COURSE TECHNOLOGY
CENGAGE Learning™

# Data Structures Using C++ 2E

*Chapter 10*
*Sorting Algorithms*

# Objectives

- Learn the various sorting algorithms
- Explore how to implement selection sort, insertion sort, Shellsort, quicksort, mergesort, and heapsort
- Discover how the sorting algorithms discussed in this chapter perform
- Learn how priority queues are implemented

# Sorting Algorithms

- Several types in the literature
  - Discussion includes most common algorithms
- Analysis
  - Provides a comparison of algorithm performance
- Functions implementing sorting algorithms
  - Included as `public` members of related class

```
template <class elemType>
class arrayListType
{
public:
    void selectionSort();
    ...
};
```

# Selection Sort: Array-Based Lists

- List sorted by selecting elements in the list
  - Select elements one at a time
  - Move elements to their proper positions
  - Either array-based list or linked list
- Selection sort operation
  - Find location of the smallest element in unsorted list portion
    - Move it to top of unsorted portion of the list
  - First time: locate smallest item in the entire list
  - Second time: locate smallest item in the list starting from the second element in the list, and so on

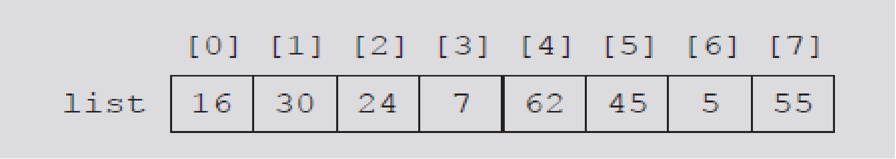



**FIGURE 10-1** List of 8 elements

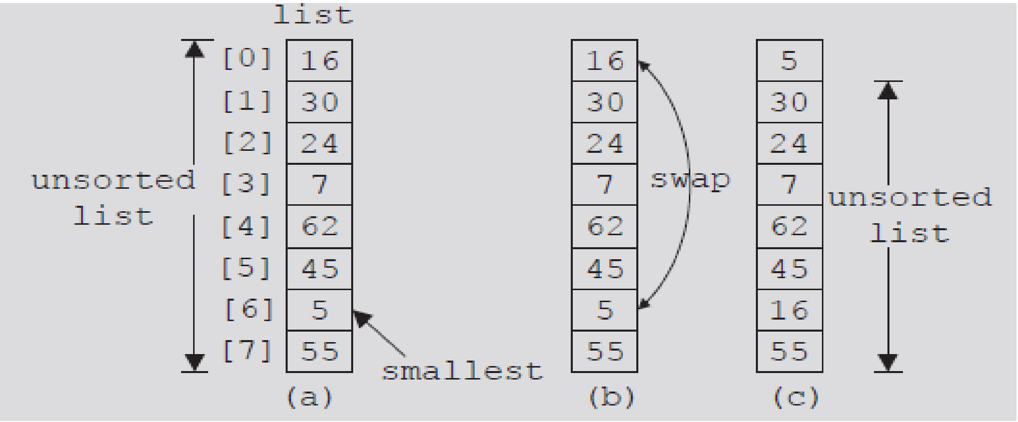



**FIGURE 10-2** Elements of list during the first iteration

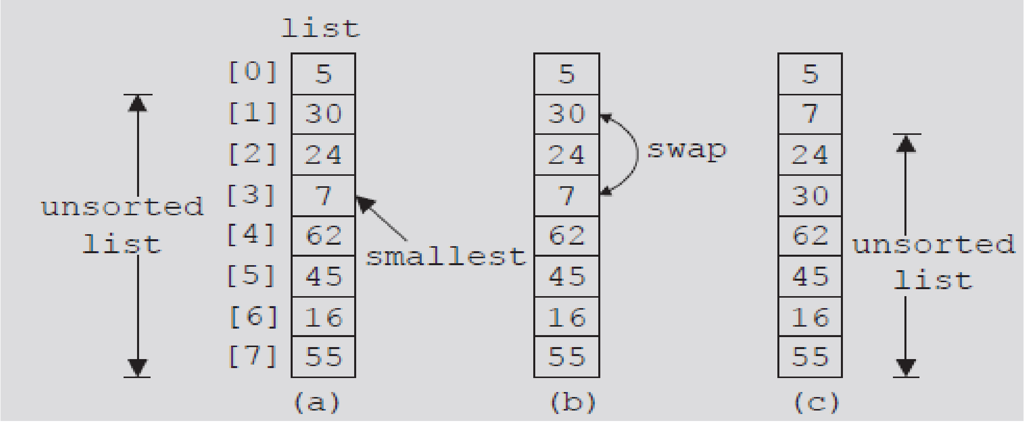



**FIGURE 10-3** Elements of `list` during the second iteration

# Selection Sort: Array-Based Lists (cont'd.)

- Selection sort steps
  - In the unsorted portion of the list
    - Find location of smallest element
    - Move smallest element to beginning of the unsorted list
- Keep track of unsorted list portion with `for` loop

```
for (index = 0; index < length - 1; index++)
{
   1. Find the location, smallestIndex, of the smallest element in
      list[index]...list[length - 1].
   2. Swap the smallest element with list[index]. That is, swap
      list[smallestIndex] with list[index].
}
```

# Selection Sort: Array-Based Lists (cont'd.)

- starting index `first`, and ending index `last`
  - C++ function returns index of the smallest element in `list[first]...list[last]`

O(n)

```
template <class elemType>
int arrayListType<elemType>::minLocation(int first, int last)
{
    int minIndex;

    minIndex = first;

    for (int loc = first + 1; loc <= last; loc++)
        if( list[loc] < list[minIndex])
            minIndex = loc;

    return minIndex;
} //end minLocation
```

# Selection Sort: Array-Based Lists (cont'd.)

- Function `swap`

O(1)

```
template <class elemType>
void arrayListType<elemType>::swap(int first, int second)
{
    elemType temp;

    temp = list[first];
    list[first] = list[second];
    list[second] = temp;
}//end swap
```

- Function `selectionSort`

$O(n^2)$

```
template <class elemType>
void arrayListType<elemType>::selectionSort()
{
    int minIndex;

    for (int loc = 0; loc < length - 1; loc++)
    {
        minIndex = minLocation(loc, length - 1);
        swap(loc, minIndex);
    }
}
```

# Selection Sort: Array-Based Lists (cont'd.)

- Add functions to implement selection sort in the definition of `class arrayListType`

```
template<class elemType>
class arrayListType
{
public:
     //Place the definitions of the function given earlier here.

    void selectionSort();
    ...

private:
      //Place the definitions of the members given earlier here.
    void swap(int first, int second);
    int minLocation(int first, int last);
};
```

```
#include <iostream>                                    //Line 1
#include "arrayListType.h"                             //Line 2

using namespace std;                                   //Line 3

int main()                                             //Line 4
{                                                      //Line 5
    arrayListType<int> list;                           //Line 6
    int num;                                           //Line 7

    cout << "Line 8: Enter numbers ending with -999"
         << endl;                                      //Line 8

    cin >> num;                                        //Line 9

    while (num != -999)                                //Line 10
    {                                                  //Line 11
        list.insert(num);                              //Line 12
        cin >> num;                                    //Line 13
    }                                                  //Line 14

    cout << "Line 15: The list before sorting:" << endl; //Line 15
    list.print();                                      //Line 16
    cout << endl;                                      //Line 17

    list.selectionSort();                              //Line 18

    cout << "Line 19: The list after sorting:" << endl;  //Line 19
    list.print();                                      //Line 20
    cout << endl;                                      //Line 21

    return 0;                                          //Line 22
}                                                      //Line 23
```



**Sample Run**: In this sample run, the user input is shaded.

```
Line 8: Enter numbers ending with -999
34 67 23 12 78 56 36 79 5 32 66 -999
Line 15: The list before sorting:
34 67 23 12 78 56 36 79 5 32 66
```

```
Line 19: The list after sorting:
5 12 23 32 34 36 56 66 67 78 79
```

# Analysis: Selection Sort

- Search algorithms
  - Concerned with number of key (item) comparisons
- Sorting algorithms
  - Concerned with number of key comparisons *and* number of data movements
- Analysis of selection sort
  - Function `swap`
    - Number of item assignments: $3(n-1)$
  - Function `minLocation` and invocation
    - Number of key comparisons of $O(n^2)$

| | Best case | Avg case | Worst case |
|---|---|---|---|
| Selection sort | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ |

# In Class Exercise

- Start with the following list of numbers: 29, 43, 14, 62, 39, 55, 11, 18, 101, 99, what is the list after the $4^{th}$ iteration of the for loop?

# Insertion Sort: Array-Based Lists

- Attempts to improve high selection sort key comparisons
- Sorts list by moving each element to its proper place
- Either array-based list or linked list
- Given list of length eight

| | [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] |
|---|---|---|---|---|---|---|---|---|
| list | 10 | 18 | 25 | 30 | 23 | 17 | 45 | 35 |

**FIGURE 10-4** `list`

# Insertion Sort: Array-Based Lists (cont'd.)

- Elements `list[0]`, `list[1]`, `list[2]`, `list[3]` in order
- Consider element `list[4]`
  - First element of unsorted list

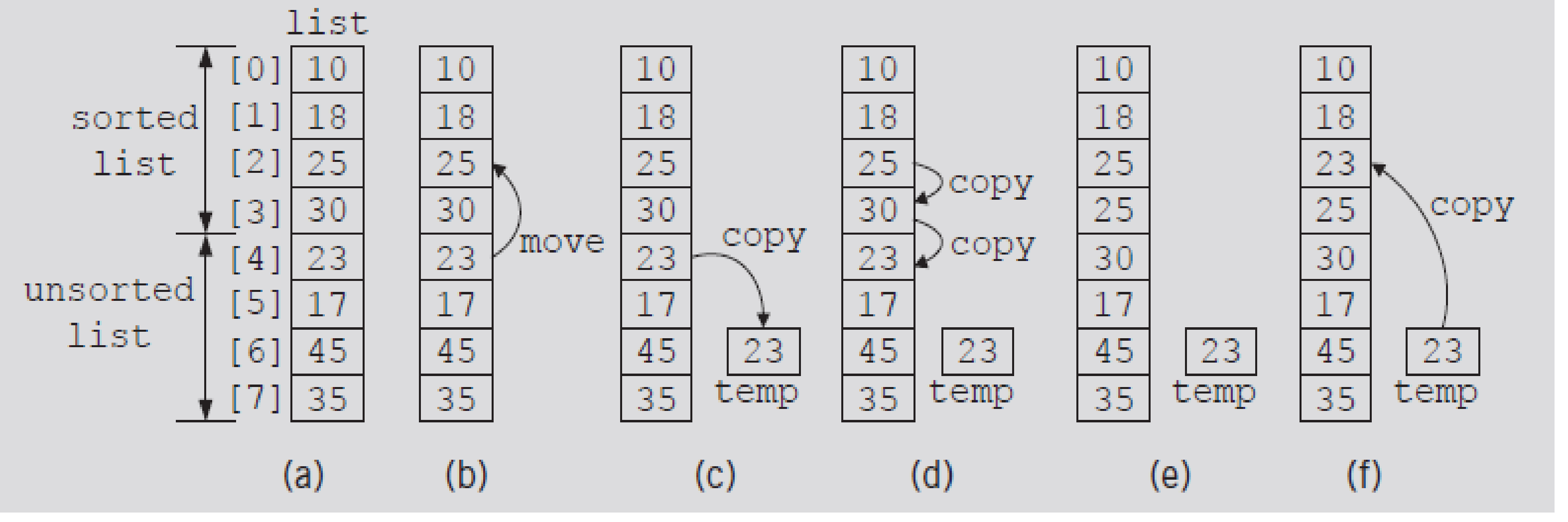



**FIGURE 10-5** `list` elements while moving `list[4]` to its proper place

# Insertion Sort: Array-Based Lists (cont'd.)

- Array containing list divided into two sublists
  - Upper (sorted) and lower (unsorted)
- Index `firstOutOfOrder`
  - Points to first element in the lower sublist

```
for (firstOutOfOrder = 1; firstOutOfOrder < length; firstOutOfOrder++)
  if (list[firstOutOfOrder] is less than list[firstOutOfOrder - 1])
  {
        copy list[firstOutOfOrder] into temp

        initialize location to firstOutOfOrder

        do
        {
            a. move list[location - 1] one array slot down
            b. decrement location by 1 to consider the next element
               sorted of the portion of the array
        }
        while (location > 0 && the element in the upper list at
                location - 1 is greater than temp)
  }
copy temp into list[location]
```

