ASSIGNMENT 2

PSTAT 160A – Winter 2019
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Release date: **Saturday, January 11th**
Due date: **Thursday, January 24th, before your lecture in class**

The first part of this assignment sheet contains exercises for the sections. The second part consists of homework problems which have to be submitted on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, your **TAs name**, and your **section's time & date** legibly on the front page of your homework sheet. Make sure that all your sheets are stapled together. Solve all homework problems. Your reasoning has to be comprehensible and complete. To receive full credit sufficient explanations need to be provided. **Two** randomly chosen questions will be graded.

Practice Exercises

1. Derive the inverse cumulative distribution function F_X^{-1} for:

   1. An exponentially distributed random variable X with density $f_X(x) = \lambda e^{-\lambda x}$ for all $x > 0$ and $\lambda > 0$;
   2. A Poisson distributed random variable X with probability mass function $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for all $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\lambda > 0$.

2. The joint probability mass function of the two random variables X and Y with values in $\{1, 2, 3\}$ and $\{0, 1, 2\}$, respectively, is given by the following table:

$X \backslash Y$	0	1	2
1	0	1/9	0
2	1/3	2/9	1/9
3	0	1/9	1/9

   (a) Find the conditional probability mass function of X given $Y = y$.

   (b) Find the conditional expectation $\mathbb{E}[X|Y = y]$ for $y = 0, 1, 2$.

3. Consider the (infinite) coin toss model with success probability $p \in (0, 1)$, i.e., the probability space $\Omega = \{0, 1\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$ with $\mathbb{P}$ such that

$$\mathbb{P}[X_1 = x_1, \ldots, X_N = x_N] = p^k(1-p)^{N-k} \text{ with } k = x_1 + \ldots + x_N$$

   and where $X_i(\omega) = x_i$ for $\omega = (x_1, \ldots, x_N) \in \Omega$.

   (a) Compute the conditional distribution given that K successes occur, i.e., determine

$$\mathbb{P}[X_1 = x_1, \ldots, X_N = x_N \mid S_N = K]$$

   where $S_N = \sum_{i=1}^N X_i$.

(b) Compute

$$\mathbb{E}[X_i \,|\, S_N = K] \quad (i = 1, \dots, N)\,.$$

4. Let the joint density of two random variables X and Y be given by

$$f(x,y) = \begin{cases} 2xe^{x^2-y}, & \text{if } 0 < x < 1 \text{ and } y > x^2 \\ 0, & \text{else.} \end{cases}$$

Find the conditional density function $f_{Y|X}(y|x)$ and the conditional probability $\mathbb{P}(Y \geqslant 1/4|X = x)$. Verify the averaging identity

$$\mathbb{P}(Y \geqslant 1/4) = \int_{-\infty}^{\infty} \mathbb{P}(Y \geqslant 1/4|X = x) f_X(x)dx.$$

Homework Problems

1. (10 points) Let X be a continuous random variable with density

$$f_X(x) = \begin{cases} \frac{\theta}{x^{1+\theta}} & \text{for } x \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

with some parameter $\theta > 0$.

(a) Verify that f_X is a proper density function.

(b) Compute the cumultative distribution function F_X of X.

(c) Describe how you can draw samples from the random variable X on a computer by using a uniformly distributed random variable U taking values in $(0, 1)$. Show all your work.

2. (10 points) Let X be a random variable with values in $\{0, 1\}$ and Y a random variable with values in $\{0, 1, 2\}$. Initially we have the following partial information about their joint probability mass function:

$X \backslash Y$	0	1	2
0			
1	1/8		1/8

Subsequently we are provided with the following information:

(i) Given $X = 1$, Y is uniformly distributed.

(ii) $p_{X|Y}(0|0) = 2/3$.

(iii) $\mathbb{E}[Y|X = 0] = 4/5$.

Use this additional information to fill out the missing values in the table. Remember that as usual you need to provide calculations and explanations, not just the numbers.

3. (10 points) Assume the number B of blossoms on an apple tree is Poisson distributed with parameter λ and suppose that, independently from all the others, each blossom will yield a fruit with probability $p \in [0, 1]$.

   (a) Compute the distribution of the number of apples A on the tree.

   (b) Given the number of apples A, how is the original number of blossoms B distributed?

   Hint: Both answers can be phrased in terms of Poisson distributions. You also need to use the fact that

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \qquad (\lambda \in \mathbb{R}).$$

4. (10 points) Let the joint density function of (X, Y) be

$$f(x, y) = \frac{x+y}{4}, \quad 0 < x < y < 2.$$

   (a) Find the conditional density of X given $Y = y$, i.e., $f_{X|Y}(x \,|\, y)$.

   (b) Compute the conditional probabilities $\mathbb{P}[X < 1/2 \,|\, Y = 1]$ and $\mathbb{P}[X < 3/2 \,|\, Y = 1]$.

   (c) Compute the conditional expectation $\mathbb{E}[X^2 \,|\, Y = y]$. Check the averaging identity

$$\int_{-\infty}^{\infty} \mathbb{E}[X^2 \,|\, Y = y] f_Y(y) dy = \mathbb{E}[X^2].$$