COURSE TECHNOLOGY
CENGAGE Learning™

Data Structures Using C++ 2E

Chapter 12
Graphs

Objectives

- Learn about graphs
- Become familiar with the basic terminology of graph theory
- Discover how to represent graphs in computer memory
- Examine and implement various graph traversal algorithms

Objectives (cont’d.)

- Learn how to implement a shortest path algorithm
- Examine and implement the minimum spanning tree algorithm
- Explore topological sort
- Learn how to find Euler circuits in a graph

Introduction

- Königsberg bridge problem
  - Given: river has four land areas
    - *A*, *B*, *C*, *D*
  - Given: land areas connected using seven bridges
    - *a*, *b*, *c*, *d*, *e*, *f*, *g*
  - Starting at one land area
    - Is it possible to walk across all the bridges exactly once and return to the starting land area?
- Euler represented problem as a graph
  - Answered question in the negative
  - Marked birth of graph theory

Introduction (cont’d.)

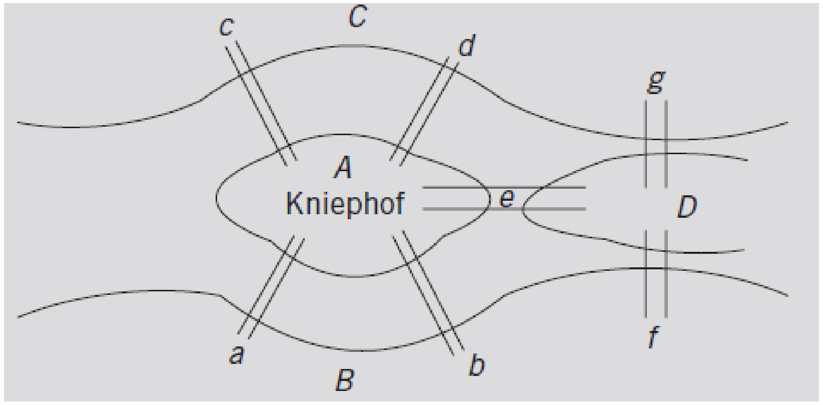


FIGURE 12-1 The Königsberg bridge problem

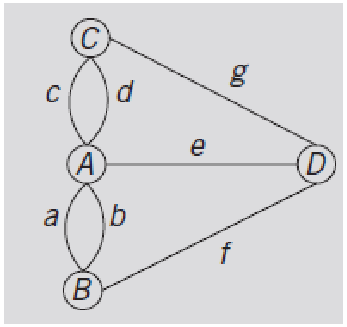


FIGURE 12-2 Graph representation of the Königsberg bridge problem

Graph Definitions and Notations

- Borrow definitions, terminology from set theory
- Subset
  - Set *Y* is a **subset** of *X*: $Y \subseteq X$
    - If every element of *Y* is also an element of *X*
- **Intersection** of sets *A* and *B*: $A \cap B$
  - Set of all elements that are in *A* and *B*
- **Union** of sets *A* and *B*: $A \cup B$
  - Set of all elements in *A* or in *B*
- **Cartesian product**: $A \times B$
  - Set of all ordered pairs of elements of *A* and *B*

Graph Definitions and Notations (cont’d.)

- **Graph** *G* is a pair
  - *G* = (*V*, *E*), where *V* is a finite nonempty set
    - Called the set of **vertices** of *G*, and $E \subseteq V$ x V
  - Elements of *E*
    - Pairs of elements of *V*
- *E*: *s*et of **edges** of *G*
  - *G* called **trivial** if it has only one vertex
- **Directed graph** (**digraph**)
  - Elements in set of edges of graph *G*: ordered
- **Undirected graph**: not ordered

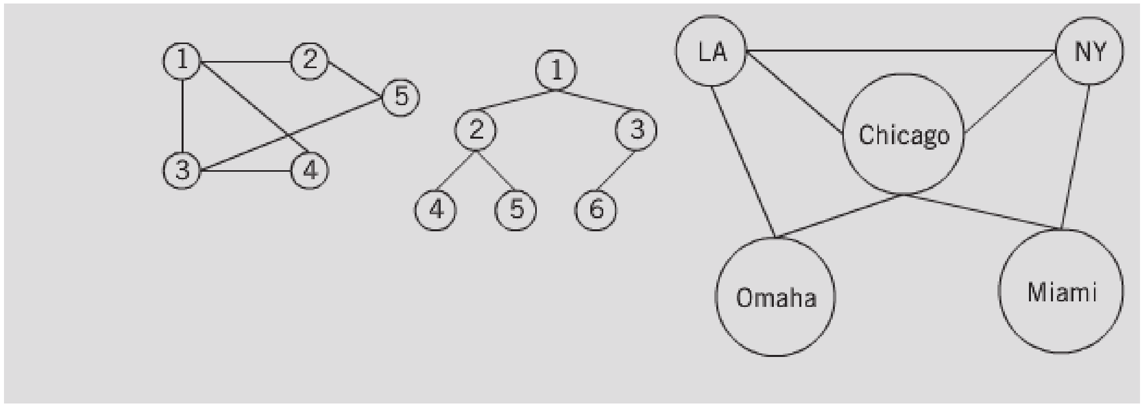


FIGURE 12-3 Various undirected graphs

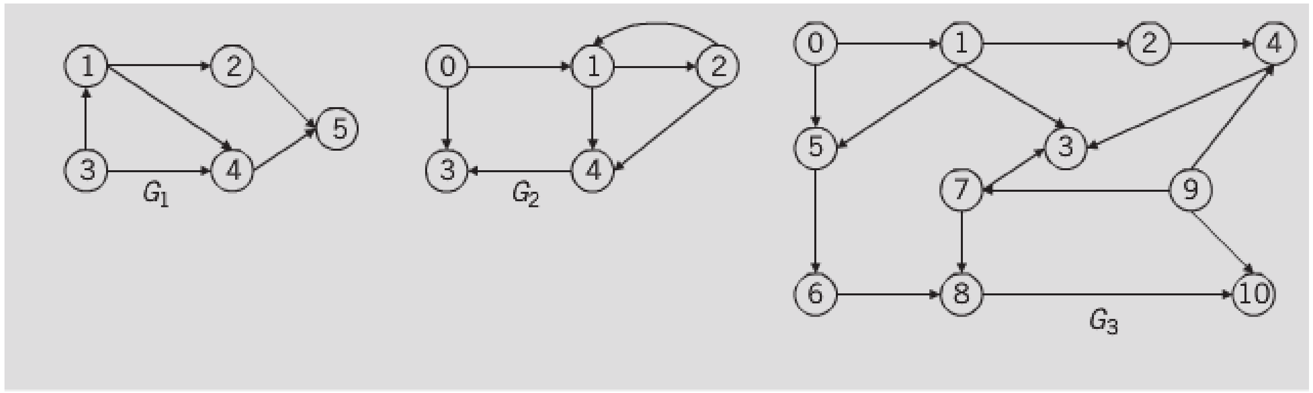


FIGURE 12-4 Various directed graphs

$V(G_1) = \{1, 2, 3, 4, 5\}$ $E(G_1) = \{(1, 2), (1, 4), (2, 5), (3, 1), (3, 4), (4, 5)\}$

$V(G_2) = \{0, 1, 2, 3, 4\}$ $E(G_2) = \{(0, 1), (0, 3), (1, 2), (1, 4), (2, 1), (2, 4), (4, 3)\}$

$V(G_3) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $E(G_3) = \{(0, 1), (0, 5), (1, 2), (1, 3), (1, 5), (2, 4), (4, 3), (5, 6), (6, 8), (7, 3), (7, 8), (8, 10), (9, 4), (9, 7), (9, 10)\}$

Graph Definitions and Notations (cont' d.)

- Graph *H* called **subgraph** of *G*
  - If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
  - Every vertex of *H*: vertex of *G*
  - Every edge in *H*: edge in *G*
- Graph shown pictorially
  - Vertices drawn as circles
    - Label inside circle represents vertex
- Undirected graph: edges drawn using lines
- Directed graph: edges drawn using arrows

Graph Definitions and Notations (cont’d.)

- Let *u* and *v* be two vertices in *G*
  - *u* and *v* **adjacent**
    - If edge from one to the other exists: $(u, v) \in E$
- **Loop**
  - Edge incident on a single vertex
- e_1 and e_2 called **parallel edges**
  - If two edges e_1 and e_2 associate with same pair of vertices $\{u, v\}$
- **Simple graph**
  - No loops, no parallel edges

Graph Definitions and Notations (cont’d.)

- Let *e* = (*u*, *v*) be an edge in *G*
  - Edge *e* is incident on the vertices *u* and *v*
  - **Degree** of *u* written deg(*u*) or d(*u*)
    - Number of edges incident with *u*
- Each loop on vertex *u*
  - Contributes two to the degree of *u*
- *u* is called an **even** (**odd**) **degree** vertex
  - If the degree of *u* is even (odd)

Graph Definitions and Notations (cont’d.)

- **Path** from *u* to *v*
  - If sequence of vertices $u_1, u_2, \ldots, u_n$ exists
    - Such that $u = u_1$, $u_n = v$ and $(u_i, u_i + 1)$ is an edge for all $i = 1, 2, \ldots, n - 1$
- Vertices *u* and *v* called **connected**
  - If path from *u* to *v* exists
- **Simple path**
  - All vertices distinct (except possibly first, last)
- **Cycle** in *G*
  - Simple path in which first and last vertices are the same

Graph Definitions and Notations (cont’d.)

