Formalisation of CW-complexes

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1. The mathematics of CW-complexes

1.1. Definition and basic properties of a CW-complexes

The modern definition of a CW-complex is the following:

Definition 1.1.1. Let X be a topological space. A *CW-complex* on X is a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ such that

(i) For every $n \geq 0$ there is a pushout of topological spaces

$$\begin{tikzcd}
\coprod\limits_{i\in I_n} S_i^{n-1} \arrow[r, "\coprod_{i\in I_n} q_i^n"] \arrow[d, hook, "\coprod_{i\in I_n} j_i"'] & X_{n-1} \arrow[d] \\
\coprod\limits_{i\in I_n} D_i^n \arrow[r, "\coprod_{i\in I_n} Q_i^n"'] & X_n
\end{tikzcd}$$

where I_n is any indexing set and $j_i\colon S_i^{n-1} \to D_i^n$ is the usual inclusion for every $i \in I_n$.

(ii) We have $X = \bigcup_{n\geq 0} X_n$.

(iii) X has weak topology, i.e. $A \subseteq X$ is open $\iff$ $A \cap X_n$ is open in X_n for every n.

X_n is called the *n-skeleton*. An element $e^n \in \pi_0(X_n \setminus X_{n-1})$ is called an *(open) n-cell*. Q_i^n is called a *characteristic map*.

In this thesis we will however focus on the historical definition of CW-complexes first presented by Whitehead which can be found in [Whi18].

Definition 1.1.2. Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of maps $(Q_i^n\colon D_i^n \to X)_{n\geq 0, i\in I_n}$ s.t.

(i) $Q_i^n|_{\mathrm{int}(D_i^n)} : \mathrm{int}(D_i^n) \to Q_i^n(\mathrm{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\mathrm{int}(D_i^n))$ an *(open) n-cell* (or a cell of dimension n) and $\overline{e}_i^n := Q_i^n(D_i^n)$ a *closed n-cell.*

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n,i) \neq (m,j)$ the cells e_i^n and e_j^m are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n\geq 0}\bigcup_{i\in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n \coloneqq Q_i^n(\partial D_i^n)$ the *frontier of the n-cell* for any i and n. Additionally we define $X_n \coloneqq \bigcup_{m<n+1}\bigcup_{i\in I_m} \overline{e}_i^m$ and call it the *n-skeleton* of X for $-1 \leq n \leq \infty$.

For the rest of the section let X be a CW-complex.

Remark 1.1.3. Property (iii) in the above definition is called *closure finiteness*. Property (iv) is called *weak topology*. Whitehead named CW-complexes or long *closure finite complexes with weak topology* after these two properties [Whi18].

Let us first answer the obvious question about the two definitions:

Proposition 1.1.4. *Definition 1.1.1 and 1.1.2 are equivalent.*

The proof to this proposition is long, tedious and not relevant to this thesis so we will skip it here. It can be found as the proof of Proposition A.2. in [Hat01]. From here on the term CW-Complex will always refer to the older definition 1.1.2. As such keep in mind that throughout this thesis any CW-complex will by definition be assumed to be Hausdorff.

Remark 1.1.5. The name *open n-cell* and the notation ∂e_i^n can be confusing as an open n-cell is not necessarily open and ∂e_i^n is not necessarily the boundary of $\overline{e}_i^n$.

But at least the notion of a closed n-cell makes sense:

Lemma 1.1.6. *$\overline{e}_i^n$ is compact and closed for every $n \in \mathbb{N}$ and $i \in I_n$. Similarly ∂e_i^n is compact and closed for every $n \in \mathbb{N}$ and $i \in I_n$.*

Proof. D_i^n is compact. Therefore its image $Q_i^n(D_i^n) = \overline{e}_i^n$ is compact as well. In a Hausdorff space any compact set is closed. Thus $\overline{e}_i^n$ is closed. The proof for ∂e_i^n works in the same way. $\square$

And luckily the following is also true:

Lemma 1.1.7. *$\overline{e_i^n} = \overline{e}_i^n$ for every $n \in \mathbb{N}$ and $i \in I_n$.*

