Applied Stochastic Analyis
Homework 03

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Problem 1

From ASA, problem 6.7

A standard *Brownian bridge* is a Gaussian process with continuous paths, mean function $m(t) = 0$, and covariance function $K(s,t) = s \wedge t - st$ for $s,t \in [0,1]$. Prove that both $X_t = W_t - tW_1$ and $Y_t = (1-t)\,W_{t/(1-t)}$ for $0 \le t < 1$, $Y_1 = 0$ give a Brownian bridge.

Solution:

First Brownian bridge. $X_t = W_t - t \cdot W_1$

1. $X_0 = X_1 = 0$.

$$\begin{aligned} X_0 =& W_0 - 0 \cdot W_1 = 0, \quad W_0 = 0 \\ X_1 =& W_1 - 1 \cdot W_1 = 0 \end{aligned}$$

2. $\mathrm{E}\left[X_t\right] = 0$.
   Using $\mathrm{E}[W_t] = 0$ and $W_1 = 0$

$$\begin{aligned} \mathrm{E}[X_t] =& \mathrm{E}\left[W_t - t \cdot W_t\right] \\ =& \mathrm{E}[W_t] - t \cdot W_1 \\ =& 0 \end{aligned} \tag{1}$$

3. $\mathrm{Cov}\left(X_s, X_t\right) = s \wedge t - st$.
   Using the previous numeral $\mathrm{E}[X_\tau] = 0$ and $W_1 = 1$

$$\begin{aligned} \mathrm{Cov}\left(X_s, X_t\right) =& \mathrm{E}\left[X_s \cdot X_t\right] - \mathrm{E}\left[X_s\right] \cdot \mathrm{E}\left[X_t\right] \\ =& \mathrm{E}\left[X_s \cdot X_t\right] \\ =& \mathrm{E}\left[\left(W_s - s \cdot W_1\right) \cdot \left(W_t - t \cdot W_1\right)\right] \\ =& \mathrm{E}\left[W_s W_t\right] - t\mathrm{E}[W_1 W_s] - s\mathrm{E}[W_1 W_t] + st\mathrm{E}\left[W_1^2\right] \\ & \text{using } \mathrm{Cov}[W_s W_t] = \mathrm{E}[W_s W_t] = \min\{s,t\} \\ =& \min\{s,t\} - ts - st + st \\ =& s \wedge t - st \end{aligned} \tag{2}$$

4. As W_t is continuous everywhere then X_t is continuous.

5. X_t is a linear combination of W_t that by definition is Gaussian, as a consequence X_t is Gaussian.

Second Brownian bridge. $Y_t = (1-t)W_{t/(1-t)}$ and $Y_1 = 0$.

1. $Y_0 = Y_1 = 0$

$$\begin{aligned} Y_0 =& W_0 = 0 \\ & \text{and by definition} \\ Y_1 =& 0 \end{aligned} \tag{3}$$

2. $\mathrm{E}[Y_t] = 0$.
   By definition $\mathrm{E}[W_t] = 0$.

$$\begin{aligned} \mathrm{E}[Y_t] =& \mathrm{E}\left[(1-t)W_{t/(1-t)}\right] \\ =& (1-t)\cdot \cancel{\mathrm{E}\left[W_{t/(1-t)}\right]} \\ =& 0 \end{aligned} \tag{4}$$

3. $\mathrm{Cov}(Y_s, Y_t) = s \wedge t - st$.
   From the previous numeral $\mathrm{E}[Y_\tau] = 0$.

$$\begin{aligned} \mathrm{Cov}(Y_s, Y_t) =& \mathrm{E}[Y_s \cdot Y_t] - \cancel{\mathrm{E}[Y_s]\cdot\mathrm{E}[Y_t]} \\ =& \mathrm{E}\left[(1-s)W_{s/(1-s)} \cdot (1-t)W_{t/(1-t)}\right] \\ =& \mathrm{E}\left[W_{s/(1-s)}W_{t/(1-t)}\,(1-t-s+st)\right] \\ =& (1-t-s+st)\,\mathrm{E}\left[W_{s/(1-s)}W_{t/(1-t)}\right] \\ & \text{using } \mathrm{Cov}\left(W_{s/(1-s)}, W_{t/(1-t)}\right) = \mathrm{E}\left[W_{s/(1-s)}W_{t/(1-t)}\right] \\ & \mathrm{E}\left[W_{s/(1-s)}W_{t/(1-t)}\right] = \min\{s/(1-s), t/(1-t)\} \\ & \text{assuming } s < t \to s/(1-s) < t/(1-t) \\ =& (1-t-s+st)\left(\frac{s}{1-s}\right) \\ =& (1-t)\cancel{(1-s)}\left(\frac{s}{\cancel{1-s}}\right) \\ =& s\,(1-t) \\ & \text{assuming } s < t \\ =& \min\{s,t\} - st \end{aligned} \tag{5}$$

4. The only discontinuity occurs at $t = 1$, where $W_{t/(1-t)}$ is undefined. However by definition $Y_1 = 0$ in consequence as $W_{t/(1-t)}$ is continuous everywhere else then Y_t is continuous.