- *G* is **connected**
  - If path from any vertex to any other vertex exists
- **Component** of *G*
  - Maximal subset of connected vertices
- Let *G* be a directed graph and let *u* and *v* be two vertices in *G*
  - If edge from *u* to *v* exists: $(u, v) \in E$
    - *u* is **adjacent to** *v*
    - *v* is **adjacent from** *u*

Graph Definitions and Notations (cont’d.)

- Definitions of paths and cycles in *G*
  - Similar to those for undirected graphs
- *G* is **strongly connected**
  - If any two vertices in *G* are connected

Graph Representation

- Graphs represented in computer memory
  - Two common ways
    - Adjacency matrices
    - Adjacency lists

Adjacency Matrices

- Let G be a graph with n vertices where n > zero
- Let $V(G) = \{v_1, v_2, \ldots, v_n\}$
  - Adjacency matrix

$$A_G(i,j) = \begin{cases} 1 & \text{if}(v_i, v_j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

$$A_{G_1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ A_{G_2} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Adjacency Lists

- Given:
  - Graph G with n vertices, where $n > zero$
  - $V(G) = \{v_1, v_2, \ldots, v_n\}$
- For each vertex v: linked list exists
  - Linked list node contains vertex u: $(v, u) \in E(G)$
- Use array A, of size n, such that $A[i]$
  - Reference variable pointing to first linked list node containing vertices to which v_i adjacent
- Each node has two components: vertex, link
  - Component vertex
    - Contains index of vertex adjacent to vertex i

Adjacency Lists (cont'd.)

- Example 12-4

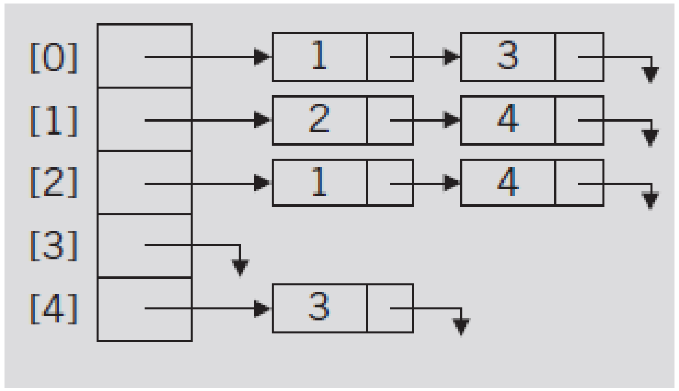


FIGURE 12-5 Adjacency list of graph *G2* of Figure 12-4

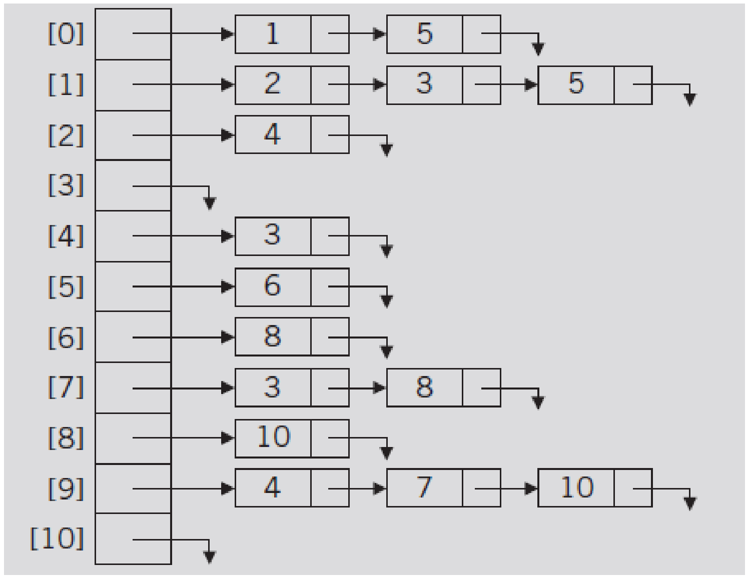


FIGURE 12-6 Adjacency list of graph *G3* of Figure 12-4

Operations on Graphs

- Commonly performed operations
  - Create graph
    - Store graph in computer memory using a particular graph representation
  - Clear graph
    - Makes graph empty
  - Determine if graph is empty
  - Traverse graph
  - Print graph

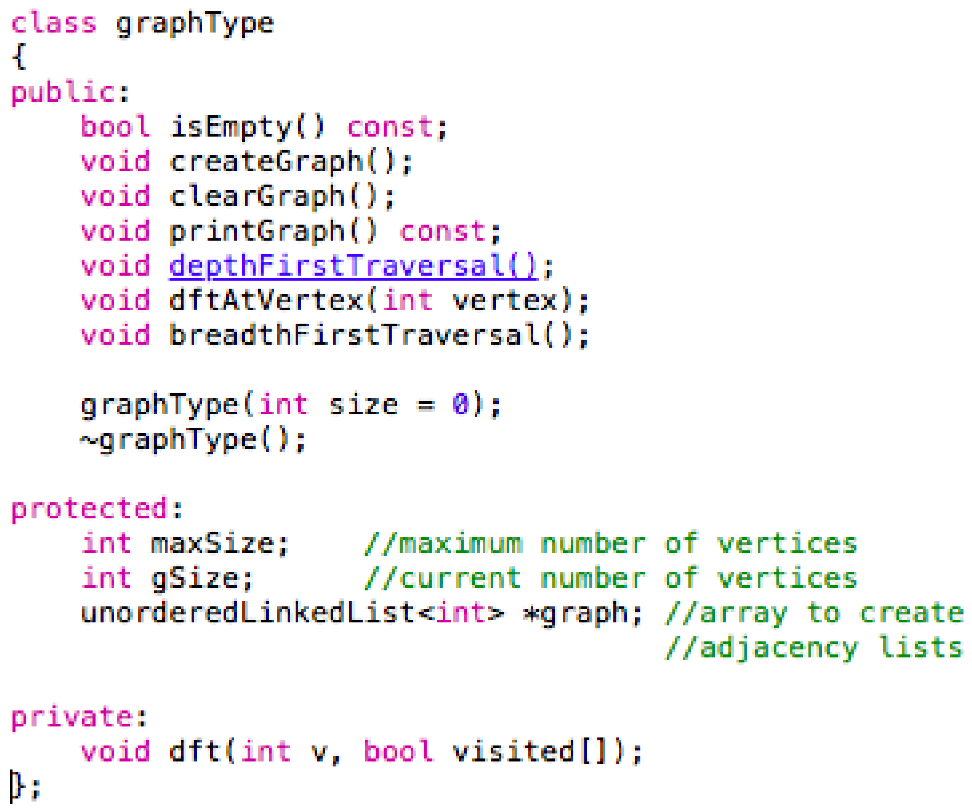
Operations on Graphs (cont'd.)

- Graph representation in computer memory
  - Depends on specific application
- Use linked list representation of graphs
  - For each vertex *v*
    - Vertices adjacent to *v* (directed graph: called **immediate successors**)
    - Stored in the linked list associated with *v*
- Managing data in a linked list
  - Use `class unorderedLinkedList`
- Labeling graph vertices
  - Depends on specific application

Graphs as ADTs

- See code on pages 692-693
  - Defines a graph as an ADT
  - Class specifying basic operations to implement a graph
- Definitions of the functions of the `class graphType`

```
bool graphType::isEmpty() const
{
    return (gSize == 0);
}
```



```
class graphType
{
public:
    bool isEmpty() const;
    void createGraph();
    void clearGraph();
    void printGraph() const;
    void depthFirstTraversal();
    void dftAtVertex(int vertex);
    void breadthFirstTraversal();

    graphType(int size = 0);
    ~graphType();

protected:
    int maxSize;    //maximum number of vertices
    int gSize;      //current number of vertices
    unorderedLinkedList<int> *graph; //array to create
                                     //adjacency lists

private:
    void dft(int v, bool visited[]);
};
```

Graphs as ADTs (cont'd.)

- Function `createGraph`
  - Implementation
    - Depends on how data input into the program
  - See code on page 694
- Function `clearGraph`
  - Empties the graph
    - Deallocates storage occupied by each linked list
    - Sets number of vertices to zero
  - See code on page 695

- Input file of the graph

```
5
0 1 3 -999
1 2 4 -999
2 1 4 -999
3 -999
4 3 -999
```

```cpp
void graphType::createGraph()
{
    ifstream infile;
    char fileName[50];

    int vertex;
    int adjacentVertex;

    if (gSize != 0) //if the graph is not empty, make it empty
        clearGraph();

    cout << "Enter input file name: ";
    cin >> fileName;
    cout << endl;

    infile.open(fileName);

    if (!infile)
    {
        cout << "Cannot open input file." << endl;
        return;
    }

    infile >> gSize;    //get the number of vertices

    for (int index = 0; index < gSize; index++)
    {
        infile >> vertex;
        infile >> adjacentVertex;

        while (adjacentVertex != -999)
        {
            graph[vertex].insertLast(adjacentVertex);
            infile >> adjacentVertex;
        } //end while
    } // end for

    infile.close();
} //end createGraph
```

```
void graphType::clearGraph()
{
    for (int index = 0; index < gSize; index++)
        graph[index].destroyList();

    gSize = 0;
} //end clearGraph

void graphType::printGraph() const
{
    for (int index = 0; index < gSize; index++)
    {
        cout << index << " ";
        graph[index].print();
        cout << endl;
    }

    cout << endl;
} //end printGraph
```

```
    //Constructor
graphType::graphType(int size)
{
    maxSize = size;
    gSize = 0;
    graph = new unorderedLinkedList<int>[size];
}

    //Destructor
graphType::~graphType()
{
    clearGraph();
}
```

Graph Traversals

- Processing a graph
  - Requires ability to traverse the graph
- Traversing a graph
  - Similar to traversing a binary tree
    - A bit more complicated
- Two most common graph traversal algorithms
  - Depth first traversal
  - Breadth first traversal

