Formalisation of CW-complexes

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Born 21st July, 2003 in Bonn, Germany

22nd August, 2024

Bachelor's Thesis Mathematics

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1. The mathematics of CW-complexes

1.1. Definition and basic properties of a CW-complexes

The modern definition of a CW-complex is the following:

Definition 1.1.1. Let X be a topological space. A *CW-complex* on X is a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ such that

(i) For every $n \geq 0$ there is a pushout of topological spaces

$$\begin{tikzcd}
\coprod_{i \in I_n} S_i^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d, hook, "\coprod_{i \in I_n} j_i"'] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} D_i^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"'] & X_n
\end{tikzcd}$$

where I_n is any indexing set and $j_i \colon S_i^{n-1} \to D_i^n$ is the usual inclusion for every $i \in I_n$.

(ii) We have $X = \bigcup_{n \geq 0} X_n$.

(iii) X has weak topology, i.e. $A \subseteq X$ is open $\iff$ $A \cap X_n$ is open in X_n for every n.

X_n is called the *n-skeleton*. An element $e^n \in \pi_0(X_n \setminus X_{n-1})$ is called an *(open) n-cell*. Q_i^n is called a *characteristic map*.

In this thesis we will however focus on the historical definition of CW-complexes first presented by Whitehead which can be found in [Whi18].

Definition 1.1.2. Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n \in \mathbb{N}}$ and a family of maps $(Q_i^n \colon D_i^n \to X)_{n \geq 0, i \in I_n}$ s.t.

(i) $Q_i^n|_{\mathrm{int}(D_i^n)} : \mathrm{int}(D_i^n) \to Q_i^n(\mathrm{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\mathrm{int}(D_i^n))$ an *(open) n-cell* (or a cell of dimension n) and $\overline{e}_i^n := Q_i^n(D_i^n)$ a *closed n-cell*.

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n, i) \neq (m, j)$ the cells e_i^n and e_j^m are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n\geq 0}\bigcup_{i\in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n := Q_i^n(\partial D_i^n)$ the *frontier of the n-cell* for any i and n. Additionally we define $X_n := \bigcup_{m<n+1}\bigcup_{i\in I_m} \overline{e}_i^m$ and call it the *n-skeleton* of X for $-1 \leq n \leq \infty$.

For the rest of the section let X be a CW-complex.

Remark 1.1.3. Property (iii) in the above definition is called *closure finiteness*. Property (iv) is called *weak topology*. Whitehead named CW-complexes or long *closure finite complexes with weak topology* after these two properties [Whi18].

Let us first answer the obvious question about the two definitions:

Proposition 1.1.4. *Definition 1.1.1 and 1.1.2 are equivalent.*

The proof to this proposition is long, tedious and not relevant to this thesis so we will skip it here. It can be found as the proof of Proposition A.2. in [Hat01]. From here on the term CW-Complex will always refer to the older definition 1.1.2. As such keep in mind that throughout this thesis any CW-complex will by definition be assumed to be Hausdorff.

Remark 1.1.5. The name *open n-cell* and the notation ∂e_i^n can be confusing as an open n-cell is not necessarily open and ∂e_i^n is not necessarily the boundary of $\overline{e}_i^n$.

But at least the notion of a closed n-cell makes sense:

Lemma 1.1.6. *$\overline{e}_i^n$ is compact and closed for every $n \in \mathbb{N}$ and $i \in I_n$. Similarly ∂e_i^n is compact and closed for every $n \in \mathbb{N}$ and $i \in I_n$.*

Proof. D_i^n is compact. Therefore its image $Q_i^n(D_i^n) = \overline{e}_i^n$ is compact as well. In a Hausdorff space any compact set is closed. Thus $\overline{e}_i^n$ is closed. The proof for ∂e_i^n works in the same way. □

And luckily the following is also true:

Lemma 1.1.7. *$\overline{e_i^n} = \overline{e}_i^n$ for every $n \in \mathbb{N}$ and $i \in I_n$.*

Proof. Since $e_i^n \subseteq \overline{e}_i^n$ and $\overline{e}_i^n$ is closed by the lemma above, the left inclusion is trivial. So let us show now that $\overline{e}_i^n \subseteq \overline{e_i^n}$. This statement can be rewritten as $Q_i^n\left(\overline{D_i^n}\right) \subseteq \overline{Q_i^n(D_i^n)}$. It is generally true for any continuous map that the closure of the image is contained in the image of the closure. Thus we are done. □

Now let us define what it means for a CW-complex to be finite:

Definition 1.1.8. Let X be a CW-complex. We call X *of finite type* if there are only finitely many cells in each dimension, i.e. if I_n is finite for all $n \in \mathbb{N}$. X is said to be *finite dimensional* if there is an $n \in \mathbb{N}$ such that $X = X_n$. Finally, X is called *finite* if it is of finite type and finite dimensional.

If we already know that the CW-complex we want to construct will be finite or of finite type we can relax some of the conditions:

Remark 1.1.9.

(i) For a CW-complex of finite type condition (iii) in definition 1.1.2 follows from the following: For each $n \in \mathbb{N}$, $i \in I_n$ $Q_i^n(\partial D_i^n)$ is contained in $\bigcup_{m \leq n-1} \bigcup_{i \in I_m} e_i^m$.

(ii) Additionally for a finite CW-complex condition (iv) in definition 1.1.2 is follows from the other conditions.