# Insertion Sort: Array-Based Lists (cont'd.)

- `length` = 8
- Initialize `firstOutOfOrder` to one

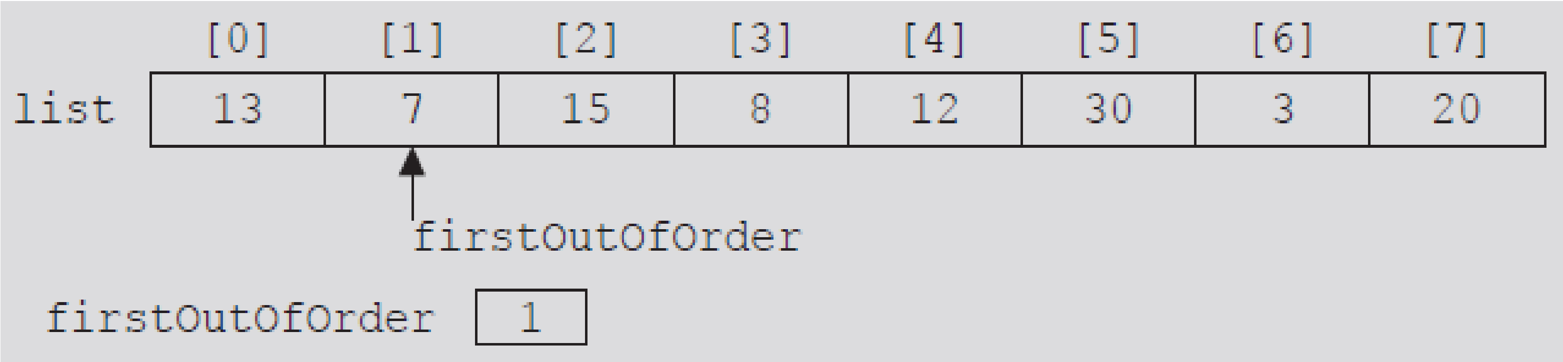



**FIGURE 10-6** `firstOutOfOrder` = 1

# Insertion Sort: Array-Based Lists (cont'd.)

- `list[firstOutOfOrder] = 7`
- `list[firstOutOfOrder - 1] = 13`

  ➔ 7 < 13

- Expression in `if` statement evaluates to true
  - Execute body of `if` statement
    - `temp = list[firstOutOfOrder] = 7`
    - `location = firstOutOfOrder = 1`
  - Execute the `do...while` loop
    - `list[1] = list[0] = 13`
    - `location = 0`

# Insertion Sort: Array-Based Lists (cont'd.)

- `do...while` loop terminates
  - Because `location` = 0
    - Copy `temp` into `list[location]` (`list[0]`)

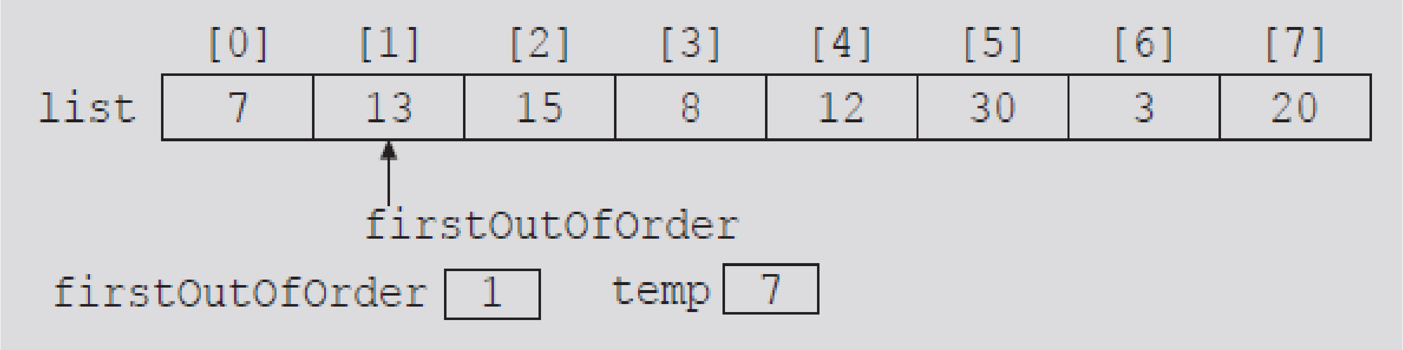



**FIGURE 10-7** `list` after the first iteration of insertion sort

# Insertion Sort: Array-Based Lists (cont'd.)

- Suppose `list` given in Figure 10-8(a)
  - Walk through code

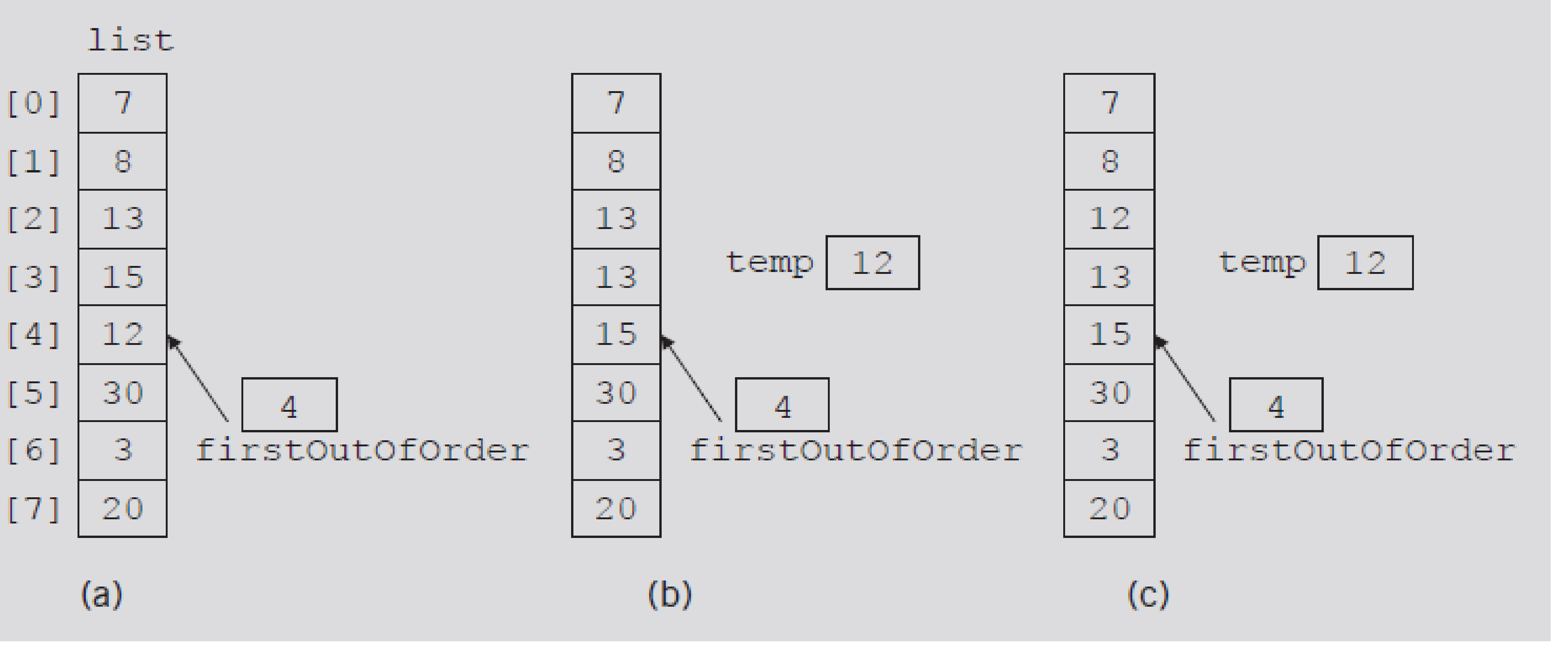



**FIGURE 10-8** `list` elements while moving `list[4]` to its proper place

# Insertion Sort: Array-Based Lists (cont'd.)

- Suppose `list` given in Figure 10-9
  - Walk through code

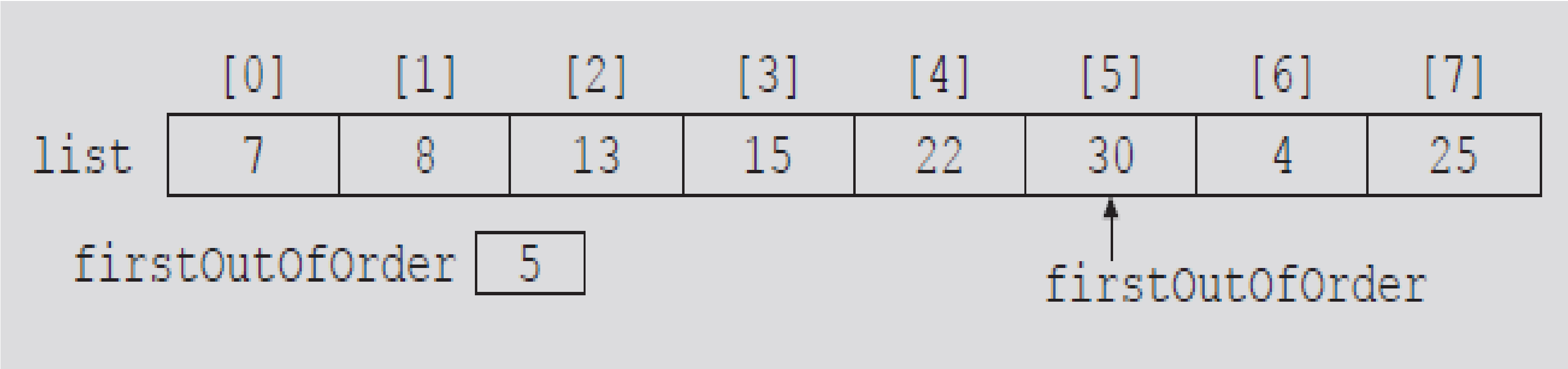



**FIGURE 10-9** First out-of-order element is at position 5

# Insertion Sort: Array-Based Lists (cont'd.)

- C++ function implementing previous algorithm

```
template <class elemType>
void arrayListType<elemType>::insertionSort()
{
    int firstOutOfOrder, location;
    elemType temp;

    for (firstOutOfOrder = 1; firstOutOfOrder < length;
                              firstOutOfOrder++)
        if (list[firstOutOfOrder] < list[firstOutOfOrder - 1])
        {
            temp = list[firstOutOfOrder];
            location = firstOutOfOrder;

            do
            {
                list[location] = list[location - 1];
                location--;
            }
            while (location > 0 && list[location - 1] > temp);

            list[location] = temp;
        }
} //end insertionSort
```

# Insertion Sort: Linked List-Based Lists

- If list stored in an array
  - Traverse list in either direction using index variable
- If list stored in a linked list
  - Traverse list in only one direction
    - Starting at first node: links only in one direction

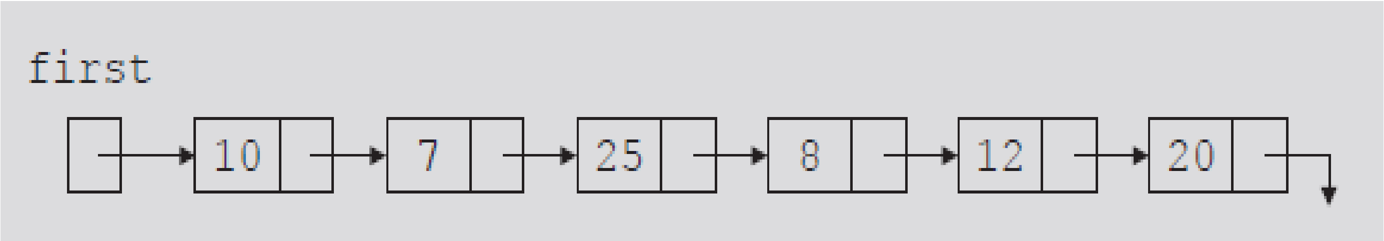



**FIGURE 10-10** Linked list

# Insertion Sort: Linked List-Based Lists (cont'd.)

- `firstOutOfOrder`
  - Pointer to node to be moved to its proper location
- `lastInOrder`
  - Pointer to last node of the sorted portion of the list

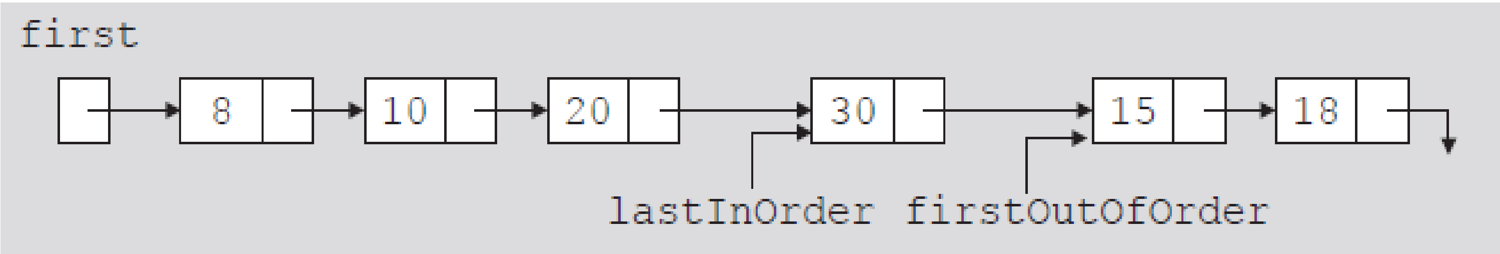



**FIGURE 10-11** Linked list and pointers `lastInOrder` and `firstOutOfOrder`

# Insertion Sort: Linked List-Based Lists (cont'd.)

- Compare `firstOutOfOrder` info with `first` node info
  - If `firstOutOfOrder` info smaller than `first` node info
    - `firstOutOfOrder` moved before `first` node
  - Otherwise, search list starting at second node to find location where to move `firstOutOfOrder`
- Search list using two pointers
  - `current`
  - `trailCurrent`: points to node just before `current`
- Handle any special cases