Proof. Since $e_i^n \subseteq \overline{e}_i^n$ and $\overline{e}_i^n$ is closed by the lemma above, the left inclusion is trivial. So let us show now that $\overline{e}_i^n \subseteq \overline{e_i^n}$. This statement can be rewritten as $Q_i^n\left(\overline{D_i^n}\right) \subseteq \overline{Q_i^n(D_i^n)}$. It is generally true for any continuous map that the closure of the image is contained in the image of the closure. Thus we are done. $\square$

Now let us define what it means for a CW-complex to be finite:

Definition 1.1.8. Let X be a CW-complex. We call X *of finite type* if there are only finitely many cells in each dimension, i.e. if I_n is finite for all $n \in \mathbb{N}$. X is said to be *finite dimensional* if there is an $n \in \mathbb{N}$ such that $X = X_n$. Finally, X is called *finite* if it is of finite type and finite dimensional.

If we already know that the CW-complex we want to construct will be finite or of finite type we can relax some of the conditions:

Remark 1.1.9.

(i) For a CW-complex of finite type condition (iii) in definition 1.1.2 follows from the following: For each $n \in \mathbb{N}$, $i \in I_n$ $Q_i^n(\partial D_i^n)$ is contained in $\bigcup_{m \leq n-1} \bigcup_{i \in I_m} e_i^m$.

(ii) Additionally for a finite CW-complex condition (iv) in definition 1.1.2 is follows from the other conditions.

Proof. Let us begin with statement (i). Take $n \in \mathbb{N}$ and $i \in I_n$. We need to show that $Q_i^n(\partial D_i^n)$ is contained in a finite number of cells of a lower dimension. But by assumption we have $Q_i^n(\partial D_i^n) \subseteq \bigcup_{m \leq n-1} \bigcup_{i \in I_m} e_i^m$ which in this case is made up of finitely many cells. Now we can move on to statement (ii). We need to prove condition (iv) of definition 1.1.2, i.e.

$$A \subseteq X \text{ is closed } \iff \overline{e}_i^n \cap A \text{ is closed for all } n \in \mathbb{N} \text{ and } i \in I_n.$$

For the forward direction notice that $\overline{e}_i^n \cap A$ is just the intersection of two closed sets by assumption and lemma 1.1.6. As such it is closed. For the backward direction take an $A \subseteq X$ such that $\overline{e}_i^n$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$. We need to show that A is closed. But using condition (v) of definition 1.1.2 we get

$$A = A \cap \bigcup_{n \geq 0} \bigcup_{i \in I_n} \overline{e}_i^n = \bigcup_{n \geq 0} \bigcup_{i \in I_n} (A \cap \overline{e}_i^n)$$

which by assumption is a finite union of closed sets, making A closed. □

We can also think about the n-skeletons as being made up of open cells:

Lemma 1.1.10. $X_n = \bigcup_{m < n+1} \bigcup_{i \in I_m} e_i^m$ *for every* $-1 \leq n \leq \infty$.

Proof. We show this by induction over $-1 \leq n \leq \infty$. For the base case assume that $n = -1$. Then we get $X_n = \bigcup_{m<0} \bigcup_{i \in I_m} \overline{e}_i^m = \varnothing = \bigcup_{m<0} \bigcup_{i \in I_m} e_i^m$.

For the induction step assume that that the statement is true for n. We now show that it also holds for $n+1$.

$$\begin{aligned}
X_{n+1} &= \bigcup_{m<n+2} \bigcup_{i \in I_m} \overline{e}_i^m \\
&= \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} \overline{e}_i^m \\
&= \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup X_n \\
&\overset{(1)}{=} \bigcup_{i \in I_{n+1}} \overline{e}_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&= \bigcup_{i \in I_{n+1}} e_i^{n+1} \cup \bigcup_{i \in I_{n+1}} \partial e_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&\overset{(2)}{=} \bigcup_{i \in I_{n+1}} e_i^{n+1} \cup \bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m \\
&= \bigcup_{m<n+2} \bigcup_{i \in I_m} e_i^m
\end{aligned}$$

Where (1) holds by induction and (2) holds by closure finiteness (property (iii) in definition 1.1.2).

