Examples & proofs lecture 3

Example 3.1. Expectation Bernoulli RV

$$\mathbb{P}(X=1)=p \ , \ \mathbb{P}(X=0)=1-p$$

and thus $\mathbb{E}[X]=1\cdot p+0\cdot(1-p)=p$

Example 3.2. Expectation Poisson RV

$\mathbb{P}(X=k)=e^{-\lambda}\cdot\frac{\lambda^k}{k!}$ and thus

$$\mathbb{E}[X]=\sum_{k\in\mathbb{N}} e^{-\lambda}\cdot\frac{\lambda^k}{k!}\cdot k=\sum_{k=1}^{\infty}e^{-\lambda}\frac{\lambda^k}{k!}\cdot k=\lambda\sum_{k=1}^{\infty}e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}$$

$$=\lambda\cdot e^{-\lambda}\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}_{e^{\lambda}}=\underline{\underline{\lambda}}$$

Proof that $F_X^{-1}(U)\overset{d}{=}X$ for $U\in \mathrm{Unif}((0,1))$

Let $Z:=F_X^{-1}(U)$, then

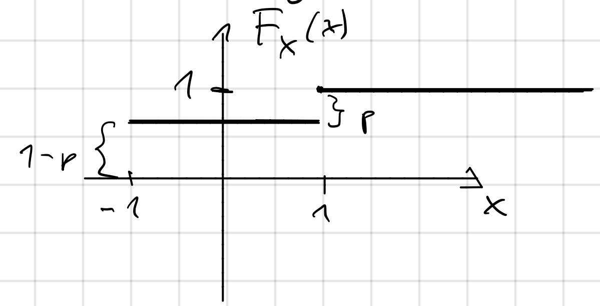
$$F_Z(z)=\mathbb{P}(Z\le z)=\mathbb{P}(F_X^{-1}(U)\le z)\overset{\text{mon. fct. on both sides of "}\le\text{"}}{=}\mathbb{P}(\underbrace{F_X(F_X^{-1}(U))}_{"U"}\le F_X(z))$$

$$=\mathbb{P}(U\le F_X(z))=F_U(F_X(z))=F_X(z) \qquad \square$$

$F_U(u)=u$ for $u\in(0,1)$

Example 3.3. $\mathbb{P}(X=1)=p$, $\mathbb{P}(X=-1)=1-p$, then clearly

$$F_X(x)=\begin{cases}0 & x<-1\\ 1-p & -1\le x<1\\ 1 & 1\le x\end{cases}$$



Inverse of F_X:

$$F_X^{-1}(u) = \begin{cases} -1 & \text{if } 0 < u \leq 1-p \\ +1 & \text{if } 1-p < u \leq 1 \end{cases}$$

and $F_X^{-1}(U) = 1 \cdot \mathbb{1}_{\{1-p < U \leq 1\}} - 1 \cdot \mathbb{1}_{\{0 < U \leq 1-p\}}$

Relation conditional pmf / conditional probability

For discrete RVs by definition

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(X=x)}.$$

Define events $A = \{\omega \in \Omega : Y(\omega) = y\}$ and $B = \{\omega \in \Omega : X(\omega) = x\}$,

then $P(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(X=x)} = p_{Y|X}(y|x)$

Example 3.4. Joint pmf of X and Y is given by

$p(x,y) = \mathbb{P}(X=x, Y=y) = \frac{x+y}{18}$ $\quad x,y = 0,1,2.$

What is $p_{Y|X}(y|x)$?

Relation marginal/joint

First calculate $\mathbb{P}(X=x) = p_X(x) = \sum_{y=0}^{2} p(x,y) = \frac{x}{18} + \frac{x+1}{18} + \frac{x+2}{18} = \frac{x+1}{6}$ $\quad x = 0,1,2$

Then $p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{x+y}{3 \cdot (x+1)}$