Radial Basis Function
Mesh Morphing Tool in OpenFOAM

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Outline

Objective

- Definition, numerical background and examples of use for the Radial Basis Function (RBF) mesh morphing tool in OpenFOAM

Topics

- Background: Parametrisation of geometry for shape optimisation
- Definition and mathematics of Radial Basis Function
- Radial Basis Function interpolation
- RBF mesh morphing tool
- Examples in 2-D and 3-D

Background

Geometry Parametrisation

- In shape optimisation, the optimiser will seek to minimise the penalty function with respect to a **small** number of control parameters
- A CAD description or a CFD mesh carries a very large number of shape parameters: for the optimiser to work effectively, changes in geometry must be described in a simpler manner
- Defining motion of individual vertices or returning to the CAD description is out of the question: smoothness criteria, re-meshing concerns etc.
- Even automatic mesh motion is unsatisfactory: boundary motion should be defined on all vertices, which is already too much information

Radial Basis Function Mesh Morphing

- Geometry morphing tools are defined without reference to mesh or CAD: deforming space as a function of motion of control points
- Control motion will be parametrised (for the optimiser) and used to move the computational mesh directly
- The main concern is **mesh motion function smoothness**: mesh must remain valid after morphing: using parametrisation based on FEM-like background shape functions or **Radial Basis Function**

Radial Basis Function

Radial Basis Function Interpolation

- RBF interpolation defines the interpolation directly from the sufficient smoothness criterion on the interpolation (positive weighting factors):

$$s(\mathbf{x}) = \sum_{j=1}^{N_b} \gamma_j \phi(|\mathbf{x} - \mathbf{x}_{b,j}|) + q(\mathbf{x})$$

  where
  - $\mathbf{x}$ is the interpolant location
  - $\mathbf{x}_b$ is the set of N_b locations carrying the data
  - $\phi(x)$ is the basis function, dependent on point distance
  - $q(\mathbf{x})$ is the (usually linear) polynomial function, depending on choice of basis function and γ_j
- Consistency of interpolation is achieved by requiring that all polynomials of the order lower than q disappear at data points

$$\sum_{j=1}^{N_b} \gamma_j p(\mathbf{x}_{b,j}) = 0$$

Radial Basis Function

Radial Basis Function Interpolation

- Upon choosing the basis function, coefficients of $q = b_0 + b_1 x + b_2 y + b_3 z$ and γ_j are determined by solving the system:

$$\begin{bmatrix} s(\mathbf{x}_{b,j}) \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_{bb} & Q_b \\ Q_b^T & 0 \end{bmatrix} \begin{bmatrix} \gamma \\ \beta \end{bmatrix}$$

  where
  - $s(\mathbf{x}_{b,j})$ is the function value at interpolant locations (source data)
  - γ carries all γ_i coefficients and β carries $b_0 - b_3$
  - Φ_{bb} carries the evaluation of the basis function for pairs of interpolation points $(\mathbf{x}_{b,i}, \mathbf{x}_{b,j})$ and acts as a dense connectivity matrix:

$$\Phi_{(i,j)} = s(|\mathbf{x}_{b,i} - \mathbf{x}_{b,j}|)$$

  - Q_b is the rectangular matrix with $[1 \; \mathbf{x}_b]$ in each row
- The system is a dense matrix and needs to be solved for γ and β using QR decomposition (direct solver), thus defining the interpolation

Radial Basis Function

Choice of Radial Basis Function

- **Functions with local support** disappear beyond the radius r and are typically polynomial. This eliminates some entries in Φ_{bb}, making the system easier to solve. Defined in terms of

$$\xi = x/r$$

  with the condition to equal zero for $\xi > 1$
- **Functions with global support** cover the whole interpolation space, and usually require a smoothing function to make the system easier to solve

RBF Name	**Abbrev.**	$s(x)$
Wendland, second order	W2	$pos(r-\xi)\,(1-\xi)^4\,(4\xi+1)$
Thin plate spline	TPS	$x^2\,log(x)$
Inverse multi-quadratic bi-harmonic	IMQB	$\frac{1}{\sqrt{a^2+x^2}}$
Quadratic bi-harmonic	QB	$1+x^2$
Gaussian	Gauss	e^{-x^2}

