On Brownian motion governed by telegraph process

Doctoral Dissertation Defense

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Overview

- On occupation time for on-off processes with multiple off-states.
- On estimation for Brownian motion governed by telegraph process with multiple off-states.
- Moving-resting process with measurement error in animal movement modeling.
- Density and distribution evaluation for convolution of independent gamma variables.
- Softwares: R package `smam`, and R package coga.

Motivation

Brownian motion governed by telegraph process with multiple off-states

- In ecology, Brownian motion and random walk are often used to model animal movement (Preisler et al., 2004; Horne et al., 2007; Lonergan et al., 2009; Willems and Hill, 2009; Farmer et al., 2010; Takekawa et al., 2010).
- Moving-resting process, introduced by Yan et al. (2014); Pozdnyakov et al. (2019), allows animal to have two different states, moving state and motionless state (resting).
- It is reasonable to assume that there are two very different explanations why an animal is not moving. One, as before, is resting, and the second one is a prolonged period, which is called handling time. The duration of these two motionless activities are different.
- We built the Brownian motion governed by telegraph process with multiple motionless states.
- Estimation of model parameter is challenging because the movement states are hidden in real data and the observed location sequence is not Markov.

For construction of on-off process with two off-states, we introduce

1. $\{M_k\}_{k\geq 1}$ are independent identically distributed (iid) random variables with cumulative distribution function (cdf) F_M and probability density function (pdf) f_M (these random variables will be used as the holding times of on-state (state 0) or moving state in animal movement problem);
2. $\{R_k\}_{k\geq 1}$ are iid random variables with cdf F_R and pdf f_R (holding times of 1st off-state (state 1) or resting state);
3. $\{H_k\}_{k\geq 1}$ are iid random variables with cdf F_H and pdf f_H (holding times of 2nd off-state (state 2) or handling state);
4. $\{\xi_k\}_{k\geq 1}$ are iid random variables with $\Pr(\xi_k = 1) = p_1 > 0$ and $\Pr(\xi_k = 0) = p_2 = 1 - p_1$.

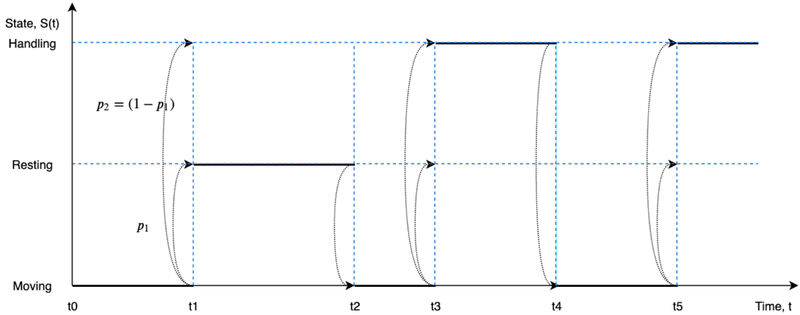
On-off process with multiple off-states

Process construction

Algorithmic construction of the on-off process with two off-states, $S(t)$.

1. Initialize with $S(0) = 0$ and $T_0 = 0$.
2. For cycles $i = 1, 2, \ldots$:
   2.1 Let $T_{2i-1} = M_i + T_{2i-2}$, and $S(t) = 0$ for all $t \in [T_{2i-2}, T_{2i-1})$.
   2.2 If $\xi_i = 1$ then $T_{2i} = T_{2i-1} + R_i$, and $S(t) = 1$ for all $t \in [T_{2i-1}, T_{2i})$; otherwise, $T_{2i} = T_{2i-1} + H_i$, and $S(t) = 2$ for all $t \in [T_{2i-1}, T_{2i})$.

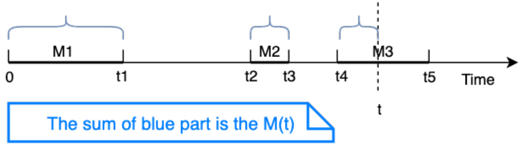
(State process $S(t)$ can also start from state 1 or state 2.)