Proof. Let us begin with statement (i). Take $n \in \mathbb{N}$ and $i \in I_n$. We need to show that $Q_i^n(\partial D_i^n)$ is contained in a finite number of cells of a lower dimension. But by assumption we have $Q_i^n(\partial D_i^n) \subseteq \bigcup_{m \leq n-1} \bigcup_{i \in I_m} e_i^m$ which in this case is made up of finitely many cells. Now we can move on to statement (ii). We need to prove condition (iv) of definition 1.1.2, i.e.

$$A \subseteq X \text{ is closed} \iff \overline{e}_i^n \cap A \text{ is closed for all } n \in \mathbb{N} \text{ and } i \in I_n.$$

For the forward direction notice that $\overline{e}_i^n \cap A$ is just the intersection of two closed sets by assumption and lemma 1.1.6. As such it is closed. For the backward direction take an $A \subseteq X$ such that $\overline{e}_i^n$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$. We need to show that A is closed. But using condition (v) of definition 1.1.2 we get

$$A = A \cap \bigcup_{n \geq 0} \bigcup_{i \in I_n} \overline{e}_i^n = \bigcup_{n \geq 0} \bigcup_{i \in I_n} (A \cap \overline{e}_i^n)$$

which by assumption is a finite union of closed sets, making A closed. ☐

We can also think about the n-skeletons as being made up of open cells:

Lemma 1.1.10. $X_n = \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m$ *for every* $-1 \leq n \leq \infty$.

Proof. We show this by induction over $-1 \leq n \leq \infty$. For the base case assume that $n = -1$. Then we get $X_n = \bigcup_{m<0} \bigcup_{i \in I_m} \overline{e}_i^m = \varnothing = \bigcup_{m<0} \bigcup_{i \in I_m} e_i^m$.

For the induction step assume that that the statement is true for n. We now show that it also holds for $n+1$.

$$\begin{aligned}
X_{n+1} &= \bigcup_{m<n+2} \bigcup_{i \in I_m} \overline{e}_i^m \\
&= \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} \overline{e}_i^m \\
&= \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup X_n \\
&\overset{(1)}{=} \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&= \bigcup_{i \in I_{n+1}} e_i^{n+1} \cup \bigcup_{i \in I_{n+1}} \partial e_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&\overset{(2)}{=} \bigcup_{i \in I_{n+1}} e_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&= \bigcup_{m<n+2} \bigcup_{i \in I_m} e_i^m
\end{aligned}$$

Where (1) holds by induction and (2) holds by closure finiteness (property (iii) in definition 1.1.2).

Now we can move on to the case $n = \infty$.

$$\begin{aligned} X_\infty &= \bigcup_{m<\infty+1} \bigcup_{i \in I_m} \overline{e}_i^m \\ &= \bigcup_{m<\infty+1} \bigcup_{l<m+1} \bigcup_{i \in I_l} \overline{e}_i^l \\ &= \bigcup_{m<\infty+1} X_m \\ &\overset{(1)}{=} \bigcup_{m<\infty+1} \bigcup_{l<m+1} \bigcup_{i \in I_l} e_i^l \\ &= \bigcup_{m<\infty+1} \bigcup_{i \in I_m} \overline{e}_i^m \end{aligned}$$

Where (1) holds by induction. □

This also enables us to write X as a union of open cells:

Corollary 1.1.11. $\bigcup_{n \geq 0} \bigcup_{i \in I_n} e_i^n = X$.

When we want to show that a set $A \subseteq X$ is closed the weak topology (property (iv) in 1.1.2) lets us reduce that question to an individual cell. It is then often convenient to do strong induction over the dimension of the cell. We now want to prove a lemma that makes this repeated process easier. We first need the following:

Lemma 1.1.12. *Let $A \subseteq X$ be a set and n a natural number. Assume that for every $m \leq n$ and $j \in I_m$ the intersection $A \cap \overline{e}_j^m$ is closed. Then $A \cap \partial e_j^{n+1}$ is closed for every $j \in I_{n+1}$.*

Proof. By closure finiteness of X (property (iii) in 1.1.2) there is a set E of cells of dimension lower than $n+1$ such that $\partial e_j^{n+1} \subseteq \bigcup_{e \in E} \overline{e}$. This gives us

$$A \cap \partial e_j^{n+1} = A \cap \bigcup_{e \in E} \overline{e} \cap \partial e_j^{n+1} = \bigcup_{e \in E} (A \cap \overline{e}) \cap \partial e_j^{n+1}.$$

$\bigcup_{e \in E}(A \cap \overline{e})$ is closed as a finite union of sets that are by assumption closed and ∂e_j^{n+1} is closed by lemma 1.1.6. Therefore the intersection is also closed. □

Now we can move on to the lemma that we actually want. We can think of this lemma as being a weaker condition than the weak topology i.e. property (iv) in 1.1.2.

Lemma 1.1.13. *Let $A \subseteq X$ be a set such that for every $n > 0$ and $j \in I_n$ either $A \cap e_j^n$ or $A \cap \overline{e}_j^n$ is closed. Then A is closed.*

Proof. Since X has weak topology, it is enough to show that $A \cap \overline{e}_j^n$ is closed for every $n \in \mathbb{N}$ and $i \in I_n$. We show this by doing strong induction over n. For the base case $n = 0$ notice that $\overline{e}_j^0$ is a singleton and the intersection with a singleton is either that singleton or empty. As such the intersection is closed in both cases.

Now let us move on to the induction step. Assume that for every $m \leq n$ the statement already holds. We now need to show it for $n+1$. By assumption either $A \cap e_j^{n+1}$ or $A \cap \overline{e}_j^{n+1}$ is closed. The second case is just immediately what we wanted to show.