Depth First Traversal

- Similar to binary tree preorder traversal
- General algorithm
- Depth-first ordering: 0 1 2 4 3 5 6 8 10 7 9

```
for each vertex, v, in the graph
    if v is not visited
        start the depth first traversal at v
```

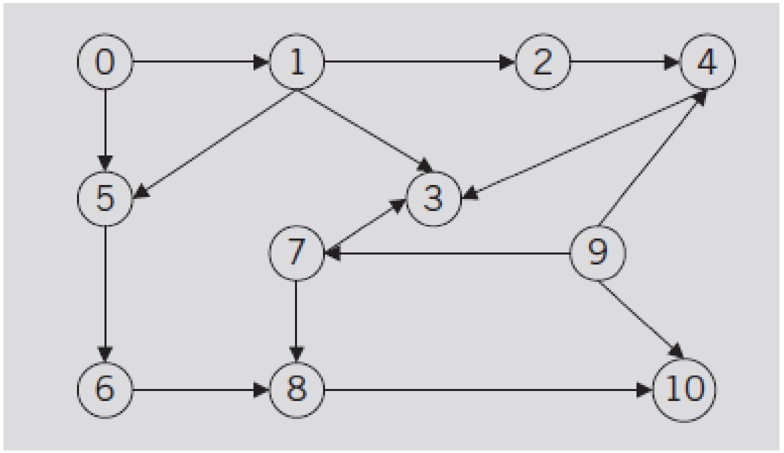


FIGURE 12-7 Directed graph G_3

Depth First Traversal (cont’d.)

- General algorithm for depth first traversal at a given node *v*
  - Recursive algorithm

```
1. mark node v as visited
2. visit the node
3. for each vertex u adjacent to v
         if u is not visited
            start the depth first traversal at u
```

Depth First Traversal (cont’d.)

- Function `dft` implements algorithm

```
void graphType::dft(int v, bool visited[])
{
    visited[v] = true;
    cout << " " << v << " ";  //visit the vertex

    linkedListIterator<int> graphIt;

        //for each vertex adjacent to v
    for (graphIt = graph[v].begin(); graphIt != graph[v].end();
                                     ++graphIt)
    {
        int w = *graphIt;
        if (!visited[w])
            dft(w, visited);
    } //end while
} //end dft
```

Depth First Traversal (cont’d.)

- Function `depthFirstTraversal`
  - Implements depth first traversal of the graph

```
void graphType::depthFirstTraversal()
{
    bool *visited; //pointer to create the array to keep
                   //track of the visited vertices
    visited = new bool[gSize];

    for (int index = 0; index < gSize; index++)
        visited[index] = false;

        //For each vertex that is not visited, do a depth
        //first traverssal
    for (int index = 0; index < gSize; index++)
        if (!visited[index])
            dft(index,visited);
    delete [] visited;
} //end depthFirstTraversal
```

Depth First Traversal (cont’d.)

- Function `depthFirstTraversal`
  - Performs a depth first traversal of entire graph
- Function `dftAtVertex`
  - Performs a depth first traversal at a given vertex

```
void graphType::dftAtVertex(int vertex)
{
    bool *visited;

    visited = new bool[gSize];

    for (int index = 0; index < gSize; index++)
        visited[index] = false;

    dft(vertex, visited);

    delete [] visited;
} // end dftAtVertex
```

Breadth First Traversal

- Similar to traversing binary tree level-by-level
  - Nodes at each level
    - Visited from left to right
  - All nodes at any level *i*
    - Visited before visiting nodes at level *i* + one
  - Breadth first ordering: 0 1 5 2 3 6 4 8 10 7 9

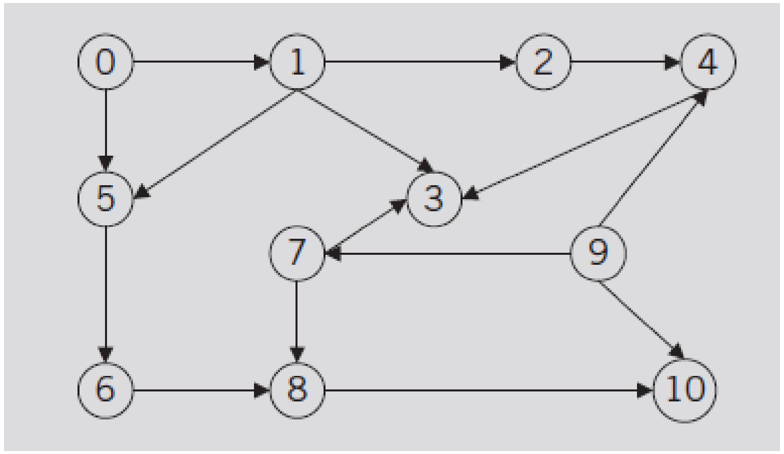
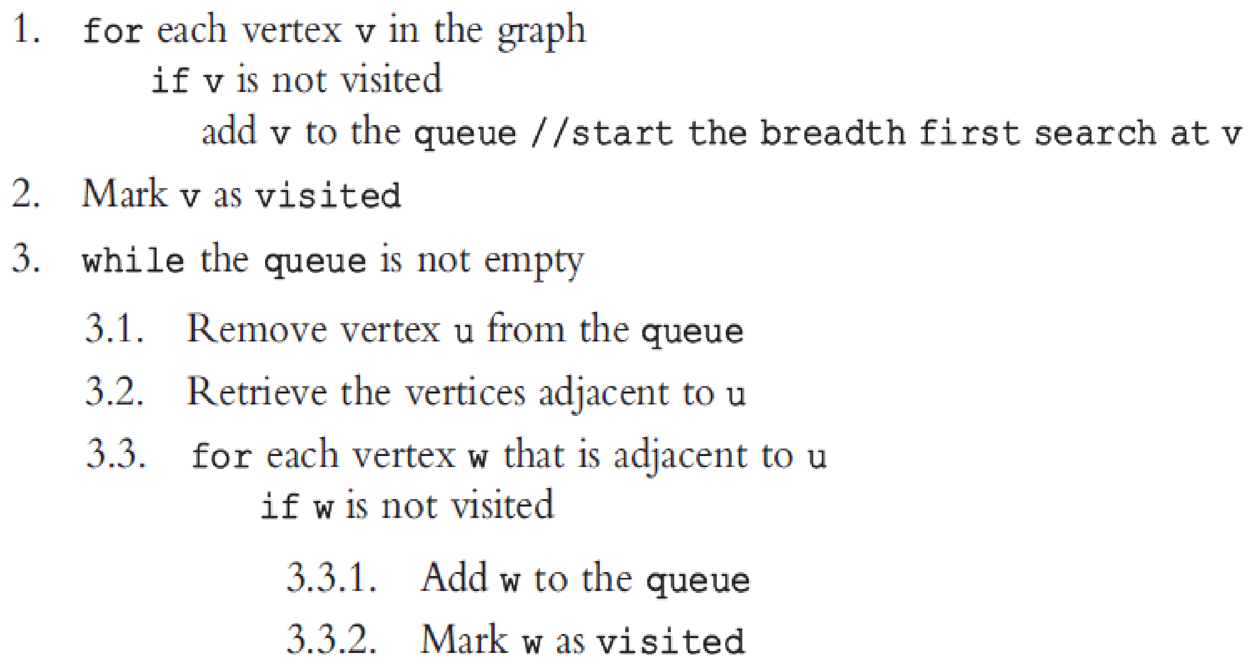


FIGURE 12-7 Directed graph G_3

Breadth First Traversal (cont’d.)

- General search algorithm
  - Breadth first search algorithm with a queue



```
1.   for each vertex v in the graph
         if v is not visited
             add v to the queue //start the breadth first search at v
2.   Mark v as visited
3.   while the queue is not empty
     3.1.   Remove vertex u from the queue
     3.2.   Retrieve the vertices adjacent to u
     3.3.   for each vertex w that is adjacent to u
                if w is not visited
                3.3.1.   Add w to the queue
                3.3.2.   Mark w as visited
```

```
void graphType::breadthFirstTraversal()
{
    linkedQueueType<int> queue;

    bool *visited;
    visited = new bool[gSize];

    for (int ind = 0; ind < gSize; ind++)
        visited[ind] = false;   //initialize the array
                               //visited to false

    linkedListIterator<int> graphIt;

    for (int index = 0; index < gSize; index++)
        if (!visited[index])
        {
            queue.addQueue(index);
            visited[index] = true;
            cout << " " << index << " ";

            while (!queue.isEmptyQueue())
            {
                int u = queue.front();
                queue.deleteQueue();

                for (graphIt = graph[u].begin();
                     graphIt != graph[u].end(); ++graphIt)
                {
                    int w = *graphIt;
                    if (!visited[w])
                    {
                        queue.addQueue(w);
                        visited[w] = true;
                        cout << " " << w << " ";
                    }
                }
            } //end while
        }

    delete [] visited;
} //end breadthFirstTraversal
```