5. Y_t is a linear combination of W_t that by definition is Gaussian, as a consequence Y_t is Gaussian.

Problem 2

From ASA, problem 6.9
Construct the Karhunen-Loève expansion for the Brownian bridge and Ornstein-Uhlenbeck process for $t \in [0,1]$.

The Karhunen-Loève eigenvalues λ_k, and eigenvectors ϕ_k expansion for a Wiener process W_t are presented below.

$$\begin{aligned}
\lambda_k =& \frac{1}{\left(k-\frac{1}{2}\right)\pi} \\
\phi_k(s) =& \sqrt{2}\sin\left(\left(k-\frac{1}{2}\right)\pi s\right)
\end{aligned} \tag{6}$$

The reconstruction of the Wiener process W_t, is given by equation below.

$$W_t = \sum_{k=1}^{\infty} \alpha_k \lambda_k \phi_k(t)$$

A Brownian bridge X_t satisfies $\mathrm{E}[X_t] = 0$ and $K(s,t) = \mathrm{Cov}(X_s, X_t) = \min\{s,t\} - st$.

The eigenvalue problem for the Karhunen-Loève expansion for a Brownian bridge is presented below. I first expand the integral defining the eigenvalue problem. Note that the first integral correspond to the eigenvalue problem of a Wiener process whose eigenvalues and eigenvectors I presented in equation 6.

$$\begin{aligned}
\int_0^1 \left(\min\{s,t\} - st\right)\phi_k(t)dt =& \lambda_k \phi(s) \\
\int_0^1 \min\{s,t\}\phi_k(t)dt - \int_0^1 st\phi_k(t)dt =& \lambda_k \phi(s) \\
\int_0^s t\phi_k(t)dt + \int_s^1 s\phi_k(t)dt - \int_0^1 st\phi_k(t)dt =& \lambda_k \phi_k(s) \\
&\text{taking the derivative respect to } s \\
\frac{d}{ds}\left(\int_0^s t\phi_k(t)dt\right) + \frac{d}{ds}\left(\int_s^1 s\phi_k(t)dt\right) - \frac{d}{ds}\left(\int_0^1 st\phi_k(t)dt\right) =& \frac{d}{ds}\left(\lambda_k \phi_k(s)\right) \\
\cancel{s\phi_k(s)} + \left(\int_s^1 \phi_k(t)dt - \cancel{s\phi_k(s)}\right) - \int_0^1 t\phi_k(t)dt =& \lambda_k \frac{d\phi_k(s)}{ds} \\
&\text{taking the derivative again respect to } s \\
\frac{d}{ds}\left(\int_s^1 \phi_k(t)dt\right) - \cancel{\frac{d}{ds}\left(\int_0^1 t\phi_k(t)dt\right)} =& \lambda_k \frac{d^2\phi_k(s)}{ds^2} \\
-\phi_k(s) =& \lambda_k \frac{d^2\phi_k(s)}{ds^2}
\end{aligned} \tag{7}$$

In fact as presented in the previous equations, the eigenvalue problem after taking the derivative two times is the same obtained for a Wiener process W_t. The function of the eigenvectors is presented in Equation below.

$$\phi_k(t) = A\sin\left(\frac{t}{\sqrt{\lambda_k}}\right) + B\cos\left(\frac{t}{\lambda_k}\right)$$

The restriction to find the values of λ_K, A and B are presented in the list below.

