MATH2043
Ordinary Differential Equations

Chapter 2
First Order Differential Equations

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First order linear equations

According to the general definition, first order linear equations are of the form

$$a_1(x)y' + a_0(x)y = g(x).$$

Let's assume $a_1(x) \neq 0$ on some interval I, then one can divide the equation by $a_1(x)$ to obtain the so called **standard form**:

First order linear ODE (standard form)

$$y' + p(x)y = q(x), \quad x \in I. \tag{1}$$

A *general principle* for solving an equation is to start from simple/special cases.

1. If $p(x) = 0$ in equation (1), then

$$y = \int q(x)\,dx + c,$$

where c is an arbitrary constant.

First order linear equations

2. If $q(x) = 0$, then

$$\frac{dy}{dx} = -p(x)y \quad \Rightarrow \quad \frac{dy}{y} = -p(x)\,dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int p(x)\,dx$$

$$\Rightarrow \quad \ln|y| = -\int p(x)\,dx + c \quad \Rightarrow \quad y = ce^{-\int p(x)\,dx}.$$

可以进行简单的分离变量即可将dy与dx分开。

Method of integrating factors

In the general case $p(x) \neq 0$ and $q(x) \neq 0$, the idea is to convert the equation to a new first order linear ODE where the corresponding $p(x) = 0$.

Method of integrating factors

The first order linear equation

$$y' + p(x)y = q(x),$$

has the **general solution**

$$y = \frac{1}{\mu(x)} \left[\int \mu(x) q(x)\, dx + c \right],$$

where c is an arbitrary constant and

$$\mu = e^{\int p(x)\, dx}$$

is called an **integrating factor**.

Method of integrating factors

Example 1

Find the general solution of the equation

$$y' - \frac{y}{x} = x^2.$$

Initial value problem

A **particular solution** is obtained once a particular value of c is given/found.

In practice, the value can be found from an **initial condition** $y(x_0) = y_0$, where (x_0, y_0) are given constants.

Example 2

Find a solution of the equation

$$(1 + \cos x)y' - (\sin x)y = 2x$$

such that $y(0) = 1$.

Separable equations

Consider the 1st order equation

$$y' = -\frac{x}{y}. \tag{2}$$

This equation is nonlinear, so we can't apply the method of integrating factors.

However, if we write $y' = dy/dx$, then we have

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad y\,dy = -x\,dx \quad \Rightarrow \quad \int y\,dy = -\int x\,dx$$
$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c \quad \Rightarrow \quad x^2 + y^2 = c,$$

where c is an arbitrary constant.

The last expression $x^2 + y^2 = c$ is called an **implicit solution** of the equation since it gives an implicit relation between the variables. An implicit solution can be verified by implicit differentiation:

$$x^2 + y^2 = c \quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Separable equations

Equation (2) is an example of **separable equations**.

Definition 3 (Separable equations)

The first order equation $y' = f(x, y)$ is called **separable** if f is the product/quotient of a function of x only and a function of y only.

To solve a separable equation, we first write it in the form (separation of variables)

$$A(y)\,dy = B(x)\,dx,$$

then integrate both sides to obtain the solution.

Separable equations

Example 4

We found the integrating factor for linear equations by solving the equation $\mu' = p(x)\mu$, which is both linear and separable.

Example 5

Solve the IVP

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1.$$

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Example 6

Solve the equation

$$y' = (1 - 2x)y^2.$$

Separable equations

Remark

When we solve an equation by separation of variables, we may miss some solutions by making implicit assumptions. In the above example, can you find a solution such that $y(1) = 0$?

