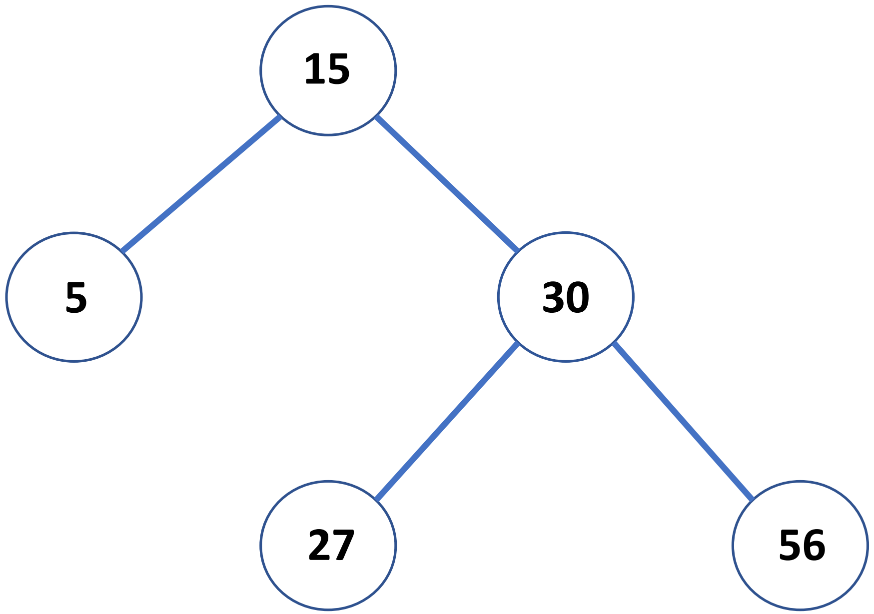

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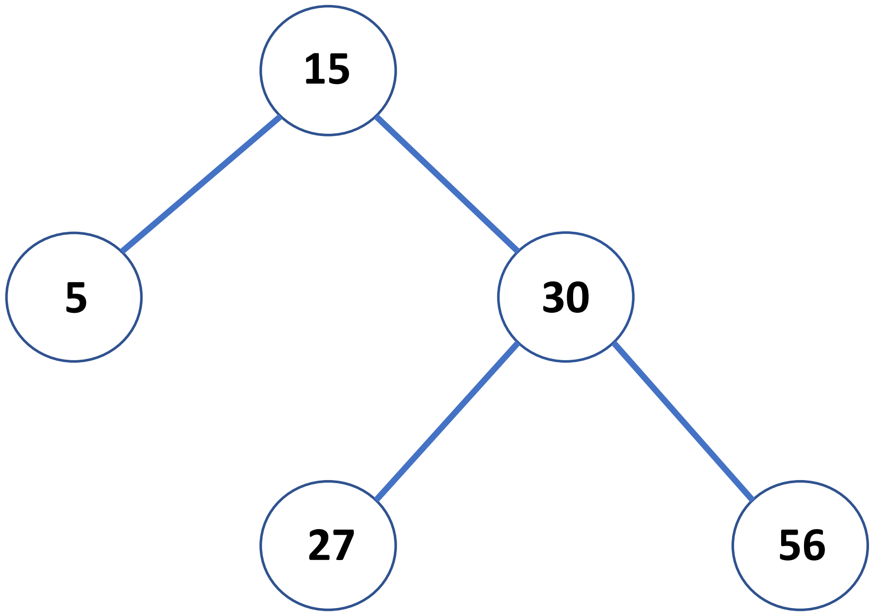

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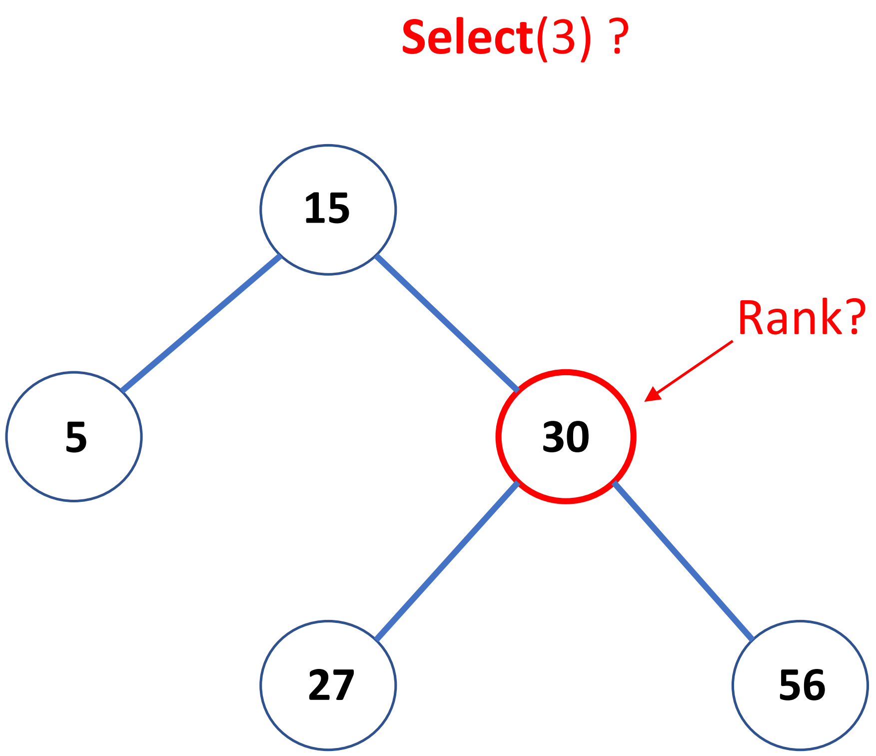

What data structure do we use?

- For efficient **Search, Insert, Delete:**

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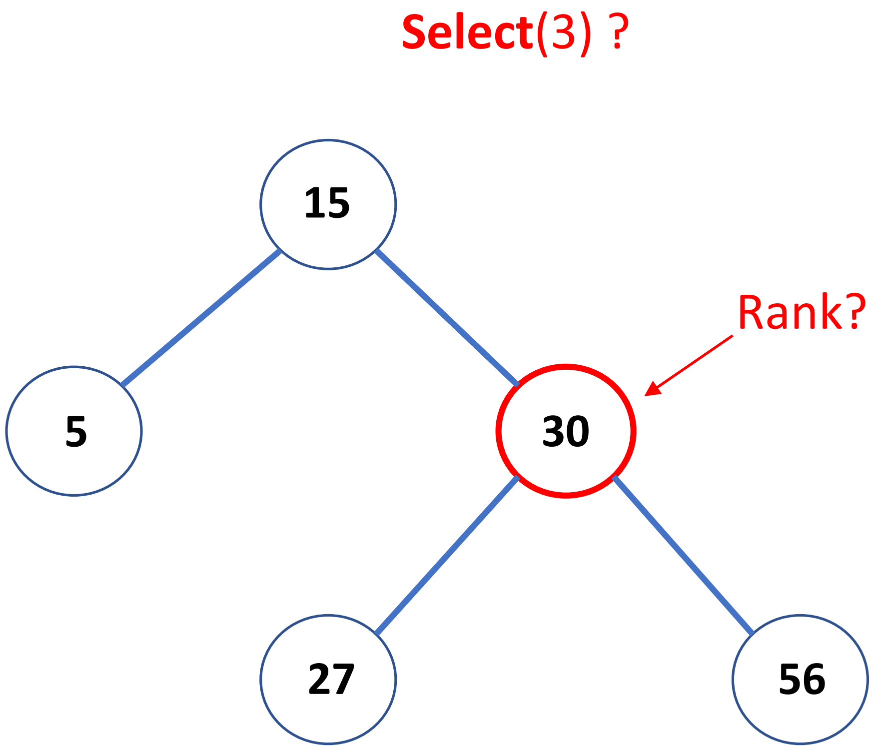
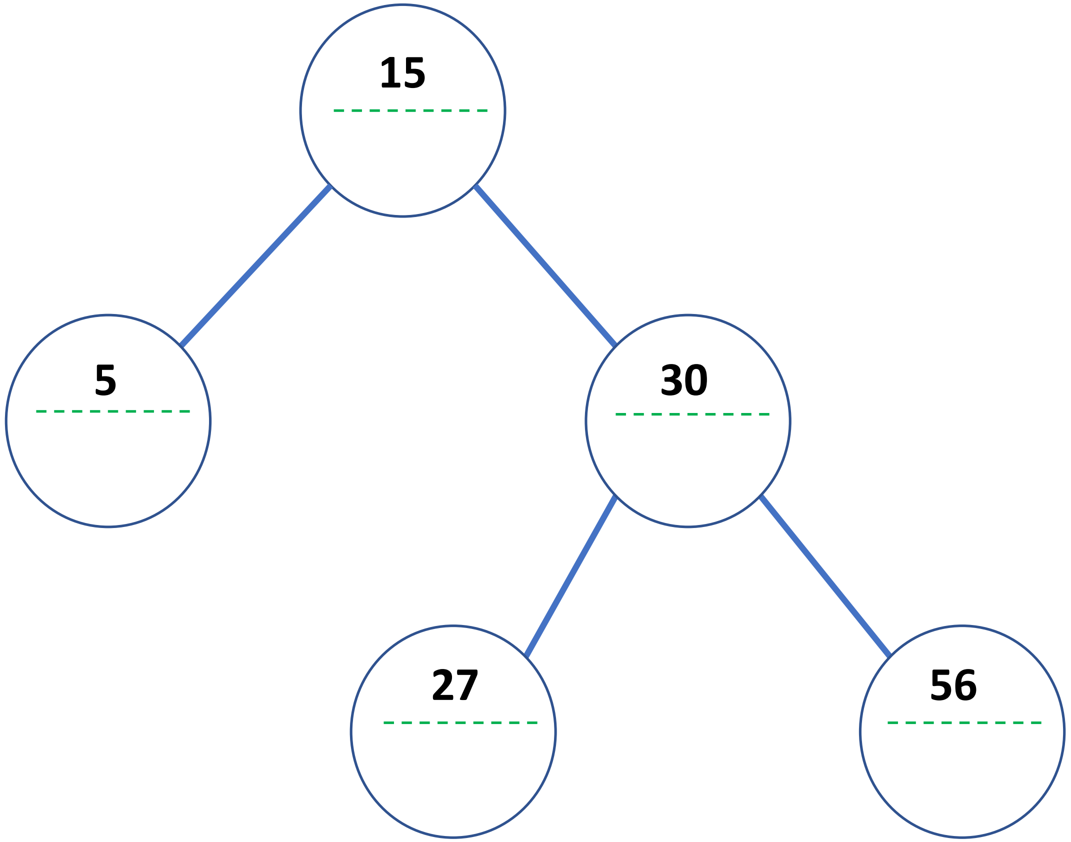
- How to efficiently implement **Select, Rank** ?
  - Augment the AVL tree!



Naïve Augmentation



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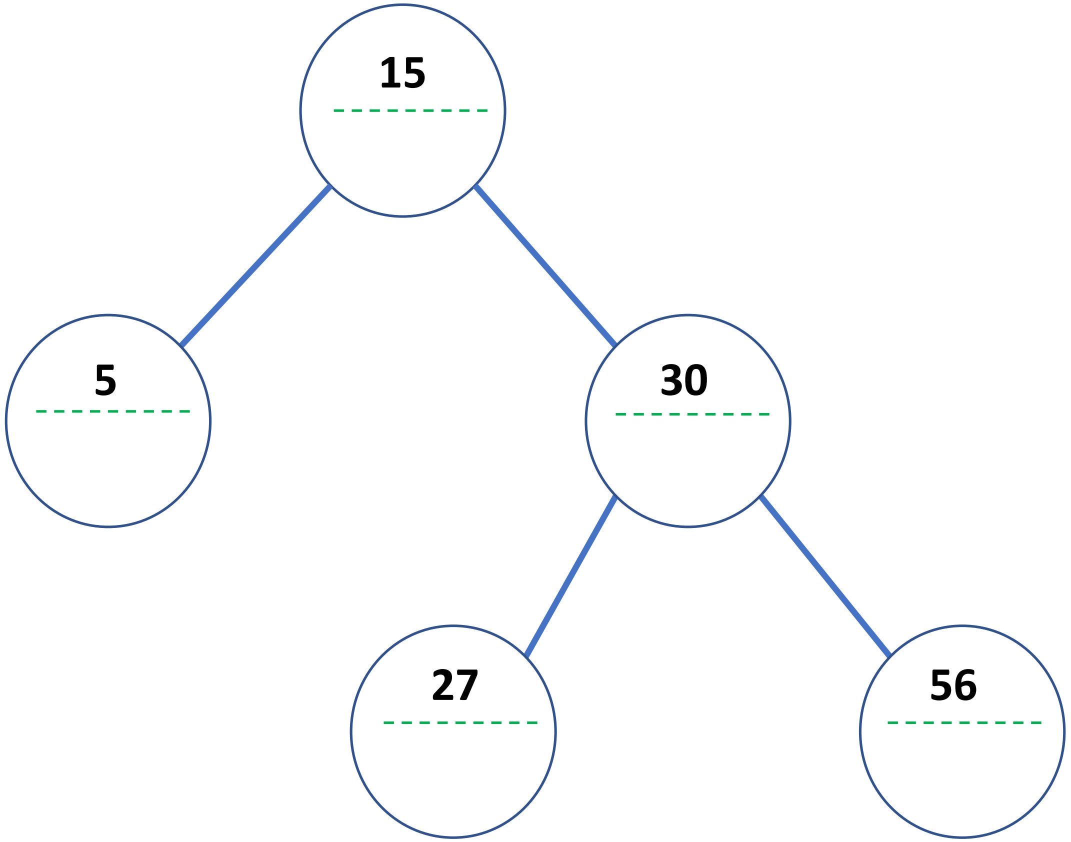

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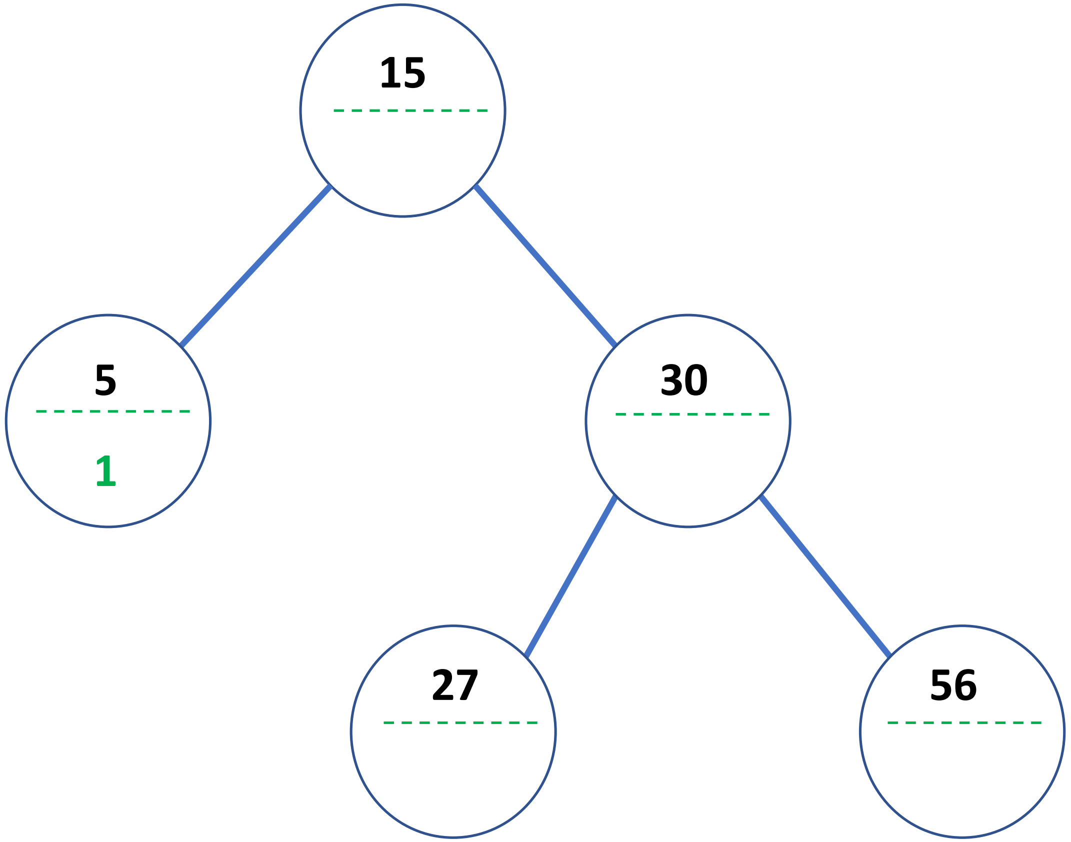

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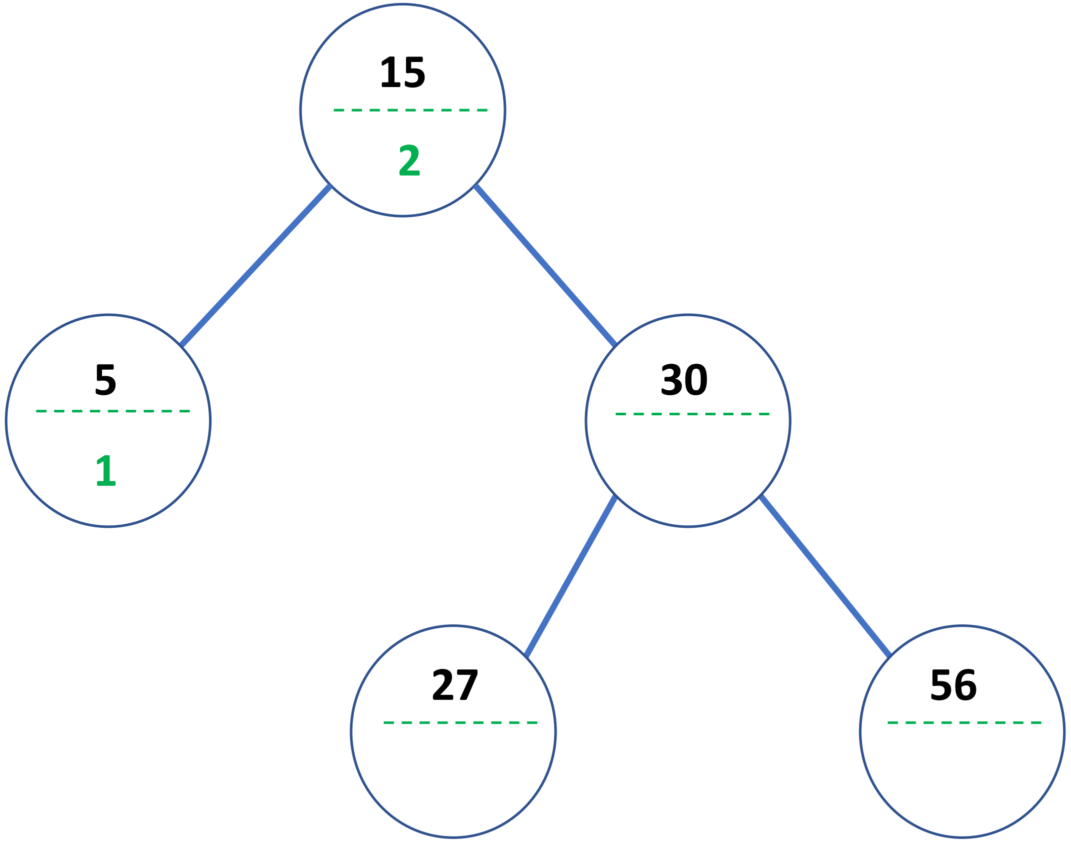

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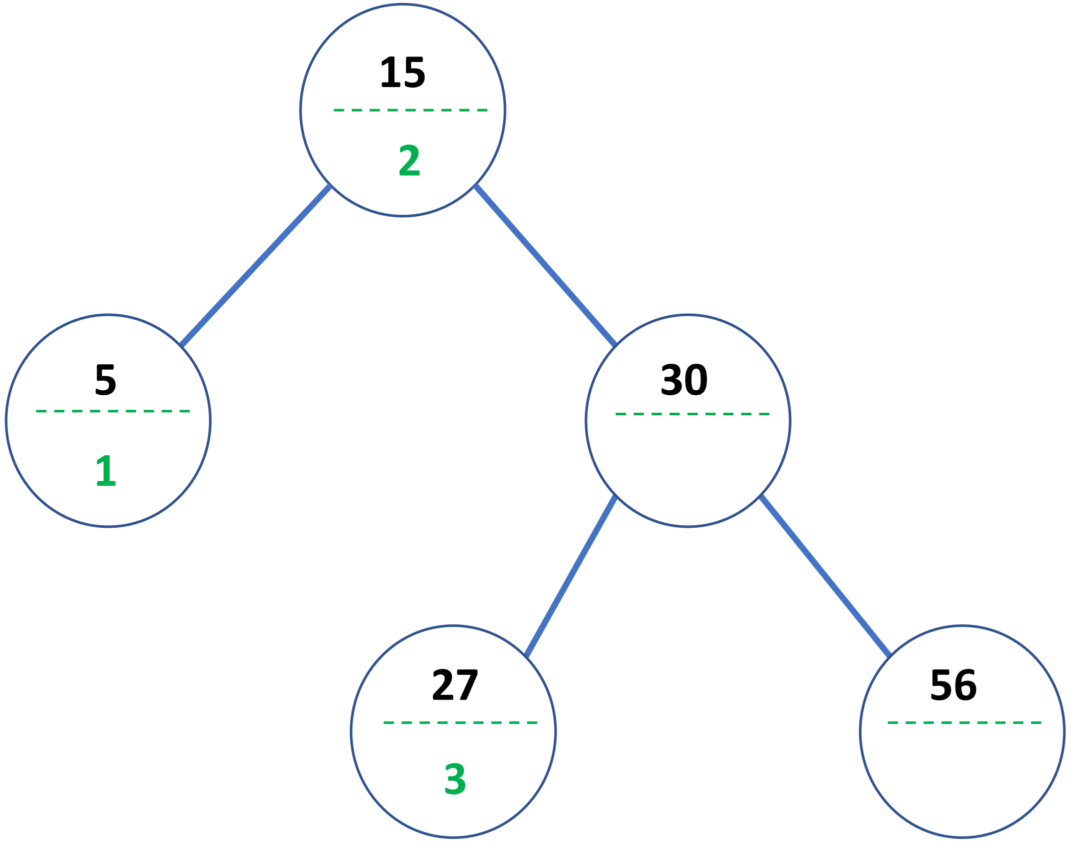

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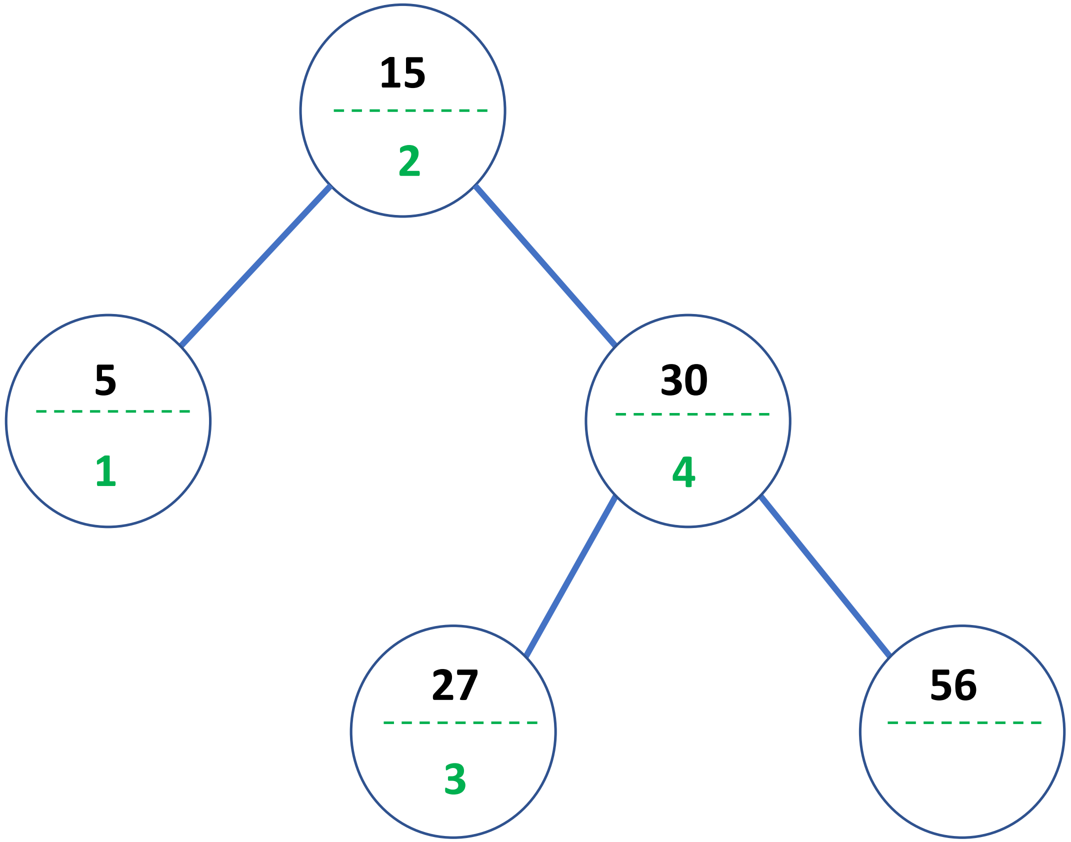

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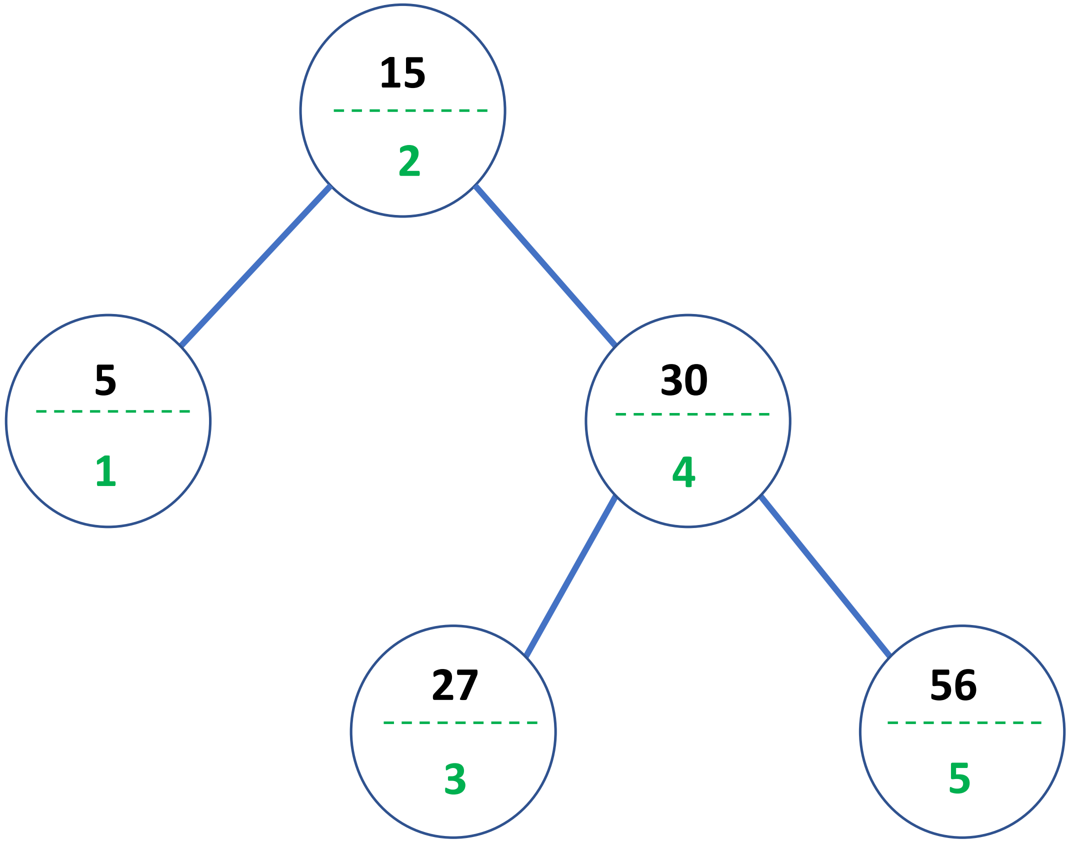

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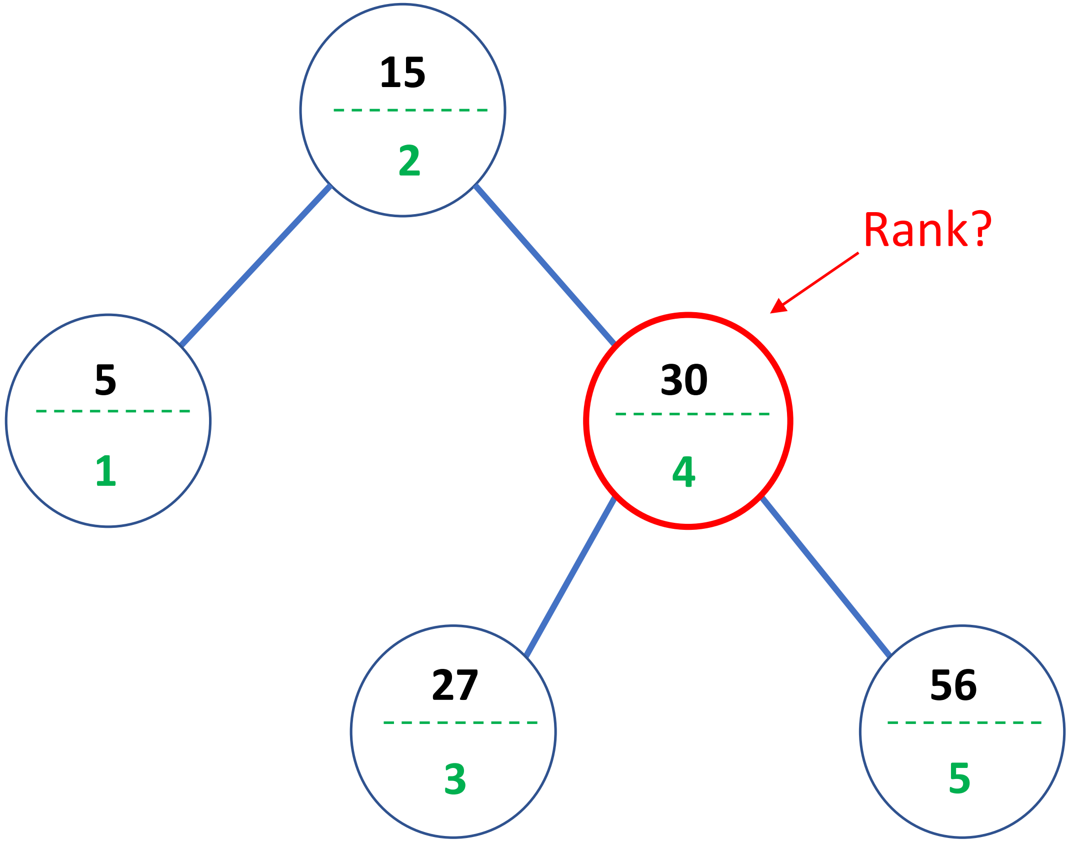

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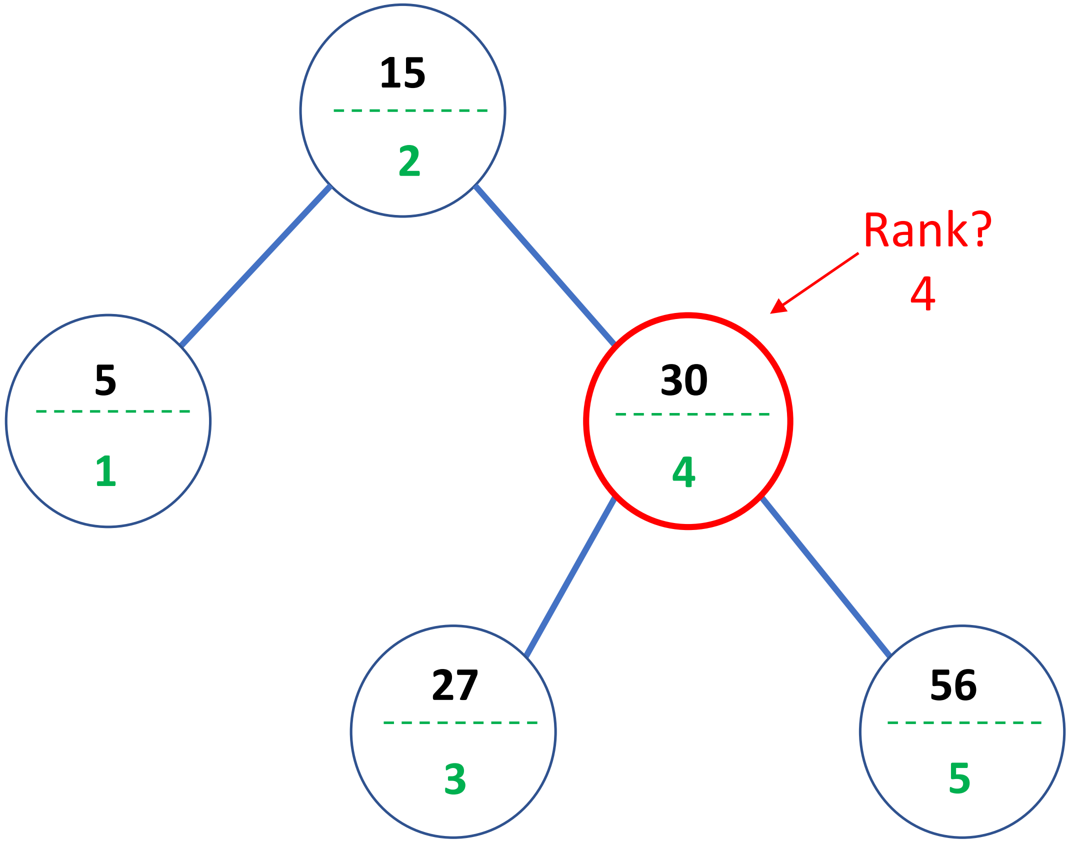

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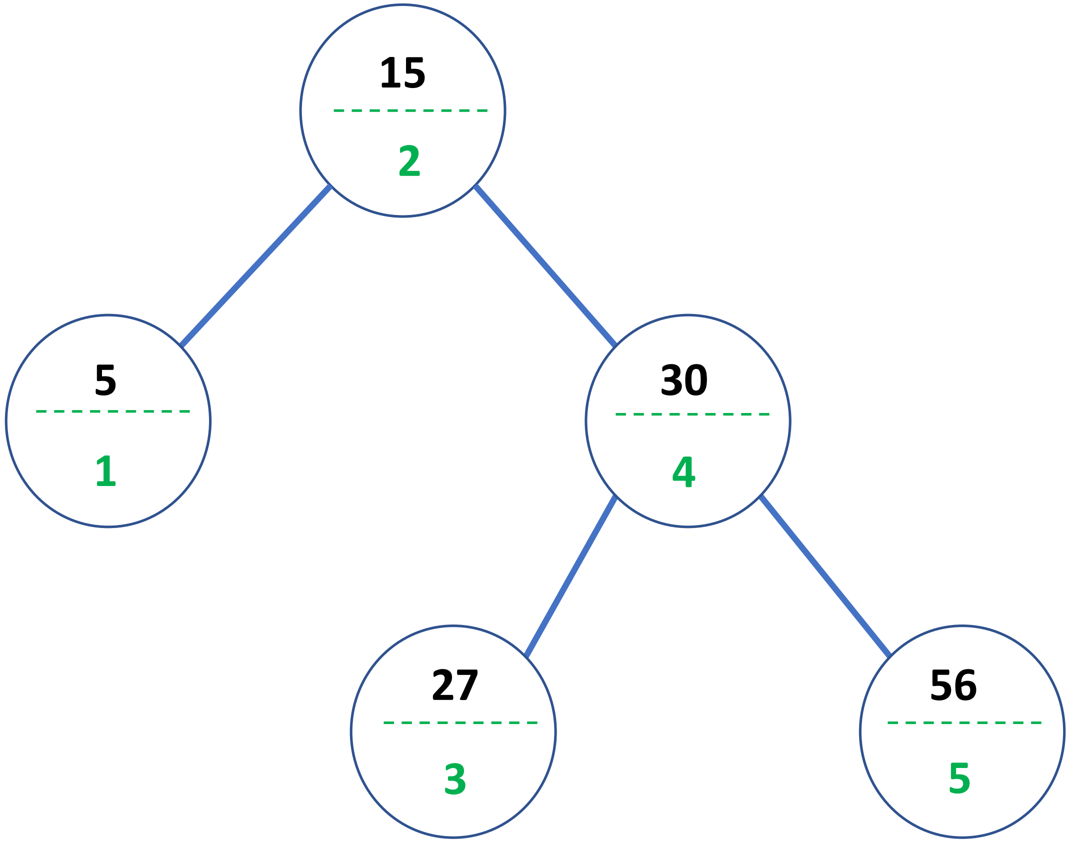

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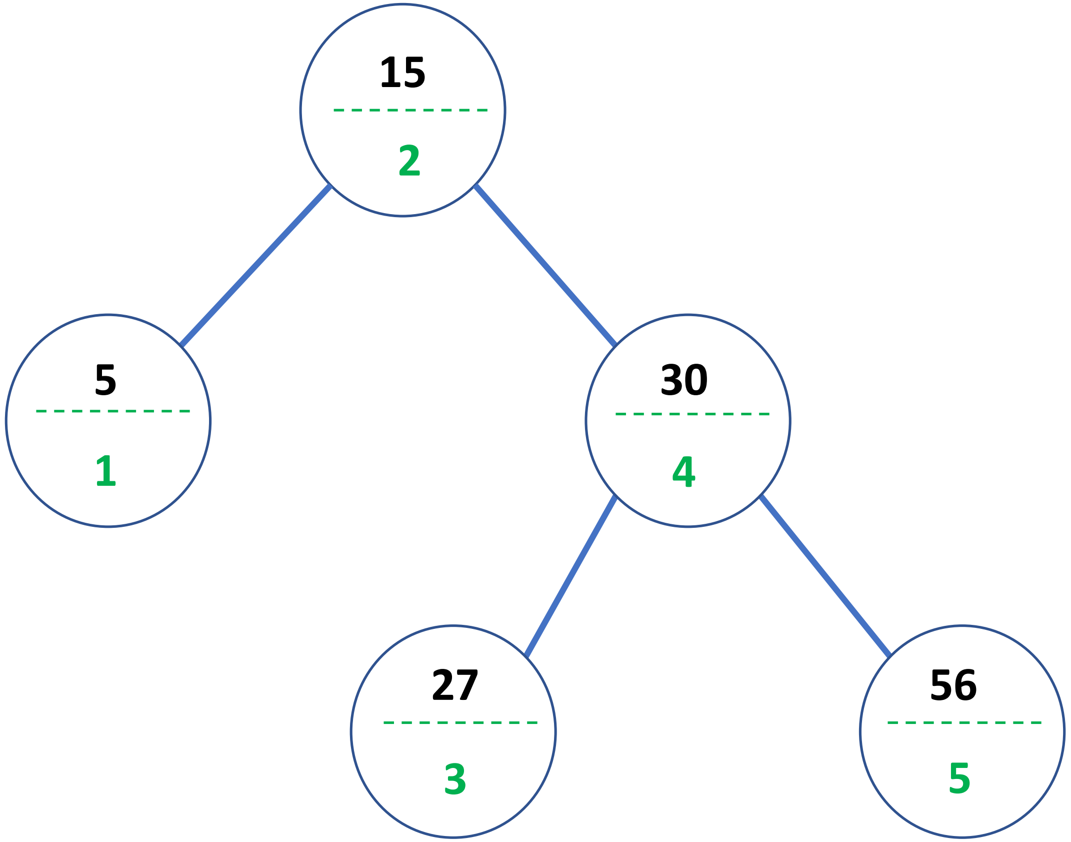

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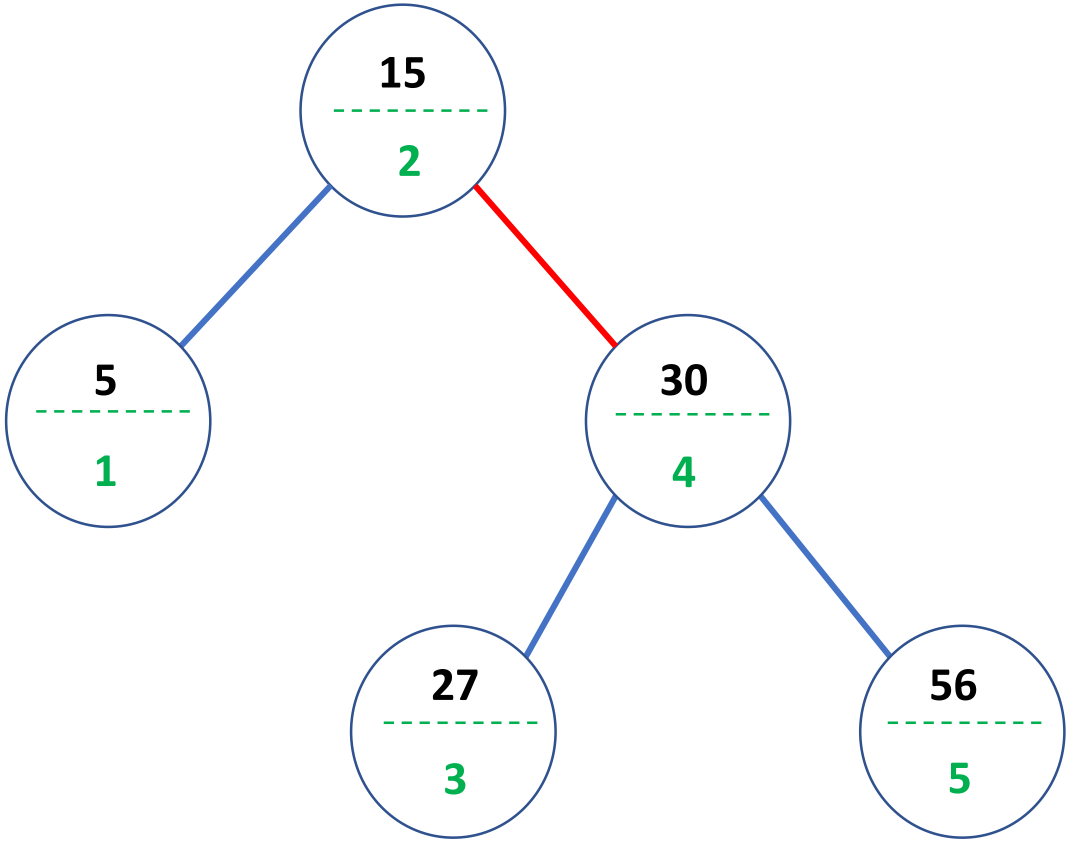

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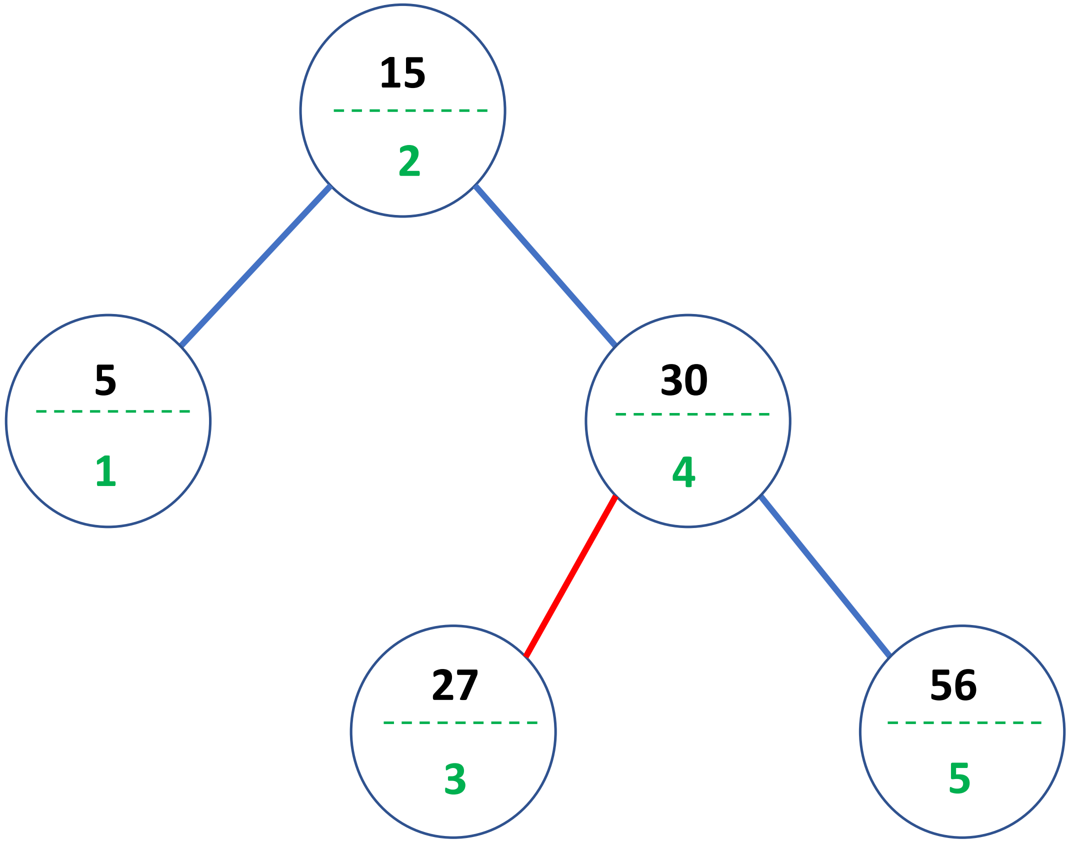

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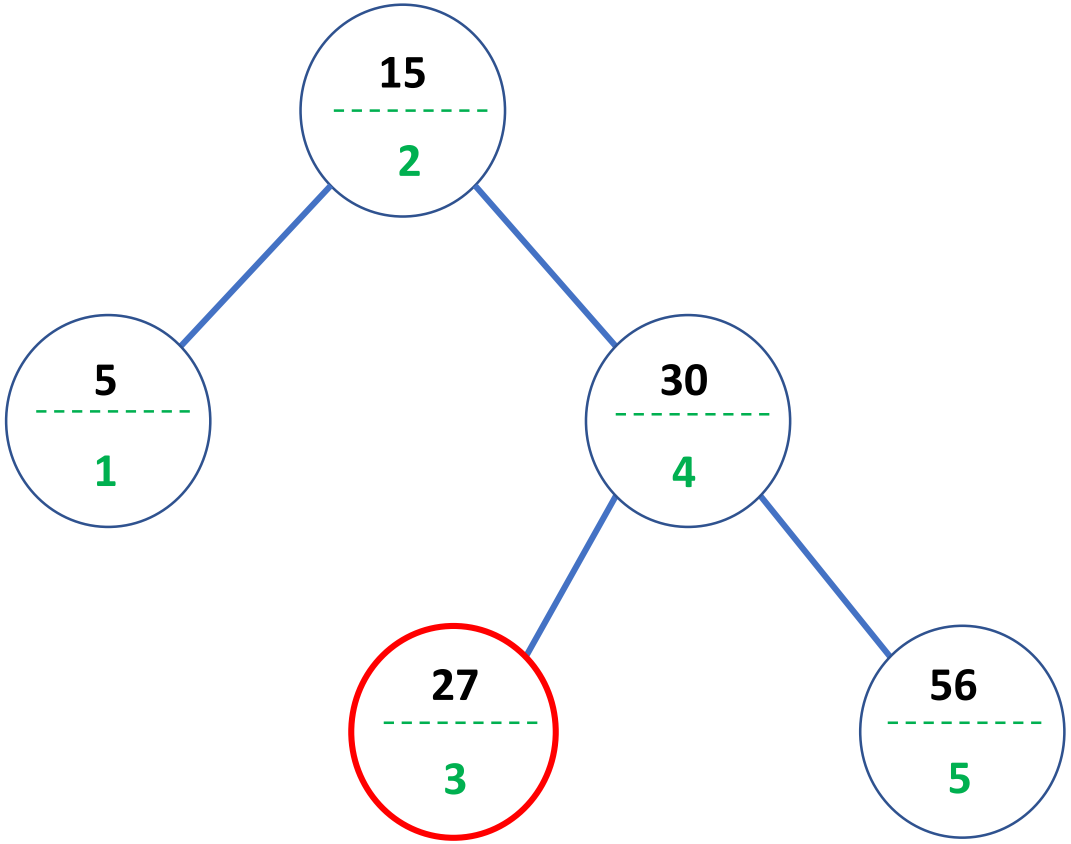