- Replacing $s = 0$ in the eigenvalue problem.

$$\begin{aligned} \lambda_k\phi_k(0) =& 0 \\ \lambda_k\left[A\sin\left(\frac{t}{\sqrt{\lambda_k}}\right) + B\cos\left(\frac{t}{\lambda_k}\right)\right] =& 0 \end{aligned}$$

  If $\lambda_k = 0$, the eigenvalue problem is not satisfied as ϕ_k is undefined and thus $\lambda_k \neq 0$. In consequence for $t = 0$, $\phi_k(0) = 0$

$$\begin{aligned} A\sin\left(\frac{0}{\sqrt{\lambda_k}}\right) + B\cos\left(\frac{0}{\lambda_k}\right) =& 0 \\ B =& 0 \end{aligned}$$

  The eigenvectors simplify to the equation presented below.

$$\phi_k(t) = A\sin\left(\frac{t}{\sqrt{\lambda_k}}\right)$$

- Replacing $s = 1$ in the eigenvalue problem, and assuming again $\lambda_k \neq 0$

$$\begin{aligned} \phi_k(s) =& 0 \\ A\sin\left(\frac{0}{\sqrt{\lambda_k}}\right) =& 0 \end{aligned}$$

- Replacing in after the first derivative respect to s. And

$$\int_s^1 \phi_k(t)dt - \int_0^1 t\phi_k(t)dt = \lambda_k \frac{d\phi_k}{ds}$$

$$\frac{d\phi_k}{ds} = \frac{A}{\sqrt{\lambda_k}}\cos\left(\frac{s}{\sqrt{\lambda_k}}\right)$$

$$\int \phi_k(t)dt = \int A\sin\left(\frac{t}{\sqrt{\lambda_k}}\right)dt = -A\sqrt{\lambda}\cos\left(\frac{s}{\sqrt{\lambda}}\right)$$

$$\int \phi_k(t)dt = -A\sqrt{\lambda}\cos\left(\frac{s}{\sqrt{\lambda}}\right)\Bigg|_{s=s}^{s=1} = -A\sqrt{\lambda}\left[\cos\left(\frac{1}{\sqrt{\lambda}}\right) - \cos\left(\frac{s}{\sqrt{\lambda}}\right)\right]$$

$$\int t\phi_k(t)dt = \int tA\sin\left(\frac{t}{\sqrt{\lambda_k}}\right)dt = A\left(\lambda\sin\left(\frac{s}{\sqrt{\lambda}}\right) - s\sqrt{\lambda}\cos\left(\frac{s}{\sqrt{\lambda}}\right)\right)$$

$$A\left(\lambda\sin\left(\frac{s}{\sqrt{\lambda}}\right) - s\sqrt{\lambda}\cos\left(\frac{s}{\sqrt{\lambda}}\right)\right)\Bigg|_{s=0}^{s=1} = A\left(\lambda\sin\left(\frac{1}{\sqrt{\lambda}}\right) - \sqrt{\lambda}\cos\left(\frac{1}{\sqrt{\lambda}}\right)\right)$$

Replacing all the expressions together, I used wolfram here.

$$\sqrt{\lambda}\cos\left(\frac{t}{\sqrt{\lambda}}\right) - \lambda\sin\left(\frac{1}{\sqrt{\lambda}}\right) = \sqrt{\lambda}\cos\left(\frac{t}{\sqrt{\lambda}}\right)$$

In consequence $\sin(1/\sqrt{\lambda}) = 0$, and thus $1/\sqrt{\lambda} = k\cdot\pi, \quad k = 0,1,\ldots$. The eigenvalues λ_k are presented below.

$$\lambda_k = \frac{1}{\pi^2 k^2}, \quad k = 1,2,3,\ldots \tag{8}$$

The eigenfunctions and eigenvalues are presented below

$$\begin{aligned}\phi_k(t) =& A\sin(tk\pi)\\ \lambda_k =& \frac{1}{k^2\lambda^2}\end{aligned}$$

The value of A is computed using the normalization $\int \phi_k(t)^2 dt = 1$. I present this calculation below.

$$\begin{aligned}\int_0^1 \sin^2(tk\pi)dt =& \frac{1}{A^2}\\ \frac{1}{k\pi}\int_0^{k\pi}\sin^2(z)dz =& \frac{1}{k\pi}\int_0^{k\pi}\frac{1-\cos(2z)}{2}dz\\ \frac{1}{4k\pi}\int_0^{2k\pi}(1-\cos(u))du =& \frac{1}{4k\pi}(2k\pi\sin(2k\pi))\\ \frac{1}{4k\pi}(2k\pi\sin(2k\pi)) =& \frac{1}{2}\\ \frac{1}{A^2} =& \frac{1}{2}\\ A =& \sqrt{2}\end{aligned} \tag{9}$$

The eigenfunctions and eigenvalues are presented below.

$$\begin{aligned}\phi_k(t) =& \sqrt{2}\sin(tk\pi)\\ \lambda_k =& \frac{1}{k^2\pi^2}\end{aligned}$$

The Brownian bridge X_t can be reconstructed as shown below.

