# Binomial Heaps

# Abstract Data Types

| Abstract Data Types | | Insert | Min | Extract_Min | Union |
|---|---|---|---|---|---|
| Mergeable Priority Queues | | ✓ | ✓ | ✓ | ✓ |

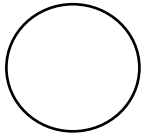
# Data Structures

| Abstract Data Types | Data Structures | Insert | Min | Extract_Min | Union |
|---|---|---|---|---|---|
| Mergeable Priority Queues | Min Binomial Heap | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(\log n)$ |

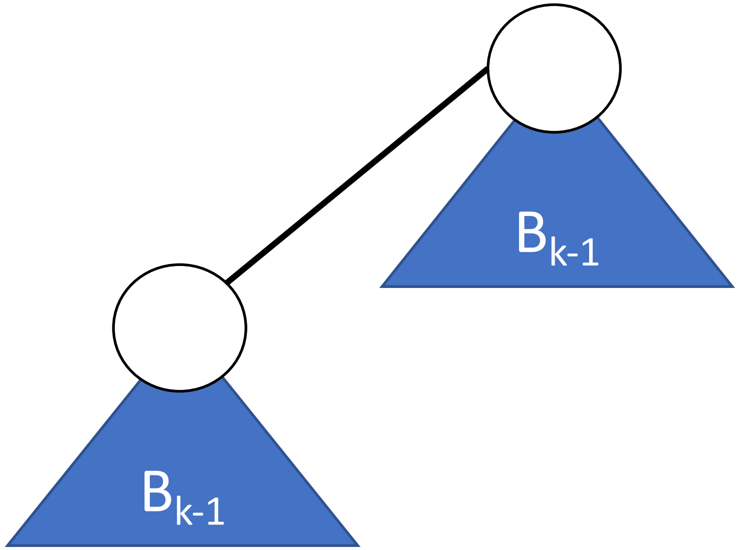
# Binomial Trees

$B_k$ tree: defined recursively

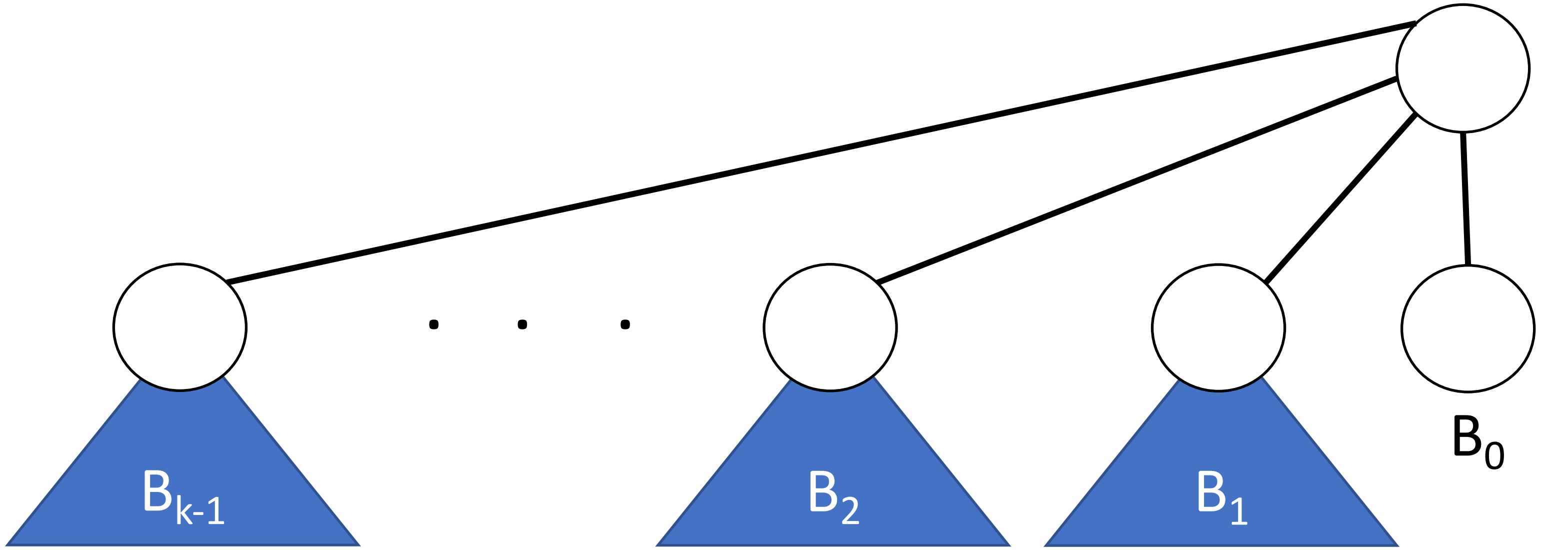
k = 0 $B_0$ :



k > = 1 $B_k$ :



# Binomial Tree $B_k$



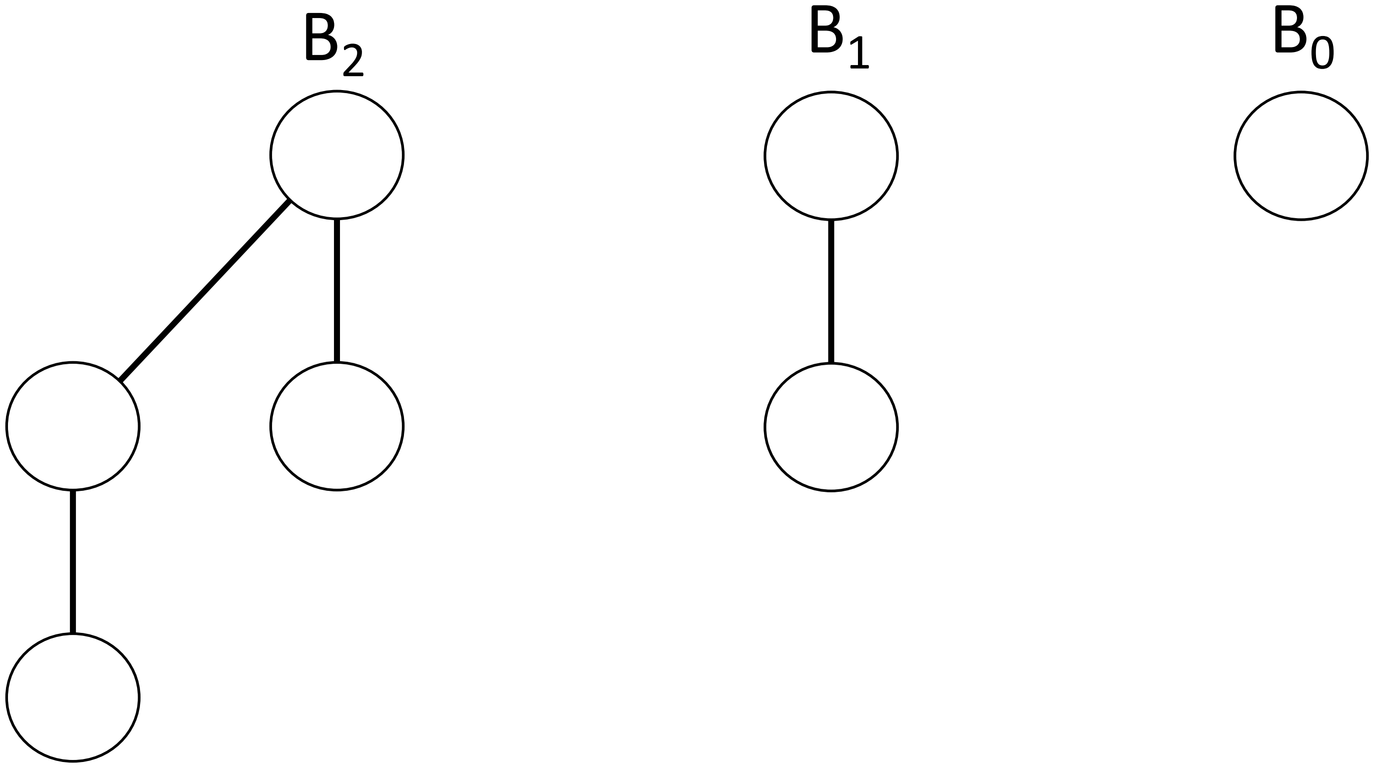
# Binomial Forest $F_n$ of size $n$

Sequence of $B_k$ trees with *strictly decreasing* $k$'s and a total of $n$ nodes.

# Example: Binomial Forest $F_7$ of n = 7 nodes

$n = 7 = \langle 1\ 1\ 1 \rangle_2 = 2^2 + 2^1 + 2^0$

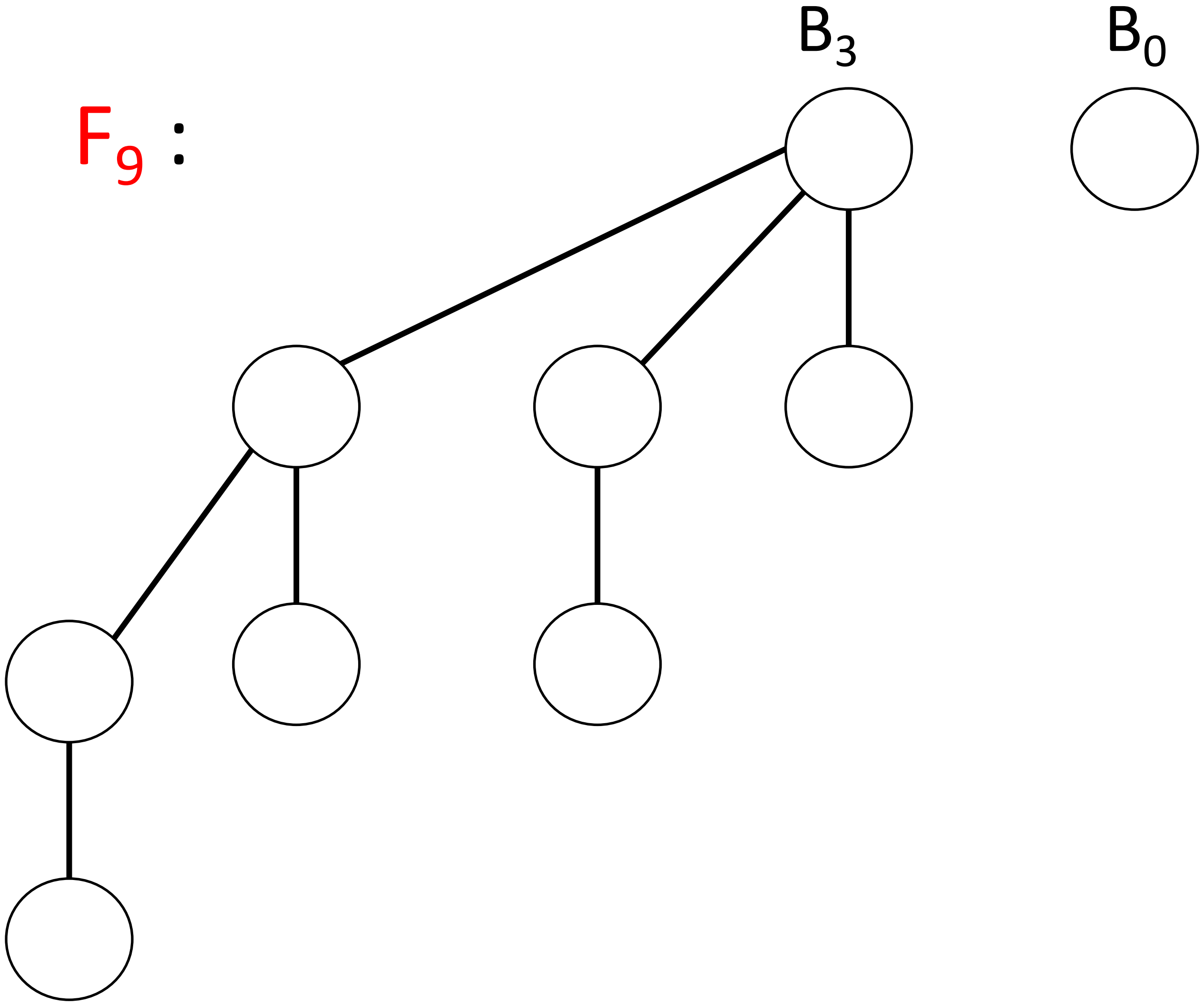
$F_7$ :



