ASSIGNMENT 3

PSTAT 160A – W19
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Release date: **Tuesday, January 22nd**
Due date: **Thursday, January 31st, before your lecture in class**

The first part of this assignment sheet contains exercises for the sections. The second part consists of homework problems which have to be submitted on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, your **TAs name**, and your **section's time & date** legibly on the front page of your homework sheet. Make sure that all your sheets are stapled together. Solve all homework problems. Your reasoning has to be comprehensible and complete. To receive full credit sufficient explanations need to be provided. **Two** randomly chosen questions will be graded.

Exercises for the Sections and Self-Study

1. The *characteristic function* $\varphi_X(t)$ of a random variable X is defined as the function $\varphi_X(t) = \mathbb{E}[e^{itX}]$ for all $t \in \mathbb{R}$, where $i = \sqrt{-1}$ denotes the imaginary unit. The expected value of a complex-valued random variable Z is simply defined through the real and imaginary parts $\mathbb{E}[Z] = \mathbb{E}[\mathrm{Re}(Z)] + i\mathbb{E}[\mathrm{Im}(Z)]$.

   The characteristic function $\varphi_X(t)$ shares many of the useful properties of the moment generating function $m_X(t)$ and formally we have the relation $\varphi_X(t) = m_X(it)$. In addition, characteristic functions have the added advantage of being well defined for *all* values of $t \in \mathbb{R}$ and *all* random variables X (why?). Especially, if the n-th moment exists, then it can be calculated via

   $$\varphi_X^{(n)}(0) = i^n \mathbb{E}[X^n].$$

   Derive the characteristic function for a Bernoulli(p) random variable and a Binomial(n, p) random variable. Then use the derivative of the characteristic function to compute the first and second moment for both random variables.

2. A fair coin is flipped repeatedly. Find the expected number of flips needed to get two heads in a row.

3. Let X and Y be two independent, uniformly distributed random variables on $]0,1[$. Compute the random variable $\mathbb{E}[X \mid \max\{X, Y\}]$ as a function of $\max\{X, Y\}$.

4. The so-called *Wald's Identity* states the following: Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with finite mean. Let further N be a nonnegative integer-valued random variable independent of the $X_1, X_2, \ldots$ with finite mean. We define the random sum $S_N := X_1 + \cdots + X_N$. Then

   $$\mathbb{E}[S_N] = \mathbb{E}[N] \cdot \mathbb{E}[X_1].$$

   Prove Wald's identity by conditioning on $\{N = n\}$ and by using the law of total expectation.

Homework Problems

1. (10 points) The density of a continuous random variable X is $f(x) = xe^{-x}$ for $x > 0$. Given $X = x$, Y is uniformly distributed on $(0, x)$. Find $\mathbb{P}[Y < 2]$ by conditioning on X.

2. (10 points) Let X and Y be two independent, geometrically distributed random variables with success probability $p \in]0, 1[$. Compute the random variable $\mathbb{E}[X^2 \mid X + Y]$ as a function in $X + Y$.

   Hint: You may use the fact that
$$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)n(2(n-1)+1)}{6}.$$

3. (10 points) Let X be a random variable with moment generating function
$$m_X(t) = \frac{1}{2} + \frac{1}{3}e^{-4t} + \frac{1}{6}e^{5t}.$$
   a) Use the derivatives of $m_X(t)$ to find the mean and variance of X.
   b) Find the probability mass function of X and use it to check your results for the mean and variance you derived in a).

4. (10 points) Let $X_1, X_2, X_3, \dots$ be a sequence of independent random variables, where X_n is Poisson distributed with parameter n for all $n \geqslant 1$.
   a) Does the sequence $(\frac{X_n}{n})_{n \geqslant 1}$ converge as $n \to \infty$ with probability 1?
   b) Show that
$$\lim_{n\to\infty} \mathbb{P}\left[\frac{X_n - n}{\sqrt{n}} \leqslant x\right] = \mathbb{P}\left[X \leqslant x\right] \quad \text{for all } x \in \mathbb{R}$$
   where X is a $N(0, 1)$-distributed random variable.
   c) Use b) to show that
$$\lim_{n\to\infty} e^{-n}\left(1 + n + \frac{n^2}{2!} + \ldots + \frac{n^n}{n!}\right) = \frac{1}{2}.$$