$$X_t = \sum_k^{\infty} \alpha_k \sqrt{\frac{1}{k^2\pi^2}} \sqrt{2} \sin(tk\pi)$$
$$X_t = \frac{\sqrt{2}}{\pi} \sum_k^{\infty} \frac{\alpha_k}{k} \sin(tk\pi)$$

The Ornstein-Uhlenbeck process for $t \in [0,1]$ covariance function is presented below.

$$K(s,t) = \sigma^2 e^{-\frac{|t-s|}{\eta}}$$

The eigenvalue problem is presented in equation below. For simplicity I don't consider the term σ^2, however it will be absorbed in the eigenvalues λ.

$$\int_0^1 \exp\left(-\frac{|t-s|}{\eta}\right) \phi(s)ds = \lambda\phi(t)$$

Expanding the left hand side for $s < t$ and $s \geq t$.

$$\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds = \lambda\phi(t)$$

Taking the derivative respect to t, I use here $\phi(0) = 0$.

$$\frac{d}{dt}\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds = \frac{-1}{\eta}\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \phi(t)$$

$$\frac{d}{dt}\int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds = \left(\frac{1}{\eta}\right)\int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds - \phi(t)$$

Putting together both expressions we have.

$$-\frac{1}{\eta}\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \frac{1}{\eta}\int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds = \lambda\frac{d\phi(t)}{dt}$$
$$\frac{1}{\eta}\left(-\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds\right) = \lambda\frac{d\phi(t)}{dt}$$

Taking the derivative respect to t again. Note here that the sign of the first integral changed so a factor of $-2\phi(t)$ will appear when summing the derivative of both integrals.

$$\frac{1}{\eta}\left(-2\phi(t) + \frac{1}{\eta}\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \frac{1}{\eta}\int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds\right) = \lambda\frac{d^2\phi(t)}{dt^2}$$

$$\int_0^t \exp\left(-\frac{t-s}{\eta}\right) \phi(s)ds + \int_t^1 \exp\left(-\frac{s-t}{\eta}\right) \phi(s)ds = \eta^2\lambda\frac{d^2\phi(t)}{dt^2} + 2\eta\phi(t)$$

Note that the left side of this equation was exactly the eigenvalue problem after splitting $|t-s|$. Replacing this left side with $\lambda\phi(t)$ results in differential equation presented below.

$$\begin{aligned}\lambda\phi(t) =&\eta^2\lambda\frac{d^2\phi(t)}{dt^2}+2\eta\phi(t)\\ 0 =&\frac{d^2\phi(t)}{dt^2}+\phi(t)\left(\frac{2\eta-\lambda}{\lambda\eta^2}\right)\end{aligned} \tag{10}$$

Defining $\omega^2 = (\eta-\lambda)/(\lambda\eta^2)$ and again I try solutions of the form presented below.

$$\phi(t) = A\cos(\omega t)+B\sin(\omega t)$$

The first and second derivative of $\phi(t)$ are presented below.

$$\begin{aligned}\frac{d\phi}{dt} =&\omega(-A\sin(\omega t)+B\cos(\omega t))\\ \frac{d^2\phi}{dt^2} =&\omega^2(-A\cos(\omega t)-B\sin(\omega t))\end{aligned} \tag{11}$$

The boundary conditions are obtaining from the first derivative and the original eigenfunction as shown below.

$$\begin{aligned}\lambda\phi(0) =& \int_0^1 \exp\left(-\frac{s}{\eta}\right)\phi(s)ds\\ \lambda\phi(1) =& \int_0^1 \exp\left(-\frac{1-s}{\eta}\right)\phi(s)ds\\ \lambda\frac{d\phi(0)}{dt} =&\frac{1}{\eta}\int_0^1 \exp\left(-\frac{s}{\eta}\right)\phi(s)ds\\ \lambda\frac{d\phi(1)}{dt} =&\frac{1}{\eta}\int_0^1 \exp\left(-\frac{1-s}{\eta}\right)\phi(s)ds\end{aligned} \tag{12}$$

This correspond to the system of equations presented below.

$$\begin{aligned}\frac{d}{dt}\phi(0)-\frac{1}{\eta}\phi(0) =&0\\ \frac{d}{dt}\phi(1)+\frac{1}{\eta}\phi(1) =&0\end{aligned}$$

Replacing the derivatives is shown below.

$$\begin{aligned}\frac{A}{\eta}-B\omega =&0\\ A(1/\eta-\omega\tan\omega)+B(1/\eta\tan\omega+\omega) =&0\end{aligned}$$

The system can be expressed in Matrix form as shown below.