Motivation

Let $\phi(x, y)$ be a function defined in a rectangle $R = (a, b) \times (c, d)$ and the partial derivatives ϕ_x, ϕ_y exists in R. Consider the equation

$$\phi(x, y) = c$$

for some constant c. Taking implicit partial differentiation on x to both sides of the above equation, we obtain

$$\phi_x(x, y) + \phi_y(x, y)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \phi_x(x, y)\, dx + \phi_y(x, y)\, dy = 0.$$

Now, if we have an ODE written in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0.$$

Then $\phi(x, y) = c$ is an implicit solution to the equation if

$$M = \phi_x, \quad N = \phi_y.$$

Exact equations

Definition 7 (Exact equations)

The ODE

$$M(x,y)dx + N(x,y)dy = 0, \tag{3}$$

is called **exact** if there exists a function $\phi(x,y)$ such that

$$M = \phi_x, \quad N = \phi_y.$$

If (3) is exact, then $M_y = \phi_{xy}$ and $N_x = \phi_{yx}$. From multivariate calculus, we know

$$\phi_{xy} = \phi_{yx}$$

if $\phi_x, \phi_y, \phi_{xy}, \phi_{yx}$ are continuous in a neighborhood, i.e. M, N, M_x, N_y are continuous. Hence $M_y = N_x$. The converse is also true, so we have the theorem.

Exact equations

Theorem 8

Let $M(x,y), N(x,y)$ *be functions defined on a rectangle* $R = (a,b) \times (c,d)$ *such that* M, N, M_x, N_y *are continuous on* R*. Then the ODE*

$$M(x,y)\,dx + N(x,y)\,dy = 0, \quad (x,y) \in R$$

is exact if and only if

$$M_y = N_x \quad \text{on } R.$$

The proof of the theorem is a constructive and gives a solution to (3) as

$$\phi(x,y) = c,$$

where

$$\phi(x,y) = \int_{x_0}^{x} M(t,y)\,dt + \int N(x_0,y)\,dy.$$

In practice, we will not choose an x_0 and solve the equation as follows.

Exact equations

Solve an exact equation

If the equation $M\,dx + M\,dy = 0$ is exact, i.e. $M_y = N_x$, then

$$\begin{aligned}
\phi &= \int M\,dx + h(y) \\
\phi_y &= \frac{\partial}{\partial y}\int M\,dx + h'(y) = N \\
h'(y) &= N - \frac{\partial}{\partial y}\int M\,dx \\
h(y) &= \int\left(N - \frac{\partial}{\partial y}\int M\,dx\right)
\end{aligned}$$

and the solution is $\phi(x, y) = c$ for an arbitrary constant c.

Exact equations

Example 9

Solve the equation

$$\left(3x^2 + 4xy\right) dx + \left(2x^2 + 2y\right) dy = 0.$$

Example 10

Find a solution of the equation

$$\left(-x^3 + x^2 \sin y + y\right) \frac{dy}{dx} = \left(2x \cos y + 3x^2 y\right).$$

with the initial condition $y(0) = 2$.

Remark

Separable equations are special examples of exact equations.

Integrating factors

The idea of integrating factors for linear equations can be adopted to convert non-exact equations into exact ones.

Integrating factors for making exact equations

An **integrating factor** for the equation $M\,dx + N\,dy = 0$ is a function μ such that the equation

$$\mu M\,dx + \mu N\,dy = 0.$$

is exact.

1. If $(M_y - N_x)/N$ is a function of x only, then we have the integrating factor

$$\mu(x) = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right).$$

2. If $(N_x - M_y)/M$ is a function of y only, then we have the integrating factor

$$\mu(y) = \exp\left(\int \frac{N_x - M_y}{M}\,dy\right).$$

Integrating factors

Example 11

Solve the equation

$$(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0.$$

Example 12

$$1 + \left(\frac{x}{y} - \sin y\right) y' = 0.$$

Homogeneous equations

Some equations can not be solved by any of the usual methods, but can be converted to one of the familiar types by using substitutions.

We consider some special type of equations that can be solved in this way.

Definition 13 (Homogeneous equations (nonlinear))

A 1st order ODE $y' = f(x, y)$ is called **homogeneous** if f can be written as a function of y/x only. That is, we consider equations in the form

$$y' = f\left(\frac{y}{x}\right). \tag{4}$$

Terminology

The term *homogeneous* has a different meaning in the context of linear equations.