# Example: Binomial Forest $F_9$ of $n = 9$ nodes

$n = 9 = < 1\ 0\ 0\ 1 >_2 = 2^3 + 2^0$

$F_9$ :

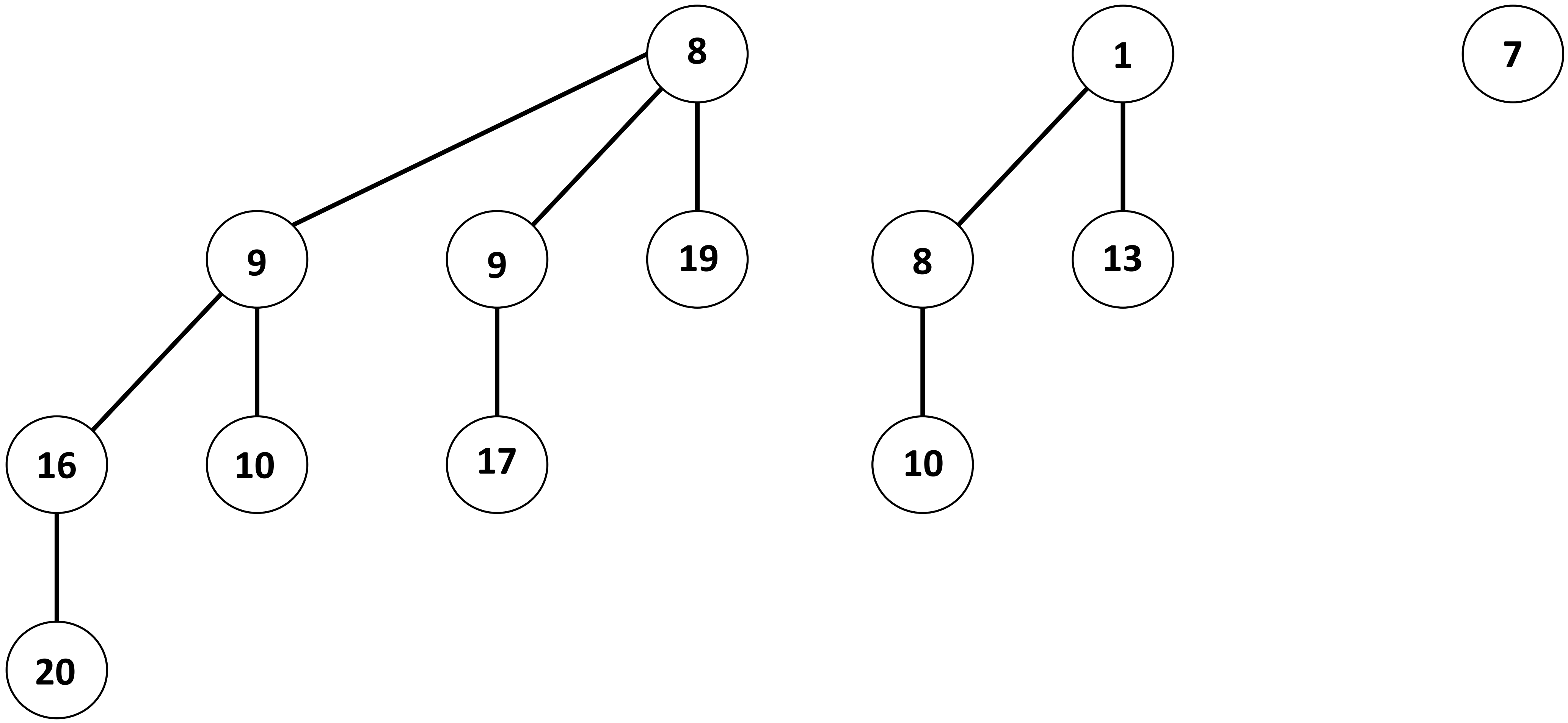


A Min Binomial Heap of $n$ elements is a Binomial Forest $F_n$ such that

1. Each node of $F_n$ stores one element

2. Each $B_k$ tree of $F_n$ is Min-Heap ordered

# Min Binomial Heap of size $n = 13 = < 1101 >_2$



# Binomial Heap Operations

# Must implement the following operations:

- **Union(T, Q)**
- **Insert(T, x)**
- **Min(T)**
- **Extract_Min(T)**

# High Level Ideas

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**Lemma 1**: Can merge two min heap-ordered $B_k$ trees into a single min heap-ordered $B_{k+1}$ tree with just one key-comparison.

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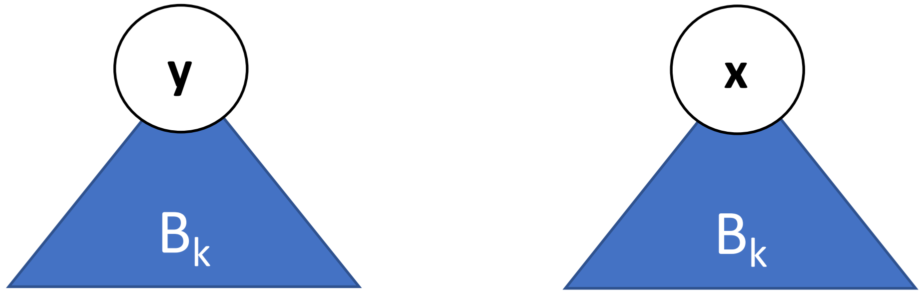

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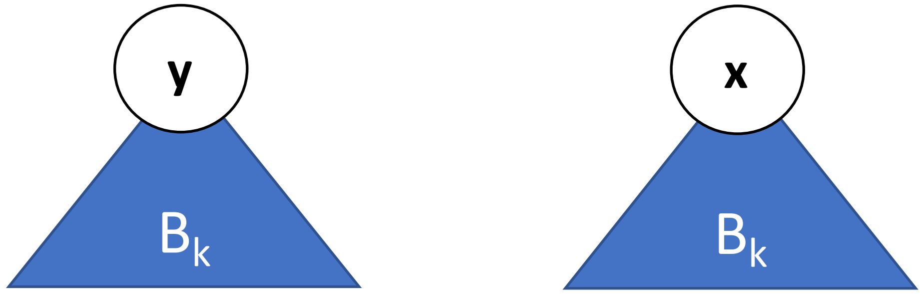

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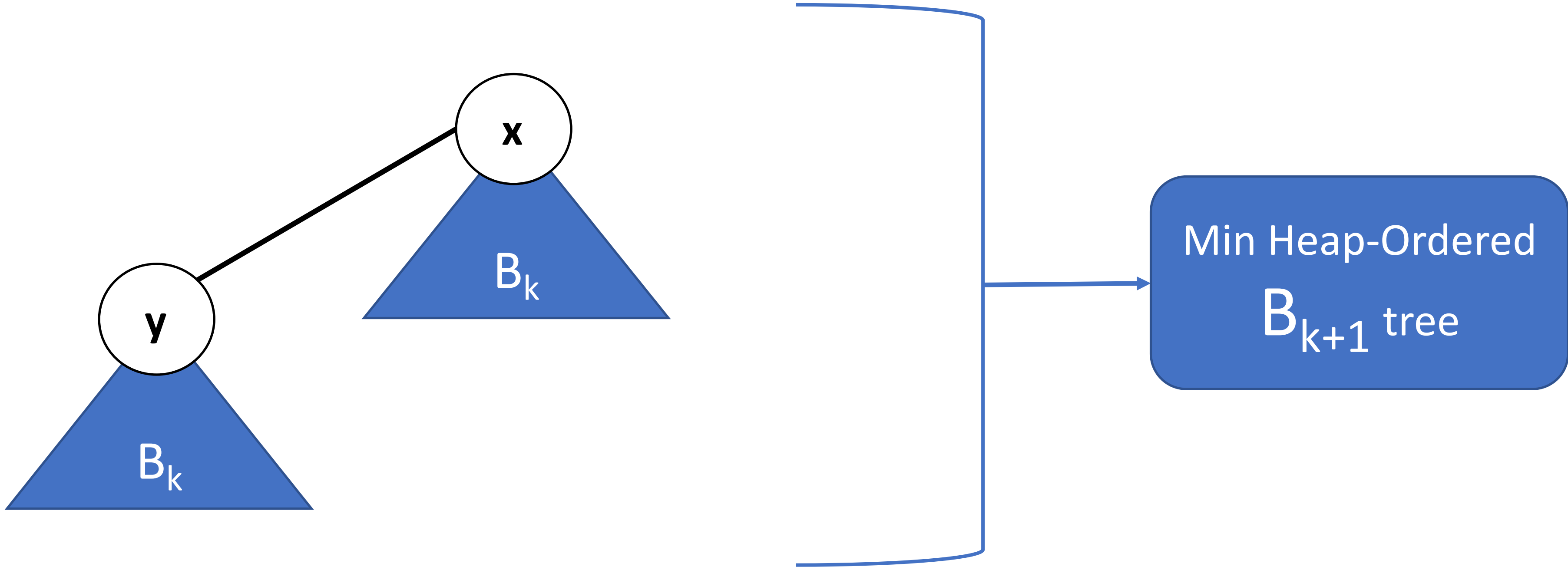

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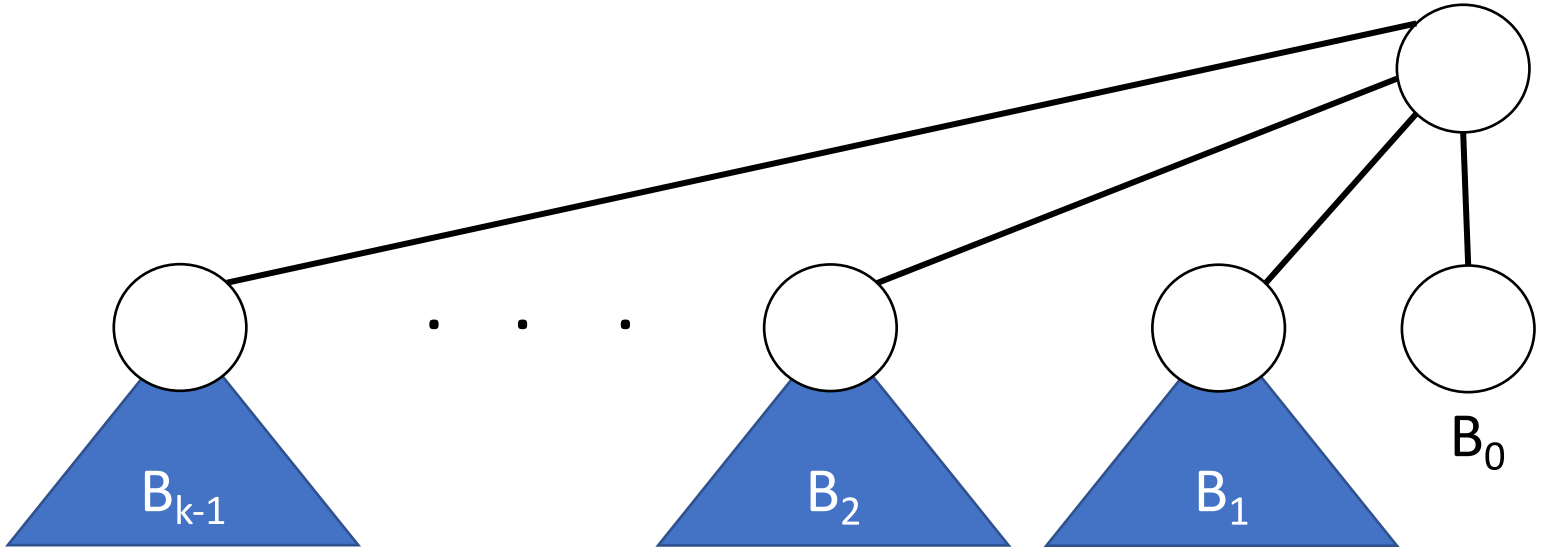