The occupation times in state 0, state 1, and state 2 are, respectively,

$$M(t) = \int_0^t 1_{\{S(s)=0\}} \mathrm{d}s,$$
$$R(t) = \int_0^t 1_{\{S(s)=1\}} \mathrm{d}s,$$
$$H(t) = \int_0^t 1_{\{S(s)=2\}} \mathrm{d}s.$$

For example, the occupation times in state 0, $M(t)$, means the total time spending in state 0 by the time t.



The defective marginal densities of $M(t)$, $R(t)$ and $H(t)$ are denoted as

$$
\begin{aligned}
p_{Mj}(s,t) &= \Pr(M(t) \in \mathrm{d}s, S(t) = j | S(0) = 0)/\mathrm{d}s, \\
p_{Rj}(s,t) &= \Pr(R(t) \in \mathrm{d}s, S(t) = j | S(0) = 0)/\mathrm{d}s, \\
p_{Hj}(s,t) &= \Pr(H(t) \in \mathrm{d}s, S(t) = j | S(0) = 0)/\mathrm{d}s,
\end{aligned}
$$

where $t \geq 0$, $0 < s < t$, $j = 0, 1, 2$.
The densities are defective because of the existence of atoms. Therefore, $\sum_{j=0}^{2} \int_0^t p_{Rj}(s,t)\mathrm{d}s$ is less than 1. For example,

$$
\Pr(R(t) = 0, S(t) = 0) = \sum_{n=0}^{\infty} \left[F_M^{(n)} * F_H^{(n)}(t) - F_M^{(n+1)} * F_H^{(n)}(t) \right] p_2^n.
$$

The detailed formulas for marginal distribution can be found in Hu et al. (2019b).

The defective joint two-dimensional density of $R(t)$ and $H(t)$ for $u, v > 0$, $u + v < t$ is denoted as

$$p_{RHj}(u, v, t) = \frac{1}{\mathrm{d}u\mathrm{d}v} \Pr(R(t) \in \mathrm{d}u, H(t) \in \mathrm{d}v, S(t) = j | S(0) = 0),$$

where $j = 0, 1, 2$.
The joint density here is also defective that means the sum of following multiple integrals

$$\sum_{j=0}^{2} \iint_{u,v>0,\ u+v<t} p_{RHj}(u, v, t)\mathrm{d}u\mathrm{d}v$$

is less than 1. One example of atom could be

$$\Pr(R(t) = 0, H(t) = 0, S(t) = 0) = 1 - F_M(t).$$

The detailed formulas for joint distribution can be found in Hu et al. (2019b).

1. One popular choice of holding times is Lévy distribution (with location parameter 0) because it is stable distribution with positive support.
2. Another popular distribution of holding times is exponential distribution that yields Markov property.
3. Both marginal distribution and joint distribution of occupation times can be generalized to multiple off-states case.
4. The result of this section is arranged as Hu et al. (2019b).

The different phases of Brownian motion are modeled by a on-off process with exponentially distributed holding times. After that, the Brownian motion governed by telegraph process with multiple off-states model (moving-resting-handling model) can be built accordingly.

Consider a state process $S(t)$, $t \geq 0$, which is an on-off process with two off-states whose holding times $\{M_k\}_{k\geq 1}$, $\{R_k\}_{k\geq 1}$, $\{H_k\}_{k\geq 1}$ following exponential distributions with rate parameters λ_0, λ_1, λ_2 respectively.

Properties of state process $S(t)$:

1. State process $S(t)$, $t \geq 0$, is a continuous-time Markov Chain with the state space $\{0, 1, 2\}$ and the transition rate matrix
$$\mathbf{Q} = \begin{pmatrix} -\lambda_0 & \lambda_0 p_1 & \lambda_0 p_2 \\ \lambda_1 & -\lambda_1 & 0 \\ \lambda_2 & 0 & -\lambda_2 \end{pmatrix}$$
where $p_1, p_2, \lambda_0, \lambda_1, \lambda_2 > 0$ and $p_1 + p_2 = 1$.
2. In animal movement modeling, the mean duration in state 0 (moving), 1 (resting), and 2 (handling) are, respectively, $1/\lambda_0$, $1/\lambda_1$, and $1/\lambda_2$.
3. We assume that the initial distribution ν_0 of $S(0)$ is stationary, that is, $\nu_0 = \boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2) = \frac{1}{1/\lambda_0 + p_1/\lambda_1 + p_2/\lambda_2} \left(\frac{1}{\lambda_0}, \frac{p_1}{\lambda_1}, \frac{p_2}{\lambda_2} \right)$. Recall that $\boldsymbol{\pi}$ has to satisfy $0 = \boldsymbol{\pi}\mathbf{Q}$.