# Insertion Sort: Linked List-Based Lists (cont'd.)

```
if (firstOutOfOrder->info is less than first->info)
    move firstOutOfOrder before first
else
{
    set trailCurrent to first
    set current to the second node in the list first->link;

     //search the list
    while (current->info is less than firstOutOfOrder->info)
    {
        advance trailCurrent;
        advance current;
    }

    if (current is not equal to firstOutOfOrder)
    {    //insert firstOutOfOrder between current and trailCurrent
        lastInOrder->link = firstOutOfOrder->link;
        firstOutOfOrder->link = current;
        trailCurrent->link = firstOutOfOrder;
    }
    else   //firstOutOfOrder is already at the first place
        lastInOrder = lastInOrder->link;
}
```

# Insertion Sort: Linked List-Based Lists (cont’d.)

- Case 1
  - `firstOutOfOrder`->info less than `first`->info
    - Node `firstOutOfOrder` moved before `first`
    - Adjust necessary links

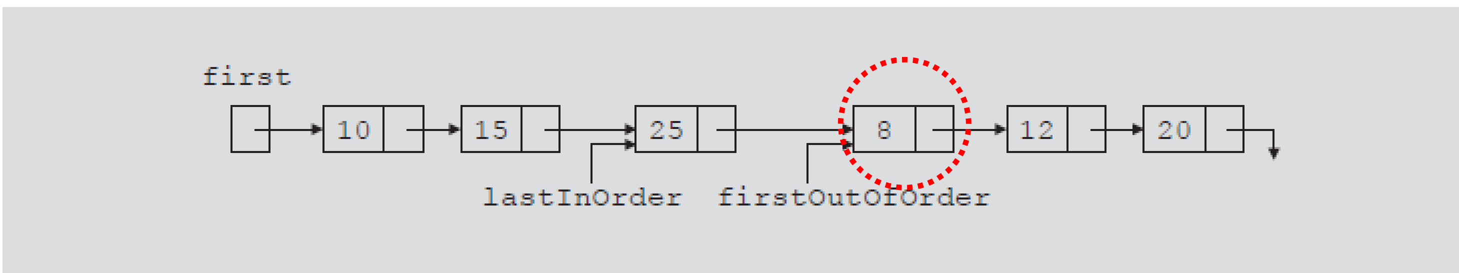



**FIGURE 10-12** Linked list and pointers `lastInOrder` and `firstOutOfOrder`

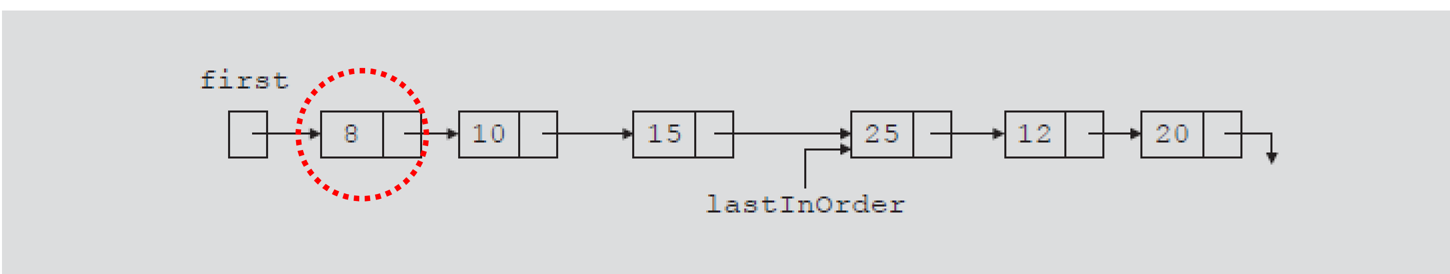



**FIGURE 10-13** Linked list after moving the node with `info` 8 to the beginning

**Case 2:** Consider the list shown in Figure 10-14.

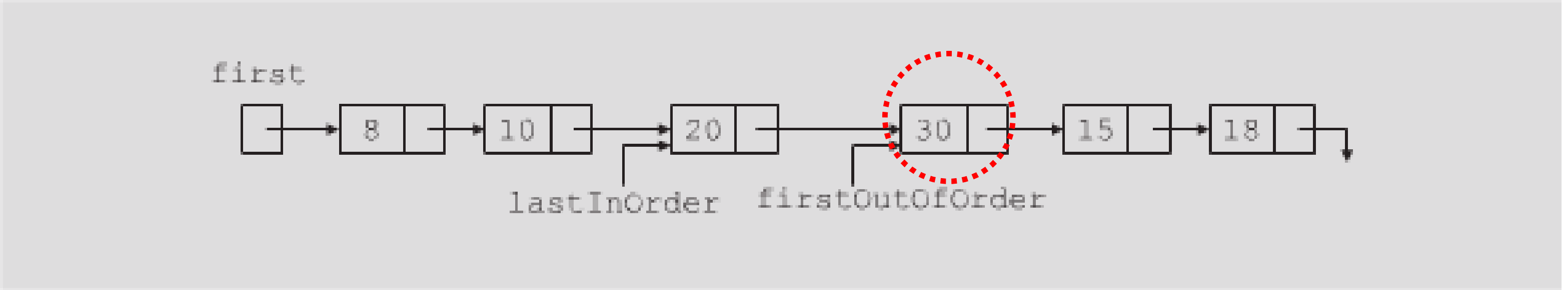



**FIGURE 10-14** Linked list and pointers `lastInOrder` and `firstOutOfOrder`

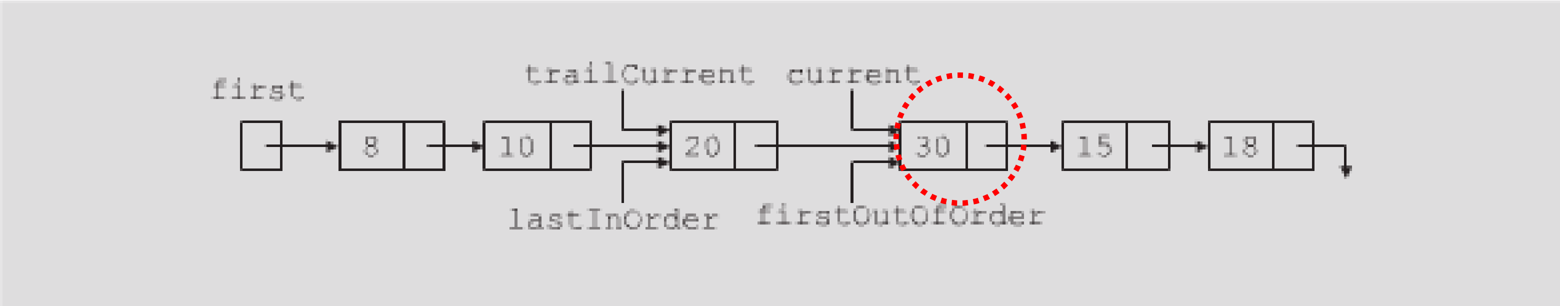



**FIGURE 10-15** Linked list and pointers `trailCurrent` and `current`

No adjustment – already in the right place

**Case 3:** Consider the list in Figure 10-16.

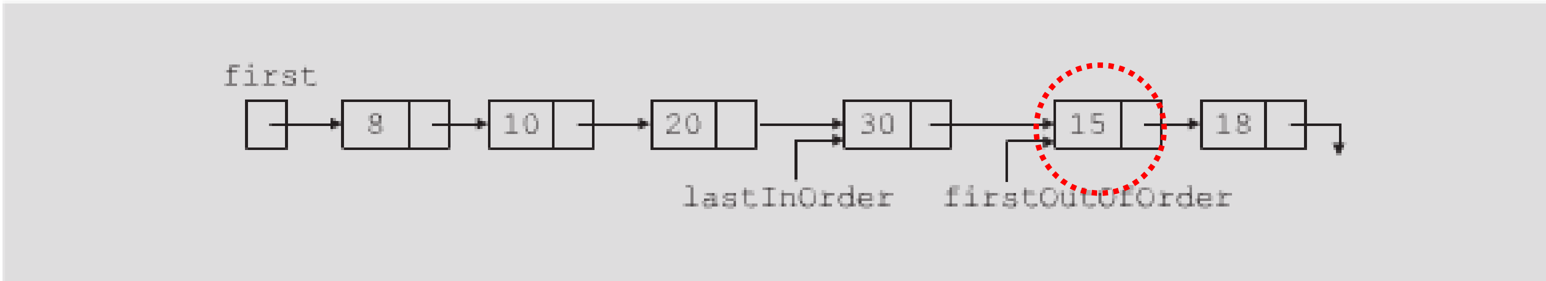



**FIGURE 10-16** Linked list and pointers `lastInOrder` and `firstOutOfOrder`

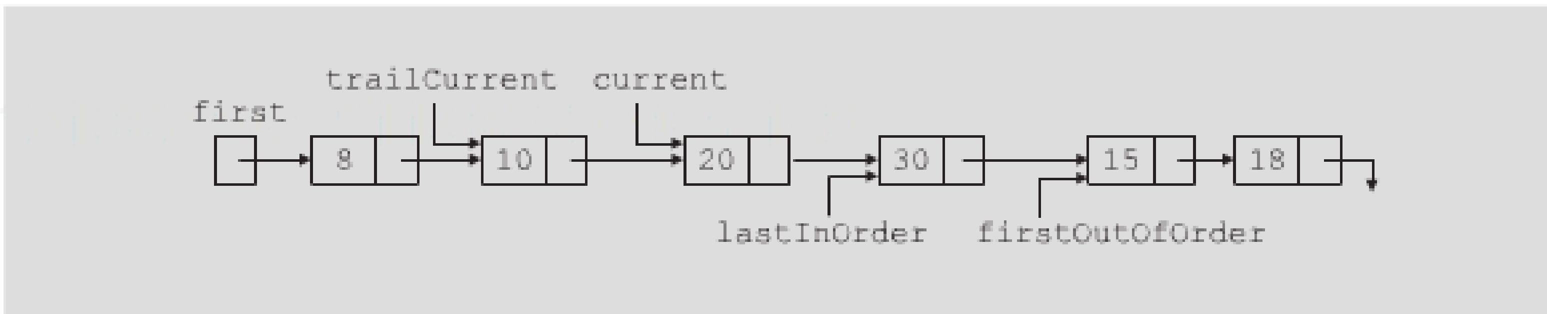



**FIGURE 10-17** Linked list and pointers `trailCurrent` and `current`

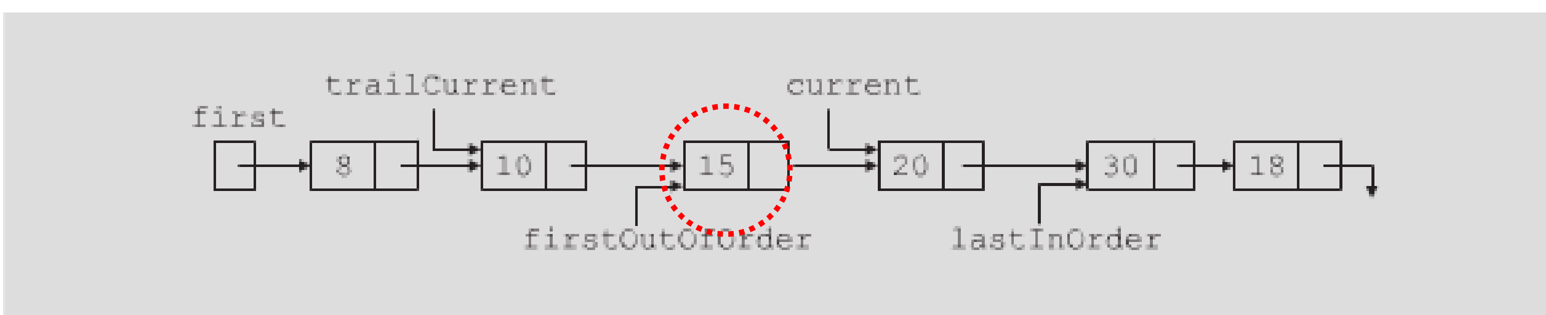



**FIGURE 10-18** Linked list after moving `firstOutOfOrder` between `trailCurrent` and `current`

# Insertion Sort: Linked List-Based Lists (cont'd.)

- Review Case 2 on page 546
- Review Case 3 on page 546
- Review function `linkedInsertionSort` on page 547
  - Implements previous algorithm

```
template <class elemType>
void unorderedLinkedList<elemType>::linkedInsertionSort()
{
    nodeType<elemType> *lastInOrder;
    nodeType<elemType> *firstOutOfOrder;
    nodeType<elemType> *current;
    nodeType<elemType> *trailCurrent;

    lastInOrder = first;

    if (first == NULL)
       cerr << "Cannot sort an empty list." << endl;
    else if (first->link == NULL)
        cout << "The list is of length 1. "
             << "It is already in order." << endl;
    else
        while (lastInOrder->link != NULL)
        {
            firstOutOfOrder = lastInOrder->link;

            if (firstOutOfOrder->info < first->info)
            {
                lastInOrder->link = firstOutOfOrder->link;
                firstOutOfOrder->link = first;
                first = firstOutOfOrder;
            }
            else
            {
                trailCurrent = first;
                current = first->link;

                while (current->info < firstOutOfOrder->info)
                {
                    trailCurrent = current;
                    current = current->link;
                }

                if (current != firstOutOfOrder)
                {
                    lastInOrder->link = firstOutOfOrder->link;
                    firstOutOfOrder->link = current;
                    trailCurrent->link = firstOutOfOrder;
                }
                else
                    lastInOrder = lastInOrder->link;
            }
        } //end while
} //end linkedInsertionSort
```

# Analysis: Insertion Sort

**TABLE 10-1** Average-case behavior of the selection sort and insertion sort for a list of length *n*

| Algorithm | Number of comparisons | Number of swaps/item assignments |
|---|---|---|
| Selection sort | $(1/2)n(n-1) = O(n^2)$ | $3(n-1) = O(n)$ |
| Insertion sort | $(1/4)n^2 + O(n) = O(n^2)$ | $(1/4)n^2 + O(n) = O(n^2)$ |

- Insertion sort reduces # of comparisons, at the cost of more data movements

| | Best case | Avg case | Worst case |
|---|---|---|---|
| Selection sort | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ |
| Insertion sort | $O(n)$ | $O(n^2)$ | $O(n^2)$ |

When?

