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Master's Thesis Mathematics

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1. Introduction

Here you have your introduction. This template is mainly based on Felix Boes Master thesis you can find it here https://github.com/felixboes/masters_thesis/tree/master

1.1. First section

Congratulations you have created your first section

1.1.1. Subsections

If you need to divide your thesis use subsections.
You can iterate this with subsub-sections if you want. If you want to do this you have to change the depth of the subsub-sections in the main.
If you want to create a subsection, which does not appear in the contents table use "subsection*"

1.2. How to cite

In this section we will discuss how to cite.
Just add your reference in masterthesis_your_name_bibliography.bib
Then use this line [Abh05] or [Dis06]

2. One

In this chapter we will learn some useful tools.

2.1. How to cross reference

In this section we will learn to cross reference.
For this just use command 1.1. You previously need to create a label.
You can also reference equations in this way

$$\langle v, \mathrm{Re} Av\rangle = \langle v, U^* DUv\rangle = \langle Uv, DUv\rangle \geq \lambda_1 \|Uv\|^2 = \lambda_1 \|v\|^2 \tag{2.1}$$

Lemma 2.1.1. *Let $A \in \mathbb{C}^{M\times M}$ be diagonally dominant then A is invertible.*

This equations can be accsessed by 2.1 and 2.1.1

2.2. How to create Index

In this section we will learn to add elements to the index.
Just use the command as in the example.

Definition 2.2.1. A vectorspace is...

2.3. How to create symbol index

In this section we will learn to add elements to the symbol index.
Just use the command as in the example.

3. Product

In this chapter we will talk about the product.

3.1. K-spaces and the k-ification

Before we can move on to discuss the product of CW-complexes we need to discuss its topology. Therefore we will study k-spaces and the k-ification in this section.

A k-space or also called a compactly generated space is defined for our purposes as follows. Note that we mean quasi-compactness when talking about compactness.

Definition 3.1.1. Let X be a topological space. We call X a k-space if

$$A \subseteq X \text{ is open } \iff \text{ for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

There are a lot of different definitions in the literature. The most popular ones all agree on Hausdorff spaces. An overview of these different notions can be found on Wikipedia [Wik24].

It will also be helpful to characterise closed sets in the same way as the open sets:

Lemma 3.1.2. *Let X be a k-space. Then*

$$A \subseteq X \textit{ is closed } \iff \textit{ for all compact sets } C \subseteq X \textit{ the intersection } A \cap C \textit{ is closed in } C.$$

Proof. The forward direction is trivial. So let $A \subseteq X$ be a set such that for all compact sets $C \subseteq X$ the intersection $A \cap C$ is closed in C. It is enough to show that A^c is open. By definition of the k-space that is the case if for every compact set $C \subseteq X$ the intersection $A^c \cap C$ is open in C. Take any compact $C \subseteq X$. By assumption $A \cap C$ is closed in C. Since $A \cap C$ is the complement of $A^c \cap C$ in C, this immediately gives us that $A^c \cap C$ is open in C. □

We also define a way to make any topological space into a k-space which we call the k-ification:

Definition 3.1.3. Let X be a topological space. We can define another topological space X_c on the same set by setting

$$A \subseteq X_c \text{ is open } \iff \text{ for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

We call X_c the k-ification of X.

It is easy to see that this gives us a finer topology:

Lemma 3.1.4. *$A \subseteq X$ is open $\implies$ $A \subseteq X_c$ is open.*

Again it it useful to characterise the closed sets in the k-ification:

Lemma 3.1.5. *$A \subseteq X_c$ is closed $\iff$ $A \cap C$ is closed in C for all compact sets $C \subseteq X$.*

Proof. Completely analogue to the proof of lemma 3.1.2. □

To show that the k-ification actually fulfils its purpose of turning any space into a k-space, we first need the following lemma:

Lemma 3.1.6. *$A \subseteq X$ is compact $\iff$ $A \subseteq X_c$ is compact.*

Proof. For the backward direction notice that lemma 3.1.4 is another way of stating that the map $\mathrm{id}\colon X_c \to X$ is continuous. As the image of a compact set under a continuous map, that makes $A \subseteq X$ compact.