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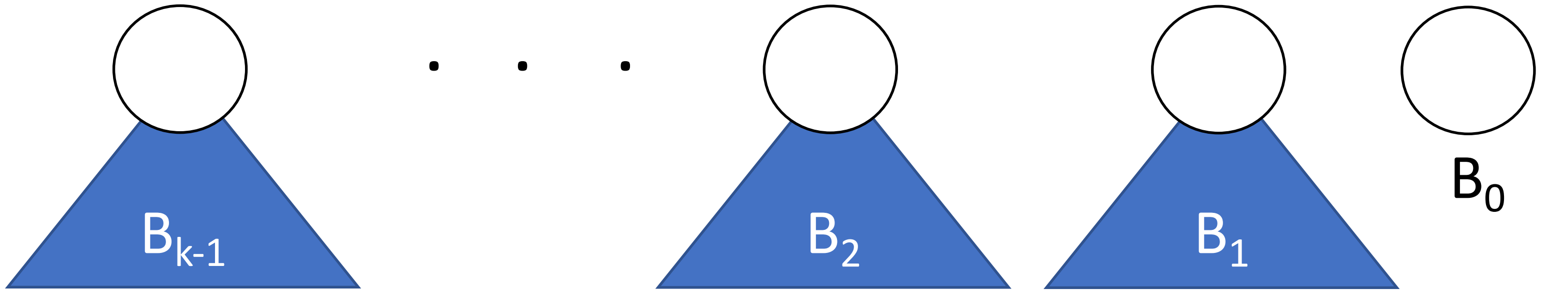

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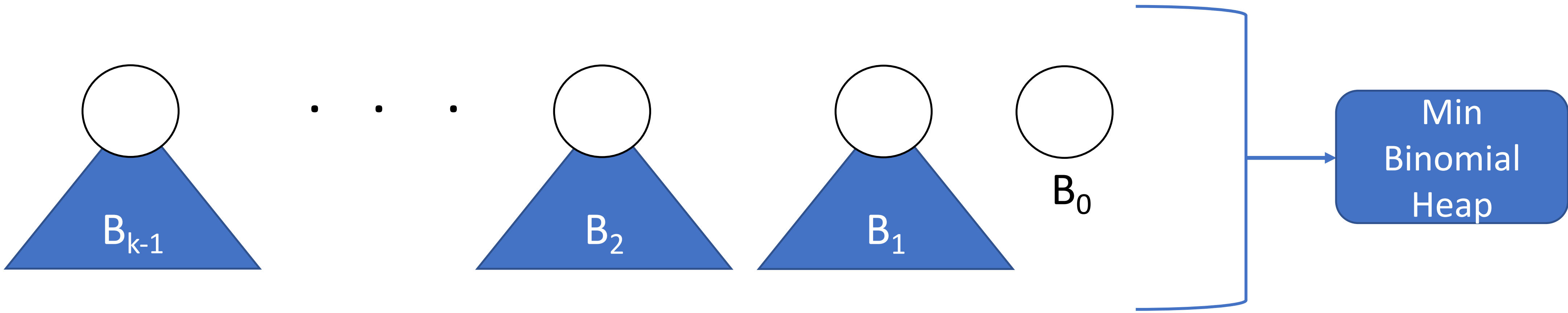

$S \leftarrow \text{Union}(T, Q)$

**S ← Union(T, Q)**

T is a Binomial Heap of size n = 3 = <   1 1>$_2$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | | $B_1$ | $B_0$ | : | T |

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|---|---|---|---|---|---|---|---|---|
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|---|---|---|---|---|---|---|---|---|---|---|
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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | | | $B_1$ | | **Carry** | |
| | 1 | 1 | + | | $B_1$ | $B_0$ | : T | + |
| 1 | 1 | 1 | | $B_2$ | $B_1$ | $B_0$ | : Q | |
| | | 0 | | | | X | : S | |

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| 1 | 0 | 1 | 0 | | $B_3$ | X | $B_1$ | X | : | S |

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How many new edges were added?

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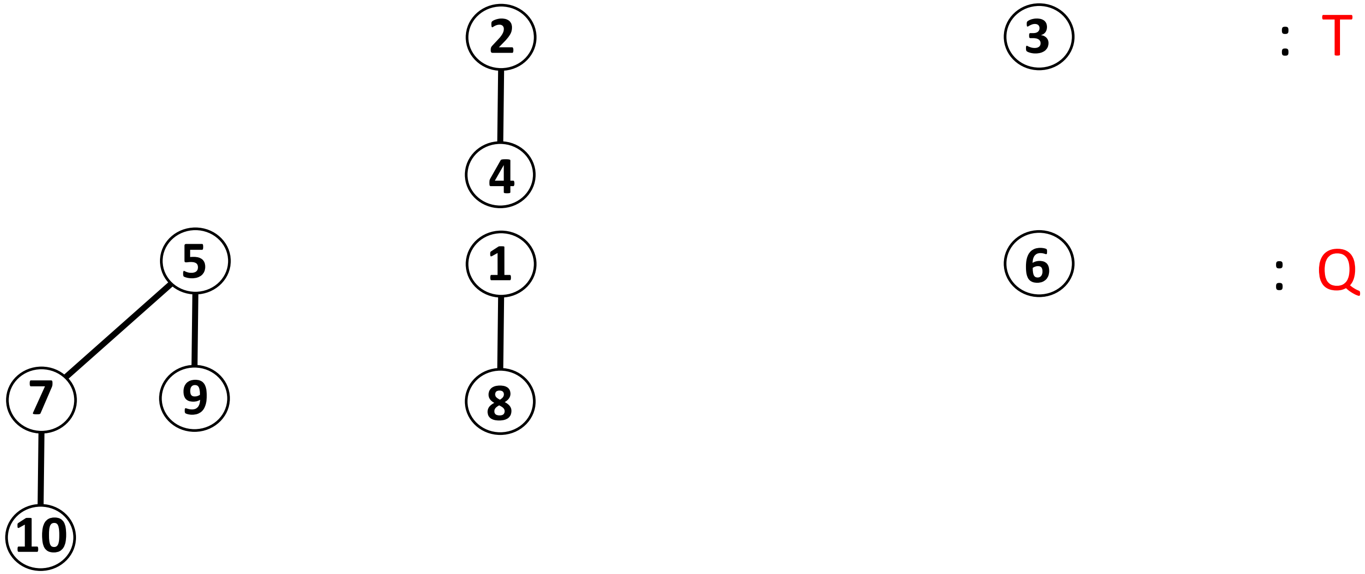
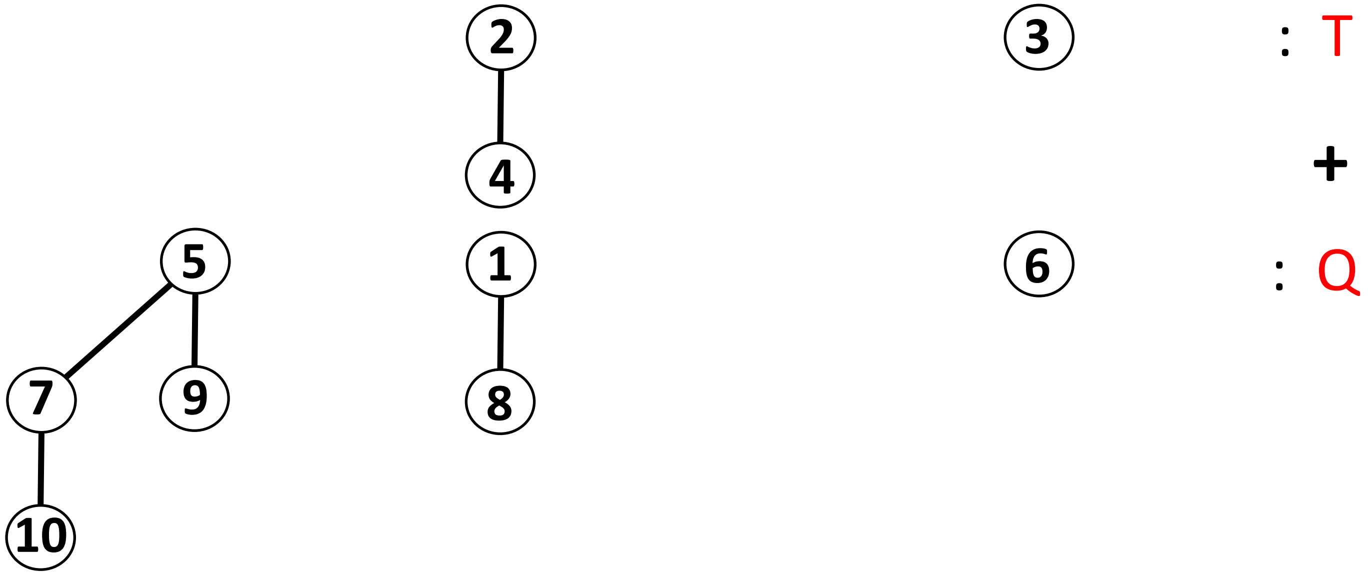
| | | | | + | | | | | + |
| | 1 | 1 | 1 | | | $B_2$ | $B_1$ | $B_0$ | : Q |
| 1 | 0 | 1 | 0 | | $B_3$ | X | $B_1$ | X | : S |

How many new edges were added?

3


2
3
: T
4


2
3
: T
4
5
1
6
: Q
7
9
8
10


2
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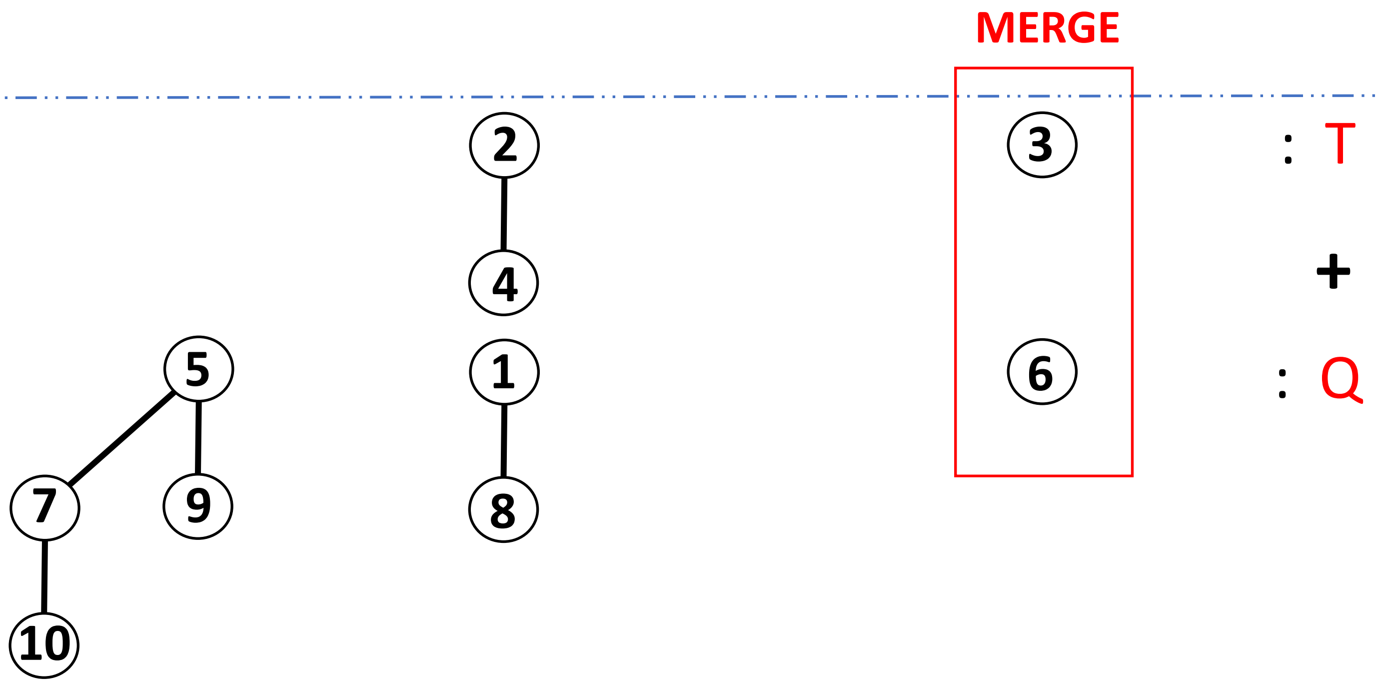
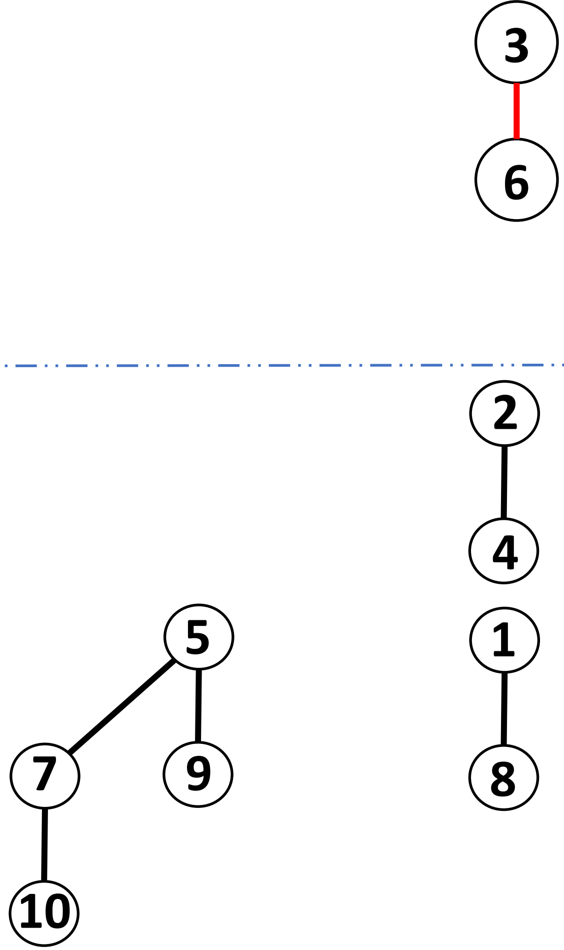