Shortest Path Algorithm

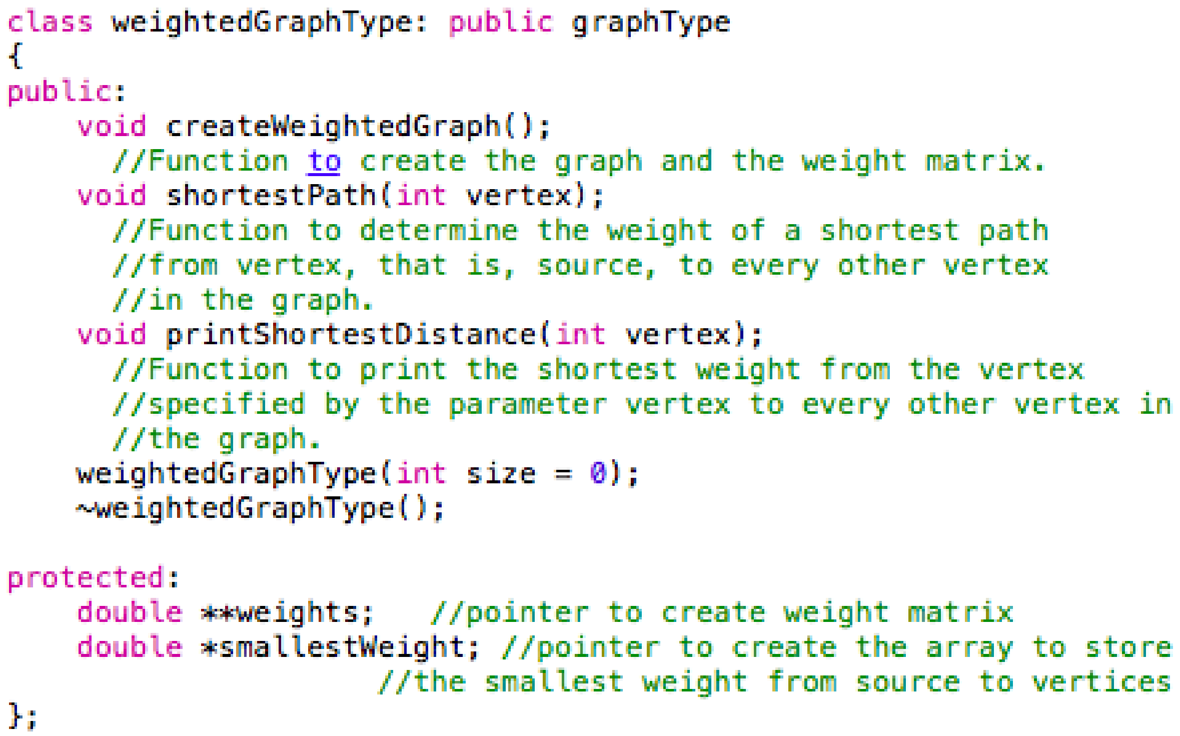
- **Weight** of the graph
  - Nonnegative real number assigned to the edges connecting to vertices
- **Weighted graphs**
  - When a graph uses the weight to represent the distance between two places
- **Weight of the path *P***
  - Given *G* as a weighted graph with vertices *u* and *v* in *G* and *P* as a path in *G* from *u* to *v*
    - Sum of the weights of all the edges on the path
- **Shortest path**: path with the smallest weight

Shortest Path Algorithm (cont'd.)

- **Shortest path algorithm** space (**greedy algorithm**)
- See code on page 700
  - `class weightedGraphType`
    - Extend definition of `class graphType`
    - Adds function `createWeightedGraph` to create graph and weight matrix associated with the graph

Let G be a graph with n vertices, where $n \geq 0$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let W be a two-dimensional $n \times n$ matrix such that

$$W(i,j) = \begin{cases} w_{ij} & \text{if } (v_i, v_j) \text{ is an edge in } G \text{ and } w_{ij} \text{ is the weight of the edge } (v_i, v_j) \\ \infty & \text{if there is no edge from } v_i \text{ to } v_j \end{cases}$$



```
class weightedGraphType: public graphType
{
public:
    void createWeightedGraph();
      //Function to create the graph and the weight matrix.
    void shortestPath(int vertex);
      //Function to determine the weight of a shortest path
      //from vertex, that is, source, to every other vertex
      //in the graph.
    void printShortestDistance(int vertex);
      //Function to print the shortest weight from the vertex
      //specified by the parameter vertex to every other vertex in
      //the graph.
    weightedGraphType(int size = 0);
    ~weightedGraphType();

protected:
    double **weights;    //pointer to create weight matrix
    double *smallestWeight; //pointer to create the array to store
                     //the smallest weight from source to vertices
};
```

Shortest Path

- General algorithm
  - Initialize array `smallestWeight`

    `smallestWeight[u] = weights[vertex, u]`
  - Set `smallestWeight[vertex] = zero`
  - Find vertex *v* closest to vertex where shortest path is not determined
  - Mark *v* as the (next) vertex for which the smallest weight is found

Shortest Path (cont’d.)

- General algorithm (cont’d.)
  - For each vertex *w* in *G*, such that the shortest path from vertex to w has not been determined and an edge (*v*, *w*) exists
    - If weight of the path to *w* via *v* smaller than its current weight
    - Update weight of *w* to the weight of *v* + weight of edge (*v*, *w*)

Shortest Path (cont’d.)

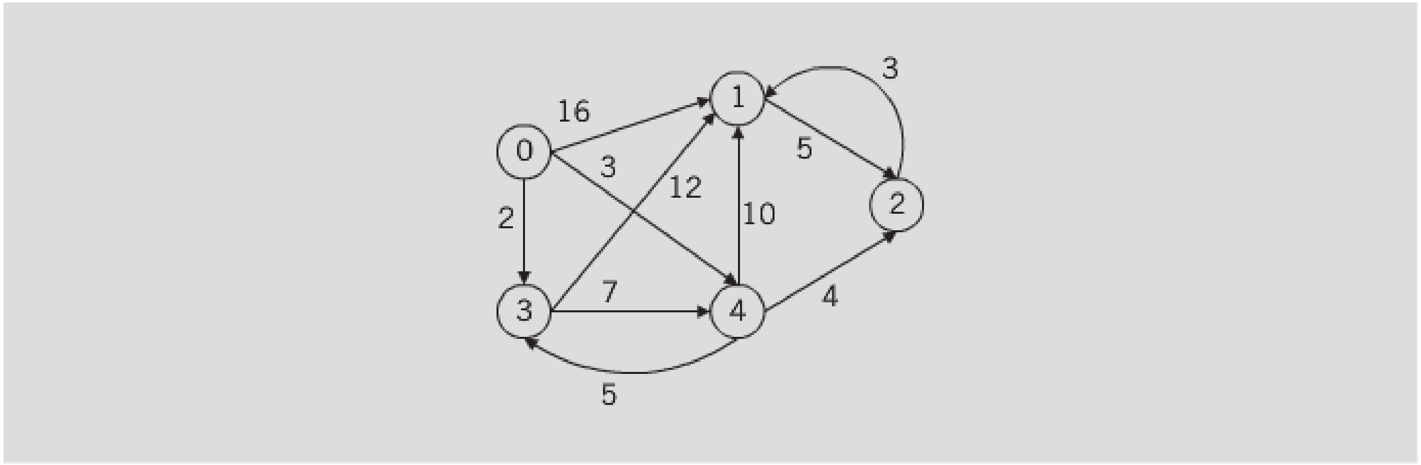
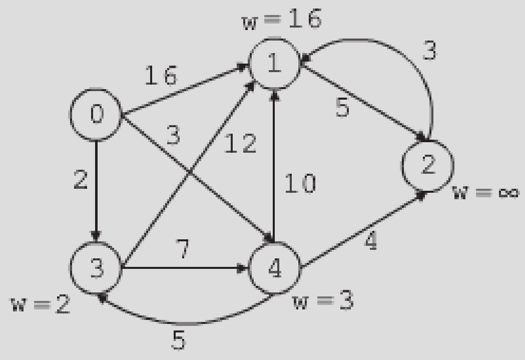