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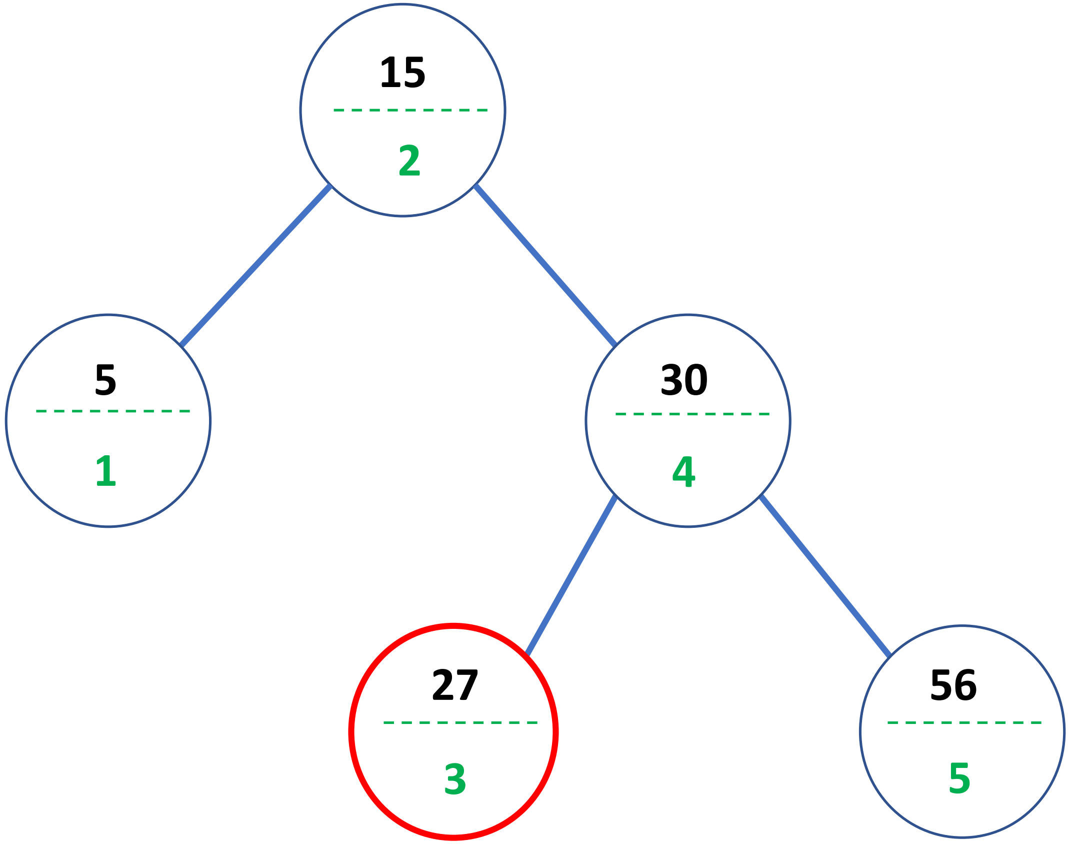
27



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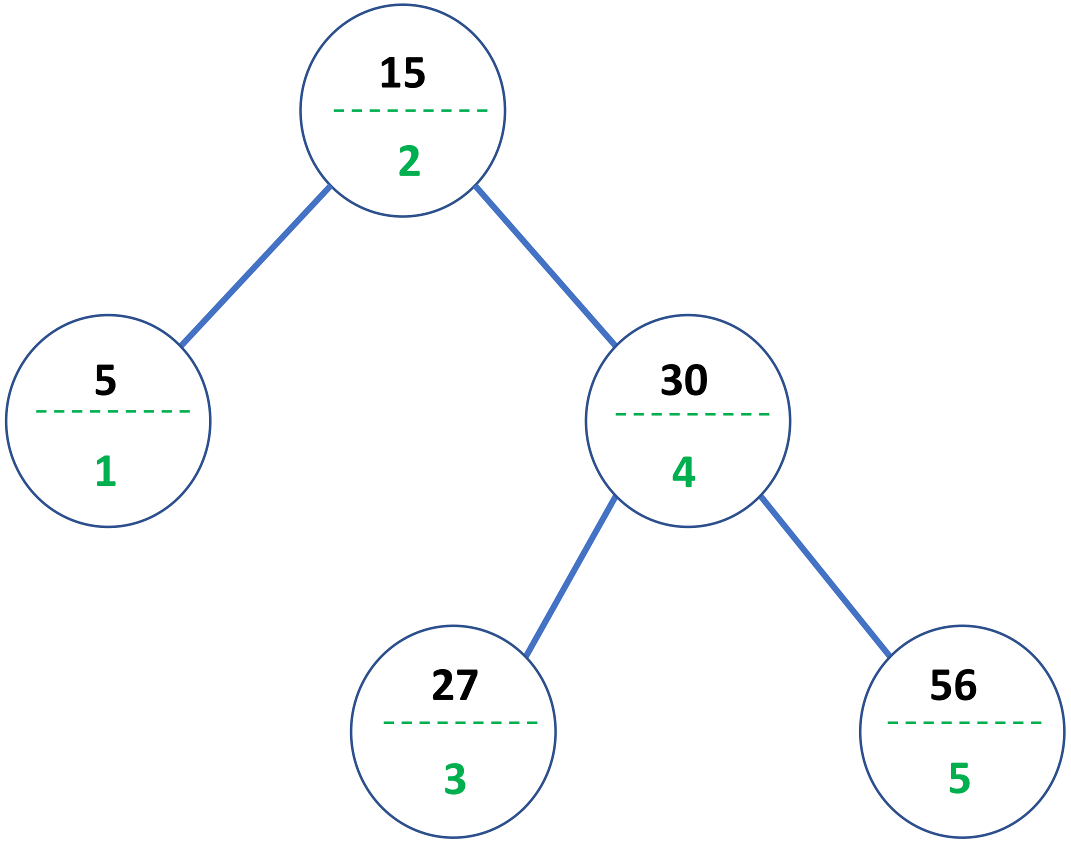
- At each node x, also store the rank of x
- <u>Good:</u> Efficient **Rank**(x) and **Select**(k)



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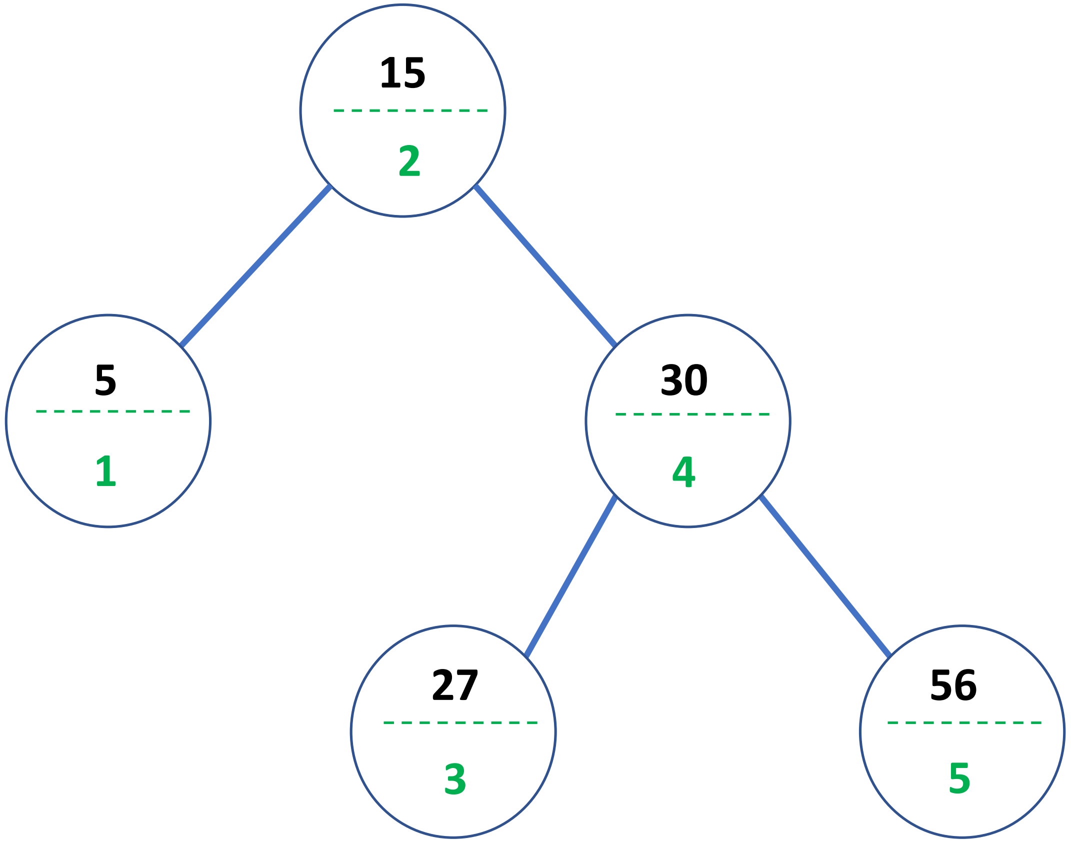
- Good: Efficient **Rank**(x) and **Select**(k)
- Are we done?



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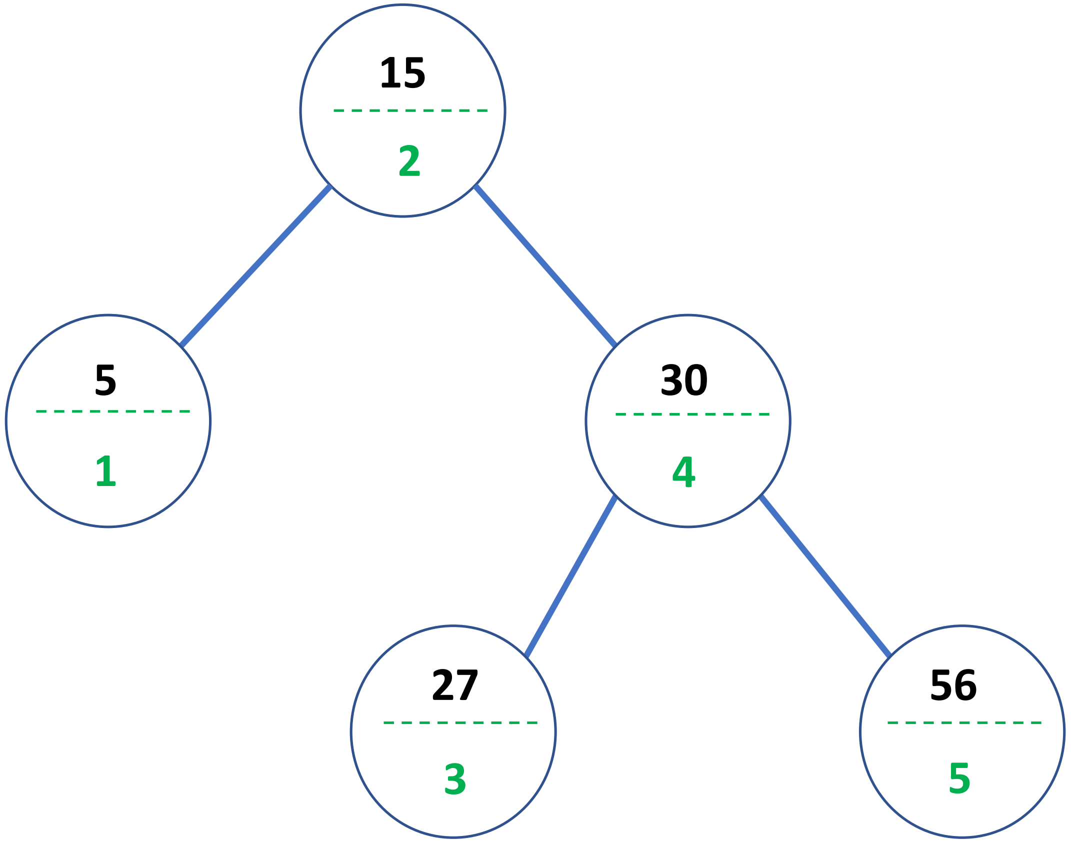
- At each node x, also store the rank of x
- <u>Good:</u> Efficient **Rank**(x) and **Select**(k)
- Are we done? No: maintaining the rank field is expensive!



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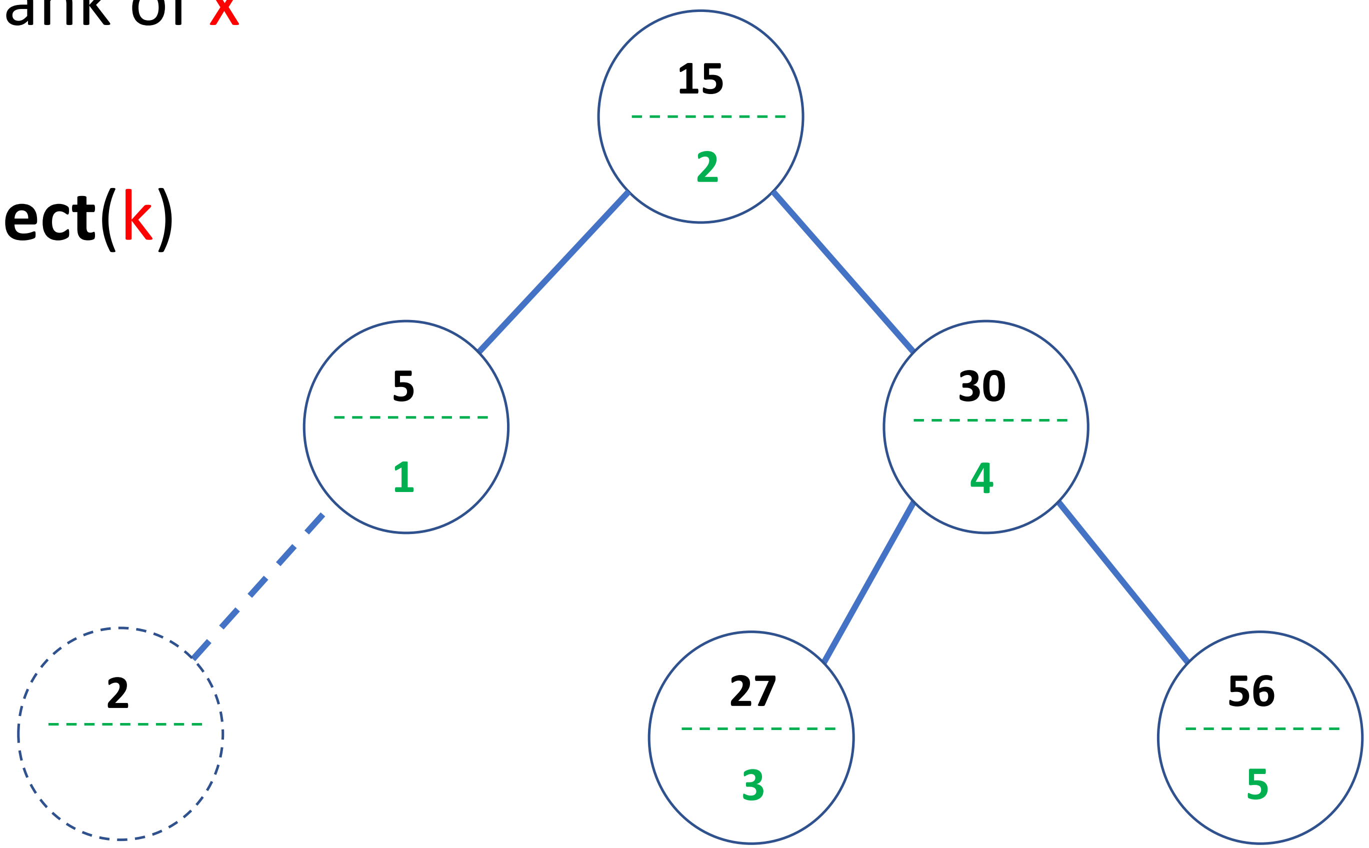

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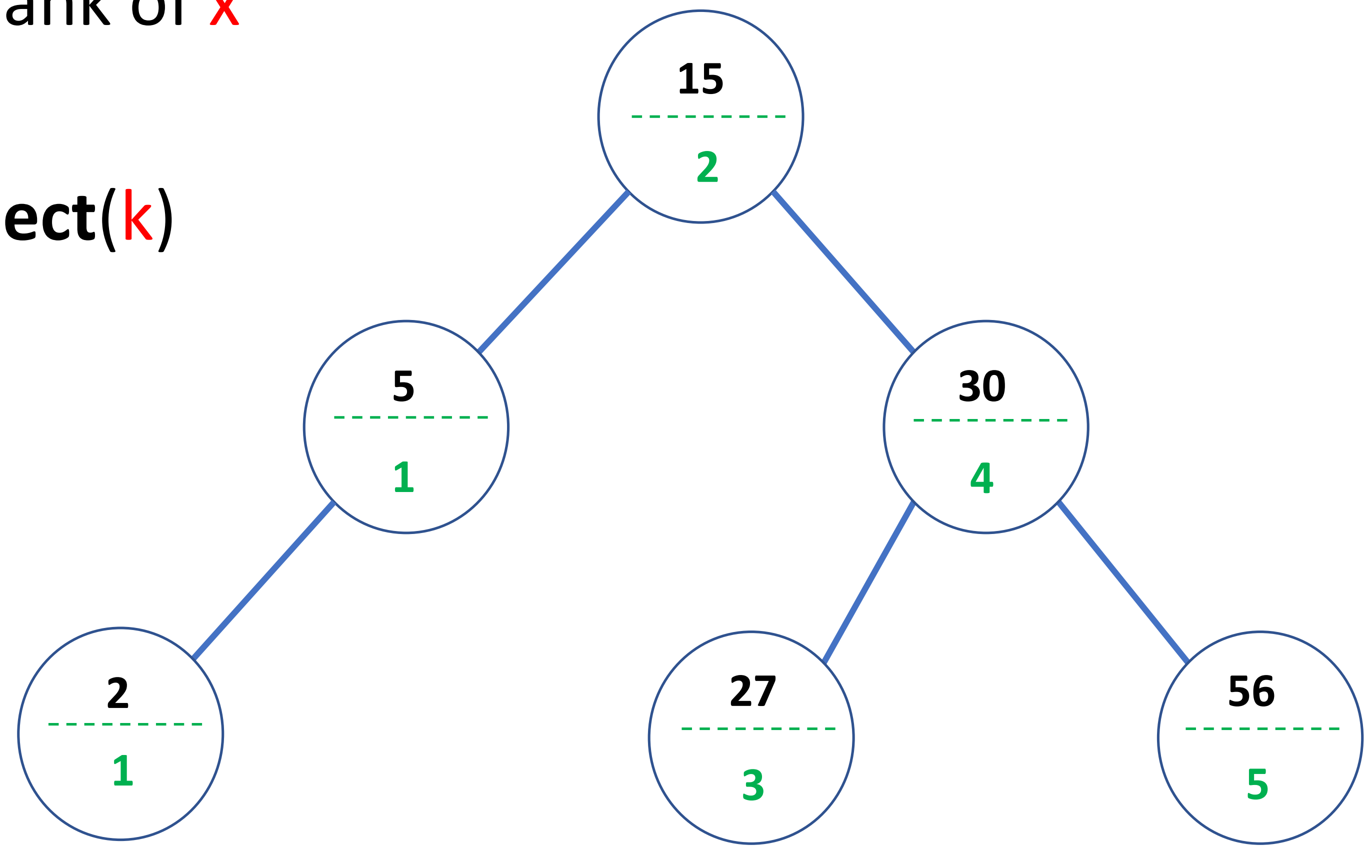

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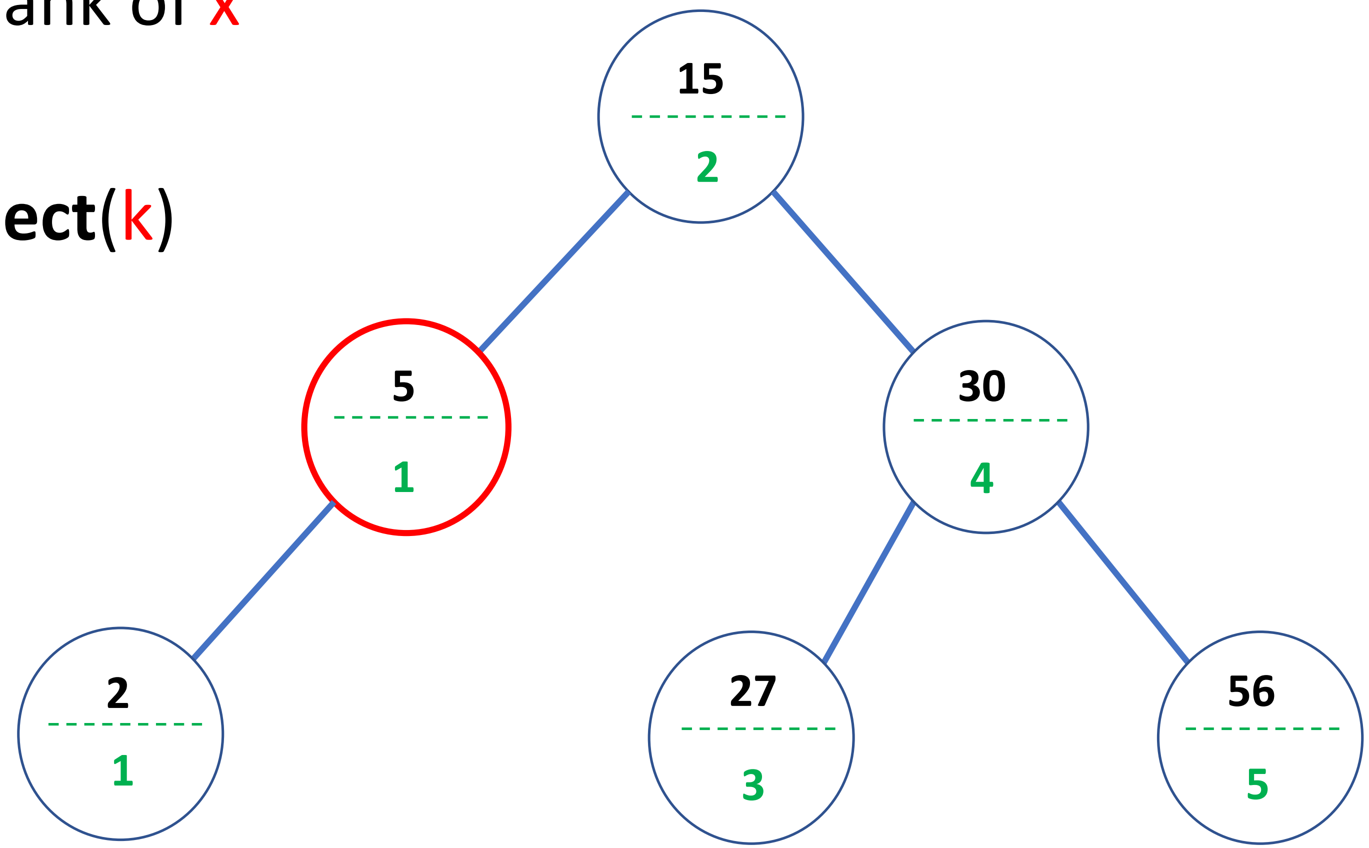

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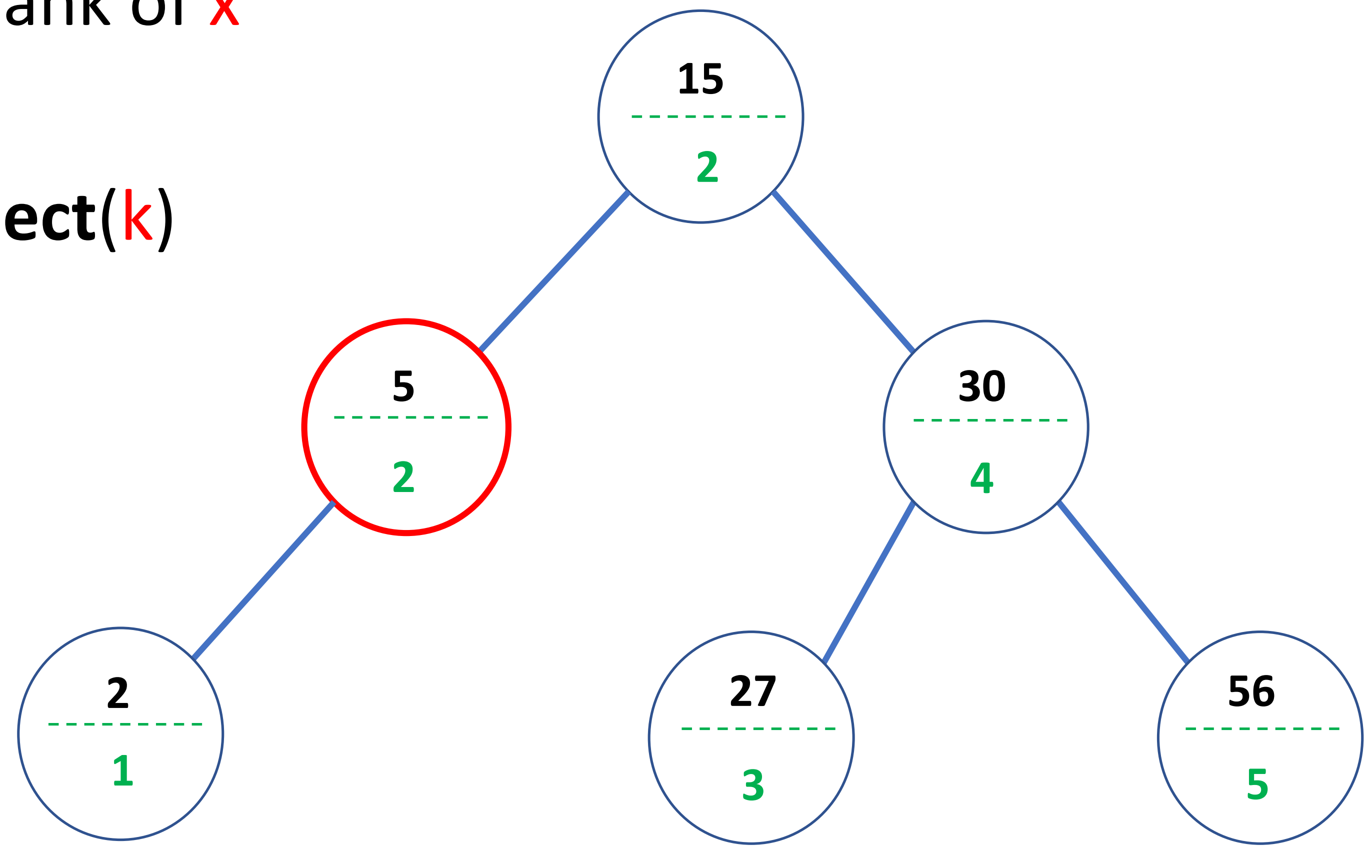

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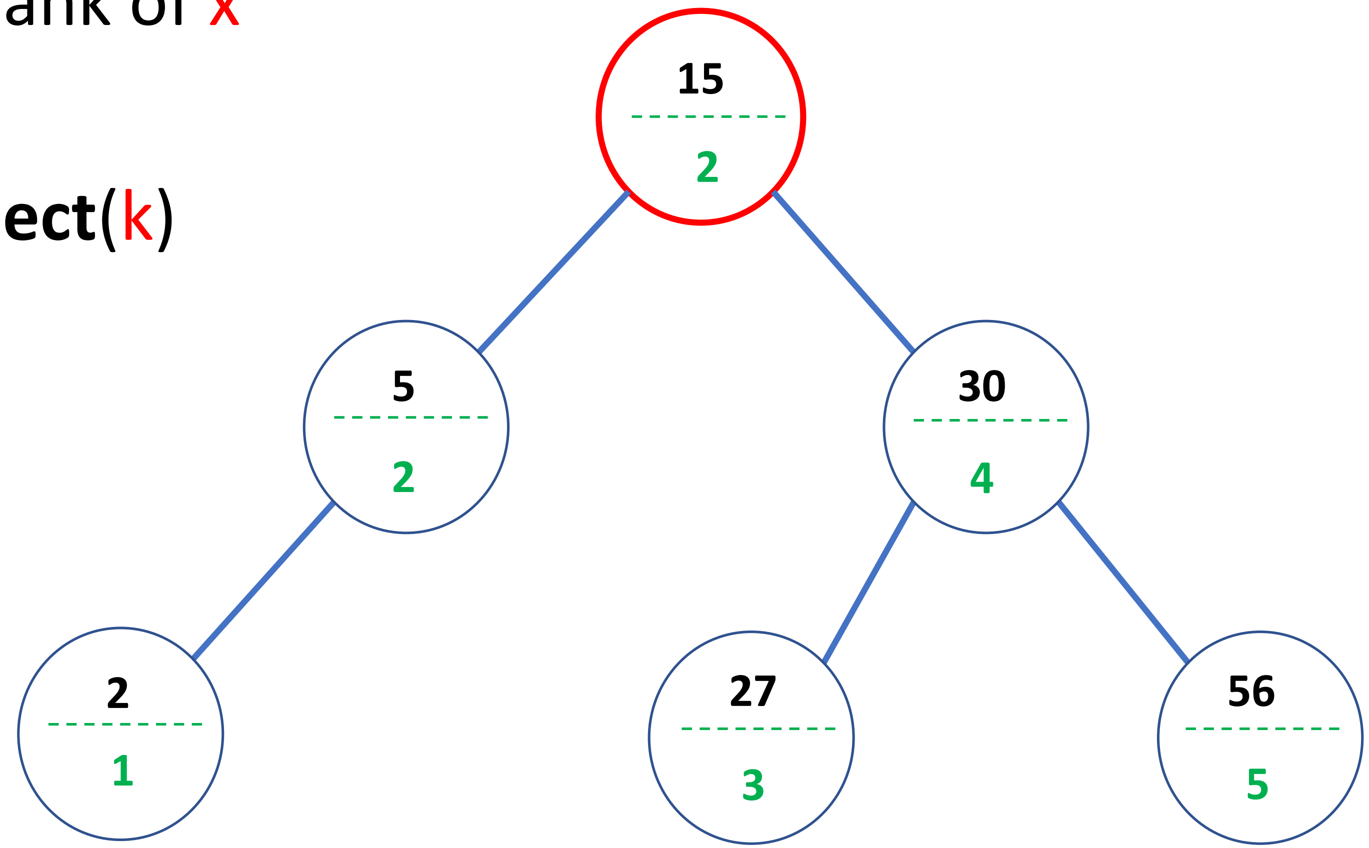

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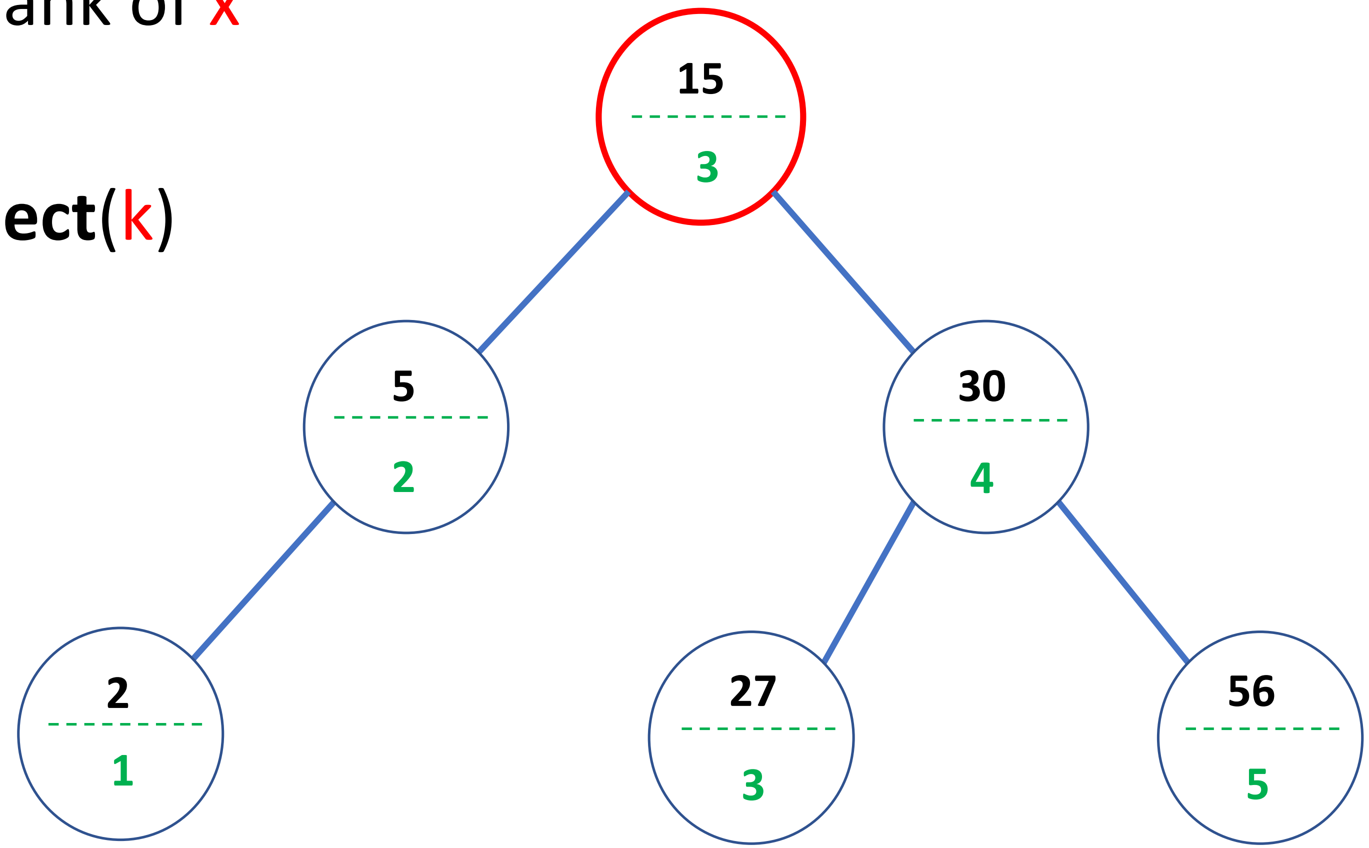

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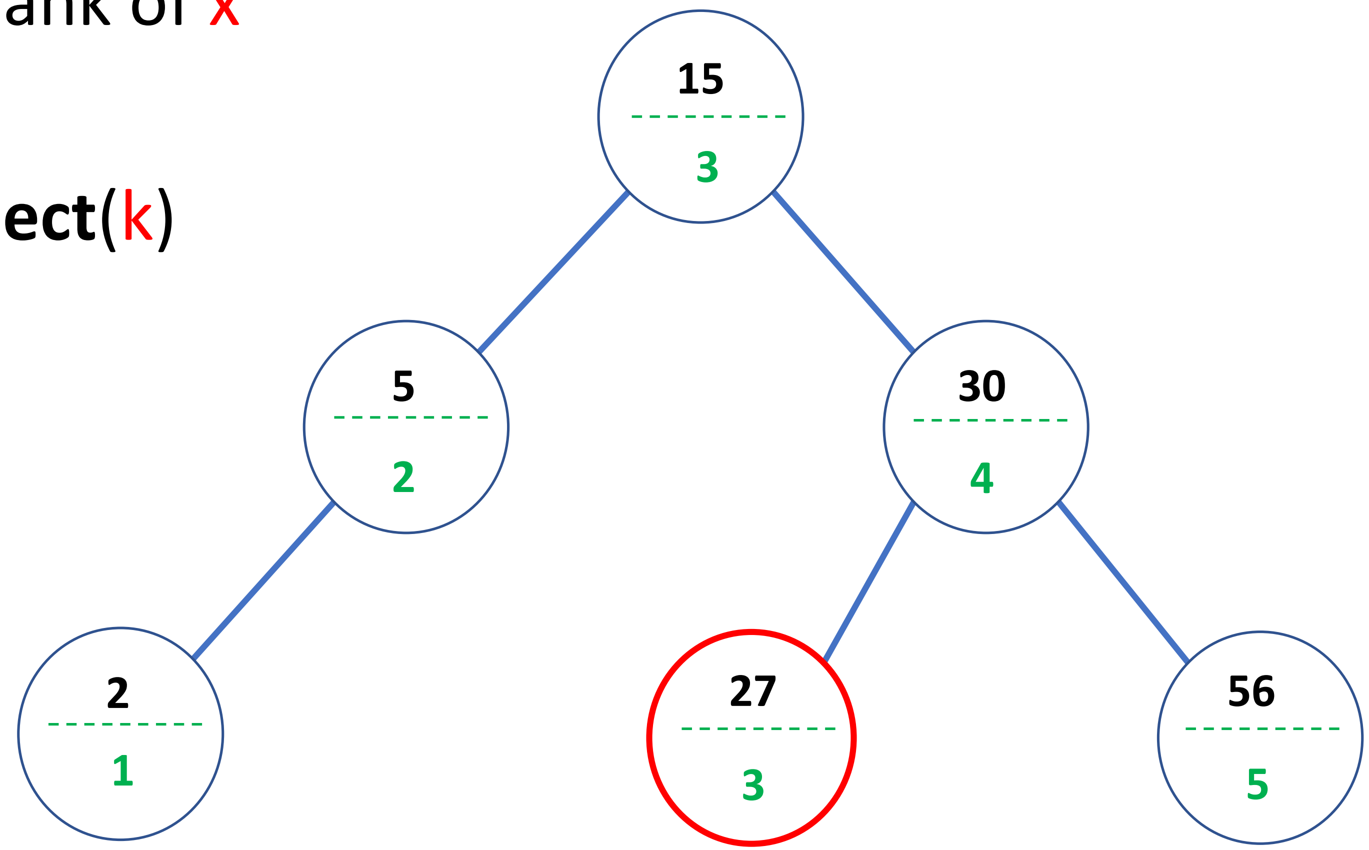

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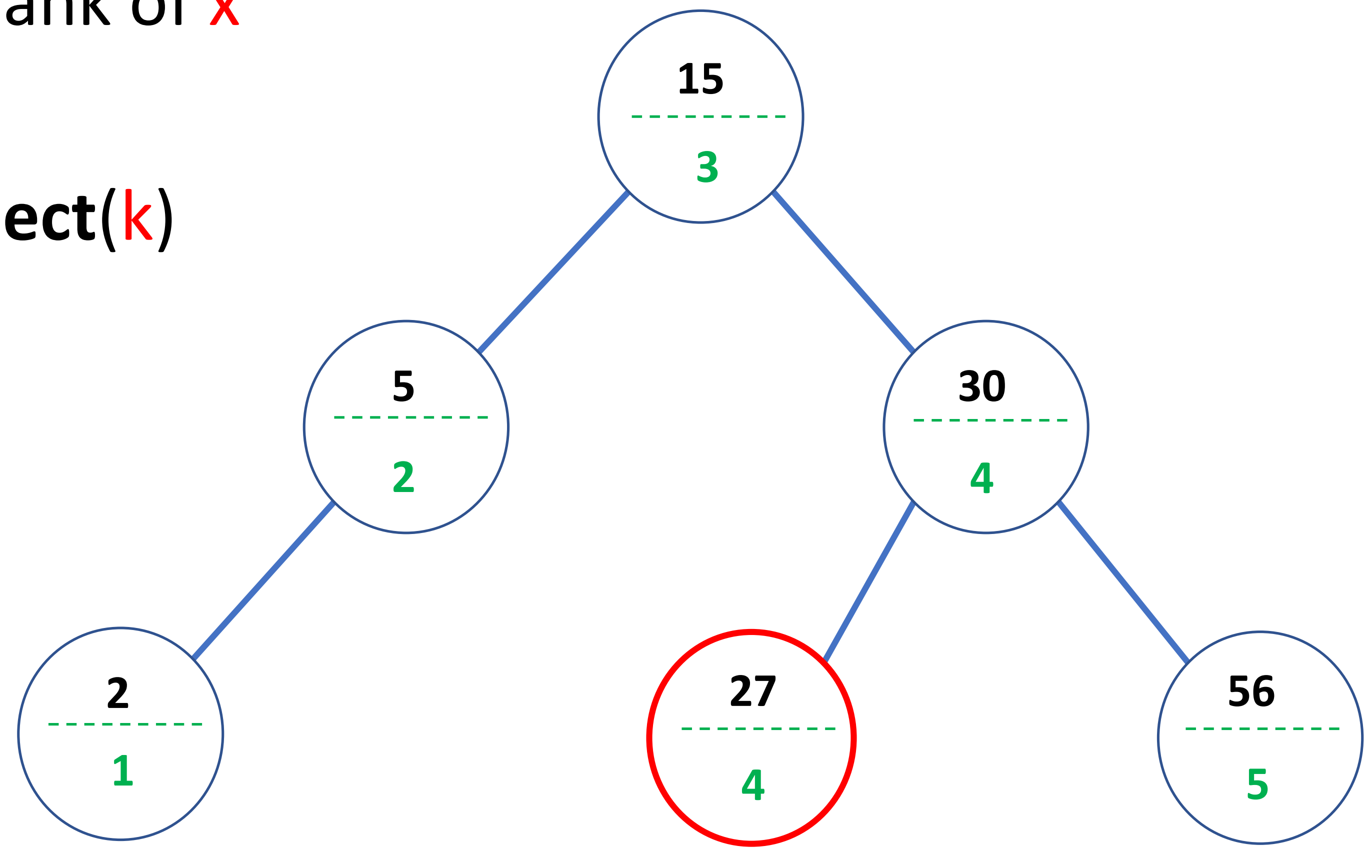

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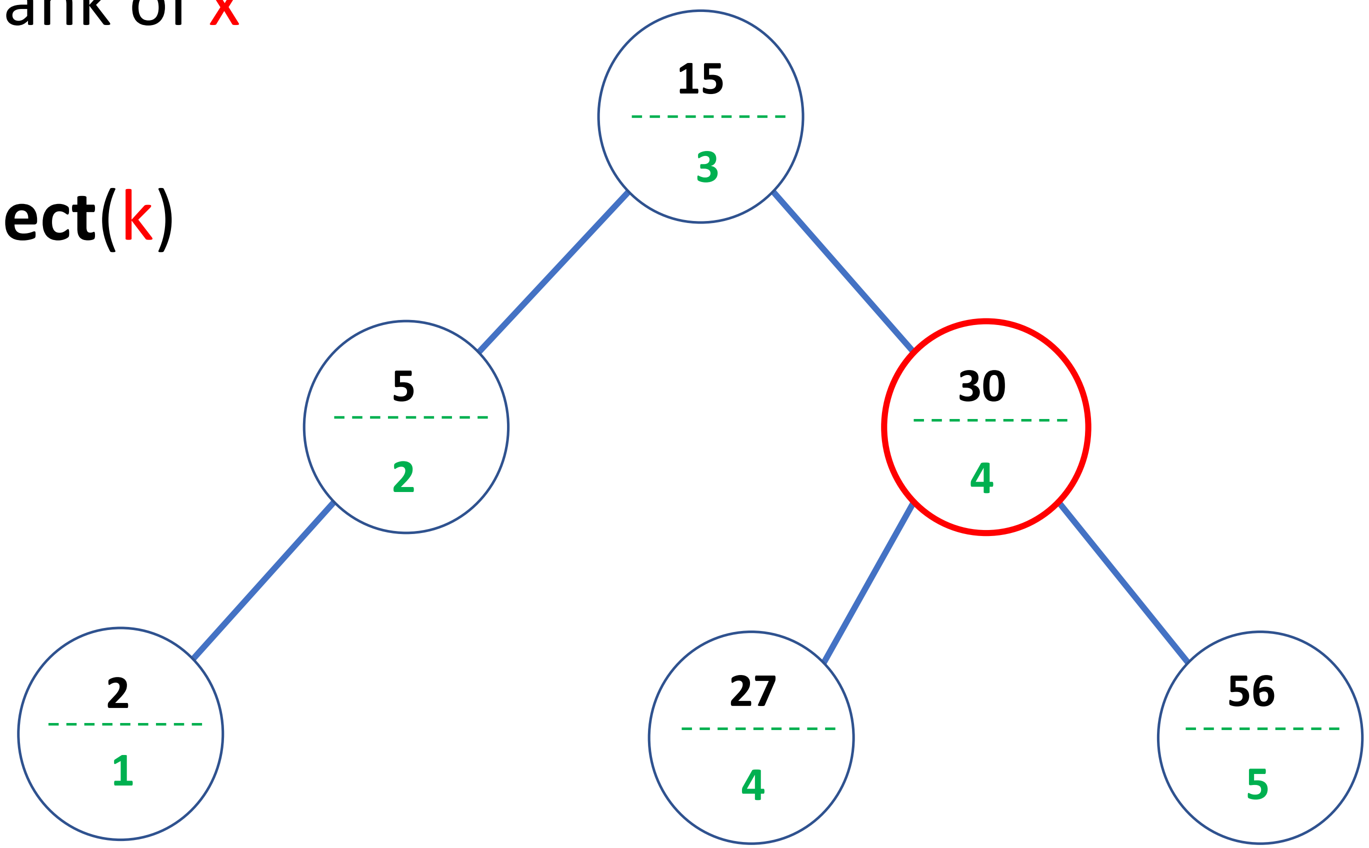

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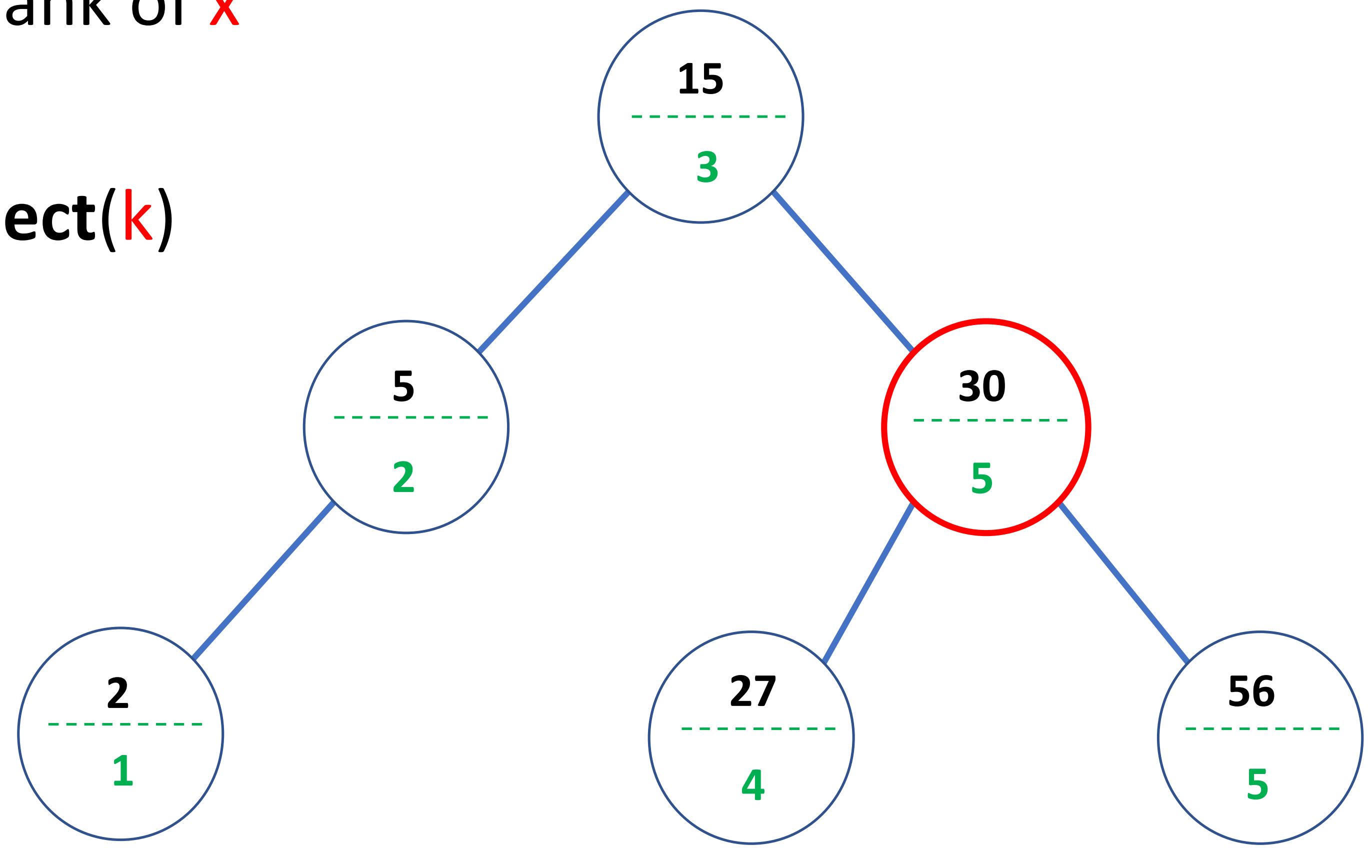

