`FindZerom` : A MATLAB package for computing the roots of analytic functions

Benoit Nennig

Institut supérieur de mécanique de Paris (SUPMECA),
Laboratoire Quartz EA 7393,
3 rue Fernand Hainaut, 93407 Saint-Ouen, France.

`benoit.nennig@supmeca.fr`

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Résumé

This package is suitable to compute the all the roots of an analytic function present inside a closed contour without initial guess. There are numerous root finding algorithm like Newton-Raphson method, Muller's method, the Secant method or the Nelder-Mead simplex method. All these techniques have in common that they require initial approximations for the zeros to start the algorithm. The proposed implementation is based on the method called the Cauchy Integration Method or the Argument Principle Method and allows to compute the number of zeros (including its multiplicity) of a function from contour integral.

This program was initially developed for poroelastic silencer applications[8] in order to solved dispersion equation. The algorithm has been already used in other application fields (see [4, 2, 7] and the references therein) and we proposed here a basic numerical implementation in matlab language.

This *short* documentation explains quickly the theoretical background, shows the calling sequence and presents some examples of applications and validations.

1 The winding number

The root-finding technique presented here is based on integration in the complex plan that does not require knowledge of initial guesses. The other advantages of the method is to avoid to evaluate at the function close to its root where round-off error may be important because of large dynamic of function obtained from matrix determinant. The roots are deduced from the regularity of the function away from the roots.

This is possible because of the Cauchy's Theorem, that allows to compute the number of zeros (including its multiplicity) of f from the following integral relation

$$S_0 = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(\beta)}{f(\beta)} \mathrm{d}\beta = \sum_{k=1}^{N_\beta} n_k, \tag{1}$$

where N_β is the number of zeros and n_k the corresponding multiplicity of the k-th zero lying in the interior of the closed curve C. Note f is chosen to be analytic for the result to hold but the presence of poles may be included by modifying the formula accordingly. This is a classical procedure based on the generalization on the previous relation to any monomial β^n, that is [7]

$$S_n = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f'(\beta)}{f(\beta)} \beta^n \mathrm{d}\beta = \sum_{k=1}^{N_\beta} n_k \beta_k^n, \tag{2}$$

where β_k denote the position of the k-th zero. To simplify further the analysis, function f is assumed to have only simple zeros so the integral (1) is exactly the number of zeros contained in the interior of C. Note this is not a stringent assumption as the occurrence of multiple zeros is highly improbable (in a somewhat different context, it is known that multiplicity may exist for specific material properties set for which simple zeros are merging [9, 11]). We now introduce the associated polynomial Π for the interior of C, that is a polynomial having the same zeros as f, i.e.

$$\Pi(\beta) = \prod_{k=1}^{N_\beta} (\beta - \beta_k) = \sum_{n=0}^{N_\beta} C_n \beta^n. \tag{3}$$

Of course the zeros are not known yet, but the polynomial coefficients C_n can be efficiently computed from the following recursive algorithm [3, 2]

$$C_n = \begin{cases} 1, & \text{if } n = S_0 \\ \dfrac{1}{n - S_0} \displaystyle\sum_{j=1}^{S_0 - n} S_j\, C_{n+j}, & \text{if } n = S_0 - 1, \ldots, 0. \end{cases} \tag{4}$$

Here S_n $(n = 0, 1, 2, \ldots, N_\beta)$, denote the values of the integral (2). Once Π is known, finding its zeros simply requires computing the eigenvalues of the companion matrix which is a very fast procedure for moderate size matrices.

More details can be found in [7] and many other applications can be found in the reference listed therein of in Refs. [2, 8]. Note that extension are possible to matrix functions[5, 6].