FIGURE 12-8 Weighted graph *G*

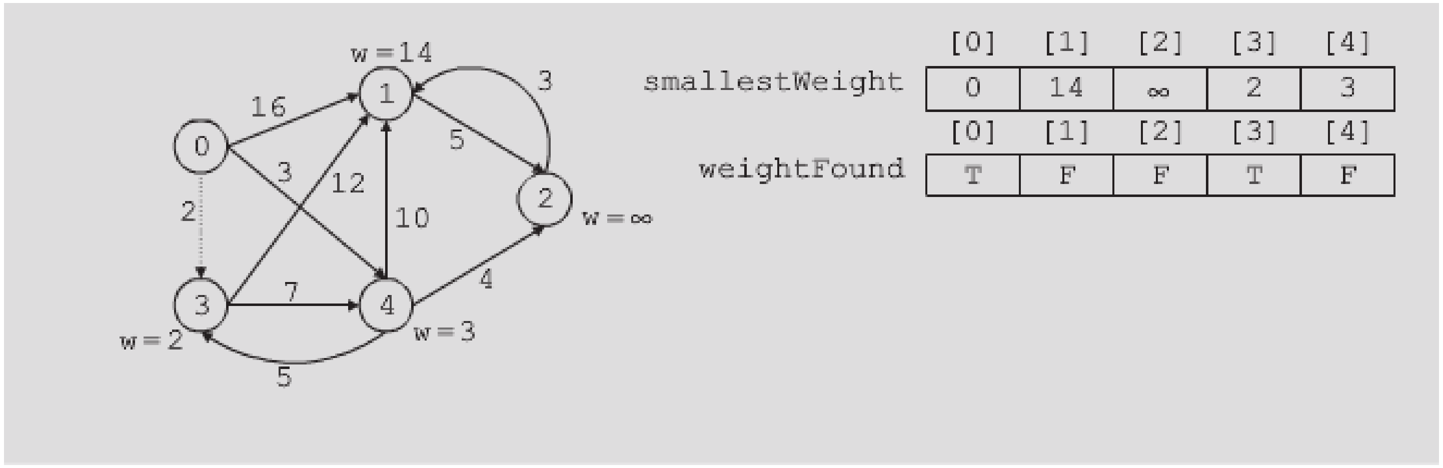


	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	16	∞	2	3

	[0]	[1]	[2]	[3]	[4]
weightFound	T	F	F	F	F

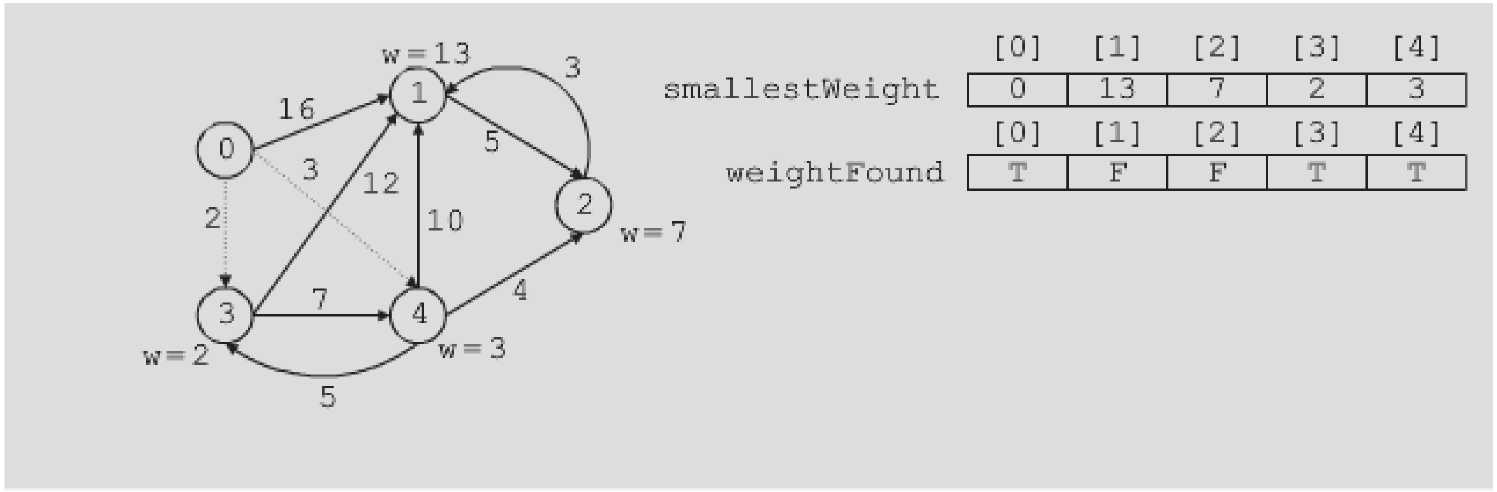
FIGURE 12-9 Graph after Steps 1 and 2 execute

Shortest Path (cont’d.)



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	14	∞	2	3
weightFound	T	F	F	T	F

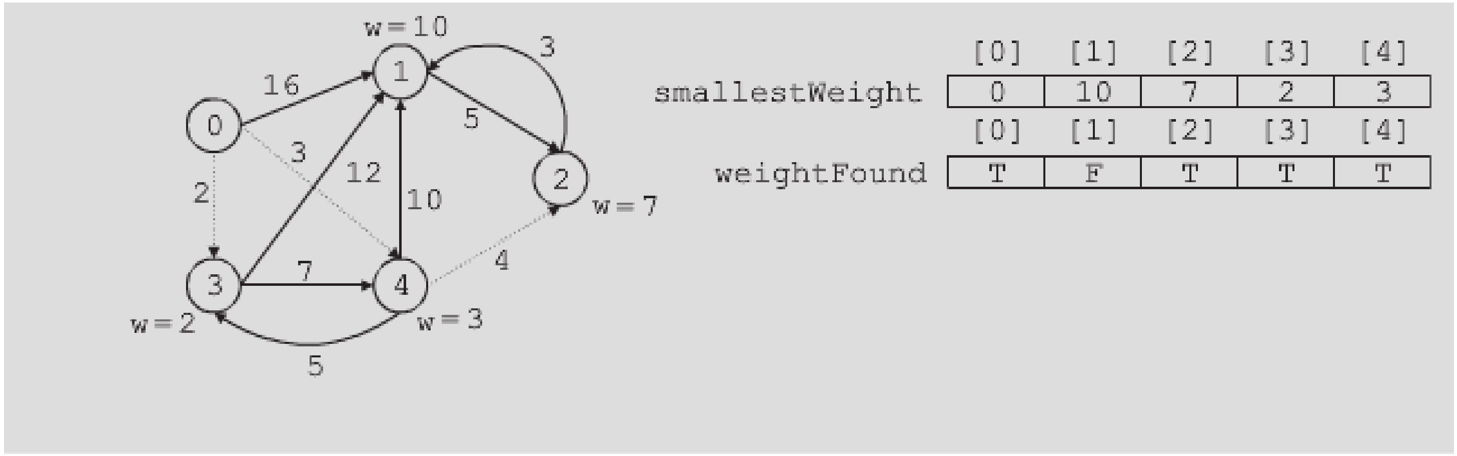
FIGURE 12-10 Graph after the first iteration of Steps 3 to 5



	[0]	[1]	[2]	[3]	[4]
smallestWeight	0	13	7	2	3
weightFound	T	F	F	T	T

FIGURE 12-11 Graph after the second iteration of Steps 3 to 5

Shortest Path (cont’d.)



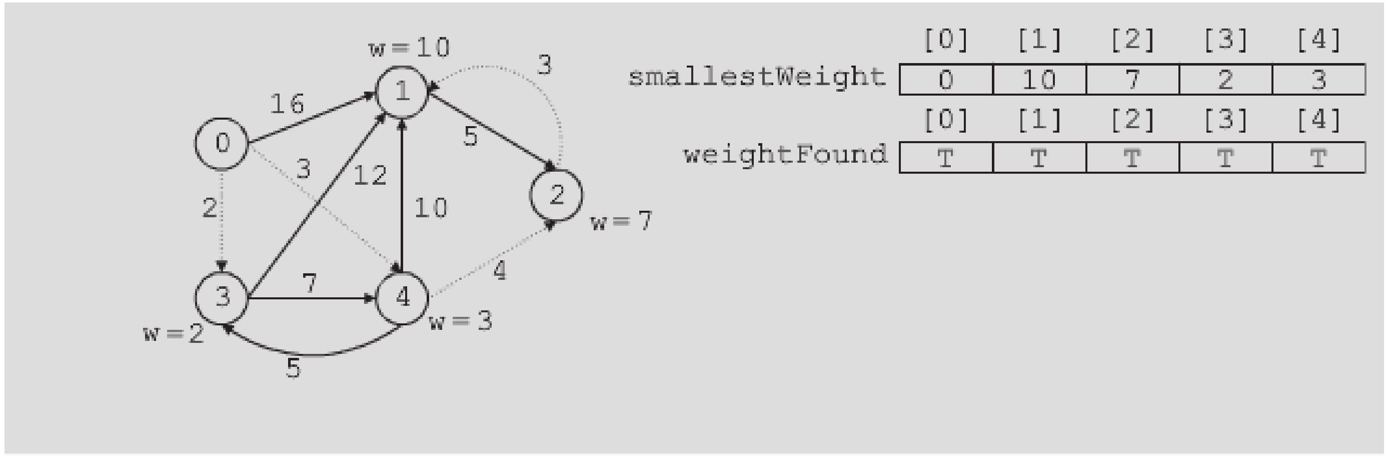
smallestWeight

[0]	[1]	[2]	[3]	[4]
0	10	7	2	3

weightFound

[0]	[1]	[2]	[3]	[4]
T	F	T	T	T

FIGURE 12-12 Graph after the third iteration of Steps 3 to 5



smallestWeight

[0]	[1]	[2]	[3]	[4]
0	10	7	2	3

weightFound

[0]	[1]	[2]	[3]	[4]
T	T	T	T	T

FIGURE 12-13 Graph after the fourth iteration of Steps 3 through 5

Shortest Path (cont’d.)

- See code on pages 704-705
  - C++ function `shortestPath` implements previous algorithm
    - Records only the weight of the shortest path from the source to a vertex
- Review the definitions of the function `printShortestDistance` and the constructor and destructor on pages 705-706

```
void weightedGraphType::shortestPath(int vertex)
{
    for (int j = 0; j < gSize; j++)
        smallestWeight[j] = weights[vertex][j];

    bool *weightFound;
    weightFound = new bool[gSize];

    for (int j = 0; j < gSize; j++)
        weightFound[j] = false;

    weightFound[vertex] = true;
    smallestWeight[vertex] = 0;

    for (int i = 0; i < gSize - 1; i++)
    {
        double minWeight = DBL_MAX;
        int v;

        for (int j = 0; j < gSize; j++)
            if (!weightFound[j])
                if (smallestWeight[j] < minWeight)
                {
                    v = j;
                    minWeight = smallestWeight[v];
                }

        weightFound[v] = true;

        for (int j = 0; j < gSize; j++)
            if (!weightFound[j])
                if (minWeight + weights[v][j] < smallestWeight[j])
                    smallestWeight[j] = minWeight + weights[v][j];
    } //end for
} //end shortestPath
```

Minimum Spanning Tree

- Airline connections of a company
  - Between seven cities

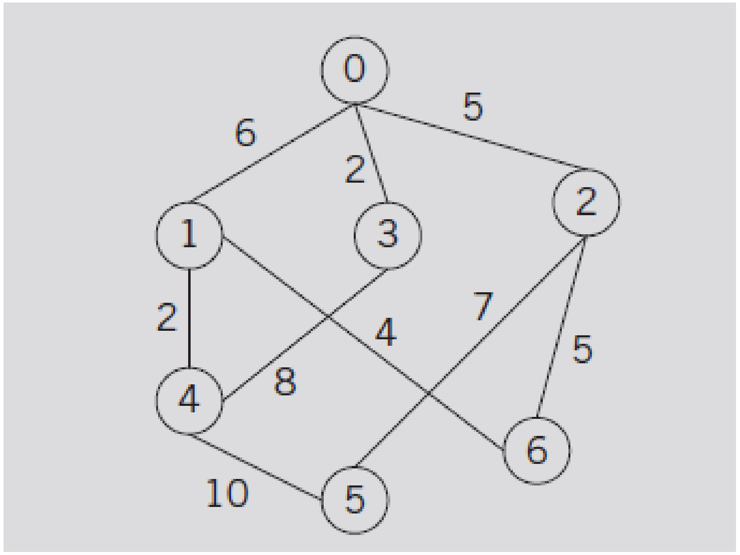


FIGURE 12-14 Airline connections between cities and the cost factor of maintaining the connections

Minimum Spanning Tree (cont’d.)