Let $X(t)$ be MRH process indexed by time $t > 0$. Conditioning on the state of the underlying renewal process, $S(t)$, the process $X(t)$ is defined by the stochastic differential equation

$$X(t) = \sigma \int_0^t 1_{\{S(s)=0\}} \mathrm{d}B(s),$$

where σ is a volatility parameter, and $B(t)$ is the standard Brownian motion independent of $S(t)$.

Let us note here that $X(t)$ is not Markov even though both $S(t)$ and $B(t)$ are Markov. Nonetheless, the joint process $\{X(t), S(t) : t \geq 0\}$ is a Markov process with stationary increments in $X(t)$. Moreover, the distribution of the increment $X(t+u) - X(t)$ depends only on $S(t)$.

The key random variable here is the occupation time spent in state 0 by the time t:

$$M(t) = \int_0^t 1_{\{S(s)=0\}} \mathrm{d}s.$$

According to the result of on-off process with multiple off-states, we have the formula for the following (defective) densities

$$p_{ij}(s,t) = P_i(M(t) \in \mathrm{d}s, S(t) = j)/\mathrm{d}s, \tag{1}$$

where $t \geq 0$, $0 < s < t$, $i, j = 0, 1, 2$. Check Pozdnyakov et al. (2020) for detail of these formulas.

Without loss of generality, we assume $X(0) = 0$. Given value of $M(t) = w$, the random variable $X(t)$ has the normal distribution with mean 0 and variance $\sigma^2 w$. Combining this observation with formulas from previous section, one can get the joint distribution of $(X(t), S(t))$.
For example, for $0 < w < t$,

$$P_1[X(t) \in \mathrm{d}x, S(t) = 1, M(t) \in \mathrm{d}w] = \phi(x; \sigma^2 w) p_{11}(w, t) \mathrm{d}x \mathrm{d}w,$$

where $\phi(\cdot; \sigma^2 w)$ is the density function of a normal variable with mean zero and variance $\sigma^2 w$. One can get the joint distribution of $(X(t), S(t))$ starting from $S(0) = 1$ by integrating w out.

Then, we can introduce joint density functions of $(S(t), X(t))$ as

$$
\begin{aligned}
h_{00}(x,t) &= P_0(X(t) \in dx, S(t) = 0)/dx \\
&= e^{-\lambda_0 t}\phi(x, \sigma^2 t) + \int_0^t \phi(x, \sigma^2 s) p_{00}(s,t)ds, \\
h_{0i}(x,t) &= P_0(X(t) \in dx, S(t) = i)/dx \\
&= \int_0^t \phi(x, \sigma^2 s) p_{0i}(s,t)ds, \quad \text{where } i = 1, 2, \\
h_{1i}(x,t) &= P_1(X(t) \in dx, S(t) = i)/dx \\
&= \int_0^t \phi(x, \sigma^2 s) p_{1i}(s,t)ds, \quad \text{if } x \neq 0 \text{ and } i = 0, 1, 2, \\
P_1(X(t) &= 0, S(t) = 1) = e^{-\lambda_1 t}, \\
h_{2i}(x,t) &= P_2(X(t) \in dx, S(t) = i)/dx \\
&= \int_0^t \phi(x, \sigma^2 s) p_{2i}(s,t)ds, \quad \text{if } x \neq 0 \text{ and } i = 0, 1, 2, \\
P_2(X(t) &= 0, S(t) = 2)f = e^{-\lambda_2 t}.
\end{aligned}
$$

