ASSIGNMENT 4

PSTAT 160A – W19
Nils Detering

Release date: **Monday, January 28**
Due date: **Thursday, February 7, before your lecture in class**

The first part of this assignment sheet contains exercises for the sections. The second part consists of homework problems which have to be submitted on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, your **TAs name**, and your **section's time & date** legibly on the front page of your homework sheet. Make sure that all your sheets are stapled together. Solve all homework problems. Your reasoning has to be comprehensible and complete. To receive full credit sufficient explanations need to be provided. **Two** randomly chosen questions will be graded.

Exercises for the Sections and Self-Study

1. Let $(S_n)_{n\geqslant 0}$ be a simple random walk starting at 0 ($S_0 = 0$) and with $p = 0.4$ and $q = 1-p = 0.6$. Compute the following probabilities:

   - $\mathbb{P}[S_4 = 2, S_m \neq 3 \text{ for all } m = 1, 2, 3]$,
   - $\mathbb{P}(S_2 = 2, S_5 = 1)$,
   - $\mathbb{P}(S_2 = 2, S_4 = 3, S_5 = 1)$.

2. For a simple symmetric random walk $(S_n)_{n=0,1,2,\dots}$, show that

$$\mathbb{P}[S_4 = 0] = \mathbb{P}[S_3 = 1].$$

3. Propose a numerical method to approximate the irrational number $\pi = 3.1415926\dots$ by using the integral representation $\pi = \int_0^1 4\sqrt{1-x^2}dx$ and a sequence of independent uniformly distributed random variables $U_1, U_2, \dots$ taking values in $(0, 1)$.

4. Let $X_1, X_2, X_3, \dots$ be a sequence of independent random variables, where X_n is Binomial distributed with parameter n, p for all n.

   a) Does the sequence $(\frac{X_n}{n})_{n\geqslant 1}$ converge as $n \to \infty$ with probability 1?

   b) Show that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{X_n - np}{\sqrt{np}} \leqslant x\right] = \mathbb{P}[X \leqslant x] \quad \text{for all } x \in \mathbb{R}$$

   where X is a $N(0, 1)$-distributed random variable.

Homework Problems

1. (10 points) (Markov's and Chebyshev's Inequality)
   a) Use Markov's inequality to show that for a sequence of positive random variables $X_1, X_2, \dots$ with values in $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\lim_{i\to\infty} \mathbb{E}[X_i] = 0$, it holds that $\lim_{i\to\infty} \mathbb{P}[X_i = 0] = 1$.
   b) Use Markov's and Chebyshev's inequality to give two upper bounds on the probability that an exponential distributed random variable with parameter λ (rate) is larger than x. Calculate the bounds explicitly for $\lambda = 5$ and $x = 3$,

2. (10 points) Let $(S_n)_{n\geqslant 0}$ be a simple random walk starting at 0 ($S_0 = 0$) and with $p = 0.7$ and $q = 1 - p = 0.3$.
   a) Compute the probability mass function (p.m.f.) of S_3,
   b) In mathematical finance, the random walk is used for an elementary modeling of the movement of a stock. Define the price of a stock at day n as

   $$P_n = exp(S_n + bn), \quad \text{for } n = 0, 1, 2, \dots$$

   Compute the expectation of P_{10}.

3. (10 points) Let $(S_n)_{n\geqslant 0}$ be a simple symmetric random walk. Compute $\mathbb{P}(S_n = y | S_m = x)$ for the two cases $n > m$ and $n < m$.

4. (10 points) Let $X_1, X_2, \dots$ be independent Poisson random variables with mean λ and $S_n = X_1 + \cdots + X_n$. Show that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{n} \leqslant t\right) = \begin{cases} 1 & \text{if} \quad \lambda < t, \\ 0 & \text{if} \quad \lambda > t. \end{cases}$$