MERGE
5
7
9
10
2
4
1
8
3
6
: T
+
: Q




**Carry**

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+

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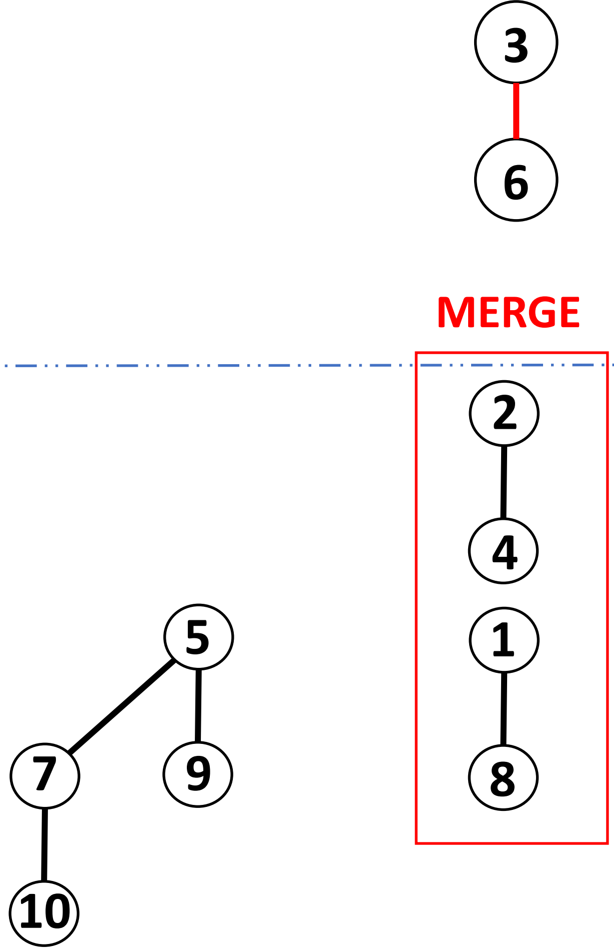


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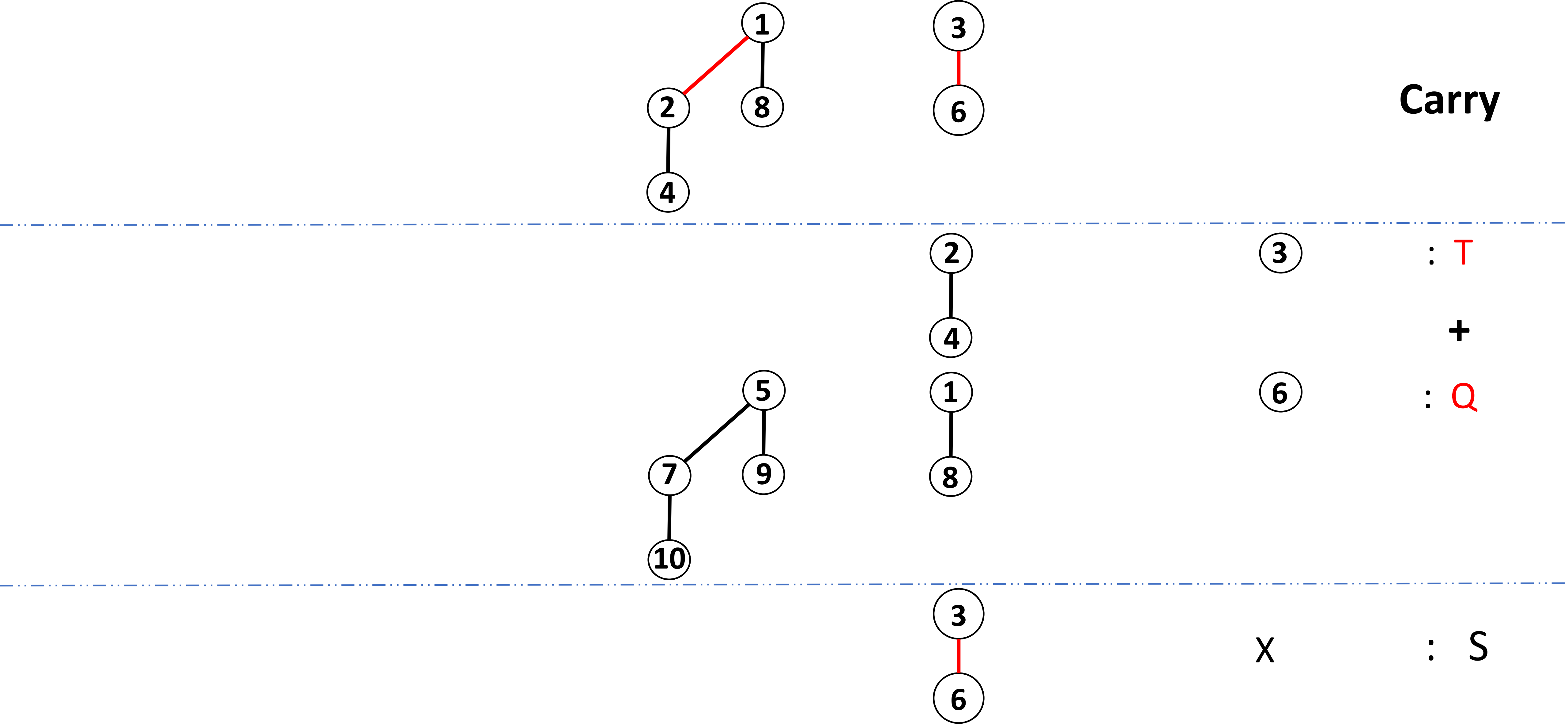
Carry
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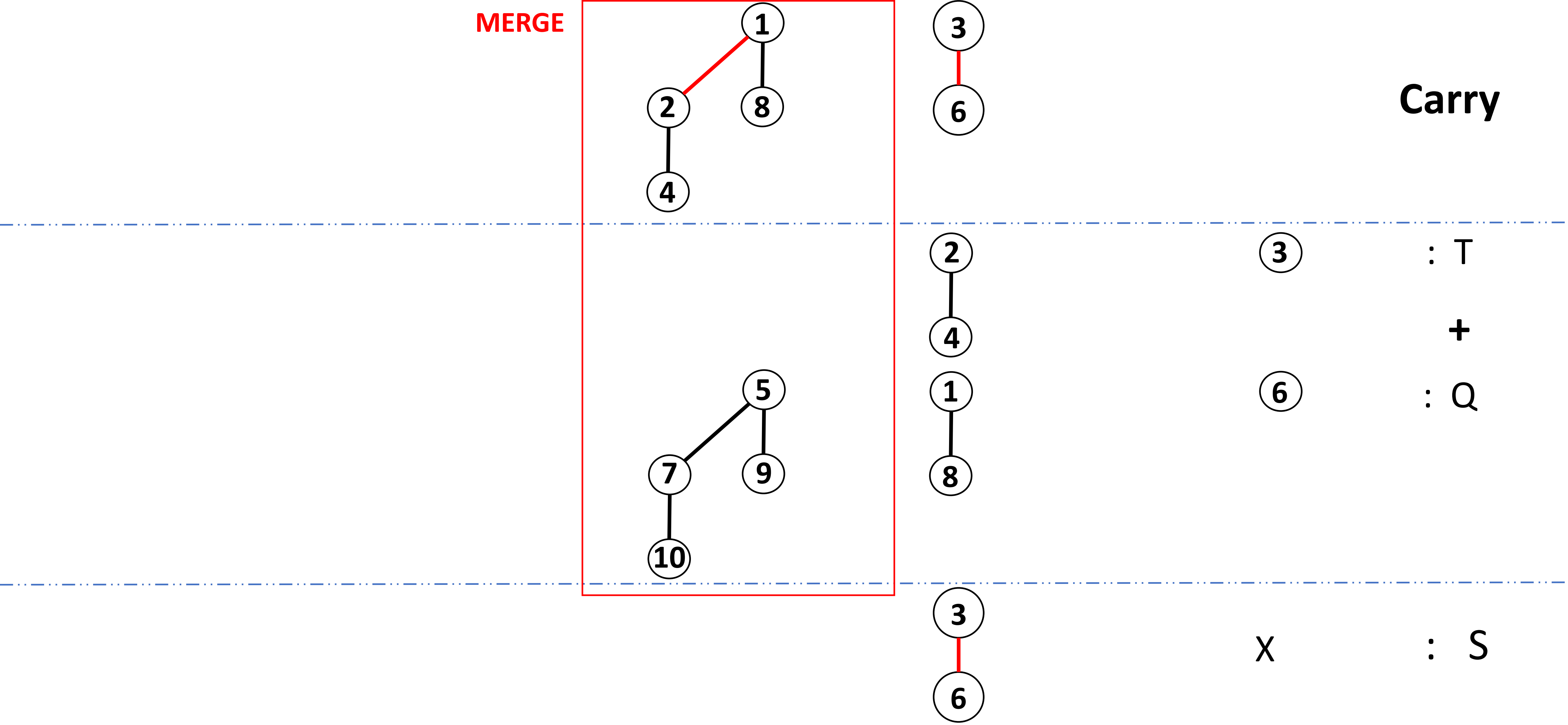
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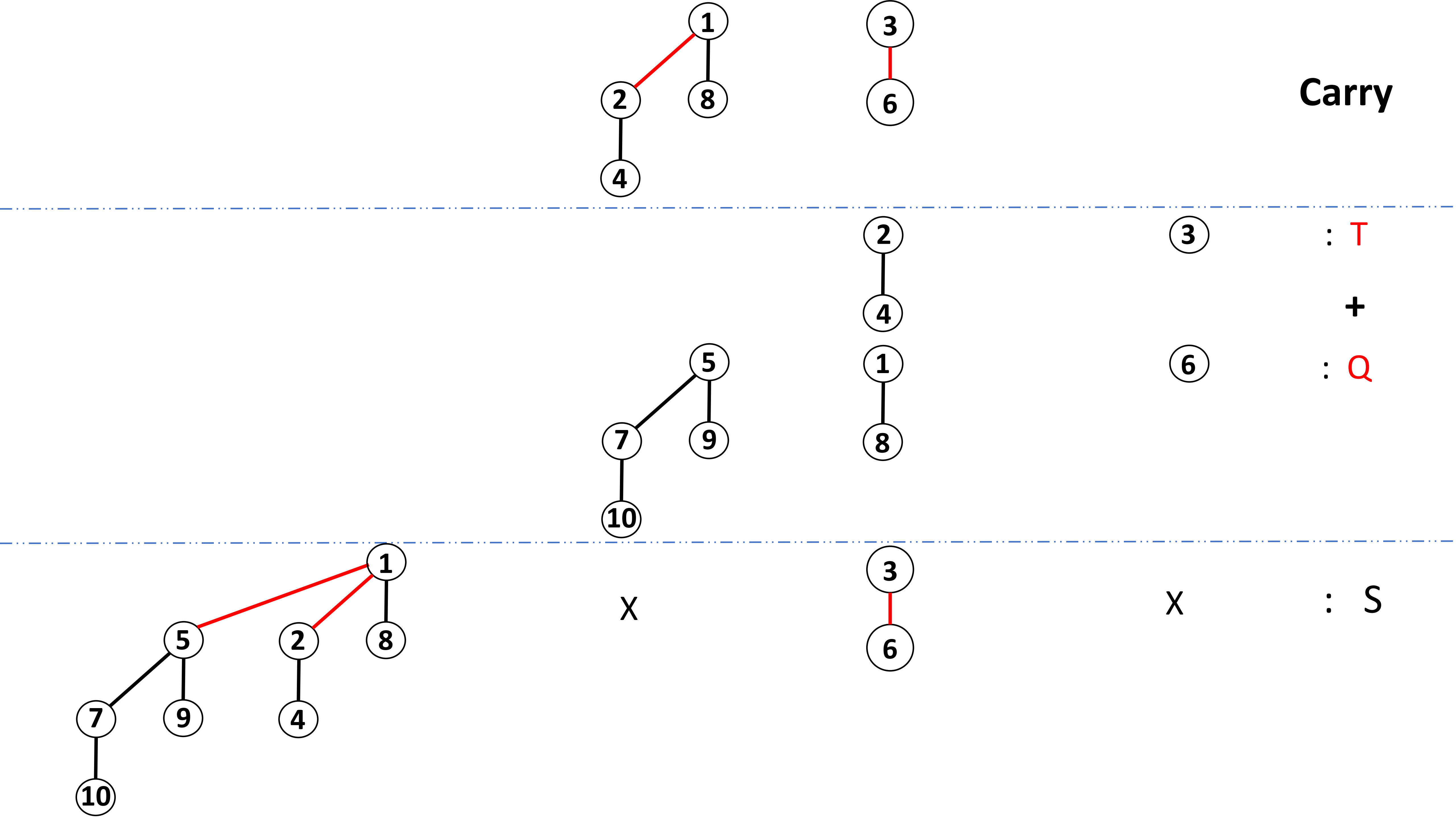
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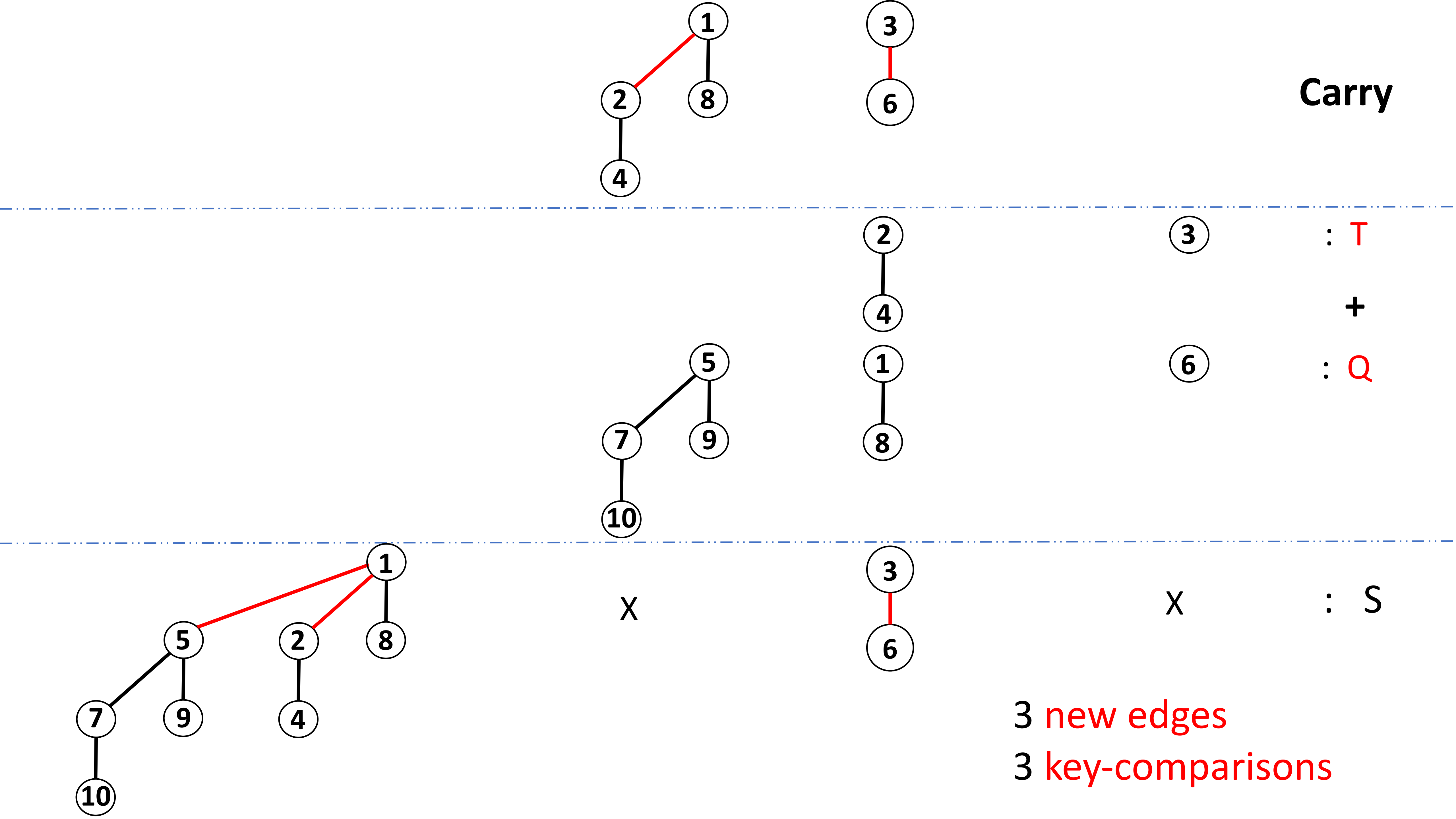


