Authors

SIMO Package User Manual

Contents

Chapter 1

Fundamentals of the IsoGeometric Analysis

The two most common methods of representing curves and surfaces in geometric modeling are implicit equations and parameteric functions. Although it is difficult to confirm that which method is always more appropriate than the other, the parametric methods show some advances in designing and representing shapes in a computer.
As mentioned in [], a parametric method that is more suitable for geometric modeling should has following properties

- capable of precisely representing all the curves the users of the system need;
- easily, efficiently and accurately processed in a computer, in particular
  - the computational of points and derivatives on the curve is efficient;
  - numerical processing of the functions is relatively insensitive to float-ting point round-off error;
  - the functions require little memory for storage;
- simple and mathematically well understood.

Possessing many of those properties, polynomials is a class of functions that is used widely in parameterizing many kinds of geometry. We can named some common methods such as

- Power basis functions
- Bezier and rational Bezier basis function

- B-Spline and NURBS basis functions

Roughly, all above methods represent shapes as a combination of basis functions with the coefficients are coordinate vectors called control points

$$\mathbf{C}\left(\xi\right) = \sum_{i=1}^{n} N_{i,p}\left(\xi\right)\mathbf{P}_i$$

The shape can be changed by modifying the location of control points or by reconstructing the basis functions.
Each method has its own advantages and disadvantages in geometric representation. However in Finite Element Analysis (FEA) the B-Spline and NURBs methods have many appropriate properties such as local support, ease in refinement. As a result, they become the main idea for a new approach for FEA, which is proposed by Hughes, the IsoGeometric Analysis (IGA).
SimoPackage is one of many packages which are developed to make IGA more simple and efficient for everyone. The package mainly supports the B-Spline and NURBS methods for solving many problems in mechanics such as elasticity, heat conducting .... To support the readers, in the first chapter of this guide, some fundamentals of B-spline geometry and IGA will be introduced briefly.

1.1 B-spline basis functions

B-spline basis functions are identified by a knot vector Ξ,

$$\Xi = \left\{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\right\}, \xi_i \in \mathbb{R}, \xi_i \le \xi_{i+1}, i = 1, 2, 3, \ldots, n+p+1$$

where i is a knot index, p is degree of B-spline and n is the number of basis functions (the number of control points). If the knot distribution is uniform, the knot vector is uniform. The ith B-spline basis functions is defined by Cox-de Boor recursive formula

$$N_{i,0}\left(\xi\right) = \begin{cases} 1 & \text{if}\, \xi_i \le \xi \le \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}\left(\xi\right) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}\left(\xi\right) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}\left(\xi\right)$$

Conveniently, If the fomular above gives the output of $0/0$ at a parametric point, the value of basis functions are define 0 at this point.
Each B-spline basis function is C^{∞} continuity inside each open interval $\left(\xi_i, \xi_{i+1}\right)$

and C^{p-m_i} at knot ξ_i where m_i is the multiplicity of the knot ξ_i.
In general, a knot vector can be written in the following form

$$\Xi = \left\{ \underbrace{\xi_1, \ldots, \xi_1}_{m_1}, \underbrace{\xi_2, \ldots, \xi_2}_{m_2}, \ldots, \underbrace{\xi_s, \ldots, \xi_s}_{m_s} \right\}, \xi_i \in \mathbb{R}, \xi_i < \xi_{i+1}, i = 1, 2, 3, \ldots, s$$

where ξ_i is ascending array of knots and m_i is the corresponding multiplicity.
There are some important of B-spline basis functions

1. $N_{i,p}\left(\xi\right)$ are piecewise p-order polynomials
2. Local support: $N_{i,p} > 0$ inside the half-open interval $\left[\xi_i, \xi_{i+p+1}\right)$ and $N_{i,p} = 0$ outside the interval
3. There are at most $p+1$ B-spline basis functions that are different from 0 in the i-th knot span, $\left[\xi_i, \xi_{i+1}\right)$
4. Non-negative and partition of unity: $N_{i,p}\left(\xi\right) \geq 0 \forall \xi$ and $\sum_{i=1}^{i=n}\left(\xi\right) = 1, \forall \xi$
5. If in a local support $\left[\xi_i, \xi_{i+p+1}\right)$ of $N_{i,p}$, the multiplicity of knot ξ_k which is also belong to this half-open interval is m_k then the continuity of $N_{i,p}$ at ξ_i is C^{p-m_i}.
6. As a result, increasing the order of the B-spline or decreasing the multiplicity of knots will increase the continuity of the basis functions.