For the forward direction take $A \subseteq X$ compact. To show that $A \subseteq X_c$ is compact, take an open cover $(U_i)_{i\in\iota}$ of A in X_c. For every $i \in \iota$ there is by definition of the k-ification an open set $V_i \subseteq X$ such that $V_i \cap A = U_i \cap A$. $(V_i)_{i\in\iota}$ is an (open) cover of A in X:

$$A = A \cap \bigcup_{i\in\iota} U_i = \bigcup_{i\in\iota}(A \cap U_i) = \bigcup_{i\in\iota}(A \cap V_i) = A \cap \bigcup_{i\in\iota} V_i \subseteq \bigcup_{i\in\iota} V_i.$$

Thus there is a finite subcover $(V_i)_{i\in\iota'}$ of A in X. $(U_i)_{i\in\iota'}$ is now a finite subcover of A in X_c:

$$A = A \cap \bigcup_{i\in\iota'} V_i = \bigcup_{i\in\iota'}(A \cap V_i) = \bigcup_{i\in\iota'}(A \cap U_i) = A \cap \bigcup_{i\in\iota'} U_i \subseteq \bigcup_{i\in\iota'} U_i.$$

□

Now we are ready to move on to the promised lemma:

Lemma 3.1.7. *X_c is a k-space for every topological space X.*

Proof. We need to show that a set $A \subseteq X_c$ is open iff $A \cap C$ is open in C for every compact set $C \subseteq X_c$. The forward direction is again trivial.

For the backward direction take a set $A \subseteq X_c$ such that for every compact set $C \subseteq X_c$ the intersection $A \cap C$ is open in C. By the definition of the k-ification it is enough to show that for every compact set $C \subseteq X$ the intersection $A \cap C$ is open in C. So let $C \subseteq X$ be a compact set. By 3.1.6 C is also compact in X. By assumption this means that $A \cap C$ is open in $C \subseteq X_c$ (in the subspace topology of the k-ification). Thus there is an open set $B \subset X_c$ such that $A \cap C = B \cap C$. By the definition of the k-ification $B \cap C$ is open in $C \subseteq X$. That means there is an open set $E \subseteq X$ such that $B \cap C = E \cap C$. But that now gives us $A \cap C = B \cap C = E \cap C$ with which we can conclude that $A \cap C$ is open in $C \subseteq X$ (in the subspace topology of the original topology of X). □

If we already have a k-space, then the k-ification just maintains the topology of our space:

Lemma 3.1.8. *Let X be a k-space. Then the topologies of X and X_c coincide.*

Proof. Notice that the characterisation of open sets in X and X_c respectively agree in this setting. □

Corollary 3.1.9. *The k-ification is idempotent.*

Now we will characterise continuous maps to and from the k-ification. Going from the k-ification is not a big issue:

Lemma 3.1.10. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y$ is continuous.*

Proof. This follows easily from lemma 3.1.4. □

A more interesting question is when a map to the k-ification is continuous. The following lemma is the first step towards the answer:

Lemma 3.1.11. *Let X be a compact space and $f\colon X \to Y$ be a continuous map. Then $f\colon X \to Y_c$ is continuous.*

Lemma 3.1.12. *Let X be an anti-compact T_1 space. Then X_c has discrete topology.*

Proof. Let $A \subseteq X_c$ be any set. We need to show that it is open. By the definition of the k-ification it is enough to show that $A \cap C$ is open in C for every compact set $C \subseteq X$. Since X is anti-compact C is finite. And by T_1 every finite set has discrete topology. Thus $A \cap C$ is open in C and X_c has discrete topology. □

Corollary 3.1.13. *Let X be a non-discrete anti-compact T_1 space. Then X is not a k-space.*

Proof. This follows easily from the previous lemma and lemma 3.1.8. □

3.2. The product of CW-complexes

Lemma 3.2.1. *$(X \times Y)_c$ has weak topology, i.e. $A \subseteq (X \times Y)_c$ is closed iff $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed for all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.*

Proof.

"⇒" Since D^{n+m} is compact, its image is compact and therefore closed. As the intersection of two closed sets $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed as well.

"⇐" We know by definition of the k-ification that A is closed if for every compact set $C \subseteq X \times Y$ $A \cap C$ is closed in C. Take such a compact set C. The projections $\mathrm{pr}_1(C)$ and $\mathrm{pr}_2(C)$ are compact as images of a compact set. By ? there are finite sets $E \subseteq \{e_i^n \mid n \in \mathbb{N}, i \in I_n\}$ and $F \subseteq \{f_j^m \mid m \in \mathbb{N}, j \in J_m\}$ s.t $\mathrm{pr}_1(C) \subseteq \bigcup_{e \in E} e$ and $\mathrm{pr}_2(C) \subseteq \bigcup_{f \in F} f$. Thus

$$C \subseteq \mathrm{pr}_1(C) \times \mathrm{pr}_2(C) \subseteq \bigcup_{e \in E} e \times \bigcup_{f \in F} f = \bigcup_{e \in E} \bigcup_{f \in F} e \times f.$$

So C is included in a finite union of cells of $(X \times Y)_c$. Therefore

$$A \cap C = A \cap \left(\bigcup_{e \in E} \bigcup_{f \in F} e \times f \right) \cap C = \left(\bigcup_{e \in E} \bigcup_{f \in F} A \cap (e \times f) \right) \cap C$$

is closed since by assumption $A \cap (e \times f)$ is closed for every e and f and the union is finite. Thus $A \cap C$ is in particular closed in C. □

Appendix

A. a

B. b

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Bibliography

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