To solve equation (4), we adopt the following steps.

Homogeneous equations

1. Consider the substitution
$$\frac{y}{x} = v,$$
where v is considered as a function of x only. Then $y = xv(x)$ and $y' = v + xv'$. The original equation (4) becomes
$$v + xv' = f(v),$$
or equivalently,
$$xv' = f(v) - v.$$
2. This new equation is separable and can be solved for v in x by separation of variables.
3. Back substitute v with y/x to obtain the solution in terms of x, y.

Homogeneous equations

Example 14

Solve the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

Example 15

Solve the equation

$$(x^2 - 3y^2)dx + 2xydy = 0.$$

Remark

The equation in the last example can also be solved using integrating factors for exact equations (exercise).

Bernoulli equations

Definition 16 (Bernoulli equations)

Bernoulli equations are ODEs in the form

$$y' + p(x)y = q(x)y^r,$$

where r is a constant.

If $r = 0$ or $r = 1$, then the Bernoulli equation reduces to a linear equation.

Hence we only need consider the cases when $r \neq 0$ and $r \neq 1$. In this case, then equation is nonlinear, non-separable and can not be made exact by the usual methods.

We can solve Bernoulli equations in the following steps.

Bernoulli equations

1. Consider the substitution
$$v = y^{1-r}.$$
Using the chain rule, we obtain
$$\frac{dv}{dx} = (1-r)y^{-r}\frac{dy}{dx} = (1-r)y^{-r}(qy^n - py) = (1-r)(q-pv),$$
or equivalently,
$$\frac{dv}{dx} + (1-r)p(x)v = (1-r)q(x).$$
2. This is a linear equation for $v(x)$. Solve v using the method of integrating factors.
3. Obtain y by back substitution.

Example 17

Solve the equation
$$y' + y = xy^3.$$

Initial value problems

We now turn to the general first order equations and investigate the theoretical properties of the solutions.

Definition 18 (Initial value problems (1st order equations))

Given a first order equation $y' = f(x, y)$, a point $(x_0, y_0) \in \mathbb{R}^2$, the **initial value problem (IVP)**

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{5}$$

is to find solutions of $y' = f(x, y)$ satisfying $y(x_0) = y_0$. The condition $y(x_0) = y_0$ is called an **initial condition**.

Existence and uniqueness

There are two fundamental questions concerning the IVP (5).

Existence: Do the solutions exist (at least one solution)?

Uniqueness: Is the solution unique (at most one solution)?

If both existence and uniqueness hold, then we say the problem has a unique solution.

Linear equations

We begin with the simpler case of linear equations.

Theorem 19 (Existence and Uniqueness of Solutions (Linear Equations))

Suppose the functions p and q are continuous in an interval $I = (a, b)$ containing x_0. Then there exists a unique solution to the IVP

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

in the interval I.

The proof is based on the method of integrating factors with minor modifications.

Example 20

Investigate the existence and uniqueness of solutions for the IVP

$$xy' + 2y = 4x^2, \quad y(1) = 2.$$

General case

We now turn to general equations $y' = f(x, y)$, where f may be linear or nonlinear in y.

Theorem 21 (Existence and Uniqueness of Solutions (General Case))

Suppose the functions f and f_y are continuous in a closed rectangle

$$R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b].$$

Then there exists an unique solution to the IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

on an interval $I = [x_0 - \delta, x_0 + \delta]$ with some $\delta \leq a$

Remark

For linear equations, we have $f = -p(x)y + q(x)$ and $f_y = -p(x)$. Hence f, f_y are continuous if and only if p, q are continuous. That is, the hypotheses in the general theorem reduces to those in the theorem for linear equations.

General case

Before proving the theorem, we demonstrate how one can apply the result of the theorem by several simple examples.