In the first case we can use that $A \cap \overline{e}_j^{n+1} = (A \cap \partial e_j^{n+1}) \cup (A \cap e_j^{n+1})$. The left part of the union is closed by lemma 1.1.12 applied to the induction hypothesis. The right part of the union is closed by the assumption of our case. The union is therefore also closed. □

We can use the lemma we just proved to show that the n-skeletons are closed:

Lemma 1.1.14. *X_n is closed for every $n \in \mathbb{N}$.*

Proof. By the previous lemma 1.1.13 it is enough to show that for every $m \in \mathbb{N}$ and $j \in I_m$ either $X_n \cap e_j^m$ or $X_n \cap \overline{e}_j^m$ is closed. We differentiate two cases bases on whether $m < n+1$. First assume that $m < n+1$ holds. Then by the definition of X_n we get $X_n \cap \overline{e}_j^m = (\bigcup_{m<n+1} \bigcup_{i \in I_m} \overline{e}_i^m) \cap \overline{e}_j^m = \overline{e}_j^m$ which is closed by lemma 1.1.6. Now assume that it does not hold. Then by lemma 1.1.10 we get $X_n \cap e_j^m = (\bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m) \cap e_j^m = \varnothing$ where the last equality holds because the open cells are pairwise disjoint by property (ii) in definition 1.1.2. The empty set is obviously closed. □

Another fact that can be quite helpful is a version of closure finiteness using open cells:

Lemma 1.1.15. *For each $n \in \mathbb{N}$ and $i \in I_n$ ∂e_i^n is contained in the union of a finite number of open cells of dimension less than n.*

Proof. We show this by doing strong induction on n. For the base case $n = 0$ notice that ∂e_i^0 is empty.

Moving on to the induction step assume that the statement holds for all $m \leq n$. We need to show that it also holds for $n+1$. By closure finiteness there is a finite set E of cells of dimension less than $n+1$ such that $\partial e_i^{n+1} \subseteq \bigcup_{e \in E} \overline{e}$. If we can show that for every $e \in E$ there is a finite set E_e of cells of dimension less than $n+1$ such that $\overline{e} \subseteq \bigcup_{e' \in E} e'$, we would then be done since $\partial e_i^{n+1} \subseteq \bigcup_{e \in E} \overline{e} \subseteq \bigcup_{e \in E} \bigcup_{e' \in E} e'$.

So take $e \in E$. By the induction hypothesis there is a finite set E'_e of cells of a lesser dimension than that of e such that $\partial e \subseteq \bigcup_{e' \in E'_e} e'$. This gives us $\overline{e} = \partial e \cup e \subseteq (\bigcup_{e' \in E'_e} e') \cup e$ which finishes the proof. □

Let us now look at some more ways to show that sets in X are closed.

Lemma 1.1.16. *$A \subseteq X$ is closed iff $A \cap X_n$ is closed for every $n \in \mathbb{N}$.*

Proof. The forward direction follows directly from lemma 1.1.14. For the backward direction take $A \subseteq X$ such that $A \cap X_n$ is closed for every n. Since X has weak topology we need to show that $A \cap \overline{e}_i^n$ is closed for every $n \in \mathbb{N}$ and $i \in I_n$. But $A \cap \overline{e}_i^n = A \cap X_n \cap \overline{e}_i^n$ which is closed by assumption and lemma 1.1.6. □

When we use this lemma with induction we might want the following for the induction step:

Lemma 1.1.17. *Let $A \subseteq X$. $A \cap X_{n+1}$ is closed iff $A \cap X_n$ and $A \cap \overline{e}_j^{n+1}$ are closed for every $j \in I_{n+1}$.*

Proof. For the forward direction notice that $A \cap X_n = A \cap X_{n+1} \cap X_n$ which is closed by assumption and lemma 1.1.14 and $A \cap \overline{e}_j^{n+1} = A \cap X_{n+1} \cap \overline{e}_j^{n+1}$ which is closed by assumption and lemma 1.1.6. For the backwards direction we apply lemma 1.1.13. We now need to show that for every $m \in \mathbb{N}$ and $j \in I_m$ either $A \cap e_j^m$ or $A \cap \overline{e}_j^m$ is closed. We differentiate three different cases. First let us look at the case $m \leq n$. Then $A \cap X_{n+1} \cap \overline{e}_j^m = A \cap \overline{e}_j^m = A \cap X_n \cap \overline{e}_j^m$ which is closed by assumption and lemma 1.1.6. No we consider $m = n+1$. Then $A \cap X_{n+1} \cap \overline{e}_j^{n+1} = A \cap \overline{e}_j^{n+1}$ which is closed by assumption. Lastly we show the claim for $m > n+1$. Here we get $A \cap X_{n+1} \cap e_j^m = A \cap (\bigcup_{l<n+1} \bigcup_{i \in I_l} e_j^l) \cap e_j^m = \varnothing$ where we used lemma 1.1.10 and the fact that different open cells are disjoint (property (ii) in definition 1.1.2). The empty set is obviously closed. ☐

With that we can write a new strong induction principle for showing that sets in a CW-complex are closed:

Lemma 1.1.18. *Let $A \subseteq X$ be a set such that for all $n \in \mathbb{N}$ if for all $m \leq n$ the intersection $A \cap X_m$ is closed then for all $j \in I_{n+1}$ the intersection $A \cap \overline{e}_j^{n+1}$ is closed. Then A is closed.*

Proof. By lemma 1.1.16 it is enough to show that for all $n \in \mathbb{N}$ the set $A \cap X_n$ is closed. We do strong induction over n starting at -1. For the base case notice that $X_{-1} = \varnothing$. Now for the induction step assume that $A \cap X_m$ is closed for all $m \leq n$. We need to show that $A \cap X_{n+1}$ is closed as well. By the previous lemma it is enough to show that $A \cap X_n$ and $A \cap \overline{e}_j^{n+1}$ are closed for all $j \in I_{n+1}$. But the first one is closed by induction hypothesis and the second one is closed by our assumption applied to the induction hypothesis. ☐

We can now use all these new techniques to show some important properties of CW-complexes:

Lemma 1.1.19. *X_0 is discrete.*

Proof. We want to show that every set $A \subseteq X_0$ is closed in X_0. It is enough if A is closed in X. We apply lemma 1.1.13. Take $n > 0$ and $i \in I_n$. We show that $A \cap e_i^n$ is closed. But using 1.1.10 that different open cells are disjoint we have $A \cap e_i^n = A \cap X_0 \cap e_i^n = A \cap (\bigcup_{m<1} \bigcup_{j \in I_m} e_j^m) \cap e_i^n = \varnothing$ which is closed. ☐

The proof of the following lemma is based on the proof of Proposition A.1. in [Hat01].