When?

# Shellsort

- Reduces number of item movements in insertion sort by modifying it
  - Introduced in 1959 by D.E. Shell
  - Also known as *diminishing-increment sort*
- List elements viewed as sublists at a particular distance
  - Each sublist sorted
    - Elements far apart move closer to their final position
  - Next sequence, repeat
- Given increment (gap) sequence 1, 4, 7
  - Sort elements at a distance of 7, 4, 1

# Shellsort (cont'd.)

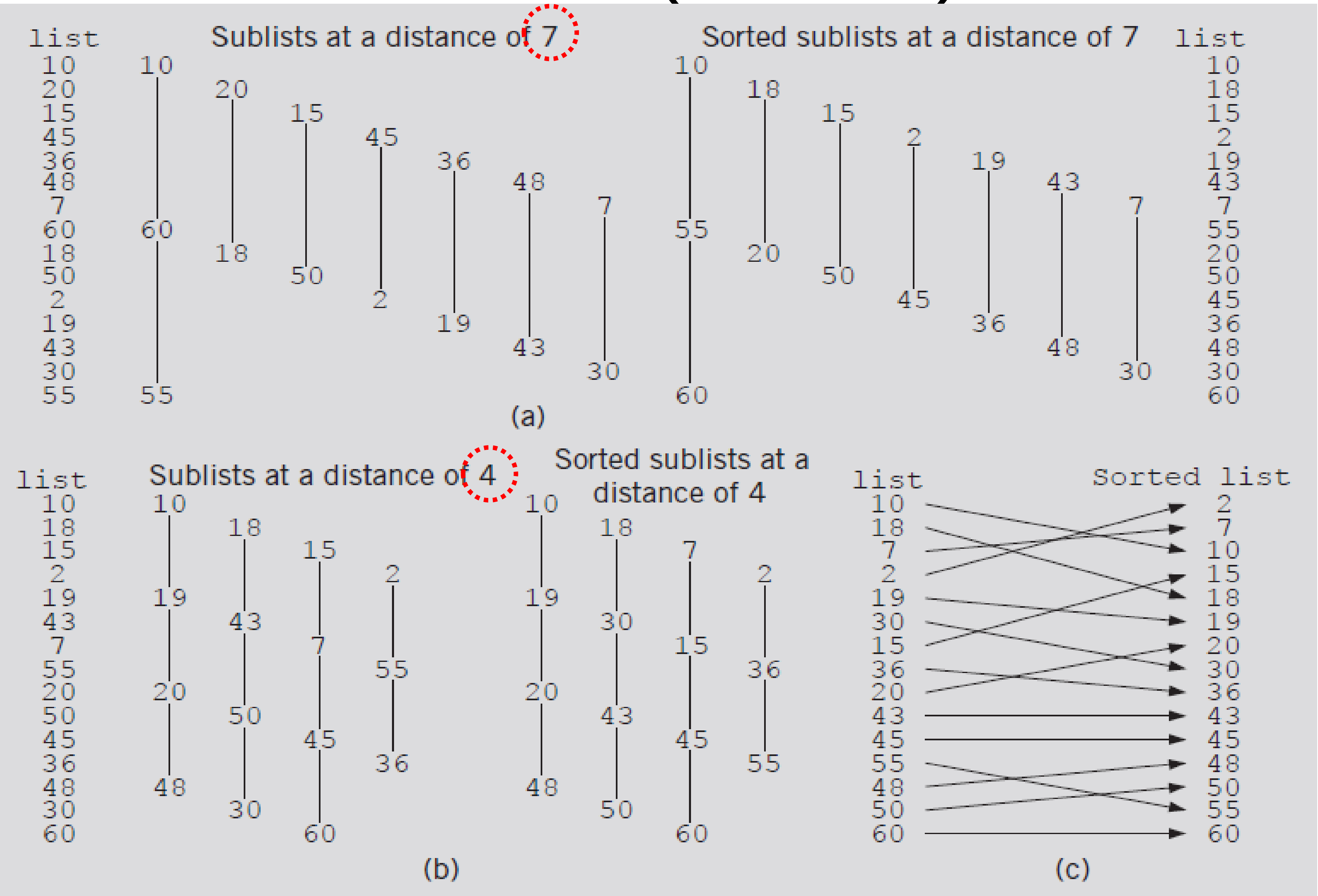



**FIGURE 10-19** Lists during Shellsort (7-4-1)

# Shellsort (cont'd.)

- Figure 10-19
  - Increment (gap) sequence 1, 4, 7
  - Sort elements at a distance of 7, 4, 1
- Desirable to use as few increments as possible
- D.E. Knuth recommended increment sequence
  - 1, 4, 13, 40, 121, 364, 1093, 3280. . . .
    - Ratio between successive increments: about one-third
    - $i^{th}$ increment = 3 • ($i$ – 1)*th* increment + 1
- Certain increment sequences must be avoided: bad performance
  - 1, 2, 4, 8, 16, 32, 64, 128, 256. . . .

# Shellsort (cont'd.)

- Function implementing Shellsort algorithm

Max value in the increment sequence determined by this loop. Find the largest inc that is less than length/3

```
template <class elemType>
void arrayListType<elemType>::shellSort()
{
    int inc;

    for (inc = 1; inc < (length - 1) / 9; inc = 3 * inc + 1);

    do
    {
        for (int begin = 0; begin < inc; begin++)
            intervalInsertionSort(begin, inc);

        inc = inc / 3;
    }
    while (inc > 0);
} //end shellSort
```

# Shellsort (cont'd.)

- Function `shellSort`
  - Uses function `intervalInsertionSort`
    - Modified version of insertion sort for array-based lists
- `intervalInsertionSort`
  - Sublist starts at variable `begin`
  - Increment between successive elements given by variable `inc` instead of one
- Analysis of Shellsort
  - Difficult to obtain, # of moves $O(n^{1.25}) \sim O(1.6n^{1.25})$
  - Better than insertion sort

# Lower Bound on Comparison-Based Sort Algorithms

- Comparison tree
  - Graph tracing comparison-based algorithm execution
    - Node: comparison drawn as a circle
    - Leaf: rectangle representing final node ordering
    - Root node: top node in the figure
    - Branch: straight line connecting two nodes
    - Path: sequence of branches between nodes

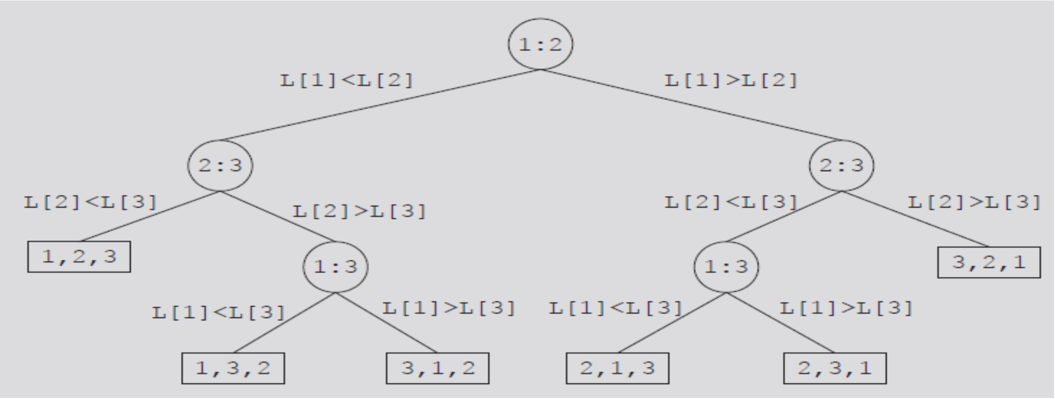



**FIGURE 10-20** Comparison tree for sorting three items

# Lower Bound on Comparison-Based Sort Algorithms (cont'd.)

- Unique permutation of the elements
  - Associated with each path from the root to a leaf
  - The tree has $n$! leaves
  - Depth of the tree: $\log_2 n$
  - Key comparisons at each level: O($n$)
- Theorem
  - Let $L$ be a list of $n$ distinct elements. Any sorting algorithm that sorts $L$ by comparison of the keys only, in its worst case, makes at least $O(n\log_2 n)$ key comparisons
    - Better than selection/insert sort O($n^2$)

# Quicksort: Array-Based Lists

- Uses the divide-and-conquer technique to sort a list
  - List partitioned into two sublists
    - Two sublists sorted and combined into one list
    - Combined list then sorted using quicksort (recursion)
- Trivial to combine sorted `lowerSublist` and `upperSublist`
- *All sorting work done in partitioning the list (i.e., selecting the pivot)*

```
if (the list size is greater than 1)
{
   a. Partition the list into two sublists, say lowerSublist and upperSublist.
   b. Quicksort lowerSublist.
   c. Quicksort upperSublist.
   d. Combine the sorted lowerSublist and sorted upperSublist.
}
```

# Quicksort: Array-Based Lists (cont'd.)

- Pivot divides list into two sublists
  - `lowerSublist`: elements smaller than pivot
  - `upperSublist`: elements greater than pivot
  - The pivot ends up in the correct place in the total order
  - Two sublists can be sorted independently
- Choosing the pivot
  - Ideally, `lowerSublist` and `upperSublist` nearly equal.
  - For simplicity, choose the *middle* element of the list

| | [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] |
|---|---|---|---|---|---|---|---|---|---|
| list | 45 | 82 | 25 | 94 | 50 | 60 | 78 | 32 | 92 |

**FIGURE 10-21** List before the partition

Two sublists can be sorted independently

| | [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] |
|---|---|---|---|---|---|---|---|---|---|
| list | 32 | 25 | 45 | 50 | 82 | 60 | 78 | 94 | 92 |
| | lowerSublist | | | | upperSublist | | | | |

**FIGURE 10-22** List after the partition

# Quicksort Partition Algorithm

- Partition algorithm

1. Determine the `pivot`, and swap the `pivot` with the first element of the list.

   Suppose that the index `smallIndex` points to the last element smaller than the `pivot`. The index `smallIndex` is initialized to the first element of the list.

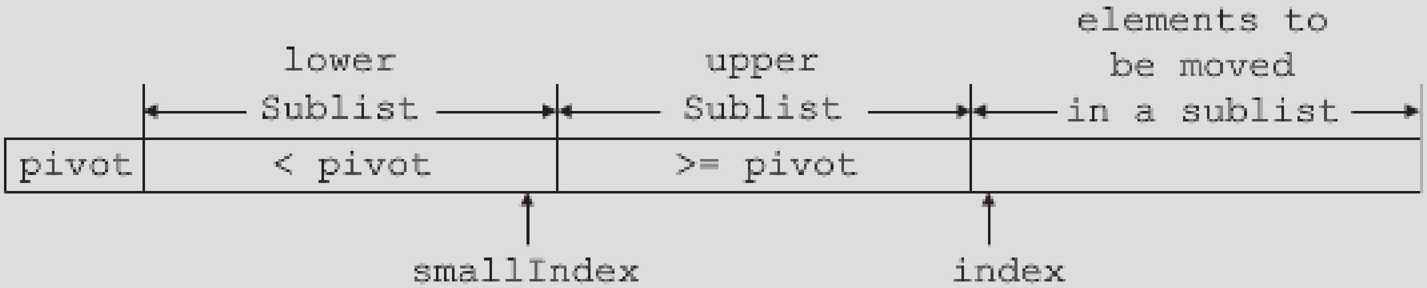
2. For the remaining elements in the list (starting at the second element) If the current element is smaller than the `pivot`

   a. Increment `smallIndex`.

   b. Swap the current element with the array element pointed to by `smallIndex`.