1. The GPS location data of mountain lion contains three columns, time points, east-west coordinator, south-north coordinator.
2. Estimation of the MRH process parameters $\boldsymbol{\theta} = (\lambda_0, \lambda_1, \lambda_2, p_1, \sigma)$ is based on observations at discrete, possibly irregularly spaced time points.
3. Assume that we observe the MRH process $\big(X(0), X(t_1), \ldots, X(t_n)\big)$ at times $0 = t_0 < t_1 < \cdots < t_n$. Let $\mathbf{X} = \big(X_1, X_2, \ldots, X_n\big)$, where $X_i = X(t_i) - X(t_{i-1})$, $i = 1, \ldots, n$ are the observed increments of the MHR process.
4. Let $\mathbf{S} = \big(S(0), S(t_1), \ldots, S(t_n)\big)$ be the states of underlying telegraph process (that are not observable) and $\Delta_i = t_i - t_{i-1}$, $i = 1, \ldots, n$.

The location-state process $\{X(t), S(t)\}$ is Markov, so the likelihood function of $(\mathbf{X}, \mathbf{S})$ is available in closed-form. More specifically, it is given by

$$L(\mathbf{X}, \mathbf{S}, \boldsymbol{\theta}) = \nu(S(0)) \prod_{i=1}^{n} f\left(X_i, S(t_i) | S(t_{i-1}), \Delta_i, \boldsymbol{\theta}\right),$$

where

$$f\left(x, u | v, t, \boldsymbol{\theta}\right) = \begin{cases} 0 & v \neq u, \;\; x = 0, \\ 0 & v = u = 0, \;\; x = 0, \\ e^{-\lambda_1 t} & v = u = 1, \;\; x = 0, \\ e^{-\lambda_2 t} & v = u = 2, \;\; x = 0, \\ h_{ij}\left(x, t\right) & v = i, \;\; u = j, \;\; x \neq 0, \end{cases}$$

with $x \in \mathbf{R}$, $u, v = 0, 1, 2$, $t > 0$, and $\boldsymbol{\theta} = (\lambda_0, \lambda_1, \lambda_2, p_1, \sigma)$.

1. Since the state vector $\mathbf{S}$ is not observed, the likelihood of the increment vector $\mathbf{X}$ can be computed using

$$L(\mathbf{X}, \boldsymbol{\theta}) = \sum_{s_0, \ldots, s_n} L(\mathbf{X}, (s_0, \ldots, s_n), \boldsymbol{\theta}),$$

   where the summation is taken over all possible trajectories of $\mathbf{S} = (s_0, \ldots, s_n)$.
2. However, the likelihood given by above formula is not practical since the number of trajectories grows exponentially as sample size $n \to \infty$. This difficulty is addressed with help of the forward algorithm.

1. First, we need to introduce forward variables:

$$\alpha(\mathbf{X}_k, s_k, \boldsymbol{\theta}) = \sum_{s_0,\ldots,s_{k-1}} \nu(s_0) \prod_{i=1}^{k} f\left(X_i, s_i | s_{i-1}, \Delta_i, \boldsymbol{\theta}\right),$$

   where $\mathbf{X}_k = \left(X_1, X_2, \ldots, X_k\right)$, and $1 \leq k \leq n$.

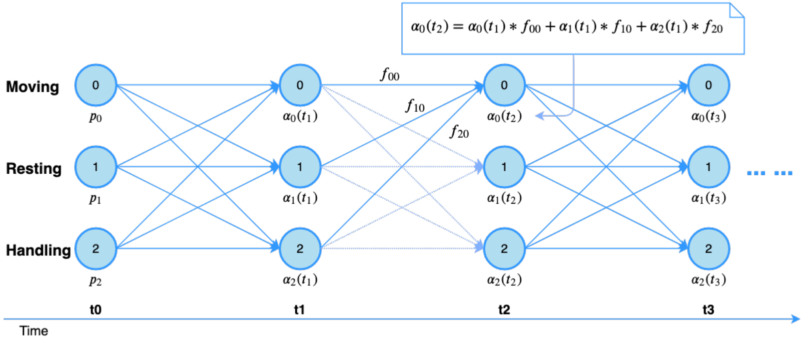
2. Then one can show that

$$\alpha(\mathbf{X}_{k+1}, s_{k+1}, \boldsymbol{\theta}) = \sum_{s_k} f\left(X_{k+1}, s_{k+1} | s_k, \Delta_{k+1}, \boldsymbol{\theta}\right) \alpha(\mathbf{X}_k, s_k, \boldsymbol{\theta}).$$

3. Since $L(\mathbf{X}, \boldsymbol{\theta}) = \sum_{s_n} \alpha(\mathbf{X}_n, s_n, \boldsymbol{\theta})$, we get an algorithm that finds $L(\mathbf{X}, \boldsymbol{\theta})$ with computational complexity that is linear with respect to sample size n.