Lemma 1.1.20. *For every compact set $C \subseteq X$ the set of all open cells e_i^n such that $e_i^n \cap C \neq \varnothing$ is finite.*

Proof. Assume towards a contradiction that the set $S := \{n \in \mathbb{N}, i \in I_n \mid e_i^n \cap C \neq \varnothing\}$ is infinite. For every pair $(n,i) \in S$ pick a point $p_{n,i} \in e_i^n \cap C$. Since the open cells are pairwise disjoint we know that the set $P := \{p_{n,i} \mid (n,i) \in S\}$ is also infinite. We will now show that P is discrete and compact. Then P must be finite which is a contradiction. For both compactness and discreteness we will need that every set $A \subseteq P$ is closed in X.

So let $A \subseteq P$. We apply lemma 1.1.18. Assuming that for all $m \leq n$ the intersection $A \cap X_m$ is closed, we need to show that $A \cap \overline{e}_j^{n+1}$ is closed for every $j \in I_{n+1}$. Since $A \cap \overline{e}_j^{n+1} = (A \cap \partial e_j^{n+1}) \cup (A \cap e_j^{n+1})$ and $A \cap \partial e_j^{n+1} = A \cap X_n \cap \partial e_j^{n+1}$ is closed by lemma 1.1.6 and the assumption, it is enough to show that $A \cap e_j^{n+1}$ is closed. If the intersection $A \cap e_j^{n+1}$ is empty then we are done. So assume that there is an $x \in A \cap e_j^{n+1}$. Since $x \in A \subseteq P$ there is $(m, i) \in S$ such that $p_{m,i} = x$. But the open cells of X are pairwise disjoint so it must be that $(m, i) = (n + 1, j)$ and therefore $p_{n+1,j} = x$. Thus $A \cap e_j^{n+1} = \{p_{n+1,j}\}$ which is closed since every singleton in a Hausdorff space is closed.

This directly gives us that the subspace topology on P is discrete. For compactness notice that by what we just did P is closed and as a closed subset of the compact set C it is also compact. This is a contradiction to the fact that P is infinite as explained above. □

This lemma helps us prove the following characterisation of finite CW-complexes:

Lemma 1.1.21. *X is a finite CW-complex iff X compact.*

Proof. For the forward direction we know that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I_n} \overline{e}_i^n$ which by assumption and lemma 1.1.6 is compact as a finite union of compact sets.

The backward direction follows from lemma 1.1.20 and corollary 1.1.11. □

1.2. Constructions

In this section we will discuss how to get new CW-complexes from existing ones.

1.2.1. Subcomplexes

One important way to get a new CW-complex from an existing one is to consider subcomplexes which we will discuss in this section.

Let X be a CW-complex. A subcomplex of X is defined as follows:

Definition 1.2.1. A subcomplex of X is a set $E \subseteq X$ together with a set $J_n \subseteq I_n$ for every $n \in \mathbb{N}$ such that:

(i) E is closed.

(ii) $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} e_i^n = E$.

Note that here we want E to be the union of the open cells instead of the union of the closed cells as in definition 1.1.2. But we can prove the other version easily:

Lemma 1.2.2. *Let $E \subseteq X$ be a subcomplex. Then $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} \overline{e}_i^n = E$.*

Proof. Let $n \in \mathbb{N}$ and $i \in J_n$. It is enough to show that $\overline{e}_i^n \subseteq E$. By lemma 1.1.7 $\overline{e}_i^n = \overline{e_i^n}$. Since E is closed by property (i) $\overline{e_i^n} \subseteq E$ is equivalent to $e_i^n \subseteq E$ which is true by property (ii). □

Here are some alternative ways to define subcomplexes. These are taken from chapter 7.4 in [Jän01]. The proof that these three notions are equivalent can be found in there. We will just show the direction that is useful to us.

Lemma 1.2.3. *Let $E \subseteq X$ and $J_n \subseteq I_n$ for $n \in \mathbb{N}$ be such that*

(i) For every $n \in \mathbb{N}$ and $i \in I_n$ we have $\overline{e}_i^n \subseteq E$.

(ii) $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} e_i^n = E$.

Then E is a subcomplex of X.

Proof. Property (ii) in definition 1.2.1 is clear immediately. So we only need to show that E is closed. We apply lemma 1.1.13 which means we only need to show that for every $n \in \mathbb{N}$ and $i \in I_n$ either $E \cap \overline{e}_i^n$ or $E \cap e_i^n$ is closed. So let $n \in \mathbb{N}$ and $i \in I_n$. We differentiate the cases $i \in J_n$ and $i \notin J_n$. For the first one notice that by property (i) E can be expressed as a union of closed cells: $E = \bigcup_{m \in \mathbb{N}} \bigcup_{j \in J_n} e_j^m \subseteq \bigcup_{m \in \mathbb{N}} \bigcup_{j \in J_n} \overline{e}_j^m \subseteq E$. This gives us $E \cap \overline{e}_i^n = \overline{e}_i^n$ which is closed by lemma 1.1.6. Now for the case $i \notin J_n$ the disjointness of the open cells of X gives us that $E \cap e_i^n = (\bigcup_{m \in \mathbb{N}} \bigcup_{j \in J_n} e_j^m) \cap e_i^n = \varnothing$ which is obviously closed. □

And here is a third way to express the property of being a subcomplex:

Lemma 1.2.4. *Let $E \subseteq X$ and $J_n \subseteq I_n$ for $n \in \mathbb{N}$ be such that*

(i) E is a CW-complex with respect to the cells determined by X and J_n.