- Example basis functions

RBF Interpolation

RBF Interpolation Procedure

1. Establish locations of data-carrying points $\mathbf{x}_b$ and their values
2. Assemble and solve the equation set for γ and β using a direct solver
3. Calculate values at desired locations by evaluating $s(\mathbf{x})$

Extinguishing Far Field: Smoothing Function

- If it can be established that that far-field value of interpolated function is known and uniform, far-field data carriers may be eliminated: reduce RBF contribution to zero
- Defining inner r and outer radius R for a focal point $\mathbf{x}_f$:

$$\psi(\mathbf{x}) = \begin{cases} 1, & \widetilde{x} < 0 \\ 1 - \widetilde{x}^2\,(3 - 2\,\widetilde{x}), & 0 \le \widetilde{x} \le 1 \\ 0 & \widetilde{x} > 1 \end{cases} \quad \text{where} \quad \widetilde{x} = \frac{|\mathbf{x} - \mathbf{x}_f| - r}{R - r}$$

$$s(\mathbf{x}) = \psi(\widetilde{x}) \left[\sum_{j=1}^{N_b} \gamma_j \phi(|\mathbf{x} - \mathbf{x}_{b,j}|) + q(\mathbf{x}) \right]$$

RBF Mesh Morphing

RBF Mesh Morphing Object

- RBF morphing object defines the parametrisation of geometry (space):
  1. Control points in space, where the parametrised control motion is defined
  2. Static points in space, whose motion is blocked
  3. Range of motion at each control point: $(\mathbf{d}_0, \mathbf{d}_1)$
  4. Set of scalar parameters δ for control points, defining current motion as

$$\mathbf{d}(\delta) = \mathbf{d}_0 + \delta(\mathbf{d}_1 - \mathbf{d}_0), \quad \text{where} \quad 0 \leq \delta \leq 1$$

- For each set of δ parameters, mesh deformation is achieved by interpolating motion of control points $\mathbf{d}$ over all vertices of the mesh: new deformed state of the geometry
- Mesh in motion remains valid since RBF satisfies smoothness criteria

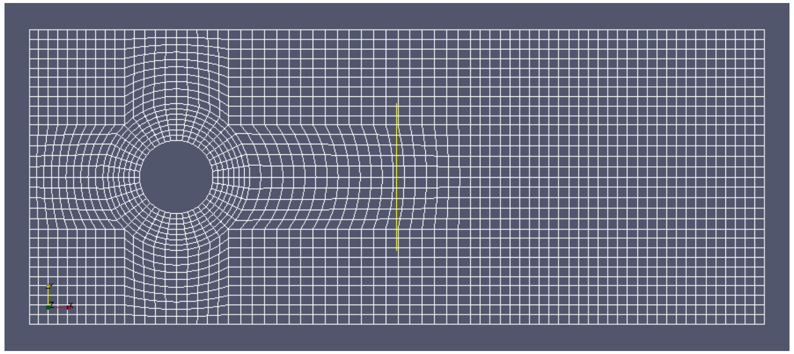
Using RBF in Optimisation

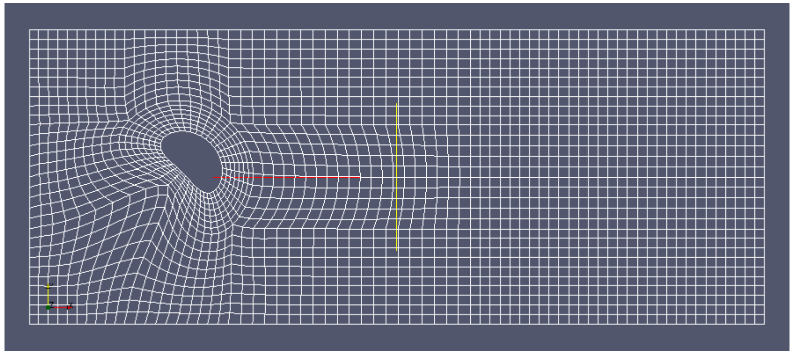
- Control points may be moved individually or share δ values: further reduction in dimension of parametrisation of space
- Mesh morphing state is defined in terms of δ parameters: to be controlled by the optimisation loop

