Lab 14
The Finite Element Method for the Heat Equation

ASC II, Fall 2014

Assigned December 1, 2014
Due December 7, 2014

In this lab we will consider using the finite element method to numerically solve a parabolic equation in one spatial dimension,

$$\frac{\partial u}{\partial t} \;=\; -D\frac{\partial^2 u}{\partial x^2} \;+\; f(x,t) \tag{1}$$

on the domain $x \in [0,1]$, $t > 0$, where $u(x,t)$ is the distribution of heat at location x and time t, D is the diffusion coefficient ($D > 0$). For our study, let us consider a problem where $x \in [0,1]$, $t > 0$, and $u(0,t) = u(1,t) = 0$. We pose the weak (or *variational*) form of the problem over a suitable function space, V. To do this, we multiply the differential equation by a test function $v(x) \in V$, integrate over the spatial domain, and use integration by parts to redistribute the spatial derivatives on the higher-order term:

$$\text{seek } u \in V \text{ such that} \tag{2}$$

$$\int_0^1 \frac{\partial u}{\partial t} v \,\mathrm{dx} + D\frac{\partial u}{\partial x} v\Big|_0^1 - \int_0^1 D\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{dx} = \int_0^1 v f \,\mathrm{dx} \tag{3}$$

$$\text{for all } v \in V \tag{4}$$

This problem is *weak* in the sense that it admits solutions which are not solutions to the classical problem, because the weak problem only requires u to be once-differentiable with respect to x, but the classical problem requires u to have two spatial derivatives. Any solution to the classical problem will also satisfy the weak problem, but there may be solutions to the weak problem that do not satisfy the classical problem.

In order to solve the weak problem, we see that the space V should be a Sobolev space containing functions that have at least one spatial derivative (at least in a weak sense) and which are zero on the boundary (i.e., at $x = 0$ and $x = 1$). This space is commonly denoted as H_0^1. Additionally, since u and v are both in H_0^1, the integrated term vanishes from the weak problem. However, we are still left with an infinite-dimensional problem. In order to approach this problem numerically, we must pose the problem over a finite-dimensional subspace of H_0^1, which we will call V_h. We partition the spatial domain into $n+1$ discrete elements using $n+2$ nodes, $\{x_i\}_{i=0}^{n+1}$. Let us use a uniform partitioning with characteristic grid size h such that $x_i = x_0 + ih$. In finite elements, we choose the space V^h to be a subspace of H_0^1 containing all of the piecewise-continuous polynomials of degree less than or equal to some chosen degree, defined over the grid of elements. The simplest feasible choice, and the one one make here, is to choose V^h to be the space of all piecewise-continuos polynomials of degreee 1 or less defined on our grid. We know that these functions are linear between the nodes, and that they are zero at $x = 0$ and at $x = 1$. Thus, we can intuitively see that there are n degrees of freedom defining one of these functions over our domain.

We can verify this in choosing a basis for V^h. There are many choices, but the simplest (and very advantageous) choice is to construct a basis using nodal functions: functions which have a value of 1 at a specific node, and a value of 0 at all other nodes. We can describe this basis as $\{\phi_i(x)\}_{i=1}^n$ where

$$\phi_i(x) = \begin{cases} 0 & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{h} & x_{i-1} \leq x < x_i \\ \dfrac{x_{i+1} - x}{h} & xi \leq x < x_{i+1} \\ 0 & x \geq x_{i+1} \end{cases} \tag{5}$$

These are the popular piecewise-linear "hat" basis functions, and it's easy to see that any function in the V^h (as we've specified V^h here) can be written as a linear combination of these functions. We seek an approximation to u as $u^h \in V^h$, which can be written

$$u^h = \sum_{j=1}^{n} \xi_j \phi_j \tag{6}$$

where the ξ_j are time-dependent coefficients.