(ii) $\bigcup_{n \in \mathbb{N}} \bigcup_{i \in J_n} e_i^n = E$.

Then E is a subcomplex of X.

Proof. We will show that this satisfies the properties of the construction above in lemma 1.2.3. Property (ii) is again immediate. Property (i) combined with the definition 1.1.2 of a CW-complex immediately gives us property (i) of lemma 1.2.3. □

Now we can show that a subcomplex is indeed again a CW-complex:

Lemma 1.2.5. *Let $E \subseteq X$ together with $J_n \subseteq I_n$ for every $n \in \mathbb{N}$ be a subcomplex of the CW-complex X. Then E is again a CW-complex with respect to the cells determined by J_n and X.*

Proof. We show this by verifying the properties in the definition 1.1.2 of a CW-complex. Properties (i) and (ii) are immediate and we already covered property (v) in lemma 1.2.2.

Let us consider property (iii) i.e. closure finiteness. So let $n \in \mathbb{N}$ and $i \in J_n$. By closure finiteness of X we know that there is a finite set $E \subseteq \bigcup_{m<n} I_n$ such that $\partial e_i^n \subseteq \bigcup_{e \in E} e$. We define $E' := \{e_j^m \in E \mid j \in J_m\}$. We want to show that $\partial e_i^n \subseteq \bigcup_{e \in E'} e$. Take $x \in \partial e_i^n$. By $\partial e_i^n \subseteq \bigcup_{e \in E} e$ there is an $e_j^m \in E$ such that $x \in e_j^m$. It is obviously enough to show that $j \in J_m$. By lemma 1.2.2 we know that $x \in \partial e_i^n \subseteq \overline{e}_i^n \subseteq E$. But since $E = \bigcup_{m' \in \mathbb{N}} \bigcup_{j' \in J_{m'}} e_{j'}^{m'}$ there is $m' \in \mathbb{N}$ and $j' \in J_{m'}$ such that $x \in e_{j'}^{m'}$. We know that the open cells of X are disjoint which gives us $(m, j) = (m', j')$. That directly implies $j \in J_m$ which we wanted to show.

Lastly we need to show property (iv), i.e. that E has weak topology. Like in a lot of our other proofs $A \subseteq E$ being closed implies that $A \cap \overline{e}_i^n$ is closed for every $n \in \mathbb{N}$ and $i \in J_n$. So now take $A \subseteq E$ such that $A \cap \overline{e}_i^n$ is closed in E for every $n \in \mathbb{N}$ and $i \in J_n$. We need to show that A is closed in E. It is enough to show that A is closed in X. We apply lemma 1.1.13 which means we only need to show that for every $n \in \mathbb{N}$ and $j \in I_n$ either $A \cap \overline{e}_j^n$ or $A \cap e_j^n$ is closed. We look at two cases. Firstly consider $j \in J_n$. Then $A \cap \overline{e}_i^n$ is closed in E by assumption. By the definition of the subspace topology this means that there exists a closed set $B \subseteq X$ such that $A \cap \overline{e}_i^n = E \cap B$. But since E is closed by assumption (i) of definition 1.2.1 of a subcomplex that means that $A \cap \overline{e}_i^n$ is the intersection of two closed sets in X making it also closed. Now let us cover the case $j \notin J_n$. This gives us $A \cap e_j^n \subseteq E \cap e_j^n = (\bigcup_{m \in \mathbb{N}} \bigcup_{i \in J_m} e_i^m) \cap e_j^n = \varnothing$ where the last equality holds since the open cells of X are pairwise disjoint. Thus $A \cap e_j^n = \varnothing$ which is obviously closed. □

Now let us look at some properties of subcomplexes:

Lemma 1.2.6. *A union of subcomplexes $(E_i)_{i \in \iota}$ of X with indexing sets $(I_{i,n})_{i \in \iota, n \in \mathbb{N}}$ is again a subcomplex of X with the indexing set $\bigcup_{i \in \iota} I_{i,n}$ for every $n \in \mathbb{N}$.*

Proof. We show that this construction satisfies the assumptions of lemma 1.2.3. Property (ii) follows easily from that the fact that each of the subcomplexes E_i is the union of its open cells. So let us look at property (i). Take $n \in \mathbb{N}$ and $j \in \bigcup_{i \in \iota} I_{i,n}$. Then there is a $i \in \iota$ such that $j \in I_{i,n}$. With lemma 1.2.2 we get $\overline{e}_j^n \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{j \in I_{i,n}} \overline{e}_j^n = E_i \subseteq \bigcup_{i \in \iota} E_i$ which means we are done. □

Remark 1.2.7. It is easy to see that taking a finite union of finite subcomplexes of X yields again a finite subcomplex of X.

1.2.2. Product of CW-complexes

In this subsection we will talk about the product of CW-complexes.

1.2.2.1. K-spaces and the k-ification

Before we can move on to discuss the product of CW-complexes we need to discuss its topology. Therefore we will now study k-spaces and the k-ification.

A k-space or also called a compactly generated space is defined for our purposes as follows. Note that we mean quasi-compactness when talking about compactness.

Definition 1.2.8. Let X be a topological space. We call X a *k-space* if

$$A \subseteq X \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

There are a lot of different definitions in the literature. The most popular ones all agree on Hausdorff spaces. An overview of these different notions can be found on Wikipedia [Wik24].

