Topological Optimization: Compliance gradient computation

Development

Albert Torres Rubio

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1. Problem Definition

In order to solve a topological optimization problem the gradient of the cost function must be computed, performing this calculation in order to minimize the compliance (as an example), the following steps must be executed to find out which is the gradient of the function. Firstly, defining:

$$l(v) = \int_{\Gamma_N} t \cdot v$$

$$a(\rho, u, v) = \int_{\Omega} J'u : C(\rho) : \nabla^s u$$

In fact, the initial problem we have is the following:

$$\text{mín } l(u)$$

subject to

$$\left.\begin{array}{c} \int_{\Omega} \rho \leq V \\ a(\rho, u, v) = l(v) \end{array}\right\} \quad \forall v \in \gamma \quad and \quad \forall \rho \in \chi$$

Where,

$$\gamma = \{\phi \in H^1(\rho) : \phi \mid_{\Gamma_0} = 0\}$$
$$\chi = L^{\infty}(\Omega, \{0, 1\})$$

2. Differential definition

Performing the calculation of the differential of the cost function:

$$l(u(\rho + \tilde{\rho})) - l(u(\rho)) = l[u(\rho + \tilde{\rho}) - u(\rho)] =$$

$$l[D_\rho u(\rho)\tilde{\rho} + O(\rho^2)] = l[D_\rho u(\rho)\tilde{\rho}] + O(\rho^2)$$

Therefore,

$$Dl(\rho)\tilde{\rho} = l[D_\rho u(\rho)\tilde{\rho}] \tag{1}$$

Where D_ρ is the derivative respect to ρ.

3. Gradient Computation

Defining,

$$F(\rho) = a(\rho, u(\rho), v) - l(v) = 0 \qquad \forall v,\ \forall \rho$$

Therefore, computing the differential:

$$\begin{gathered}
\underbrace{F(\rho+\tilde{\rho})}_{=0} - \underbrace{F(\rho)}_{=0} = a(\rho+\tilde{\rho}, u(\rho+\tilde{\rho}), v) - a(\rho, u(\rho), v) = \\
a(\rho+\tilde{\rho}, u(\rho+\tilde{\rho}), v) - a(\rho+\tilde{\rho}, u(\rho), v) + a(\rho+\tilde{\rho}, u(\rho), v) - a(\rho, u(\rho), v) = \\
a(\rho+\tilde{\rho}, Du(\rho)\tilde{\rho}, v) + D_\rho a(\rho, u(\rho), v)\tilde{\rho} + O(\tilde{\rho}^2) = \\
a(\rho, Du(\rho)\tilde{\rho}, v) + D_\rho a(\rho, u(\rho), v)\tilde{\rho} + O(\tilde{\rho}^2) = 0
\end{gathered}$$

Which lasts in

$$a(\rho, Du(\rho)\tilde{\rho}, v) = \boxed{-D_\rho a(\rho, u(\rho), v)\tilde{\rho}} + O(\tilde{\rho}^2) \qquad \forall v \in \gamma, \forall \rho \in \mathbb{R} \tag{2}$$

Continuing with Equation 1:

$$\begin{gathered}
l(u(\rho+\tilde{\rho})) - l(u(\rho)) = l[D_\rho u(\rho)\tilde{\rho}] + O(\tilde{\rho}^2) = a(\rho, u(\rho), D_\rho u(\rho)\tilde{\rho}) + O(\tilde{\rho}^2) = \\
a(\rho, D_\rho u(\rho)\tilde{\rho}, u(\rho)) + O(\tilde{\rho}^2) = -D_\rho a(\rho, u(\rho)\tilde{\rho}, u(\rho)) + O(\tilde{\rho}^2) = \\
-[a(\rho+\tilde{\rho}, u(\rho), v) - a(\rho, u(\rho), v)] + O(\tilde{\rho}^2) = \\
-\int_\Omega \nabla^s u : [C(\rho+\tilde{\rho}) - C(\rho)] : \nabla^s u + O(\tilde{\rho}^2) = \\
-\int_\Omega \nabla^s u : C'(\rho)\tilde{\rho} : \nabla^s u + O(\tilde{\rho}^2) = \\
= Dl(\rho)\tilde{\rho} + O(\rho^2)
\end{gathered}$$

Therefore,

$$Dl(\rho)\tilde{\rho} = \int_\Omega \underbrace{(\nabla^s u : C'(\rho) : \nabla^s u)}_{\text{g}} \tilde{\rho} = (g, \tilde{\rho})_{L^2}$$

Finally, the computed gradient is:

$$\boxed{g = \nabla^s u : C'(\rho) : \nabla^s u)} \tag{3}$$

Where,

$$C(\rho) = 2\mu(\rho)I + [\kappa(\rho) - \mu(\rho)]I \otimes I$$