Now we can move on to the case $n = \infty$.

$$\begin{aligned} X_\infty &= \bigcup_{m<\infty+1} \bigcup_{i\in I_m} \overline{e}_i^m \\ &= \bigcup_{m<\infty+1} \bigcup_{l<m+1} \bigcup_{i\in I_l} \overline{e}_i^l \\ &= \bigcup_{m<\infty+1} X_m \\ &\overset{(1)}{=} \bigcup_{m<\infty+1} \bigcup_{l<m+1} \bigcup_{i\in I_l} e_i^l \\ &= \bigcup_{m<\infty+1} \bigcup_{i\in I_m} \overline{e}_i^m \end{aligned}$$

Where (1) holds by induction. □

This also enables us to write X as a union of open cells:

Corollary 1.1.11. $\bigcup_{n\geq 0} \bigcup_{i\in I_n} e_i^n = X$.

When we want to show that a set $A \subseteq X$ is closed the weak topology (property (iv) in 1.1.2) lets us reduce that question to an individual cell. It is then often convenient to do strong induction over the dimension of the cell. We now want to prove a lemma that makes this repeated process easier. We first need the following:

Lemma 1.1.12. *Let $A \subseteq X$ be a set and n a natural number. Assume that for every $m \leq n$ and $j \in I_m$ the intersection $A \cap \overline{e}_j^m$ is closed. Then $A \cap \partial e_j^{n+1}$ is closed for every $j \in I_{n+1}$.*

Proof. By closure finiteness of X (property (iii) in 1.1.2) there is a set E of cells of dimension lower than $n+1$ such that $\partial e_j^{n+1} \subseteq \bigcup_{e\in E} \overline{e}$. This gives us

$$A \cap \partial e_j^{n+1} = A \cap \bigcup_{e\in E} \overline{e} \cap \partial e_j^{n+1} = \bigcup_{e\in E} (A \cap \overline{e}) \cap \partial e_j^{n+1}.$$

$\bigcup_{e\in E}(A \cap \overline{e})$ is closed as a finite union of sets that are by assumption closed and ∂e_j^{n+1} is closed by lemma 1.1.6. Therefore the intersection is also closed. □

Now we can move on to the lemma that we actually want. We can think of this lemma as being a weaker condition than the weak topology i.e. property (iv) in 1.1.2.

Lemma 1.1.13. *Let $A \subseteq X$ be a set such that for every $n > 0$ and $j \in I_n$ either $A \cap e_j^n$ or $A \cap \overline{e}_j^n$ is closed. Then A is closed.*

Proof. Since X has weak topology, it is enough to show that $A \cap \overline{e}_j^n$ is closed for every $n \in \mathbb{N}$ and $i \in I_n$. We show this by doing strong induction over n. For the base case $n = 0$ notice that $\overline{e}_j^0$ is a singleton and the intersection with a singleton is either that singleton or empty. As such the intersection is closed in both cases.

Now let us move on to the induction step. Assume that for every $m \leq n$ the statement already holds. We now need to show it for $n+1$. By assumption either $A \cap e_j^{n+1}$ or $A \cap \overline{e}_j^{n+1}$ is closed. The second case is just immediately what we wanted to show.

In the first case we can use that $A \cap \overline{e}_j^{n+1} = (A \cap \partial e_j^{n+1}) \cup (A \cap e_j^{n+1})$. The left part of the union is closed by lemma 1.1.12 applied to the induction hypothesis. The right part of the union is closed by the assumption of our case. The union is therefore also closed. ☐

We can use the lemma we just proved to show that the n-skeletons are closed:

Lemma 1.1.14. *X_n is closed for every $n \in \mathbb{N}$.*

Proof. By the previous lemma 1.1.13 it is enough to show that for every $m \in \mathbb{N}$ and $j \in I_m$ either $X_n \cap e_j^m$ or $X_n \cap \overline{e}_j^m$ is closed. We differentiate two cases bases on whether $m < n+1$. First assume that $m < n+1$ holds. Then by the definition of X_n we get $X_n \cap \overline{e}_j^m = (\bigcup_{m<n+1} \bigcup_{i \in I_m} \overline{e}_i^m) \cap \overline{e}_j^m = \overline{e}_j^m$ which is closed by lemma 1.1.6. Now assume that it does not hold. Then by lemma 1.1.10 we get $X_n \cap e_j^m = (\bigcup_{m<n+1} \bigcup_{i \in I_m} e_i^m) \cap e_j^m = \varnothing$ where the last equality holds because the open cells are pairwise disjoint by property (ii) in definition 1.1.2. The empty set is obviously closed. ☐

Another fact that can be quite helpful is a version of closure finiteness using open cells:

Lemma 1.1.15. *For each $n \in \mathbb{N}$ and $i \in I_n$ ∂e_i^n is contained in the union of a finite number of open cells of dimension less than n.*

Proof. We show this by doing strong induction on n. For the base case $n = 0$ notice that ∂e_i^0 is empty.

Moving on to the induction step assume that the statement holds for all $m \leq n$. We need to show that it also holds for $n+1$. By closure finiteness there is a finite set E of cells of dimension less than $n+1$ such that $\partial e_i^{n+1} \subseteq \bigcup_{e \in E} \overline{e}$. If we can show that for every $e \in E$ there is a finite set E_e of cells of dimension less than $n+1$ such that $\overline{e} \subseteq \bigcup_{e' \in E} e'$, we would then be done since $\partial e_i^{n+1} \subseteq \bigcup_{e \in E} \overline{e} \subseteq \bigcup_{e \in E} \bigcup_{e' \in E} e'$.

So take $e \in E$. By the induction hypothesis there is a finite set E'_e of cells of a lesser dimension than that of e such that $\partial e \subseteq \bigcup_{e' \in E'_e} e'$. This gives us $\overline{e} = \partial e \cup e \subseteq (\bigcup_{e' \in E'_e} e') \cup e$ which finishes the proof. ☐

Let us now look at some more ways to show that sets in X are closed.

Lemma 1.1.16. *$A \subseteq X$ is closed iff $A \cap X_n$ is closed for every $n \in \mathbb{N}$.*

Proof. The forward direction follows directly from lemma 1.1.14. For the backward direction take $A \subseteq X$ such that $A \cap X_n$ is closed for every n. Since X has weak topology we need to show that $A \cap \overline{e}_i^n$ is closed for every $n \in \mathbb{N}$ and $i \in I_n$. But $A \cap \overline{e}_i^n = A \cap X_n \cap \overline{e}_i^n$ which is closed by assumption and lemma 1.1.6. ☐

When we use this lemma with induction we might want the following for the induction step:

Lemma 1.1.17. *Let $A \subseteq X$. $A \cap X_{n+1}$ is closed iff $A \cap X_n$ and $A \cap \overline{e}_j^{n+1}$ are closed for every $j \in I_{n+1}$.*

Proof. For the forward direction notice that $A \cap X_n = A \cap X_{n+1} \cap X_n$ which is closed by assumption and lemma 1.1.14 and $A \cap \overline{e}_j^{n+1} = A \cap X_{n+1} \cap \overline{e}_j^{n+1}$ which is closed by assumption and lemma 1.1.6. For the backwards direction we apply lemma 1.1.13. We now need to show that for every $m \in \mathbb{N}$ and $j \in I_m$ either $A \cap e_j^m$ or $A \cap \overline{e}_j^m$ is closed. We differentiate three different cases. First let us look at the case $m \leq n$. Then $A \cap X_{n+1} \cap \overline{e}_j^m = A \cap \overline{e}_j^m = A \cap X_n \cap \overline{e}_j^m$ which is closed by assumption and lemma 1.1.6. No we consider $m = n + 1$. Then $A \cap X_{n+1} \cap \overline{e}_j^{n+1} = A \cap \overline{e}_j^{n+1}$ which is closed by assumption. Lastly we show the claim for $m > n+1$. Here we get $A \cap X_{n+1} \cap e_j^m = A \cap (\bigcup_{l<n+1} \bigcup_{i \in I_l} e_j^l) \cap e_j^m = \varnothing$ where we used lemma 1.1.10 and the fact that different open cells are disjoint (property (ii) in definition 1.1.2). The empty set is obviously closed. ☐