$$\begin{pmatrix} 1/\eta & \omega \\ 1/\eta-\omega\tan\omega & 1/\eta\tan\omega+\omega \end{pmatrix}\begin{pmatrix} A\\ B\end{pmatrix} = \mathbf{0}$$

I assume solution for $\omega = k\pi$, $k = 1, 2, 3, \dots$ as in the previous problem. This result in a solution for B as presented below

$$B = A\frac{1/\eta}{k\pi}$$

This also assume the eigenvalues follow the form presented below.

$$\omega^2 = k^2\pi^2 = \frac{2\eta - \lambda}{\lambda\eta^2}$$
$$\lambda = \frac{2\eta}{k^2\pi^2\eta^2 + 1}$$

Using the normalization condition

$$\int_0^1 \phi^2(t)dt = 1$$

Solving for A it results in the expression below

$$A = \sqrt{\frac{2k^2\pi^2\eta^2}{k^2\pi^2\eta^2 + 2\eta + 1}}$$
$$B = \frac{A}{k\pi\eta}$$
$$B = \sqrt{\frac{2}{2\eta + k^2\pi^2\eta^2 + 1}}$$

The eigen-function and eigenvalues are presented below

$$\phi_k(t) = \sqrt{\frac{2k^2\pi^2\eta^2}{k^2\pi^2\eta^2 + 2\eta + 1}}\sin(k\pi t) + \sqrt{\frac{2}{2\eta + k^2\pi^2\eta^2 + 1}}\cos(k\pi t)$$
$$\lambda_k = \frac{2\eta}{k^2\pi^2\eta^2 + 1}$$

Problem 3

From ASA, problem 6.10. There's a typo in 6.10(c). I think they meant to ask for

$$\boldsymbol{E}\left[\left(\frac{W_s - sW_T}{T}\right)\left(\frac{W(t) - tW_T}{T}\right)\middle| W_T\right] \tag{13}$$

with $s \leq t \leq T$.

For the standard Wiener process $\{W_t\}$, we can define the conditional distribution of W_t for $0 \leq t \leq T$. Prove the following assertions:

(a) The distribution of $(W_{t_1}, W_{t_2}, \dots, W_{t_k} | W_T)$ is Gaussian.

A Wiener process satisfy independence of increments, $W_v - W_u$, $W_t - W_s$ are independent whenever $u \leq v \leq s \leq t$, such that intervals (u,v), (s,t) are disjoint.

Using Bayes the density $p\,(W_{t_1}, W_{t_2}, \dots, W_{t_k} | W_T)$ can be written as shown below.

$$p\,(W_{t_1}, W_{t_2}, \dots, W_{t_k} | W_T) = \frac{1}{p(W_T)} p(W_T | W_{t_1}, \dots, W_{t_k}) \cdot p(W_{t_1}, \dots, W_{t_k}) \tag{14}$$

As $t_1 \leq t_2 \leq t_3 \leq \dots T$, and by definition increments of disjoint time intervals are independent one can write W_T as the sum of independent increases. The independent increases are the set defined by $[W_0, W_{t_1} - W_{t_0}, W_{t_1} - W_{t_1}, \dots, W_{t_k} - W_{t_{k-1}}]$.

$$W_T = (W_{t_k} - W_{t_{k-1}}) + (W_{t_{k-1}} - W_{t_{k-2}}) + \dots + (W_{t_1} - W_{t_0}) + W_{t_0}$$

$$W_T = \sum_{k=0}^{T} \left(W_{t_{k+1}} - W_{t_k}\right) + W_{t_0}$$

By definition $(W_{t_{k+1}} - W_{t_k}) \sim \mathrm{N}\,(0, \delta t)$. Also $W_T \sim \mathrm{N}(0,T)$.

From equation 15 $p(W_T | W_{t_k}, \dots, W_{t_1}) = p(W_T | W_{t_k})$ by the Markov property and joint transition density $P(W_{t_k}, W_{t_{k-1}}, \dots, W_{t_1})$ can be written as shown below, again as increments are independent

$$P(W_{t_k}, W_{t_{k-1}}, \dots, W_{t_1}) = P(W_{t_k} | W_{t_k-1}) P(W_{t_k-1} | W_{t_k-2}) \times \dots \times P(W_{t_2} | W_{t_1}) P(W_{t_1} | 0)$$

Replacing this expression Eq. 15 we have

$$\begin{aligned} p\,(W_{t_1}, \dots, W_{t_k} | W_T) =& \frac{1}{p(T)} p(W_T | W_{t_k}) \times p(W_{t_k} | W_{t_k-1}) \times \dots \times p(W_{t_2} | W_{t_1}) p(W_{t_1} | W_{t_0}) \\ p\,(W_{t_1}, \dots, W_{t_k} | W_T) =& \frac{1}{p(T)} \prod_{t=t_0}^{T} p(W_{t+1} | W_t) \end{aligned} \tag{15}$$