3. Swap the first element, that is, the `pivot`, with the array element pointed to by `smallIndex`.

# Quicksort Example




| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] | [13] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 32 | 55 | 87 | 13 | 78 | 96 | 52 | 48 | 22 | 11 | 58 | 66 | 88 | 45 |

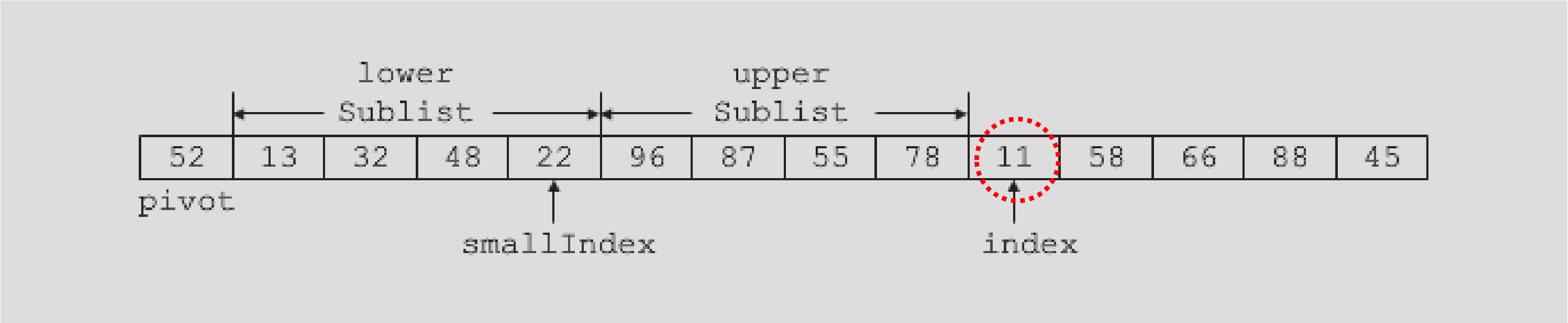
pivot

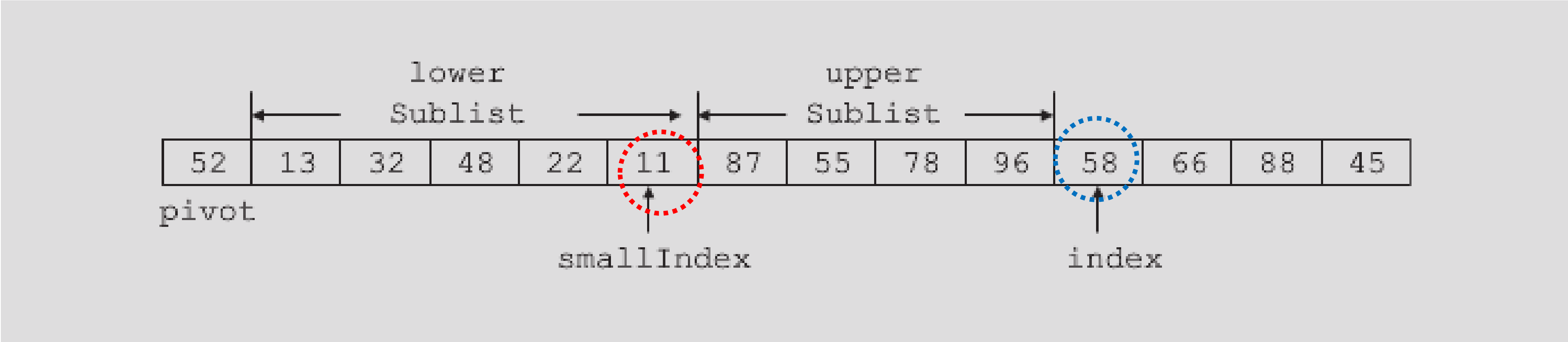
| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] | [13] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 52 | 55 | 87 | 13 | 78 | 96 | 32 | 48 | 22 | 11 | 58 | 66 | 88 | 45 |

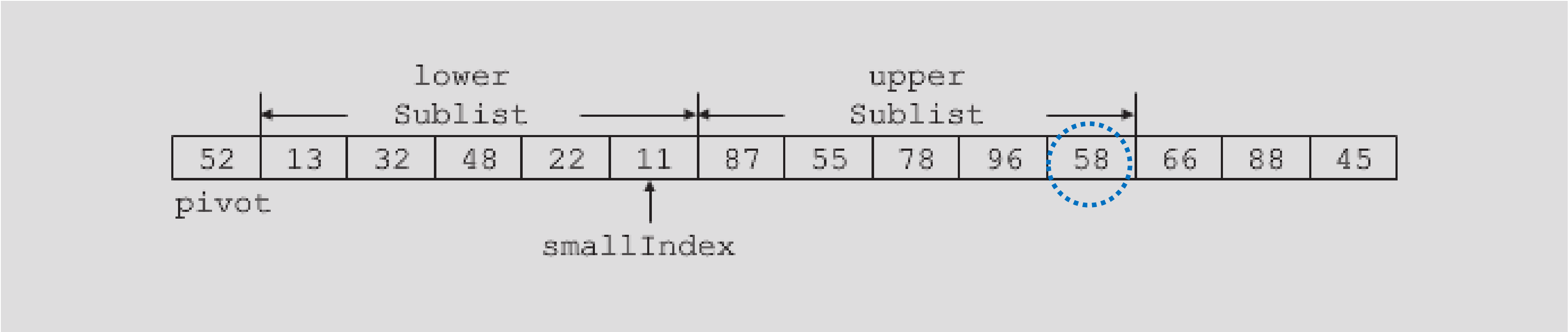
pivot

**FIGURE 10-25** List after moving `pivot` at the first array position

# Quicksort Example



# Quicksort Example

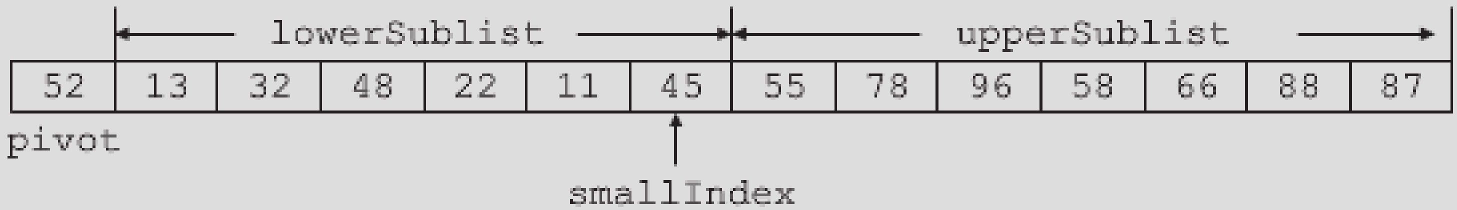



**FIGURE 10-30** List elements after arranging into `lowerSublist` and `upperSublist`

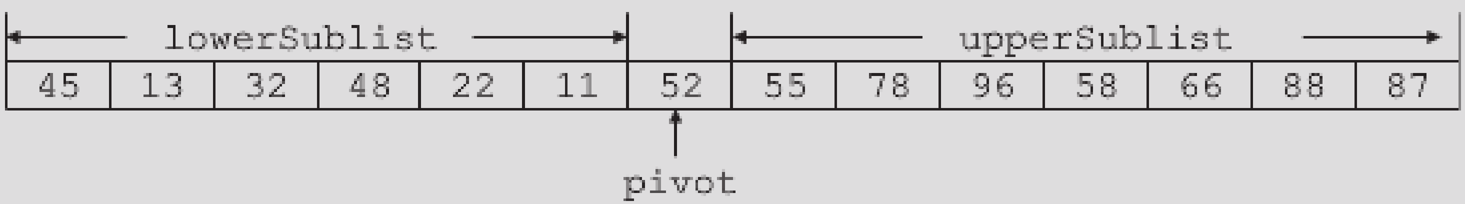



**FIGURE 10-31** List after swapping `52` with `45`

Step 3

# Quicksort: Array-Based Lists (cont'd.)

- Recap: Partition algorithm
  1. Determine pivot; swap pivot with first list element
     - Suppose index `smallIndex` points to last element smaller than pivot. `smallIndex` initialized to first list element
  2. For the remaining list elements (starting at second element): If current element smaller than pivot
     - Increment `smallIndex`
     - Swap current element with array element pointed to by `smallIndex`
  3. Swap first element (pivot) with array element pointed to by `smallIndex`

# Quicksort: Array-Based Lists (cont'd.)

- Function `partition`
  - Passes starting and ending list indices
  - Swaps certain elements of the list

```
template <class elemType>
int arrayListType<elemType>::partition(int first, int last)
{
    elemType pivot;

    int index, smallIndex;

    swap(first, (first + last) / 2);

    pivot = list[first];
    smallIndex = first;

    for (index = first + 1; index <= last; index++)
        if (list[index] < pivot)
        {
            smallIndex++;
            swap(smallIndex, index);
        }

    swap(first, smallIndex);

    return smallIndex;
}
```

```
template <class elemType>
void arrayListType<elemType>::swap(int first, int second)
{
    elemType temp;

    temp = list[first];
    list[first] = list[second];
    list[second] = temp;
}
```

# Quicksort: Array-Based Lists (cont'd.)

- Given starting and ending list indices
  - Function `recQuickSort` implements the recursive version of quicksort
- Function `quickSort` calls `recQuickSort`

```
template <class elemType>
void arrayListType<elemType>::recQuickSort(int first, int last)
{
    int pivotLocation;

    if (first < last)
    {
        pivotLocation = partition(first, last);
        recQuickSort(first, pivotLocation - 1);
        recQuickSort(pivotLocation + 1, last);
    }
}
template <class elemType>
void arrayListType<elemType>::quickSort()
{
    recQuickSort(0, length -1);
}
```

# Quicksort: In Class Exercise

- Assume pivot is always the middle element
  - Left to pivot: values less than pivot
  - Right to pivot: values greater than pivot
  - The element order in each sublist does not matter

[0]: 16 38 54 80 22 65 55 48 64 95 5 100 58 25 36

[1]: 16 38 22 5 25 36 **48** 54 80 65 55 64 95 100 58

[2]: 16 5 **22** 38 25 36 **48** 54 **55** 80 65 64 95 100 58

[3]: 5 **16 22 25** 38 36 **48 54 55** 58 **64** 80 65 95 100

[4]: **5 16 22 25** 36 **38 48 54 55 58 64 65** 80 95 100

[5]: **5 16 22 25 36 38 48 54 55 58 64 65** 80 **95** 100

[6]: **5 16 22 25 36 38 48 54 55 58 64 65 80 95 100**

# Quicksort: In Class Exercise 2

- Fix two elements of the input array. How many times can these two elements get compared with each other during the execution of Quicksort?
  A. 1
  B. 0 or 1
  C. 0, 1, or 2
  D. Any integer between 0 and n-1

# Quicksort: Linked List-Based

- Similar algorithm
- Programming Exercise 10.7

# Analysis: Quicksort

| | Number of comparisons | Number of swaps |
|---|---|---|
| Average case | $1.39n\log_2 n + O(n) = O(n\log_2 n)$ | $0.69n\log_2 n + O(n) = O(n\log_2 n)$ |
| Worst case | $(1/2)(n^2 - n) = O(n^2)$ | $(1/2)n^2 + (3/2)n - 2 = O(n^2)$ |

**TABLE 10-2** Analysis of quicksort for a list of length *n*

| | Best case | Avg case | Worst case |
|---|---|---|---|
| Quicksort | $O(n\log_2 n)$ | $O(n\log_2 n)$ | $O(n^2)$ |

When?

