Last time we showed that if $\exists\, \alpha \in (0,1)$ and a prob measure ν s.t.

$$P[X^{(1)} \in A \mid X^{(0)} = x] \geq \alpha\, \nu(A) \quad \forall x$$

Then $\mu T^t \to \pi$ where $\pi T = \pi$

But the Doeblin condition is too strong.

We might prefer

$$P[X^{(1)} \in A \mid X^{(0)} = x] \geq \alpha\, \nu(A) \quad \forall x \in S$$

That works but you need to control the time spent outside of S.

One way to do that is with a Lyapunov condition.

For example, if for some $V: \mathbb{R} \to [0,\infty)$, $g: \mathbb{R} \to [1,\infty)$ and some constant $b < \infty$

if

$$TV(x) - V(x) \leq \begin{cases} -g(x) + b & \text{for } x \in S \\ -g(x) & \text{for } x \notin S \end{cases}$$

and if $P[X^{(1)} \in A \mid X^{(0)} = x] \geq \alpha\, \nu(A) \quad \forall x \in S$

Then $\eta T^k \to \pi$ in a norm that depends on g

In fact, if in addition $\pi[V^2] < \infty$ then for f w/ $|f| < g$

$$\sqrt{N}\left(\frac{1}{N}\sum_{k=1}^{N} f(X^{(k)}) - \pi[f]\right) \to N(0, \tau_f\, \sigma_f^2)$$

where $\sigma_f^2 = \mathrm{var}_\pi(f)$ and

$$\tau_f = 1 + 2\sum_{k=1}^{\infty} \mathrm{corr}_\pi\left(f(X^{(0)}), f(X^{(k)})\right)$$

as long as $\tau_f > 0$.

τ_f is called the integrated autocorrelation time

Detailed Balance

Let $B_1, B_2, \ldots$ be a partition of state space

Invariance of π requires that if $X^{(0)} \sim \pi$ then $X^{(1)} \sim \pi$

i.e. $P[X^{(1)} \in B_j] = P[X^{(0)} \in B_j] = \pi(B_j)$

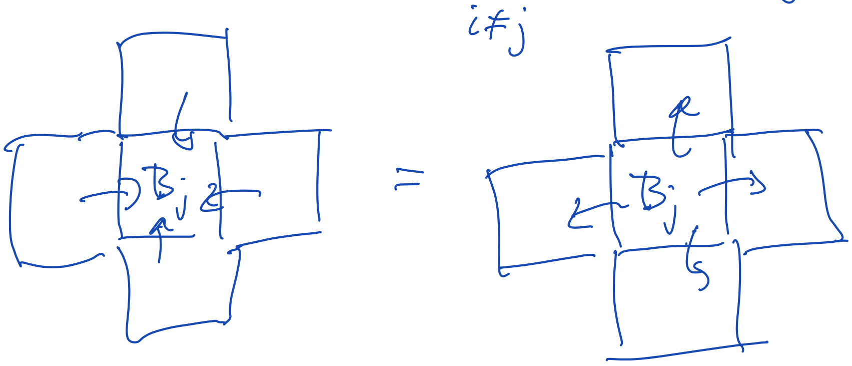
that equality can be rewritten as

$$P[X^{(1)} \in B_j \text{ and } X^{(0)} \notin B_j] + \cancel{P[X^{(1)} \in B_j \text{ and } X^{(0)} \in B_j]}$$

$$= P[X^{(0)} \in B_j \text{ and } X^{(1)} \notin B_j] + \cancel{P[X^{(0)} \in B_j \text{ and } X^{(1)} \in B_j]}$$

or $$\sum_{i \neq j} P[X^{(1)} \in B_j \text{ and } X^{(0)} \in B_i]$$

$$= \sum_{i \neq j} P[X^{(0)} \in B_j \text{ and } X^{(1)} \in B_i]$$



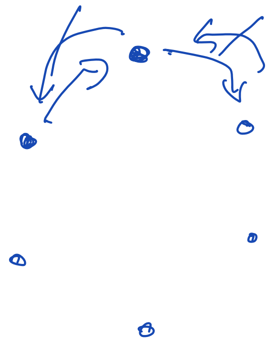
Probability going into B_j equals the Probability going out

"Global" or "Net" balance

"Detailed" balance requires

$$P[X^{(1)} \in B_j \text{ and } X^{(0)} \in B_i]$$
$$= P[X^{(0)} \in B_j \text{ and } X^{(1)} \in B_i]$$

