1 Homework1

1. Given the symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, suppose the $k(k \leq n)$ eigenvalues $\{\lambda_1, \cdots, \lambda_k\}$ of $\mathbf{A}$ are distinct and take any corresponding eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$, (i.e., $\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j = 1, \cdots, k$). Then, prove that $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ are orthogonal.

2. Given the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, prove that the sum of the n eigenvalues of $\mathbf{A}$ is the same as the trace of $\mathbf{A}$ (i.e, $tr(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$).

3. Given the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, prove that the product of the n eigenvalues of $\mathbf{A}$ is the same as the determinant of $\mathbf{A}$ (i.e, $|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$).

4. Prove that if matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive definite, then the eigenvalues of $\mathbf{A}$ are positive.

5. For any real invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, the product $\mathbf{A}^T\mathbf{A}$ is a positive definite matrix.

6. Do you agree with the statement that "If matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible, then $\mathbf{A}$ can be eigendecomposed. ". If no, please given the reasons/examples.

7. Do you agree with the statement that "If matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be eigendecomposed, then $\mathbf{A}$ is invertible. ". If no, please given the reasons or examples.

8. Given arbitrary matrices $\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{n\times p}, \mathbf{C} \in \mathbb{R}^{p\times m}$, prove that $tr(\mathbf{ABC}) = tr(\mathbf{BCA})$.