Lecture 5
Tail bounds and limit theorems

Nils Detering

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Tail bounds

- Often interested in probability $\mathbb{P}(X \geq a)$ but precise distribution is not known.
- Instead information about $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ are at hand

Recall: If X, Y RVs with $\mathbb{P}(X \geq Y) = 1$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$

Proposition (Markov's inequality)

Let X be a random variable with $X \geq 0$. Then for any $c > 0$,

$$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}[X]}{c}.$$

Proof.
See whiteboard $\square$

Tail bounds

Example

Bernoulli RV: See whiteboard

Tail bounds

Often we can do better if we know the variance of X.

Proposition (Chebyshev's inequality)

Let X be a random variable with finite mean $\mu = \mathbb{E}[X]$ and variance $\sigma^2 = \mathrm{Var}(X)$. Then for any $c > 0$,

$$\mathbb{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}.$$

Chebyshev's inequality is often used to get estimate $\mathbb{P}(X \geq c + \mu) \leq \frac{\sigma^2}{c^2}$ or $\mathbb{P}(X \geq \mu - c) \leq \frac{\sigma^2}{c^2}$.

$\leq$

Proof.

See whiteboard

$$\{\omega : X(\omega) \leq \mu - c\} \subseteq \{\omega : |X(\omega) - \mu| \geq c\}$$

$\square$

Tail bounds

Example

See whiteboard

Limit theorems

Consider **i.i.d. Bernoulli** RVs $X_1, X_2, \dots$ with success probability p. Set

$$S_n := X_1 + \dots + X_n.$$

What would you expect S_n/n to be for large n? **Of course p!** Otherwise our notion of probability would be bad (relative frequency).

Theorem ((LLN) Law of large numbers (weak))

Let $X_1, X_2, \dots$ i.i.d. RVs with finite mean $\mathbb{E}[X_1] = \mu$ and finite variance $Var(X_1) = \sigma^2$. Let $S_n = X_1 + \dots + X_n$, then for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}\left(|\frac{S_n}{n} - \mu| < \varepsilon \right) = 1$$

Proof.

See whiteboard $\square$

Limit theorems

Theorem (Strong law of large numbers)

Let $X_1, X_2, \ldots$ *i.i.d. RVs with finite mean* $\mathbb{E}[X_1] = \mu$ *and finite variance* $Var(X_1) = \sigma^2$*. Let* $S_n = X_1 + \cdots + X_n$*, then*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1$$

Remark: Weak law follows from strong law.

Proof.

Skipped $\square$

Limit theorems

Example (Application 1)

See whiteboard

Example (Application 2)

See whiteboard

Limit theorems

Theorem (Central limit theorem)

Let $X_1, X_2, \ldots$ i.i.d. RVs with finite mean $\mathbb{E}[X_1] = \mu$ and finite variance $Var(X_1) = \sigma^2$. Let again $S_n = X_1 + \cdots + X_n$, then for any fixed $-\infty \leq a \leq b \leq \infty$

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b) &= \Phi(b) - \Phi(a) \\ &= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

Proof.

Sketch (see whiteboard) $\square$