- Due to financial hardship
  - Company must shut down maximum number of connections
    - Still be able to fly (maybe not directly) from one city to another

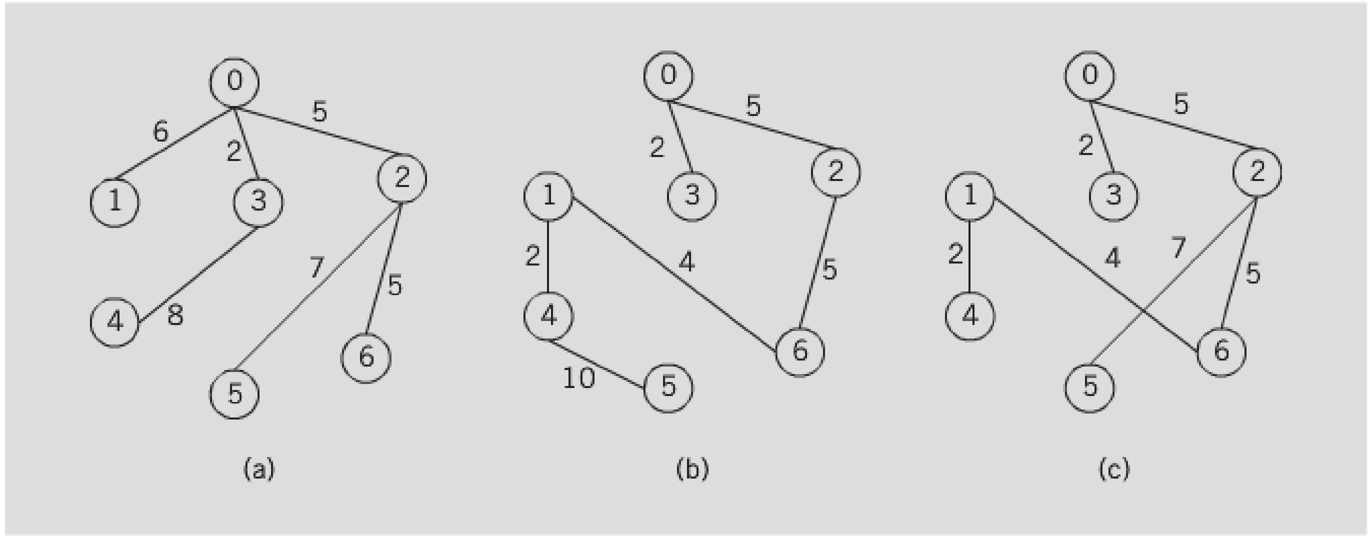


FIGURE 12-15 Possible solutions to the graph of Figure 12-14

Minimum Spanning Tree (cont'd.)

- **Free tree** *T*
  - Simple graph
  - If *u* and *v* are two vertices in *T*
    - Unique path from *u* to *v* exists
- **Rooted tree**
  - Tree with particular vertex designated as a root

Minimum Spanning Tree (cont'd.)

- **Weighted tree** *T*
  - Weight assigned to edges in *T*
  - Weight of *T* denoted by *W*(*T*): sum of weights of all the edges in *T*
- **Spanning tree** *T* of graph *G*
  - *T* is a subgraph of *G* such that *V*(*T*) = *V*(*G)*

Minimum Spanning Tree (cont’d.)

- Theorem 12-1
  - A graph *G* has a spanning tree if and only if *G* is connected
  - From this theorem, it follows that to determine a spanning tree of a graph
    - Graph must be connected
- Minimum (minimal) spanning tree of *G*
  - Spanning tree with the minimum weight

Minimum Spanning Tree (cont’d.)

- Two well-known algorithms for finding a minimum spanning tree of a graph
  - Prim’s algorithm
    - Builds the tree iteratively by adding edges until a minimum spanning tree obtained
  - Kruskal’s algorithm

Minimum Spanning Tree (cont'd.)

- General form of Prim's algorithm

```
1.  Set V(T) = {source}
2.  Set E(T) = empty
3.  for i = 1 to n
    3.1.  minWeight = infinity;
    3.2.  for j = 1 to n
              if v_j is in V(T)
                  for k = 1 to n
                      if v_k is not in T and weight[v_j, v_k] < minWeight
                      {
                          endVertex = v_k;
                          edge = (v_j, v_k);
                          minWeight = weight[v_j, v_k];
                      }
    3.3.  V(T) = V(T) ∪ {endVertex};
    3.4.  E(T) = E(T) ∪ {edge};
```

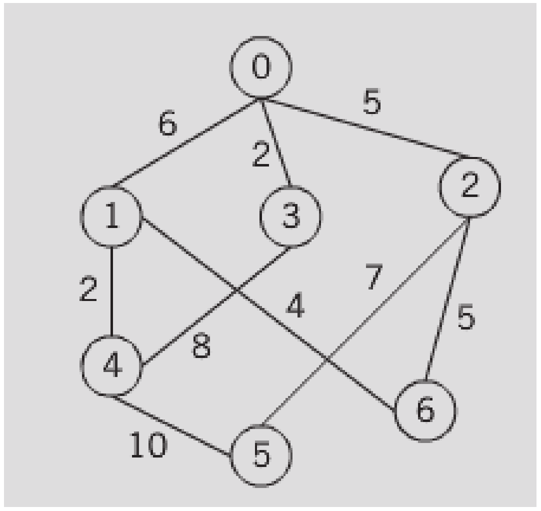


FIGURE 12-16 Weighted graph G

Minimum Spanning Tree (cont'd.)

- See code on page 710
  - `class msTreeType` defines spanning tree as an ADT
- See code on page 712
  - C++ function `minimumSpanning` implementing Prim's algorithm
  - Prim's algorithm given in this section: $O(n^3)$
    - Possible to design Prim's algorithm order $O(n^2)$
- See function `printTreeAndWeight` code
- See constructor and destructor code

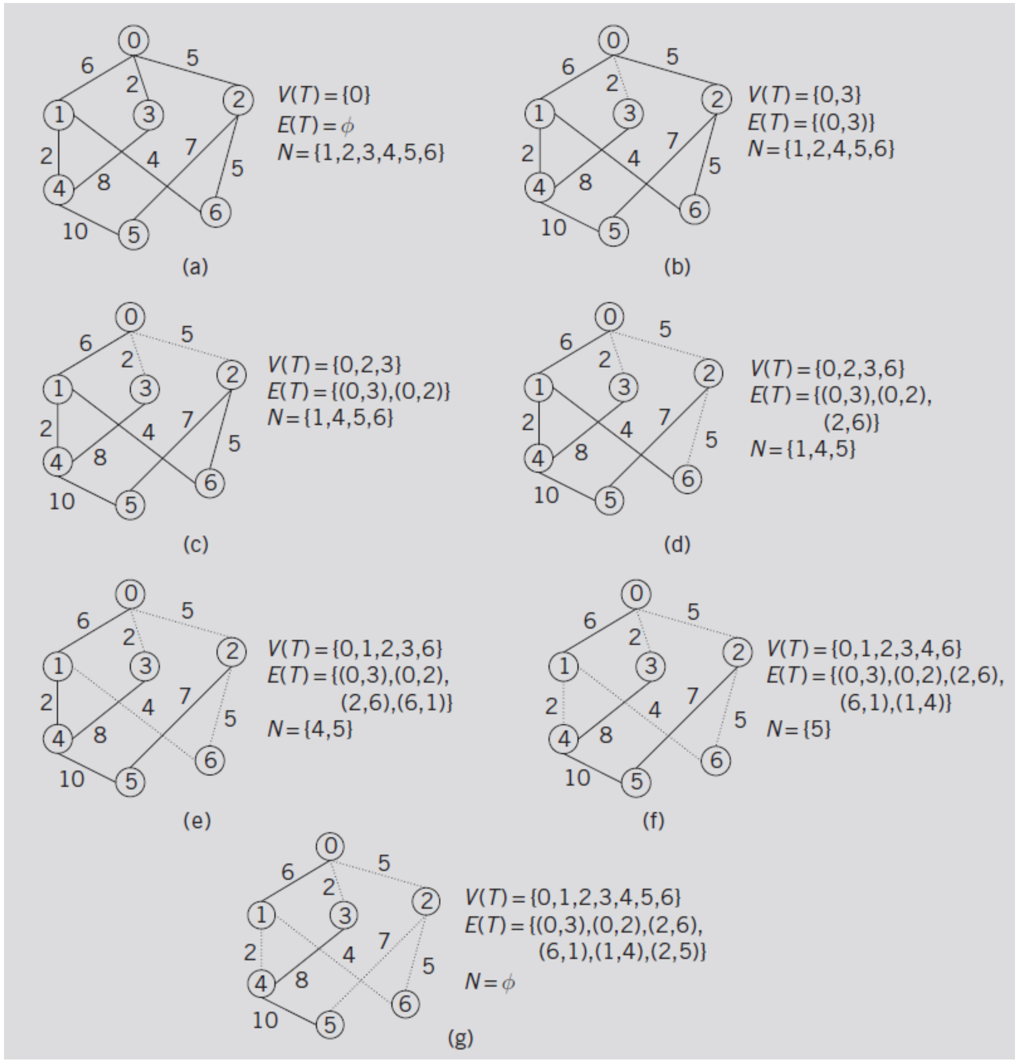


FIGURE 12-17 Graph *G*, *V*(*T*), *E*(*T*), and *N* after Steps 1 and 2 execute

Topological Order

- Topological ordering of $V(G)$
  - Linear ordering $v_{i1}, v_{i2}, \ldots, v_{in}$ of the vertices such that
    - If v_{ij} is a predecessor of v_{ik}, $j \neq k$, $1 \leq j \leq n$, $1 \leq k \leq n$
    - Then v_{ij} precedes v_{ik}, that is, $j < k$ in this linear ordering
- Algorithm topological order
  - Outputs directed graph vertices in topological order
  - Assume graph has no cycles
    - There exists a vertex v in G such that v has no successor
    - There exists a vertex u in G such that u has no predecessor

Topological Order (cont’d.)