The two point transition density from $t_{k-1} \to t_k$ is by definition Gaussian with mean zero and variance $\delta t = t_k - t_{k-1}$.

$$p(W_{t_k}|W_{t_{k-1}}) = \frac{1}{\sqrt{2\pi\delta t}} \exp\left(-\frac{1}{2}\frac{(W_{t_k} - W_{t_{k-1}})^2}{\delta t}\right)$$

Thus $p(W_T|W_{t_k}) \times p(W_{t_k}|W_{t_k-1}) \times \ldots \times p(W_{t_2}|W_{t_1})p(W_{t_1}|W_{t_0})$ is a product of Gaussians. Also note that $p(W_T)$ is a Gaussian. In expression below I present the expression for the joint distribution $\prod_{t=t_0}^{T} p(W_{t+1}|W_t)$ ($p(W_{t+1}|W_t) \sim \mathrm{N}(0, \delta t_k)$ and for $p(T) \sim \mathrm{N}(0,T)$. Define the set $\mathscr{T} = \{t_0, t_1, t_2, \ldots, T\}$ and let $t+1$ be the position after position t in the set, such that if $t = t_k$ then $t+1 = t_{k+1}$.

$$\begin{aligned}
p(T) =& \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2}\frac{W_T^2}{T}\right) \\
\prod_{t\in\mathscr{T}} p(W_{t+1}|W_t) =& \prod_{t\in\mathscr{T}} \frac{1}{\sqrt{2\pi\delta t}} \exp\left(-\frac{1}{2}\frac{(W_{t+1} - W_t)^2}{\delta t}\right) \\
=& \frac{1}{Z} \exp\left(-\frac{1}{2}\sum_{t\in\mathscr{T}} \frac{(W_{t+1} - W_t)^2}{\delta t}\right)
\end{aligned}$$

Where Z is the normalization factor, presented below. I assumed the δt, that are the differences between consecutive timesteps are not equal so $t_{i+1} - t_i \neq t_{k+1} - t_k$ if $k \neq i$.

$$Z = \left(\frac{1}{\sqrt{2\pi}}\right)^{|\mathscr{T}|} \frac{1}{\prod_{t\in\tau} \delta t}$$

$$\begin{aligned}
p\left(W_{t_1}, \ldots, W_{t_k}|W_T\right) =& \frac{1}{p(T)} \prod_{t=t_0}^{T} p(W_{t+1}|W_t) \\
=& \frac{1}{\frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2}\frac{W_T^2}{T}\right)} \cdot \frac{1}{Z} \exp\left(-\frac{1}{2}\sum_{t\in\mathscr{T}} \frac{(W_{t+1} - W_t)^2}{\delta t}\right) \\
=& \frac{\sqrt{2\pi T}}{Z} \exp\left(-\frac{1}{2}\sum_{t\in\mathscr{T}} \frac{(W_{t+1} - W_t)^2}{\delta t} + \frac{1}{2}\frac{W_T^2}{T}\right)
\end{aligned} \tag{16}$$

(b) The condition mean function.

$$\mathrm{E}\left(W_t|W_T\right) = \frac{t}{T} W_T$$

The density $p(W_t|W_T)$ was computed in the previous numeral and is presented below.

$$\begin{aligned}
p(W_t|W_T) =& \frac{1}{p(W_T)} \cdot p(W_T|W_t)p(W_t) \\
=& \frac{\sqrt{2\pi T}}{2\pi\sqrt{(T-t)(t-0)}} \exp\left(-\frac{1}{2}\left(\frac{(W_T - W_t)^2}{(T-t)} + \frac{(W_t - W_0)^2}{(t-0)} - \frac{W_T^2}{T}\right)\right) \\
=& \frac{1}{\sqrt{2\pi}}\sqrt{\frac{T}{t(T-t)}} \exp\left[-\frac{1}{2}\left(\frac{(W_T - W_t)^2}{T-t} + \frac{W_t^2}{t} - \frac{W_T^2}{T}\right)\right]
\end{aligned} \tag{17}$$

Expanding the term inside the exponential to obtain the conditional mean we have. For simplicity denote $x = W_t$ and $y = W_T$.