Moving-resting-handling model

Forward algorithm



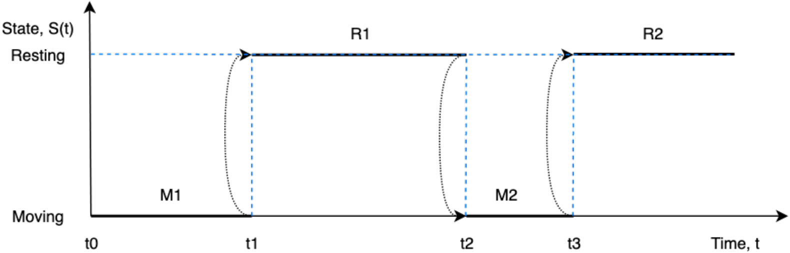
For large k, forward variables $\alpha(\mathbf{X}_k, s_k, \boldsymbol{\theta})$ might be numerically indistinguishable from zero, that is named as underflow problem. An improved version of forward algorithm can be developed by introducing normalized forward variables,

$$\bar{\alpha}(\mathbf{X}_k, s_k, \boldsymbol{\theta}) = \frac{\alpha(\mathbf{X}_k, s_k, \boldsymbol{\theta})}{L(\mathbf{X}_k, \boldsymbol{\theta})},$$

where $L(\mathbf{X}_k, \boldsymbol{\theta}) = \sum_{s_k} \alpha(\mathbf{X}_k, s_k, \boldsymbol{\theta})$, the likelihood of vector $\mathbf{X}_k$. Detailed demonstration of normalized forward algorithm is given in Pozdnyakov et al. (2020).

Moving-resting (MR) process was introduced by Yan et al. (2014). Comparing to MRH process, its underlying state process only have one off-state.

1. Let $\{M_i\}_{i\geq 1}$ and $\{R_i\}_{i\geq 1}$ be two series of independent and identically exponential random variables with mean $1/\lambda_1$ and $1/\lambda_0$ respectively.
2. Then, we can build a telegraph process with probability $p_1 \in (0, 1)$ starts with a 1-cycle, $(M_1, R_1, M_2, M_2, \dots)$, and probability $p_0 = 1 - p_1$ starts with a 0-cycle, $(R_1, M_1, R_2, M_2, \dots)$.
3. State variable, $S(t)$, $t \geq 0$, is defined as the state (moving, resting) at time t, with state space, $\{0 - \textit{resting}, 1 - \textit{moving}\}$.



4. Now, we define moving-resting process, $X(t)$, $t \geq 0$, as stochastic differential equation, $\mathrm{d}X(t) = \sigma S(t)\mathrm{d}B(t)$, where $B(t)$ is the standard Brownian motion and σ is the volatility parameter.
5. Note that, $\{X(t), S(t)\}$ is Markov, but $\{X(t)\}$ itself is not Markov.
6. In case of real world data sets, however, we may never observe exact values of $X(t)$, but $X(t)$ with added measurement errors. Ignoring the added noise does not work even when the noise is relatively small.
7. Addressing this problem via rounding can help. But it is not a trivial task to come up with an appropriate choice for rounding accuracy.

Influence of measurement error on moving-resting process parameter estimation. The true parameter of moving-resting process is $\lambda_1 = 1/hr$, $\lambda_0 = 0.5/hr$, $\sigma = 1km/hr^{1/2}$. The measurement error is set as Gaussian noise with standard deviation 0.05 and 0.01. The length of the time intervals between consecutive observations is 5. The number of observations is 200. The number of replication is 100. The mean and empirical standard deviation of estimators under different setups are recorded.