It will also be helpful to characterise k-spaces via closed sets:

Lemma 1.2.9. *Let X be a topological space. X is a k-space iff*

$$A \subseteq X \textit{ is closed} \iff \textit{for all compact sets } C \subseteq X \textit{ the intersection } A \cap C \textit{ is closed in } C.$$

Proof. We only show that forward direction as the backward direction follows in the same way. Of the equivalence that we now need to show the forward direction is trivial. Thus let $A \subseteq X$ be a set such that for all compact sets $C \subseteq X$ the intersection $A \cap C$ is closed in C. It is enough to show that A^c is open. By definition of the k-space that is the case if for every compact set $C \subseteq X$ the intersection $A^c \cap C$ is open in C. Take any compact $C \subseteq X$. By assumption $A \cap C$ is closed in C. Since $A \cap C$ is the complement of $A^c \cap C$ in C, this immediately gives us that $A^c \cap C$ is open in C. □

We also define a way to make any topological space into a k-space which we call the k-ification:

Definition 1.2.10. Let X be a topological space. We can define another topological space X_c on the same set by setting

$$A \subseteq X_c \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

We call X_c the *k-ification* of X.

It is easy to see that this gives us a finer topology:

Lemma 1.2.11. *$A \subseteq X$ is open $\implies$ $A \subseteq X_c$ is open.*

Again it it useful to characterise the closed sets in the k-ification:

Lemma 1.2.12. *$A \subseteq X_c$ is closed $\iff$ $A \cap C$ is closed in C for all compact sets $C \subseteq X$.*

Proof. Completely analogue to the proof of lemma 1.2.9. □

To show that the k-ification actually fulfils its purpose of turning any space into a k-space, we first need the following lemma:

Lemma 1.2.13. *$A \subseteq X$ is compact $\iff$ $A \subseteq X_c$ is compact.*

Proof. For the backward direction notice that lemma 1.2.11 is another way of stating that the map $\mathrm{id}\colon X_c \to X$ is continuous. As the image of a compact set under a continuous map, that makes $A \subseteq X$ compact.

For the forward direction take $A \subseteq X$ compact. To show that $A \subseteq X_c$ is compact, take an open cover $(U_i)_{i\in\iota}$ of A in X_c. For every $i \in \iota$ there is by definition of the k-ification an open set $V_i \subseteq X$ such that $V_i \cap A = U_i \cap A$. $(V_i)_{i\in\iota}$ is an (open) cover of A in X:

$$A = A \cap \bigcup_{i\in\iota} U_i = \bigcup_{i\in\iota}(A \cap U_i) = \bigcup_{i\in\iota}(A \cap V_i) = A \cap \bigcup_{i\in\iota} V_i \subseteq \bigcup_{i\in\iota} V_i.$$

Thus there is a finite subcover $(V_i)_{i\in\iota'}$ of A in X. $(U_i)_{i\in\iota'}$ is now a finite subcover of A in X_c:

$$A = A \cap \bigcup_{i\in\iota'} V_i = \bigcup_{i\in\iota'}(A \cap V_i) = \bigcup_{i\in\iota'}(A \cap U_i) = A \cap \bigcup_{i\in\iota'} U_i \subseteq \bigcup_{i\in\iota'} U_i.$$

□

Now we are ready to move on to the promised lemma:

Lemma 1.2.14. *X_c is a k-space for every topological space X.*

Proof. We need to show that a set $A \subseteq X_c$ is open iff $A \cap C$ is open in C for every compact set $C \subseteq X_c$. The forward direction is again trivial.

For the backward direction take a set $A \subseteq X_c$ such that for every compact set $C \subseteq X_c$ the intersection $A \cap C$ is open in C. By the definition of the k-ification it is enough to show that for every compact set $C \subseteq X$ the intersection $A \cap C$ is open in C. So let $C \subseteq X$ be a compact set. By 1.2.13 C is also compact in X. By assumption this means that $A \cap C$ is open in $C \subseteq X_c$ (in the subspace topology of the k-ification). Thus there is an open set

$B \subset X_c$ such that $A \cap C = B \cap C$. By the definition of the k-ification $B \cap C$ is open in $C \subseteq X$. That means there is an open set $E \subseteq X$ such that $B \cap C = E \cap C$. But that now gives us $A \cap C = B \cap C = E \cap C$ with which we can conclude that $A \cap C$ is open in $C \subseteq X$ (in the subspace topology of the original topology of X). $\square$

If we already have a k-space, then the k-ification just maintains the topology of our space:

Lemma 1.2.15. *Let X be a k-space. Then the topologies of X and X_c coincide.*

Proof. Notice that the characterisation of open sets in X and X_c respectively agree in this setting. $\square$

Corollary 1.2.16. *The k-ification is idempotent.*

Now we will characterise continuous maps to and from the k-ification. Going from the k-ification is not a big issue:

Lemma 1.2.17. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y$ is continuous.*

Proof. This follows easily from lemma 1.2.11. $\square$

More interesting questions are when a map to the k-ification or a map from a k-ification to a k-ification is continuous. The following two lemmas and proofs that answer these questions are based on lemma 46.4 of [Mun14]. The next lemma is the first step towards the answer:

Lemma 1.2.18. *Let X be a compact space and $f\colon X \to Y$ be a continuous map. Then $f\colon X \to Y_c$ is continuous.*

Proof. We want to show that for every closed $A \subseteq Y_c$ the preimage $f^{-1}(A)$ is closed in X. Take any closed set $A \subseteq Y_c$. We know by lemma 1.2.12 that $A \cap C$ is closed in C for every compact $C \subseteq Y$. As the image of a compact set $f(X)$ is compact. Thus $A \cap f(X)$ is closed in $f(X) \subseteq Y$. By the definition of the subspace topology there is a closed set $B \subseteq Y$ such that $A \cap f(X) = B \cap f(x)$. Now we have

$$f^{-1}(A) = f^{-1}(A \cap f(X)) = f^{-1}(B \cap f(X)) = f^{-1}(B)$$

which is closed as the preimage of a closed set under a continuous map. $\square$

Now this helps us get the following lemma:

Lemma 1.2.19. *Let $f\colon X \to Y$ be a map of topological spaces such that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y$ is continuous. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. The last lemma together with our assumption tells us that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y_c$ is continuous. To show the claim take any open $A \subseteq Y_c$. We need to show that $f^{-1}(A) \subseteq X_c$ is open. By definition of the k-ification this set is open if for all compact sets $C \subseteq X$ the intersection $f^{-1}(A) \cap C$ is open in C. Take any compact set $C \subseteq X$. As noted above we now know that $f|_C : C \to Y_c$ is continuous. Or in other words we know that for every open $B \subseteq Y_c$ there is an open set $E \subseteq X$ such that $f^{-1}(B) \cap C = E \cap C$. Applying this to the set $A \subseteq Y_c$ gives us an open set $E \subseteq X$ such that $f^{-1}(A) \cap C = E \cap C$. But that is just another way of stating that $f^{-1}(A)$ is open in $C \subseteq X$. $\square$

That yields the following corollary:

Corollary 1.2.20. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. This situation trivially fulfils the conditions of the previous lemma. □

If you look at the discussion of the product of CW-complexes in some topology books, for example [Hat01] and [Lüc05], you will notice that the k-ification rarely gets discussed in detail. One possible reason for this is that most common spaces that you encounter are already k-spaces. Lemma 1.2.15 then allows you to ignore the k-ification entirely. We will therefore discuss in the remainder of this section which spaces are k-spaces and which are not. The first example are weakly locally compact spaces.

Definition 1.2.21. Let X be a topological space. We call X *weakly locally compact* if every point $x \in X$ has some compact neighbourhood.

This property is in some sources just called locally compact. The following proof is from Lemma 46.3 in [Mun14].

Lemma 1.2.22. *Weakly locally compact spaces are k-spaces.*

Proof. Let X be a weakly locally compact space. Let $A \subseteq X$. We need to show that A is open iff $A \cap C$ is open in C for every compact set C. The forward direction is trivial. So assume that that for every compact set C the intersection $A \cap C$ is open in C. A is open if it is a neighbourhood of every point $x \in A$. So fix any $x \in A$. Since X is weakly locally compact, x has a compact neighbourhood C. By definition of neighbourhoods there is an open set $U \subseteq C$ such that $x \in U$ and we need to find an open set $V \subseteq A$ such that $x \in V$. We show that $A \cap U$ fulfils these conditions. It is obvious that $A \cap U \subseteq A$ and $x \in A \cap U$. So it is left to show that $A \cap U$ is open. By assumption $A \cap C$ is open in C. That means that there is an open set B such that $A \cap C = B \cap C$. This now gives us

$$A \cap U = A \cap C \cap U = B \cap C \cap U = B \cap U$$

which is open as the intersection of two open sets. □

Another big class of spaces which are k-spaces are sequential spaces.

Definition 1.2.23. A set A in a topological space X is *sequentially closed* if for every convergent sequence contained in A its limit point is also in A. The *sequential closure* of a set A in X is defined as $\mathrm{scl}(A) = \{x \in X \mid \text{there is a sequence } (a_n)_{n\in\mathbb{N}} \subseteq A \text{ such that} (a_n)_{n\in\mathbb{N}} \text{ converges to } x\}$. A *sequential space* is a space in which all sequentially closed sets are closed.

We will need the following characterisation of sequentially closed sets:

Lemma 1.2.24. *A set $A \subseteq X$ is sequentially closed iff $A = scl(A)$.*

Proof. This is easy to see from the definitions. □

The following proof is based on [Sco16] and Lemma 46.3 in [Mun14].

Lemma 1.2.25. *Sequential Spaces are k-spaces.*

Proof. Let X be a Sequential Space. By lemma 1.2.9 it is enough to show that

$$A \subseteq X \text{ is closed } \iff \text{ for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is closed in } C.$$

The forward direction is trivial. Let A be a set such that $A \cap C$ is closed in C for every compact set C. Since X is a sequential space it is enough to show that A is sequentially closed or by the previous lemma $A = \mathrm{scl}(A)$. The inclusion $A \subseteq \mathrm{scl}(A)$ is obvious. For the backward inclusion take $x \in \mathrm{scl}(A)$. We need to show that $x \in A$. By definition there is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ that converges to x. It is well known (and can be shown directly from the definition of compactness) that the set $\{a_n \mid n \in \mathbb{N}\} \cup x$ is compact as the set of terms of a sequence together with the limit point of that sequence. By assumption that gives us that $A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x)$ is closed in $\{a_n \mid n \in \mathbb{N}\} \cup x$. In other words there is a closed set B such that

$$A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x).$$

With that we get

$$x \in A \iff x \in A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) \iff x \in B$$

and for all $n \in \mathbb{N}$ we get $a_n \in B$ in the exact same way. Thus $(a_n)_{n\in\mathbb{N}} \subseteq B$. Since B is in particular sequentially closed this gives us $x \in B$ which is enough by the above equivalence. □

In particular sequential spaces include metric spaces:

Lemma 1.2.26. *Metric spaces are sequential spaces.*

Proof. Let X be a metric space and A be a sequentially closed set. We need to show that that A is closed which is equivalent to A^c being open. Assume towards a contradiction that A^c is not open. Then there is a point $x \in A^c$ such that for every $n \in \mathbb{N}$ the open ball $\mathrm{B}_{1/n}(x)$ contains a point $x_n \in A$. But then we have a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ that converges to $x \in A^c$. Thus A is not sequentially closed. Contradiction. □

Corollary 1.2.27. *Metric spaces are k-spaces.*

Lastly we will discuss spaces that are not k-spaces:

Lemma 1.2.28. *Let X be an anti-compact T_1 space. Then X_c has discrete topology.*

Proof. Let $A \subseteq X_c$ be any set. We need to show that it is open. By the definition of the k-ification it is enough to show that $A \cap C$ is open in C for every compact set $C \subseteq X$. Since X is anti-compact C is finite. And by T_1 every finite set has discrete topology. Thus $A \cap C$ is open in C and X_c has discrete topology. □

Corollary 1.2.29. *Let X be a non-discrete anti-compact T_1 space. Then X is not a k-space.*

Proof. This follows easily from the previous lemma and lemma 1.2.15. □

That leads us to our first concrete example of a space that is not a k-space:

Example 1.2.30. Let X be any uncountable set. Equip X with the cocountable topology, i.e. let a set $A \subseteq X$ be open iff $A = \varnothing$ or A^c is countable. Then X is not a k-space.