Now we return to the weak problem. Once we pose the problem over the finite-dimensional space V^h we obtain the discrete weak problem,

$$\text{seek } u^h \in V^h \text{ such that} \tag{7}$$

$$\int_0^1 \frac{\partial u^h}{\partial t} v^h \, \mathrm{dx} - \int_0^1 D \frac{\partial u^h}{\partial x} \frac{\partial v^h}{\partial x} \, \mathrm{dx} = \int_0^1 v^h f \, \mathrm{dx} \tag{8}$$

$$\text{for all } v^h \in V^h \tag{9}$$

Fortunately, in order to ensure that this equation holds for all $v^h \in V^h$, it is enough to test against each of the functions in the basis for V^h. So, we let v be each ϕ_i in turn, transforming the discrete weak problem into a system of n equations:

$$\int_0^1 \frac{\partial u^h}{\partial t} \phi_i \, \mathrm{dx} - \int_0^1 D \frac{\partial u^h}{\partial x} \frac{\partial \phi_i}{\partial x} \, \mathrm{dx} = \int_0^1 \phi_i f \, \mathrm{dx} \quad i = 1 \dots n \tag{10}$$

We may now substitute the linear combination for u^h into these equations:

$$\int_0^1 \sum_{j=1}^{n} \frac{\xi_j}{\partial t} \phi_j \phi_i \, \mathrm{dx} - \int_0^1 D \sum_{j=1}^{n} \xi_j \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, \mathrm{dx} = \int_0^1 \phi_i f \, \mathrm{dx} \quad i = 1 \dots n \tag{11}$$

Assuming that D is constant, we can move the terms not involving the spatial variables (including the summation) out of the integrals:

$$\sum_{j=1}^{n} \frac{\xi_j}{\partial t} \int_0^1 \phi_j \phi_i \, \mathrm{dx} - D \sum_{j=1}^{n} \xi_j \int_0^1 \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, \mathrm{dx} = \int_0^1 \phi_i f \, \mathrm{dx} \quad i = 1 \dots n \tag{12}$$

This is a system of time-dependent ODEs, where the only unknown quantities in this system are the coefficients, $\{\xi_j\}_{j=1}^{n}$. These ODEs may be discretized by any time discretization scheme. Also, because the functions ϕ_i have compact support over the domain, we find that the integrals on the left-hand side of the equation are 0 for all $|i - j| > 1$. This results in a tridiagonal system, which makes finite elements competitive with other methods such as finite difference methods.

If we apply backward Eulter to discretize the time derivative, letting ξ_j^k be the j^{th} coefficient at time t_k, then we can move the known terms (at time t_{k-1}) to the right hand side:

$$\frac{1}{\Delta t}\sum_{j=1}^{n}\xi_j^k\int_0^1\phi_j\phi_i\,\mathrm{dx}-D\sum_{j=1}^{n}\xi_j\int_0^1\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\,\mathrm{dx}=\int_0^1\phi_i f\,\mathrm{dx}+\frac{1}{\Delta t}\sum_{j=1}^{n}\xi_j^{k-1}\int_0^1\phi_j\phi_i\,\mathrm{dx}\qquad i=1\dots n \tag{13}$$

The fully-discrete equations can be expressed as:

$$\left[\frac{1}{\Delta t}M-DS\right]\vec{\xi^k}=R\vec{H}S \tag{14}$$

where $M, S \in \mathbb{R}^{n\times n}$, with

$$m_{i,j}=\int_0^1\phi_j\phi_i\,\mathrm{dx} \tag{15}$$

$$s_{i,j}=\int_0^1\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\,\mathrm{dx} \tag{16}$$

Your mission:

1. For row i, what are the entries $m_{i,j}$ and $s_{i,j}$? Use the midpoint rule to evaluate the integrals. (Hint: Recall that $m_{i,j}$ and $s_{i,j}$ are nonzero in this problem only for $|i-j|\leq 1$.)

2. For $D=1$, $u(x,t)=e^t\sin(\pi x)$, what should $f(x,t)$ be so that the PDE is satisfied?

3. Use the Thomas algorithm you had to write for the previous lab to solve this system for 10 elements with $\Delta t = 0.05$ from $t=0$ to $t=1$. Note that M and S will not change between iterations, but the right hand side vector will have to be recomputed since f depends on the current time. Compare your solution to the exact solution to the problem.

Submission and Grading

To get credit for this assignment, you must submit the following information to the lab instructor and course instructor by 11:59pm, December 2, 2014:

- your source code files

- Your report as a single file in pdf format, including responses to the questions in the lab, results from your work, and relevant discussion of your observations, results, and conclusions.

This information must be received by 11:59pm, December 2, 2014. Send the required documents as email attachments to both the course instructor and the lab TA. As stated in the course syllabus, late assignment submissions will be subject to a 10% point penalty per 24 hours past the due date at time of submission, to a maximum reduction of 50%, according to the formula:

*[final score] = [raw score] - min(0.5, 0.1 * [# of days past due]) * [maximum score]*