Gaussian noise	Rounding	$\hat{\lambda}_1$		$\hat{\lambda}_0$		$\hat{\sigma}$	
s.d. (km)	(km)	mean	s.d.	mean	s.d.	mean	s.d
—	—	1.58	2.29	0.50	0.08	1.08	0.38
0.05	—	21.16	12.21	0.89	0.08	2.73	0.75
	0.01	21.58	9.92	0.89	0.08	2.78	0.68
	0.05	15.08	10.00	0.85	0.08	2.34	0.76
	0.10	9.72	7.70	0.75	0.10	1.98	0.74
0.01	—	22.00	10.82	0.83	0.07	2.89	0.75
	0.01	17.61	9.29	0.80	0.07	2.62	0.74
	0.05	4.95	4.60	0.61	0.09	1.57	0.65
	0.10	2.24	3.15	0.54	0.09	1.18	0.47

Suppose the observations are recorded at times $t_0 = 0, t_1, \dots, t_n$. Let $\{\epsilon_k\}_{k=0,\dots,n}$ be independent and identically normally distributed random variables with mean 0 and variance σ_ϵ^2. An MR process with measurement error (MRME) $Z(t)$ at each time point is the superimposition of a measurement error and the exact location. That is, the observed location at t_k, $Z(t_k)$, is

$$Z(t_k) = X(t_k) + \epsilon_k.$$

Moving-resting model with measurement error

Real data v.s. MR v.s. MRME

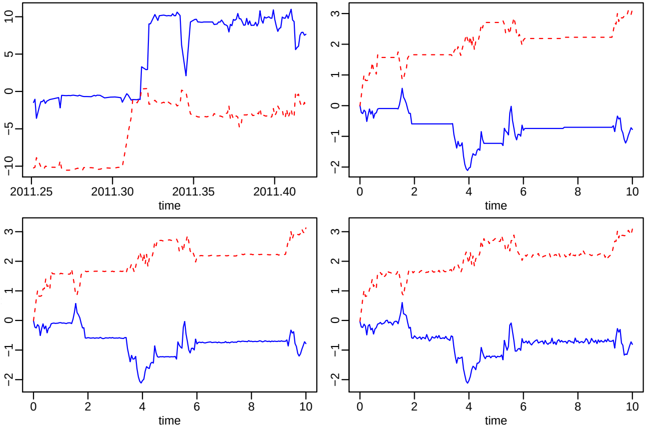


Figure: *Up left*: Actual coordinates of a female mountain lion. The solid blue line is UTM easting (*km*) and the dashed red line is UTM northing (*km*). *Up right*: Realization from a two-dimensional MR process. *Bottom left*: Same realization after adding Gaussian noise with s.d. 0.01. *Bottom right*: Same realization after adding Gaussian noise with s.d. 0.05.

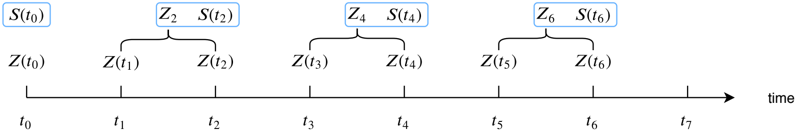
Moving-resting model with measurement error

Markov property in our process

1. Both $\{Z(t_k)\}_{k=0,\ldots,n}$ and location-state process $\{Z(t_k), S(t_k)\}_{k=0,\ldots,n}$ are not Markov.
2. The discrete time process, $\{Z(t_{2k}) - Z(t_{2k-1}), S(t_{2k})\}_{k=1,\ldots,[n/2]}$, that is

$$\begin{pmatrix} S(t_{2k}) \\ Z(t_{2k}) - Z(t_{2k-1}) \end{pmatrix} \to \begin{pmatrix} S(t_{2k+2}) \\ Z(t_{2k+2}) - Z(t_{2k+1}) \end{pmatrix},$$

   is Markov.
3. More specifically, the distribution of $(Z(t_{2k+2}) - Z(t_{2k+1}), S(t_{2k+2}))$ depends only on state $S(t_{2k})$.