- Topological sort algorithm
  - Implemented with the depth first traversal or the breadth first traversal
- Extend `class graphType` definition (using inheritance)
  - Implement breadth first topological ordering algorithm
    - Called `class topologicalOrderType`
  - See code on pages 714-715
    - Illustrating class including functions to implement the topological ordering algorithm

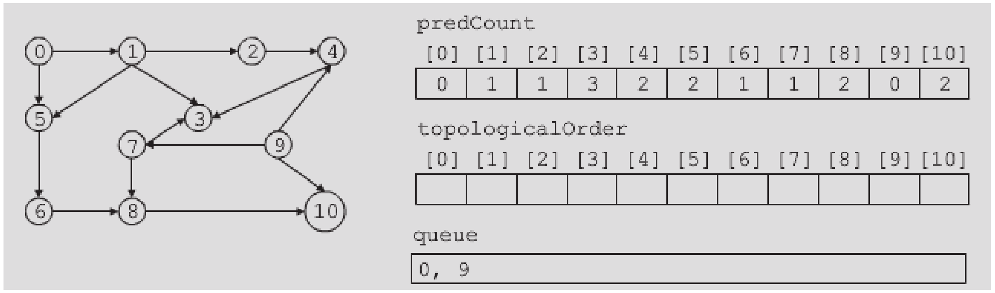
Breadth First Topological Ordering

- General algorithm

```
1. Create the array predCount and initialize it so that predCount[i] is
   the number of predecessors of the vertex $v_i$.
2. Initialize the queue, say queue, to all those vertices $v_k$ so that
   predCount[k] is 0. (Clearly, queue is not empty because the graph has
   no cycles.)
3. while the queue is not empty
   3.1. Remove the front element, u, of the queue.
   3.2. Put u in the next available position, say
        topologicalOrder[topIndex], and increment topIndex.
   3.3. For all the immediate successors w of u,
        3.3.1. Decrement the predecessor count of w by 1.
        3.3.2. if the predecessor count of w is 0, add w to queue.
```

Breadth First Topological Ordering (cont’d.)

- `Breadth First Topological order`
  - 0 9 1 7 2 5 4 6 3 8 10



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	1	1	3	2	2	1	1	2	0	2

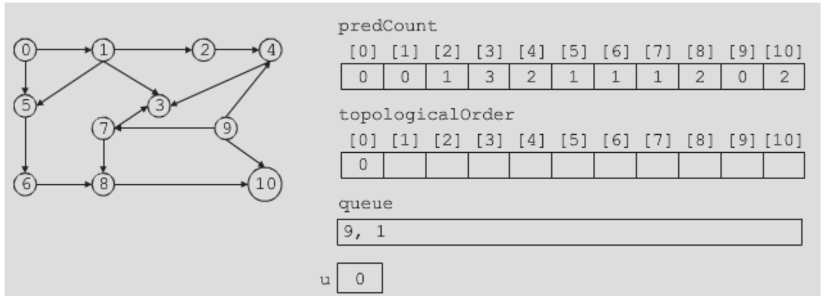
topologicalOrder

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

queue

```
0, 9
```

FIGURE 12-18 Arrays `predCount`, `topologicalOrder`, and `queue` after Steps 1 and 2 execute



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	1	3	2	1	1	1	2	0	2

topologicalOrder

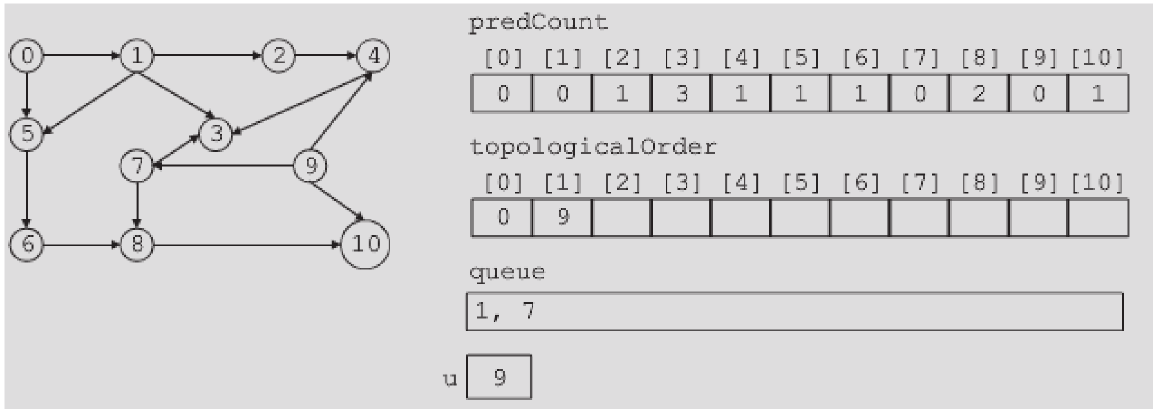
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0										

queue

```
9, 1
```

u `0`

FIGURE 12-19 Arrays `predCount`, `topologicalOrder`, and `queue` after the first iteration of Step 3



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	1	3	1	1	1	0	2	0	1

topologicalOrder

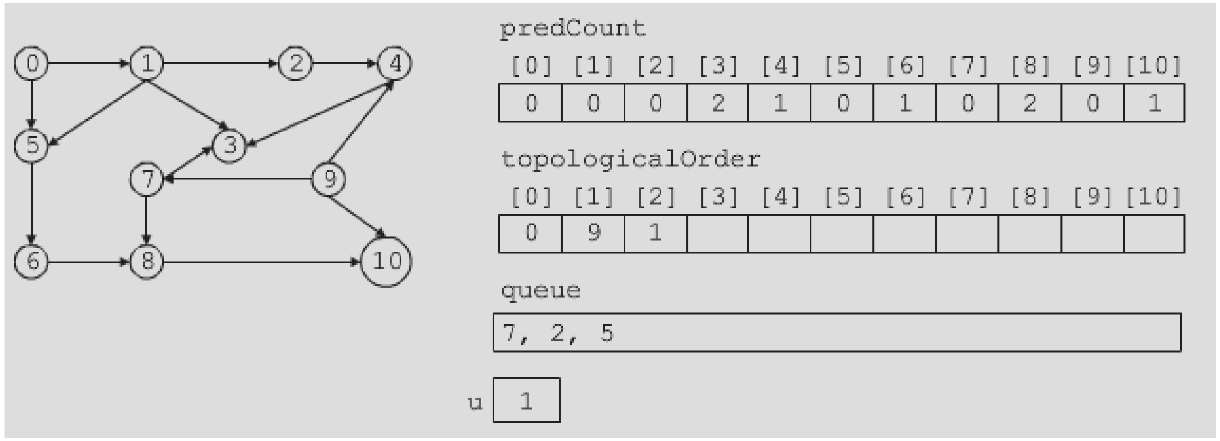
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	9									

queue

1, 7

u 9

FIGURE 12-20 Arrays `predCount`, `topologicalOrder`, and `queue` after the second iteration of Step 3



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	0	2	1	0	1	0	2	0	1

topologicalOrder

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	9	1								

queue

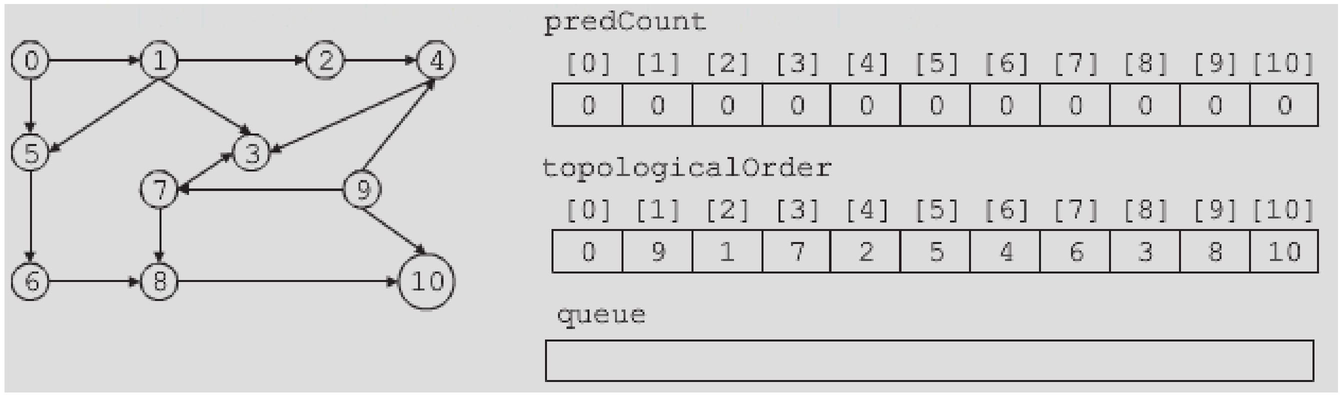
7, 2, 5

u 1

FIGURE 12-21 Arrays `predCount`, `topologicalOrder`, and `queue` after the third iteration of Step 3

Breadth First Topological Ordering (cont' d.)