Examples

Simple Examples: Morphing of a Cylinder

- Control points in motion: (-0.02 -0.02 0); motion range: ((0 0 0) (0.02 0.02 0))





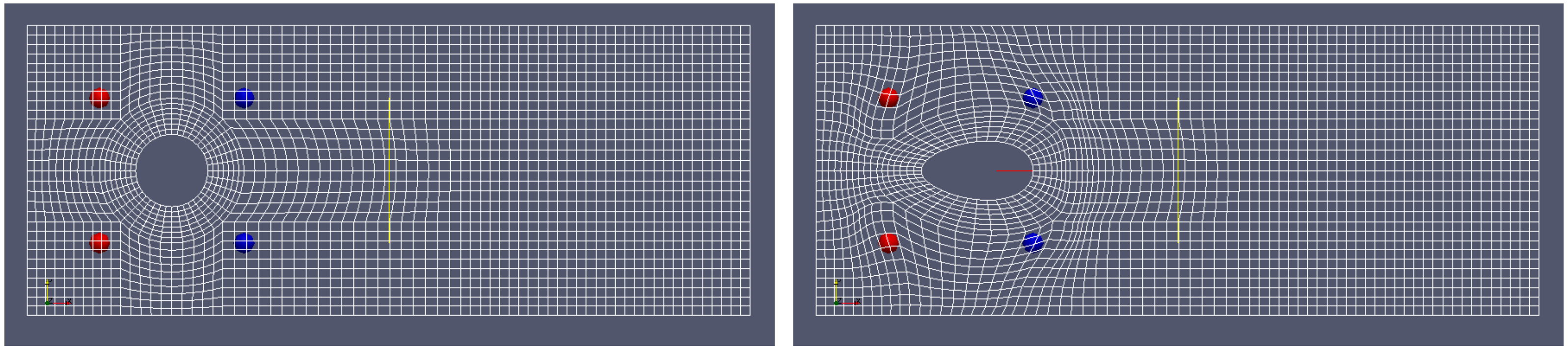
Examples

Simple Examples: Morphing of a Cylinder

- Control points in motion:

Point	Motion range
(-0.02 -0.02 0)	((0 0 0) (-0.01 0.01 0))
(0.02 -0.02 0)	((0 0 0) (0.2 0.0 0))
(-0.02 0.02 0)	((0 0 0) (-0.01 -0.01 0))
(0.02 0.02 0)	((0 0 0) (0.02 0 0))