3 new edges

3 key-comparisons

# Worst-Case Complexity of **Union(T, Q)**

Say |T| <= n and |Q| <= n (i.e. each contains at most n elements)

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$\Rightarrow$ **Union**(T, Q) takes at most O(log n) key-comparisons

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Trivial Binomial Heap containing just x

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Scan the roots of the $B_k$ trees of T and return the smallest key.

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If $|T| <= n$, Min(T) takes at most $O(\log n)$ key-comparisons

# Extract_Min(T)

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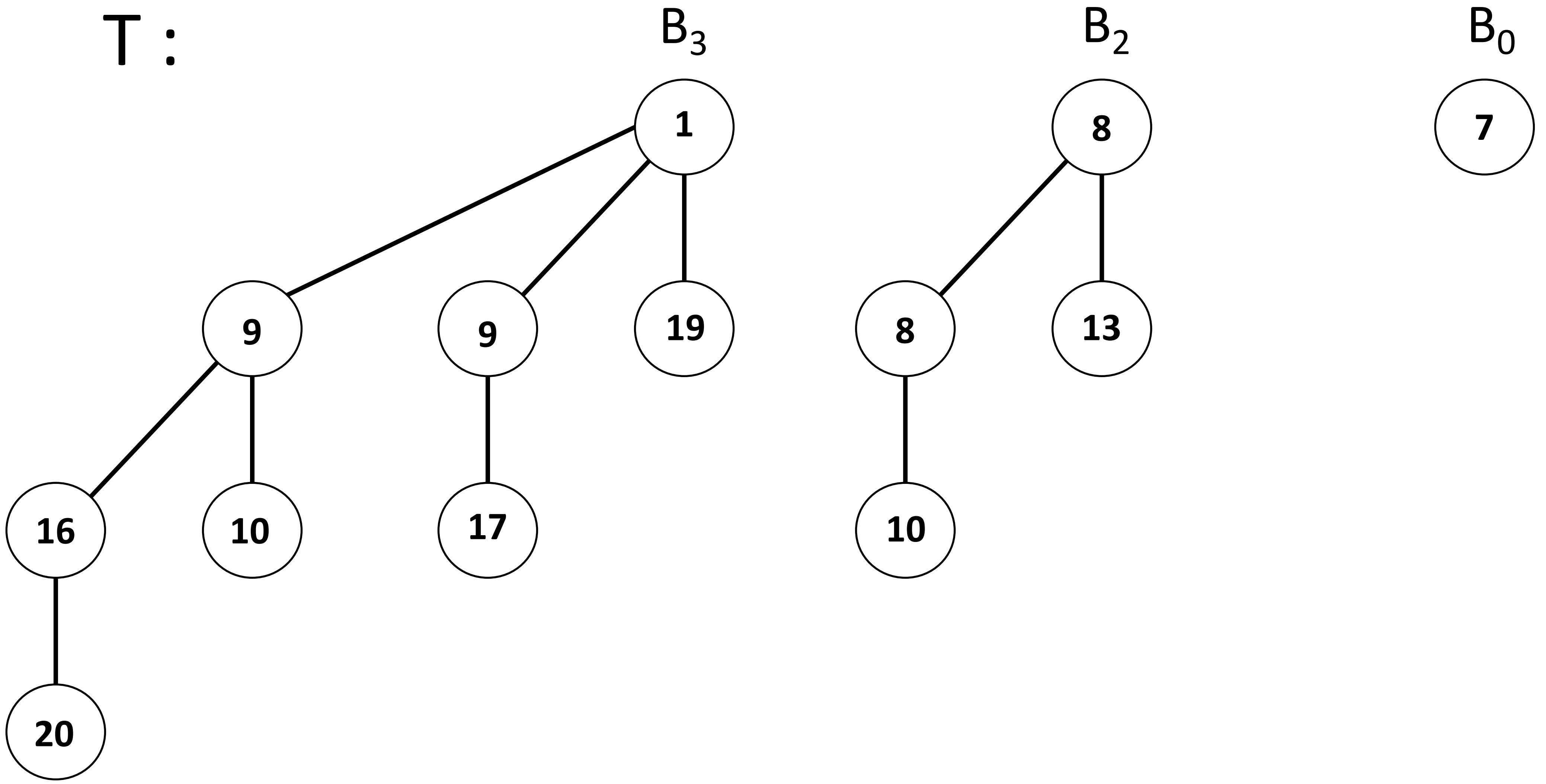
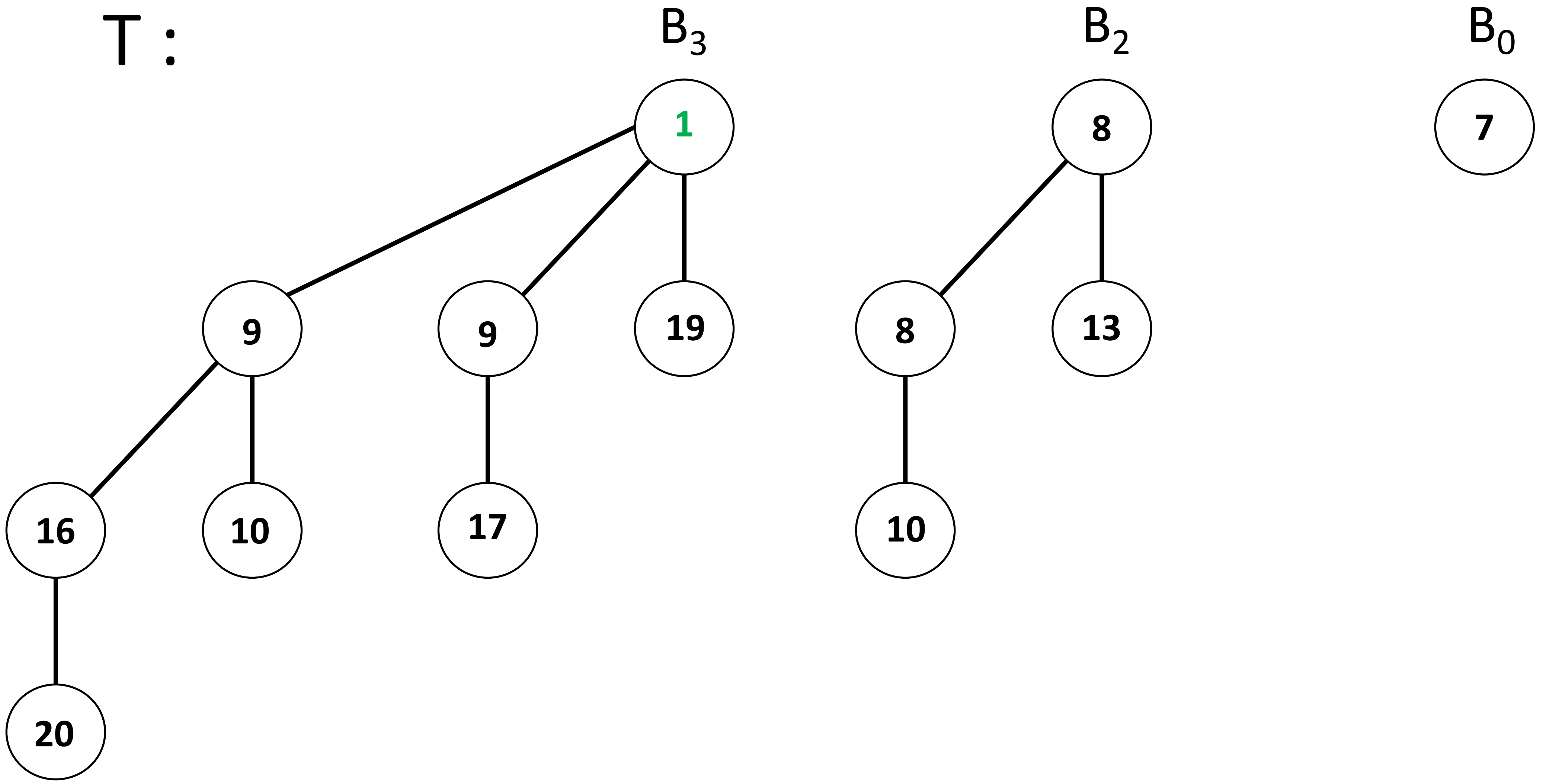
T :