# Mergesort: Linked List-Based Lists

- Quicksort
  - Uses divide-and-conquer technique to sort a list
  - Average-case behavior: $O(n\log_2 n)$
  - Worst-case behavior: $O(n^2)$
- Mergesort behavior: always $O(n\log_2 n)$
  - Uses divide-and-conquer technique to sort a list
  - Partitions list into two sublists
    - Sorts sublists
    - Combines sorted sublists into one sorted list
- Difference between mergesort and quicksort
  - How list is partitioned

# Mergesort: Linked List-Based Lists (cont'd.)

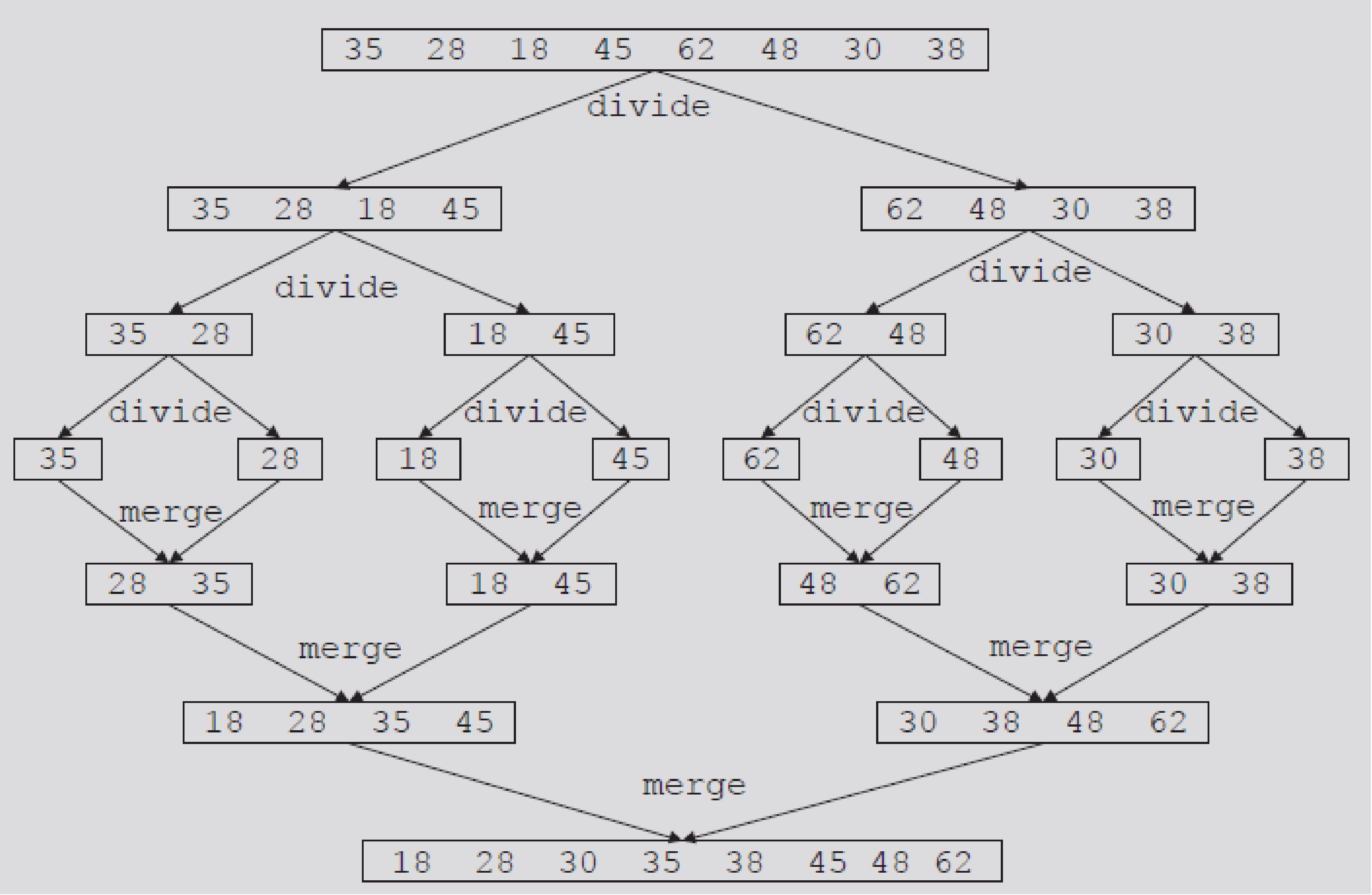



**FIGURE 10-32** Mergesort algorithm

# Mergesort: Linked List-Based Lists (cont'd.)

- Most sorting work done in merging sorted sublists
- General algorithm for mergesort

```
if the list is of a size greater than 1
{
   1. Divide the list into two sublists.
   2. Mergesort the first sublist.
   3. Mergesort the second sublist.
   4. Merge the first sublist and the second sublist.
}
```

# Divide

- To divide list into two sublists
  - Need to find middle node
  - Use two pointers: `middle` and `current`
    - Loop: whenever `middle` advances by one, `current` advances by two
    - `current` becomes NULL; `middle` points to last node of 1st sublist
  - Divide list into two sublists
    - Using the link of `middle`: assign pointer to node following `middle`
    - Set link of `middle` to `NULL`
- See function `divideList` on page 561

```
template <class Type>
void unorderedLinkedList<Type>::
               divideList(nodeType<Type>* first1,
                          nodeType<Type>* &first2)
{
    nodeType<Type>* middle;
    nodeType<Type>* current;

    if (first1 == NULL)   //list is empty
        first2 = NULL;
    else if (first1->link == NULL)  //list has only one node
        first2 = NULL;
    else
    {
        middle = first1;
        current = first1->link;

        if (current != NULL)    //list has more than two nodes
            current = current->link;
        while (current != NULL)
        {
            middle = middle->link;
            current = current->link;
            if (current != NULL)
                current = current->link;
        } //end while

        first2 = middle->link; //first2 points to the first
                               //node of the second sublist
        middle->link = NULL;   //set the link of the last node
                               //of the first sublist to NULL
    } //end else
} //end divideList
```

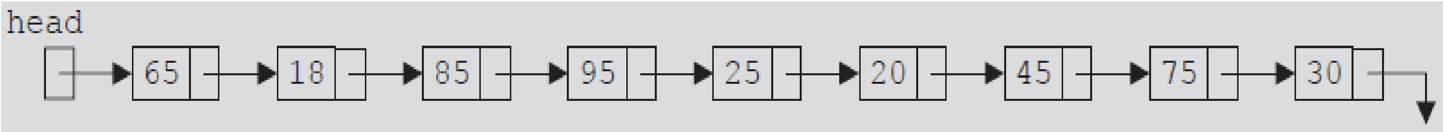



**FIGURE 10-33** Unsorted linked list

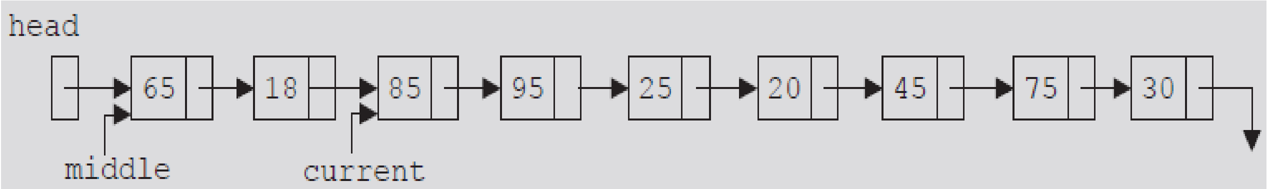



**FIGURE 10-34** `middle` and `current` before traversing the list

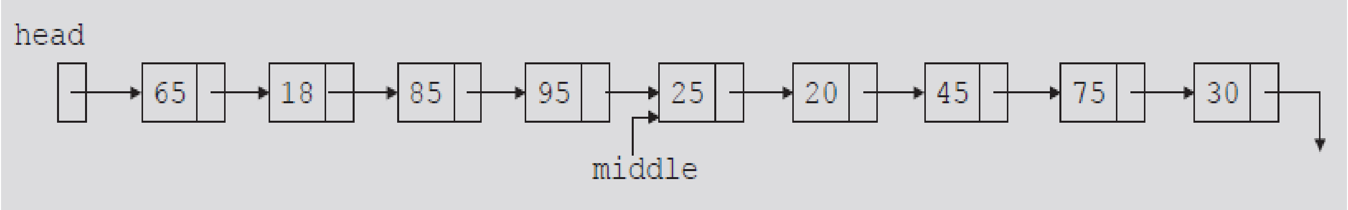



**FIGURE 10-35** `middle` after traversing the list

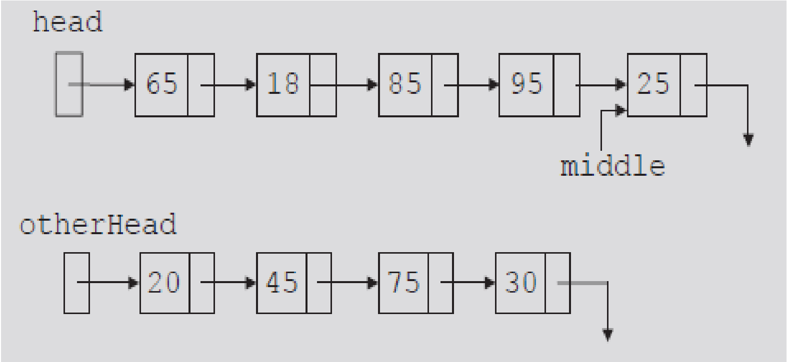



**FIGURE 10-36** List after dividing it into two lists

# Merge

- Once sublists sorted
  - Next step: merge the sorted sublists
- Merge process
  - Compare elements of the sublists
  - Adjust references of nodes with smaller info
- See code on page 564 and 565

