Lecture 5
Moment generating functions

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Moment generating functions

- Useful tool to derive results about sums and limits of RVs.
- Let X be a random variable. The moment generating fct. (m.g.f.) of X is the function $m_X : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$m_X(t) = m(t) = \mathbb{E}[e^{tX}],$$

if it exists.

Why the name?
Because

$$m'(t) = \frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[\frac{de^{tX}}{dt}] = \mathbb{E}[Xe^{tX}]$$

Now

$$m'(0) = \mathbb{E}[X]$$

and

$$m^k(0) = \mathbb{E}[X^k].$$

Moment generating functions

Example (Bernoulli)

See whiteboard.

Example (Normal)

See whiteboard.

Properties Moment generating functions

Important properties:

1. If X and Y indep., then the m.g.f. of $X+Y$ is the product of the m.g.f. of X and the m.g.f. of Y:

$$m_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX}e^{tY}]$$

$$= \mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}] = m_X(t) \cdot m_Y(t)$$

2. For constant c:

$$m_{cX}(t) = m_X(ct)$$

Moment generating functions

Example (Bernoulli)

See whiteboard.

Example (Normal)

See whiteboard.

Properties Moment generating functions

3. Moment generating fcts. uniquely determine the probability distribution of a RV. This means that if $m_X(t) = m_Y(t)$ for all t, then the distribution of X and Y equal.
4. Continuity: Let $X_1, X_2, \dots$ sequence of RVs with m.g.f. $m_{X_1}, m_{X_2}, \dots$. Let further X be RV s.t. for all t, $m_{X_n}(t) \to m_X(t)$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \mathbb{P}(X \leq x)$$

for every x for which $\mathbb{P}(X \leq x)$ continuous.

Example (Sum of Poisson)

See whiteboard.

Characteristic function

- Alternative to m.g.f. is the characteristic function defined by

$$\varphi_X(t) = \mathbb{E}[e^{itX}]$$

  with $i = \sqrt{-1}$.
- The expectation $\mathbb{E}[e^{itX}]$ is defined by
  $\mathbb{E}[e^{itX}] = \mathbb{E}[\mathrm{Re}(e^{itX})] + i\mathbb{E}[\mathrm{Im}(e^{itX})]$
- Similar properties as moment-generating function but well defined for any RV X and any value of t
  $e^{itX} = \cos(tX) + i\sin(tX)$
- if $\mathbb{E}[X^n] < \infty$, then $\varphi_X^{(n)}(0) = i^n\mathbb{E}[X^n]$