A knot vector is open if the multiplicities of the first and the last knots are same and equal $p+1$.

$$\Xi = \left\{ \underbrace{a, \ldots, a}_{p+1}, \xi_{p+2}, \xi_{p+3}, \ldots, \xi_n, \underbrace{b, \ldots, b}_{p+1} \right\}$$

In this case, a curve defined using the result B-spline basis functions will be interpolated at the first and the last control points. This is an important property that make B-spline suitable for geometric representation.

To demonstrate some important properties of B-spline basis functions and B-spline curve, let's consider the following examples

Example 1.1.1 (A second-order B-spline basis). Let $\Xi = \left\{0, 0, 0, 1, 2, 2, 3, 4, 5, 5, 5\right\}$ is a knot vector of a second-order B-spline basis functions. We can see that

1. The order $p = 2$ and the length of knot vector is $p + n + 1 = 11$ then there are $n = 8$ B-spline basis functions.

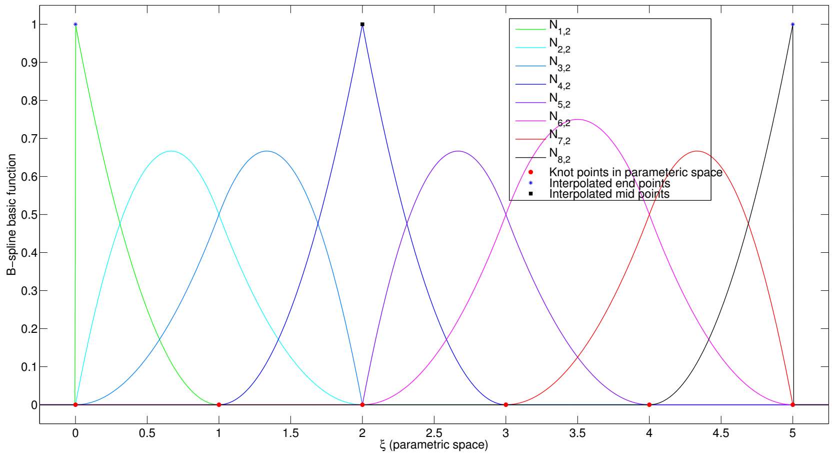

Figure 1.1: All second-order B-spline basic functions

2. This is an open knot vector then the first and the last basis functions, $N_{1,2}$ and $N_{8,2}$, are interpolated respectively at two end points. Moreover, they are C^{-1} or discontinuous at these end points.

3. The forth basis function, the support of $N_{4,2}$ is $[\xi_4, \xi_7]$ or the knot span $\{1,2,2,3\}$. Therefore this basis function is C^1 at $\xi = 1$ and $\xi = 3$ and C^0 at $\xi = 2$.

4. Using the unit participation, we can be sure that $N_{4,2}(2) = 1$.

1.2 B-spline Curves, Surfaces

With B-spline basis functions, we can represent curves, surface and body.

B-spline Curves

A pth-order B-spline curve is defined by

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{P}_i \qquad a \le \xi \le b$$

where $\{P_i\}$ are control points and $\{N_{i,p}\}$ are the pth-degree B-spline basis functions defined on the open knot vectors

$$\Xi = \left\{ \underbrace{a,\ldots,a}_{p+1}, \xi_{p+2}, \xi_{p+3}, \ldots, \xi_n, \underbrace{b,\ldots,b}_{p+1} \right\}$$

Usually, we choose that $a = 0$ and $b = 1$. The polygon formed by $\{\mathbf{P}_i\}$ is called the control polygon.