Distribution of increment

1. Let us calculate the distribution of $Z(t) - Z(0)$ (the increment of $\{Z(t)\}_{t\geq 0}$ from time 0 to time t).
2. Consider $\Delta Z(t) = Z(t) - Z(0) = X(t) - X(0) + \xi$, where $\xi \sim N(0, 2\sigma_\epsilon^2)$ independent of process $(X(t), S(t))$.
3. Denote
$$g_{ij}(z,t) = \mathbf{P}_i(\Delta Z(t) \in dz, S(t) = j)/\mathrm{d}z,$$
where $i, j = \{1, 0\}$.

Transition probability of state process

Next, let us denote
$$\tau_{ij}(t) = \mathbf{P}_i(S(t) = j),$$
where $i, j = \{1, 0\}$. Check Hu et al. (2020b) for the detail of $g_{ij}(z,t)$ and $\tau_{ij}(t)$.

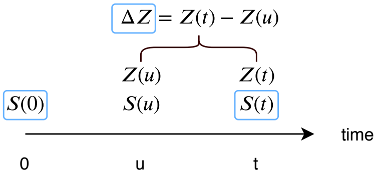
Moving-resting model with measurement error

Transition density

1. Now, we are ready to present the transition density from $S(0)$ to $(Z(t) - Z(u), S(t))$, where $0 < u < t$.
2. Using Markov property of the location-state process $(X(t), S(t))$ and independence of the added noise, one can get that

$$f(Z(t) - Z(u), S(t) = j | S(0) = i) = \sum_{k=0}^{1} \tau_{ik}(u) g_{kj}(Z(t) - Z(u), t - u),$$

where $i, j = \{0, 1\}$.



- The likelihood of increment-state observations at even time points

$$\mathbf{Z}_{even} = \left(Z_2, \dots, Z_{2[n/2]}\right),$$
$$\mathbf{S}_{even} = \left(S(t_0), S(t_2), \dots, S(t_{2[n/2]})\right),$$

where $Z_k = Z(t_k) - Z(t_{k-1})$ is given by

$$L\left(\mathbf{Z}_{even}, \mathbf{S}_{even}; \boldsymbol{\theta}\right) = \nu^*(S(t_0)) \prod_{k=1}^{[n/2]} f\left(Z_{2k}, S(t_{2k}) | S(t_{2k-2})\right),$$

where $[\cdot]$ is the integer function, $\boldsymbol{\theta} = (\lambda_1, \lambda_0, \sigma, \sigma_\epsilon)$ and $\nu^*(S(t_0))$ is the initial distribution that is assumed to be stationary.

- In practice, $\mathbf{S}_{even}$ are hidden states. The likelihood of the increment process $\mathbf{Z}_{even}$ is obtained by taking sum over all possible state trajectories:

$$L(\mathbf{Z}_{even}; \boldsymbol{\theta}) = \sum_{S(t_0), S(t_2), \ldots, S(t_{2[n/2]}) \in \{0,1\}} L\left(\mathbf{Z}_{even}, \mathbf{S}_{even}; \boldsymbol{\theta}\right).$$

- The cardinality of the set of the state trajectories is $2^{[n/2]+1}$. Fortunately, it can be efficiently evaluated with the forward algorithm. Check Hu et al. (2020b) for detail.
- Similarly, we have the likelihood of observed increments at odd time points $\mathbf{Z}_{odd} = \left(Z_1, Z_3, \ldots, Z_{2[(n+1)/2]-1}\right)$. Adding two log-likelihoods together we get the following composite log-likelihood:

$$cl\left((Z(t_0), \ldots, Z(t_n)); \boldsymbol{\theta}\right) = \log\left(L(\mathbf{Z}_{even}; \boldsymbol{\theta})\right) + \log\left(L(\mathbf{Z}_{odd}; \boldsymbol{\theta})\right). \qquad (2)$$

- The maximum composite likelihood estimator (MCLE) of $\boldsymbol{\theta}$ is the maximizer $\hat{\boldsymbol{\theta}}$ of (2).

