ASSIGNMENT 6

PSTAT 160A – W19
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Release date: **Monday, February 18**
Due date: **Thursday, February 28, before your lecture in class**

The first part of this assignment sheet contains exercises for the sections. The second part consists of homework problems which have to be submitted on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, your **TAs name**, and your **section's time & date** legibly on the front page of your homework sheet. Make sure that all your sheets are stapled together. Solve all homework problems. Your reasoning has to be comprehensible and complete. To receive full credit sufficient explanations need to be provided. **Two** randomly chosen questions will be graded.

Exercises for the Sections and Self-Study

1. Let $X_0, X_1, X_2, \dots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition matrix

$$P = \begin{pmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

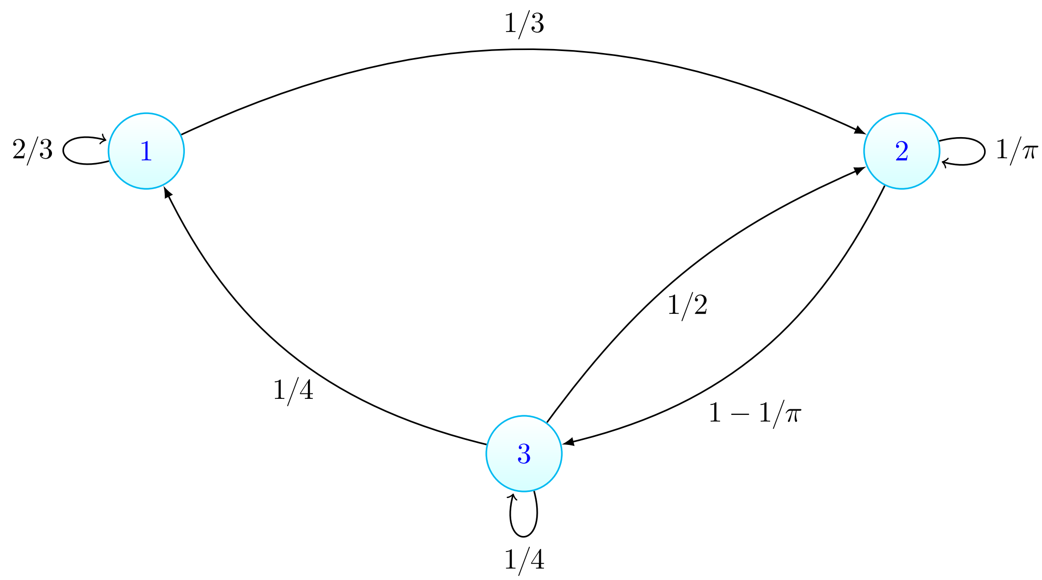
and initial distribution $\alpha^T = (0.1, 0.6, 0.3)$. Compute

   (a) $\mathbb{P}[X_7 = 1 \,|\, X_6 = 3]$,
   (b) $\mathbb{P}[X_9 = 3 \,|\, X_1 = 1, X_5 = 2, X_7 = 1]$,
   (c) $\mathbb{P}[X_0 = 2 \,|\, X_1 = 3]$,
   (d) $\mathbb{E}[X_2]$.

2. Consider again the Markov chain from Problem 1.

   (a) Draw the transition graph for the Markov chain.
   (b) Define the Markov chain $Y_0, Y_1, Y_2, \dots$ by $Y_k = X_{2k}$ for all $k \geqslant 0$. Draw the transition graph for this new Markov chain.

3. A Markov chain $X_0, X_1, X_2, \dots$ has the following transition graph:



(a) Provide the transition matrix for the Markov chain.

(b) Given the initial distribution $\alpha^T = (0, 0, 1)$ calculate $\mathbb{E}[X_2]$.

4. Let P be a $k \times k$ stochastic matrix. Show that $\sum_{j=1}^{k} P_{ij}^n = 1$ for all $1 \leqslant i \leqslant k$. Here P_{ij}^n denotes the entry in the i-th row and j-th column of the n-th power of matrix P.

Homework Problems

1. (10 points) The PSTAT Department at UCSB has developed a Markov model to simulate graduation rates in their programs. Students might drop out, repeat a year, or move on to the next year until they graduate. Students have a 4% chance of repeating their current year. First-years and sophomores have a 10% chance of dropping out. For juniors and seniors, the drop-out rate is 5%.

   Provide the state space $\mathcal{S}$ and the transition probability matrix for this Markov chain. Also draw the associated transition graph.

2. (10 points) Let $X_0, X_1, X_2, \ldots$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$, transition probabilities

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/5 & 1/10 & 7/10 \end{pmatrix}$$

   and initial distribution $\alpha^T = (1, 0, 0)$. Find the following probabilities:

   (a) $\mathbb{P}[X_5 = 2 \mid X_4 = 1]$,

   (b) $\mathbb{P}[X_1 = 1,\ X_2 = 2]$,

   (c) $\mathbb{P}[X_1 = 2 \mid X_2 = 1]$,

(d) $\mathbb{P}[X_5 = 1 \mid X_1 = 2, X_2 = 3, X_3 = 2]$.

3. (10 points) Consider a Markov chain $(X_n)_{n=0,1,2,...}$ with state space $\mathcal{S} = \{1, 2, 3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/10 & 7/10 & 0 \end{pmatrix}.$$

The initial distribution is given by $\alpha^T = (1/2, 1/6, 1/3)$. Compute

(a) $\mathbb{P}[X_2 = k]$ for all $k = 1, 2, 3$;

(b) $\mathbb{E}[X_2]$.

Does the distribution of X_2 computed in (a) depend on the initial distribution α? Does the expected value of X_2 computed in (b) depend on the initial distribution α? Give a reason for both of your answers.

4. (10 points) A stochastic matrix is called *doubly stochastic* if its columns sum to 1. Let $X_0, X_1, ...$ be a Markov chain on the state space $\mathcal{S} = \{1, ..., k\}$ with doubly stochastic transition matrix P and initial distribution that is uniform on $\mathcal{S}$.

Show that the distribution of X_n is uniform on $\mathcal{S}$ for all $n \geqslant 0$.