Example 1.2.1 (A second-order B-spline curve)**.** Using this B-spline basis in Ex.1.1.1 we can construct a picewise second-order curve with control points

$$\begin{array}{cc} P_1(0,0) & P_2(1,-1) \\ P_3(2,-1) & P_4(3,2) \\ P_5(4,-1) & P_6(5,1) \\ P_7(6,-2) & P_8(7,1) \end{array}$$

as in Fig.1.2. We can see that the curve is interpolated at control points P_1, P_8 and P_5 as well as heritage the continuity from the B-spline basis functions.

B-spline Surfaces

A B-spline surface is defined as a tensor product of two (one dimensional) B-spline basis functions

$$\mathbf{S}(\xi,\eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) N_{j,q}(\eta) \mathbf{P}_{i,j}$$

where $\{\mathbf{P}_{i,j}\}$ are a net of $n \times m$ control points. The one dimensional basis functions of p-order $\{N_{i,p}\}$ and of q-order $\{M_{j,q}\}$ are constructed respectively from two following knot vectors

$$\Xi = \left\{ \underbrace{0,\ldots,0}_{p+1}, \xi_{p+2}, \xi_{p+3}, \ldots, \xi_n, \underbrace{1,\ldots,1}_{p+1} \right\}$$

$$\mathbf{H} = \left\{ \underbrace{0,\ldots,0}_{q+1}, \eta_{q+2}, \eta_{q+3}, \ldots, \eta_m, \underbrace{1,\ldots,1}_{q+1} \right\}$$

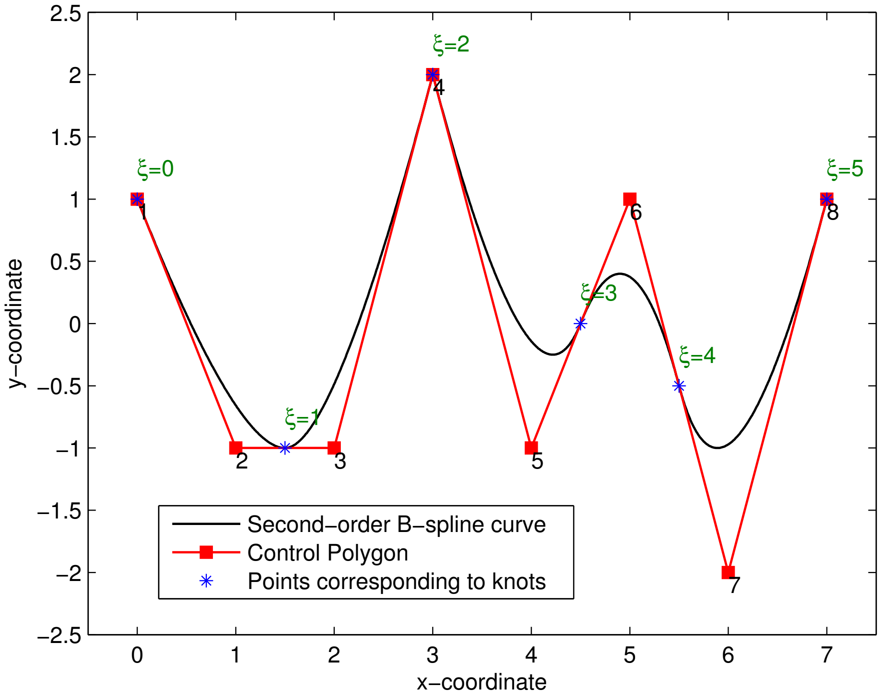

Figure 1.2: A second-order B-spline curve

Example 1.2.2 (A 2-dimensional (3-order × 2-order) B-spline basis functions)**.** Let $\Xi = \{0,0,0,0,1/4,2/4,3/4,1,1,1,1\}$ and $\mathbf{H} = \{0,0,0,1/5,2/5,3/5,3/5,4/5,1,1,1\}$ are two knot vectors of a 3-order and 2-order B-spline basis respectively. Fig.1.3 and Fig.1.4 represent two basis function of the result 2 dimensional B-spline.