Detailed balance implies global balance



No cycles with detailed balance

In terms of a transition density detailed balance means

$$p(y|x)\pi(x) = p(x|y)\pi(y)$$

not $\cancel{p(y|x) = p(x|y)}$

For a transition matrix detailed balance means

$$\pi_i T_{ij} = \pi_j T_{ji}$$

note that in the inner product $\langle v, w\rangle = \sum v_i w_i \pi_i$

then $\langle Tf, g \rangle = \sum_i g_i \left(\sum_j T_{ij} f_j \right) \pi_i$

$$= \sum_j \left(\sum_i g_i T_{ji} \right) \pi_j \, f_j$$

$$= \langle f, Tg \rangle$$

T is "self adjoint" in $\langle \cdot, \cdot \rangle$

For transition operator detailed balance means

$$\langle Tf, g \rangle = \langle f, Tg \rangle$$

where $\langle f, g \rangle = \int f(x)\, g(x)\, \pi(dx)$

i.e. T is self adjoint in $\langle \cdot, \cdot \rangle$

MCMC (Metropolis - Hastings)

you give me π

I want to construct Markov transition operator T

so $\pi T = \pi$

(we really want $\frac{1}{N} \sum_{k=1}^{N} f(X^{(k)}) \longrightarrow \int f(x)\, \pi(dx)$)

Metropolis Hastings

Pick some transition density $q(y|x)$

If q preserves π then you're done

$$\left(\int q(y|x)\pi(dx) = \pi(y) \right)$$

Usually you won't be able to pick q preserving π

How can we use q to generate a Markov chain that does preserve π?

Given $X^{(k)}$ generate $X^{(k+1)}$ as follows:

1. Generate $Y^{(k+1)} \sim q(y|X^{(k)})$

2. with probability $p_{acc} = \min\left\{1, \frac{\pi(Y^{(k+1)})\, q(X^{(k)}|Y^{(k+1)})}{\pi(X^{(k)})\, q(Y^{(k+1)}|X^{(k)})}\right\}$

   set $X^{(k+1)} = Y^{(k+1)}$

   otherwise set
   $X^{(k+1)} = X^{(k)}$

Note that if q is in detailed balance w.r.t. π then $p_{acc} = 1$

(recall detailed balance is $q(y|x)\pi(x) = q(x|y)\pi(y)$)

Simple choice: $q(y|x) = \frac{e^{-(y-x)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} = q(x|y)$

$$y = x + N(0,\sigma^2)$$

in this case

$$P_{acc} = \min\left\{1, \frac{\pi(Y^{(k+1)})}{\pi(X^{(k)})}\right\}$$

Notice the symmetry

$$q(y|x)\, P_{acc}(x,y)\,\pi(x) = q(y|x) \min\left\{1, \frac{\pi(y)\, q(x|y)}{\pi(x)\, q(y|x)}\right\} \pi(x)$$

$$= \min\left\{ q(y|x)\pi(x);\ \pi(y) q(x|y) \right\}$$

$$= q(x|y)\, P_{acc}(y,x)\, \pi(y)$$

$$Tf(x) = E\left[f(X^{(1)}) \mid X^{(0)} = x \right]$$

$$= f(x)\, P_{rej}(x) + \int f(y)\, P_{acc}(x,y)\, q(y|x)\, dy$$

$$P_{rej}(x) = \int (1 - P_{acc}(x,z))\, q(z|x)\, dz$$

(could write

$$p(y|x) = \delta(y-x)\, P_{rej}(x) + P_{acc}(x,y)\, q(y|x)$$

$$Tf(x) = \int f(y)\, p(y|x)\, dx$$
)

Recall in terms of transition operator detailed balance requires

$$\int g(x)\, Tf(x)\, \pi(dx) = \int f(x)\, Tg(x)\, \pi(dx) \quad \forall\, g, f$$

$$\int g(x)\, Tf(x)\, \pi(dx) = \int g(x) f(x)\, p_{rej}(x)\, \pi(dx)$$

$$+ \iint f(y)\, q(y|x)\, p_{acc}(x,y)\, g(x)\, \pi(x)\, dx\, dy$$

$$= \int g(x) f(x)\, p_{rej}(x)\, \pi(dx)$$

$$+ \int f(y) \int g(x)\, q(x|y)\, p_{acc}(y,x)\, dx\, \pi(y)\, dy$$

$$= \int f(x)\, Tg(x)\, \pi(dx)$$

So M-H is in detailed balance w.r.t. π.