ASSIGNMENT 1

PSTAT 160A – W19
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Release date: **Tuesday, January 8th**
Due date: **Tuesday, January 22nd, before the lecture in class**

The first part of this assignment sheet contains exercises for yourself to practice. Most of them will be shown during the sections but it is essential that you solve them before the section. Then you are able to compare your solution and check it for mistakes. The second part consists of homework problems which have to be submitted on the due date.

Instructions for the homework: Please write your **full name**, **perm number**, your **TAs name**, and your **section's time & date** legibly on the front page of your homework sheet. Make sure that all your sheets are stapled together. Solve all homework problems. **Your reasoning has to be comprehensible and complete**. To receive full credit sufficient explanations need to be provided. **Two** randomly chosen questions will be graded.

Practice Exercises.

1. The waiting times X and Y (in minutes) of two clients A and B who are standing in line at two different check outs in the supermarket are modeled as independent, exponential random variables with parameter 1.

   (a) Find the cumulative distribution function of the random variable $M := \min\{X, Y\}$ where $\min\{x, y\}$ is just the smaller value of the two numbers.

   (b) Find the probability density function of M. Do you recognize the socalled *probability law* or *probability distribution* of the random variable M?

   (c) What is the probability that both clients wait more than 2 minutes?

2. Each throw of a 6-sided die lands on each of the even numbers 2,4,6 with probability c and on each one of the odd numbers 1,3,5 with probability $2c$.

   (a) Find c.

   Suppose the die is tossed. We introduce two random variables X and Y. Let X equal 1 if the result is greater than 3 and 0 otherwise. Let Y equal 1 if the result is odd and 0 otherwise.

   (b) Find the joint probability mass function of X and Y.

   (c) Are the random variables X and Y independent? Justify your answer.

   (d) Compute $\mathbb{E}[X^2 \cdot \sqrt{Y}]$.

3. Let X and Y be two continuous random variables with joint density function given by

$$f_{X,Y}(x, y) = \begin{cases} 3e^{-x-3y}, & 0 < x < \infty, 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the marginal densities $f_X(x)$ and $f_Y(y)$

(b) Are X and Y independent? Justify your answer.

(c) Compute the probability that $X > Y$.

4. A couple has three kids. But, we do not know whether they are boys or girls. Suppose that each gender is equally likely and that their occurence is independent of each other.

   (a) Describe the setup with a proper probability space $(\Omega, \mathbb{P})$.

   Consider the following events:

   C: All kids are either 3 girls or 3 boys.

   D: There is at most one boy.

   E: There are at least one girl and at least one boy.

   (b) Compute the probability of the events C, D and E.

   (c) Show that C and D as well as D and E are independent, but not C and E.

5. You are given a 4-sided die with each of its four sides showing a different number of dots from 1 to 4. When rolled, we assume that each value is equally likely.

   Suppose that you roll the die twice in a row.

   (a) Specify the underlying probability space $(\Omega, \mathbb{P})$ in order to describe the corresponding random experiment.

   Let X represent the maximum value from the two rolls.

   (b) Specify X explicitly as a mapping defined on the sample space Ω onto a properly determined state space S_X.

   (c) Compute the probability mass function p_X of X.

   (d) Compute the cumulative distribution function F_X of X.

6. You have a bag which contains 100 coins. 99 of these coins are normal and fair having the two sides head and tail. But, one coin is *fake* and has *two heads*!

   You pick a coin randomly out of the bag without checking whether it is a normal coin or the fake one. Suppose that each coin is equally likely to be picked. You throw the picked coin $n \in \mathbb{N}$ times in a row (still without checking whether you have a regular coin or not) and observe the coin lands heads n times.

   What is the probability that you have picked the *fake* coin?

7. Three friends A, B, and C are sitting at a table and are playing following game with a fair 6-sided die: A starts throwing the die. If she does not toss number 6, she passes the die forward to B. It is B's turn now, and she throws the die. If B does not toss number 6, she passes the die forward to C. Now, C is throwing. And, again, if C does not throw number 6, she gives the die back to A, and they restart a new round. They continue until the first 6 is thrown, then the game stops. The person who rolled the first 6 is considered the winner.

   Compute the probability that C wins the game.

Hint: You may use the fact that if $|q| < 1$, then

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

8. In a casino, you play the following game: You choose a number between 1 and 6 and then throw 3 fair six-sided dice. After tossing, for each die showing your chosen number, the casino pays you \$1.

   In order to play this game, you have to pay the casino a stake of \$1.

   Let X denote your **net** win after one round of gambling (i.e., the payment received from the casino *minus* your stake).

   (a) Determine the state space S_X of X.

   (b) Compute the probability mass function p_X of X.

   (c) Compute the expected value $\mathbb{E}[X]$ of X.

   (d) In general, a game is called a *fair* game if the net win of the gambler is 0 "on average". Is the above game a fair game? If not, what should be the stake of the gambler in order to make this a fair game? Give a reason for your answer.

Homework Problems

1. (10 points) You are given a 4-sided die with each of its four sides showing a different number of dots from 1 to 4. When rolled, we assume that each value is equally likely.

   Suppose that you roll the die twice in a row.

   (a) What is a reasonable sample space Ω for this experiment? What are the probabilities $\mathbb{P}(\{\omega\})$ for each possible outcome ω from the sample space Ω?

   Let X represent the minimum value from the two rolls. For example, if you roll a 1 and a 2, then X is 1, since $\min\{1, 2\} = 1$. Similarly, if you roll 3 and 3, then X is 3, since $\min\{3, 3\} = 3$.

   (b) What is the state space S_X of X?

   (c) Find the probability mass function p_X of X.

   (d) Find the cumulative distribution function F_X of X.

2. (10 points) Suppose there are two student assistants working as typists in the main office of the Statistics & Applied Probability Department at UCSB. The number of typos per page made by student assistant A is a Poisson random variable with parameter $\lambda_A = 1$. The number of typos per page made by student assistant B is also a Poisson random variable with an average of 10 typos per page.

   One of the professors in the department asks one of the students to type up a letter. From experience, this work will be done with 1/3 probability by student A and with 2/3 probability by student B.

(a) What is the probability that the typewritten letter will contain **exactly one typo**?

(b) It turns out that the typewritten letter does **not** contain **any** typos. Given this information, what is the probability that student B typewrote this letter?

3. (10 points) Suppose you are rolling a *fair* die 600 times independently. Let X count the number of sixes that appear.

(a) What type of random variable is X? Specify all parameters needed to characterize X as well as the state space S_X of X.

(b) Find the probability that you observe the number 6 at most 100 times.

(c) Use a famous limit theorem (which one?) to show why the probability in (b) can be approximated by the value 1/2.

Hint: Use (without proof) the fact that

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = \frac{1}{2}.$$

4. (10 points) Suppose that X is a continuous random variable whose probability density function is given by

$$f_X(t) = \begin{cases} c \cdot (4t - 2t^2), & 0 < t < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant.

(a) What is the value of c?

(b) Compute the cumulative distribution function F_X of X.

(c) Find the probabilities $\mathbb{P}(X = 1)$ and $\mathbb{P}(X > 1)$.

(d) Compute the variance of X.