Example 22

Investigate the existence and uniqueness of solutions for the IVP

$$y' = y^2, \quad y(1) = -1.$$

Example 23

Investigate the existence and uniqueness of solutions for the IVP

$$y' = y^{1/3}, \quad y(0) = 0.$$

General case

Example 24

Investigate the existence and uniqueness of solutions for the IVP

$$y' = xy - \sin(y), \quad y(1) = 2.$$

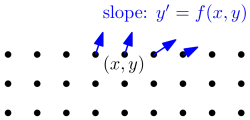
Direction field

Definition 25 (Direction Field)

A **direction field** (slope field) for the differential equation

$$y' = f(x, y)$$

can be constructed by evaluating $f(x, y)$ at each point (x_i, y_j) of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is *tangent* to the graph of the solution passing through that point.



The direction field is mostly efficiently drawn by a computer program. Try this online tool
https://homepages.bluffton.edu/ nesterd/java/slopefields.html

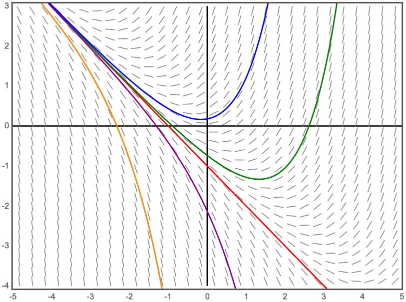
Direction field

Example 26

Investigate the qualitative behavior of the solutions for the equation

$$y' = x + y$$

from its direction field.



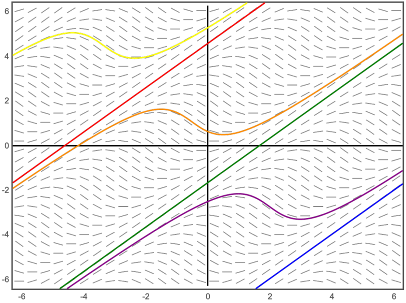
Direction field

Example 27

Investigate the qualitative behavior of the solutions for the equation

$$y' = \sin(x - y)$$

from its direction field.



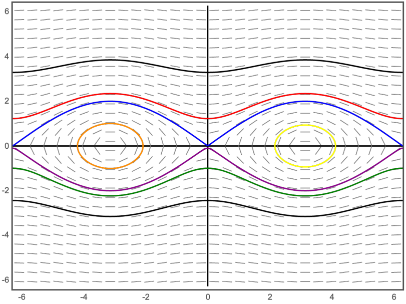
Direction field

Example 28

Investigate the qualitative behavior of the solutions for the equation

$$y' = \frac{\sin x}{y}$$

from its direction field.



Direction field

To summarize,

- A direction field of an equation drawn on a fairly fine grid gives a good picture of the *qualitative* behavior of solutions.
- The construction of a direction field is often a useful first step in the investigation of an equation.

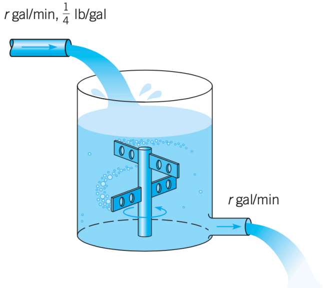
Mixing

Let's consider some examples to show how one can construct and analyze a mathematical model using (1st order) differential equations.

Example 29

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate.

1. Set up the initial value problem that describes this flow process.
2. Find the amount of salt in the tank at any time, and also find the limiting amount that is present after a very long time.



Mixing

Remark

Although this particular example has no special significance, models of this kind are often used in problems involving a *pollutant in a lake*, or a *drug in an organ* of the body, for example.

In such cases, the equation may be more complicated, but the simplified mathematical model still reflects the *essential features* of the physical process under reasonable conditions.

Compound interest

Suppose that a sum of money is deposited in a bank or money fund that pays interest at an annual rate r. The value of the investment at any time t depends on the frequency with which interest is compounded as well as on the interest rate.