1.1 Numerical Implementation

The most time consuming operation is the numerical evaluation of the functions with complex arguments at some quadrature points when computing integrals (2). Since these are integrals of an analytic periodic function over a complete period, the trapezoidal rule is the optimal quadrature rule[1, 25.4.3]. Let C be defined from a regular function $\gamma(s)$ over the interval $s \in [0, 2\pi]$, then the q-point trapezoidal rule approximation to S_n is given by

$$S_n \approx \frac{1}{\mathrm{i}q} \sum_{i=0}^{q-1} \left(\frac{\gamma^n}{\hat{f}} \frac{\mathrm{d}\hat{f}}{\mathrm{d}s} \right)(i/q), \tag{5}$$

where we put $\hat{f}(s) = f(\gamma(s))$. Though the s-derivative of $\hat{f}$ may be obtained formally via symbolic software, it is far more efficient to evaluate the derivative using high order central finite difference scheme, that is

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}s}(i/q) \approx \frac{1}{i/q} \sum_{j=-J}^{J} a_{|j|} \hat{f}\left(\frac{i+j}{q}\right). \tag{6}$$

The procedure is extremely fast since the discrete values of $\hat{f}$ on the regular grid are already calculated. Here we used either the 5^{th} or 9^{th}-order scheme. The corresponding coefficients are displayed in Table 1. The choice of the curve C must depend upon the region of the complex plane where eigenvalues are searched. When roots are expected to be symmetric with respect to the origin, choosing the circular path $\gamma = a e^{\mathrm{i}s}$ appears to be a good compromise. To avoid possible round-off errors, it is then preferable to factorize the term a^n and exclude it for the

TABLE 1 – Finite difference scheme coefficients.

	a_0	a_1	a_2	a_3	a_4
5^{th} order	0	2/3	-1/12	-	-
9^{th} order	0	4/5	-1/5	4/105	-1/280

trapezoidal summation of Eq. (2). Now, if the roots are concentrated along the real or the imaginary axis, elliptical contour integration may also be used. In this case we take $\gamma = a \cos s + ib \sin s$. To conserve good convergence properties in the trapezoidal rule, smooth contour are highly recommended (use reasonable aspect ratio in the ellipse).

The numerical results accuracy can be check because (1) provides the number of zeros. If the number is not closed to an integer, the algorithm has encountered some trouble e. g.

— Presence of non analytic functions (tangent, some bessel function ratio, square root...)
— Too few points
— A zeros is located on the contour

The algorithm change slightly the radius (see variable `RShift`) of the integration path and start again.

The proposed implementation was relevant for our application. Other approaches can also be used for instance use logarithm instead of the ratio f'/f, use orthogonal polynom basis...

remarks If needed, pole can be easily remove by creating a hole in the integration contour. The multiplicity can be obtained but it is not implemented for now. When exponential growth becomes a problem, it is sometime possible to provide directly the ratio f'/f instead of computing f and f' separately.

1.2 Convergence

Here are exposed some practical tips to set up the root-finding parameters to get cheaply the required accuracy on β's. It's worth noting that the more zeros are present in the integration path, the more the dynamic of polynom coefficients C_k is large and subject to round-off error. The repercussion on the roots location may be significant. The solution used here is to split the integration path into concentric ring ; data coming from the previous path can be obviously store. This remarks are illustrated in Table 2. A single circular/elliptical integration contour leads to poor results if more than 15 zeros are present because the round-off stop the convergence. In return, with the same number of $D(\beta)$ evaluation ($N_C/2$ for each circle), the two step approach is by far better. In our application $N_C \in [500, 1000]$ is a good compromise between time and accuracy. This method is sometime faster than doing a refinement, with a small circle or using an other algorithm, for each zeros given by a first rough calculation. Nonetheless, the local refinement is present in the package.

TABLE 2 – Converge of one wave number at fixed integration radius $R = 180$ at 1730 Hz for XFM foam. With one circular integration path (left) and with one circular and annular path (right). Bold character denotes the good digit (extract from Ref [8]).

$N_z = 20$		$N_z = 10 + 10$	
N_C	β	N_C	β
500	**1**8.1310333520 + **17**.2414041458 i	500	**17**.1619265538 + **15**.2060360731 i
1000	**17.6**094544451 + **1**4.9503659291 i	1000	**17.6293**456921 + **15.065**0726700 i
5000	**17.6**308169754 + **15.0**352734983 i	5000	**17.6294474640** + **15.065151316**3 i
20000	**17.6**362164376 + **15.0**289704442 i	20000	**17.6294474640** + **15.0651513164** i

To get 6 figures on 20 roots, around 1000 function was required. This leads to around 50 function evaluations per roots. This is a good results when compare to the complexity of the problem from Ref [8].

