MATH2043
Ordinary Differential Equations

Chapter 1
Introduction

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❶ Formulation & Classification of Differential Equations

Ordinary differential equations

Definition 1 (Ordinary differential equation (ODE))

An **ordinary differential equation (ODE)** of **order** n for the function $y(x)$ is an equation that can be put in the form

$$F\left(x, y, y', y'', ..., y^{(n)}\right) = 0. \tag{1}$$

We assume that it is always possible to solve a given ODE for its highest derivative, obtaining the form

$$y^{(n)} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right).$$

Notations

- Derivatives of y up to order 3 are denoted by y', y'', y''', respectively.
- Derivatives of order $n \geq 4$ are usually denoted by $y^{(n)}$.

Ordinary differential equations

Example 2

The equations

$$y''' - 3x^2 y' + y = e^x, \tag{2}$$

$$y'' + \sin y = 0 \tag{3}$$

are ODEs of order 3 and 2, respectively, for the function $y(x)$.

Linear vs nonlinear

Definition 3 (Linear ODEs)

An ODE of the form (1) is said to be **linear** if F is $\mathbb{C}$-linear in $\left(y, y', \ldots, y^{(n)}\right)$ (as a vector). Equivalently, linear ODEs of $y(x)$ are equations of the form

$$\sum_{m=0}^{n} a_m(x) y^{(m)}(x) = g(x),$$

where $\left\{a_m(x)\right\}_{m=0}^{n}$ and $g(x)$ are complex-valued functions.

An ODE is said to be **nonlinear** if it is not linear.

Note

The linearity is for $\left(y, y', \ldots, y^{(n)}\right)$ all together, not separately. The latter case is called *multilinear*. Consider the map $(y, y') \mapsto yy'$.

Linear vs nonlinear

Example 4

- Equation (2) is linear.
- Equation (3) is nonlinear since $\sin(y)$ is nonlinear in y.

Partial differential equations

Definition 5 (Partial differential equation (PDE))

- A **partial differential equation (PDE)** for a function $u(x_1, x_2, \ldots, x_m)$ is an equation involving $x_1, x_2, \ldots, x_n$, u, and the partial derivatives of u.
- The **order** of a PDE is the highest order partial derivative appearing in the equation.
- A PDE is called **linear** if the equation is linear in u and its partial derivatives.

Example 6

- The *Laplace equation* for $u(x, y, z)$:

$$u_{xx} + u_{yy} + u_{zz} = 0$$

  is a 2nd order linear PDE.
- The *Korteweg–de Vries (KdV)* equation for $\phi(x, t)$:

$$\phi_t + \phi_{xxx} - 6\phi\phi_x = 0$$

  is a 3rd order nonlinear PDE.

Solutions

Definition 7 (Solution of and ODE)

We say $y = \phi(x)$ is a **solution** to the ODE

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$

on the interval $I = (a, b)$ if $\phi', \dots, \phi^{(n-1)}$ exists on I and

$$\phi^{(n)}(x) = f\left(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x)\right), \quad x \in I.$$

Example 8

- The functions $y = \sin x$ and $y = \cos x$ are both solutions to the ODE $y'' + y = 0$ on $(-\infty, \infty)$.
- The function $y = -1/x$ is a solution to the ODE $y' = y^2$ on $(0, \infty)$ and $(-\infty, 0)$. Another solution is $y = 0$, which is valid on $(-\infty, \infty)$.

Historical remarks

- Differential equations first came into existence with the invention of calculus by Newton and Leibniz.
- Many classical differential equations are derived from problems in physics. For example, the Newton's law of motion can be modeled by the equation

$$m\mathbf{u}''(t) = \mathbf{f},$$

  where m is the mass, $\mathbf{u}$ is the position, $\mathbf{f}$ is the force, and t is the time. Fourier proposed the heat equation to model the heat flow:

$$u_t = k(u_{xx} + u_{yy} + u_{zz}).$$