$$\begin{aligned}
&\frac{(W_T - W_t)^2}{T-t} + \frac{W_t^2}{t} - \frac{W_T^2}{T} \\
&-\frac{x^2}{2(T-t)} + \frac{xy}{T-t} - \frac{y^2}{2(T-t)} - \frac{x^2}{2t} \\
&x^2\left(-\frac{1}{2(T-t)} - \frac{1}{2t}\right) + \frac{xy}{T-t} - \frac{y^2}{2(T-t)} \\
&\left(x^2 - \frac{2ty}{T}x\right)\frac{T}{2t^2-2tT} + \frac{ty^2}{2t^2-2tT} \\
&\text{completing the square for } x \\
&\left[\left(x - \frac{t}{T}y\right)^2 + \left(\frac{t}{T}y\right)^2\right]\frac{T}{2t^2-2tT} + \frac{ty^2}{2t^2-2tT} \\
&\left[\left(W_t - \frac{t}{T}W_T\right)^2 + \left(\frac{t}{T}W_T\right)^2\right]\frac{T}{2t^2-2tT} + \frac{tW_T^2}{2t^2-2tT}
\end{aligned}$$

All the terms that are not $\left(W_t - \frac{t}{T}W_T\right)^2$ can be taken out of the exponential and contribute to the covariance. As a consequence the conditional expectation must satisfy.

$$\left(W_t - \frac{t}{T}W_T\right)^2 = 0$$

$$\mathrm{E}[W_t|W_T] = \frac{t}{T}W_T$$

The complete distribution $p(W_t|W_T)$ is presented below.

(c) The conditional covariance function

$$\mathrm{E}\left(W_sW_t|W_T\right) = \frac{s\left(T-t\right)}{T}$$

If $T = 1$ and $W_T = 0$, we obtain exactly the Brownian bridge.

From the previous point $E_t = E[W_t|W_T] = \frac{t}{T}W_t$ and $E_s = E[W_s|W_T] = \frac{s}{T}W_s$.

$$\begin{aligned}
E\left[(W_t - E_t)(W_s - E_s)|W_T\right] =& E[W_tW_s - E_sW_t - E_tW_s + E_tE_s|W_t] \\
&= E[W_tW_s|W_T] - E_sE[W_t|W_T] - E_TE[W_s|W_T] + E[E_tE_s|W_t] \\
&= E[W_tW_s|W_T] - E_sE_t - \cancel{E_tW_s + E_tE_s} \\
&= E[W_tW_s|W_T] - E_sE_t
\end{aligned} \tag{18}$$

Below I present the calculation of $\mathrm{E}[W_s W_t|W_T]$.

$$\begin{aligned}
\mathrm{E}[W_s W_t|W_T] =& \mathrm{E}[\mathrm{E}[\mathrm{W_s W_t|W_t, W_T}]|\mathrm{W_T}] \\
=& \mathrm{E}[\mathrm{E}[\mathrm{W_s W_t|W_t}]|\mathrm{W_T}] \\
=& \mathrm{E}[\mathrm{W_t E[W_s|W_t]}|\mathrm{W_T}] \\
=& \mathrm{E}[\mathrm{W_t \frac{s}{t} W_t|W_T}] \\
=& \frac{s}{t}\mathrm{E}[W_t^2|W_T]
\end{aligned}$$

To compute the second moment I use the Bayesian approach from the last point. Where $p(W_t|W_T)$ can be expressed as shown below.

$$p(W_t|W_T) = \frac{1}{p(W_T)} p(W_T|W_t)p(W_t)$$

The second moment $\frac{s}{t}\mathrm{E}[W_t^2|W_T]$ can be computed from the mean and variance. Additionally from the previous point we know the variance is given

$$\begin{aligned}
\mathrm{Var}[W_t|W_T] =& \frac{t(T-t)}{T} \\
\mathrm{Var}[W_t|W_T] =& \mathrm{E}[W_t^2|W_T] + \mathrm{E}[W_t|W_T]^2
\end{aligned}$$

The second moment in terms of the mean an variance will be

$$\begin{aligned}
\mathrm{E}[W_t^2|W_T] =& \mathrm{Var}[W_t|W_T] - \mathrm{E}[W_t|W_T]^2 \\
=& \frac{t(T-t)}{T} - \frac{t^2}{T^2} W_T^2
\end{aligned}$$

Replacing in the expression for the covariance. $\dfrac{s}{t}\mathrm{E}[W_t^2|W_T]$ in shown below. Recall that $E_s = E[W_s|W_T]$ and $E_t = E[W_t|W_T]$.