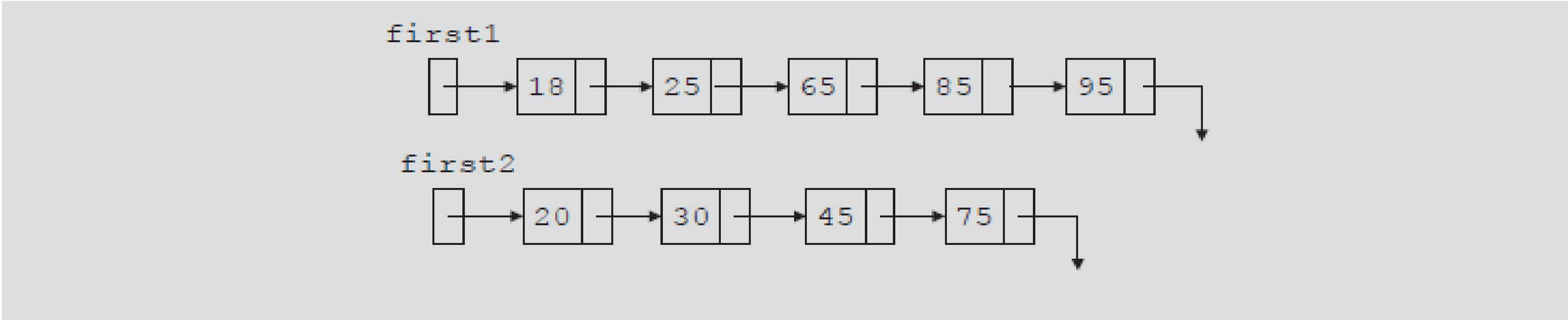



**FIGURE 10-37** Sublists before merging

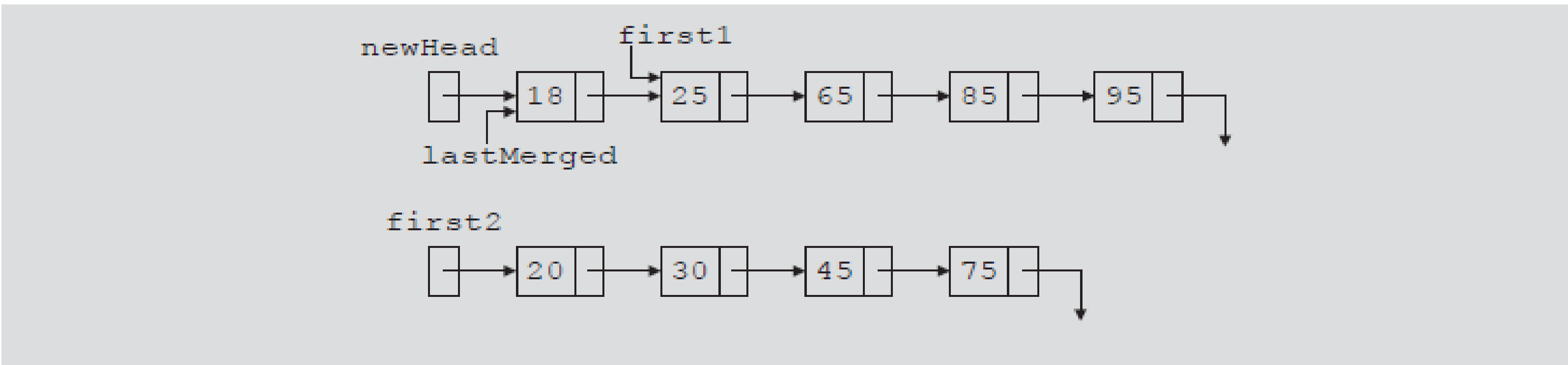



**FIGURE 10-38** Sublists after setting `newHead` and `lastMerged` and advancing `first1`

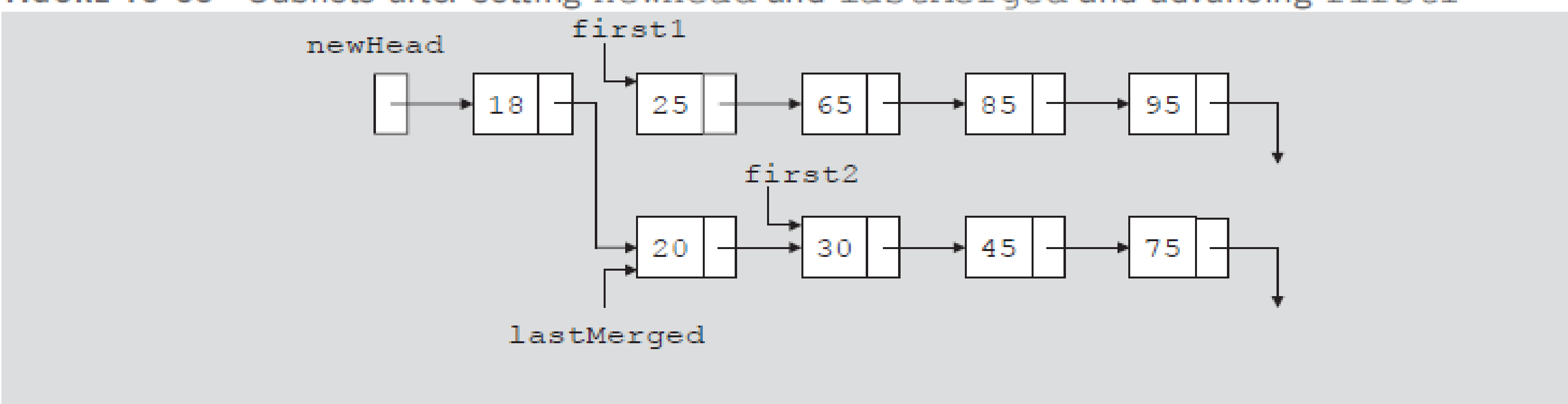



**FIGURE 10-39** Merged list after putting the node with `info` `20` at the end of the merged list

```
template <class Type>
nodeType<Type>* unorderedLinkedList<Type>::
                  mergeList(nodeType<Type>* first1,
                            nodeType<Type>* first2)
{
    nodeType<Type> *lastSmall; //pointer to the last node of
                               //the merged list
    nodeType<Type> *newHead;   //pointer to the merged list

    if (first1 == NULL)   //the first sublist is empty
        return first2;
    else if (first2 == NULL)   //the second sublist is empty
        return first1;
    else
    {
        if (first1->info < first2->info) //compare the first nodes
        {
            newHead = first1;
            first1 = first1->link;
            lastSmall = newHead;
        }
        else
        {
            newHead = first2;
            first2 = first2->link;
            lastSmall = newHead;
        }

        while (first1 != NULL && first2 != NULL)
        {
            if (first1->info < first2->info)
            {
                lastSmall->link = first1;
                lastSmall = lastSmall->link;
                first1 = first1->link;
            }
            else
            {
                lastSmall->link = first2;
                lastSmall = lastSmall->link;
                first2 = first2->link;
            }
        } //end while

        if (first1 == NULL) //first sublist is exhausted first
            lastSmall->link = first2;
        else                   //second sublist is exhausted first
            lastSmall->link = first1;

        return newHead;
    }
}//end mergeList
```

```
template <class Type>
void unorderedLinkedList<Type>::recMergeSort(nodeType<Type>* &head)
{
    nodeType<Type> *otherHead;

    if (head != NULL)  //if the list is not empty
        if (head->link != NULL) //if the list has more than one node
        {
            divideList(head, otherHead);
            recMergeSort(head);
            recMergeSort(otherHead);
            head = mergeList(head, otherHead);
        }
} //end recMergeSort
```

```
template<class Type>
void unorderedLinkedList<Type>::mergeSort()
{
    recMergeSort(first);

    if (first == NULL)
        last = NULL;
    else
    {
        last = first;
        while (last->link != NULL)
            last = last->link;
    }
} //end mergeSort
```

# Mergesort: Array-based

- Similar algorithm

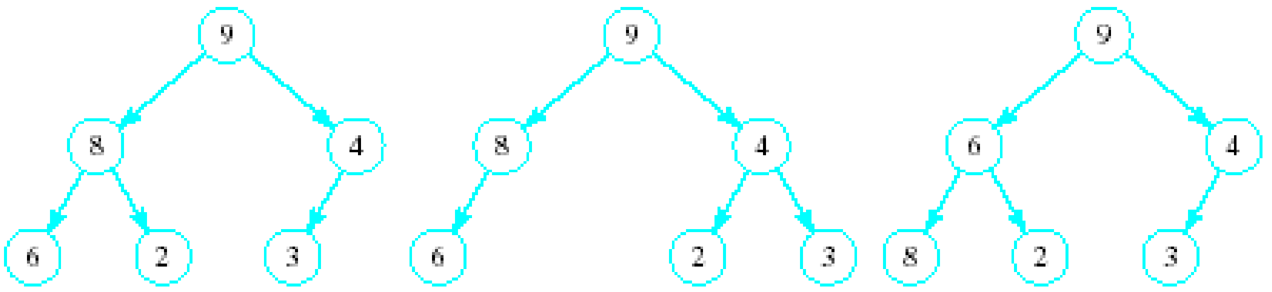
# Analysis: Mergesort

- Maximum number of comparisons made by mergesort: $O(n \log_2 n)$
- Let n = $2^m$
  - $m = \log_2 n$
  - Recursive level is $m$
  - Key comparison at each level $O(n)$
  - Maximum key comparisons: $O(nm)$, or $O(n \log_2 n)$
- $W(n)$ worst case key comparisons
  - $W(n) = O(n \log_2 n)$
- Let $A(n)$ avg case key comparisons
  - If $n$ is a power of 2, $A(n) = n \log_2 n - 1.25n = O(n \log_2 n)$

| | Best case | Avg case | Worst case |
|---|---|---|---|
| Mergesort | $O(n\log_2 n)$ | $O(n\log_2 n)$ | $O(n\log_2 n)$ |

# Heapsort: Array-Based Lists

- Overcomes quicksort worst case $O(n^2)$
- Heap: a binary tree with two properties
  - It is complete
    - Each level of tree completely filled except the bottom level
    - Nodes in bottom level are filled from left to right
  - It satisfies heap-order property
    - Data in in each node >= data in children nodes
- Which one is a heap?



# Heapsort: Array-Based Lists (cont'd)

- Heap: list in which each element contains a key
  - Key in the element at position *k* in the list
    - At least as large as the key in the element at position 2*k* + 1 (if it exists) and 2*k* + 2 (if it exists)
- C++ array index starts at zero
  - Element at position *k*: *k* + 1*th* element of the list

| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] | [11] | [12] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 85 | 70 | 80 | 50 | 40 | 75 | 30 | 20 | 10 | 35 | 15 | 62 | 58 |

**FIGURE 10-41** A heap

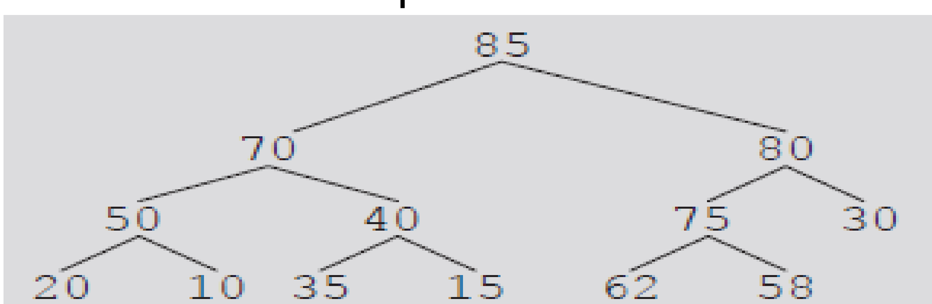





**FIGURE 10-42** Complete binary tree corresponding to the list in Figure 10-41

# Heapsort: Array-Based Lists (cont’d.)

- Heapsort
  - First step: convert list into a heap
    - Called `buildHeap`
  - After converting the array into a heap
    - Sorting phase begins

| | [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | [9] | [10] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| list | 15 | 60 | 72 | 70 | 56 | 32 | 62 | 92 | 45 | 30 | 65 |

**FIGURE 10-43** Array `list`

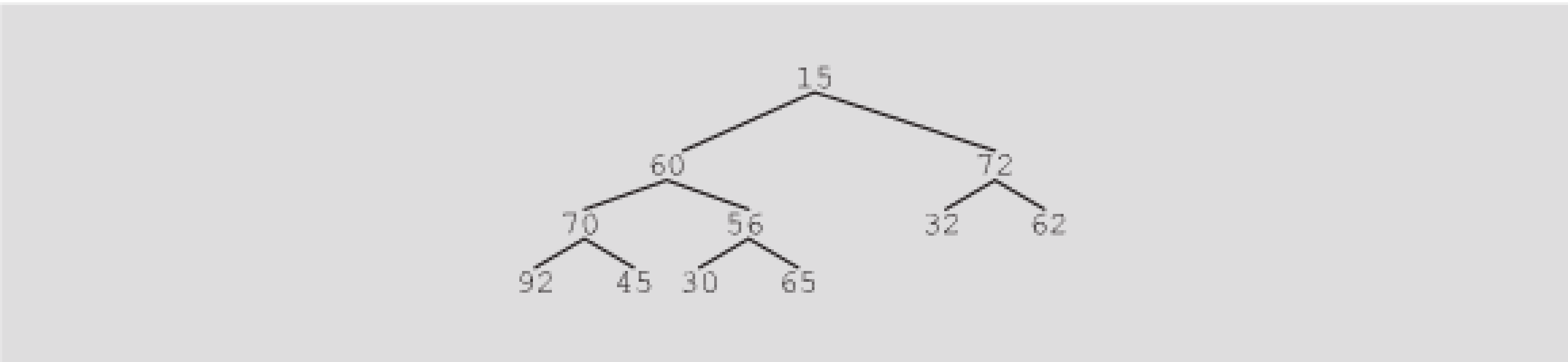
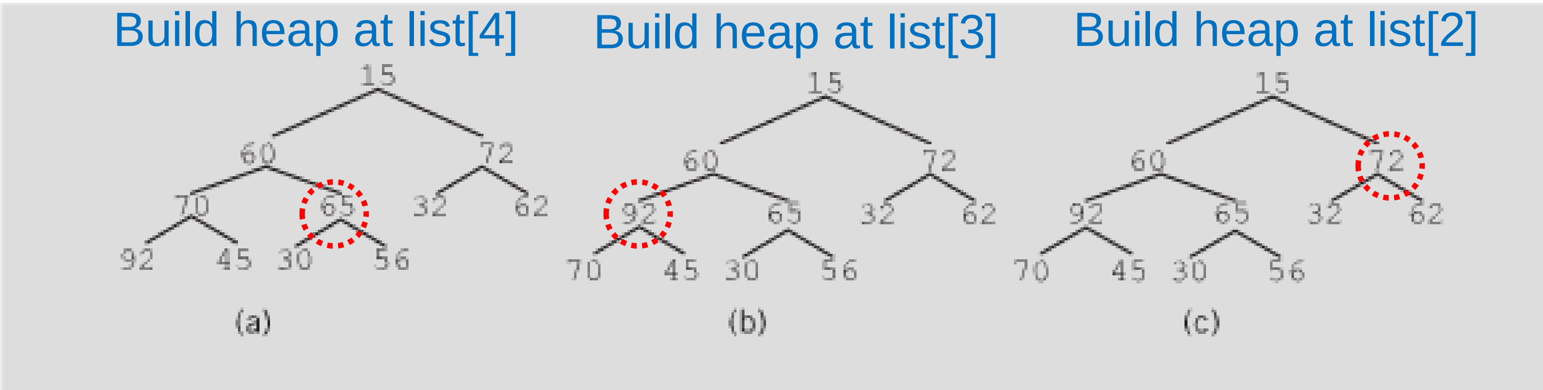



**FIGURE 10-44** Complete binary tree corresponding to the list in Figure 10-43

The position of the last element in the list which is not a leaf: (length/2 – 1).
Index = 4, val = 56

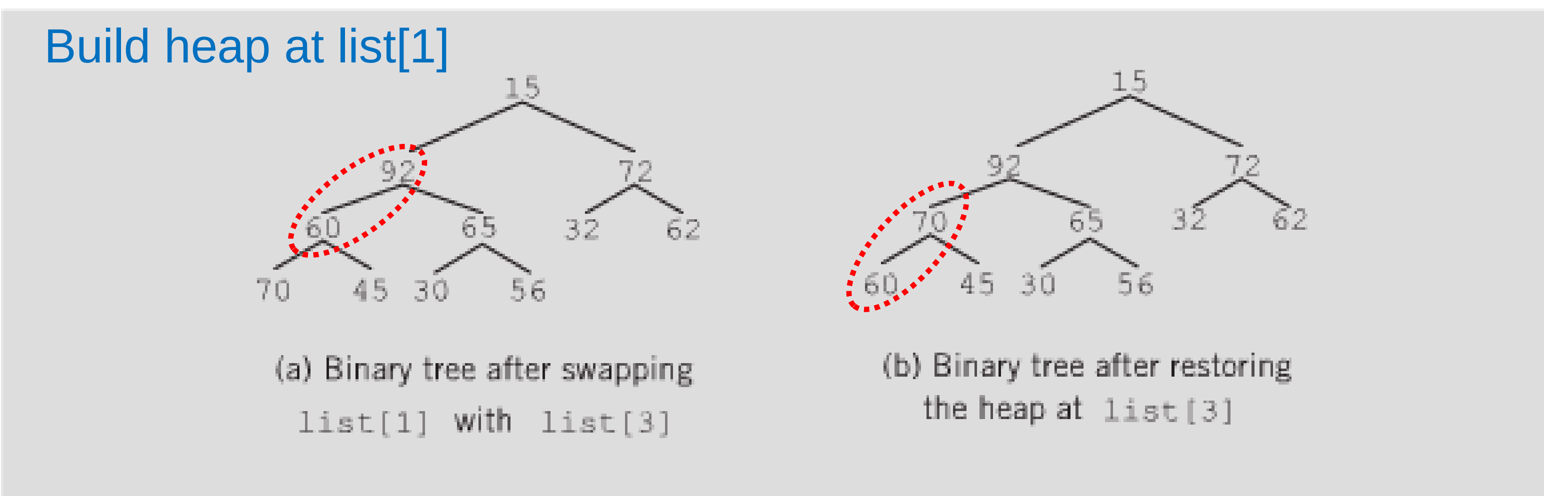
Heap building:
- Build heap for tree rooted at list[index]
- Build heap for tree rooted at list[index-1]
- …

This list has **11** elements, so the length of the list is **11**. To convert the array into a heap, we start at the list element **n/2 - 1 = 11/2 - 1 = 5 - 1 = 4**, which is the fifth element of the list.