Example 1.2.3 (A 2-dimensional (3-order × 2-order) B-spline surface)**.** Let $\Xi = \{0,0,0,1/2,1,1,1,\}$ and $\mathbf{H} = \{0,0,0,1/3,2/3,1,1,1\}$ are two knot vectors of 2-order B-spline basis. Fig.1.5 represents a B-spline surface with a control net consisting of 4×5 control points

```
% Control Points of a B-spline surface

CtrlPts(1 : 3, 1, 1) = [0; 0; 0];
CtrlPts(1 : 3, 2, 1) = [1/3; 0; 0];
CtrlPts(1 : 3, 3, 1) = [2/3; 0; 0];
CtrlPts(1 : 3, 4, 1) = [1; 0; 0];

CtrlPts(1 : 3, 1, 2) = [0; 1/4; 0];
CtrlPts(1 : 3, 2, 2) = [1/3; 1/4; 1];
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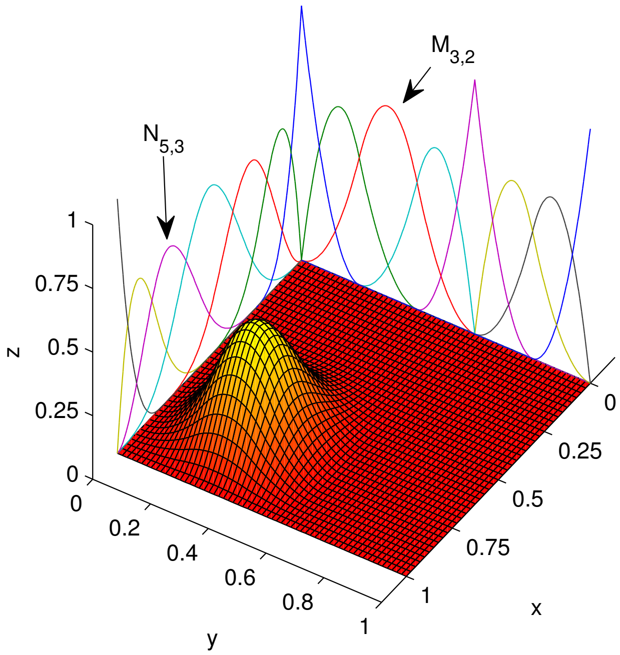

Figure 1.3: The basis function $N_{5,3} \times M_{3,2}$

```
CtrlPts(1 : 3, 3, 2) = [2/3; 1/4; -1/2];
CtrlPts(1 : 3, 4, 2) = [1; 1/4; 0];

CtrlPts(1 : 3, 1, 3) = [0; 2/4; 0];
CtrlPts(1 : 3, 2, 3) = [1/3; 2/4; 2/3];
CtrlPts(1 : 3, 3, 3) = [2/3; 2/4; -1/3];
CtrlPts(1 : 3, 4, 3) = [1; 2/4; 0];

CtrlPts(1 : 3, 1, 4) = [0; 3/4; 0];
CtrlPts(1 : 3, 2, 4) = [1/3; 3/4; 0];
CtrlPts(1 : 3, 3, 4) = [2/3; 3/4; 0];
CtrlPts(1 : 3, 4, 4) = [1; 3/4; 0];

CtrlPts(1 : 3, 1, 5) = [0; 1; 0];
CtrlPts(1 : 3, 2, 5) = [1/3; 1; 1/3];
CtrlPts(1 : 3, 3, 5) = [2/3; 1; -1/3];
CtrlPts(1 : 3, 4, 5) = [1; 1; 0];
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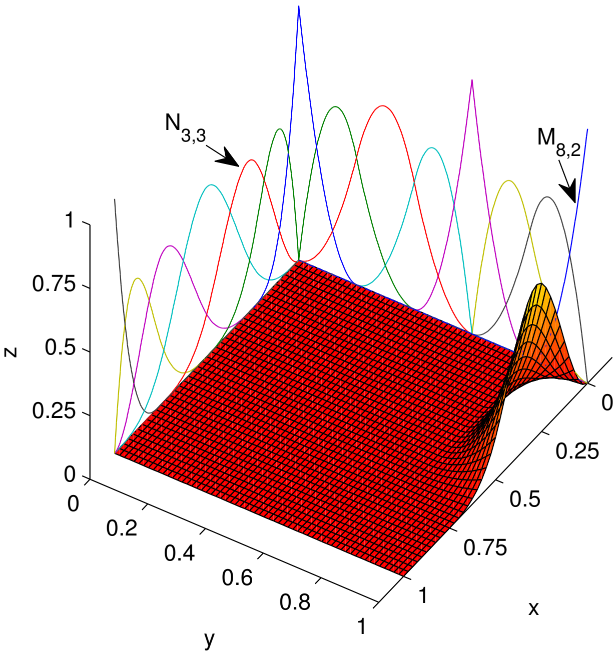