Proof. It is easy to see by going through the axioms that the cocountable topology is indeed a topology. We will now show that this space satisfies the conditions of the previous corollary. X is clearly non-discrete. To see that X is a T_1 space take two distinct points a and b. Now let A be the set $X \setminus \{b\}$. This set is open since $\{b\}$ is countable and it obviously does not contain b. We lastly need to show that that X is anti-compact. To do that take any set $A \subseteq X$. Pick an (if possible infinite) countable subset $B \subseteq A$. Now for every $b \in B$ define $U_b = (X \setminus B) \cup \{b\}$. Since $U_b^c = B \setminus \{b\}$ is countable U_b is open for every $b \in B$. It is also easy to see that $A \subseteq \bigcup_{b \in B} U_b$. Thus $(U_b)_{b \in B}$ is an open cover of A. But since for every $b \in B$ there is no $b' \in B$ with $b \neq b'$ and $b \in U_{b'}$, $(U_b)_{b \in B}$ cannot have a proper subcover. Therefore A can only be compact if all these possible covers are already finite. That can only be the case if B and with that A are finite. ☐

Other examples can be found on π-base [PiB24].

1.2.2.2. Constructing the product

We can now move on to discuss the product. We want to proof the following theorem:

Theorem 1.2.31.

Lemma 1.2.32. *$(X \times Y)_c$ has weak topology, i.e. $A \subseteq (X \times Y)_c$ is closed iff $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed for all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.*

Proof.

"⇒" Since D^{n+m} is compact, its image is compact and therefore closed. As the intersection of two closed sets $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed as well.

"⇐" We know by definition of the k-ification that A is closed if for every compact set $C \subseteq X \times Y$ $A \cap C$ is closed in C. Take such a compact set C. The projections $\mathrm{pr}_1(C)$ and $\mathrm{pr}_2(C)$ are compact as images of a compact set. By ? there are finite sets $E \subseteq \{e_i^n \mid n \in \mathbb{N}, i \in I_n\}$ and $F \subseteq \{f_j^m \mid m \in \mathbb{N}, j \in J_m\}$ s.t $\mathrm{pr}_1(C) \subseteq \bigcup_{e \in E} e$ and $\mathrm{pr}_2(C) \subseteq \bigcup_{f \in F} f$. Thus

$$C \subseteq \mathrm{pr}_1(C) \times \mathrm{pr}_2(C) \subseteq \bigcup_{e \in E} e \times \bigcup_{f \in F} f = \bigcup_{e \in E} \bigcup_{f \in F} e \times f.$$

So C is included in a finite union of cells of $(X \times Y)_c$. Therefore

$$A \cap C = A \cap \left(\bigcup_{e \in E} \bigcup_{f \in F} e \times f \right) \cap C = \left(\bigcup_{e \in E} \bigcup_{f \in F} A \cap (e \times f) \right) \cap C$$

is closed since by assumption $A \cap (e \times f)$ is closed for every e and f and the union is finite. Thus $A \cap C$ is in particular closed in C. ☐

Appendix

A. a

B. b

Symbol Index

D^n	The closed unit disk in $\mathbb{R}^n$, i.e. $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.
S^n	The boundary of the unit disk in $\mathbb{R}^n$, i.e. $S^n := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.
∂e^n	The frontier of an n-cell, i.e. $\partial e^n := Q^n(\partial D^n)$. See definition 1.1.2.
$\overline{e}^n$	A closed n-cell, i.e. $\overline{e}^n := Q^n(D^n)$. See definition 1.1.2.
e^n	An (open) n-cell, i.e. $e^n := Q^n(\text{int}(D^n))$. See definition 1.1.2.

Bibliography

[Hat01] Allen Hatcher. *Algebraic topology.* 16. printing 2016. Cambridge: Cambridge Univ. Press, 2001. ISBN: 9780521795401.

[Jän01] Klaus Jänich. *Topologie.* ger. Siebente Auflage. Springer-Lehrbuch. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001. ISBN: 9783540412847.

[Lüc05] Wolfgang Lück. *Algebraische Topologie Homologie und Mannigfaltigkeiten.* ger. 1. Aufl. Vieweg Studium : Aufbaukurs Mathematik. Wiesbaden: Vieweg, 2005. ISBN: 3528032189.

[Mun14] James Raymond Munkres. *Topology.* 2. ed., Pearson new internat. ed. Harlow: Pearson, 2014. ISBN: 9781292023625.

[PiB24] The pi-Base Community. *Search for ‘ Generated by compact subspaces‘.* π-base. Accessed: 25.07.2024. 2024. URL: `https://topology.pi-base.org/spaces?q=%7EGenerated+by+compact+subspaces`.

[Sco16] Brian M. Scott (https://math.stackexchange.com/users/12042/brian-m-scott). *A first countable Hausdorff space is compactly generated.* Mathematics Stack Exchange. Version: 24.11.2016. 2016. URL: `https://math.stackexchange.com/q/2026250`.

[Whi18] J. H. C. Whitehead. “Combinatorial homotopy. I”. In: *Bulletin (new series) of the American Mathematical Society* 55.3 (2018), pp. 213–245. ISSN: 0273-0979.

[Wik24] Wikipedia contributors. *Compactly generated space.* Wikipedia, The Free Encyclopedia. Accessed: 23.07.2024. 2024. URL: `https://en.wikipedia.org/w/index.php?title=Compactly_generated_space&oldid=1233649259`.