- Note: parametrisation uses a single parameter δ for this motion



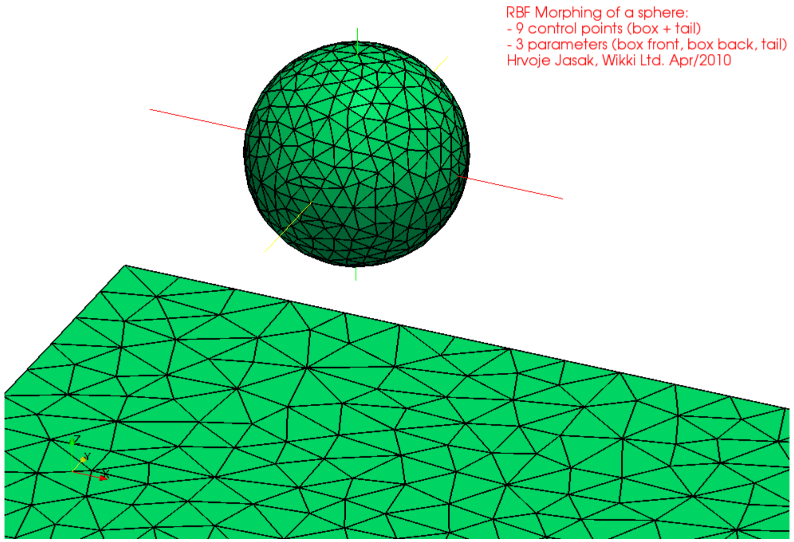
Example: Morphing

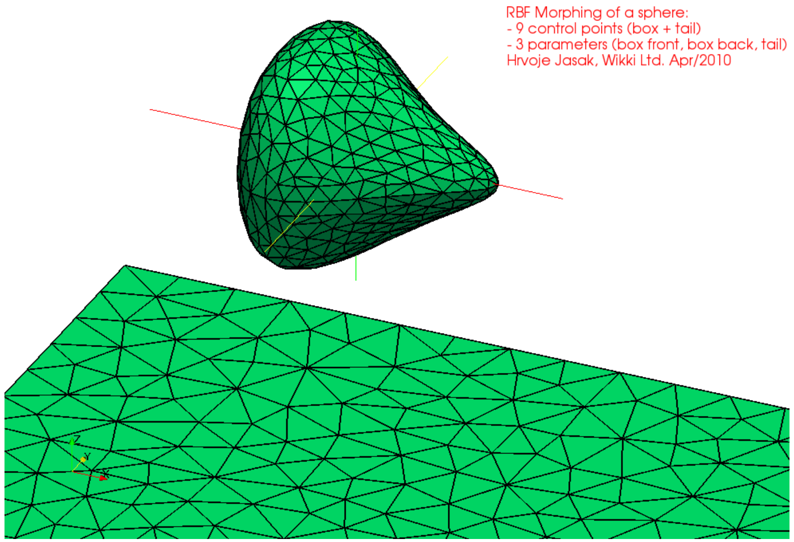
Simple Examples: Morphing of a Sphere

- Control points in motion:

Point	Motion range
(-0.3 $\pm$0.3$\pm$0.3)	((0 0 0) ($\pm$0.1 $\pm$0.1 $\pm$0.1))
(0.3 $\pm$0.3$\pm$0.3)	((0 0 0) ($\pm$0.1 $\pm$0.1 $\pm$0.1))
(0.7 0 0)	((0 0 0) (0.3 0 0))

- Note: parametrisation uses a single parameter δ for this motion
- Optimisation shall be performed with 3 parameters: front, back, tail





Example: Shape Optimisation

Geometric Shape Optimisation with Parametrised Geometry

- Specify a desired object of optimisation and use the parametrisation of geometry to explore the allowed solution space in order to find the **minimum of the optimisation objective**

$$objective = f(\mathbf{shape})$$

1. **Parametrisation of Geometry**
   - Computational geometry is complex and usually available as the computational mesh: a large amount of data
   - Parametrisation tool: **RBF mesh morphing**, defining deformation at a small number of mesh-independent points in space
2. **CFD Flow Solver** is used to provide the flow solution on the current geometry, in preparation for objective evaluation
3. **Evaluation of Objective**: usually a derived property of the flow solution
4. **Optimiser Algorithm**: explores the solution space by providing sets of $\mathbf{shape}$ coordinates and receiving the value of $objective$. The search algorithm iteratively limits the space of solutions in search of a minimum value of $objective$

Example: Shape Optimisation

Example: HVAC 90 deg Bend: Flow Uniformity at Outlet

- Flow solver: incompressible steady-turbulent flow, RANS $k - \epsilon$ model; coarse mesh: 40 000 cells; 87 evaluations of objective with CFD restart
- RBF morphing: 3 control points in motion, symmetry constraints; 34 in total
- Objective: flow uniformity at outlet plane

```
iter = 0 pos = (0.9 0.1 0.1) v = 22.914 size = 0.69282
iter = 5 pos = (0.1 0.1 0.1) v = 23.0088 size = 0.584096
iter = 61 pos = ((0.990164 0.992598 0.996147) v = 13.5433 size = 0.00095
```