$$\begin{aligned}
\mathrm{Cov(W_s, W_t|W_T)} =& E[W_s W_t|W_T] - E_s E_t. \\
=& \frac{s}{t}\left(\frac{t(T-t)}{t} - \frac{t^2}{T^2} W_T^2\right) - \frac{st W_T^2}{T^2} \\
=& \frac{s(T-t)}{T} - \cancel{\frac{st}{T^2} W_T^2} - \cancel{\frac{st W_T^2}{T^2}} \\
\mathrm{Cov(W_s, W_t|W_T)} =& \frac{s(T-t)}{T}
\end{aligned}$$

If $T = 1$ and $W_T = 0$ then the covariance is $\mathrm{Cov(W_s, W_t|W_T = 0)} = \min\{s,t\} - st$. And thus satisfy $W_0 = W_1 = 0$, and the covariance is the same one of a Brownian bridge. We already show that is Gaussian

Problem 4

From ASA, problem 7.3

Prove that with the midpoint approximation

$$\int_0^t W_s dW_s \approx \sum_j W_{t_j+\frac{1}{2}} \left(W_{t_{j+1}} - W_{t_j}\right) \to \frac{W_t^2}{2}$$

and with the rightmost approximation

$$\int_0^t W_s dW_s \approx \sum_j W_{t_j+1} \left(W_{t_{j+1}} - W_{t_j}\right) \to \frac{W_t^2}{2} + \frac{t}{2}$$

in L^2_ω as the subdivision $|\Delta| \to 0$.

Midpoint approximation

$$\int_0^t W_s dW_s \approx \sum_j W_{t_j+\frac{1}{2}} \left(W_{t_{j+1}} - W_{t_j}\right) \to \frac{W_t^2}{2}$$

$$\begin{aligned}
\int_0^t W_s dW_s \approx & \sum_j W_{t_j+\frac{1}{2}} \left(W_{t_{j+1}} - W_{t_j}\right) \\
= & \sum_j W_{t_{j+\frac{1}{2}}} \left(W_{t_{j+1}} - W_{t_{j+\frac{1}{2}}}\right) + \sum_j W_{t_{j+\frac{1}{2}}} \left(W_{t_{j+\frac{1}{2}}} - W_{t_{j+\frac{1}{2}}}\right) \\
= & \sum_j \left(W_{t_{j+\frac{1}{2}}} - W_{t_j}\right)^2 \\
& \text{by definition this increments are normal and} \\
& \mathrm{E}\left[\left(W_{t_{j+\frac{1}{2}}} - W_{t_j}\right)^2\right] = \Delta t_j^{j+1/2} = t_{j+\frac{1}{2}} - t_j \\
= & \sum_j \Delta t_j^{j+1/2} = \frac{t}{2}
\end{aligned}$$

Rightmost approximation

$$W_{t_{j+\frac{1}{2}}} = \frac{1}{2}(W_{t_{j+1}} + W_{t_j}) - \frac{1}{2}(W_{t_{j+1}} - W_{t_j})$$

$$\begin{aligned}
\int_0^t W_s dW_s \approx & \sum_j W_{t_{j+1}} \left(W_{t_{j+1}} - W_{t_j}\right) \to \frac{W_t^2}{2} + \frac{t}{2} \\
= & \frac{1}{2}\sum_j (W_{t_{j+1}} + W_{t_j})(W_{t_{j+1}} - W_{t_j}) - \frac{1}{2}\sum_j (W_{t_{j+1}} - W_{t_j})(W_{t_{j+1}} - W_{t_j})
\end{aligned}$$

The first sum is a telescopic sum in where

$$(x_0 - x_1) + (x_1 - x_2) + \ldots + (x_{n-2} - x_{n-1}) + (x_{n-1} - x_n) = x_0 - x_n$$

The first sum is is show below. Note that by definition $W_0 = 0$.

$$\frac{1}{2}(W_{t_{n+1}}^2 - W_0^2)$$

The second sum is the magnitude of the approximation.

$$\frac{1}{2}(W_{t_{j+1}} - W_j)(W_{t_{j+1}} - W_{t_j}) = (W_{t_{j+1}} - W_j)^2$$

By definition the increases $W_{t_{j+1}} - W_{t_j}$ are independent and are normally distributed, and the expected value is $\mathrm{E}[W_{t_{j+1}} - W_{t_j}] = \Delta t = t_{j+1} - t_j$, and the variance $\mathrm{Var}[W_{t_{j+1}} - W_{t_j}] = 2\Delta t^2$. The variance of the sum is presented below.

$$\mathrm{Var}\left(\sum \Delta W\right) = 2\Delta t \left(\sum \Delta t\right) = 2\Delta t t_n \leq 2t2^{-n}$$

In consequence as the $n \to \infty$ the sum $\sum \Delta W \to t$

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t$$