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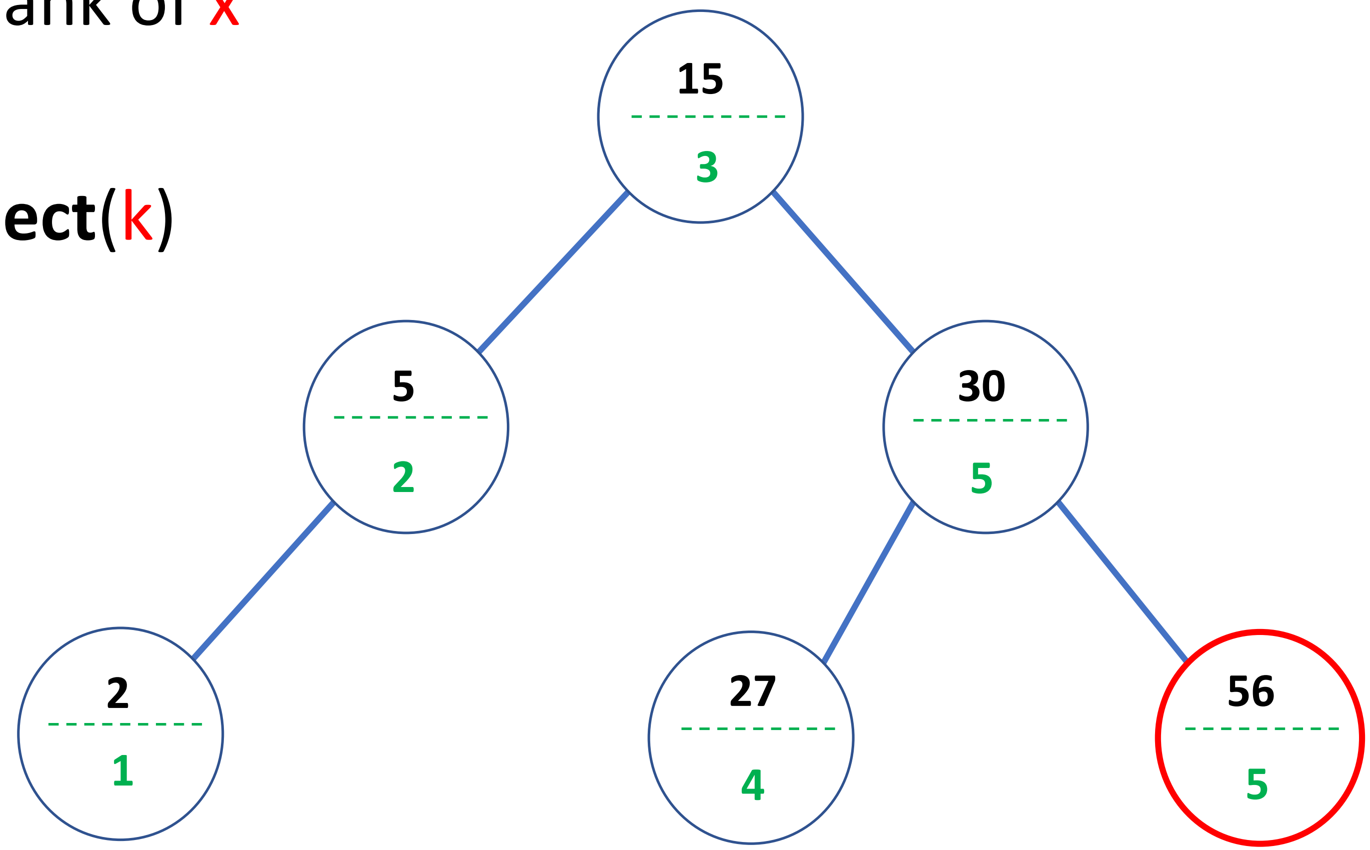

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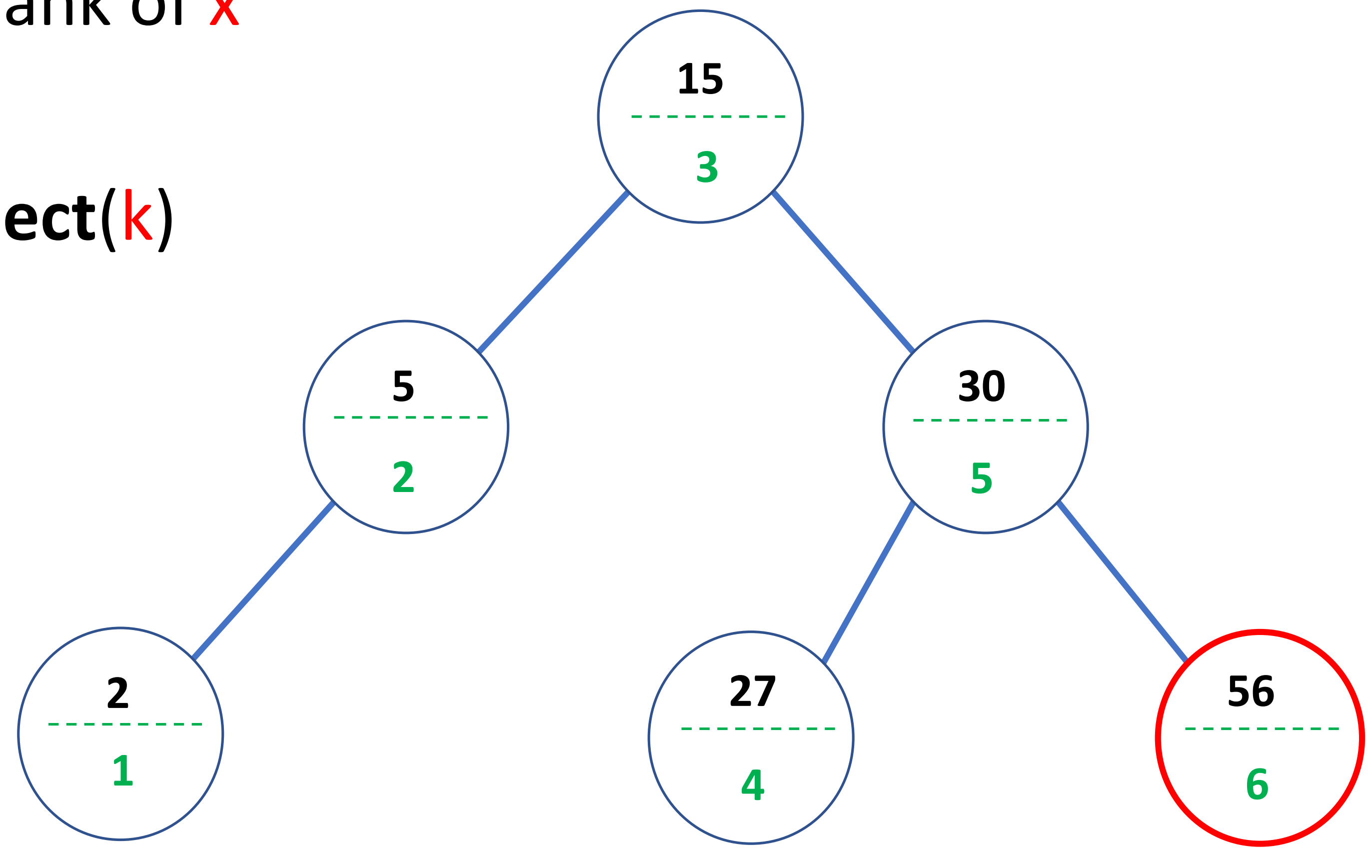

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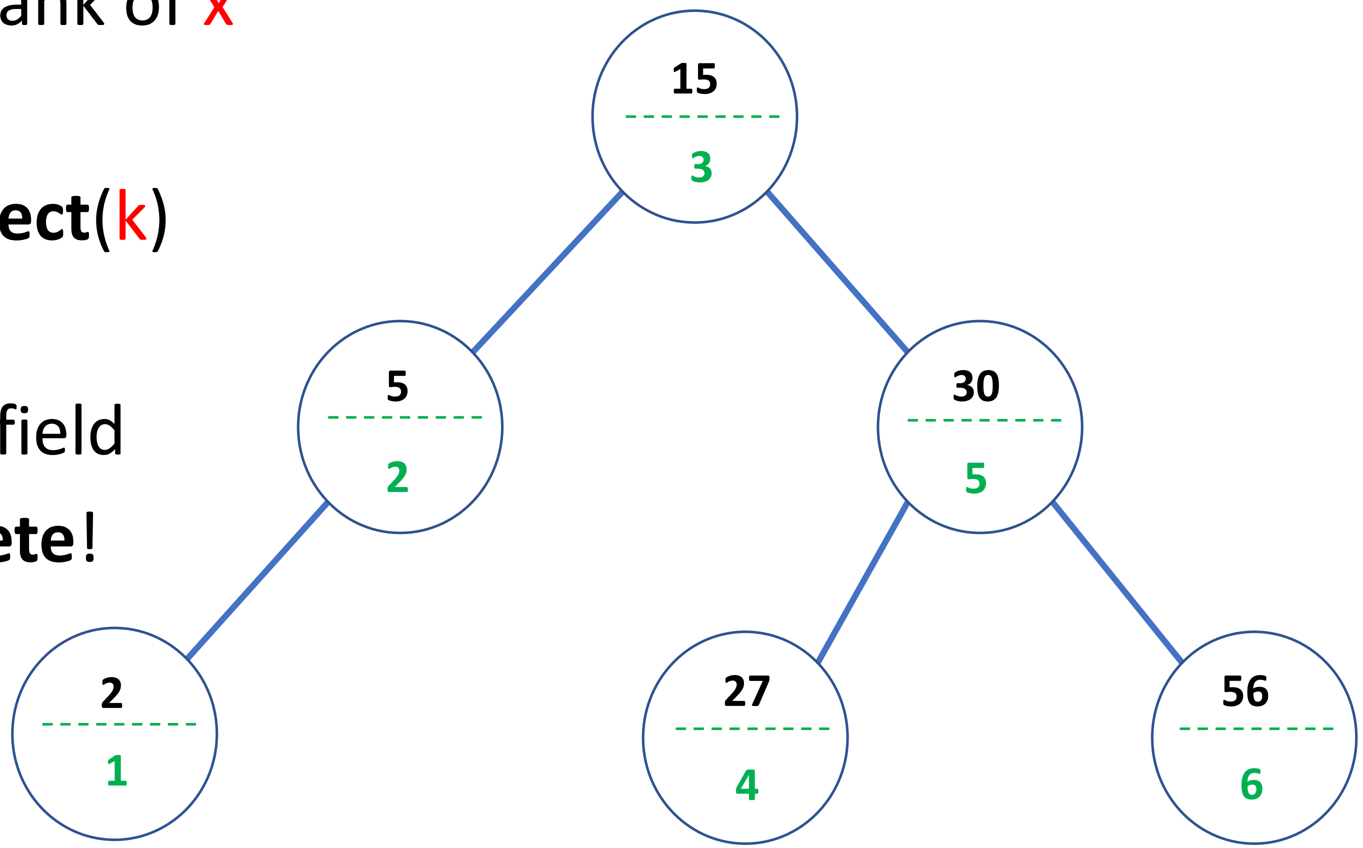
- Bad: Expensive to update rank field
  when doing **Insert** or **Delete**!



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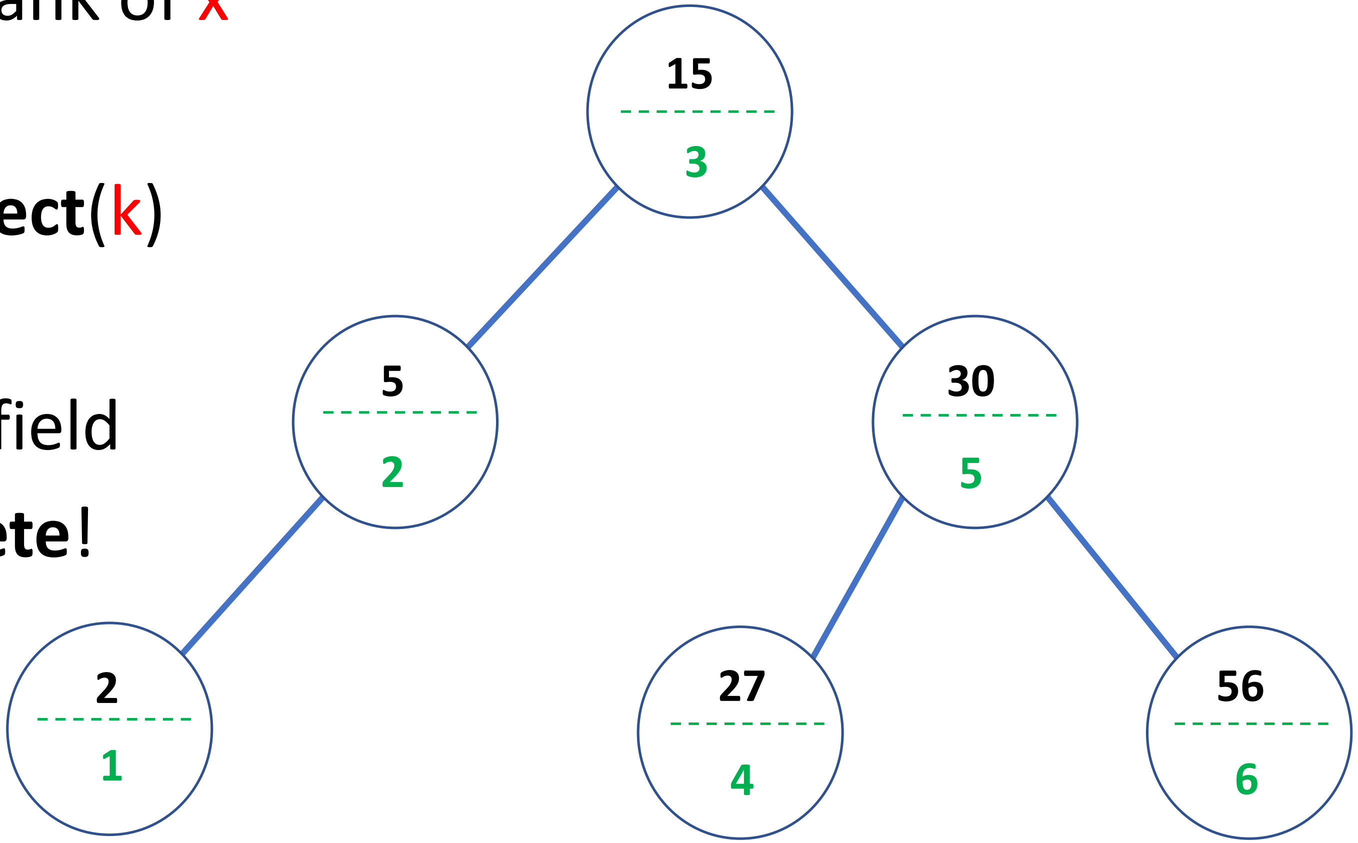
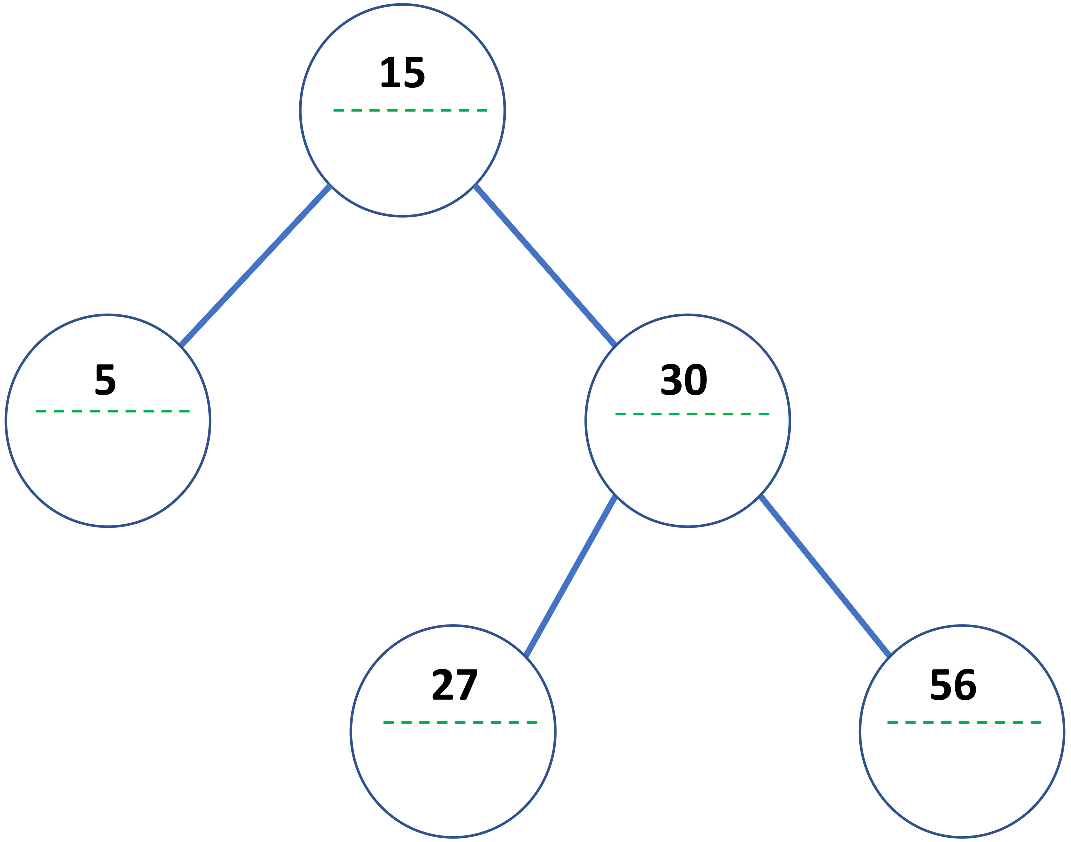
Takes O(n) per insert/delete



A better augmentation



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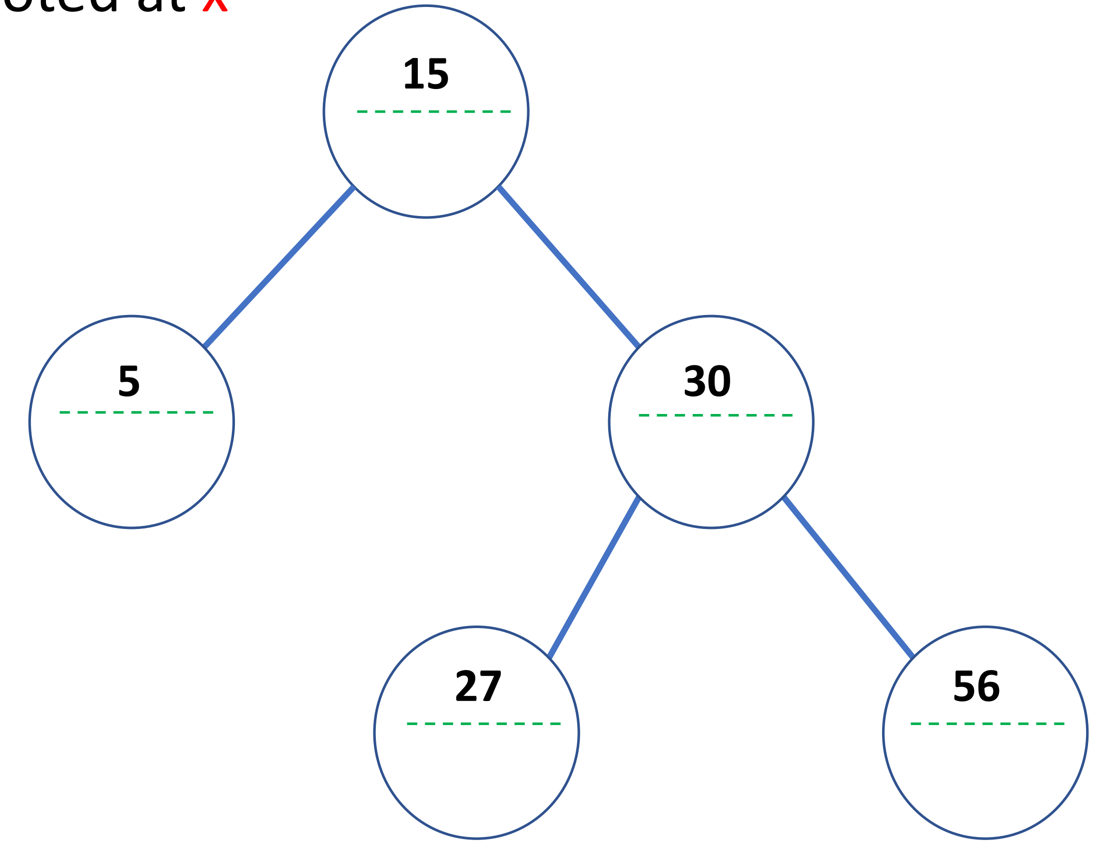
- At each node x, store the size of the subtree rooted at x



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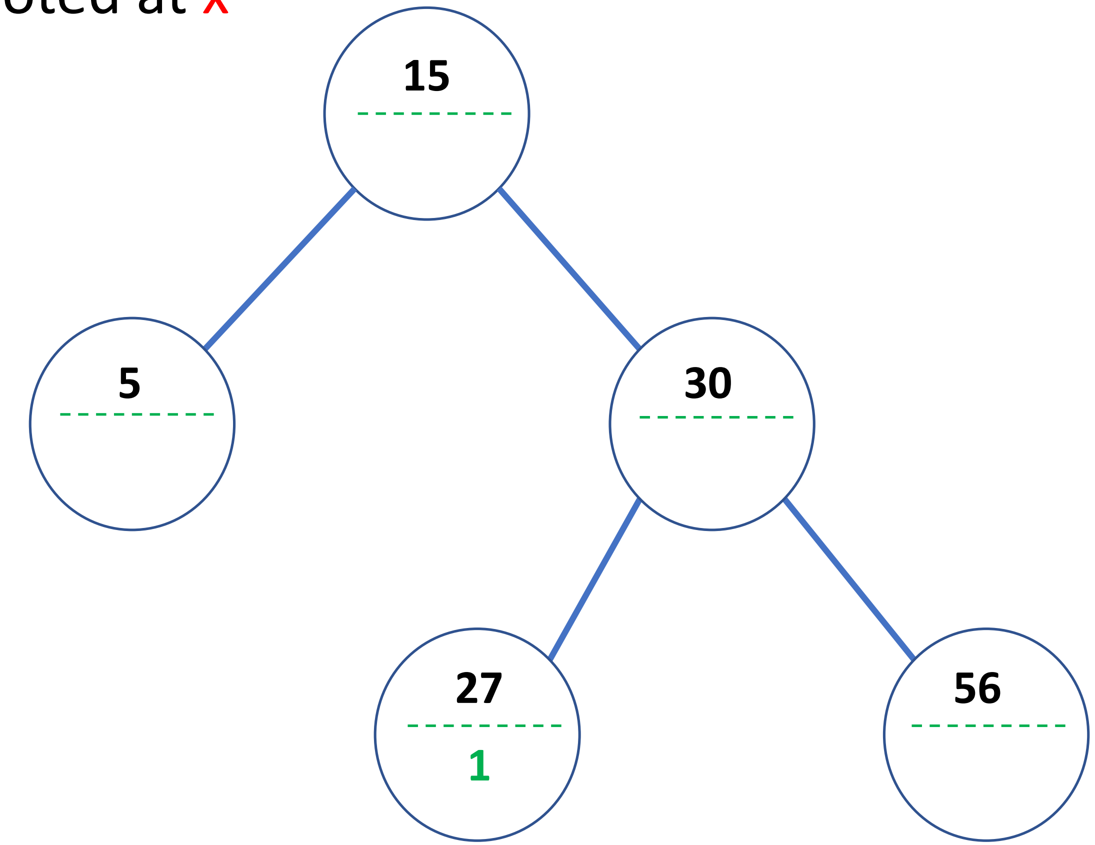

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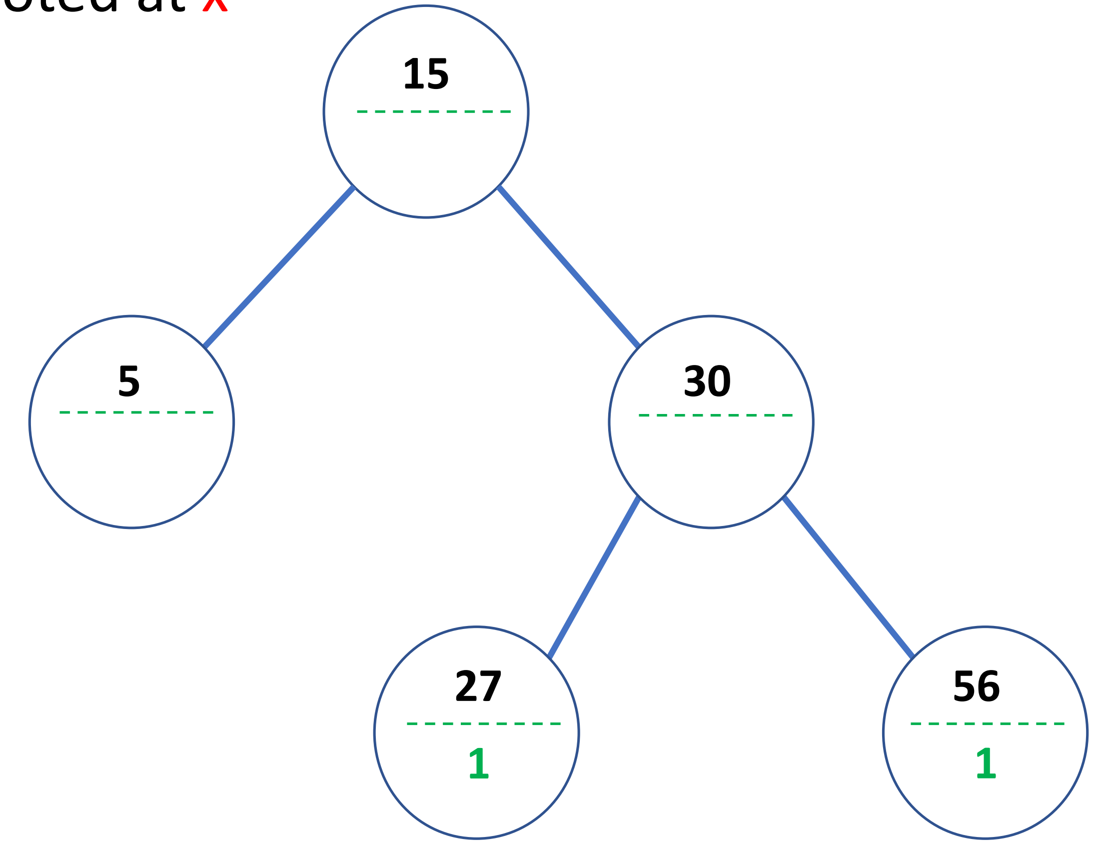

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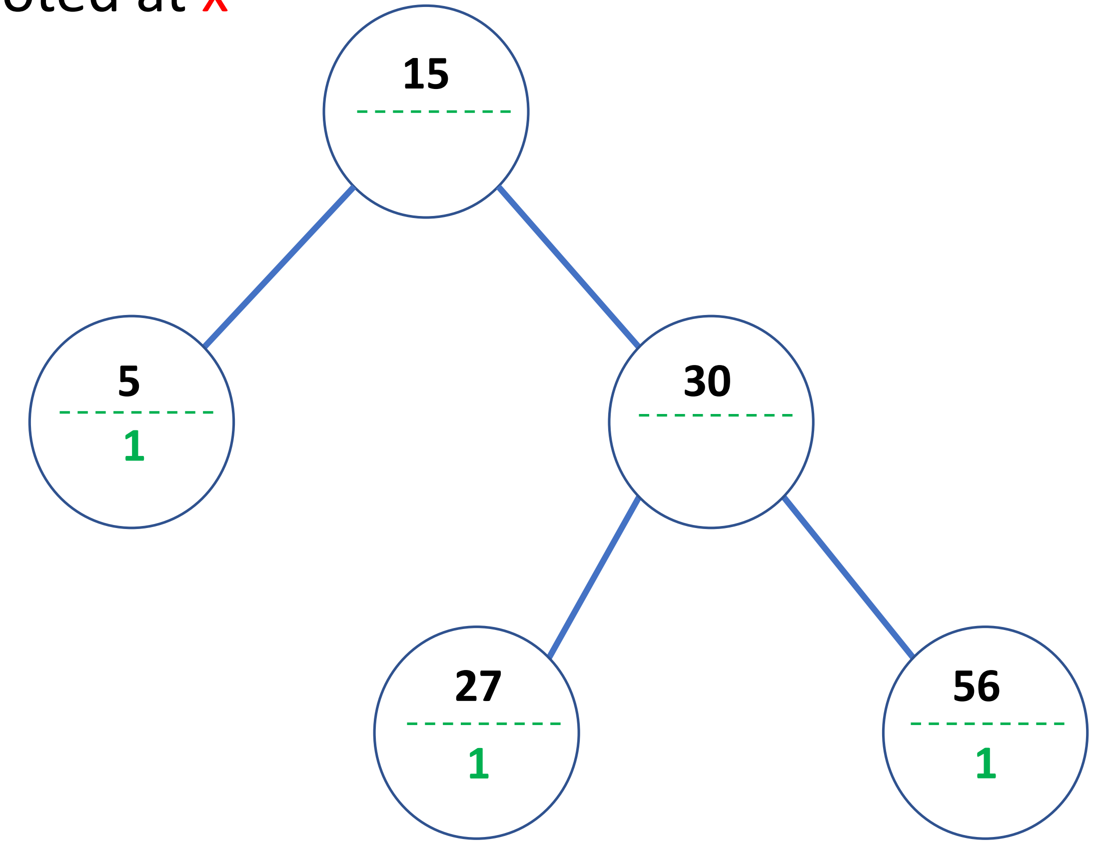

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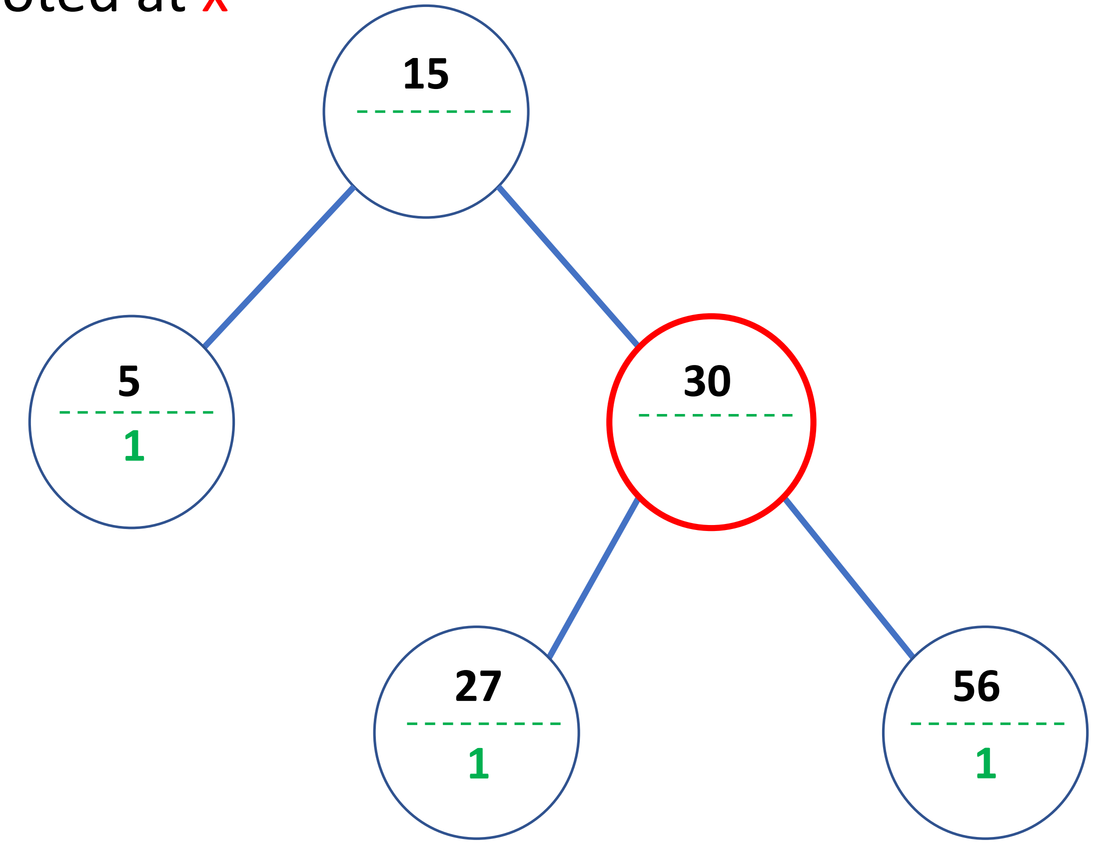

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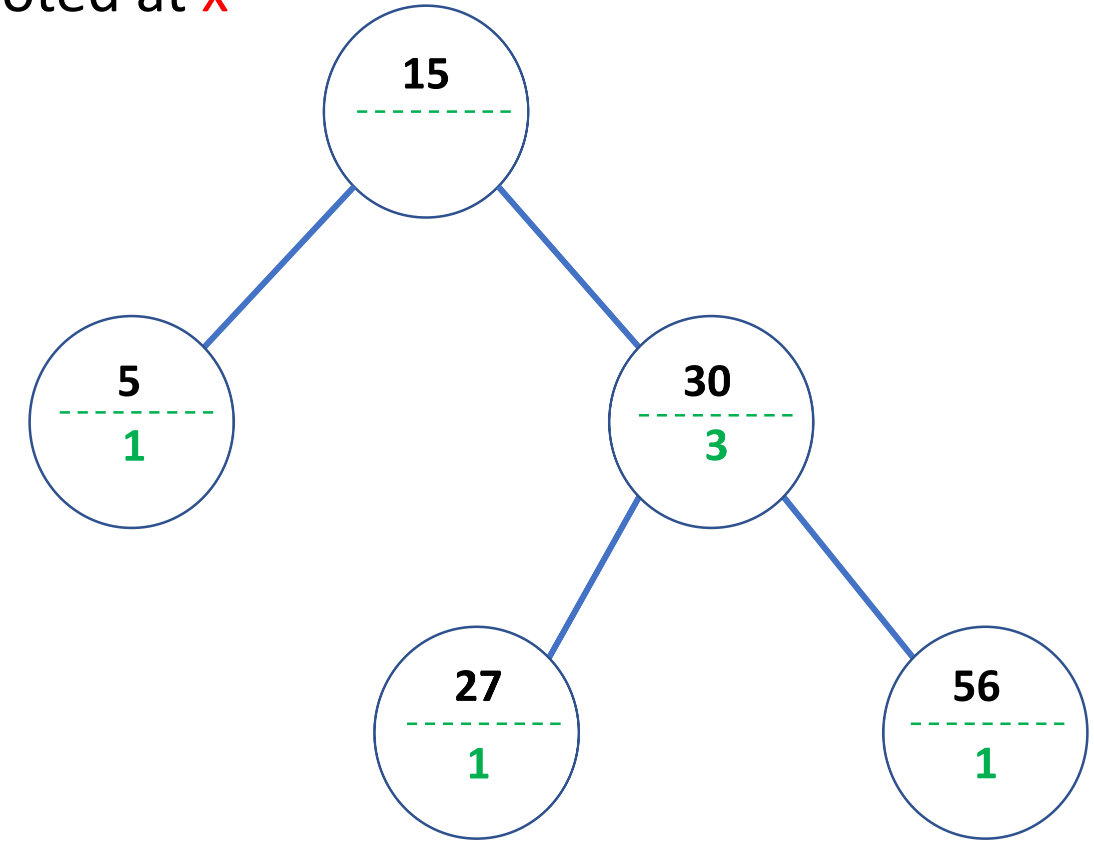

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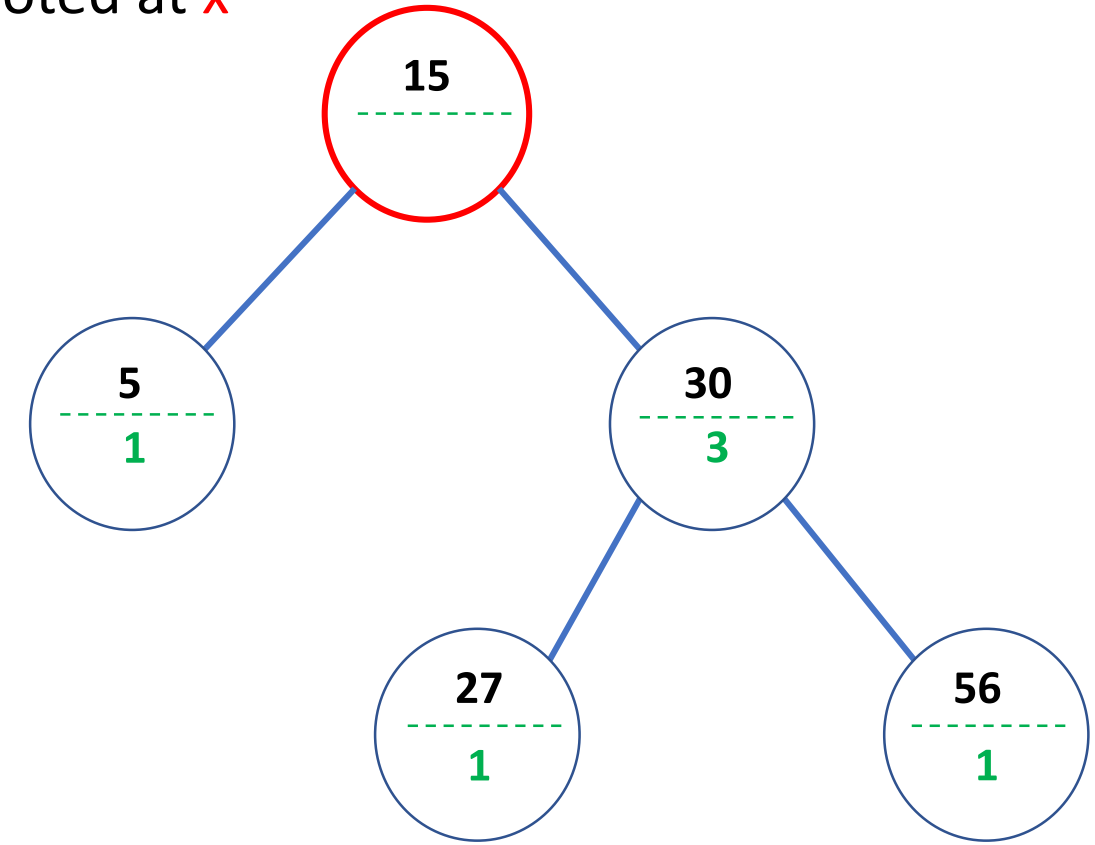

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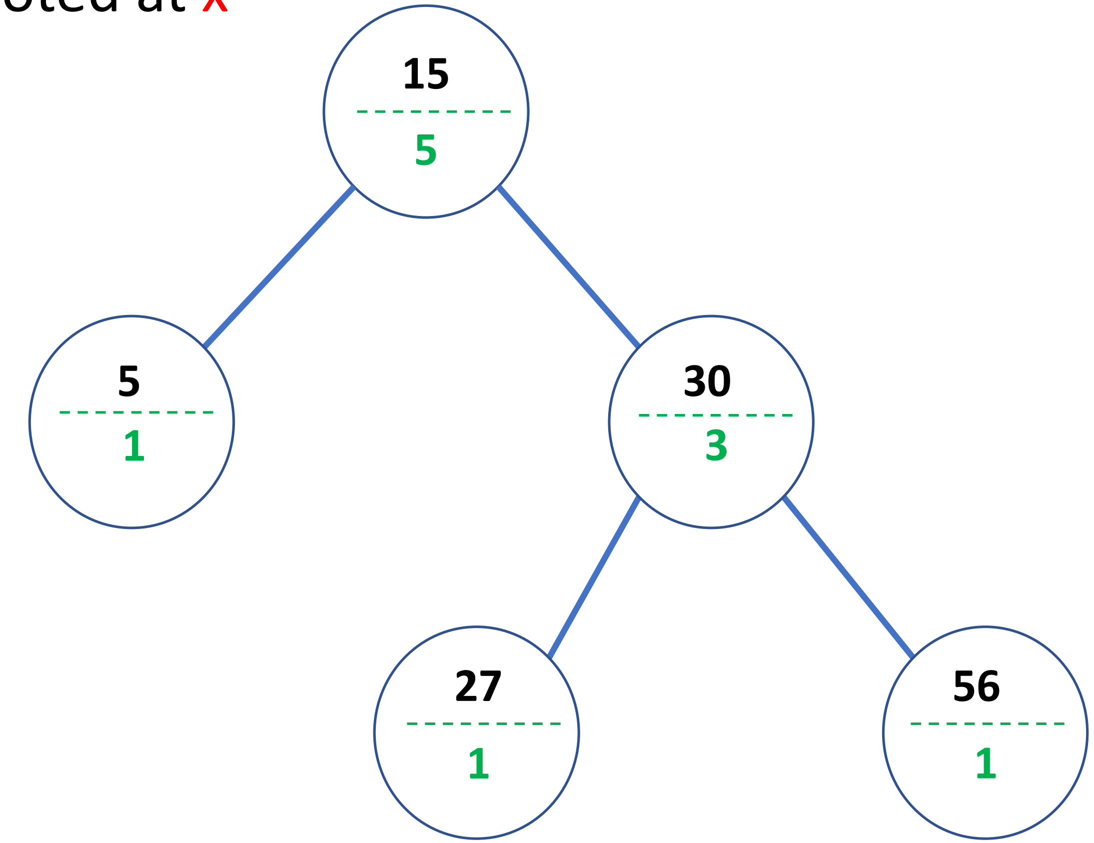

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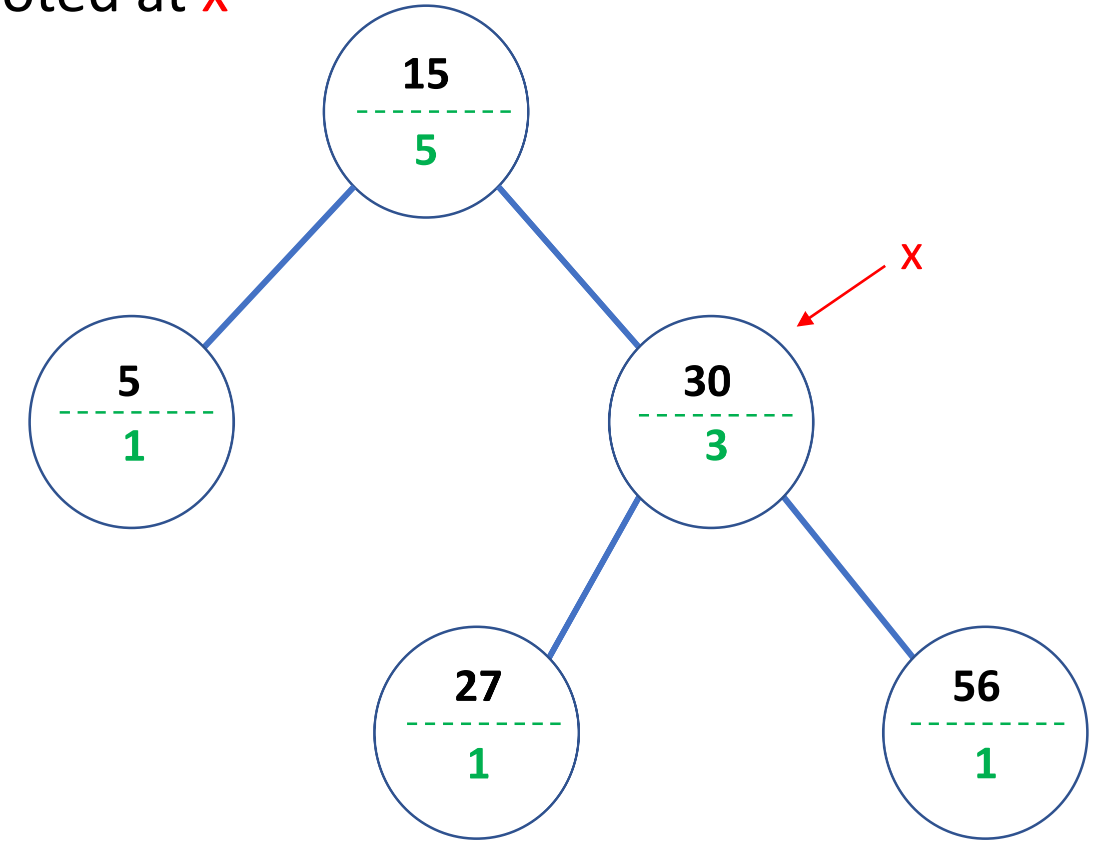

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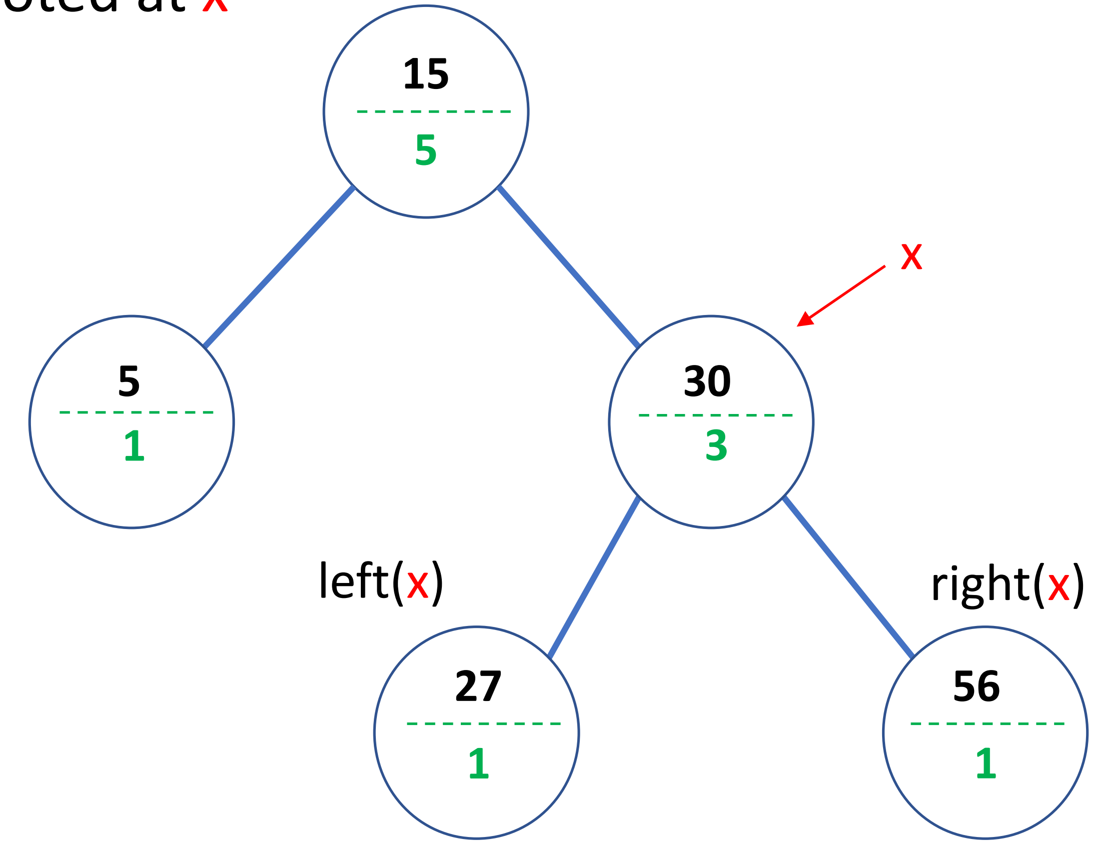