Figure 1.4: The basis function $N_{3,3} \times M_{8,2}$

1.3 NURBS basis functions

It is known from classical mathematics that all the conic curves, including the circle, can be represented using rational functions, which are defined as the ratio of two polynomials

$$x(\xi) = \frac{X(\xi)}{W(\xi)} \qquad y(\xi) = \frac{Y(\xi)}{W(\xi)}$$

where $X(\xi)$, $Y(\xi)$ and $W(\xi)$ are polynomials. Similar to non-uniform rational Bezier curve, a pth-degree Non-Uniform Rational B-Spline (NURBS) curve is defined as follows

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\,\mathbf{P}_i$$

where the Non-Uniform Rational B-Spline (NURBS) basis function, $R_{i,p}(\xi)$ is defined by

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)\,w_i}{\sum_{j=1}^{n} B_{j,p}(\xi)\,w_i}$$

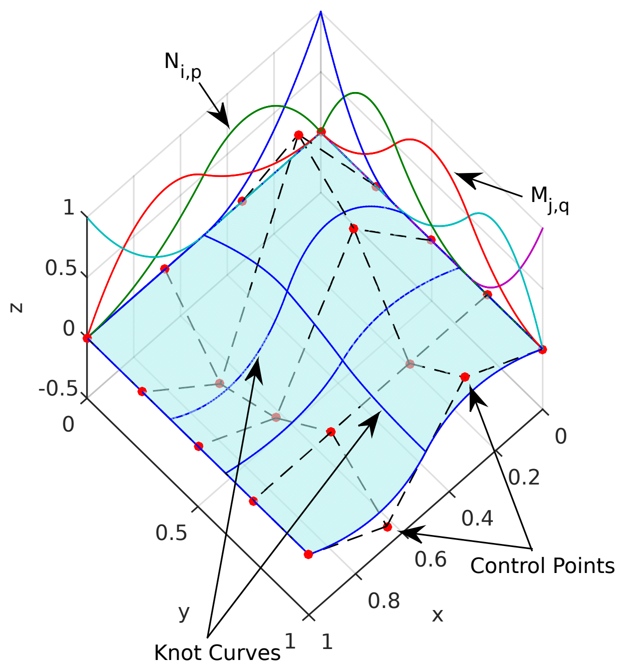

Figure 1.5: A B-spline surface with its control net

$N_{i,p}$ is p-order B-Spline basis function that is constructed from a non-uniform knot vector $\Xi = \left\{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\right\}$ and w_i are scalar called weights and usually assumed possitive i.e. $w_i > 0\ \forall i$. We can see that B-spline curve is a special NURBS where all weights, w_i equal to 1. Another way to represent a NURBS curve is using homogeneous coordinates to represent a rational curve in n-dimensional space as a polynomial curve in $n+1$-dimensional space. Consider a point $\mathbf{P}(x, y, z)$ in 3D space. This point can be written as $\mathbf{P}^w = (X, Y, Z, W) = (wx, wy, wz, w)$, $w \neq 0$ in 4D space. $\mathbf{P}$ is obtained from $\mathbf{P^w}$ by mapping $\mathbf{P}^w$ from the origin to the hyperplane $W = 1$

$$\mathbf{P} = H\left\{\mathbf{P}^w\right\} = H\left\{(X, Y, Z, W)\right\} = \left(\frac{X}{W}, \frac{Y}{W}, \frac{Z}{W}\right)$$