2 Package matlab

The package is built on three main file

— `FindZerosm.m`, the main routine

— `diffZcircleTheta.m`, use compute high order finite differences

— `CoefC`, to recast the coeficients S_n into the suitable polynomial form.

this files are adapted to handle elliptic integral path :`diffZcircleEllipse.m`. The calling syntax is recalled here

```
[K, *Ci] = FindZerosm(R,N,fhandle, *Refine, *R0)
% Mandatory input args :
%  R : integration radius
%  N : number of integration points
%  fhandle : finds the zero of the anonymous function fhandle
% Optionnal input args (indicated by *) :
%  Refine = 1 ou 0 (local refinement with small circle around each root)
%  R0 = 0 si l'origine est en O, coordonée de l'origine sinon
% Mandatory output args :
%  K : roots
% Optionnal output args (indicated by *):
%  Ci : save of contour usefull value for annular computation
```

If more than 15-20 zeros are present in C, it is often better to use annular contour to get the 15-20 roots in each closed contour.

```
[K1,Ci] = FindZerosm(R1,N,@nom, *Refine, *R0);                        % disk between 0 and R1
[K2,Ci] = FindZerosmAnnular(R2,N,@nom,Ci, *Refine, *R0);              % Annular region between R1
[K3,Ci] = FindZerosmAnnularEllipse([R3 R4],N,@nom,Ci, *Refine, *R0); % "Annular" elliptic R2 and
K = [K1 ; K2 ; K3];
```

Circular contour and elliptic contour are provided.

3 Validation and examples

3.1 Cantilever beam

The first example concern a cantilever beam of length L. We are looking for the first discrete values of the wavenumber α_n roots of

$$f(\alpha L) = \cos \alpha L \cosh \alpha L - 1. \tag{7}$$

The roots can be used for instance to compute the eigenfrequency of the beam. The usefull root are real and positive value of $\alpha_n L$.

For a complete problem description, see `https://en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory`. For an illustration of the package see `example/Cantilver.m`.

3.2 2D acoustic liner dispersion equation

The second proposed example (time convention $\mathrm{e}^{-\omega t}$) is based on acoustic propagation in a duct of height h lined with an impedance. The dispersion equation reads

$$g(\alpha) = \alpha \tan \alpha h + \frac{\mathrm{i}k}{Z}. \tag{8}$$

n	α_n	β_n^+
0	0.7897-1.1705i	7.0543+ 0.1310i
1	2.8012-0.3759i	6.4282+ 0.1638i
2	6.1213-0.1649i	3.4124+ 0.2958i
3	9.3179-0.1077i	0.1632+ 6.1512i
4	12.4865-0.0803i	0.0969+10.3399i
5	15.6441-0.0640i	0.0716+13.9907i
6	18.7964-0.0533i	0.0574+17.4443i
7	21.9456-0.0456i	0.0481+20.7993i
8	25.0929-0.0399i	0.0415+24.0968i
9	28.2389-0.0354i	0.0366+27.3576i

FIGURE 1 – Example of numerical values[10] for transversal and axial wavenumber obtained for a lined 2D acoustic duct, $k = 7$, $h = 1$, and $Z = \pm 3.5(1 + 1i)$ obtained with Newton-Raphson method.

The roots are the transverse wavenumbers α_n and are related to the axial wavenumber with the relation $k^2 = \beta_n^2 + \alpha_n^2$, where k is the free field wavenumber.

To use the proposed method, the previous equation must be modified to keep analytic function

$$f(\alpha) = \alpha \sin \alpha h + \frac{\mathrm{i}k}{Z} \cos \alpha h. \tag{9}$$

To the authors best knowledge, this always possible for closed waveguides.

The proposed approach is validated with the results from Fig. 1 extracted from [10] when $h = 1$, $k = 1$, $c_0 = 1$ and $Z = 3.5(1 + \mathrm{i})$ in `example/Liner2D.m` obtained with Newton-Raphson method.

For this particular example, $R = 10$ and 500 points are put along the contour C . All figures are the same and the computation time is around 0.02 seconds.

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