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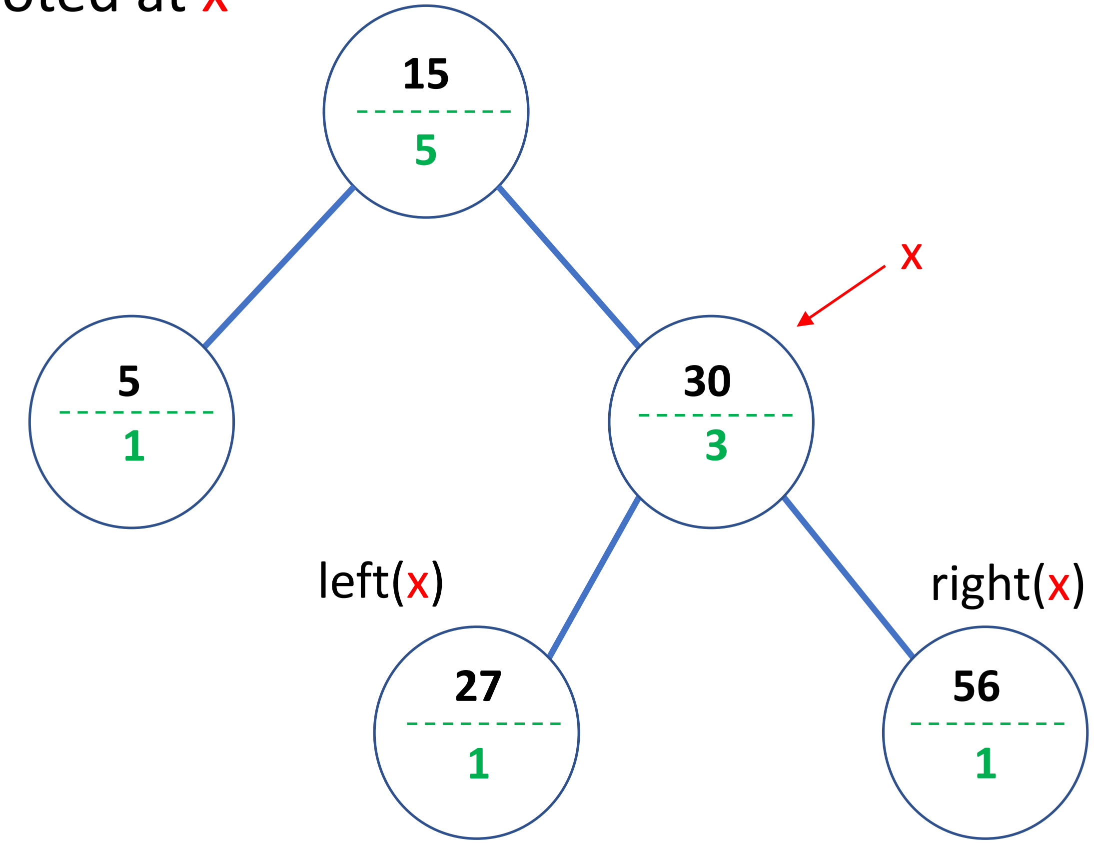
For every node x,

$$\boxed{\mathbf{size(x) = size(left(x)) + size(right(x)) + 1}}$$



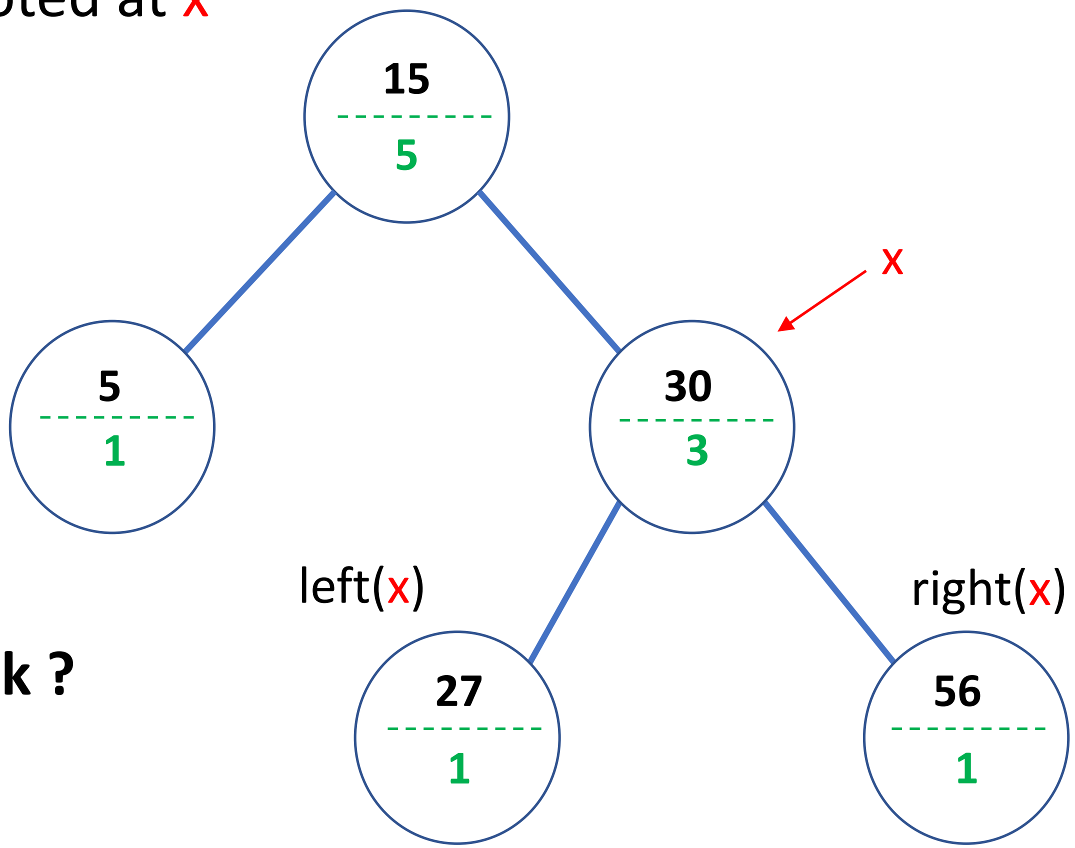
A better augmentation

- At each node x, store the size of the subtree rooted at x

For every node x,

$$\boxed{\textbf{size}(x) = \textbf{size}(\textbf{left}(x)) + \textbf{size}(\textbf{right}(x)) + 1}$$

1. How to efficiently implement **Select** and **Rank** ?

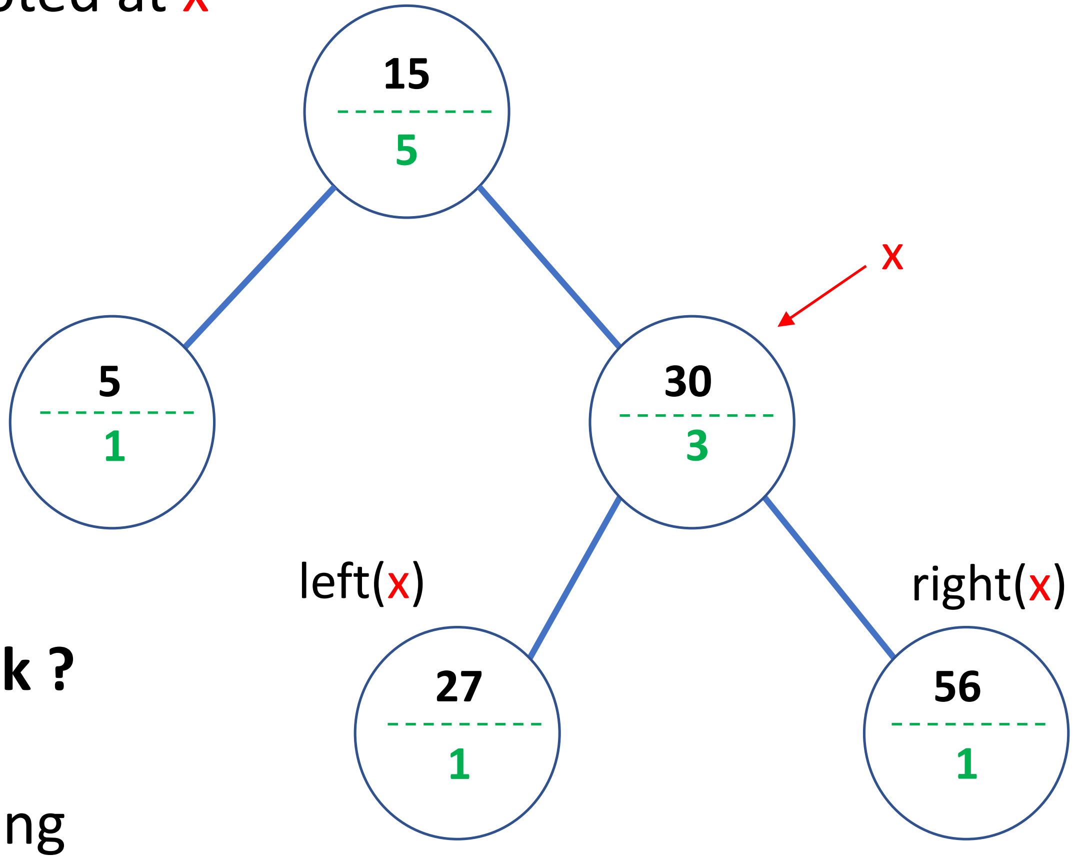


A better augmentation

- At each node x, store the size of the subtree rooted at x

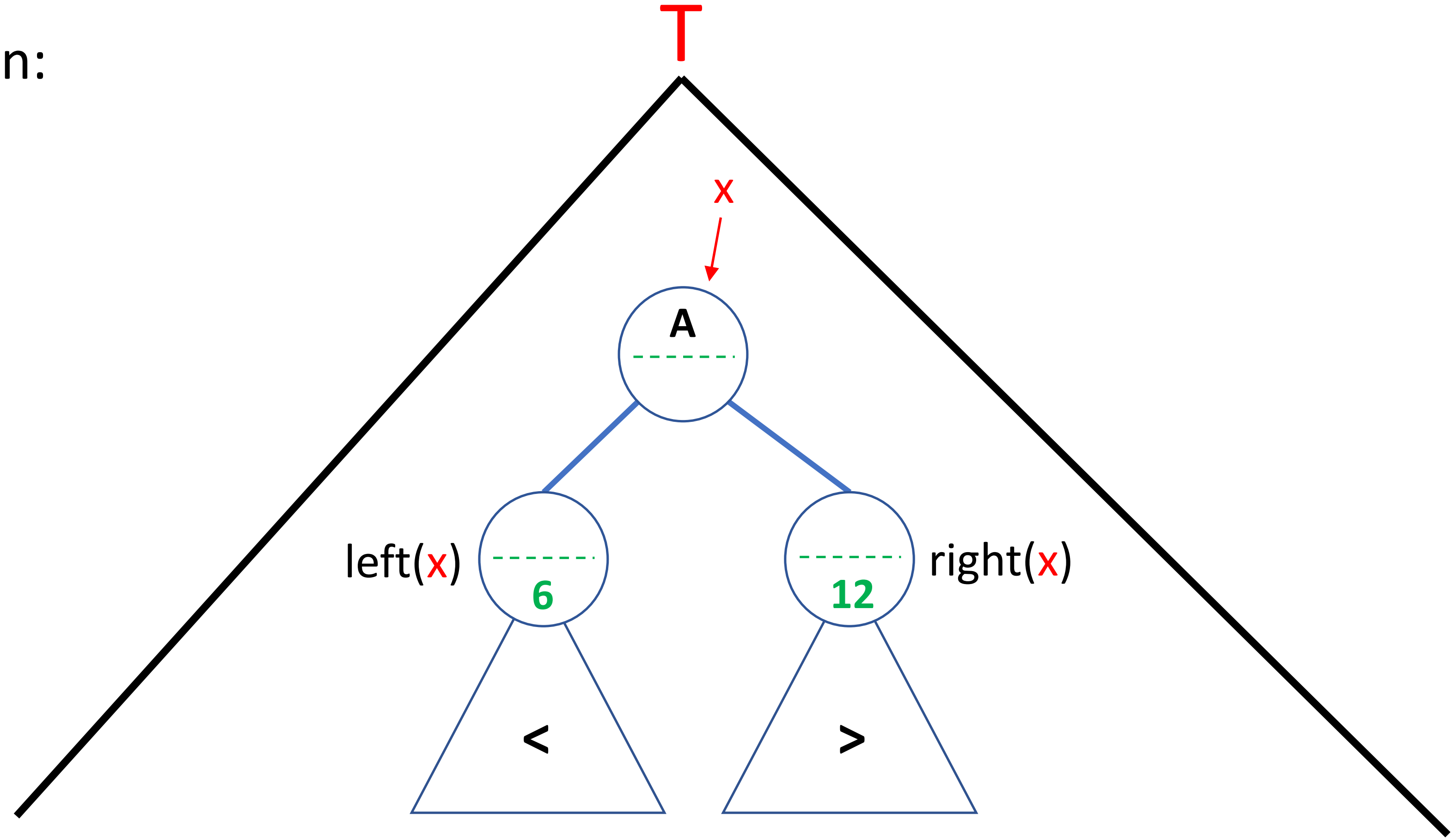
For every node x,

$$\boxed{\mathbf{size}(x) = \mathbf{size}(\mathbf{left}(x)) + \mathbf{size}(\mathbf{right}(x)) + 1}$$

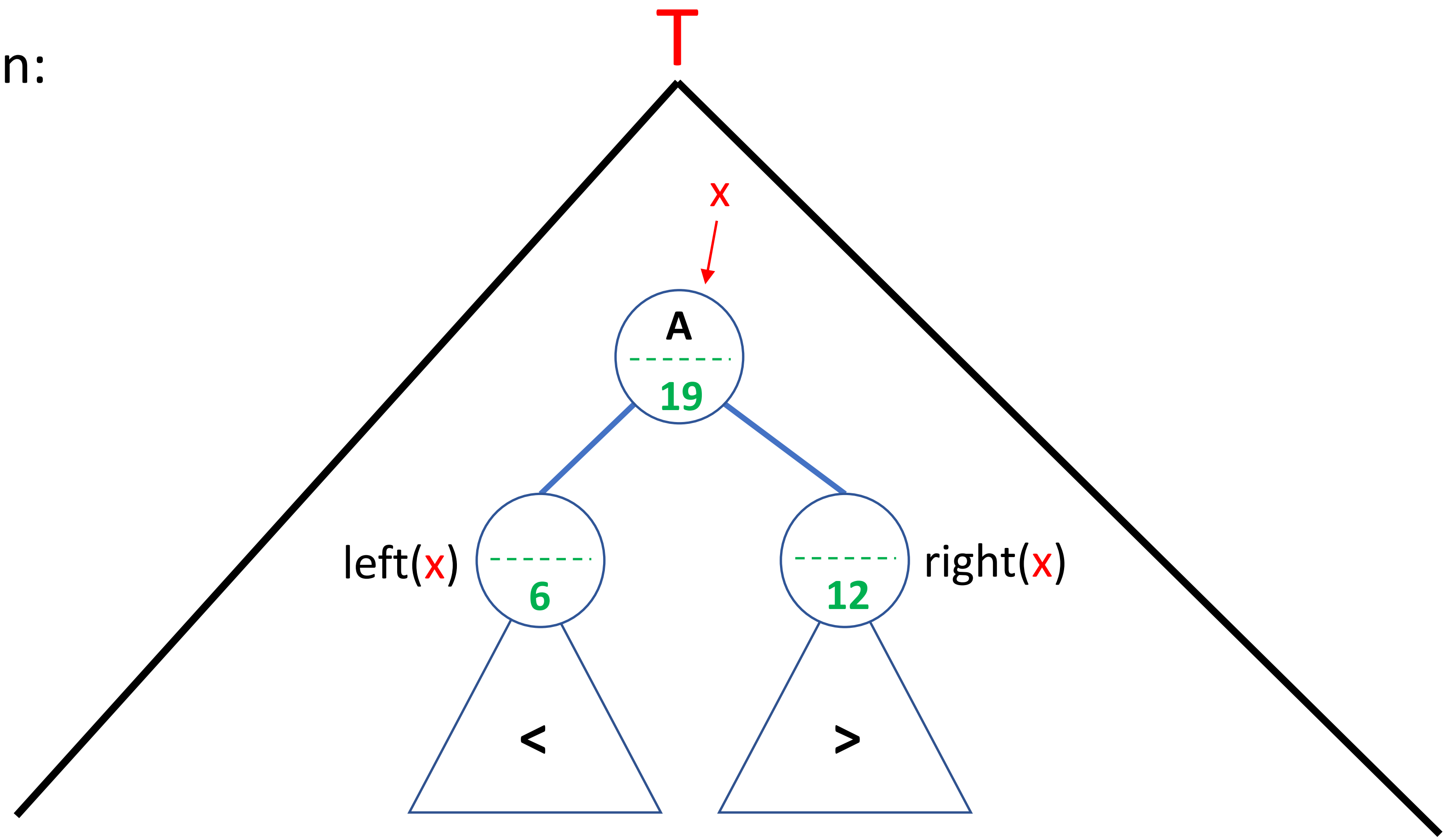


1. How to efficiently implement **Select** and **Rank ?**
2. How to efficiently maintain the size field during **Insert** and **Delete**?

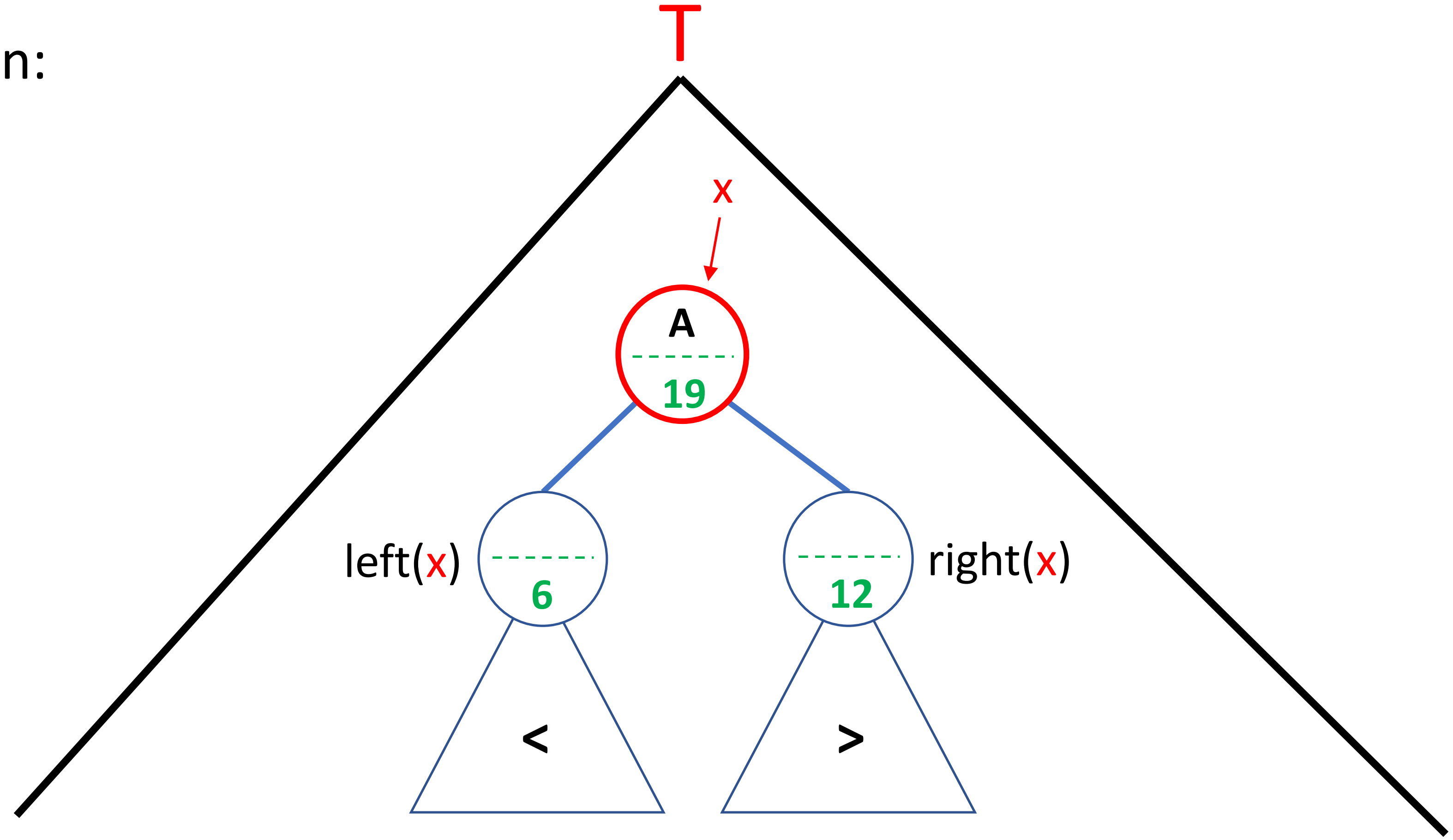
Basic Observation:



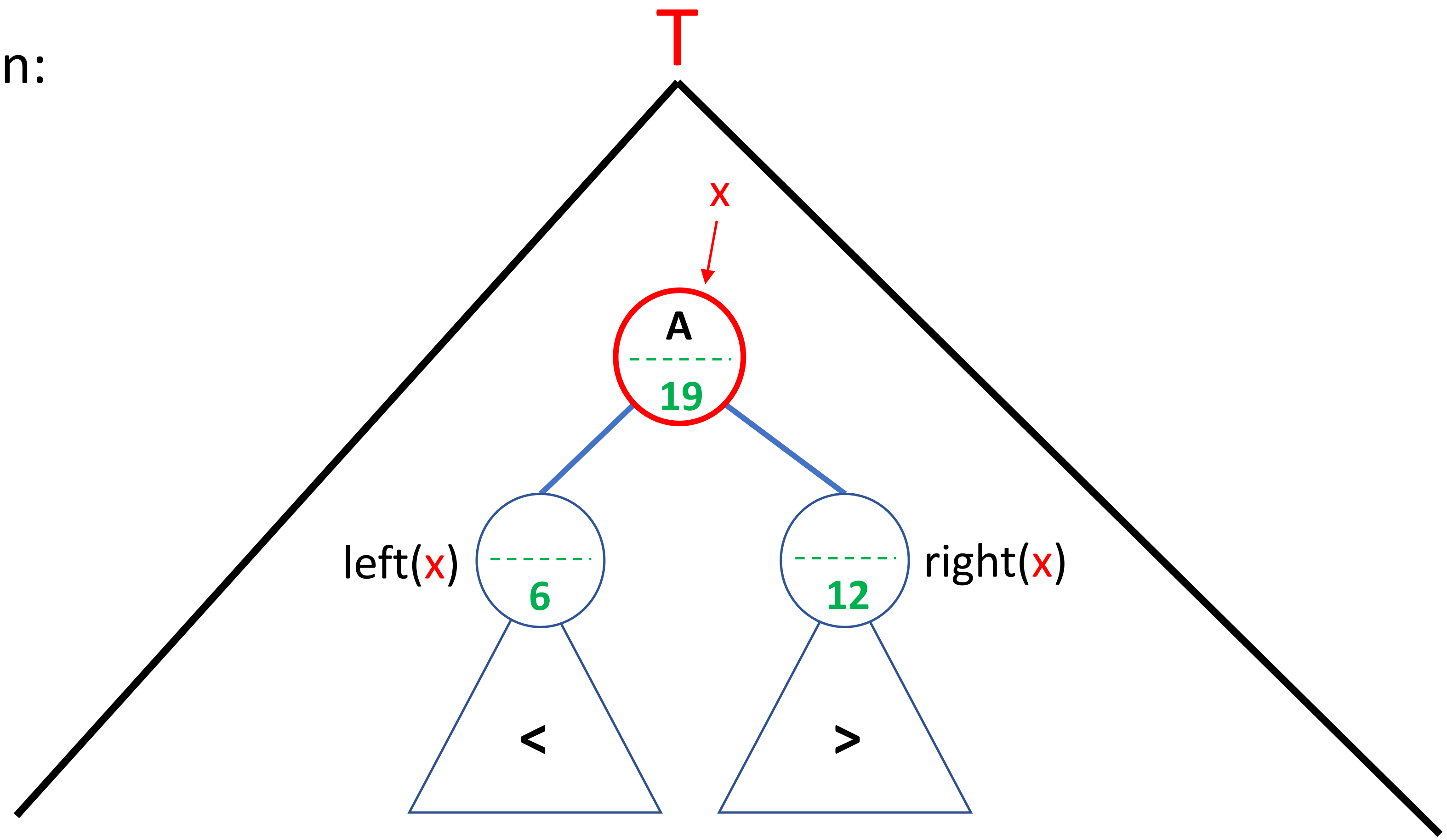
Basic Observation:



Basic Observation:

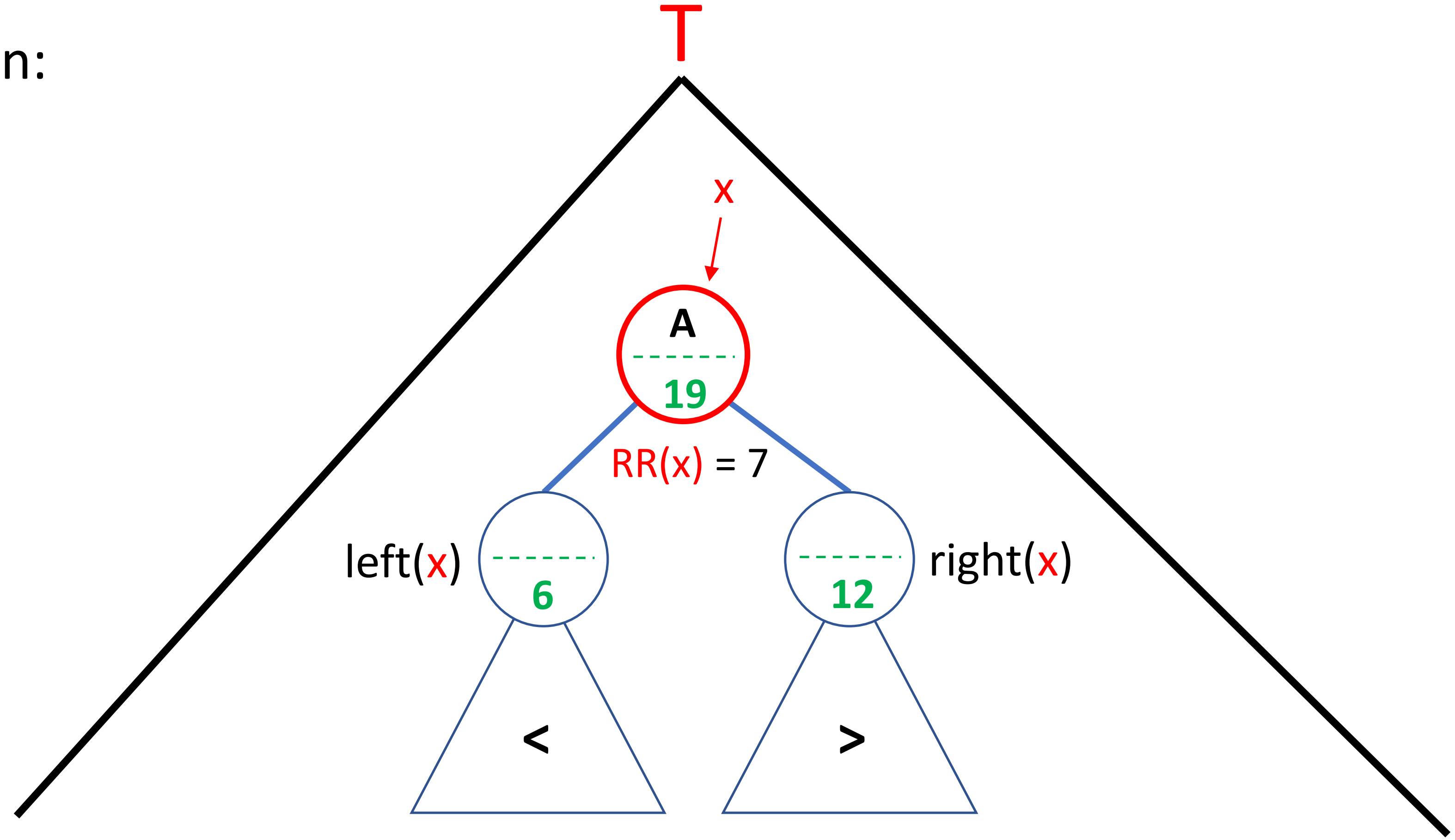


Basic Observation:



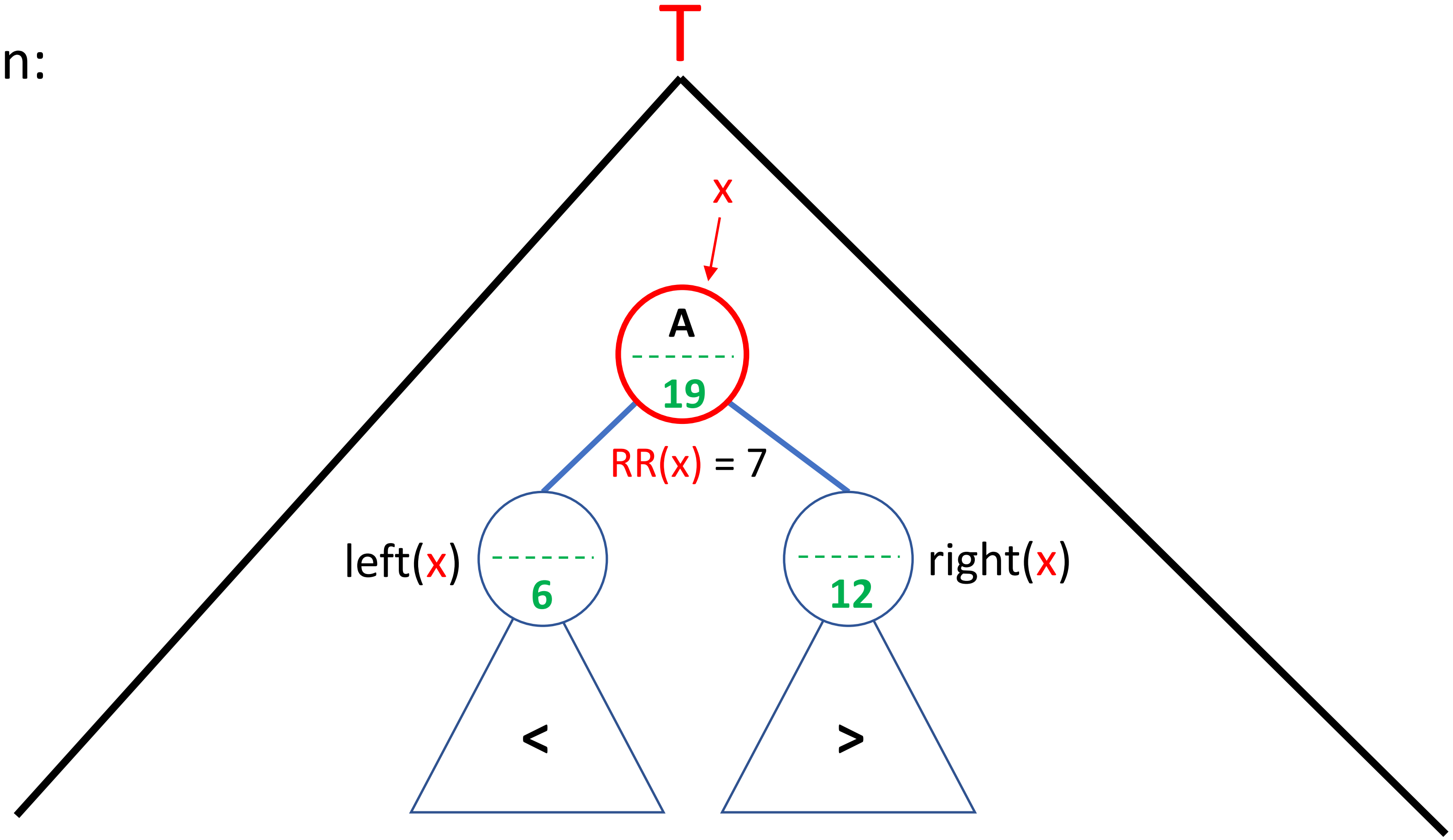
RR(x): Relative Rank of x in the subtree rooted at x

Basic Observation:



RR(x): Relative Rank of x in the subtree rooted at x

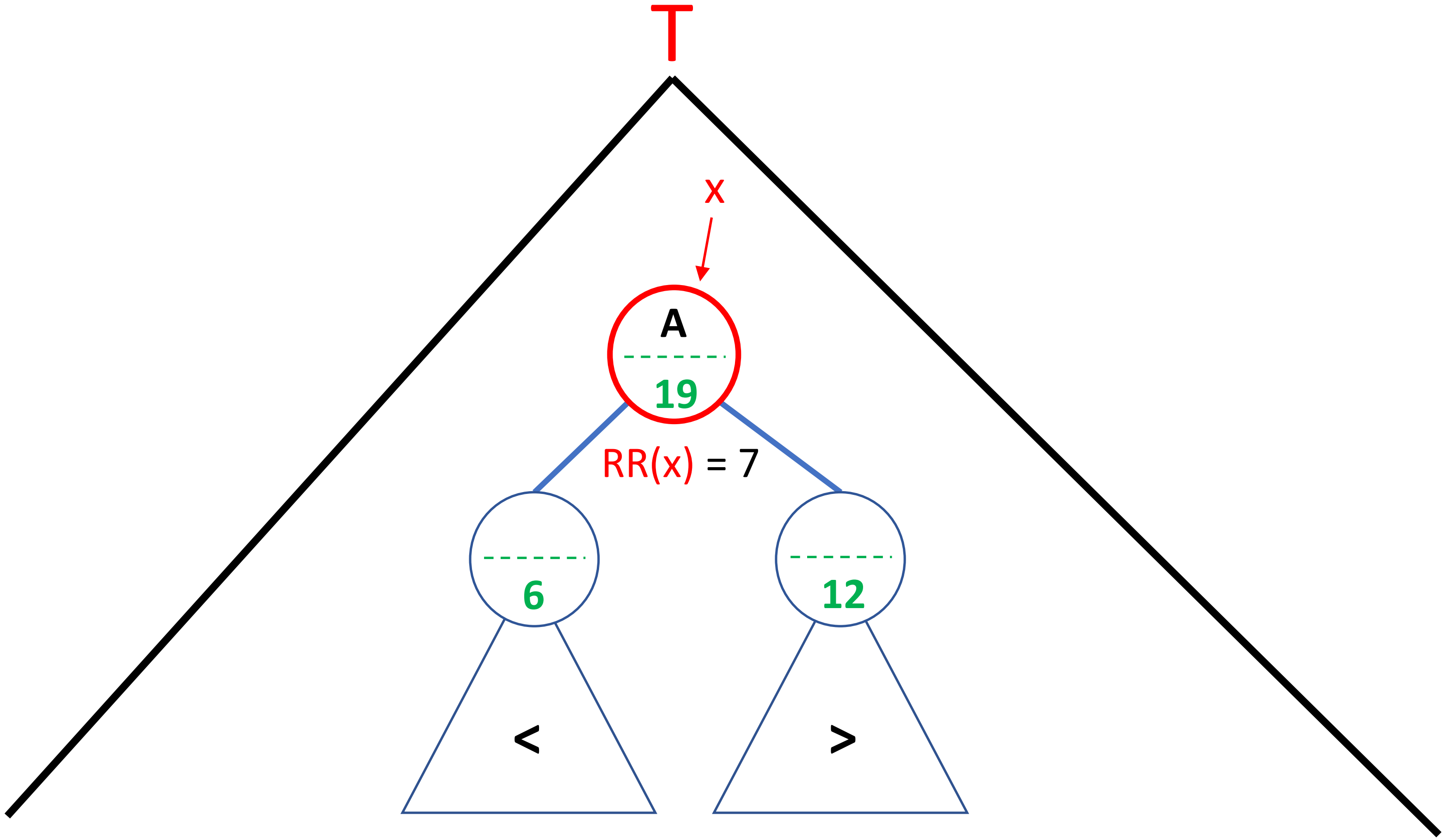
Basic Observation:



RR(x): Relative Rank of x in the subtree rooted at x

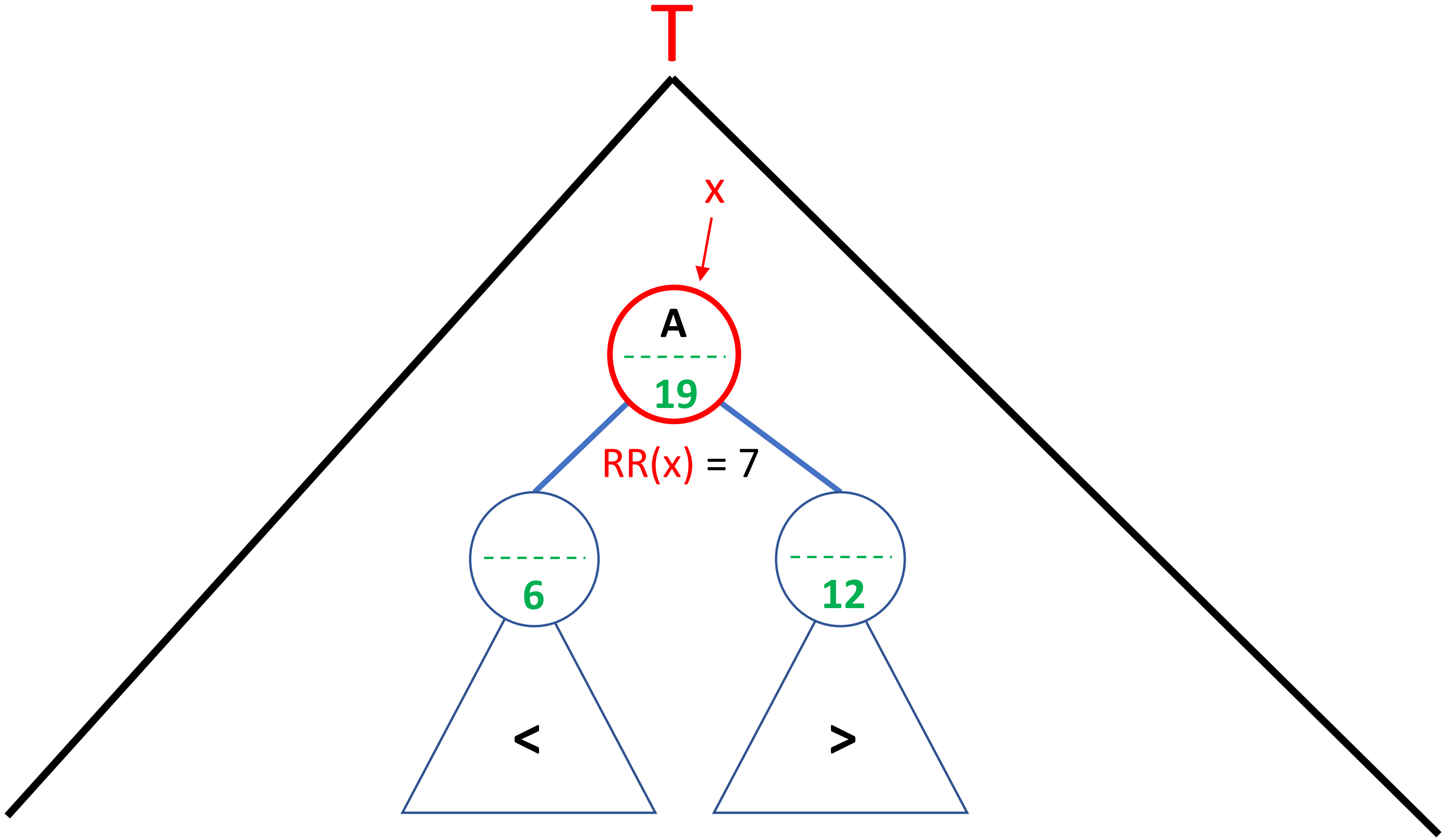
RR(x) = size(left(x)) + 1

RR(x) = 7



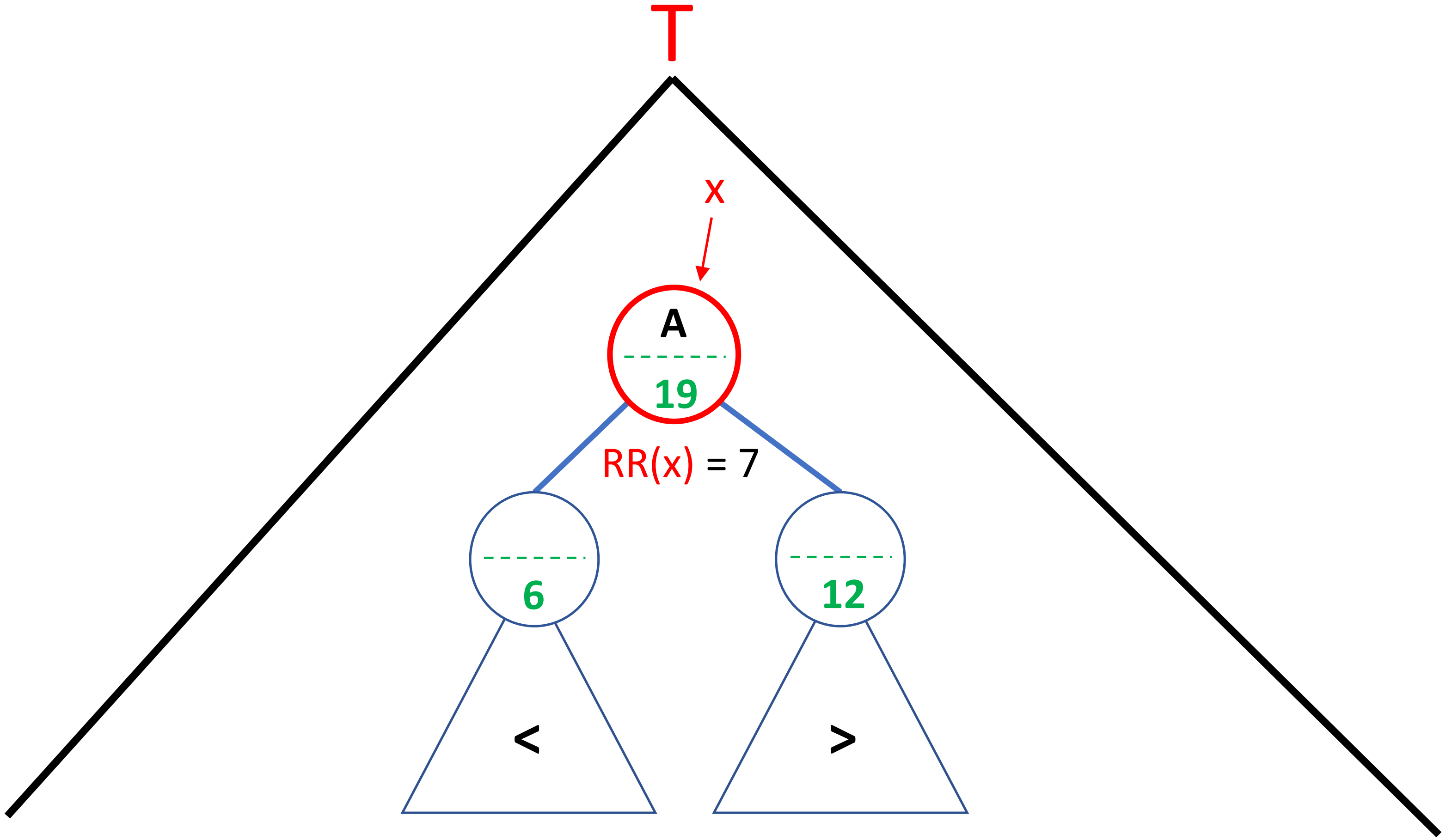
$RR(x) = 7$

Select(x, 7)



RR(x) = 7

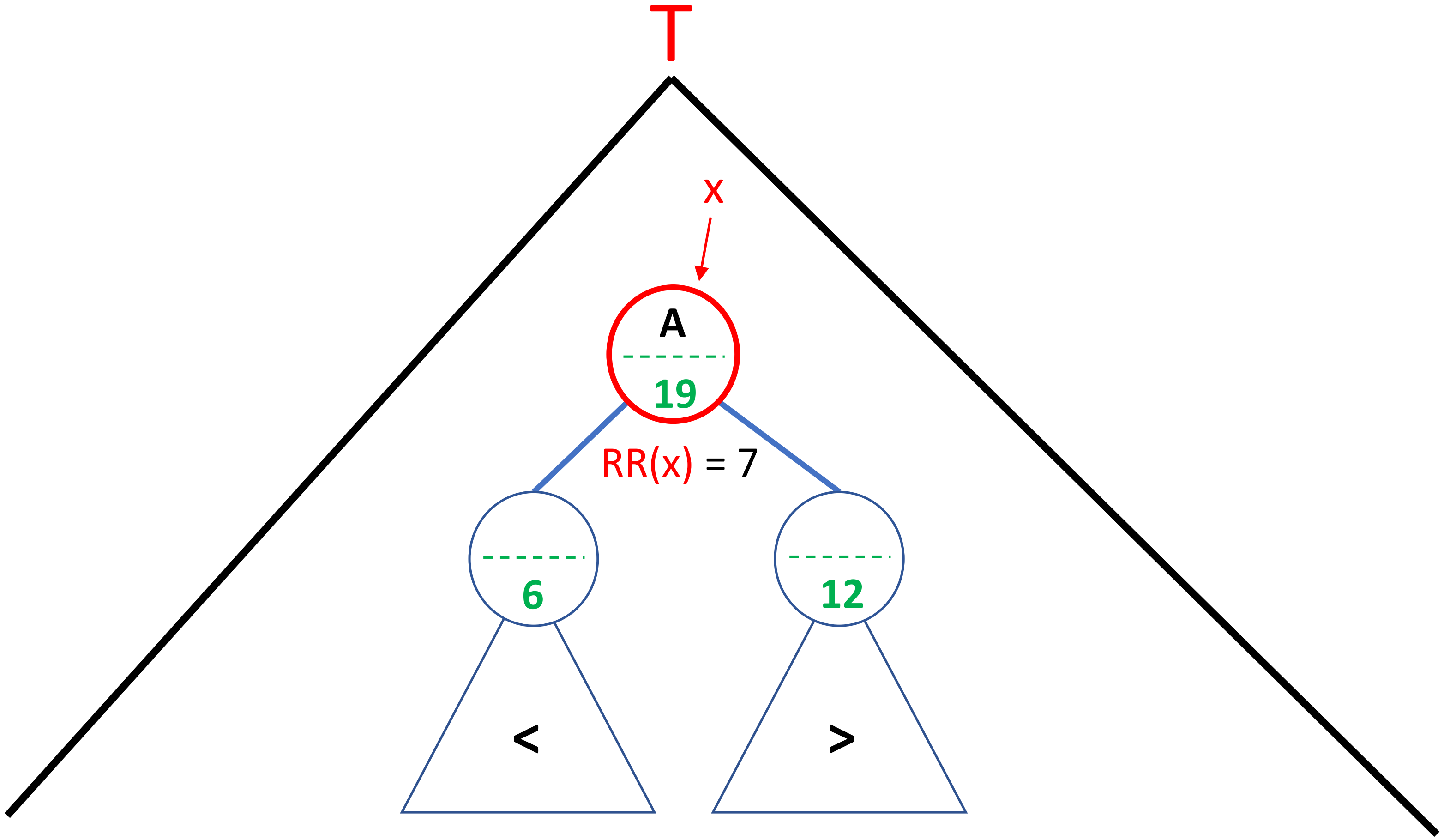
Select(x, 7) : Returns x



RR(x) = 7

Select(x, 7) : Returns x

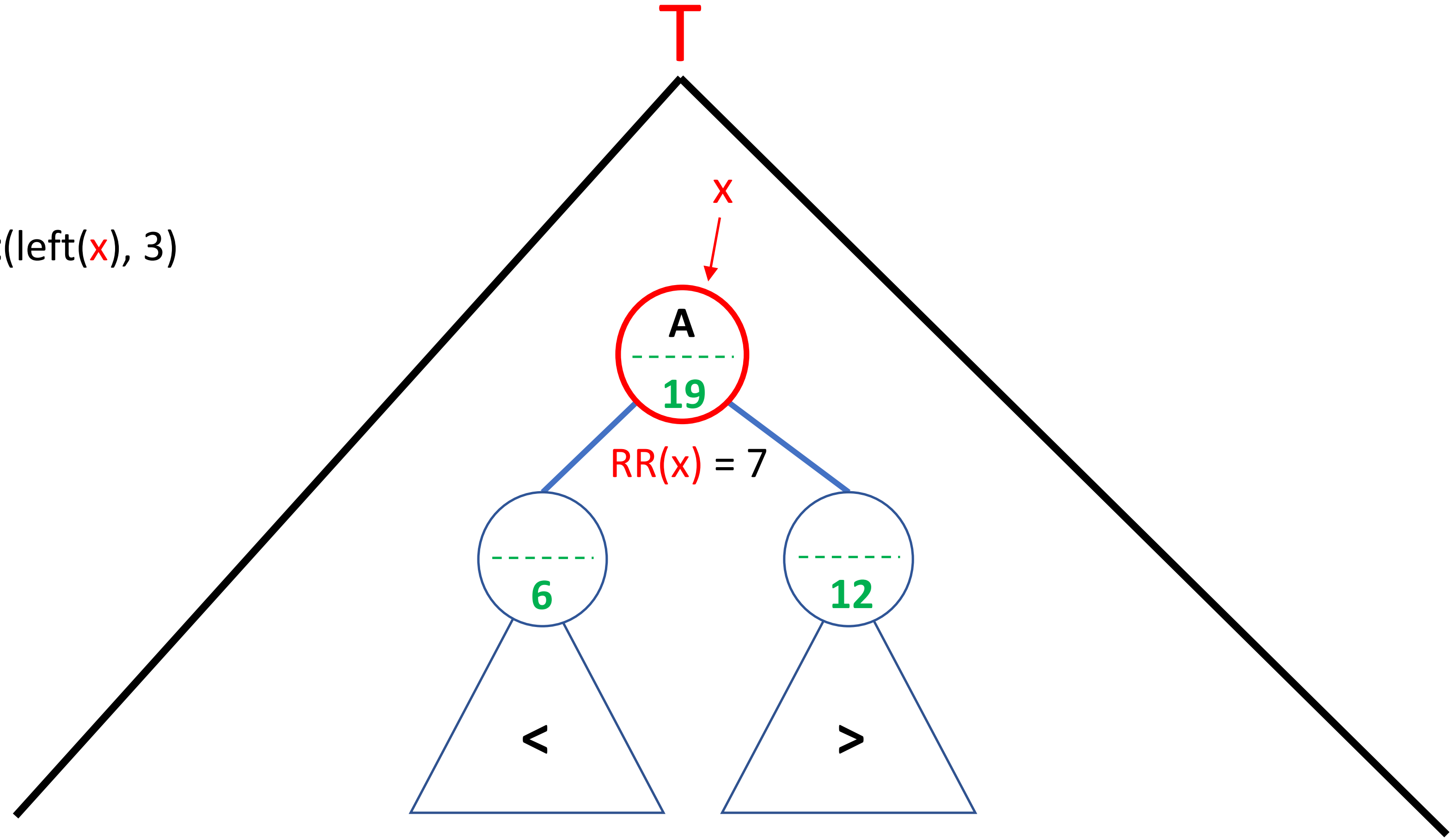
Select(x, 3)



RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

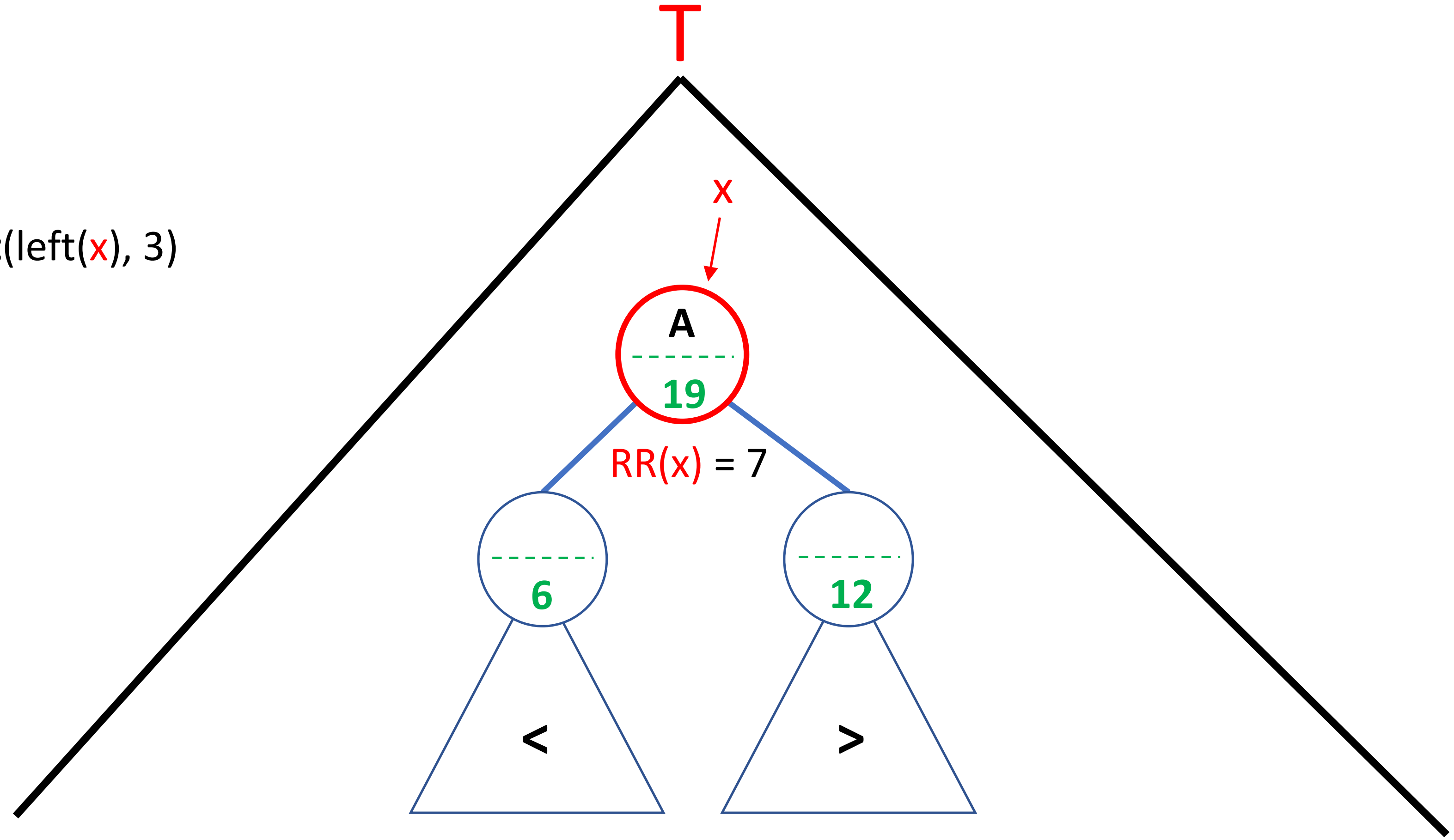


RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8)

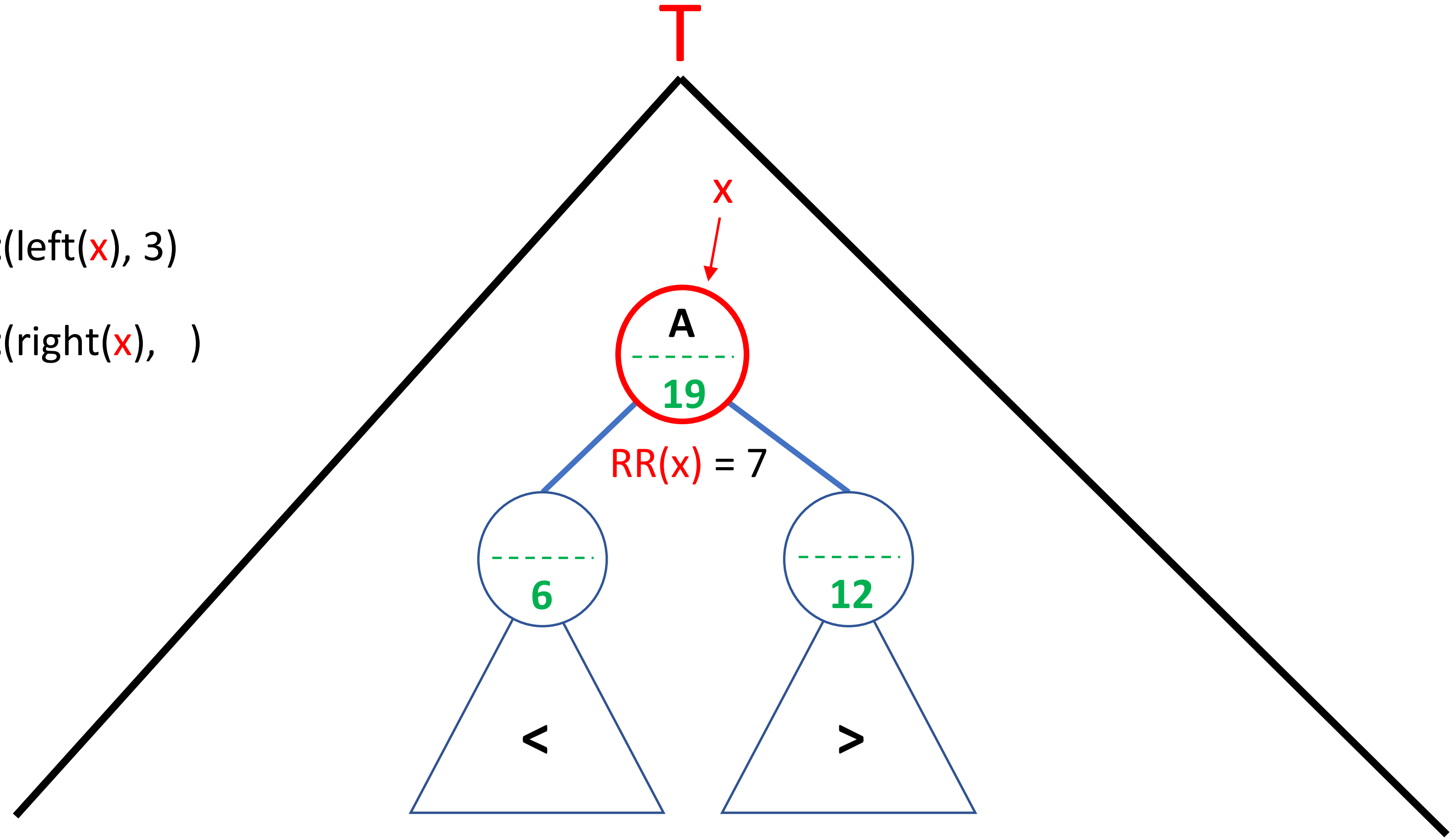


RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x),   )

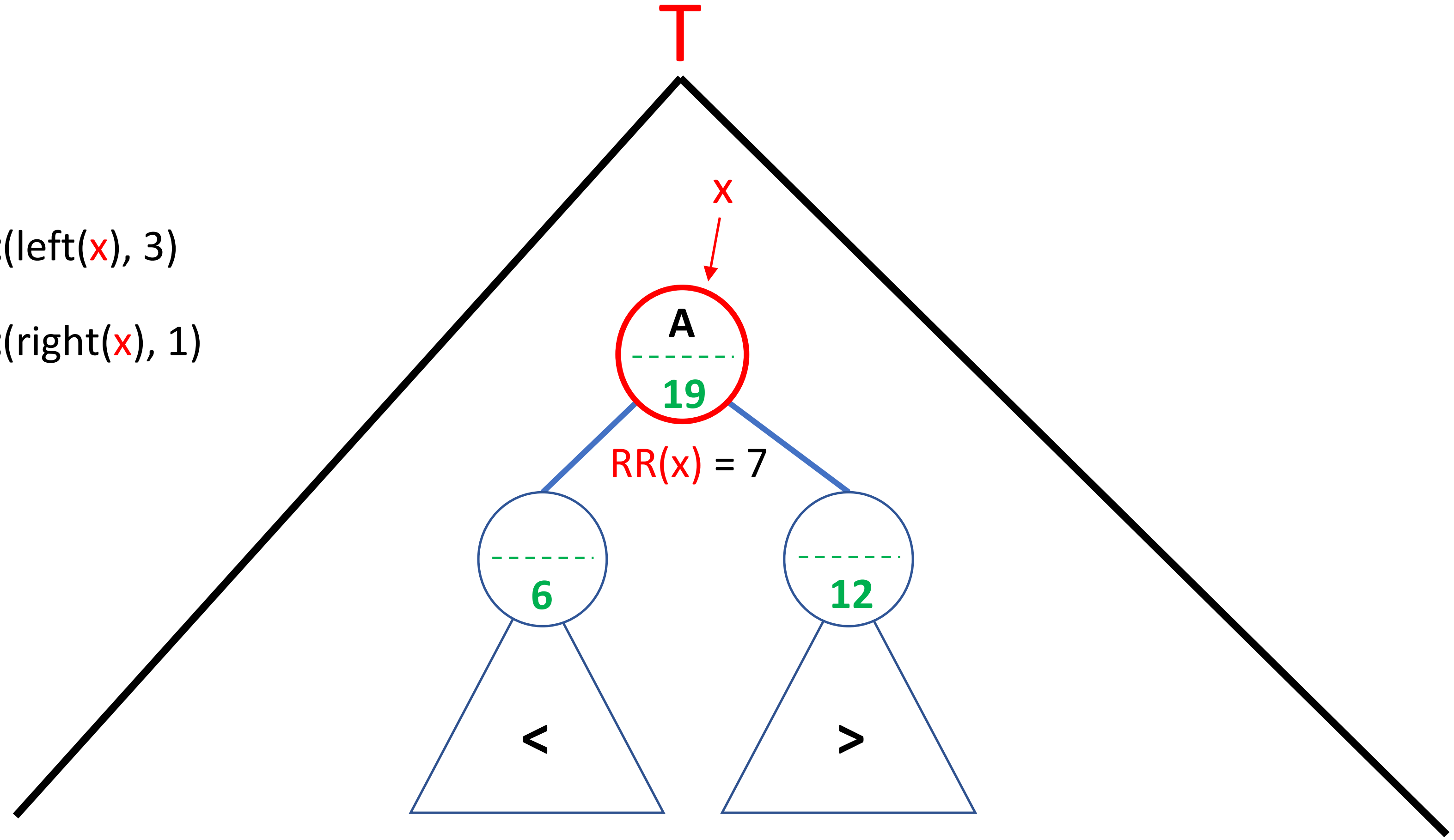


RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x), 1)



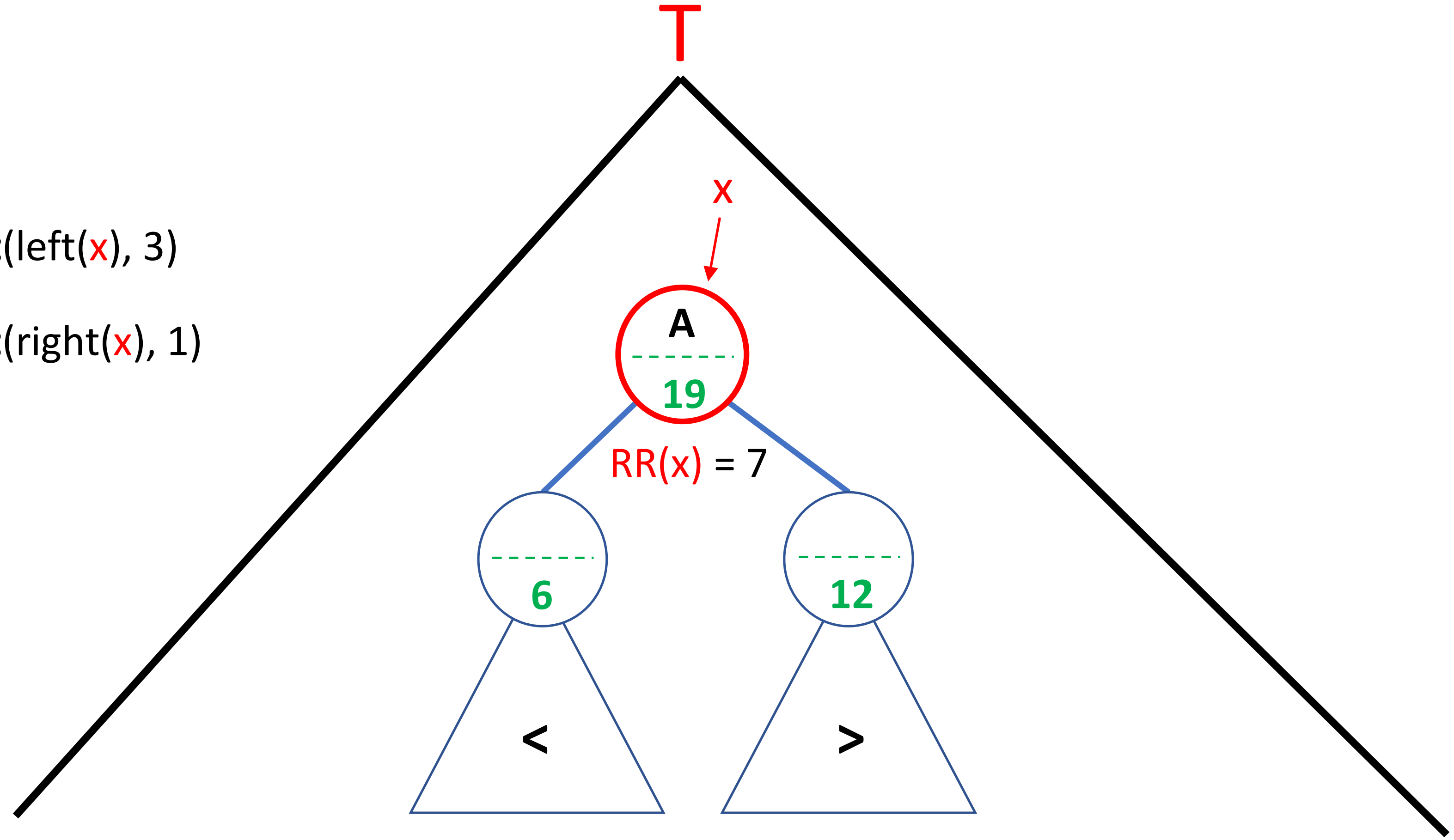
RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x), 1)

Select(x, 11)



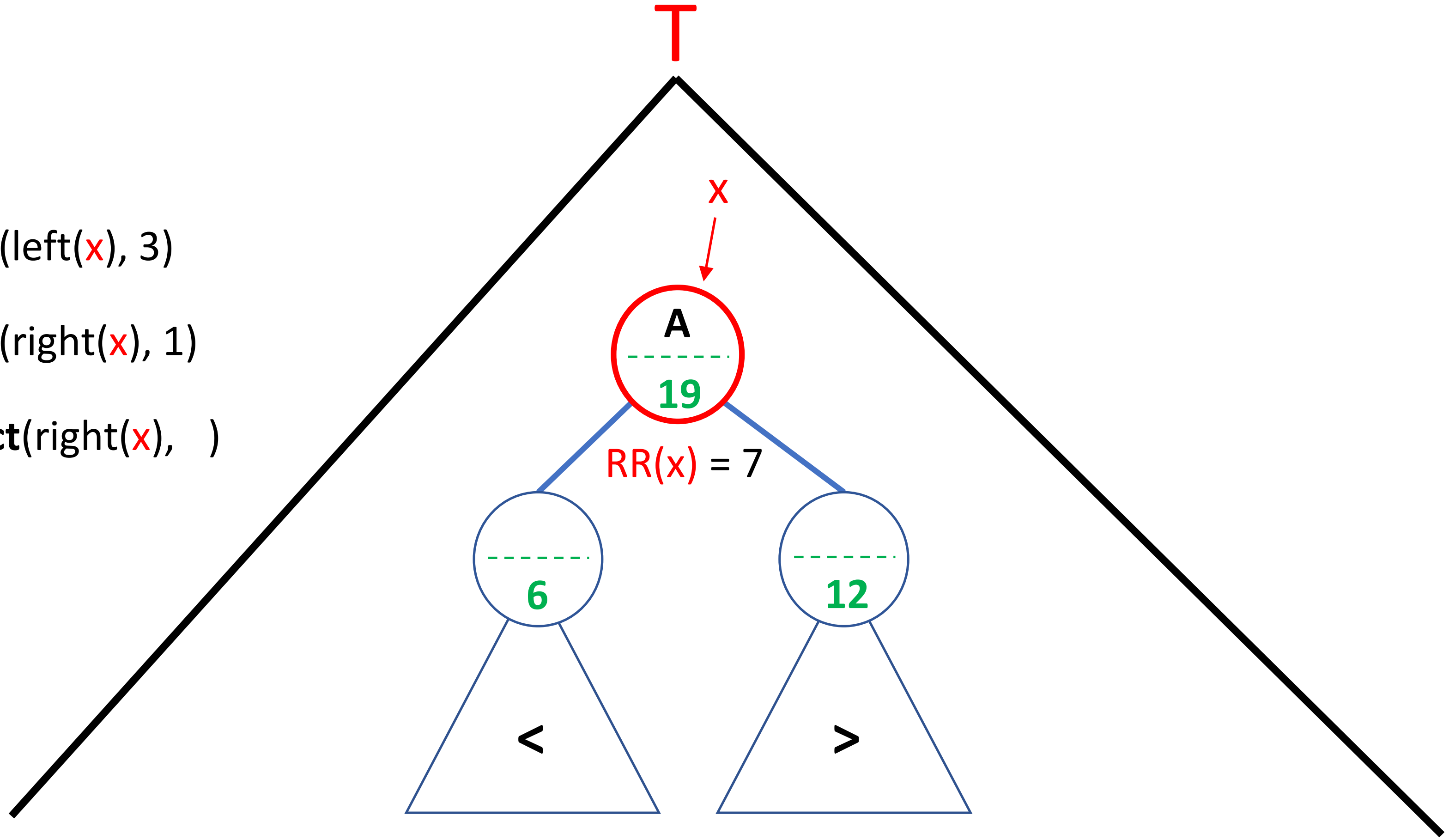
RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x), 1)

Select(x, 11) : Calls **Select**(right(x),   )



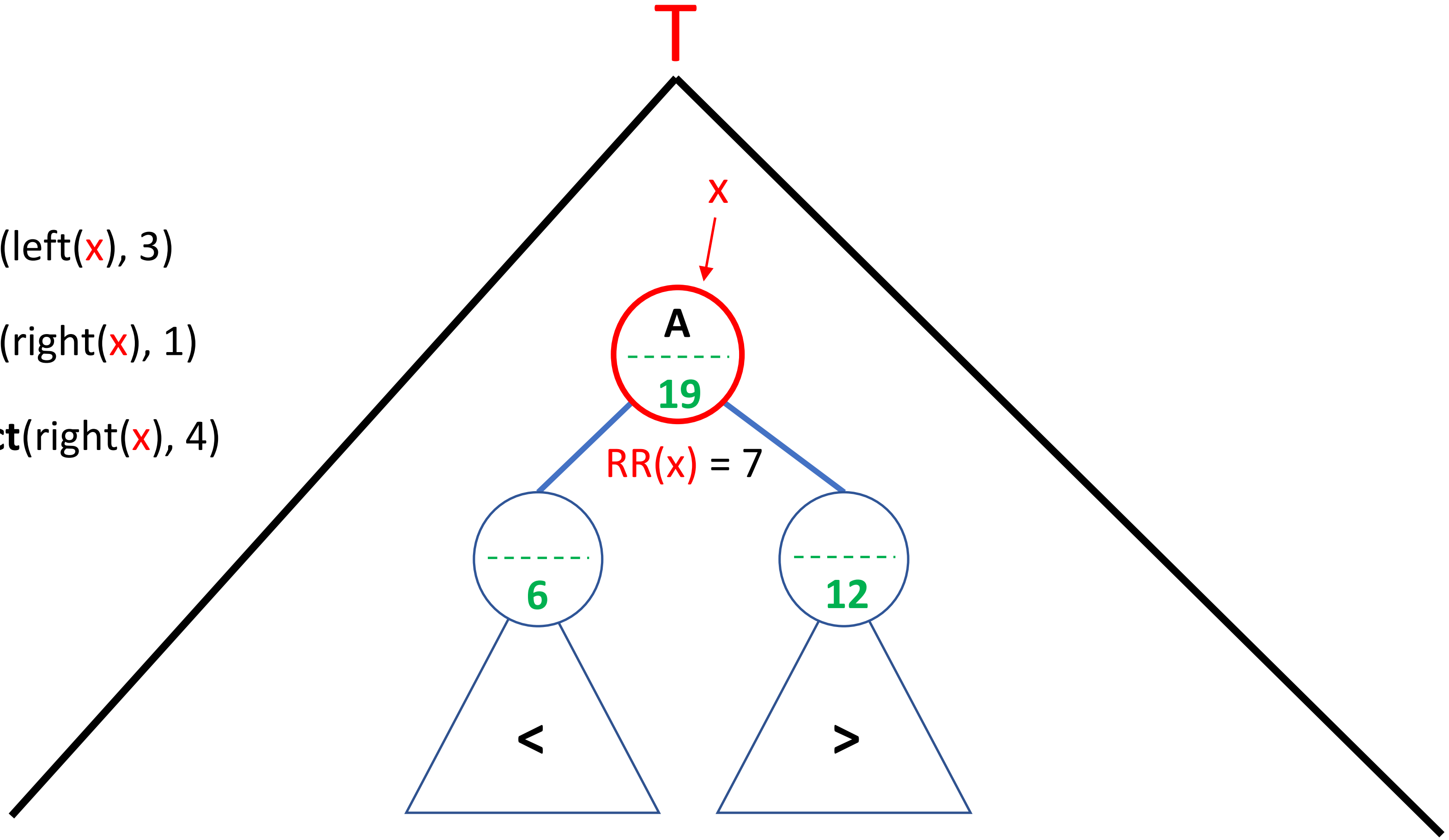
RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x), 1)

Select(x, 11) : Calls **Select**(right(x), 4)



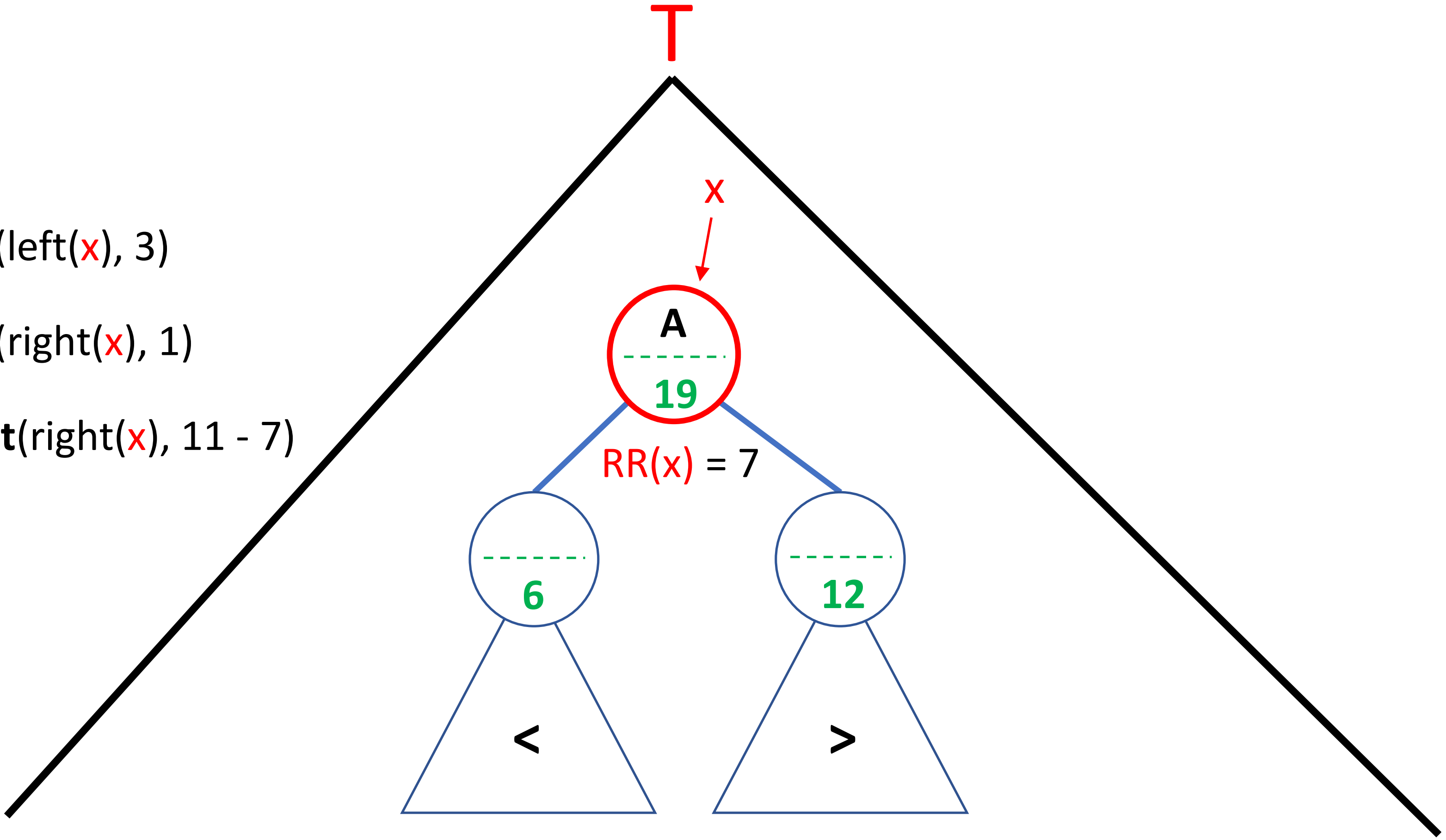
RR(x) = 7

Select(x, 7) : Returns x

Select(x, 3) : Calls **Select**(left(x), 3)

Select(x, 8) : Calls **Select**(right(x), 1)

Select(x, 11) : Calls **Select**(right(x), 11 - 7)



Select(x, k) : Return element with rank k in subtree rooted at x

Select(x, k) : Return element with rank k in subtree rooted at x

RR(x) $\leftarrow$ size(left(x)) + 1

Select(x, k) : Return element with rank k in subtree rooted at x

$RR(x) \leftarrow \text{size}(\text{left}(x)) + 1$

if $k = RR(x)$

if $k < RR(x)$

if $k > RR(x)$

Select(x, k) : Return element with rank k in subtree rooted at x

$RR(x) \leftarrow \text{size}(\text{left}(x)) + 1$

if $k = RR(x)$ then return x

if $k < RR(x)$

if $k > RR(x)$

Select(x, k) : Return element with rank k in subtree rooted at x

RR(x) ← size(left(x)) + 1

if k = RR(x) then return x

if k < RR(x) then **Select**(left(x),)

if k > RR(x)

Select(x, k) : Return element with rank k in subtree rooted at x

$RR(x) \leftarrow \text{size}(\text{left}(x)) + 1$

if $k = RR(x)$ then return x

if $k < RR(x)$ then **Select**(left(x), k)

if $k > RR(x)$

Select(x, k) : Return element with rank k in subtree rooted at x

RR(x) ← size(left(x)) + 1

if k = RR(x) then return x

if k < RR(x) then **Select**(left(x), k)

if k > RR(x) then **Select**(right(x),)

Select(x, k) : Return element with rank k in subtree rooted at x

RR(x) ← size(left(x)) + 1

if k = RR(x) then return x

if k < RR(x) then **Select**(left(x), k)

if k > RR(x) then **Select**(right(x), k – RR(x))

Select(x, k) : Return element with rank k in subtree rooted at x

$RR(x) \leftarrow size(left(x)) + 1$

if $k = RR(x)$ then return x

if $k < RR(x)$ then **Select**(left(x), k)

if $k > RR(x)$ then **Select**(right(x), $k - RR(x)$)

Select(T, k) = **Select**(x, k) where x is the root of T

Select(x, k) : Return element with rank k in subtree rooted at x

$RR(x) \leftarrow$ size(left(x)) + 1

if $k = RR(x)$ then return x

if $k < RR(x)$ then **Select**(left(x), k)

if $k > RR(x)$ then **Select**(right(x), $k - RR(x)$)

Select(T, k) = **Select**(x, k) where x is the root of T

Worst-Case Time Complexity of **Select**(T, k):

Select(x, k) : Return element with rank k in subtree rooted at x

RR(x) ← size(left(x)) + 1

if k = RR(x) then return x

if k < RR(x) then **Select**(left(x), k)

if k > RR(x) then **Select**(right(x), k – RR(x))

Select(T, k) = **Select**(x, k) where x is the root of T

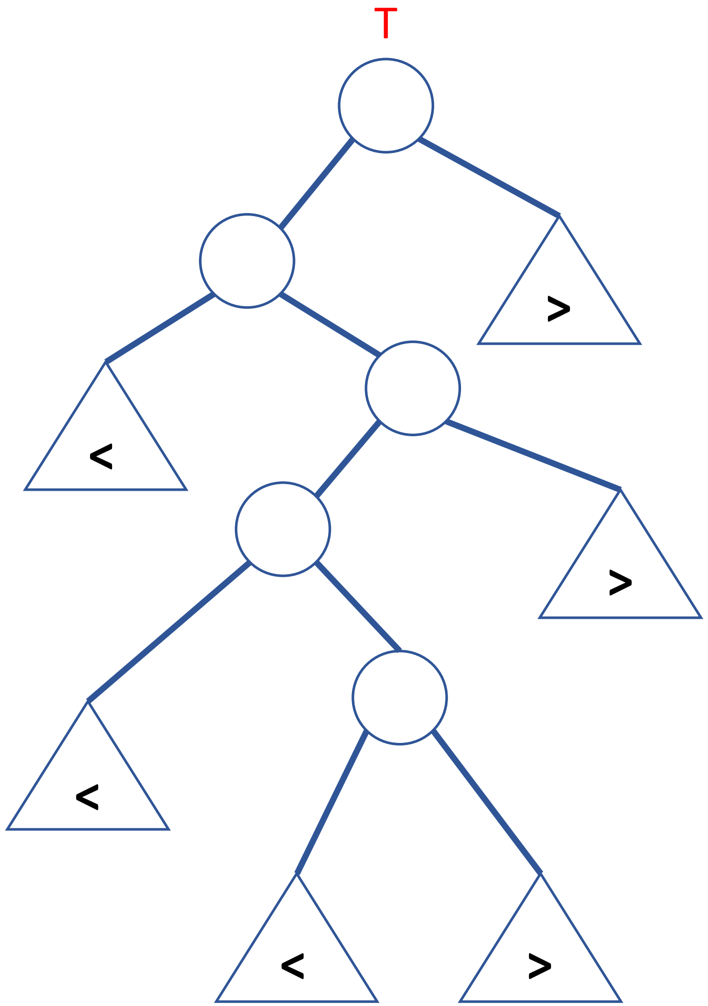
Worst-Case Time Complexity of **Select**(T, k):

- Each **Select** call goes down one level in T (or returns)
- Height(T) is O(log n)
- Hence **Select** takes O(log n)

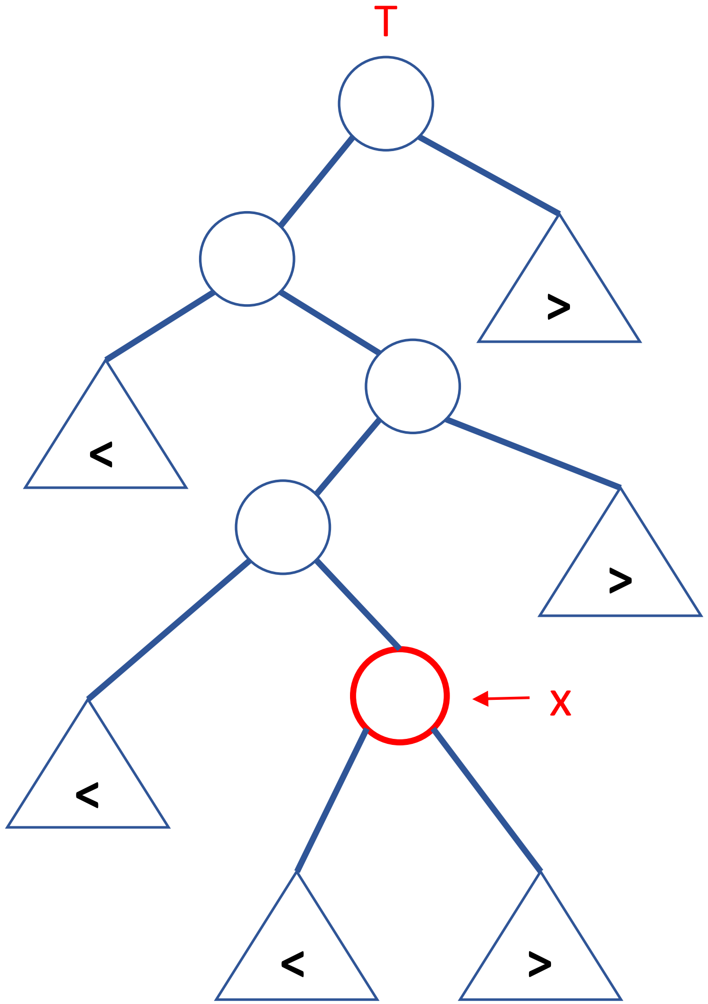
Augmenting AVL

- **Select** operation ✓
- **Rank** operation
- Maintain size() field

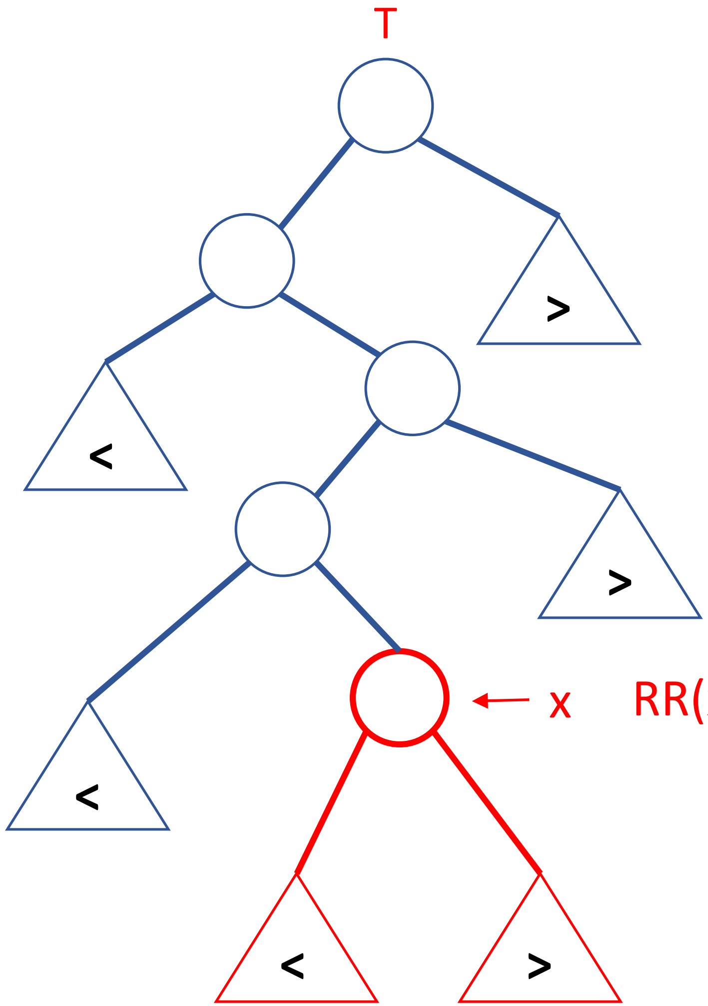
Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T