- See code on pages 718-719
  - Function implementing breadth first topological ordering algorithm



predCount

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	0	0	0	0	0	0	0	0	0	0

topologicalOrder

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0	9	1	7	2	5	4	6	3	8	10

queue

FIGURE 12-22 Arrays `predCount`, `topologicalOrder`, and `queue` after Step 3 executes

Euler Circuits

- Euler’ s solution to Königsberg bridge problem
  - Reduces problem to finding circuit in the graph
- **Circuit**
  - Path of nonzero length
    - From a vertex *u* to *u* with no repeated edges
- **Euler circuit**
  - Circuit in a graph including all the edges of the graph
- **Eulerian** graph *G*
  - If either *G* is a trivial graph or *G* has an Euler circuit

Euler Circuits (cont'd.)

- Graph of Figure 12-24: Euler circuit

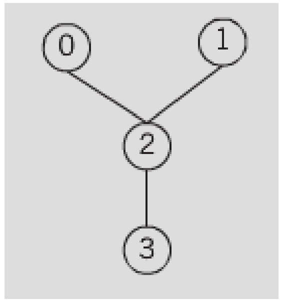


FIGURE 12-23 A graph with all vertices of odd degree

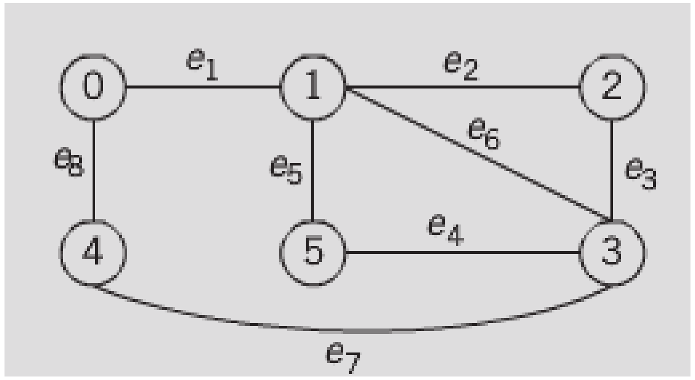


FIGURE 12-24 A graph with all vertices of even degree

Euler Circuits (cont’d.)

- Theorem 12-2
  - If a connected graph *G* is Eulerian, then every vertex of *G* has even degree
- Theorem 12-3
  - Let *G* be a connected graph such that every vertex of *G* is of even degree; then, *G* has an Euler circuit

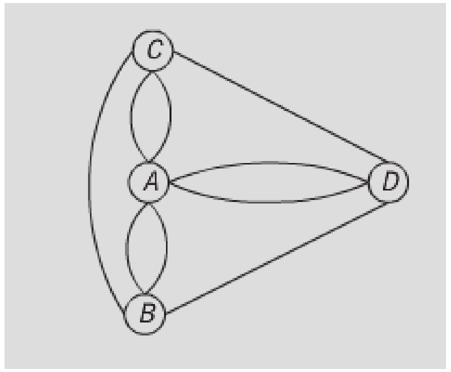
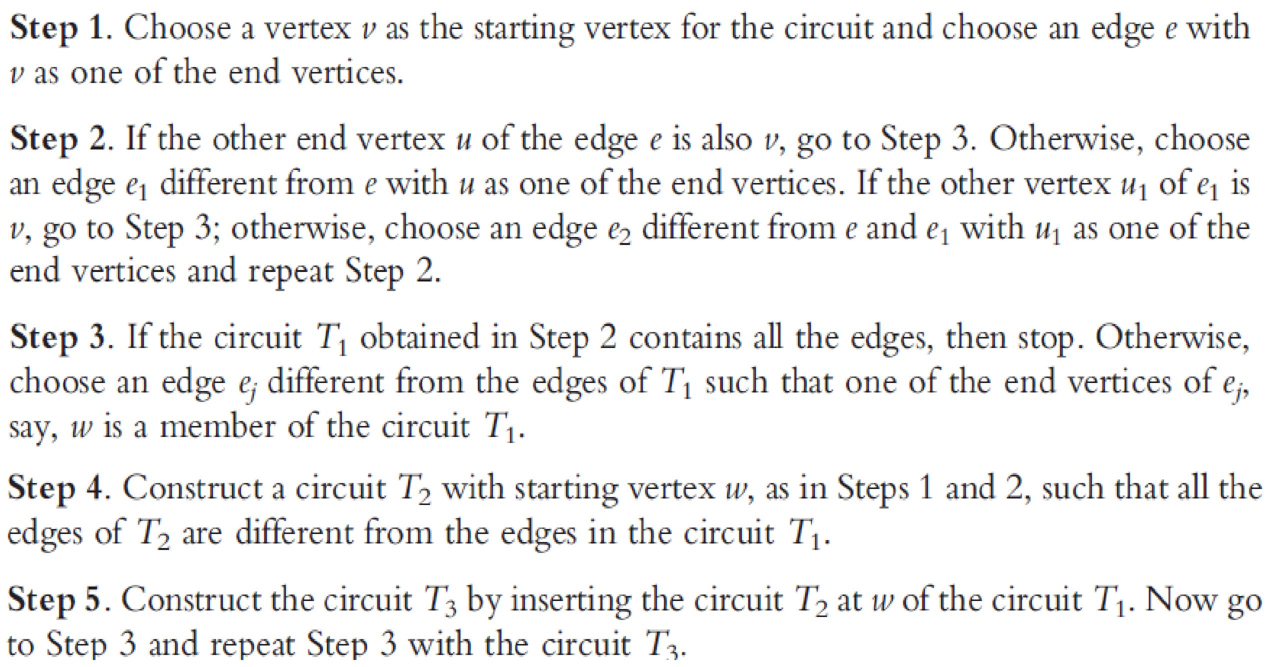


FIGURE 12-25 Graph of the Königsberg bridge problem with two additional bridges

Euler Circuits (cont’d.)

- Fleury’s Algorithm



Step 1. Choose a vertex v as the starting vertex for the circuit and choose an edge e with v as one of the end vertices.

Step 2. If the other end vertex u of the edge e is also v, go to Step 3. Otherwise, choose an edge e_1 different from e with u as one of the end vertices. If the other vertex u_1 of e_1 is v, go to Step 3; otherwise, choose an edge e_2 different from e and e_1 with u_1 as one of the end vertices and repeat Step 2.

Step 3. If the circuit T_1 obtained in Step 2 contains all the edges, then stop. Otherwise, choose an edge e_j different from the edges of T_1 such that one of the end vertices of e_j, say, w is a member of the circuit T_1.

Step 4. Construct a circuit T_2 with starting vertex w, as in Steps 1 and 2, such that all the edges of T_2 are different from the edges in the circuit T_1.

Step 5. Construct the circuit T_3 by inserting the circuit T_2 at w of the circuit T_1. Now go to Step 3 and repeat Step 3 with the circuit T_3.

Euler Circuits (cont’d.)

- Fleury’s Algorithm (cont’d.)

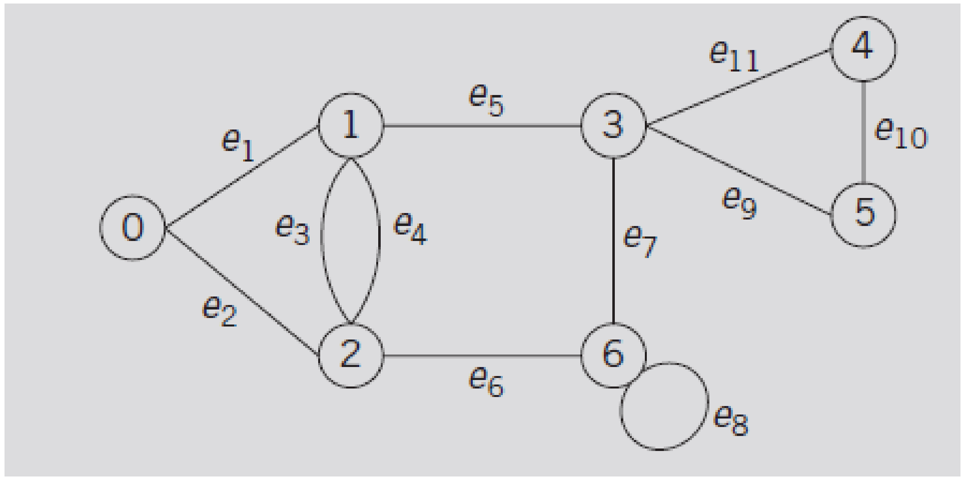


FIGURE 12-26 A graph with all vertices of even degree

Summary

- Many types of graphs
  - Directed, undirected, subgraph, weighted
- Graph theory borrows set theory notation
- Graph representation in memory
  - Adjacency matrices, adjacency lists
- Graph traversal
  - Depth first, breadth first
- Shortest path algorithm
- Prim's algorithm
- Euler circuit