(a) (b) (c)

**FIGURE 10-45** Binary tree while building heaps at `list[4]`, `list[3]`, and `list[2]`




(a) Binary tree after swapping `list[1]` with `list[3]`

(b) Binary tree after restoring the heap at `list[3]`

**FIGURE 10-46** Binary tree while building heap at `list[1]`

Build heap at list[0]

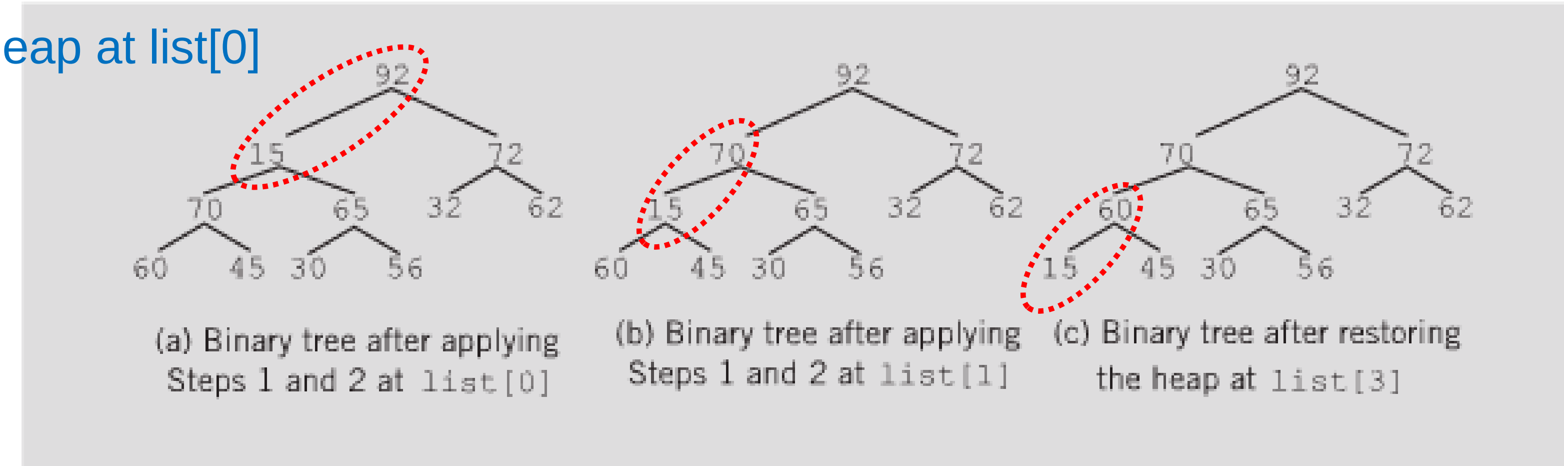



**FIGURE 10-47** Binary tree while building heap at list[0]

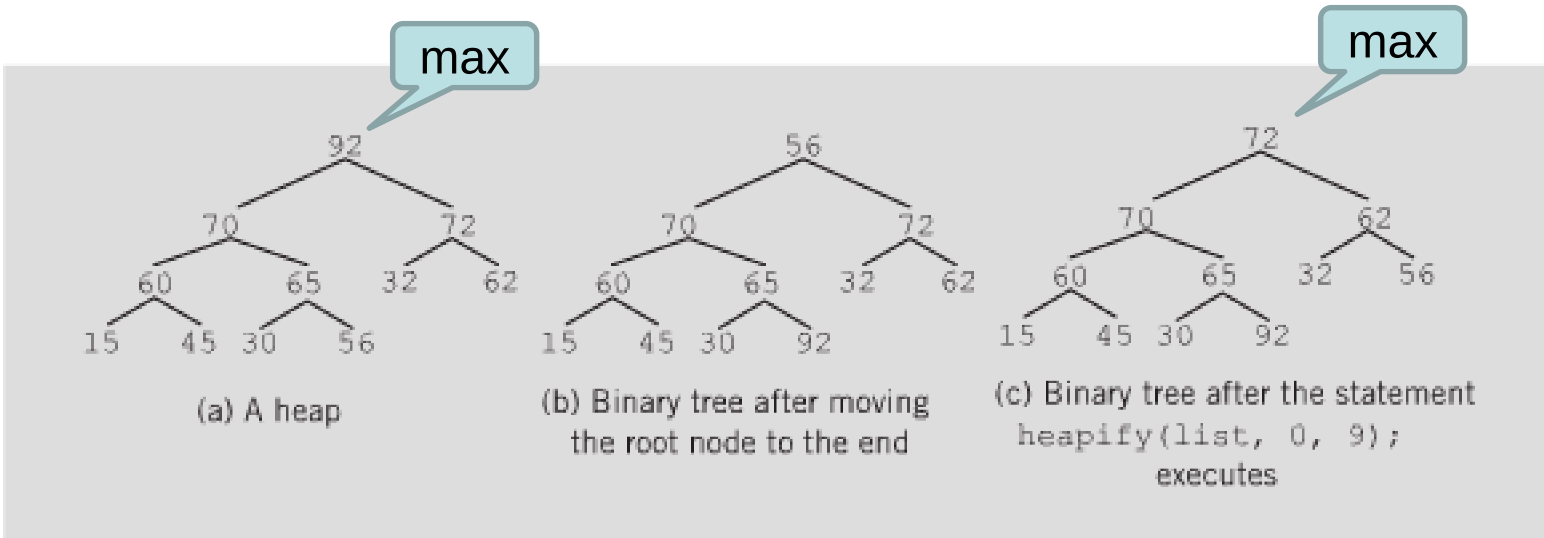



**FIGURE 10-48** Heapsort

# Build Heap

```
template <class elemType>
void arrayListType<elemType>::buildHeap()
{
   for (int index = length / 2 - 1; index >= 0; index--)
       heapify(index, length - 1);
}
```

- Function `heapify`
  - Restores the heap in a subtree
  - Implements the `buildHeap` function
    - Converts list into a heap

# Build Heap (cont'd)

`low`: index of the root node of the subtree
`high`: index of the last element in the list

```
int largeIndex = 2 * low + 1;    //index of the left child

while (largeIndex <= high)
{
   if ( largeIndex < high)
       if (list[largeIndex] < list[largeIndex + 1])
          largeIndex = largeIndex + 1;  //index of the larger child
   if (list[low] > list[largeIndex])    //the subtree is already in
                                        //a heap
       break;
   else
   {
       swap(list[low], list[largeIndex]);   //Line swap**
       low = largeIndex;  //go to the subtree to further
                          //restore the heap
       largeIndex = 2 * low + 1;
   } //end else
}//end while
```

```
template<class elemType>
void arrayListType<elemType>::heapify(int low, int high)
{
   int largeIndex;

   elemType temp = list[low]; //copy the root node of the subtree

   largeIndex = 2 * low + 1;  //index of the left child

   while (largeIndex <= high)
   {
       if (largeIndex < high)
          if (list[largeIndex] < list[largeIndex + 1])
             largeIndex = largeIndex + 1; //index of the largest
                                          //child

       if (temp > list[largeIndex]) //subtree is already in a heap
          break;
       else
       {
          list[low] = list[largeIndex]; //move the larger child
                                        //to the root
          low = largeIndex; //go to the subtree to restore the heap
          largeIndex = 2 * low + 1;
       }
   }//end while


   list[low] = temp; //insert temp into the tree, that is, list

} //end heapify
```

# Build Heap (cont'd.)

- The heapsort algorithm

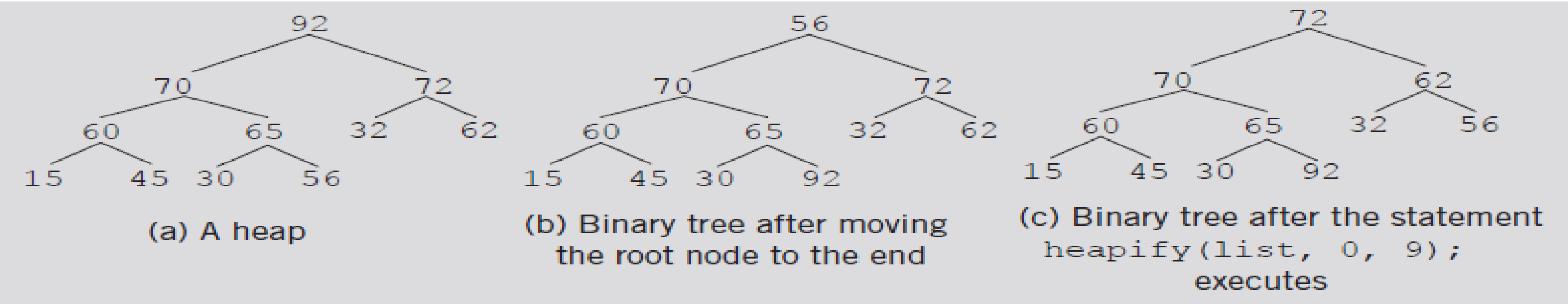



**FIGURE 10-48** Heapsort

```
template <class elemType>
void arrayListType<elemType>::heapSort()
{
    elemType temp;

    buildHeap();

    for (int lastOutOfOrder = length - 1; lastOutOfOrder >= 0;
                                          lastOutOfOrder--)
    {
        temp = list[lastOutOfOrder];
        list[lastOutOfOrder] = list[0];
        list[0] = temp;
        heapify(0, lastOutOfOrder - 1);
    }//end for
}//end heapSort
```

# Heapsort: exercise

- Given the following list of elements,

  13 31 56 45 34 22 107 99

  a. what is your heap after running function `buildHeap`?
  b. what does your tree look like after each iteration of `heapSort`?

# Analysis: Heapsort

- Given $L$ *a* list of $n$ elements where $n > 0$
- Worst case
  - Number of key comparisons to sort $L$
    - $2n\log_2 n + O(n)$
  - Number of item assignments to sort $L$
    - $n\log_2 n + O(n)$
- Average number of comparisons to sort $L$
  - $O(n\log_2 n)$
- Heapsort takes twice as long as quicksort
  - Avoids the slight possibility of poor performance

# Time-complexity of Various Sort Algorithms

| | Best case | Avg case | Worst case |
|---|---|---|---|
| Selection sort | O($n^2$) | O($n^2$) | O($n^2$) |
| Bubble sort | O($n$) | O($n^2$) | O($n^2$) |
| Insertion sort | O($n$) | O($n^2$) | O($n^2$) |
| Shellsort | depending on increment (gap) sequence | depending on increment (gap) sequence | depending on increment (gap) sequence |
| Quicksort | O($n\log_2 n$) | O($n\log_2 n$) | O($n^2$) |
| Mergesort | O($n\log_2 n$) | O($n\log_2 n$) | O($n\log_2 n$) |
| Heapsort | O($n\log_2 n$) | O($n\log_2 n$) | O($n\log_2 n$) |

# Priority Queues (Revisited)

- Customers or jobs with higher priorities
  - Pushed to front of the queue
- Assume priority of the queue elements is assigned using the relational operators
  - In a heap, largest list element is always the first element of the list
  - After removing largest list element
    - Function `heapify` restores the heap in the list
  - Implement priority queues as heaps
    - To ensure largest element of the priority queue is always the first element of the queue

# Priority Queues (Revisited) (cont'd.)

- Insert an element in the priority queue
  - Insert new element in first available list position
    - Ensures array holding the list is a complete binary tree
  - After inserting new element in the heap, the list might no longer be a heap
    - Restore the heap (might result in moving the new entry to the root node)

```
while (the parent of the new entry is smaller than the new entry)
    swap the parent with the new entry.
```

# Priority Queues (Revisited) (cont'd.)

- Remove an element from the priority queue
  - Assume priority queue implemented as a heap
    - Copy last element of the list into first array position
    - Reduce list length by one
    - Restore heap in the list
  - Other operations for priority queues
    - Can be implemented in the same way as implemented for queues

# Summary

- Sort: key comparison + data swap/movement
- Selection sort, insertion sort, Shellsort, quicksort, mergesort, and heapsort
  - How does each algorithm work
  - Time complexity
  - Comparisons among them
  - Pros and cons
  - Which algorithm should be used under what situation
  - Quicksort is mostly popular: heapsort takes twice as long as quicksort
  - Implementation
- Lower bound of time complexity

# Self Exercises

- Programming Exercises: 1, 4, 7, 8, 10, 11