# Extract_Min(T)



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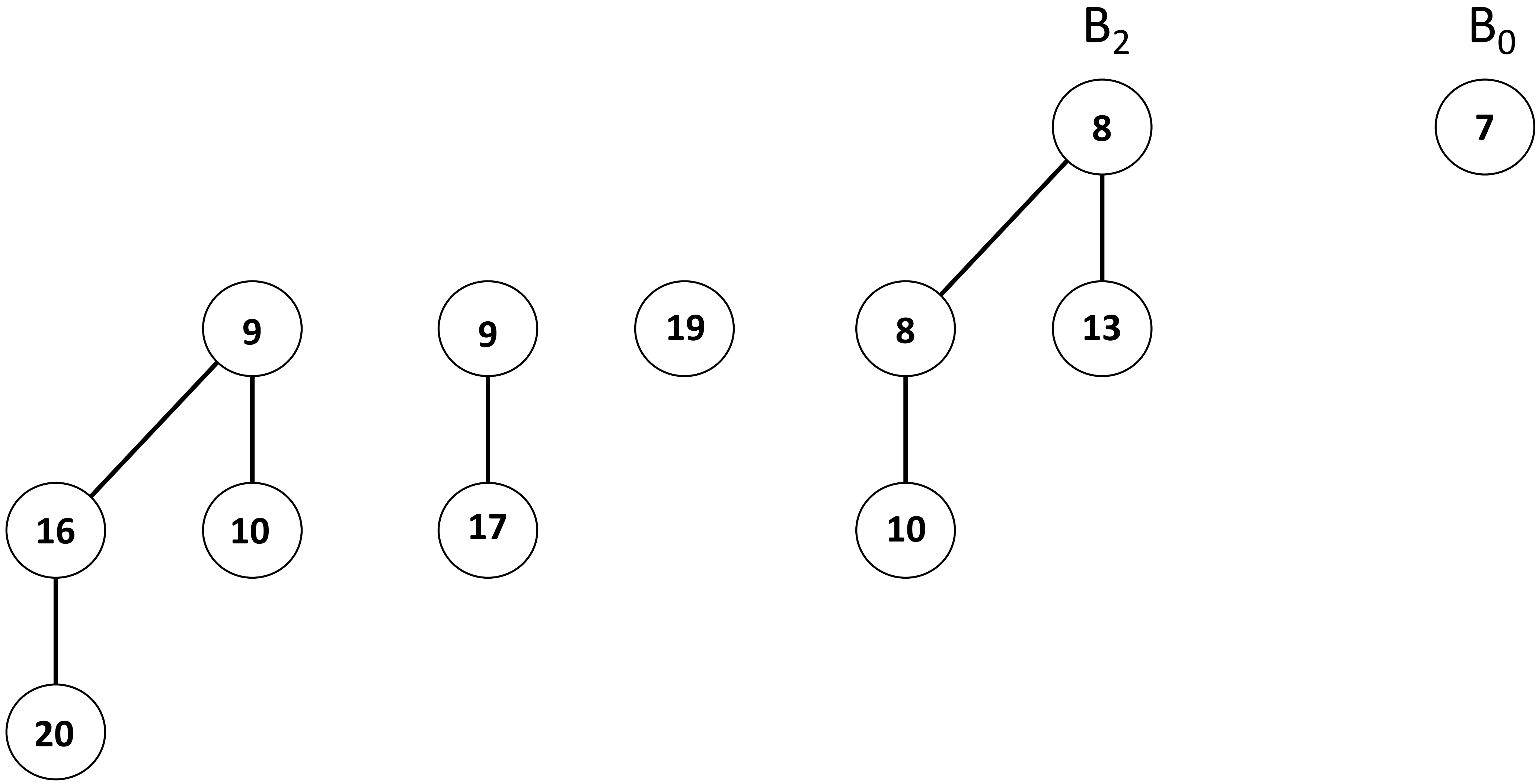

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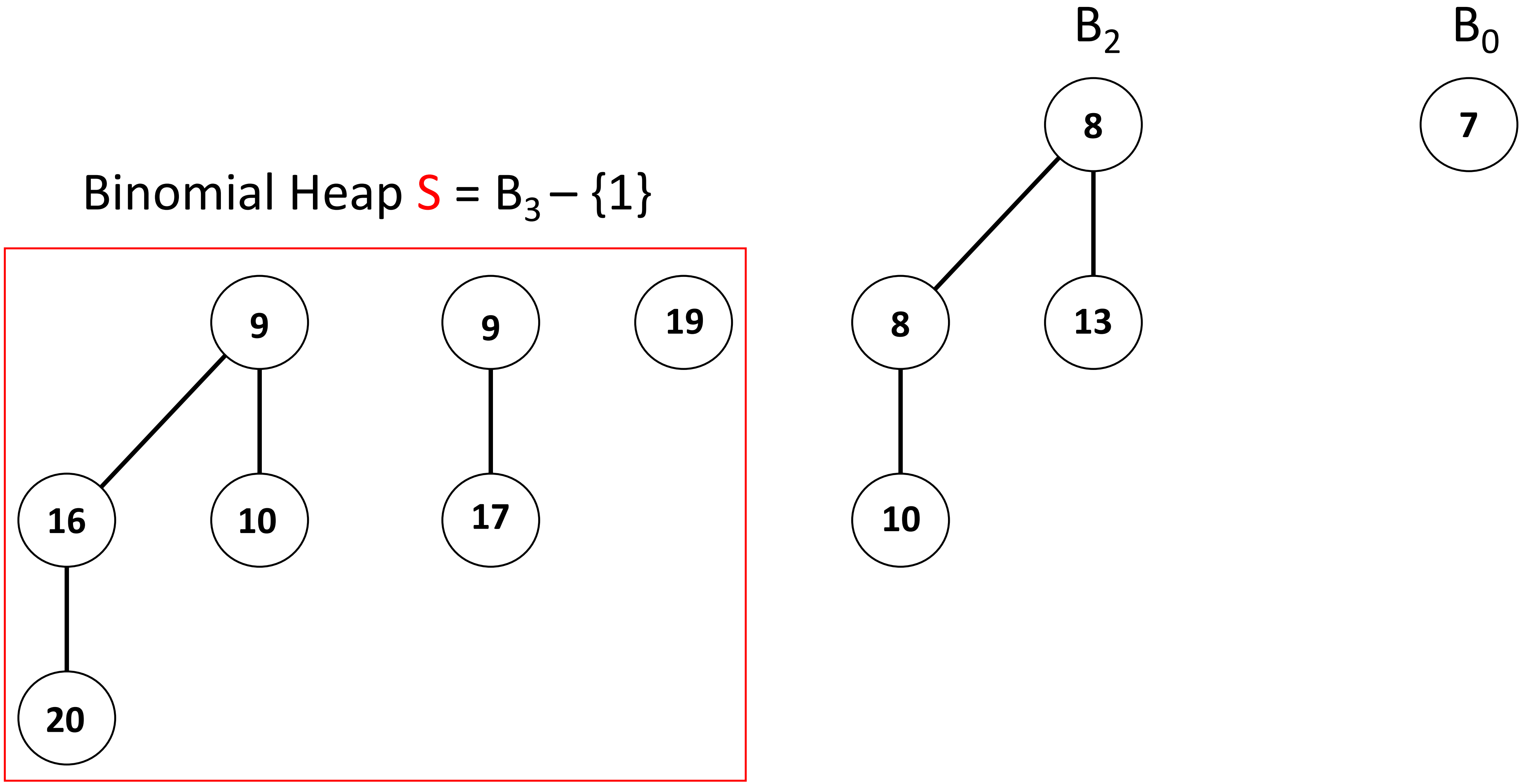
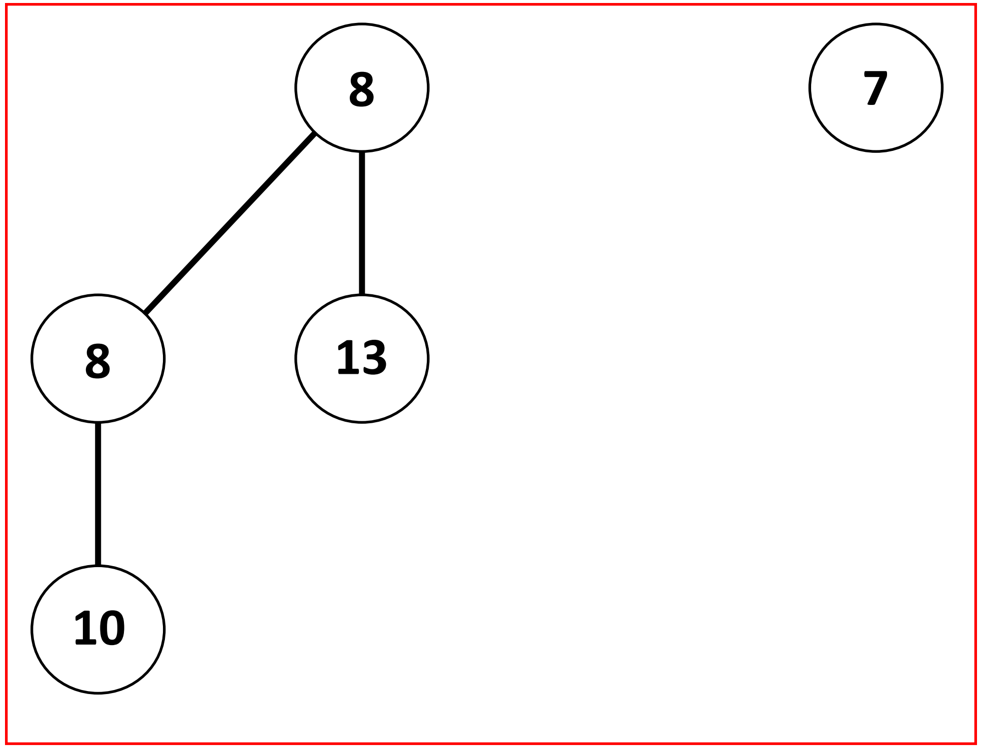


