Report of Lab 14

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December 11, 2016

Introduction

In this lab, the 1D Poisson equation was solved using the weighted Jacobi method. Multiple grid levels have been used to damping different frequency residuals. The v-cycle algorithm has been implemented. The residual of ordinary weighted Jacobi method and the weighted Jacobi with multigrid v-cycle have been calculated separately and the result have been compared.

Methodology

Firstly, the weighted Jacobi solver was implemented according to the recurrence:

$$x^{(k+1)} = \omega D^{-1}(b - Rx^{(k)}) + (1-\omega)x^{(k)}$$

where ω is a weight coefficient between 0 to 1, D is the diagonal part of coefficient matirx A, b is the right hand side, R=A-D. After successfully implemented the weighted Jacobi method. The Poisson equation was discretized as following:

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = -4e^{2x_i}$$

where i is from 2 to n-1, with the initial condition:

$$U_1 = 1$$

$$U_n = e^2$$

Therefor, we can write our system in the form of Ac=b, where:

$$A = -\frac{1}{h^2}\begin{bmatrix} -h^2 & & & & \\ 1 & -2 & 1 & & \\ & \cdots & & & \\ & & \cdots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

$$b = \begin{bmatrix} U_1 \\ -4e^{2x_2} \\ -4e^{2x_3} \\ \vdots \\ -4e^{2x_{n-1}} + \frac{U_n}{h^2} \end{bmatrix}$$

$$c = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{n-1} \end{bmatrix}$$

This system was solved via the ordinary weighted Jacobi solver with grid point number as 256(255 grids) for 2000 iterations, with the weight ω specified as

0.8. Then an v-cycle algorithm was implemented as described in lab assignment document. 250 iteration steps was done at each level of v-cycle until the total number of iterations reached 2000. The result and l2 norm of the residual of this two approaches have been calculated and compared.

Result

The solution of ordinary weighted Jacobi was calculated when repeating the weighted Jacobi method for 2000 iteration. Then the system was solved using the weigthed Jacobi with v-cycle in the same setting. The results of these two approach, together with the exact solution, was plotted in Figure 1 below:

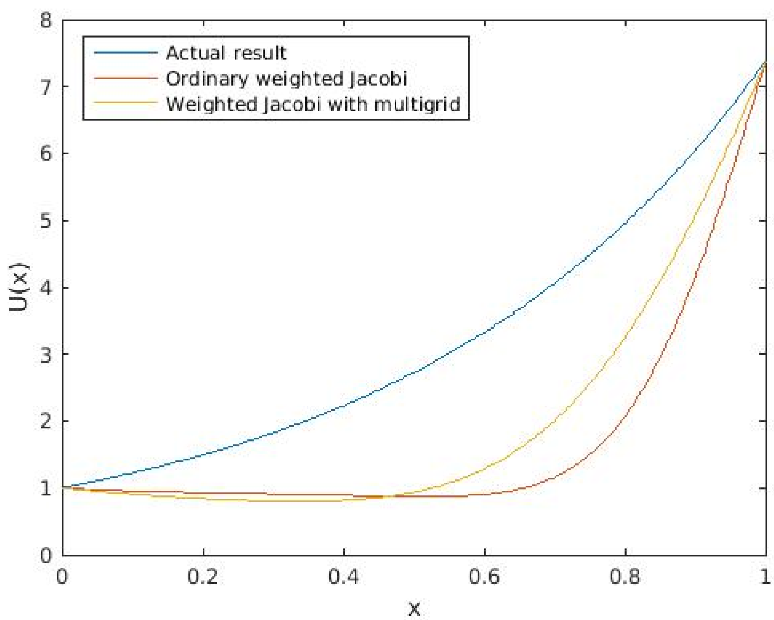

Figure 1: The result of exact solution, ordinary weighted Jacob and weighted Jacobi with v-cycle. With number of grid points to be 256 and cycle number to be 2000

Compared to the exact result, the l2 error norm of these two method have been calculated to be 28.7999 and 21.9752 separately(at 2000 iterations).
The l2 norm of residual of these two approaches has been calculated and plotted as a function of cycle number in Figure 2:

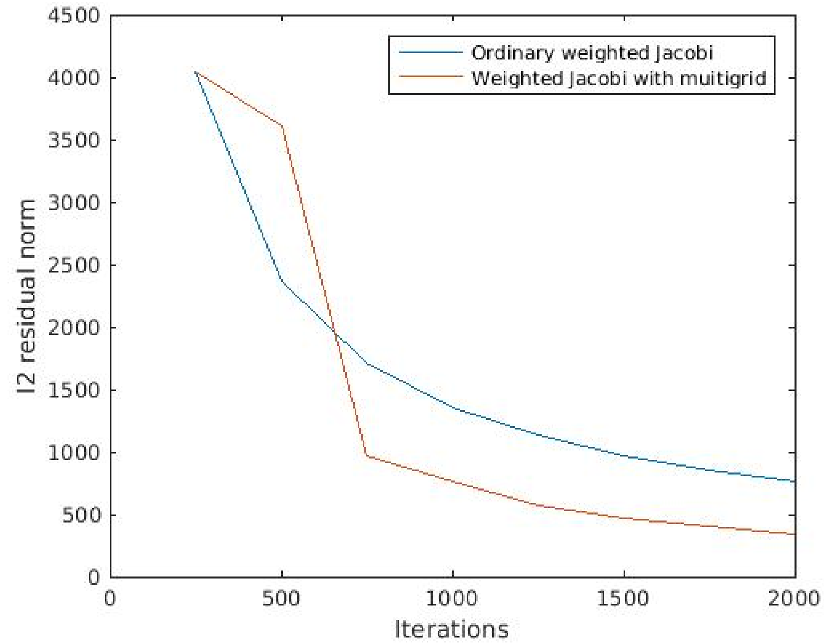

Figure 2: The residual of ordinary weighted Jacob and weighted Jacobi with v-cycle. With number of grid points to be 256 and cycle number to be 250, 500, 750, 1000, 1250, 1500, 1750 and 2000

Discussion:
From Figure 1 we can see that the result of weighted Jacobi and weighted Jacobi with v-cycle have significant difference from the exact solution(e^{2x}). So in order to justify the correctly implementation of these two method, another experiment was done using a larger number of iterations(40000), in this condition, both of the method are much more approximate to the exact solution, with l2 error norm to be 2.5157 and 0.065708 separately(see Figure 3 in supplementary data). These result suggested that both of the methods have been implemented successfully. However, the iteration number given are not big enough for the weighted Jacobi(with or without v-cycle) to converge to the exact solution.

Compared to the ordinary weighted Jacobi(l2 error norm=28.7999), the weighted Jacobi with v-cycle(l2 error norm=21.9752) are more approximate to the exact solution, which indicate the multigrid approach can reduce the global error under the setting of this problem. This phenomenon is as expected since by moving the data between fine grid and coarse grid repeatedly will help the system damping both the high frequency error and the low frequency error.

The l2 residual norms of ordinary weighted Jacobi and the weighted Jacobi with v-cycle in Figure 2 indicates the speed of noise damping of these two method. We can see that from the region 250 to 500, the ordinary weighted Jacobi damps residuals more efficiently compared to the Jacobi with v-cycle(at course grid in that region), this is because the major error in the system initially has relatively high frequency, so the approach with fine grid damps the residual faster. But after 500 iterations, the Jacobi with v-cycle soon catches up and at the iterations higher than 750, the residual of Jacobi with v-cycle is smaller than ordinary weighted Jacobi method. This is because the multigrid approach can damp both high frequency residual and low frequency residual efficiently, whereas the ordinary Jacobi can only efficiently damp the high frequency residual.

Supplementary data:

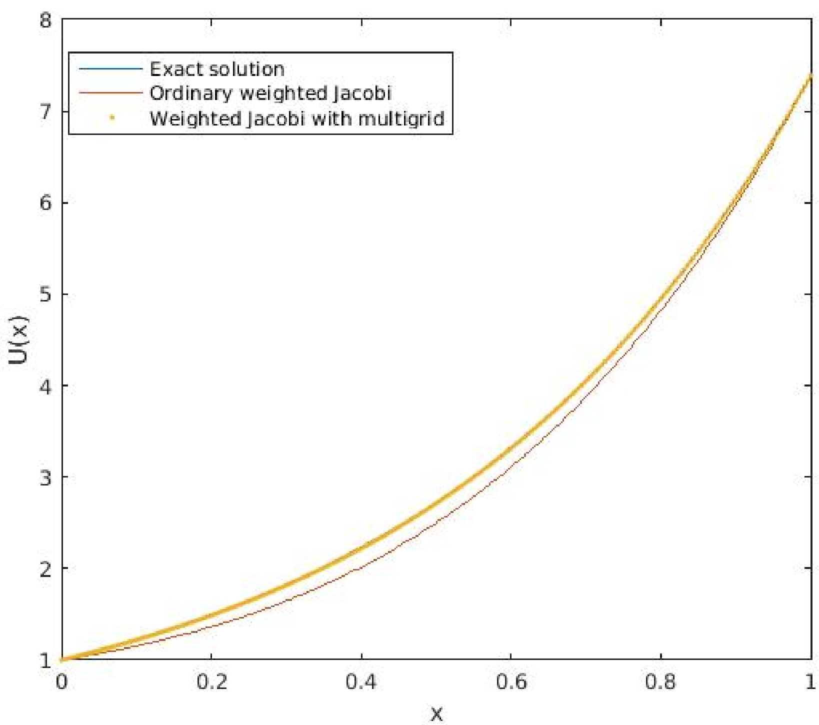

Figure 3: The result of exact solution, ordinary weighted Jacob and weighted Jacobi with v-cycle. With number of grid points to be 256 and cycle number to be 40000. The exact solution and weighted Jacobi with multigrid are too close to be separate from this figure