Particularly, for a given set of control points, $\mathbf{P}_i$, and weights, w_i, contruct the weighted control points, $\mathbf{P}_i^w = (w_i x_i, w_i y_i, w_i z_i, w_i)$. The nonrational (polynomial) B-spline curve in four-dimensional space

$$\mathbf{C}^w(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\,\mathbf{P}_i^w = \left(X(\xi), Y(\xi), Z(\xi), W(\xi)\right)$$

where

$$\begin{array}{ll} X(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, w_i x_i & Y(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, w_i z_i \\ Z(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, w_i z_i & W(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, w_i \end{array}$$

Using the map to hyperplane $W = 1$, we have the corresponding rational B-spline curve in three dimensional space is

$$\begin{aligned} x(\xi) &= \tfrac{X(\xi)}{W(\xi)} = \tfrac{\sum_{i=1}^{n} N_{i,p}(\xi)\, w_i x_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\, w_j} = \sum_{i=1}^{n} R_{i,p}(\xi)\, x_i \\ y(\xi) &= \tfrac{Y(\xi)}{W(\xi)} = \tfrac{\sum_{i=1}^{n} N_{i,p}(\xi)\, w_i y_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\, w_j} = \sum_{i=1}^{n} R_{i,p}(\xi)\, y_i \\ z(\xi) &= \tfrac{Z(\xi)}{W(\xi)} = \tfrac{\sum_{i=1}^{n} N_{i,p}(\xi)\, w_i z_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\, w_j} = \sum_{i=1}^{n} R_{i,p}(\xi)\, z_i \end{aligned}$$

where

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)}{W(\xi)} = \frac{N_{i,p}(\xi)\, w_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\, w_j}$$

is the Non-uniform Rational B-Spline (NURBS) basis functions.

Example 1.3.1 (Affect of weight to a NURBS curve)**.** Let $\Xi = \{0,0,0,1/2,1,1,1\}$ is the knot vectors of 2-order NURBS basis. Fig.1.6 represents a NURBS curve that is created using this basis and 4 control points. First, all 4 weights are equal 1 and then the second weight, w_2 is changed to 0.1, 0.5 and 3 in turns.

1.4 Bspline and NURBS constructions of some common geometry

1.5 h-, p- and k- Refinement

1.6 Fundamental steps in Isogeometric Analysis

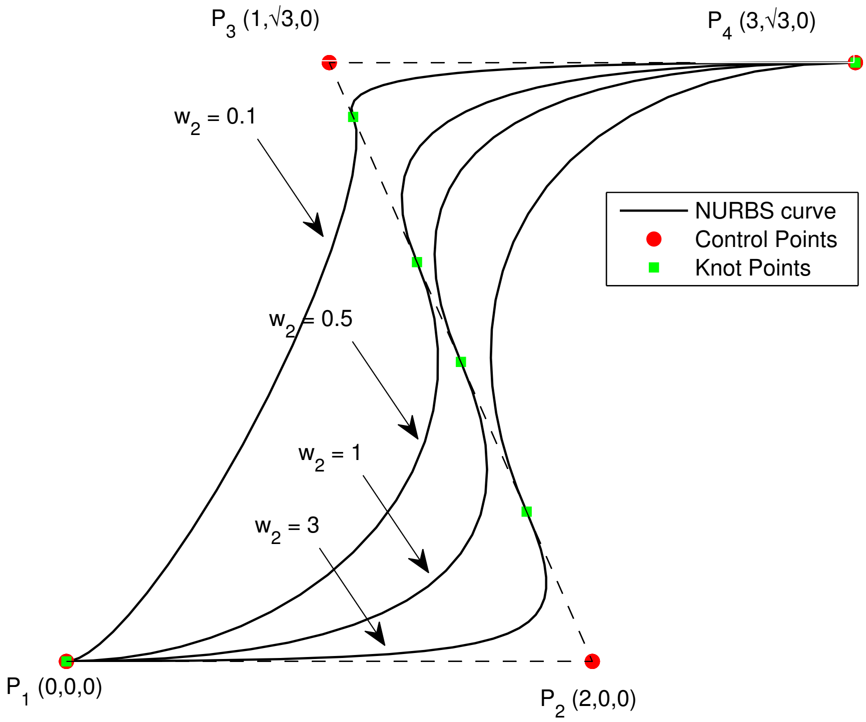

Figure 1.6: NURBS curve changed when a weight is changed