# Extract_Min(T)

Binomial Heap U = T - $B_3$

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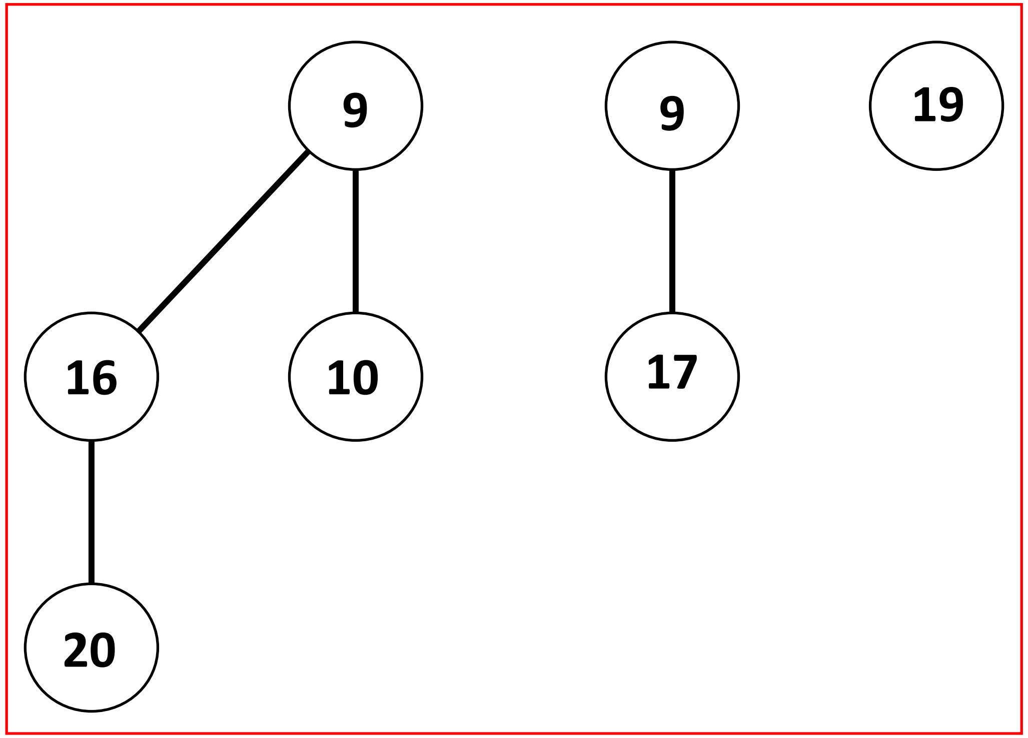

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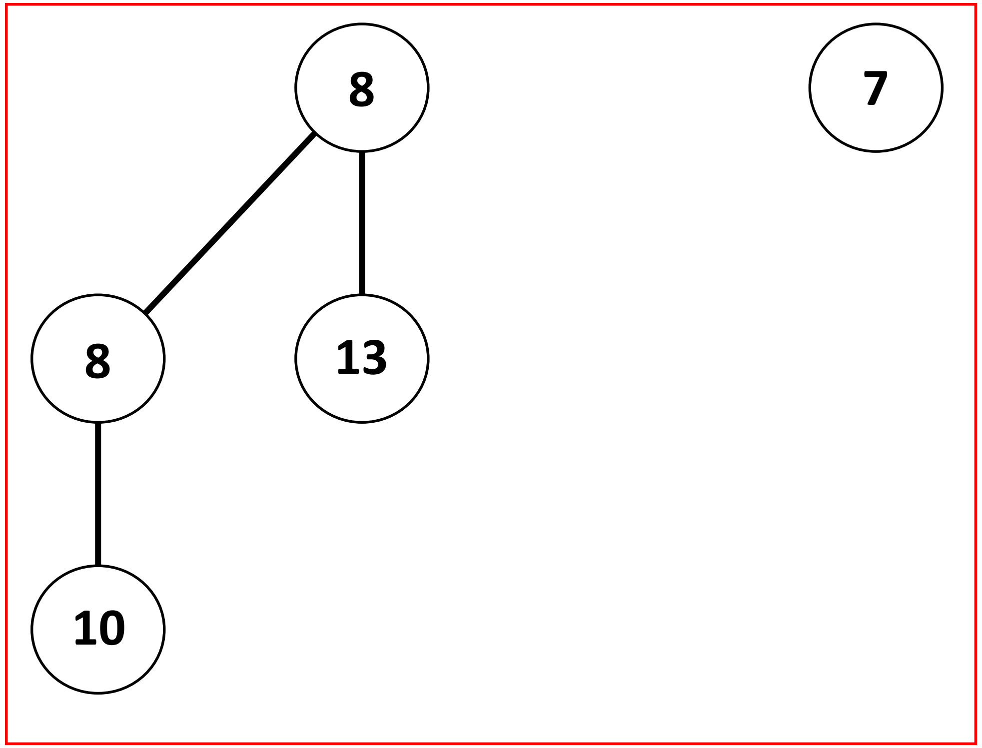
Binomial Heap U = T - $B_3$

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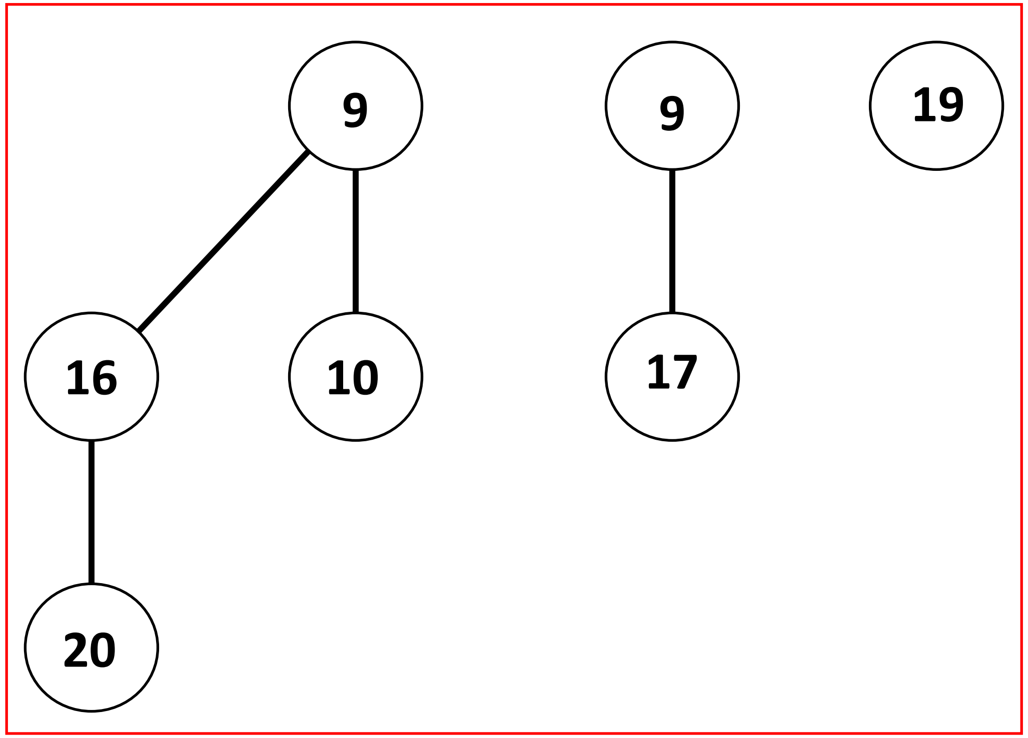


Now do: T ← **Union**(U,S)

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- Do **Min**(T) to locate the smallest element – Say it is the root of $B_i$

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- T ← **Union(**U, S**)**

If |T| <= n, **Extract_Min(T)** takes at most O(log n) key-comparisons

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  Both in O(log n) time

- How do you do **Increase_Key**(T, x, k) ?

# Cost of k successive inserts

- T: Binomial Heap with n elements.
- Cost of k successive inserts into T ?

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O( log n )

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O( log n )  O( log (n + 1) )

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O( log n )    O( log (n + 1) )   . . . . .    O( log (n + k) )

# Cost of k successive inserts

- T: Binomial Heap with $n$ elements.

- Cost of $k$ successive inserts into T ?

**Insert**(T, $x_1$) , **Insert**(T, $x_2$) , . . . . . , **Insert**(T, $x_k$)

$O(\log n)$ $O(\log (n + 1))$ . . . . . $O(\log (n + k))$

- Total: $O(k \log (n + k))$

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O( log n ) O( log (n + 1) ) . . . . . O( log (n + k) )

- Total: O( k log (n + k) )

- Is the cost of k successive inserts actually lower? **Yes!**

**Example:** Say $|T| = 27 = < 11011 >_2$ $\quad$ $T = < B_4\ B_3\ B_1\ B_0 >$

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$$\begin{array}{lrrrrr} T & 1 & 1 & 0 & 1 & 1 \\ + & & & & & \\ x_1 & & & & & 1 \end{array}$$

**Example:** Say $|T| = 27 = <11011>_2$ $T = <B_4\ B_3\ B_1\ B_0>$

$$
\begin{array}{lccccc}
 & & & & 1 & \\
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & & & & & 0
\end{array}
$$

**Example:** Say $|T| = 27 = <11011>_2$ $T = <B_4\ B_3\ B_1\ B_0>$

$$
\begin{array}{lccccc}
 & & & \color{red}{1} & \color{red}{1} & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & & & & 0 & 0
\end{array}
$$

**Example:** Say $|T| = 27 = < 11011 >_2$ $T = < B_4\ B_3\ B_1\ B_0 >$

$$
\begin{array}{lccccc}
 & & & \color{red}{1} & \color{red}{1} & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$
\begin{array}{lccccc}
 & & & \color{red}{1} & \color{red}{1} & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lccccc} & & & 1 & 1 & \\ \hline T & 1 & 1 & 0 & 1 & 1 \\ + & & & & & \\ x_1 & & & & & 1 \\ \hline & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : 2

$$\begin{array}{lccccc} \hline T & 1 & 1 & 1 & 0 & 0 \\ + & & & & & \\ x_2 & & & & & 1 \\ \hline \end{array}$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad$ $T = \langle B_4\, B_3\, B_1\, B_0 \rangle$

$$
\begin{array}{l ccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
\qquad\Bigg|\qquad
\begin{array}{l ccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 2

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$
\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

$$
\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 0

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad$ $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}$$

Cost : 2

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}$$

Cost : 0

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
\end{array}$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\, B_3\, B_1\, B_0 \rangle$

$$
\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

$$
\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lccccc}
 & & & & 1 & \\
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\, B_3\, B_1\, B_0 \rangle$

$$
\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

$$
\begin{array}{lccccc}
 & & & & & \\
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lccccc}
 & & & & 1 & \\
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Cost : 1

