Introduction:
In this lab, we studied how to use the finite difference method to solve the one dimensional Poisson's Equation with Dirichlet or Neumann boundary conditions. The finite difference method was implemented with the space discretization. The well-posed case and the ill-posed case were discussed separately.

Methodology:
The finite difference method was used based on the Lab 11 homework requirement material. When applying the finite difference method on the Poisson's Equation, the problem turned to be solving the linear system Ax=b. Since the right hand side of the Poisson's Equation is the second order derivative, which can be approximate by formula(2) in lab document, the coefficient matrix A was build as a tridiagonal matrix with diagonal to be -2/h^2, and the other two bands to be 1/h^2. The left hand side b was build based on the given information f(x)=2. The initial conditions was then considered. Based on that, the coefficient matrix and the left hand site b has been expanded. After building the linear system, if the matrix A is not singular, it has been solved using a self developed package, which calls the LAPACK routines. If the matrix A is singular, the numerical approach cannot solve this system, which indicate that the problem is ill-posted.

Result:
Condition 1:
With n=10 , h=0.1, and the given boundary conditions, the coefficient matrix is as follow:

1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

This matrix is a tridiagonal matrix and it is non-singular, thus the linear system can be solved.
The numerical solution has been plotted together with the exact solution in Figure 1:

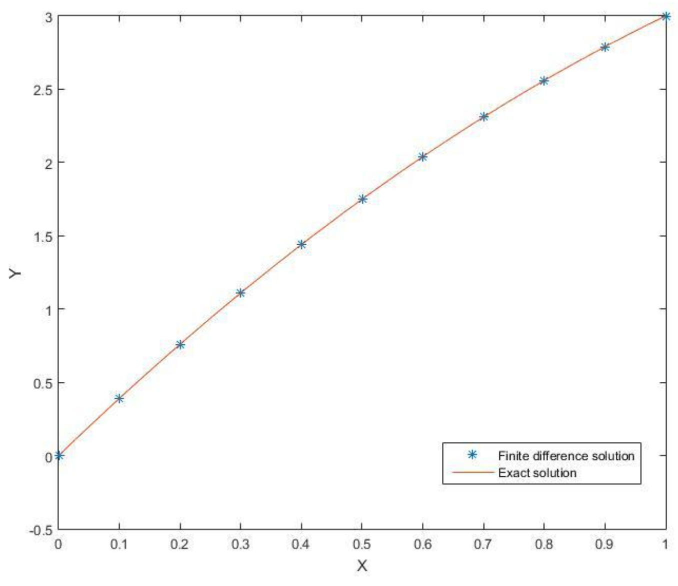

Figure 1: The finite difference solution together with the exact solution, with n=10, h=0.1, condition 1, on the domain [0 1].

Condition 2:
With n=10 , h=0.1, and the given boundary conditions, the coefficient matrix is as follow:

1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-1.0000	0.0000	1.0000

This matrix is close to a tridiagonal matrix, except there is a -1 at row 12 column 10 instead of 0, and it is non-singular, thus the linear system can be solved.
The numerical solution has been plotted together with the exact solution in Figure 2:

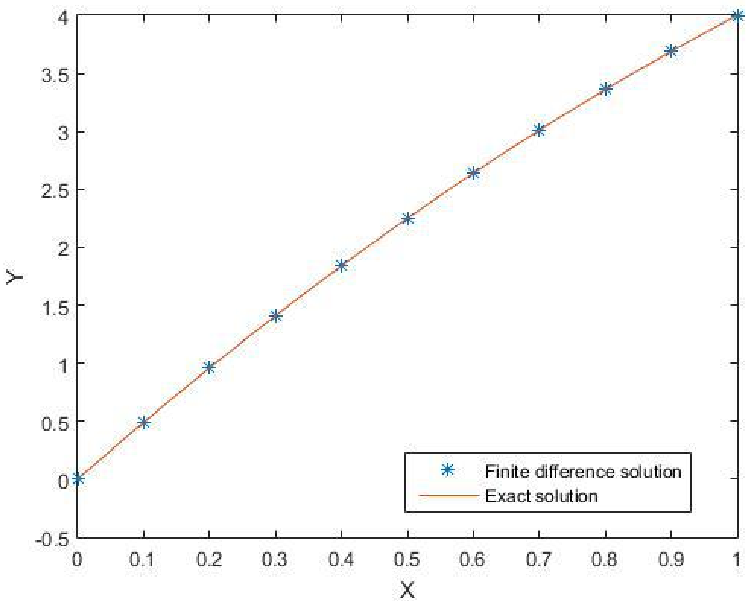

Figure 2: The finite difference solution together with the exact solution, with n=10, h=0.1, condition 2 on the domain [0 1].

Condition 3 and 4:
In this two cases, with n=10, we have the coefficient matrix:

1.0000	0.0000	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	100.0000	-200.0000	100.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-1.0000	0.0000	1.0000

This matrix is a singular matrix, so the linear system cannot be solved. With these boundary conditions, the question is ill-posed.

Discussion:

From the condition 1 and condition 2, we can see that these two groups of boundary conditions can make the question well-posed: The coefficient matrix is non-singular and the corresponding linear system is solvable, which lead to one solution. Based on Figure 1 and Figure 2, we can see that the numerical solutions are all on the curve of the exact solution, which indicate the successfully implemented finite difference method. And the solution of this method can reflect the exact trends of the theoretical underlying function.

From the condition 3 and 4, we have got the coefficient matrix to be a singular matrix, thus the corresponding linear system cannot be solved and the question under these two boundary conditions are ill-posed. The problem may have zero or infinite number of solutions based on the right hand site vector b of the linear system. This situation shows that if only two Neumann condition is specified, the problem is ill-posed and using the method in lab document cannot solve the function numerically. If the compatibility condition does not met(as in condition 3), the system has no solution. If the compatibility condition met(as in condition 4), the system has infinite solutions, additional constraints on the solution required to get the exact solution.