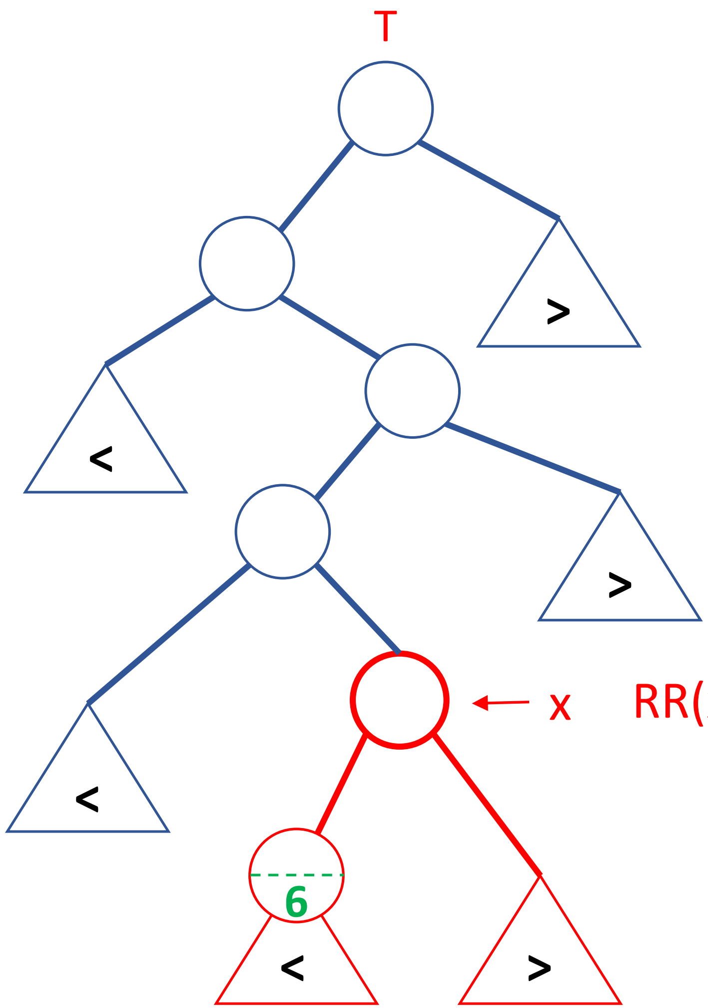


Rank(T, x): return rank of x in T



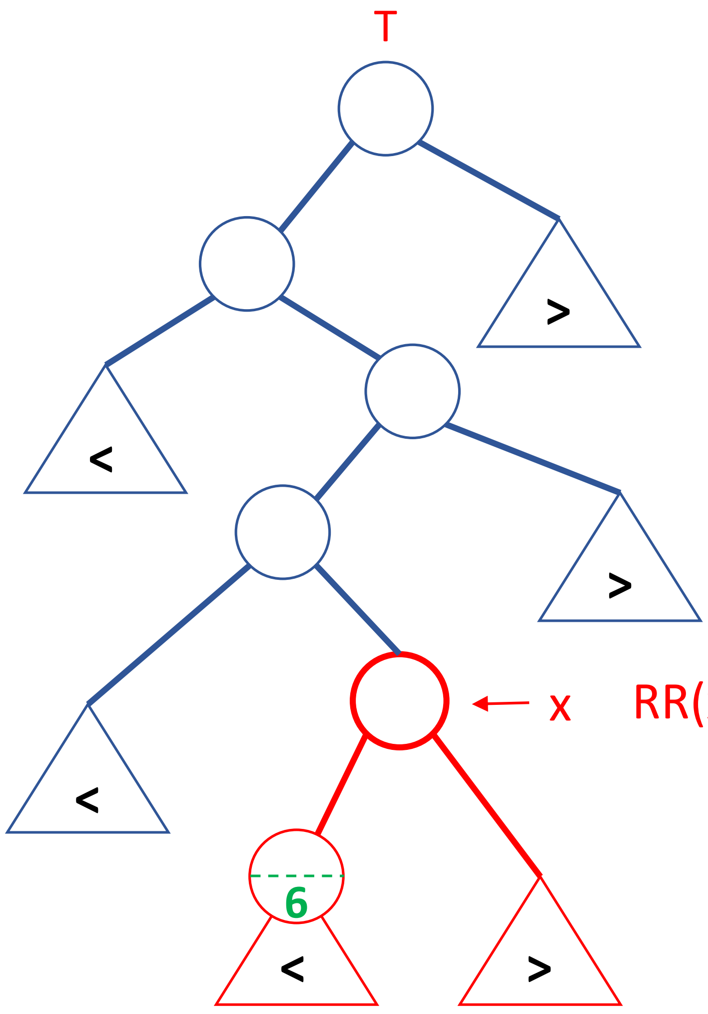
RR(x) in x-subtree =

Rank(T, x): return rank of x in T



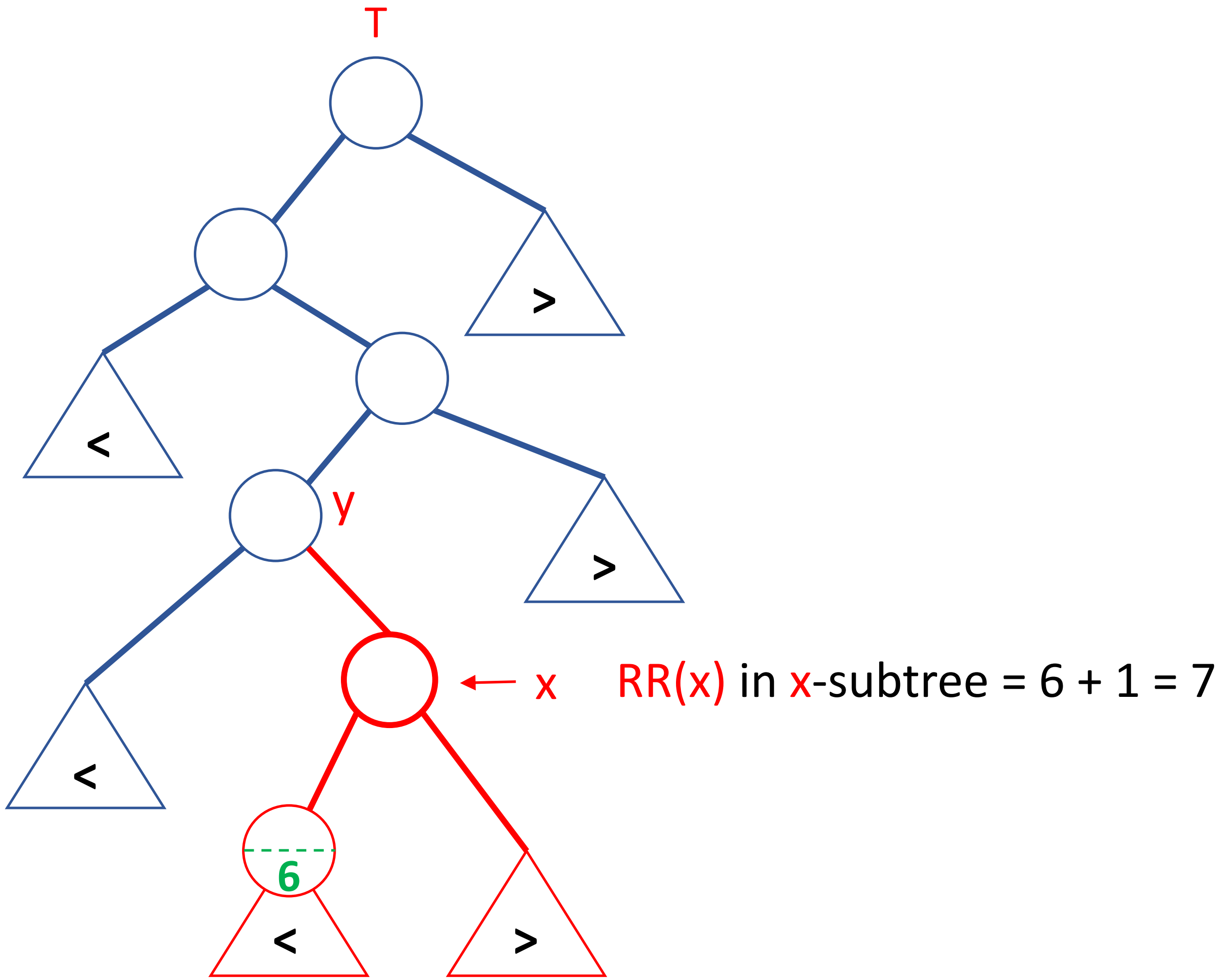
RR(x) in x-subtree =

Rank(T, x): return rank of x in T



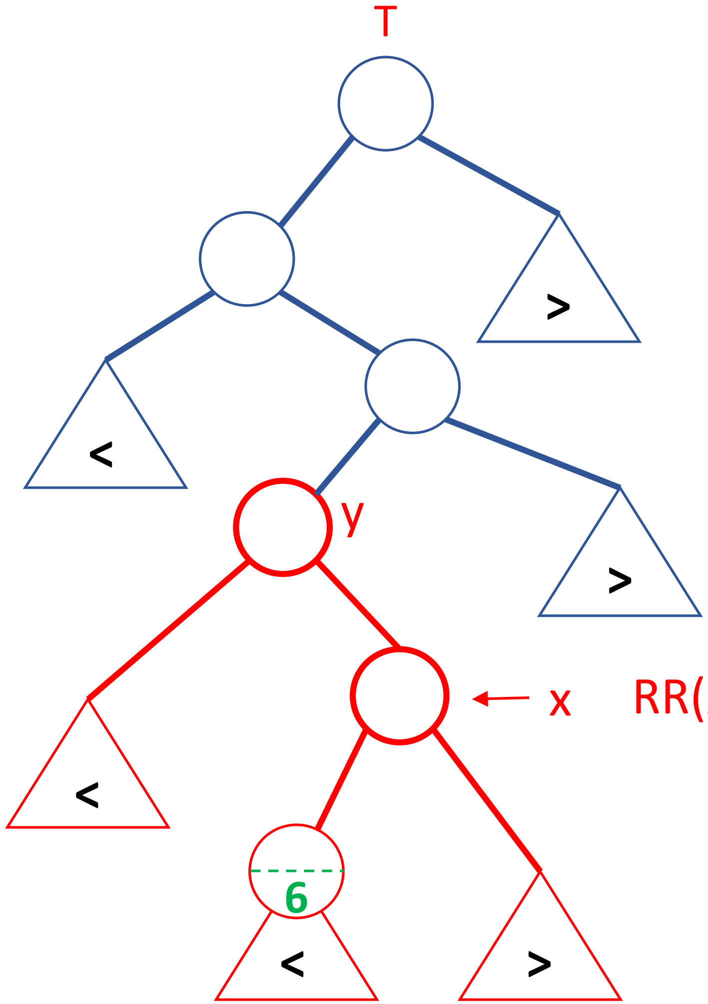
RR(x) in x-subtree = 6 + 1 = 7

Rank(T, x): return rank of x in T



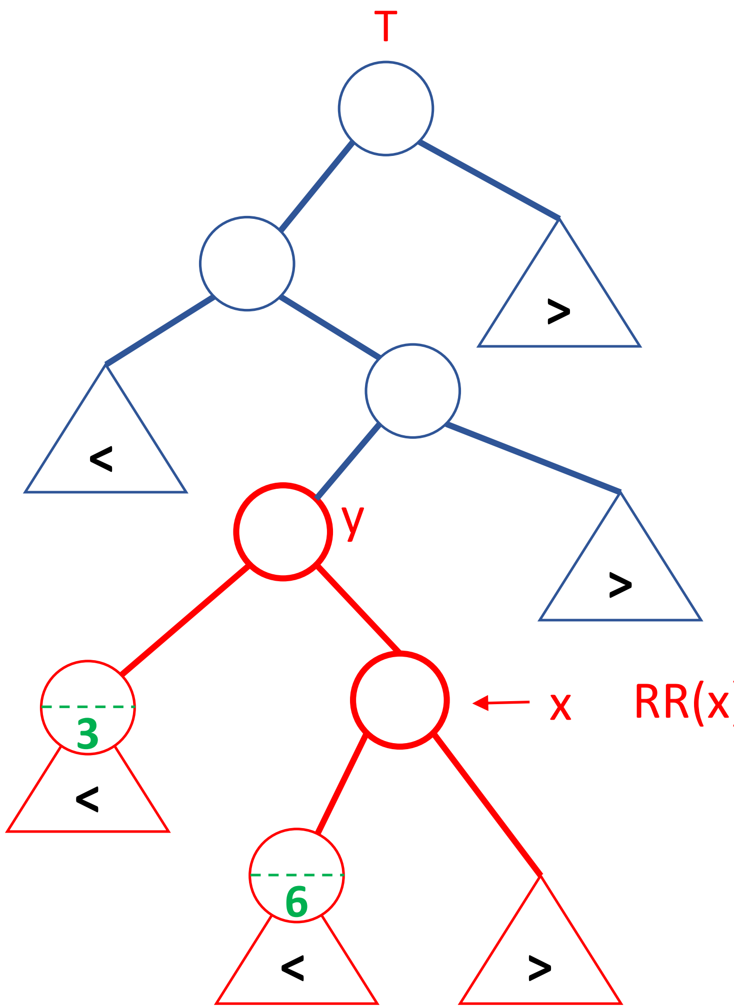
Rank(T, x): return rank of x in T

RR(x) in y-subtree =



RR(x) in x-subtree = 6 + 1 = 7

Rank(T, x): return rank of x in T

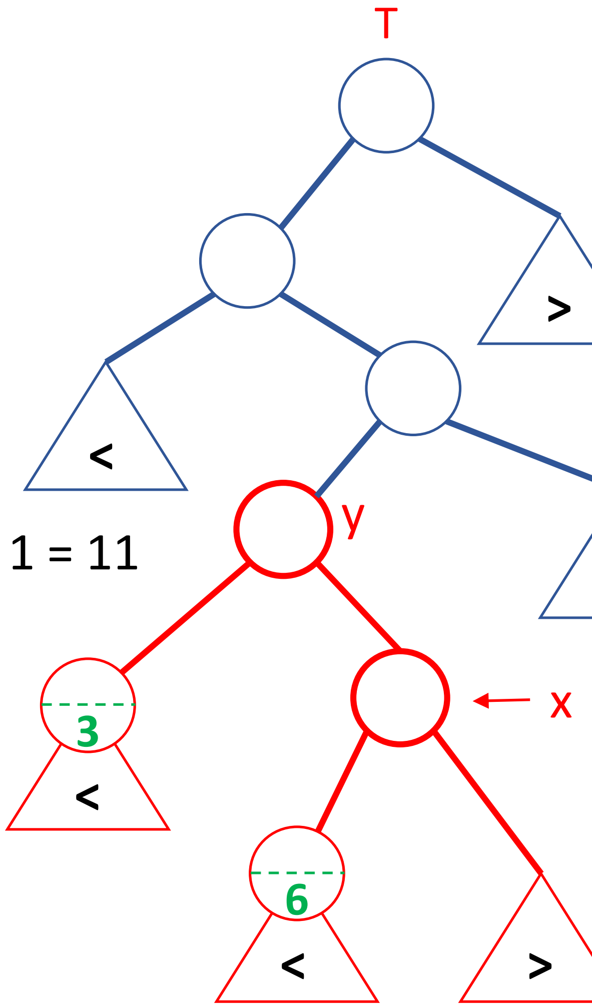


RR(x) in y-subtree =

RR(x) in x-subtree = 6 + 1 = 7

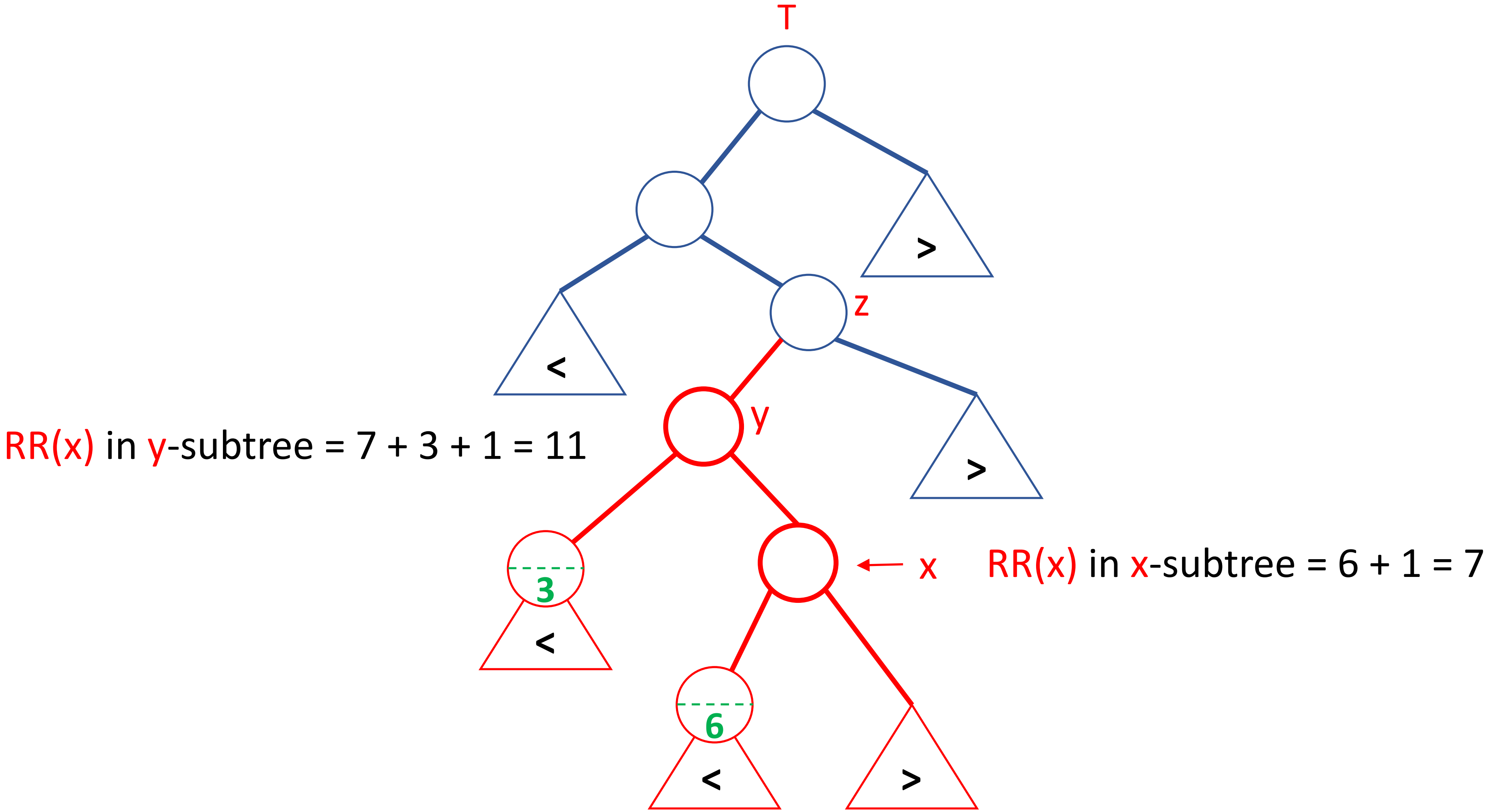
Rank(T, x): return rank of x in T

RR(x) in y-subtree = 7 + 3 + 1 = 11

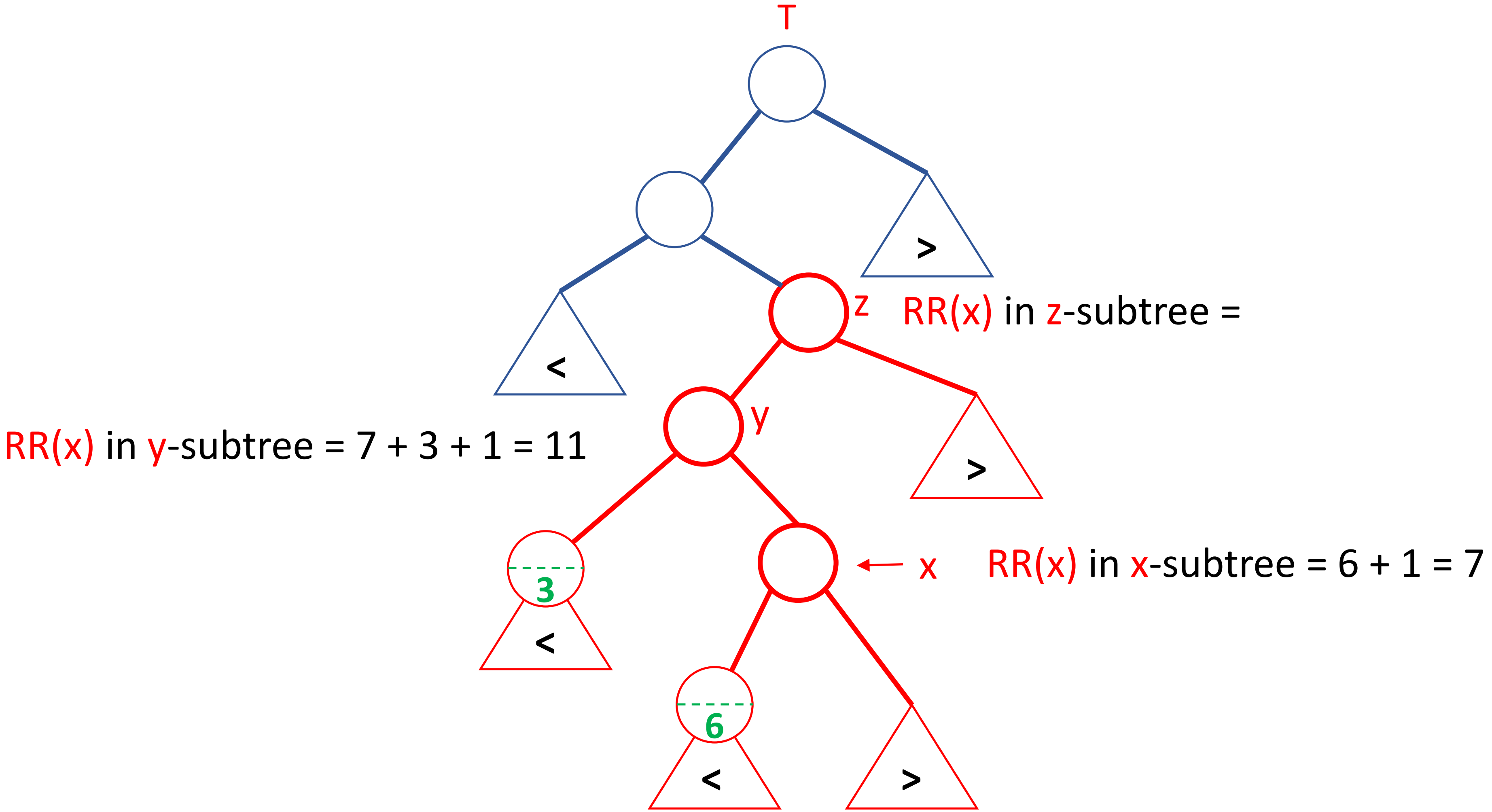


RR(x) in x-subtree = 6 + 1 = 7

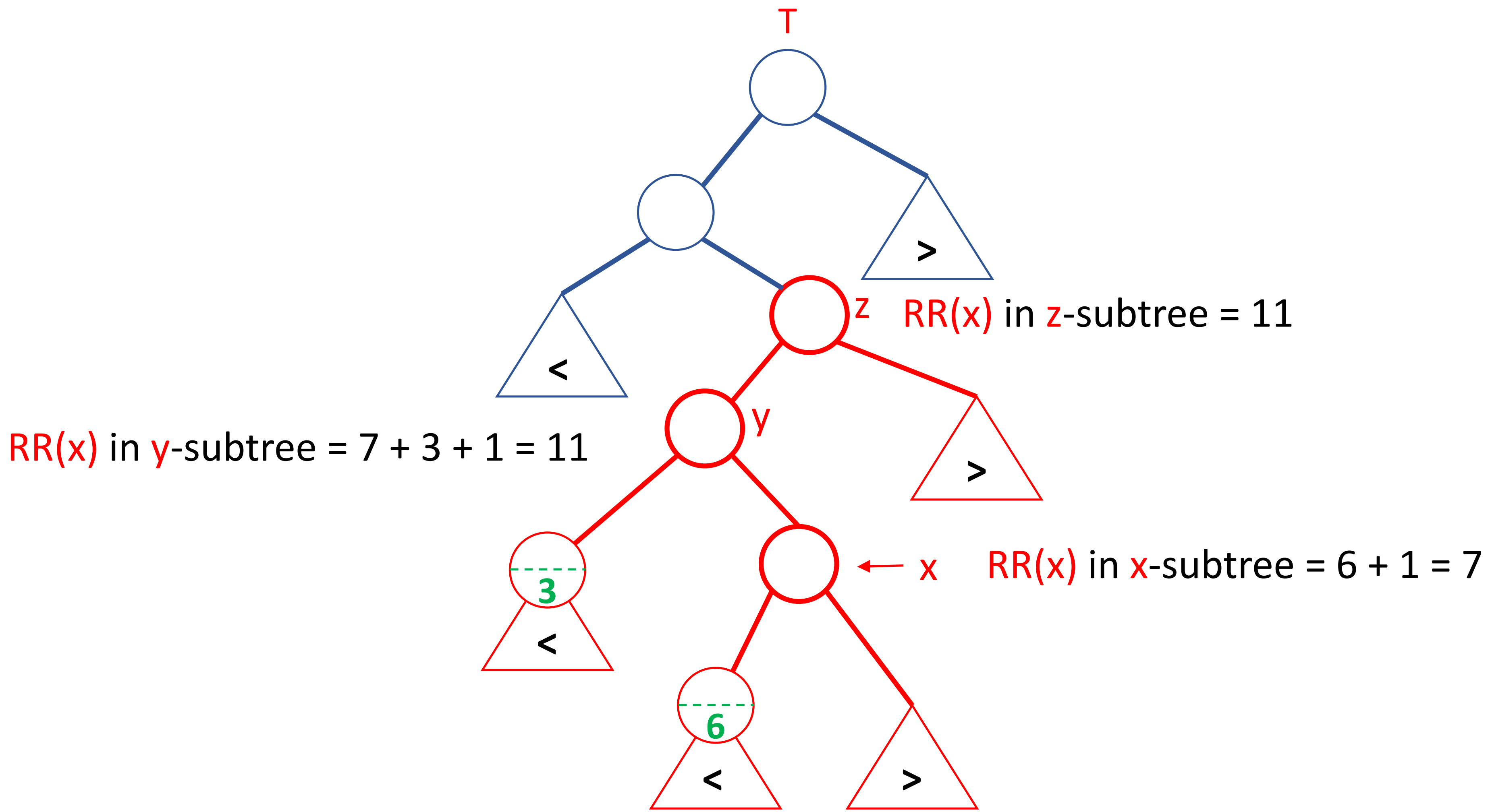
Rank(T, x): return rank of x in T



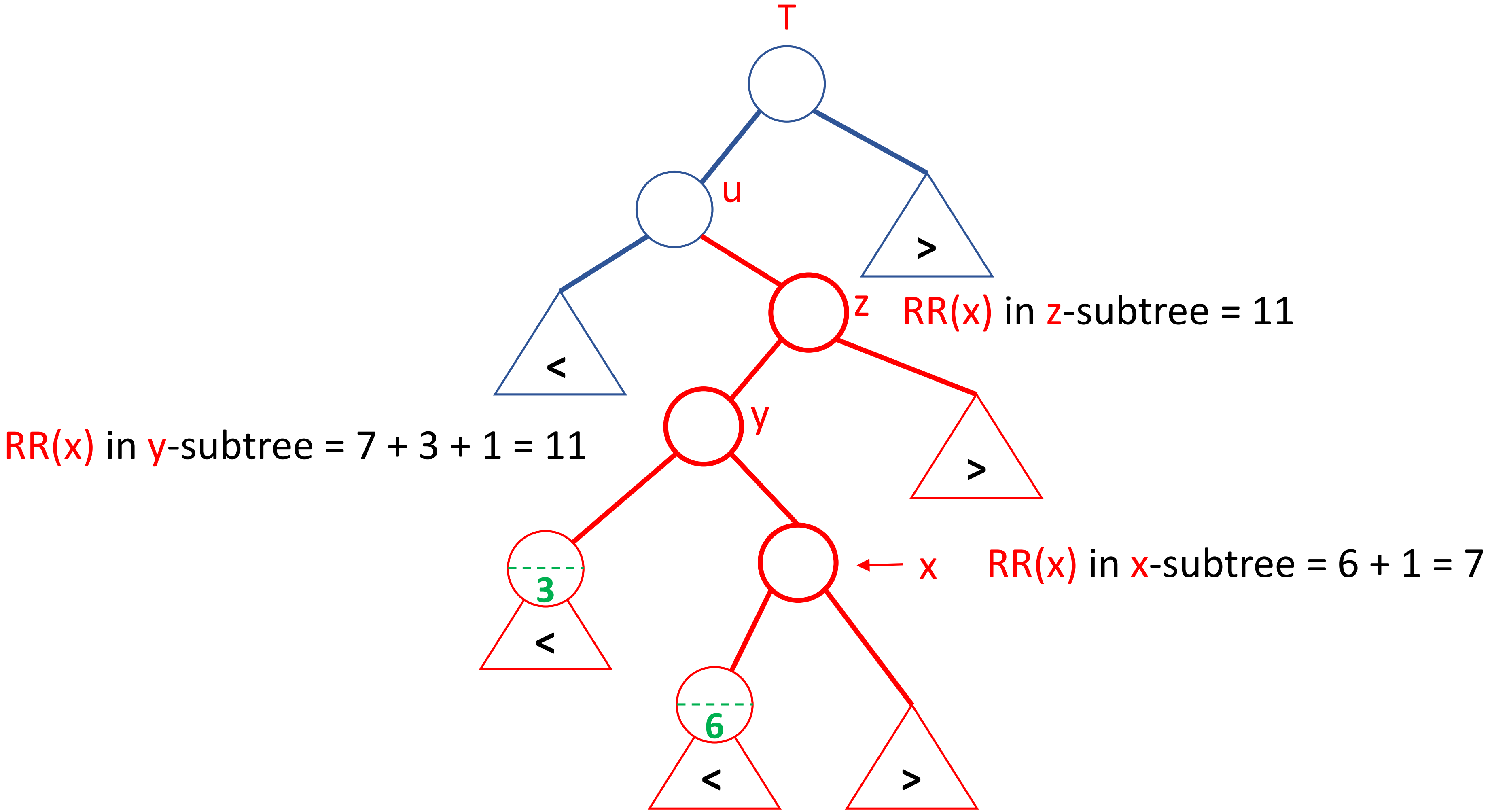
Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T

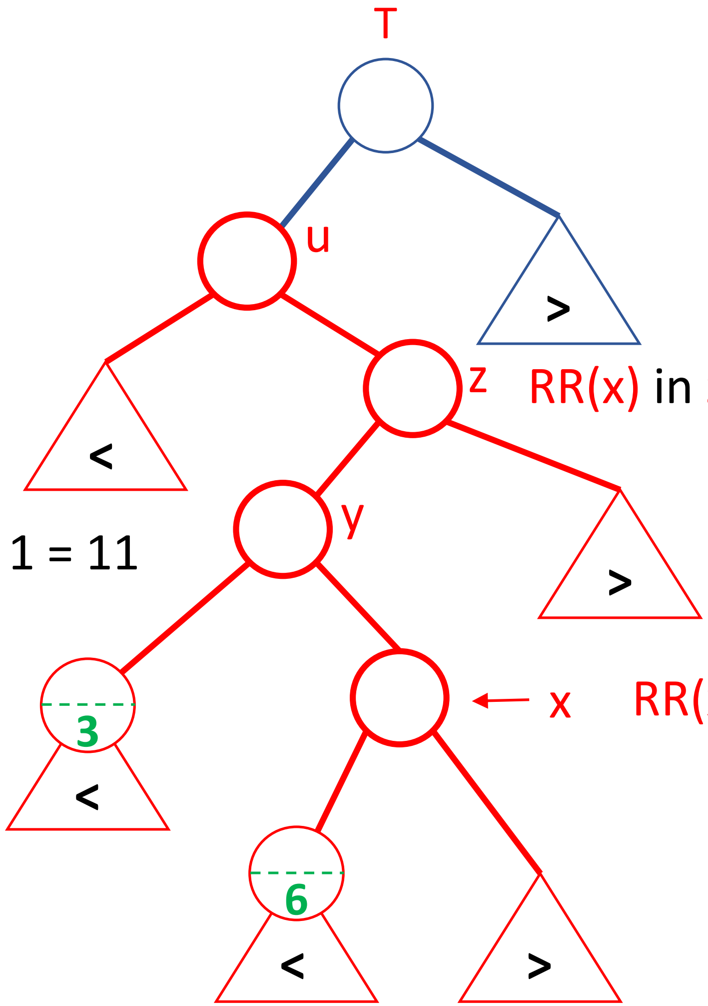


Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T

RR(x) in u-subtree =

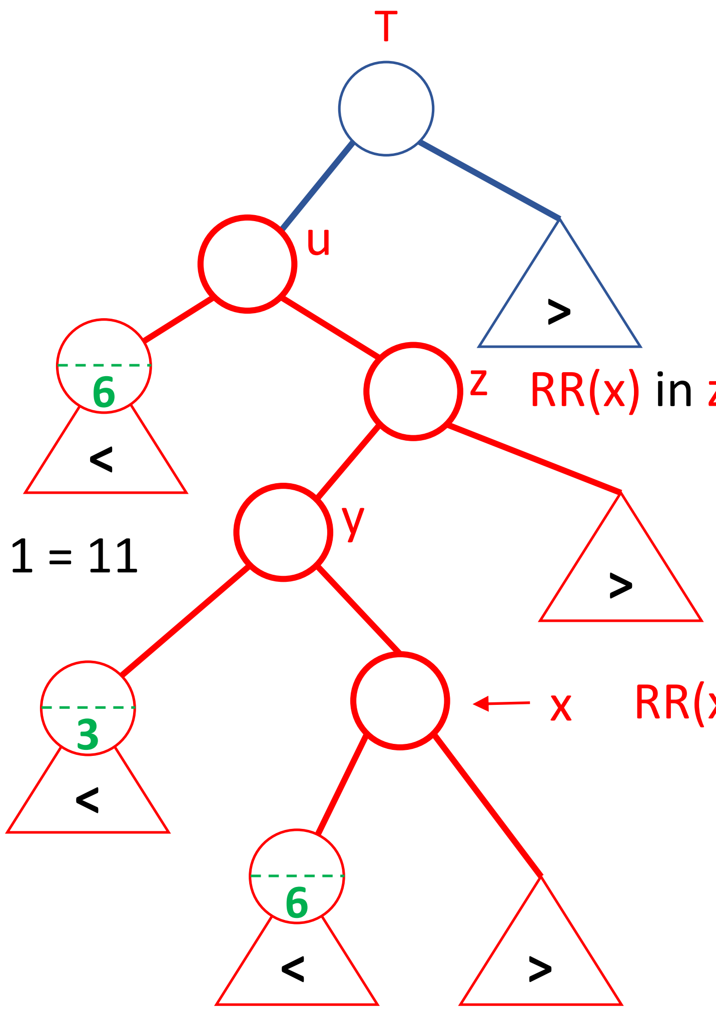


RR(x) in z-subtree = 11

RR(x) in y-subtree = 7 + 3 + 1 = 11

RR(x) in x-subtree = 6 + 1 = 7

Rank(T, x): return rank of x in T



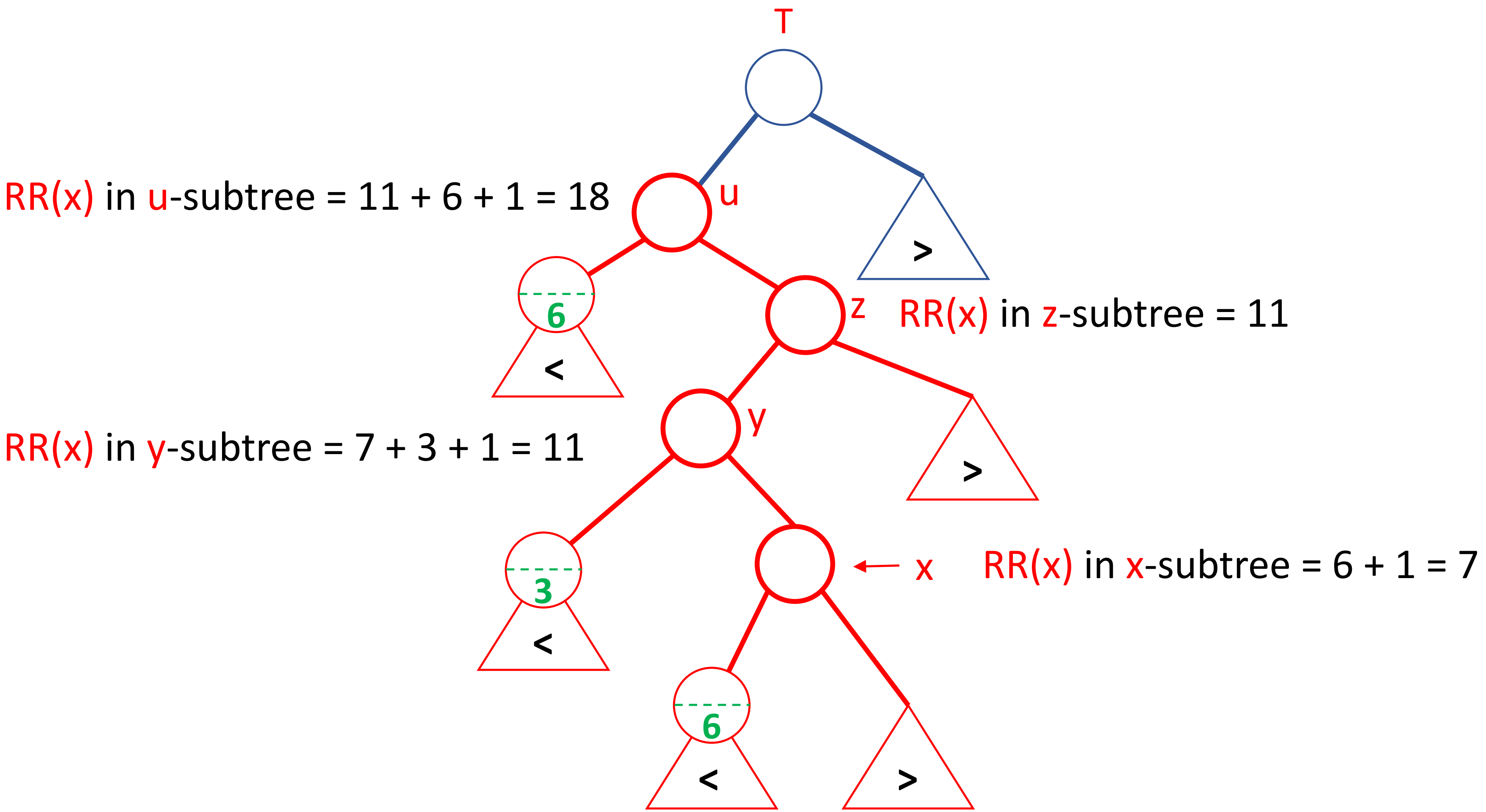
RR(x) in u-subtree =

RR(x) in z-subtree = 11

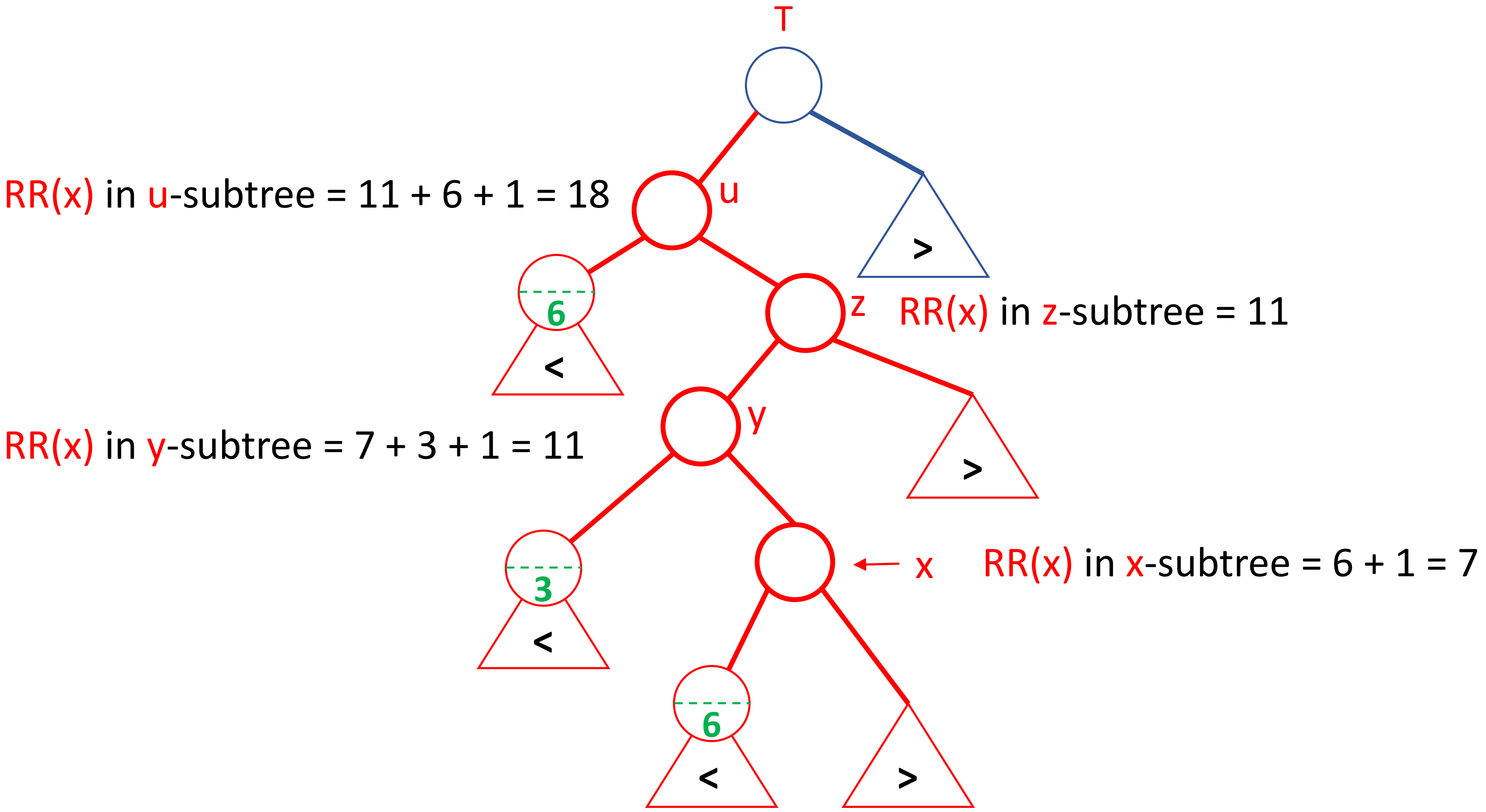
RR(x) in y-subtree = 7 + 3 + 1 = 11

RR(x) in x-subtree = 6 + 1 = 7

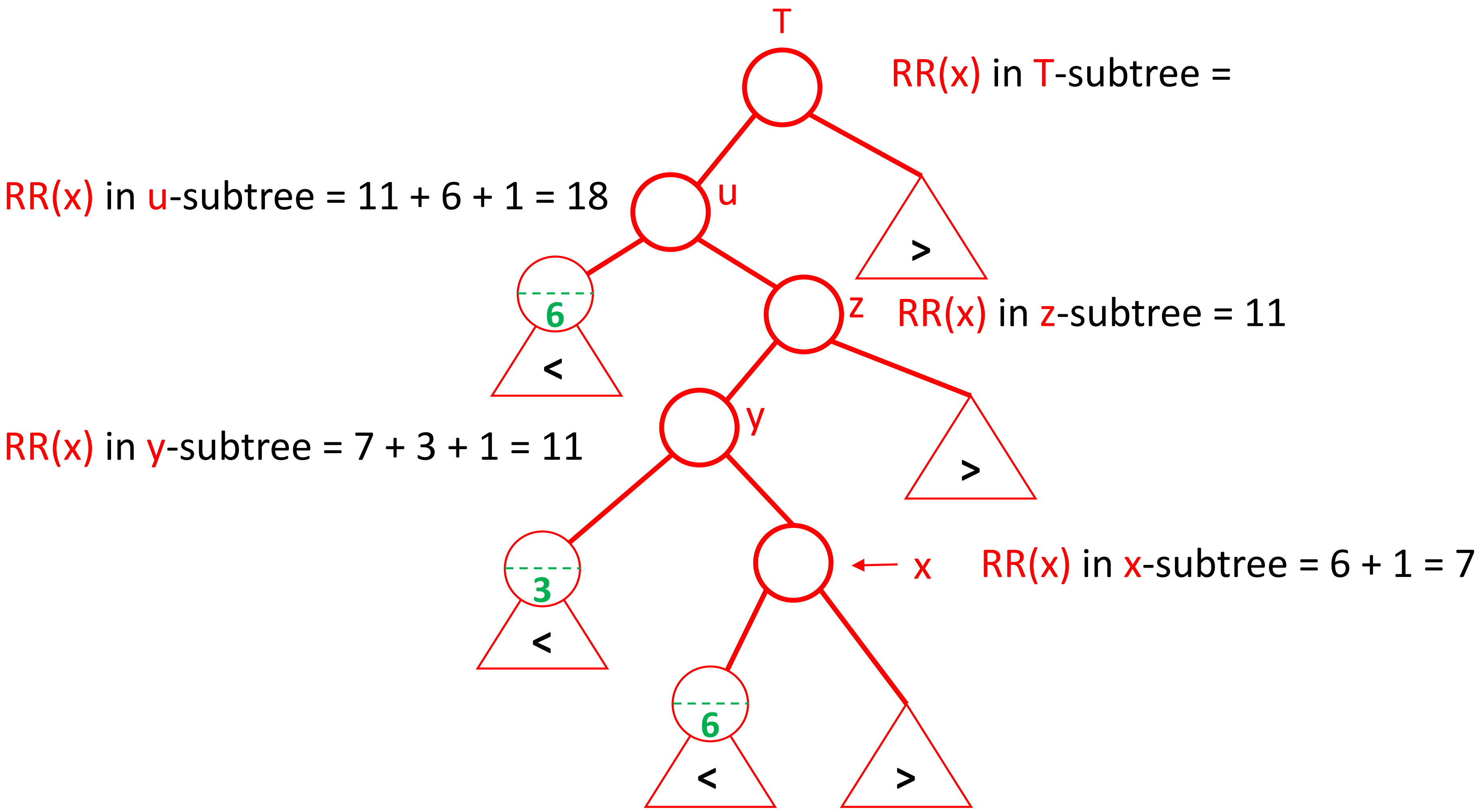
Rank(T, x): return rank of x in T



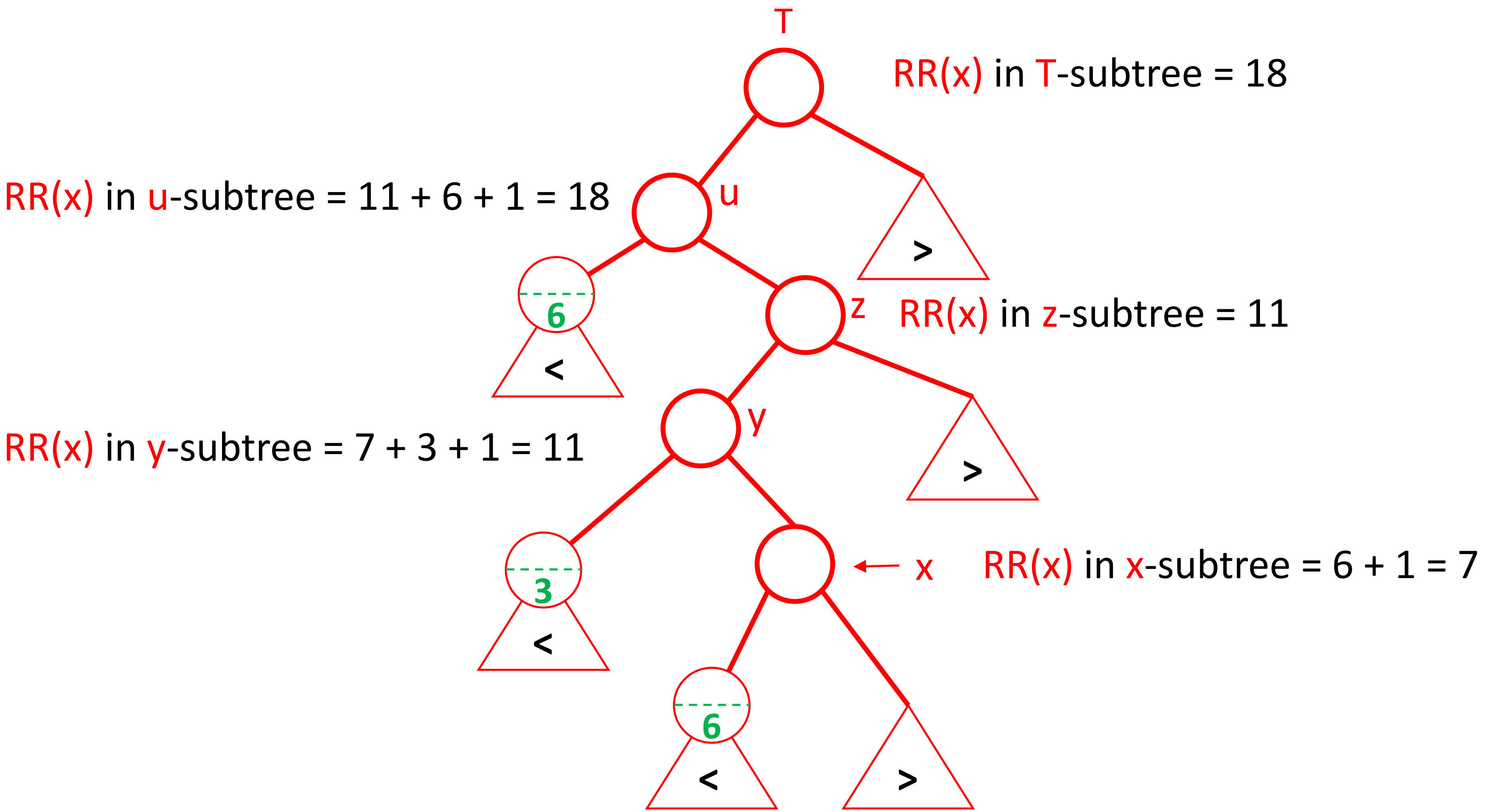
Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T



Rank(T, x): return rank of x in T

- Find the rank of x in x-subtree:

  RR(x) ← size(left(x)) + 1

Rank(T, x): return rank of x in T

- Find the rank of x in x-subtree:

  RR(x) ← size(left(x)) + 1

- For each node y in path x to root of T :

  Compute rank of x in y-subtree as shown in previous example

Rank(T, x): return rank of x in T

- Find the rank of x in x-subtree:

  RR(x) ← size(left(x)) + 1

- For each node y in path x to root of T :

  Compute rank of x in y-subtree as shown in previous example

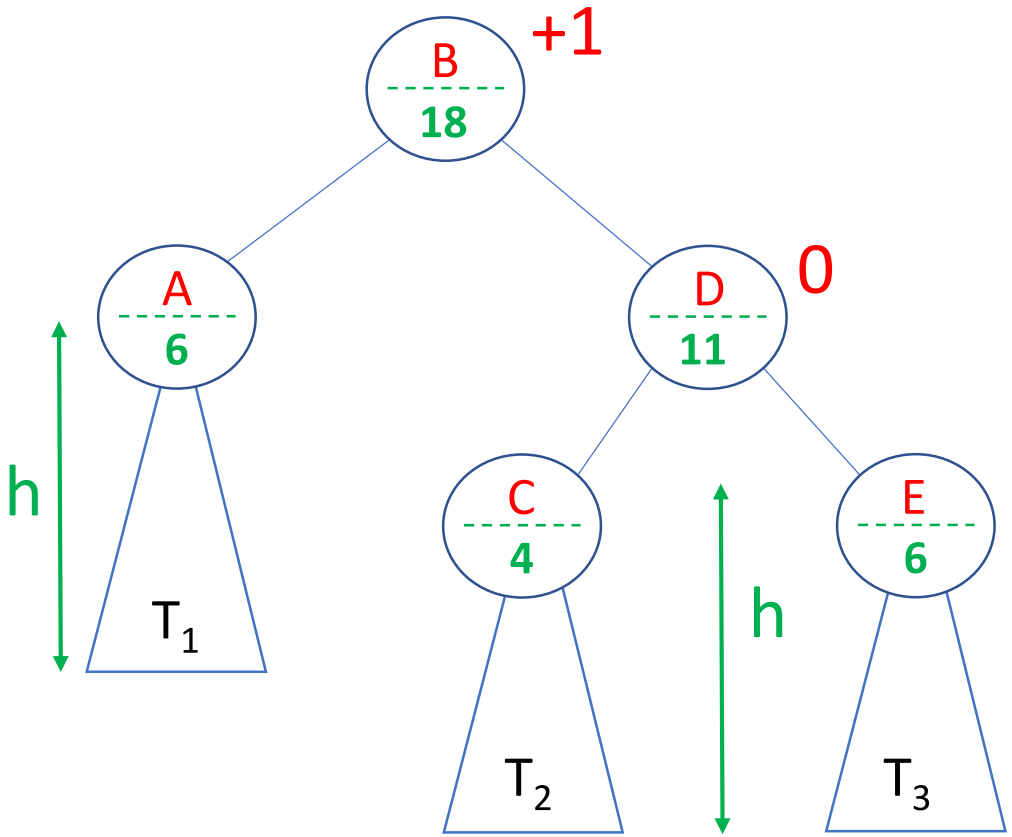
Worst-Case Time Complexity of **Rank**(T, k):

- Constant time for each level of T
- Height(T) is O(log n)
- Hence **Rank** takes O(log n)

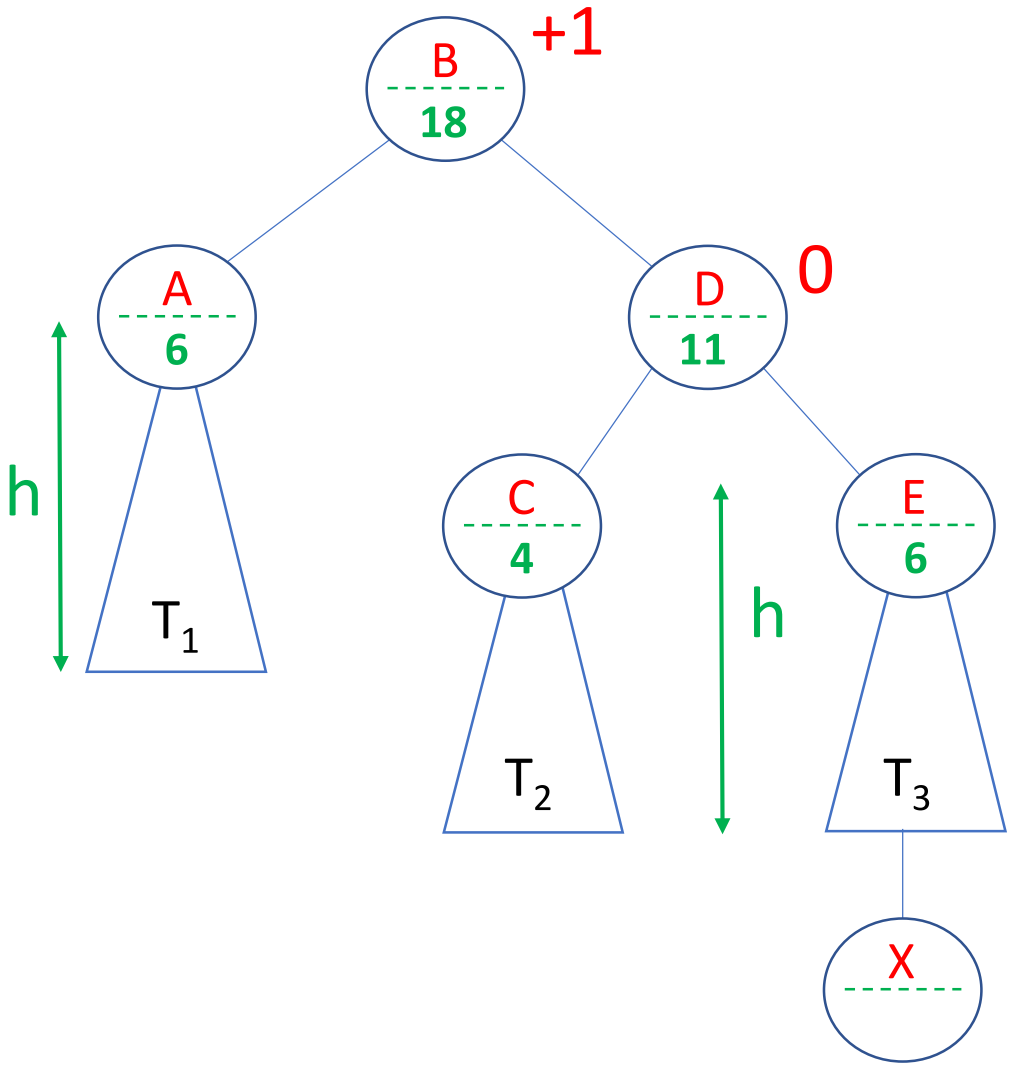
Augmenting AVL

- **Select** operation ✓
- **Rank** operation ✓
- Maintain size() field
  after **Insert or Delete**