Moving-resting model with measurement error

Real data analysis

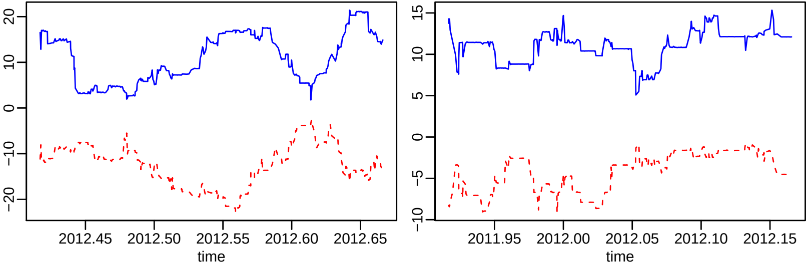


Figure: Actual coordinates of a female mountain lion in the Gros Ventre mountain range, Wyoming. The x-axis is time in years. The y-axis is departure from the starting point. The solid blue line is UTM easting (*km*) and the dashed red line is UTM northing (*km*). *Left:* Summer period data, from June 1, 2012 to August 31, 2012. *Right:* Winter period data, from December 1, 2011 to February 29, 2012.

Table: Analysis results for mountain lion movement data. Point estimates (EST) from two-piece method are reported. Standard error of point estimates are evaluated by parametric bootstrap (SE).

Parameter	Summer		Winter	
	EST	SE	EST	SE
λ_1	2.841	0.459	6.225	0.825
λ_0	0.179	0.014	0.118	0.010
σ	1.335	0.106	1.506	0.095
$\sigma_\epsilon(\times 10^{-2})$	1.854	0.087	0.908	0.036

Moving-resting model with measurement error

Remarks

1. Another approach of estimation is to use the one-step transition density with the dependence between two consecutive increments discarded.
2. Parametric bootstrap can be use to estimate standard error.
3. Alternatively, we can estimate the variance by inverting the observed Godambe information matrix (Godambe, 1960).

- The numerical evaluation of above models require the computation of density and distribution function of convolution of gamma distributions.
- With the purpose of improving the speed of estimation approach, it is worthwhile to dive into the algorithm related to convolution of gamma distributions.
- Convolution of gamma distributions is also the basic working block in many stochastic process, such as renewal process with mixture of exponential distributions.

1. In paper Hu et al. (2020a), we compared several numerical evaluations of the density and distribution of convolution of independent gamma variables in their accuracy and speed, including two exact methods (Mathai, 1982; Moschopoulos, 1985), and one approximation method (Barnabani, 2017).
   1.1 During the convolution of two gamma variables case, Mathai (1982)'s method is about $300 - 2500$ times faster than Moschopoulos (1985)'s method.
   1.2 During the convolution of more than two gamma variables case, Moschopoulos (1985)'s method performs $20 - 50$ times faster than Mathai (1982)'s method.
   1.3 The approximation from Barnabani (2017) was also included in the comparison, which is up to 30 times faster in density evaluation and 14 times faster in distribution evaluation than Moschopoulos' method, with a good accuracy.

2. A special formula for $H(x, \alpha_1, \beta_1, \alpha_2, \beta_2)$ was also derived as

$$H(x, \alpha_1, \beta_1, \alpha_2, \beta_2) = F(x; \alpha_1, \beta_1, \alpha_2, \beta_2) - F(x; \alpha_1 + 1, \beta_1, \alpha_2, \beta_2)$$
$$= \frac{x^{\alpha_1+\alpha_2} e^{-x/\beta_1}}{\beta_1^{\alpha_1} \beta_2^{\alpha_2} \Gamma(\alpha_1 + \alpha_2 + 1)} {}_1F_1(\alpha_2; \alpha_1 + \alpha_2 + 1; y(1/\beta_1 - 1/\beta_2)),$$

for any positive integers α_1, α_2 and $y, \beta_1, \beta_2 > 0$, where ${}_1F_1$ is Kummer's confluent hypergeometric function (Abramowitz and Stegun, 1972, Formula 13.1.2). For evaluation of function H, our special formula is about $60 - 200$ times faster, comparing to the naive approach by calculating distribution functions twice with Mathai (1982) method.

Software

- R package `smam` (Hu et al., 2020c): Statistical model for animal movement. (CRAN downloads: 19901)
- R package `coga` (Hu et al., 2019a): Convolution of Gamma distributions. (CRAN downloads: 15337)
- Both of these two packages are programmed with C++ and R. `smam` also supports parallel computation by C++.

Source code can be accessed in Chaoran Hu's GitHub, https://github.com/ChaoranHu.

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