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad$ $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$
\begin{array}{l|ccccc}
 & & & \color{red}{1} & \color{red}{1} & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : $\color{red}{2}$

$$
\begin{array}{l|ccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : $\color{red}{0}$

$$
\begin{array}{l|ccccc}
 & & & & \color{red}{1} & \\
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Cost : $\color{red}{1}$

$$
\begin{array}{l|ccccc}
\hline
T & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & \\
x_4 & & & & & 1 \\
\hline
\end{array}
$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$  $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}$$

Cost : 2

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}$$

Cost : 0

$$\begin{array}{lccccc}
 & & & & 1 & \\
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Cost : 1

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & \\
x_4 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1
\end{array}$$

**Example:** Say $|T| = 27 = < 11011 >_2$ $\quad$ $T = < B_4\ B_3\ B_1\ B_0 >$

$$\begin{array}{lccccc}
 & & & 1 & 1 & \\
\hline
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}$$

Cost : 2

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}$$

Cost : 0

$$\begin{array}{lccccc}
 & & & & 1 & \\
\hline
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Cost : 1

$$\begin{array}{lccccc}
\hline
T & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & \\
x_4 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Cost : 0

**Example:** Say $|T| = 27 = < 11011 >_2$ $\quad T = < B_4\ B_3\ B_1\ B_0 >$

$$\begin{array}{lccccc} & & & 1 & 1 & \\ T & 1 & 1 & 0 & 1 & 1 \\ + & & & & & \\ x_1 & & & & & 1 \\ \hline & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : 2

$$\begin{array}{lccccc} T & 1 & 1 & 1 & 0 & 0 \\ + & & & & & \\ x_2 & & & & & 1 \\ \hline & 1 & 1 & 1 & 0 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lccccc} & & & & 1 & \\ T & 1 & 1 & 1 & 0 & 1 \\ + & & & & & \\ x_3 & & & & & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

Cost : 1

$$\begin{array}{lccccc} T & 1 & 1 & 1 & 1 & 0 \\ + & & & & & \\ x_4 & & & & & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lccccc} T & 1 & 1 & 1 & 1 & 1 \\ + & & & & & \\ x_5 & & & & & 1 \\ \hline \end{array}$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\, B_3\, B_1\, B_0 \rangle$

$$\begin{array}{lcccccc} & & & & 1 & 1 & \\ \hline T & & 1 & 1 & 0 & 1 & 1 \\ + & & & & & & \\ x_1 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : 2

$$\begin{array}{lcccccc} T & & 1 & 1 & 1 & 0 & 0 \\ + & & & & & & \\ x_2 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lcccccc} & & & & & 1 & \\ \hline T & & 1 & 1 & 1 & 0 & 1 \\ + & & & & & & \\ x_3 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 0 \end{array}$$

Cost : 1

$$\begin{array}{lcccccc} T & & 1 & 1 & 1 & 1 & 0 \\ + & & & & & & \\ x_4 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lcccccc} & 1 & 1 & 1 & 1 & 1 & \\ \hline T & & 1 & 1 & 1 & 1 & 1 \\ + & & & & & & \\ x_5 & & & & & & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\, B_3\, B_1\, B_0 \rangle$

$$
\begin{array}{lcccccc}
 & & & & 1 & 1 & \\
T & & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & & \\
x_1 & & & & & & 1 \\
\hline
 & & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

$$
\begin{array}{lcccccc}
T & & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & & \\
x_2 & & & & & & 1 \\
\hline
 & & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lcccccc}
 & & & & & 1 & \\
T & & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & & \\
x_3 & & & & & & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Cost : 1

$$
\begin{array}{lcccccc}
T & & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & & \\
x_4 & & & & & & 1 \\
\hline
 & & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lcccccc}
 & 1 & 1 & 1 & 1 & 1 & \\
T & & 1 & 1 & 1 & 1 & 1 \\
+ & & & & & & \\
x_5 & & & & & & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Cost : 5

# **Example:** Say $|T| = 27 = <11011>_2$      $T = <B_4\ B_3\ B_1\ B_0>$

$$\begin{array}{l|ccccc}
 & & & \color{red}{1} & \color{red}{1} & \\
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\ \hline
 & 1 & 1 & 1 & 0 & 0
\end{array}$$

Cost : $\color{red}{2}$

$$\begin{array}{l|ccccc}
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\ \hline
 & 1 & 1 & 1 & 0 & 1
\end{array}$$

Cost : $\color{red}{0}$

$$\begin{array}{l|ccccc}
 & & & & \color{red}{1} & \\
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\ \hline
 & 1 & 1 & 1 & 1 & 0
\end{array}$$

Cost : $\color{red}{1}$

$$\begin{array}{l|ccccc}
T & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & \\
x_4 & & & & & 1 \\ \hline
 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Cost : $\color{red}{0}$

$$\begin{array}{l|cccccc}
 & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \\
T & & 1 & 1 & 1 & 1 & 1 \\
+ & & & & & & \\
x_5 & & & & & & 1 \\ \hline
 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Cost : $\color{red}{5}$

- Total for 5 insertions: $\color{red}{2} + \color{red}{0} + \color{red}{1} + \color{red}{0} + \color{red}{5} = \color{red}{8}$ key-comparisons (not 5 x 5).

# **Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lcccccc} & & & & {\color{red}1} & {\color{red}1} & \\ \hline T & & 1 & 1 & 0 & 1 & 1 \\ + & & & & & & \\ x_1 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : $\color{red}2$

$$\begin{array}{lcccccc} & & & & & & \\ \hline T & & 1 & 1 & 1 & 0 & 0 \\ + & & & & & & \\ x_2 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 1 \end{array}$$

Cost : $\color{red}0$

$$\begin{array}{lcccccc} & & & & & {\color{red}1} & \\ \hline T & & 1 & 1 & 1 & 0 & 1 \\ + & & & & & & \\ x_3 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 0 \end{array}$$

Cost : $\color{red}1$

$$\begin{array}{lcccccc} & & & & & & \\ \hline T & & 1 & 1 & 1 & 1 & 0 \\ + & & & & & & \\ x_4 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 \end{array}$$

Cost : $\color{red}0$

$$\begin{array}{lcccccc} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & \\ \hline T & & 1 & 1 & 1 & 1 & 1 \\ + & & & & & & \\ x_5 & & & & & & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Cost : $\color{red}5$

- Total for 5 insertions: $\color{red}2$ + $\color{red}0$ + $\color{red}1$ + $\color{red}0$ + $\color{red}5$ = $\color{red}8$ key-comparisons (not 5 x 5).
- Initially: T has $27 - \alpha(27) = 27 - 4 = 23$ edges

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$
\begin{array}{lccccc}
 & & & 1 & 1 & \\
T & 1 & 1 & 0 & 1 & 1 \\
+ & & & & & \\
x_1 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 0
\end{array}
$$

Cost : 2

$$
\begin{array}{lccccc}
T & 1 & 1 & 1 & 0 & 0 \\
+ & & & & & \\
x_2 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lccccc}
 & & & & 1 & \\
T & 1 & 1 & 1 & 0 & 1 \\
+ & & & & & \\
x_3 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

Cost : 1

$$
\begin{array}{lccccc}
T & 1 & 1 & 1 & 1 & 0 \\
+ & & & & & \\
x_4 & & & & & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Cost : 0

$$
\begin{array}{lcccccc}
 & 1 & 1 & 1 & 1 & 1 & \\
T & & 1 & 1 & 1 & 1 & 1 \\
+ & & & & & & \\
x_5 & & & & & & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Cost : 5

- Total for 5 insertions: 2 + 0 + 1 + 0 + 5 = 8 key-comparisons (not 5 x 5).
- Initially: T has 27 - $\alpha(27)$ = 27 – 4 = 23 edges
- After 5 insertions: T has 32 - $\alpha(32)$ = 32 – 1 = 31 edges

# **Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lcccccc} & & & & 1 & 1 & \\ \hline T & & 1 & 1 & 0 & 1 & 1 \\ + & & & & & & \\ x_1 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : 2

$$\begin{array}{lcccccc} & & & & & & \\ \hline T & & 1 & 1 & 1 & 0 & 0 \\ + & & & & & & \\ x_2 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 0 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lcccccc} & & & & & 1 & \\ \hline T & & 1 & 1 & 1 & 0 & 1 \\ + & & & & & & \\ x_3 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 0 \end{array}$$

Cost : 1

$$\begin{array}{lcccccc} & & & & & & \\ \hline T & & 1 & 1 & 1 & 1 & 0 \\ + & & & & & & \\ x_4 & & & & & & 1 \\ \hline & & 1 & 1 & 1 & 1 & 1 \end{array}$$

Cost : 0

$$\begin{array}{lcccccc} & 1 & 1 & 1 & 1 & 1 & \\ \hline T & & 1 & 1 & 1 & 1 & 1 \\ + & & & & & & \\ x_5 & & & & & & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Cost : 5

- Total for 5 insertions: 2 + 0 + 1 + 0 + 5 = 8 key-comparisons (not 5 x 5).
- Initially: T has 27 - $\alpha$(27) = 27 – 4 = 23 edges
- After 5 insertions: T has 32 - $\alpha$(32) = 32 – 1 = 31 edges
- The 5 insertions added: 31-23 = 8 new edges.

**Example:** Say $|T| = 27 = \langle 11011 \rangle_2$ $\quad$ $T = \langle B_4\ B_3\ B_1\ B_0 \rangle$

$$\begin{array}{lccccc} & & & \color{red}{1} & \color{red}{1} & \\ T & 1 & 1 & 0 & 1 & 1 \\ + & & & & & \\ x_1 & & & & & 1 \\ \hline & 1 & 1 & 1 & 0 & 0 \end{array}$$

Cost : $\color{red}{2}$

$$\begin{array}{lccccc} T & 1 & 1 & 1 & 0 & 0 \\ + & & & & & \\ x_2 & & & & & 1 \\ \hline & 1 & 1 & 1 & 0 & 1 \end{array}$$

Cost : $\color{red}{0}$

$$\begin{array}{lccccc} & & & & \color{red}{1} & \\ T & 1 & 1 & 1 & 0 & 1 \\ + & & & & & \\ x_3 & & & & & 1 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

Cost : $\color{red}{1}$

$$\begin{array}{lccccc} T & 1 & 1 & 1 & 1 & 0 \\ + & & & & & \\ x_4 & & & & & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 \end{array}$$

Cost : $\color{red}{0}$

$$\begin{array}{lcccccc} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \color{red}{1} & \\ T & & 1 & 1 & 1 & 1 & 1 \\ + & & & & & & \\ x_5 & & & & & & 1 \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Cost : $\color{red}{5}$

- Total for 5 insertions: $\color{red}{2} + \color{red}{0} + \color{red}{1} + \color{red}{0} + \color{red}{5} = \color{red}{8}$ key-comparisons (not 5 x 5).
- Initially: T has 27 - $\alpha(27)$ = 27 – 4 = 23 edges
- After 5 insertions: T has 32 - $\alpha(32)$ = 32 – 1 = 31 edges
- The 5 insertions added: 31-23 = 8 new edges.
- 8 new edges = 8 key-comparisons

Total Cost of k successive inserts
into a Binomial Heap with n nodes ?

Total Cost of $k$ successive inserts
into a Binomial Heap with $n$ nodes ?

total cost is at most $O(k \log (n + k))$ key-comparisons

Total Cost of k successive inserts
into a Binomial Heap with n nodes ?

**Claim:** If $k > \log_2 n$, total cost is at most key-comparisons

Total Cost of $k$ successive inserts
into a Binomial Heap with $n$ nodes ?

**Claim:** If $k > \log_2 n$, total cost is at most $2k$ key-comparisons

## Total Cost of $k$ successive inserts into a Binomial Heap with $n$ nodes ?

**Claim:** If $k > \log_2 n$, total cost is at most $2k$ key-comparisons

**Proof:** Do A2 – Q4 ☺

Total Cost of $k$ successive inserts
into a Binomial Heap with $n$ nodes ?

**Claim:** If $k > \log_2 n$, total cost is at most $2k$ key-comparisons

**Proof:** Do A2 – Q4 ☺

$\Rightarrow$ Average cost per insert is $\leq 2$ key-comparisons !