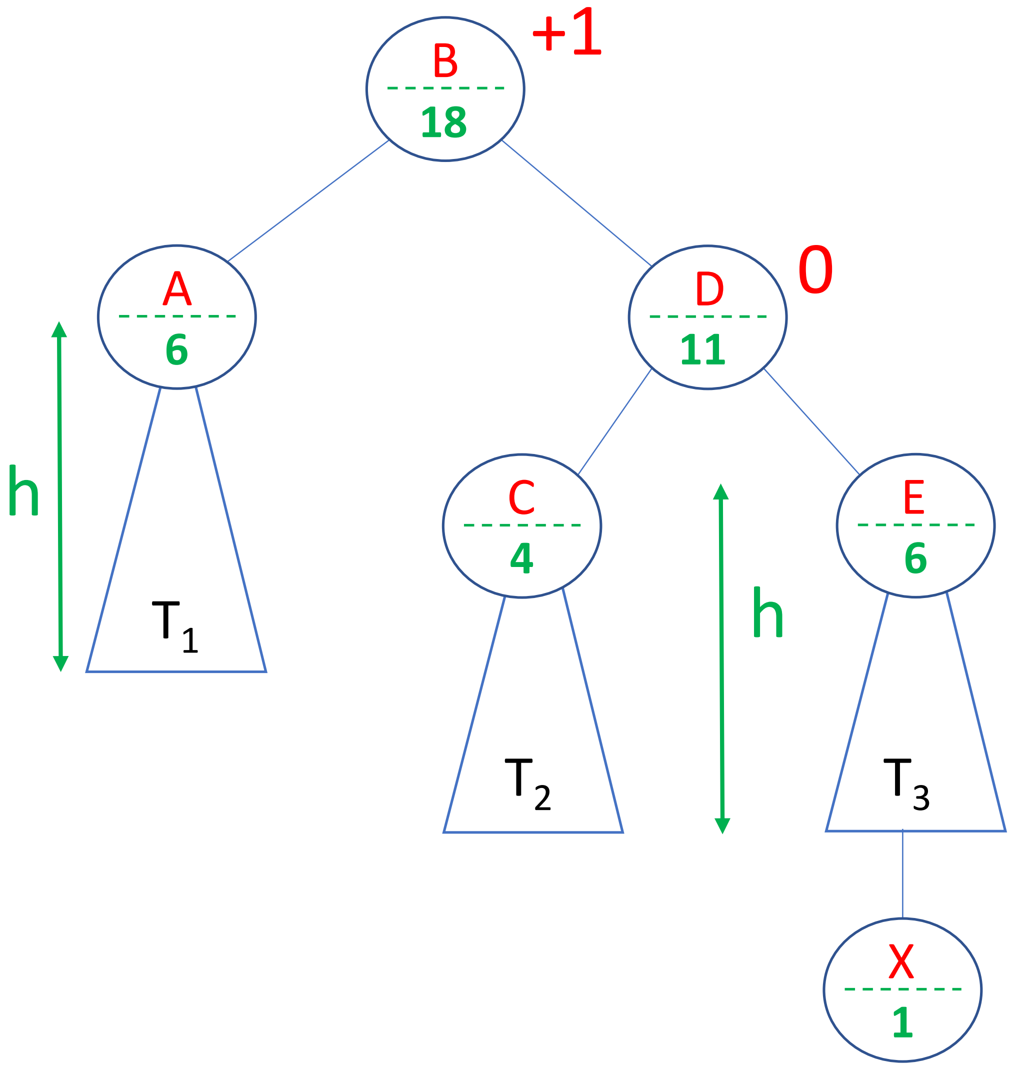
Maintaining size() field: Insert Example



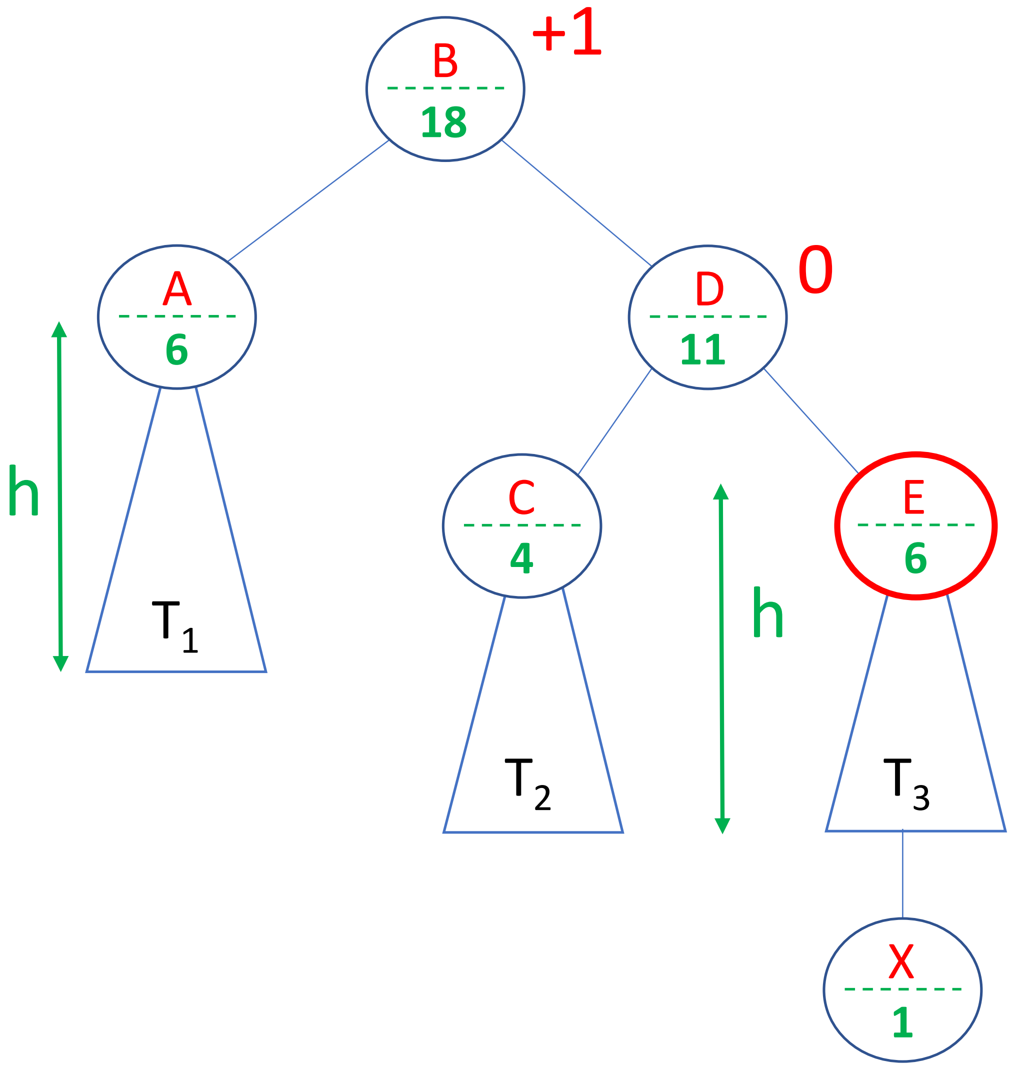
Maintaining size() field: Insert Example



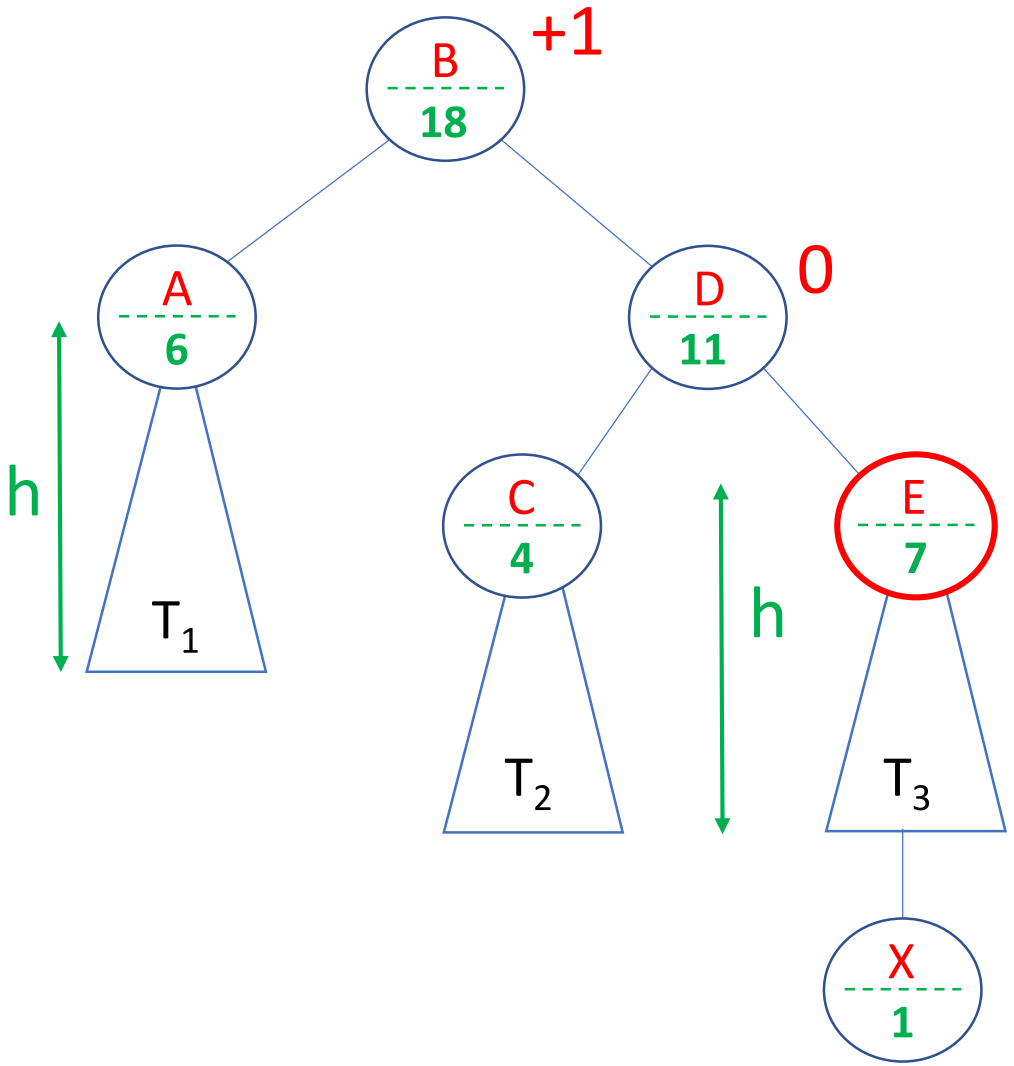
Maintaining size() field: Insert Example



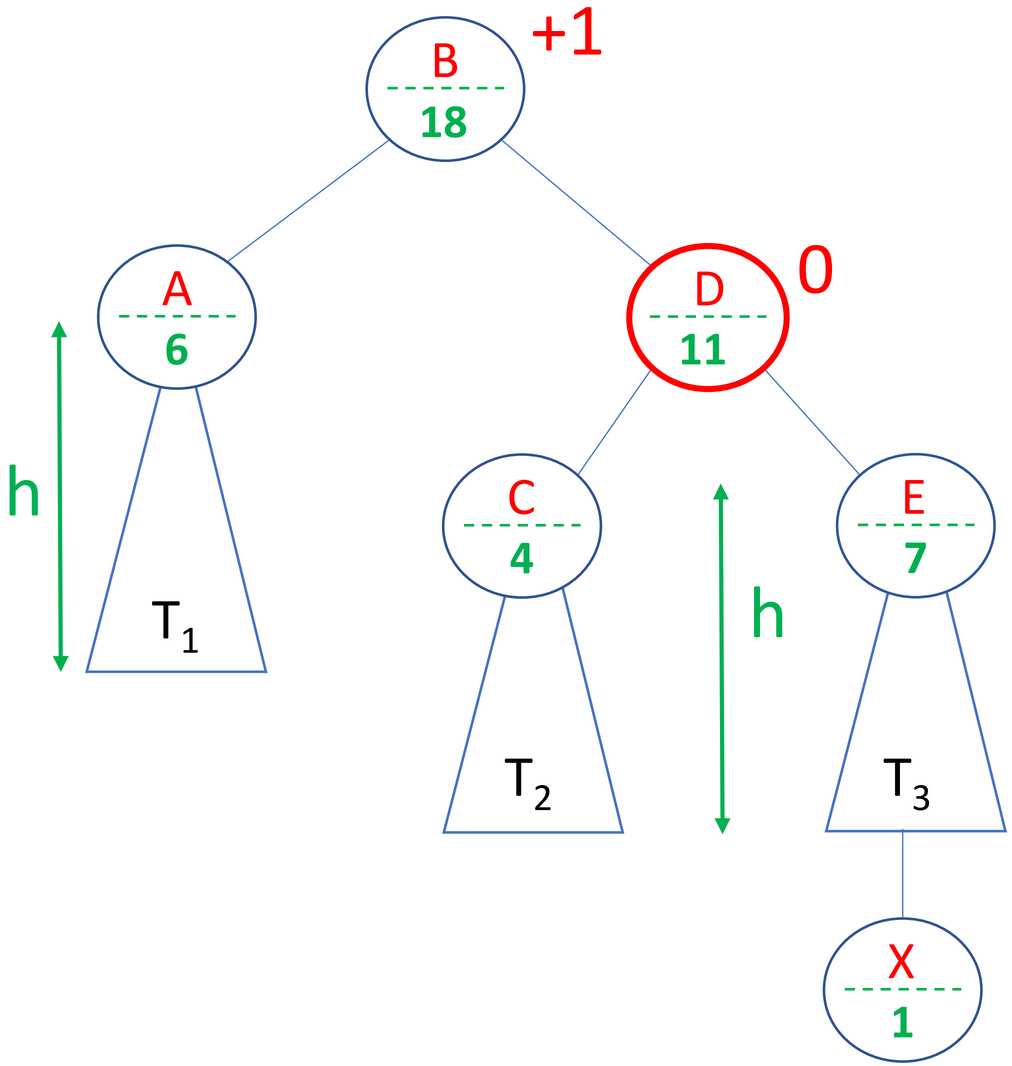
Maintaining size() field: Insert Example



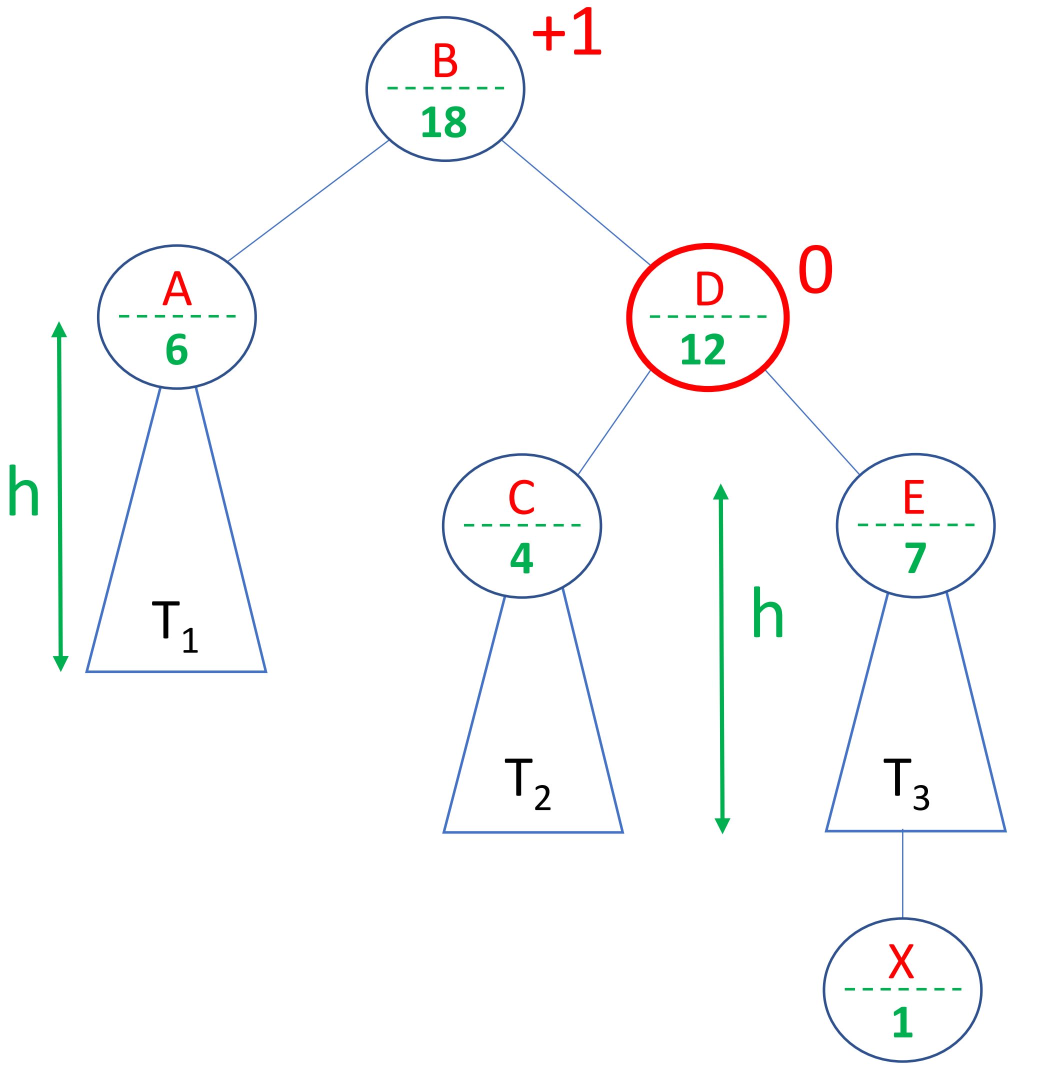
Maintaining size() field: Insert Example



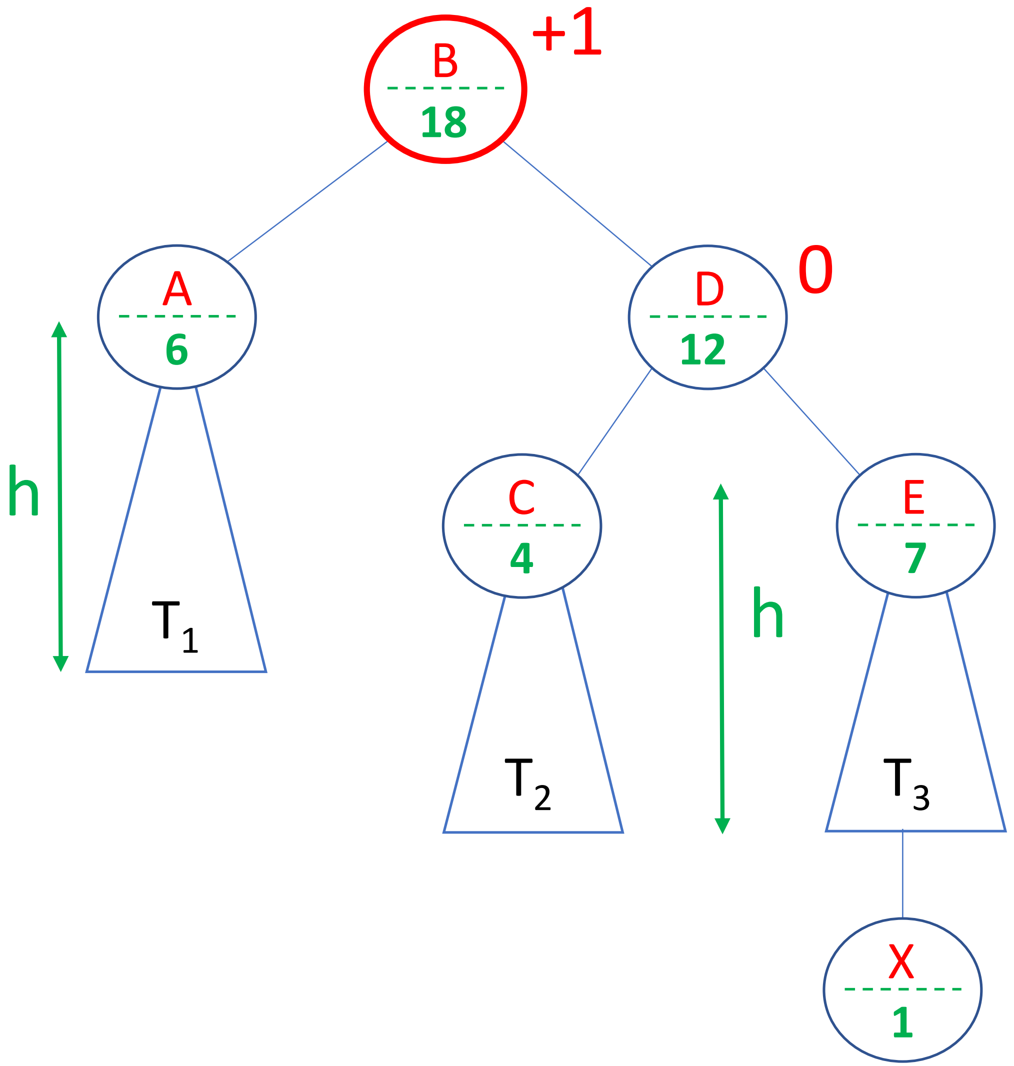
Maintaining size() field: Insert Example



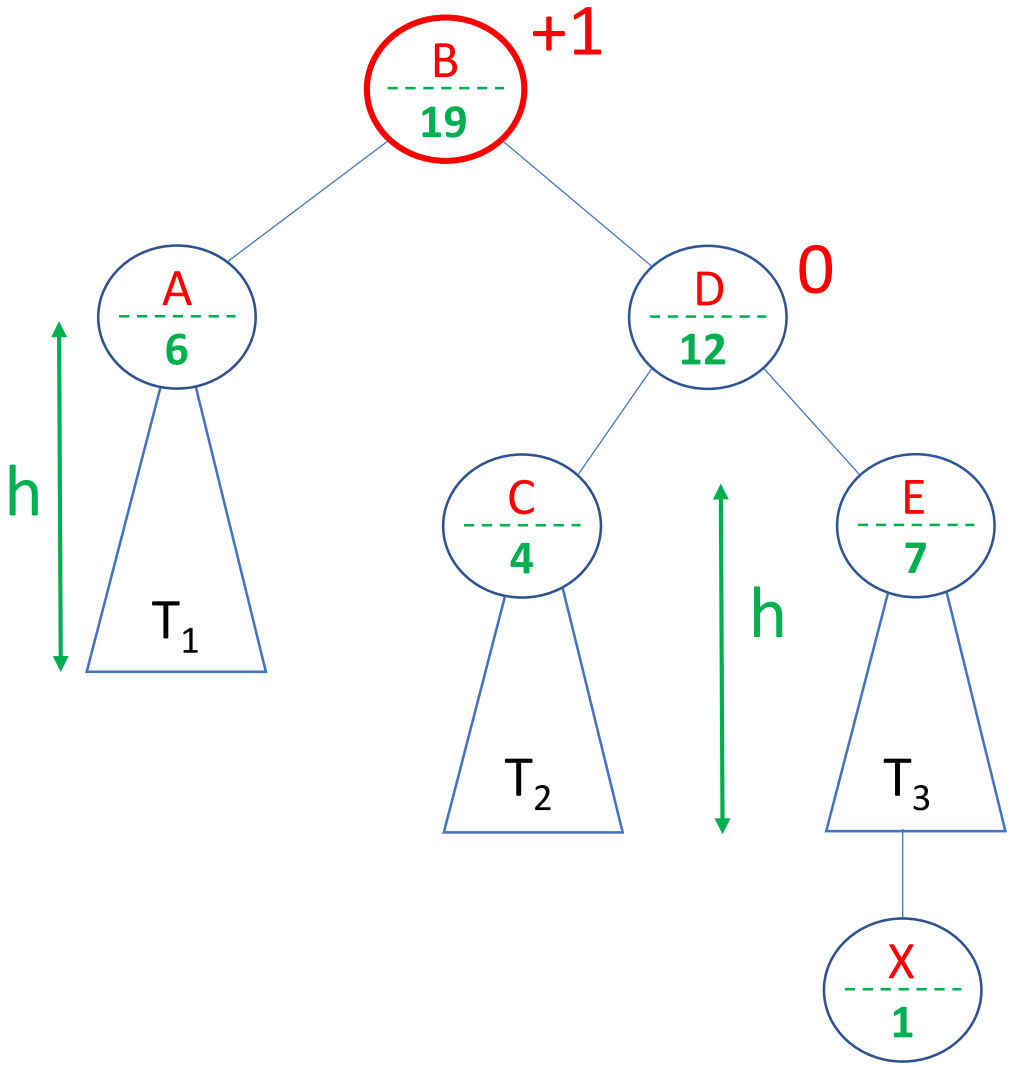
Maintaining size() field: Insert Example



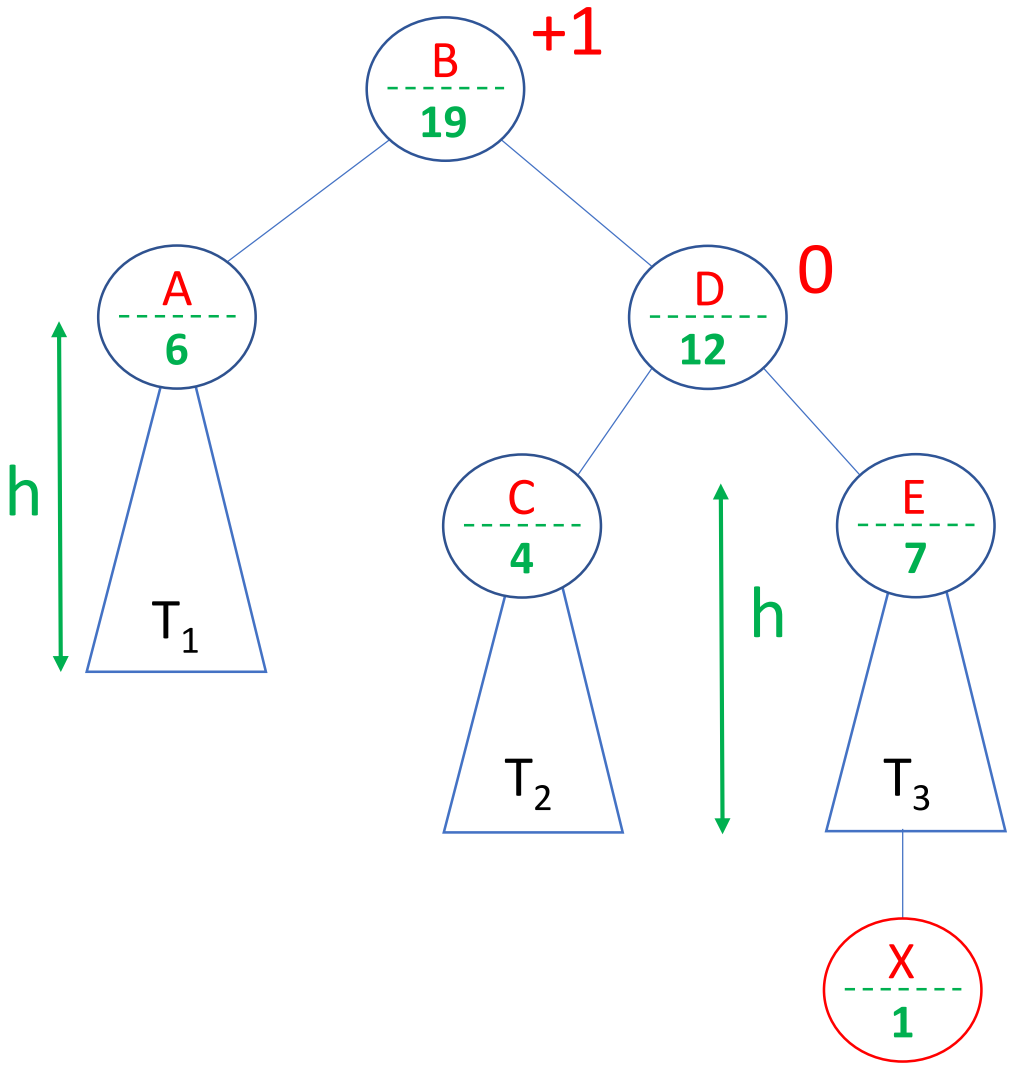
Maintaining size() field: Insert Example



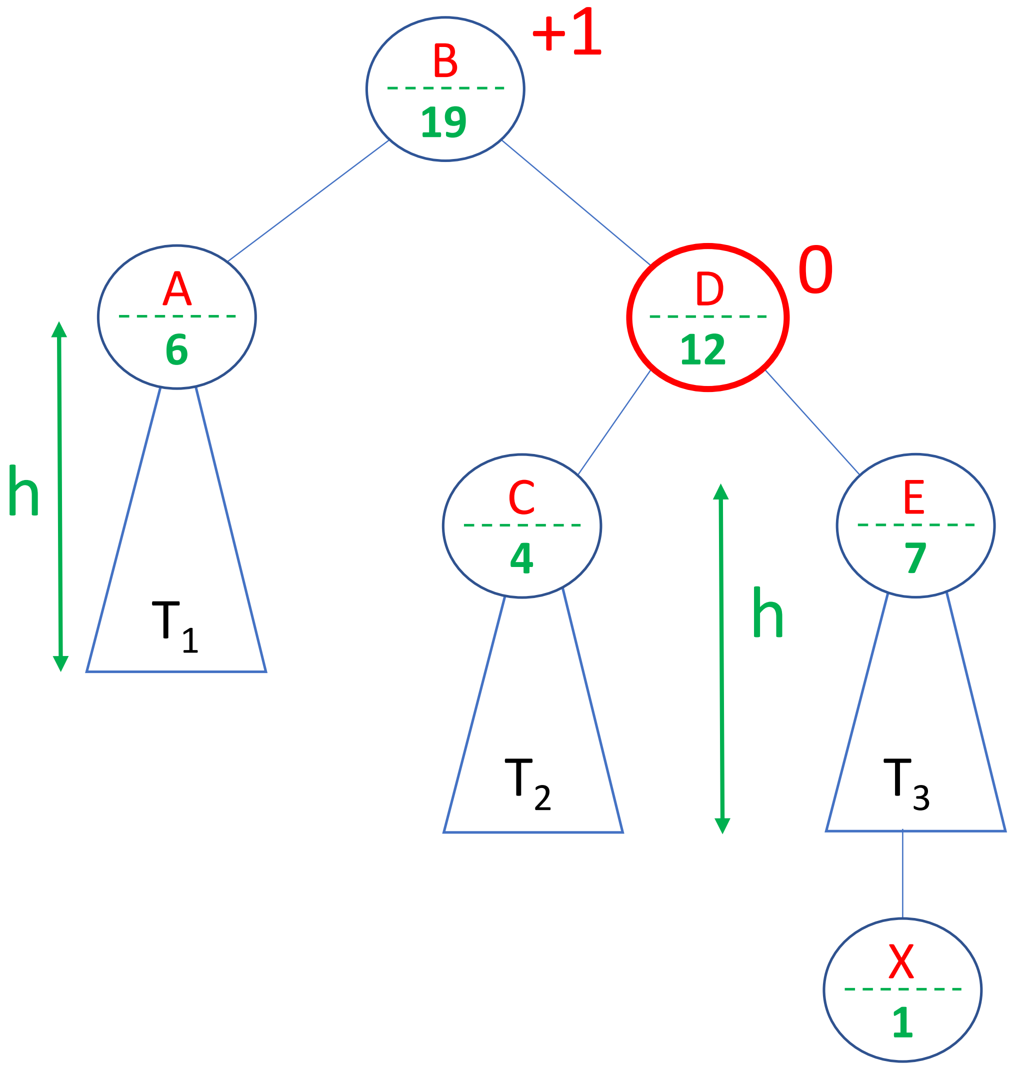
Maintaining size() field: Insert Example



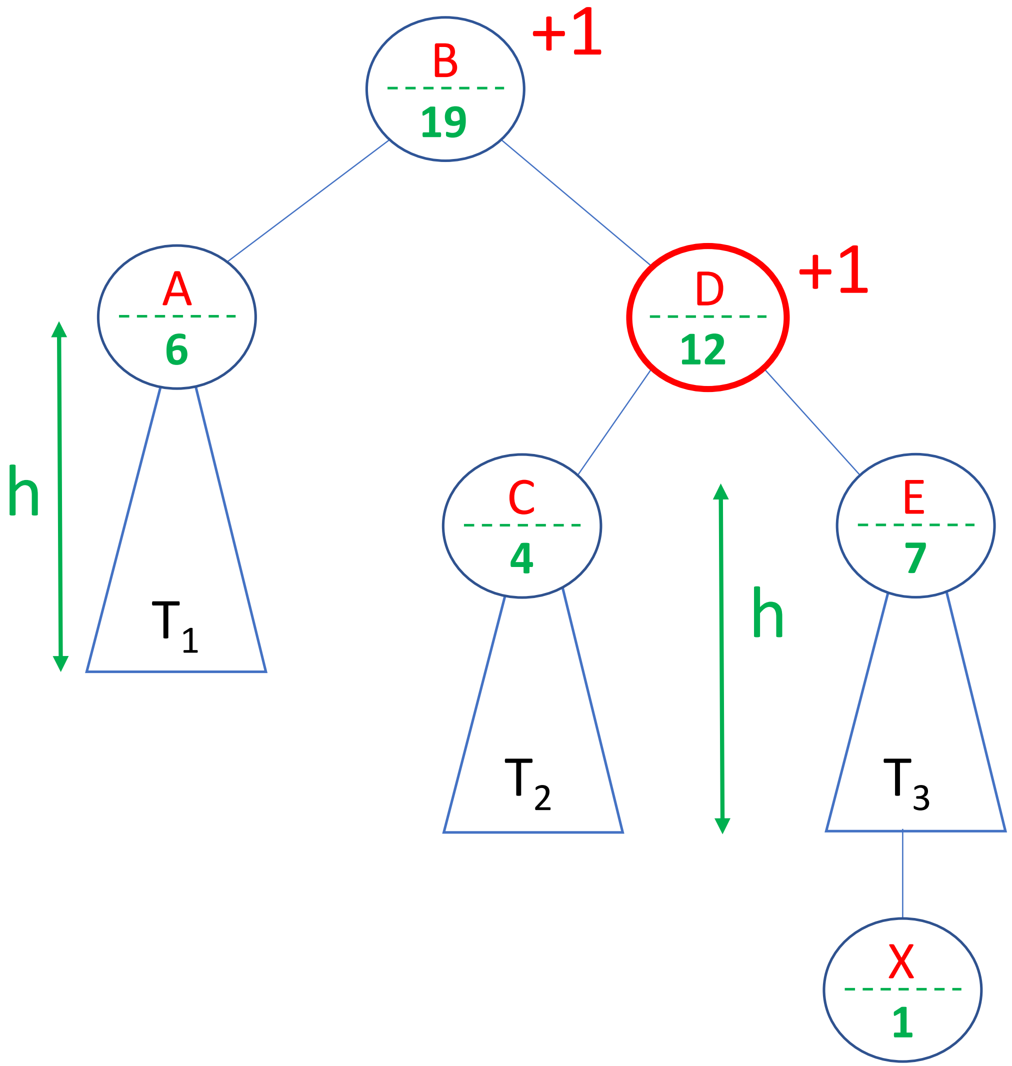
Maintaining size() field: Insert Example



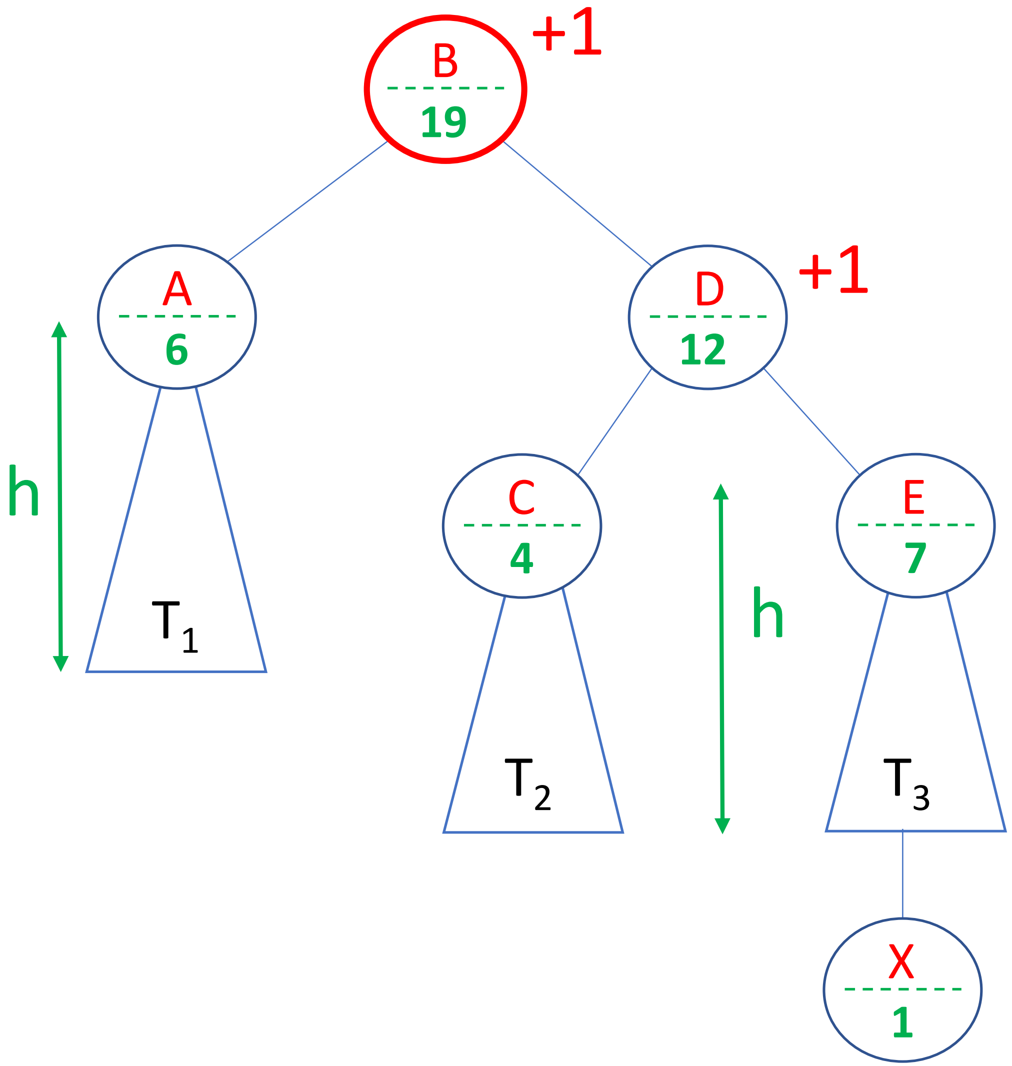
Maintaining size() field: Insert Example



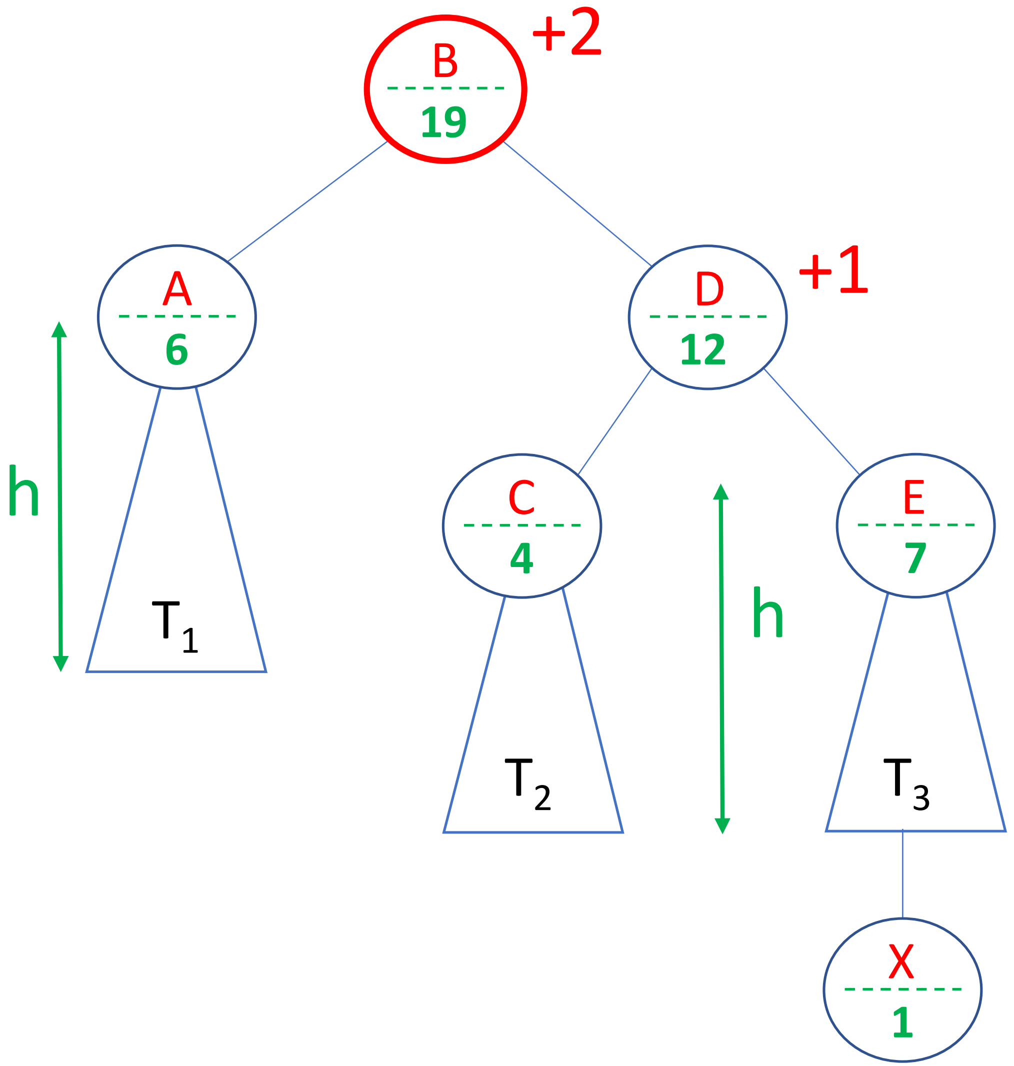
Maintaining size() field: Insert Example



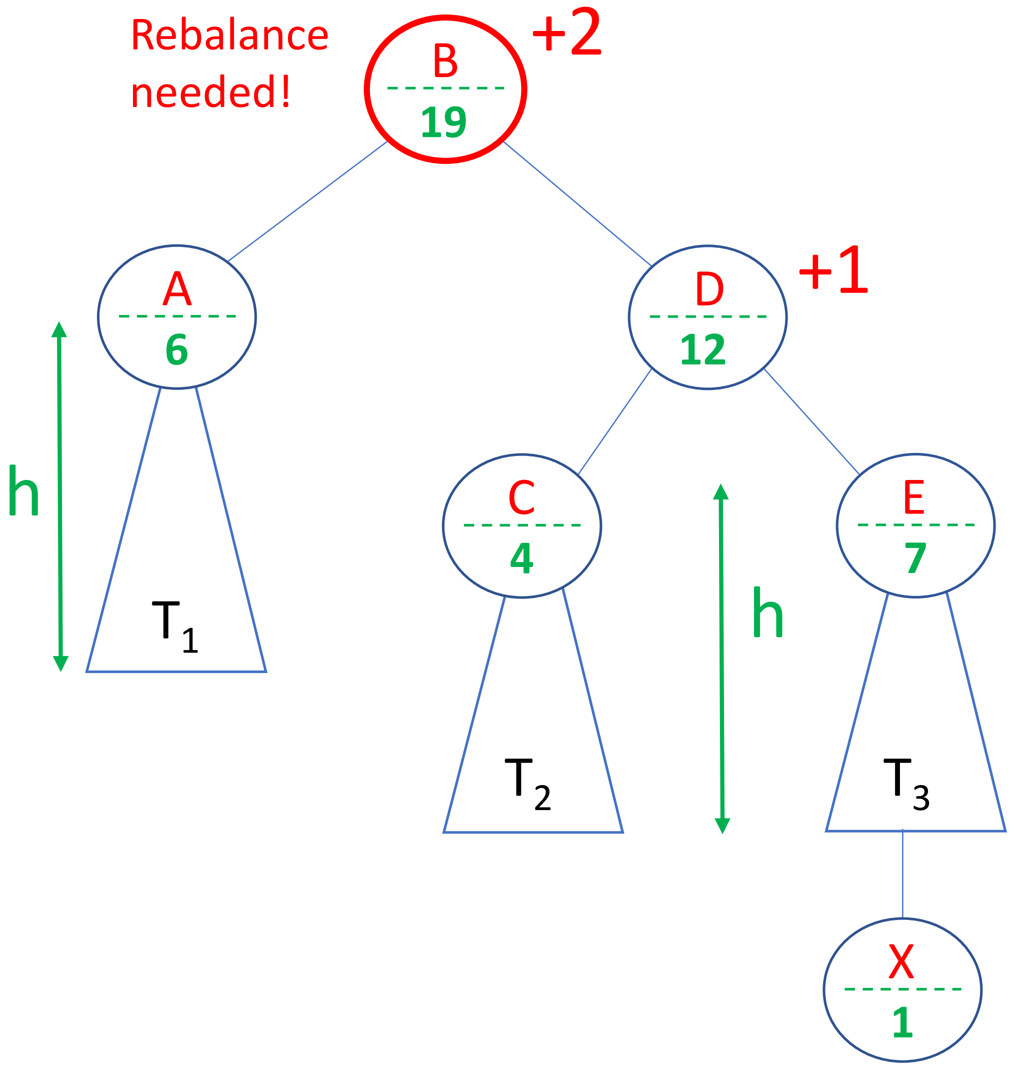
Maintaining size() field: Insert Example



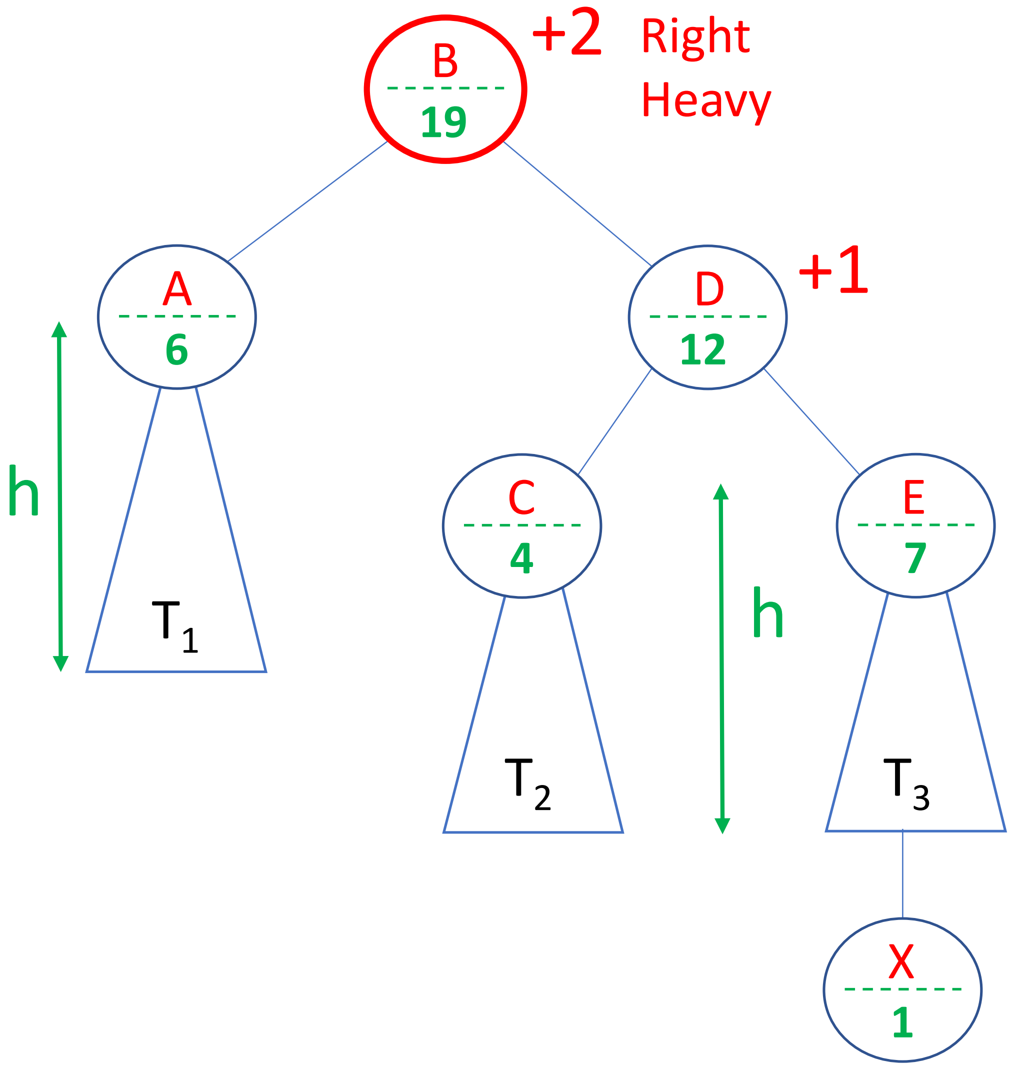
Maintaining size() field: Insert Example



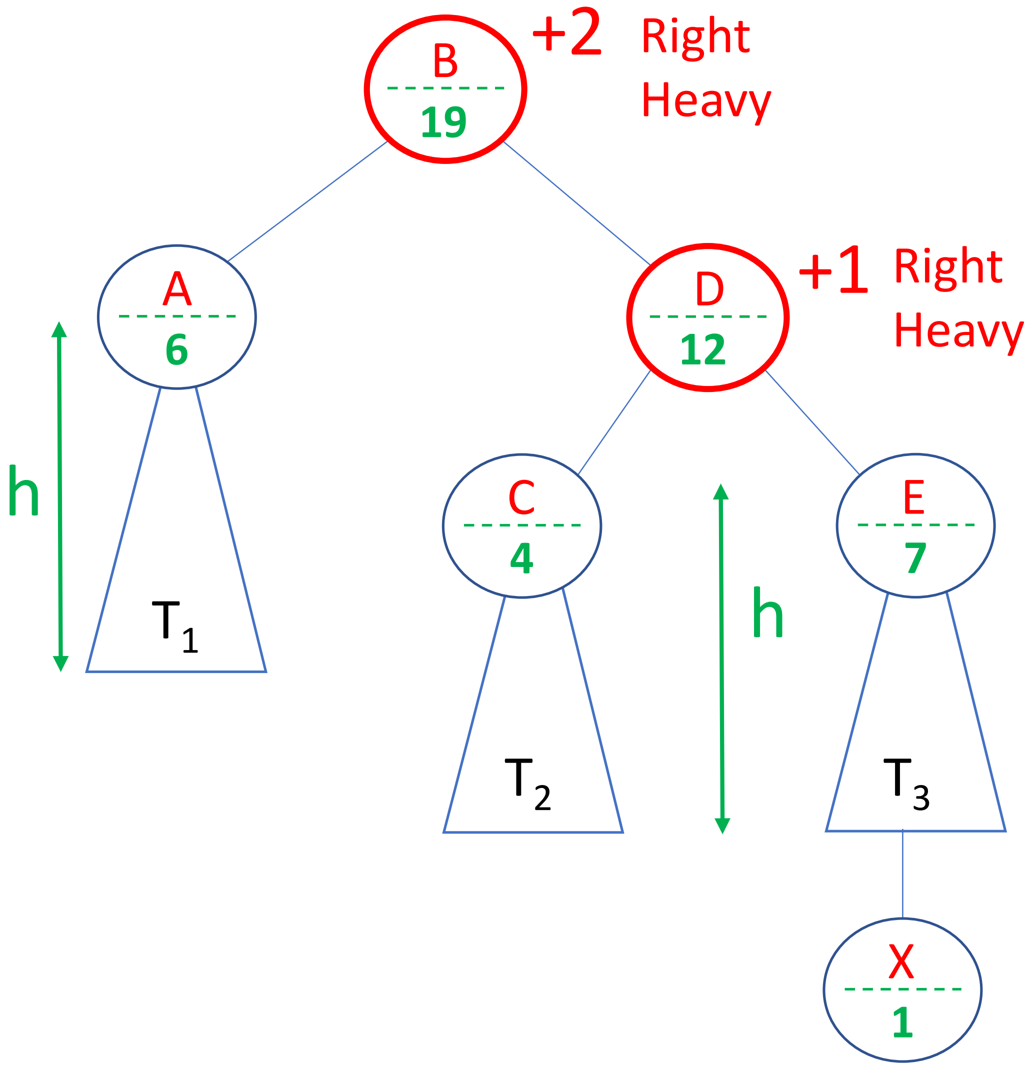
Maintaining size() field: Insert Example



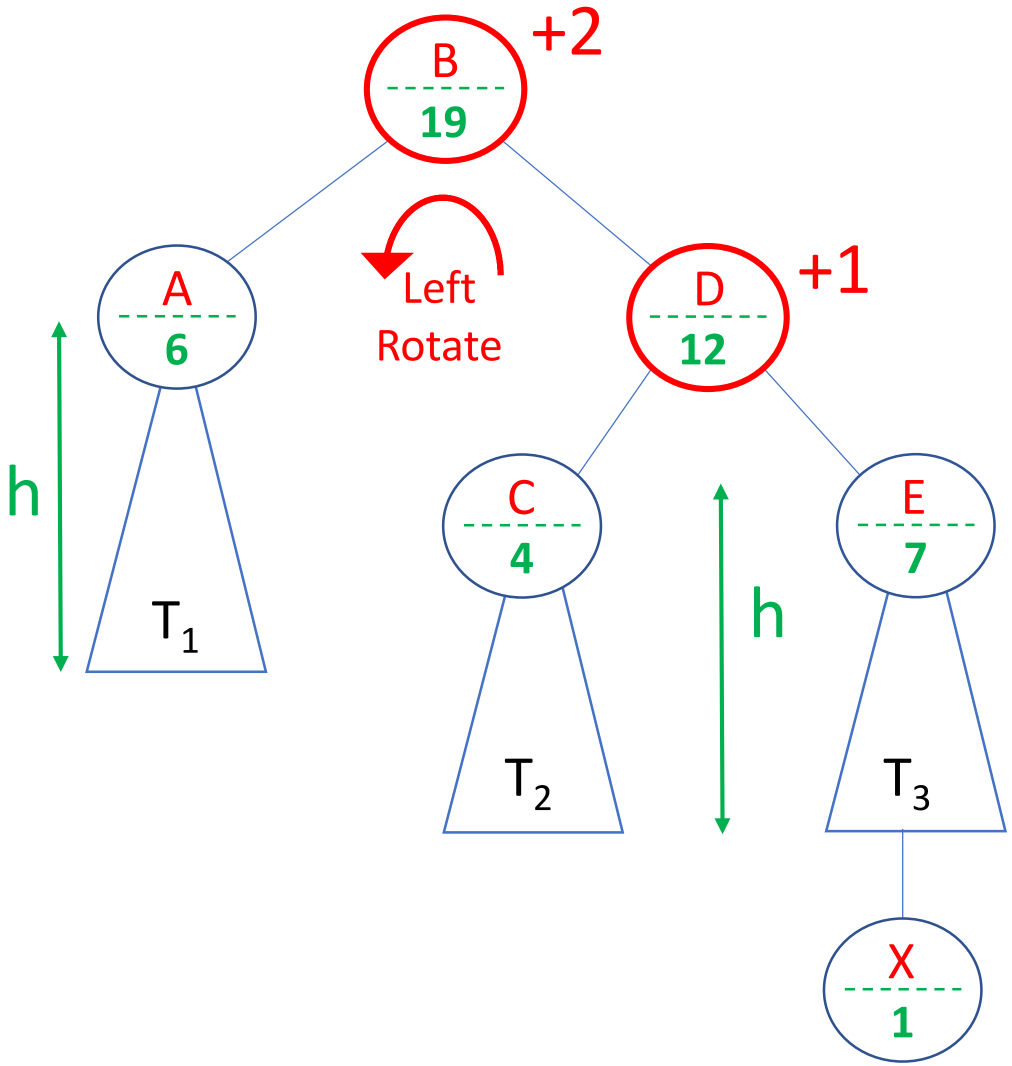
Maintaining size() field: Insert Example



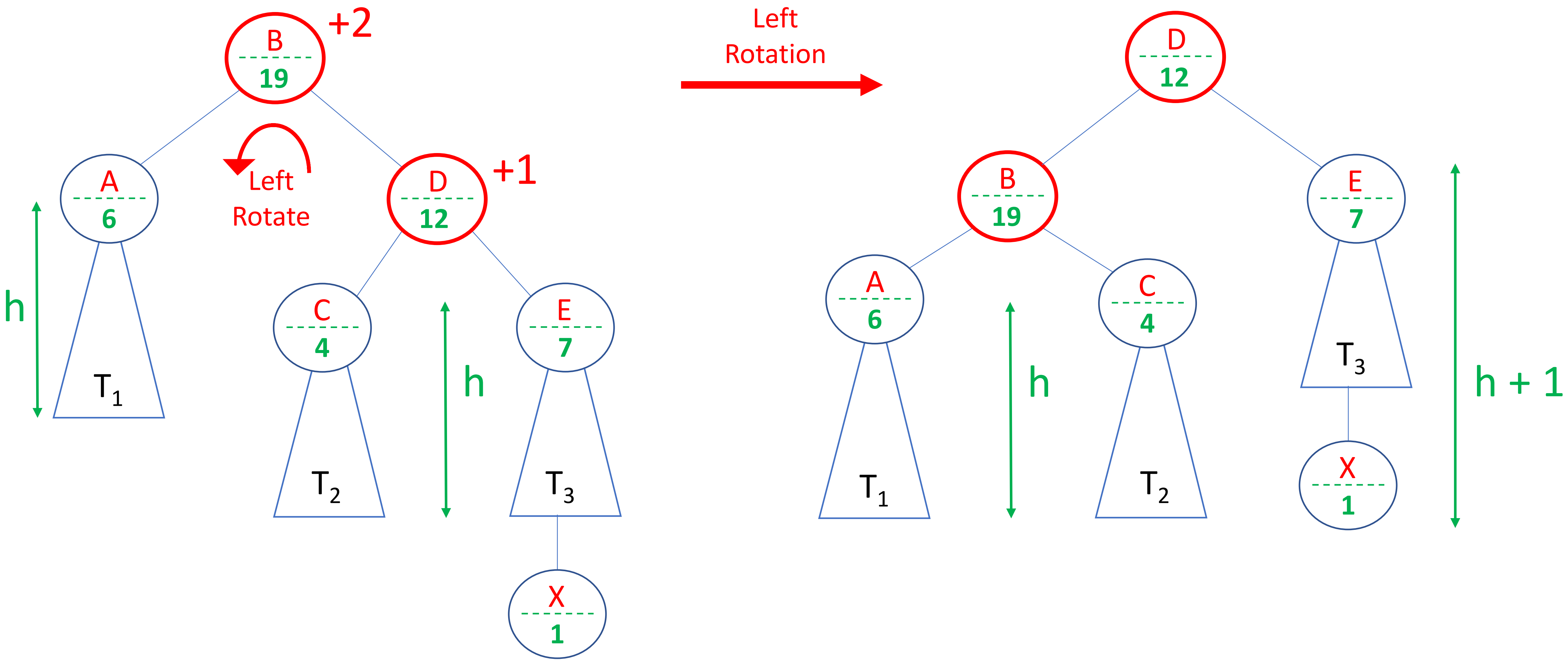
Maintaining size() field: Insert Example



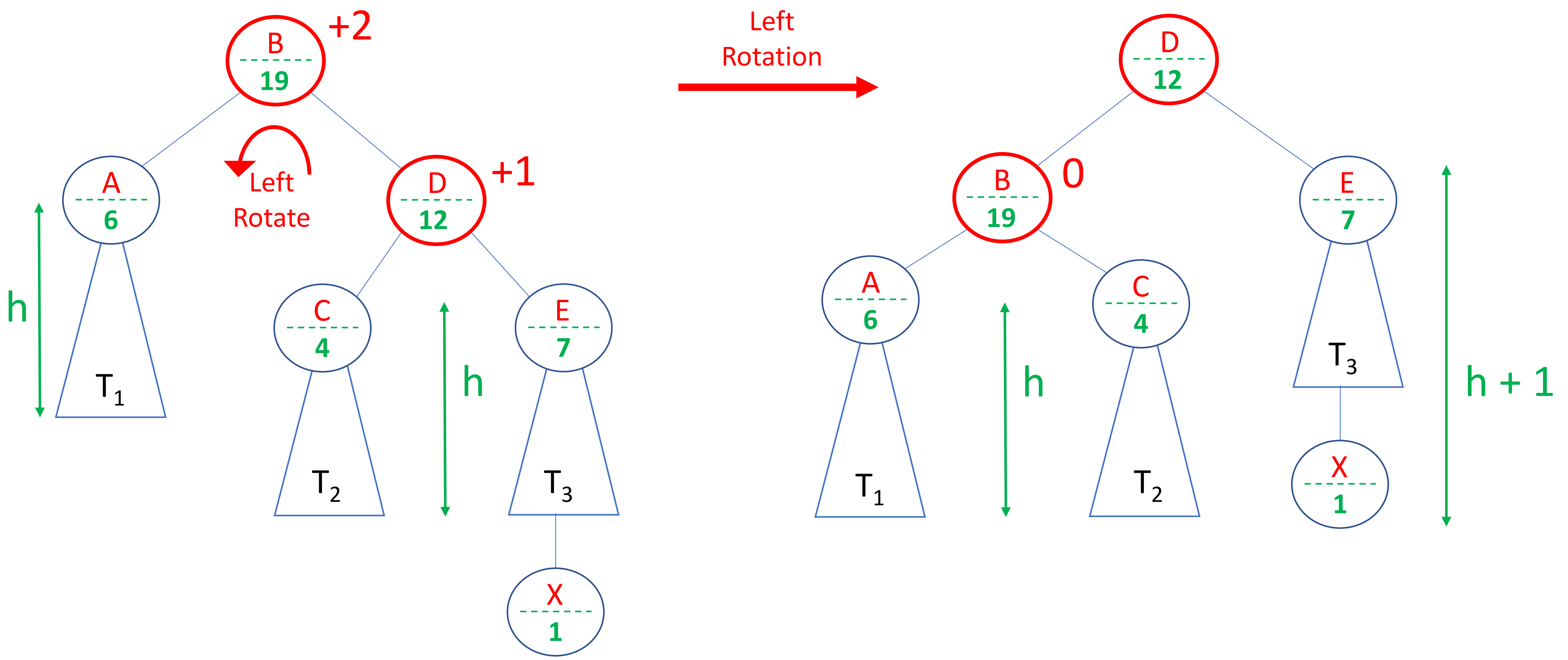
Maintaining size() field: Insert Example



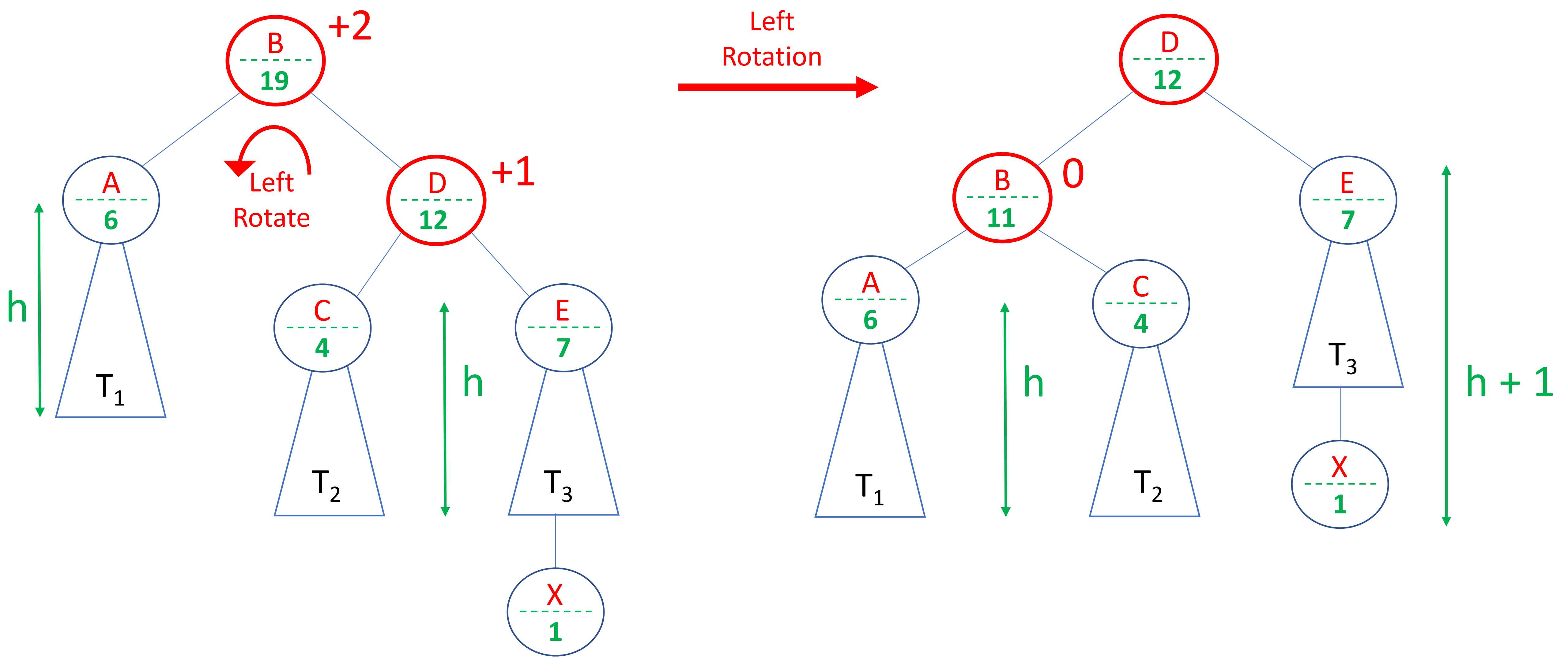
Maintaining size() field: Insert Example



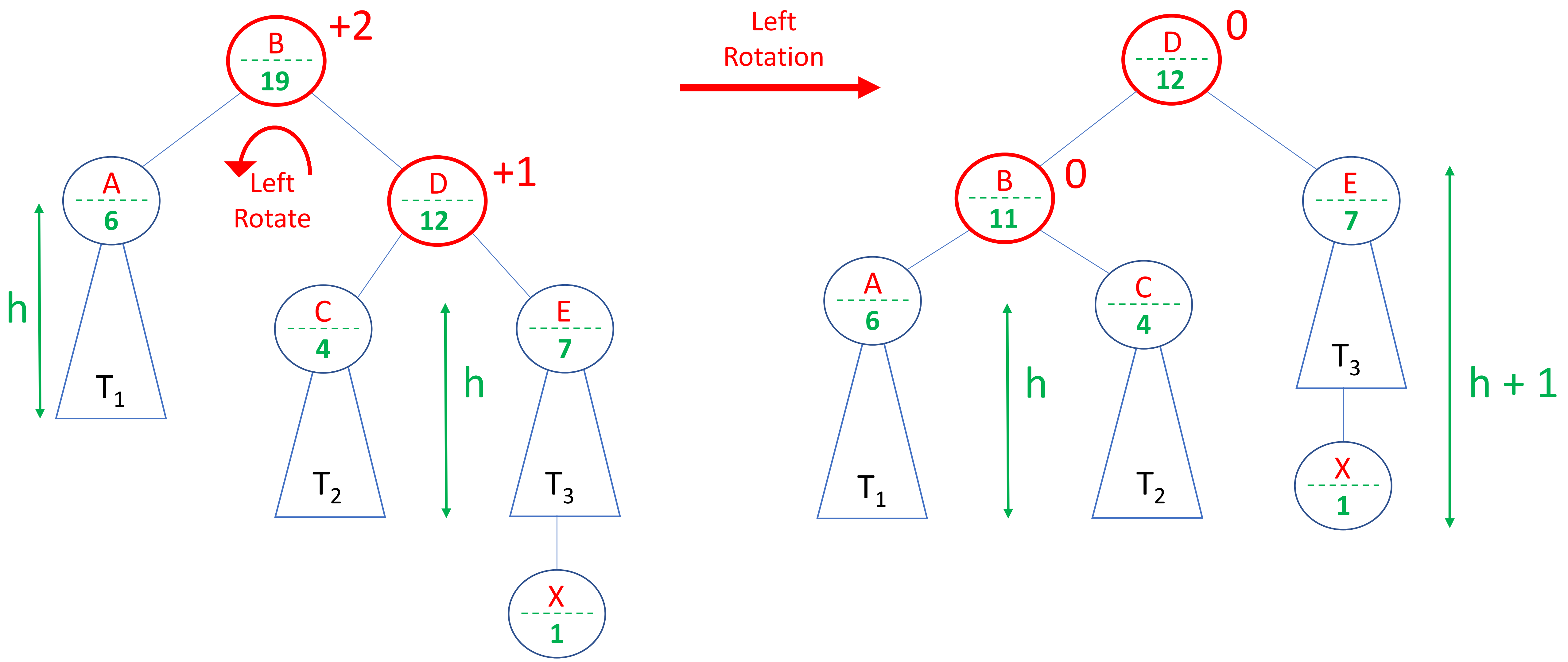
Maintaining size() field: Insert Example



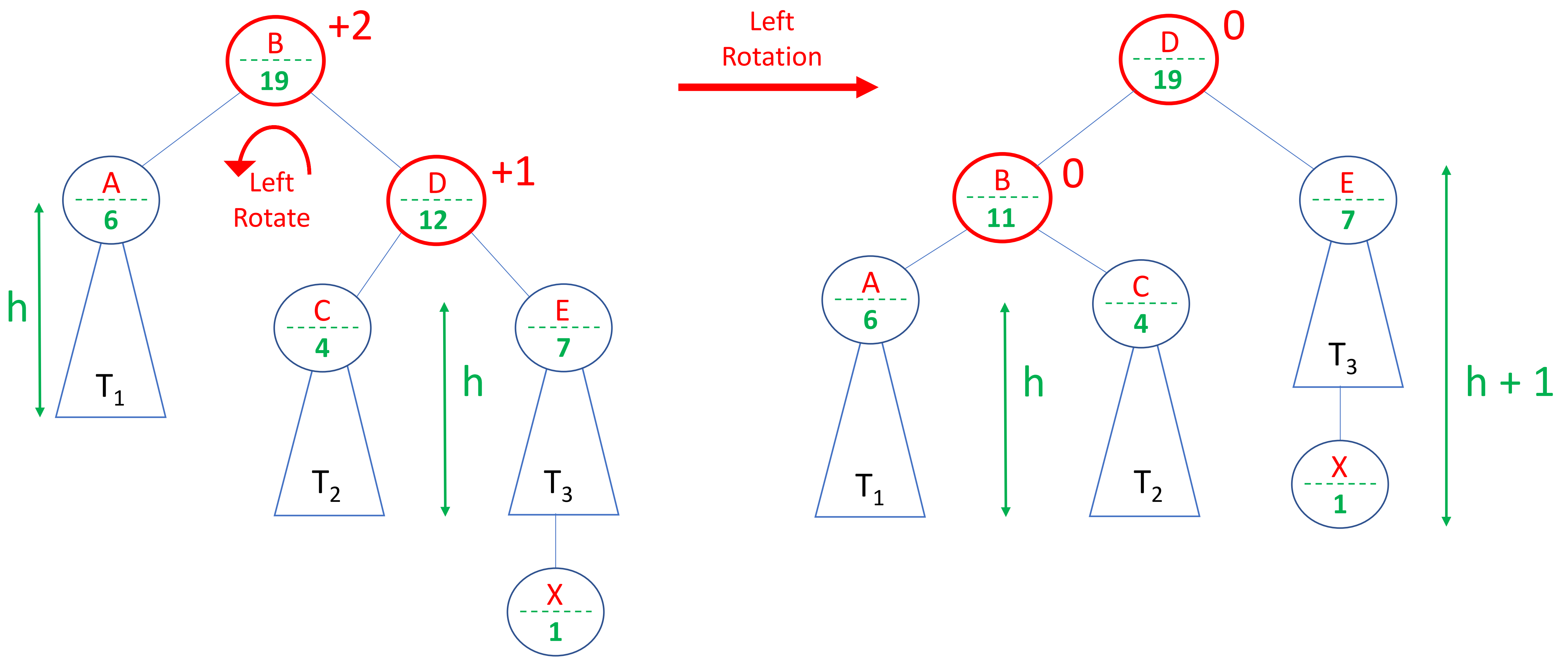
Maintaining size() field: Insert Example



Maintaining size() field: Insert Example



Maintaining size() field: Insert Example

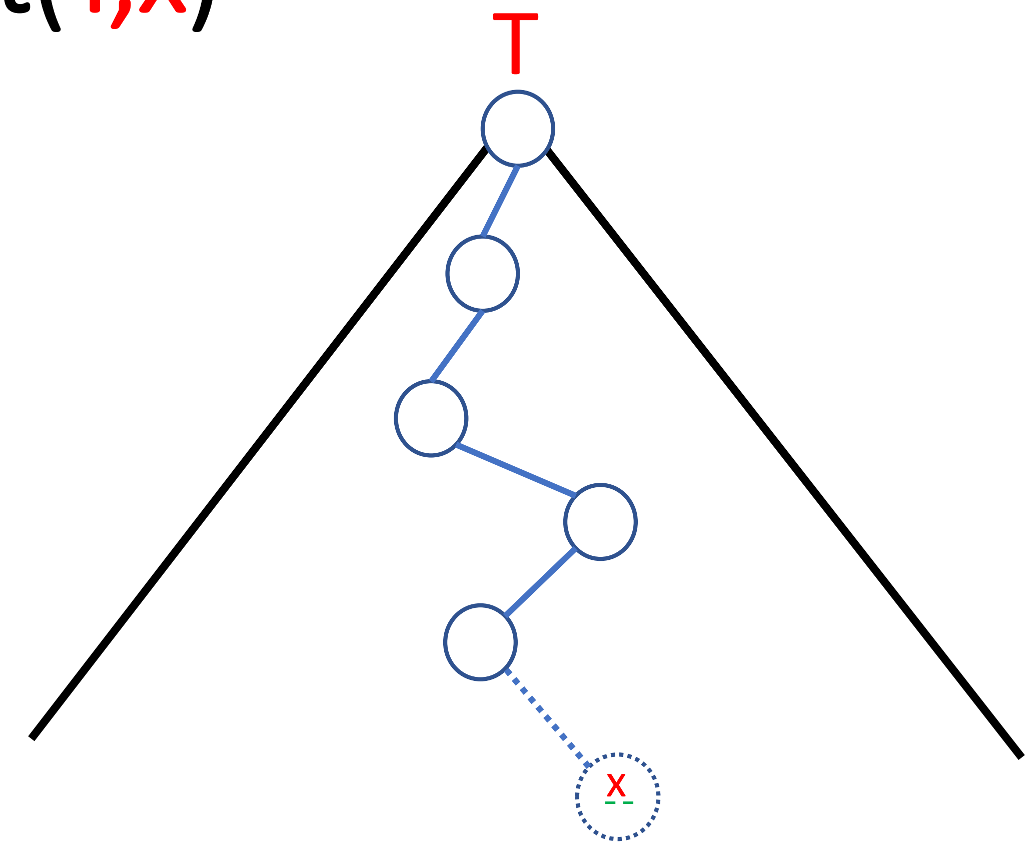


Maintaining size() field: Insert(T,x)

Maintaining size() field: Insert(T,x)

- Insert x into T as in any BST :
    - x is now a leaf

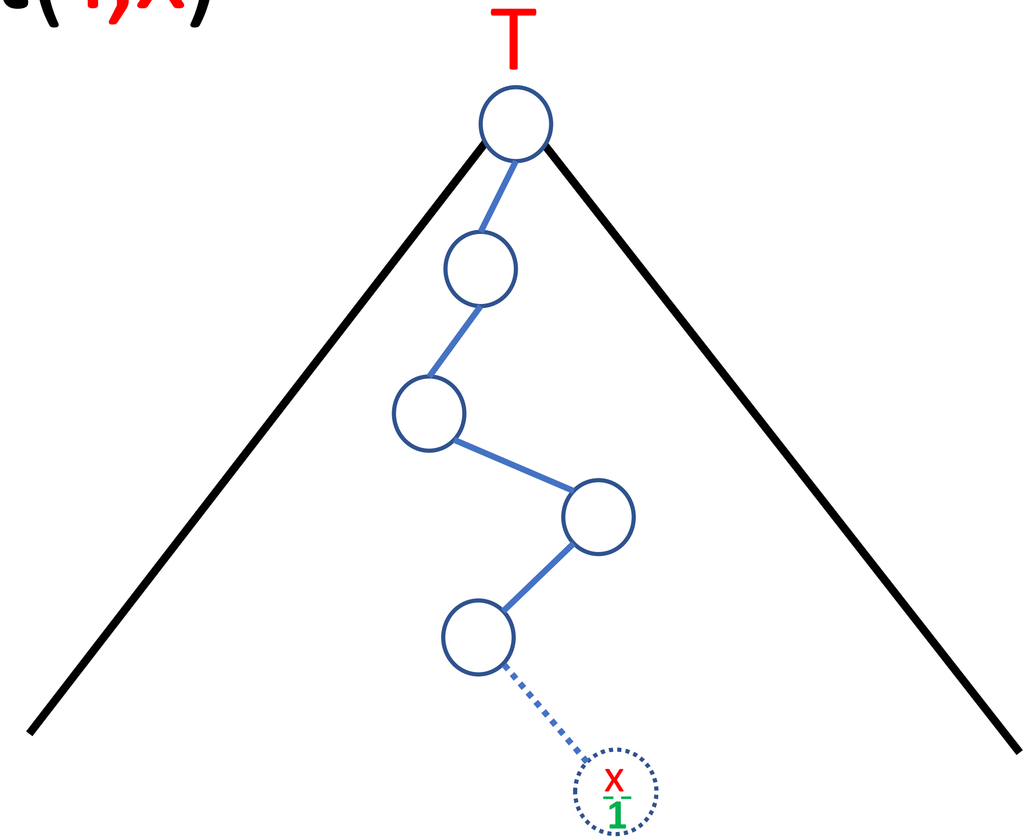
Phase 1



Maintaining size() field: Insert(T,x)

- Insert x into T as in any BST :
  - x is now a leaf
  - Set size(x) = 1

Phase 1



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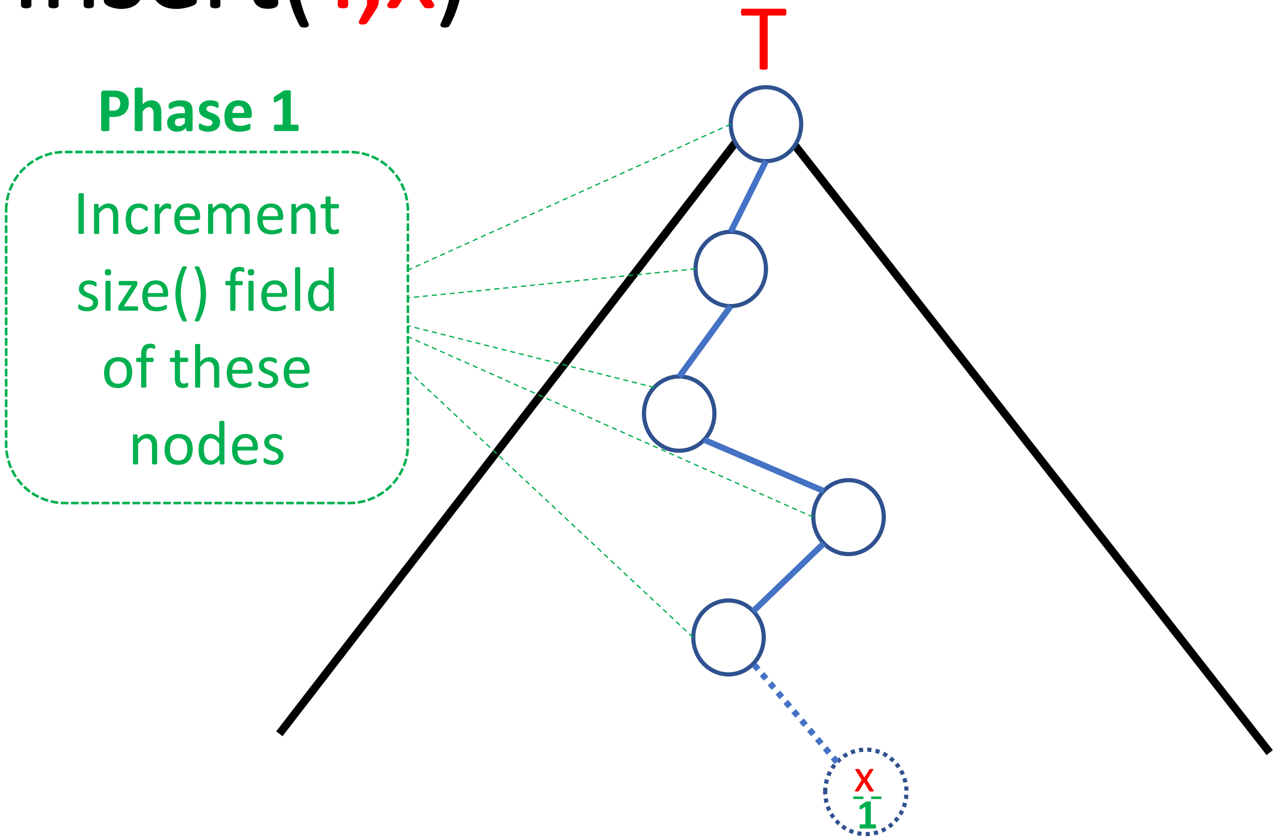
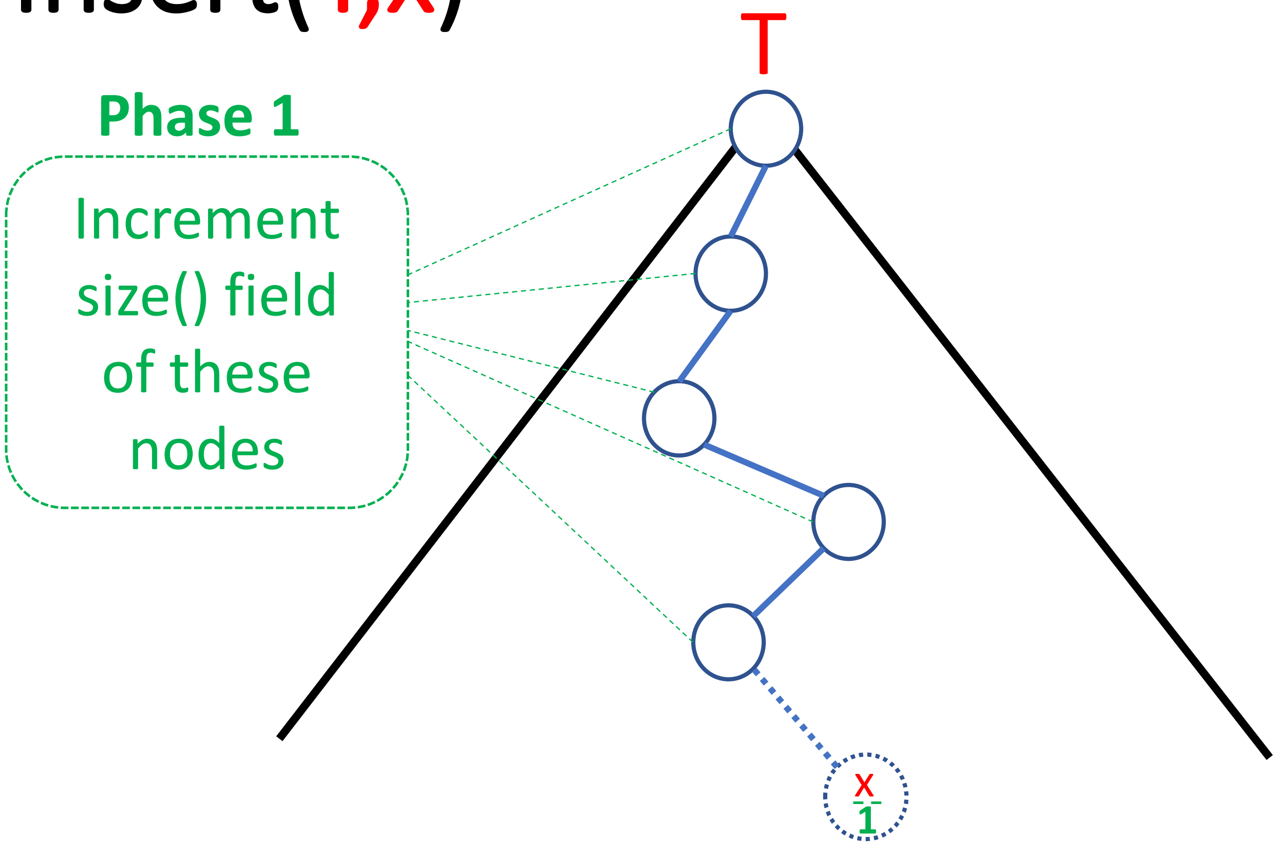
Phase 1



Maintaining size() field: Insert(T,x)

Phase 1

- Insert x into T as in any BST :
  - x is now a leaf
  - Set size(x) = 1
  - For each node y on path from x to root
    - Increment size(y)



Maintaining size() field: Insert(T,x)

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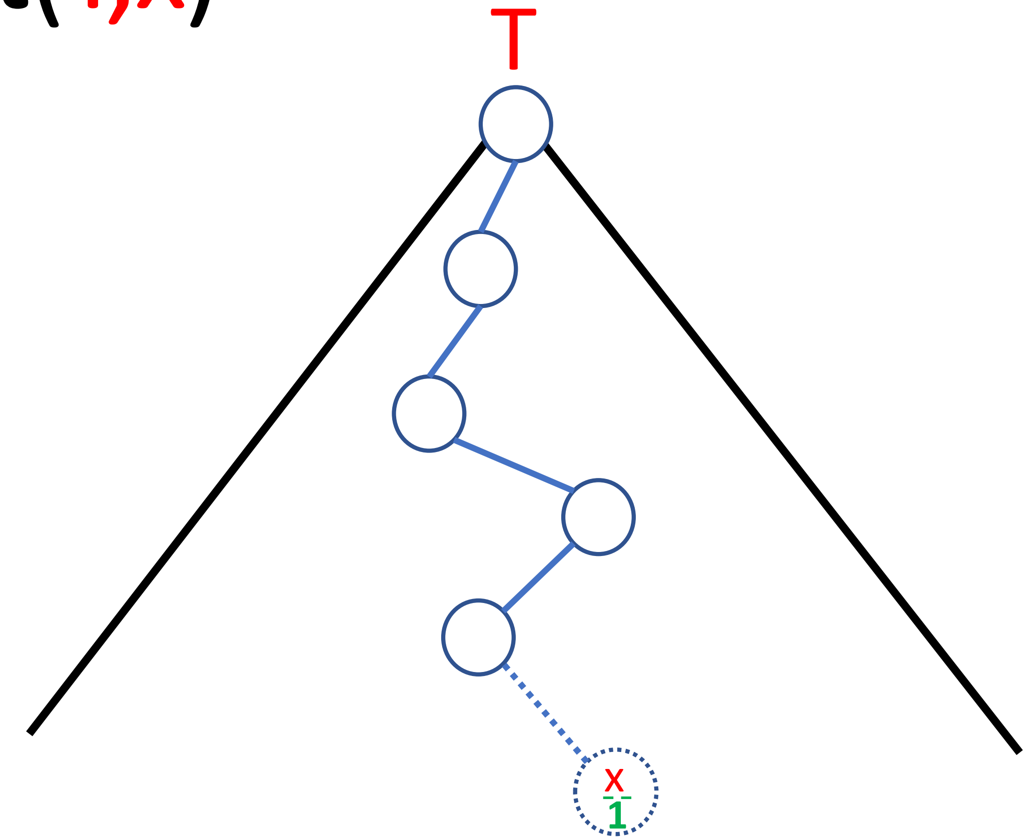
Phase 1

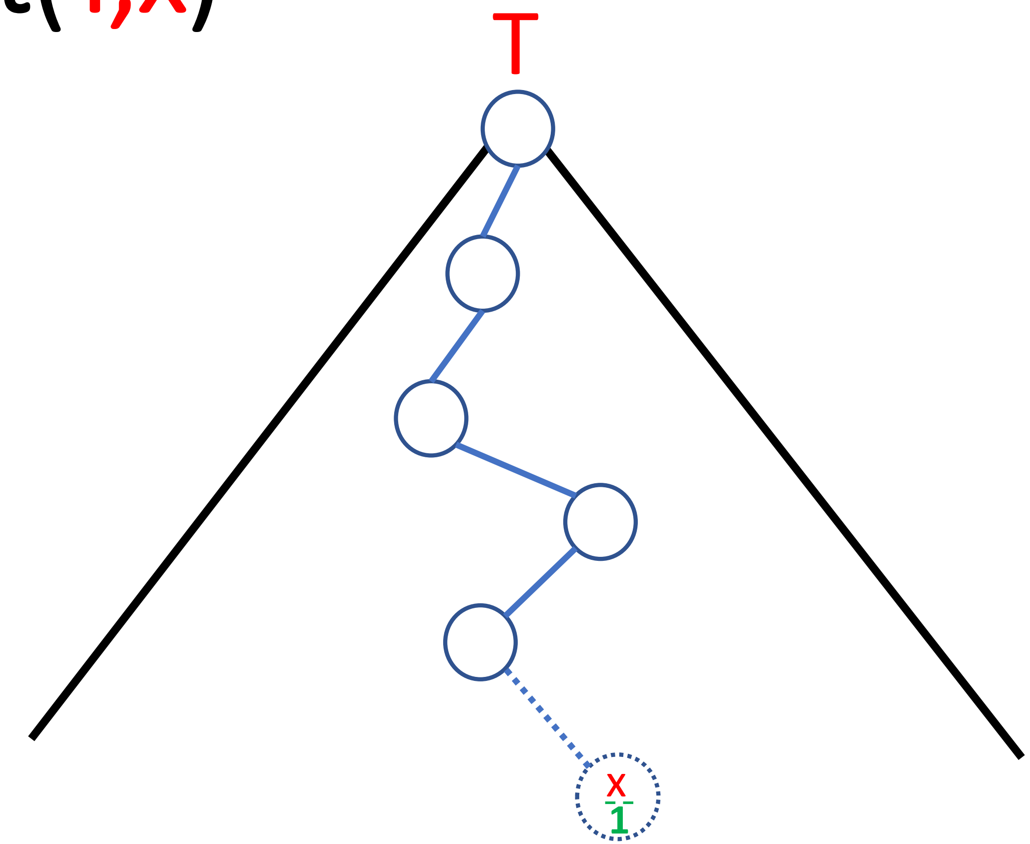


Maintaining size() field: Insert(T,x)

- Insert x into T as in any BST :
  - x is now a leaf
  - Set size(x) = 1
  - For each node y on path from x to root
    - Increment size(y)
- Go up from x to the root and for each node :
  - Adjust the BF
  - Rebalance with rotation if needed

Phase 1

Phase 2



Maintaining size() field: Insert(T,x)

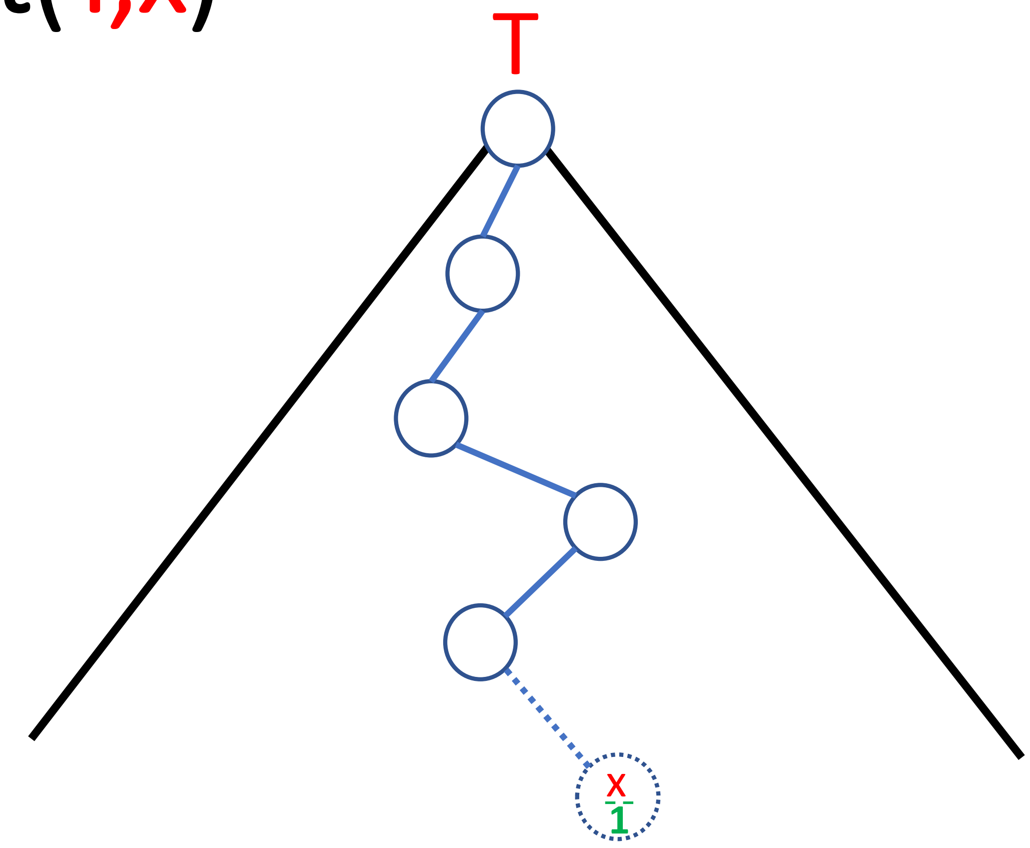
- Insert x into T as in any BST :
  - x is now a leaf
  - Set size(x) = 1
  - For each node y on path from x to root
    - Increment size(y)

Phase 1

- Go up from x to the root and for each node :
  - Adjust the BF
  - Rebalance with rotation if needed
  - If rotation is needed, update size() where necessary using the invariant:

$$\mathbf{size(z) = size(left(z)) + size(right(z)) + 1}$$

Phase 2



Maintaining size() field: Insert(T,x)

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  - Set size(x) = 1
  - For each node y on path from x to root
    - Increment size(y)

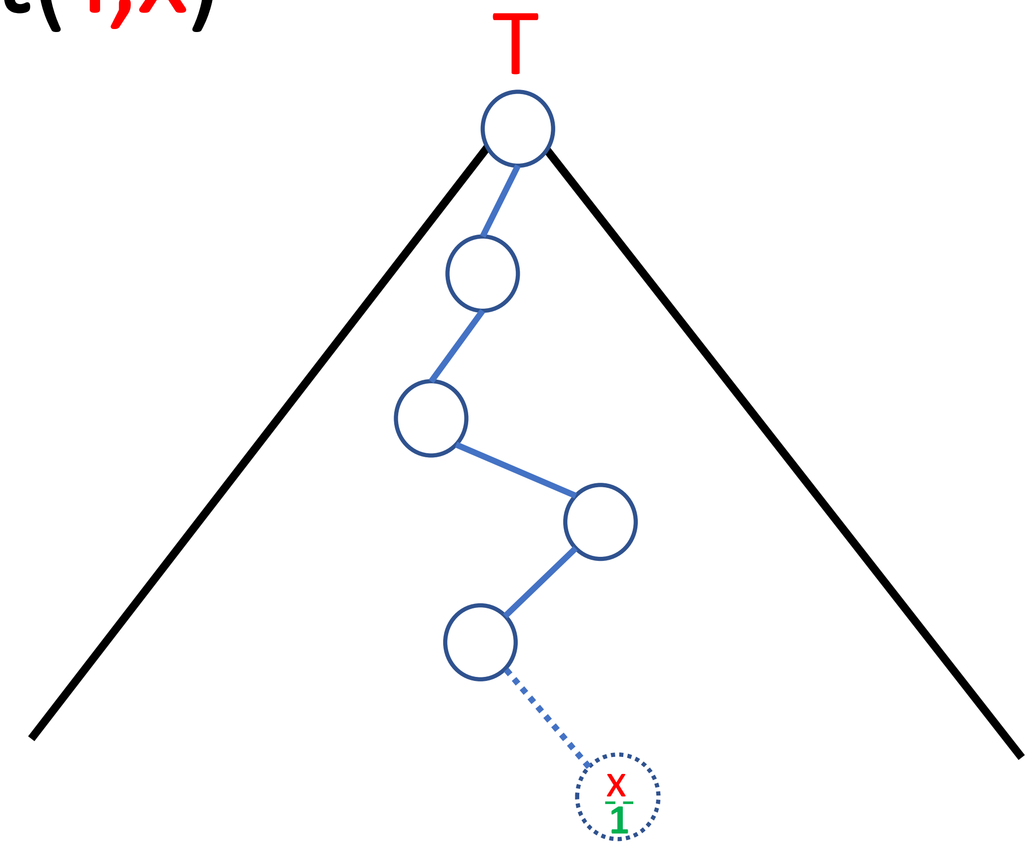
Phase 1

- Go up from x to the root and for each node :
  - Adjust the BF
  - Rebalance with rotation if needed
  - If rotation is needed, update size() where necessary using the invariant:

size(z) = size(left(z)) + size(right(z)) + 1

Phase 2

This adds constant work for each rotation



Augmenting AVL

- **Select** operation ✓
- **Rank** operation ✓
- Maintain size() field after **Insert or Delete** ✓