With that we can write a new strong induction principle for showing that sets in a CW-complex are closed:

Lemma 1.1.18. *Let $A \subseteq X$ be a set such that for all $n \in \mathbb{N}$ if for all $m \leq n$ the intersection $A \cap X_m$ is closed then for all $j \in I_{n+1}$ the intersection $A \cap \overline{e}_j^{n+1}$ is closed. Then A is closed.*

Proof. By lemma 1.1.16 it is enough to show that for all $n \in \mathbb{N}$ the set $A \cap X_n$ is closed. We do strong induction over n starting at -1. For the base case notice that $X_{-1} = \varnothing$. Now for the induction step assume that $A \cap X_m$ is closed for all $m \leq n$. We need to show that $A \cap X_{n+1}$ is closed as well. By the previous lemma it is enough to show that $A \cap X_n$ and $A \cap \overline{e}_j^{n+1}$ are closed for all $j \in I_{n+1}$. But the first one is closed by induction hypothesis and the second one is closed by our assumption applied to the induction hypothesis. ☐

We can now use all these new techniques to show some important properties of CW-complexes:

Lemma 1.1.19. *X_0 is discrete.*

Proof. We want to show that every set $A \subseteq X_0$ is closed in X_0. It is enough if A is closed in X. We apply lemma 1.1.13. Take $n > 0$ and $i \in I_n$. We show that $A \cap e_i^n$ is closed. But using 1.1.10 that different open cells are disjoint we have $A \cap e_i^n = A \cap X_0 \cap e_i^n = A \cap (\bigcup_{m<1} \bigcup_{j \in I_m} e_j^m) \cap e_i^n = \varnothing$ which is closed. ☐

The proof of the following lemma is based on the proof of Proposition A.1. in [Hat01].

Lemma 1.1.20. *For every compact set $C \subseteq X$ the set of all open cells e_i^n such that $e_i^n \cap C \neq \varnothing$ is finite.*

Proof. Assume towards a contradiction that the set $S := \{n \in \mathbb{N}, i \in I_n \mid e_i^n \cap C \neq \varnothing\}$ is infinite. For every pair $(n, i) \in S$ pick a point $p_{n,i} \in e_i^n \cap C$. Since the open cells are pairwise disjoint we know that the set $P := \{p_{n,i} \mid (n, i) \in S\}$ is also infinite. We will now show that P is discrete and compact. Then P must be finite which is a contradiction. For both compactness and discreteness we will need that every set $A \subseteq P$ is closed in X.

So let $A \subseteq P$. We apply lemma 1.1.18. Assuming that for all $m \leq n$ the intersection $A \cap X_m$ is closed, we need to show that $A \cap \overline{e}_j^{n+1}$ is closed for every $j \in I_{n+1}$. Since $A \cap \overline{e}_j^{n+1} = (A \cap \partial e_j^{n+1}) \cup (A \cap e_j^{n+1})$ and $A \cap \partial e_j^{n+1} = A \cap X_n \cap \partial e_j^{n+1}$ is closed by lemma 1.1.6 and the assumption, it is enough to show that $A \cap e_j^{n+1}$ is closed. If the intersection $A \cap e_j^{n+1}$ is empty then we are done. So assume that there is an $x \in A \cap e_j^{n+1}$. Since $x \in A \subseteq P$ there is $(m,i) \in S$ such that $p_{m,i} = x$. But the open cells of X are pairwise disjoint so it must be that $(m,i) = (n+1,j)$ and therefore $p_{n+1,j} = x$. Thus $A \cap e_j^{n+1} = \{p_{n+1,j}\}$ which is closed since every singleton in a Hausdorff space is closed.

This directly gives us that the subspace topology on P is discrete. For compactness notice that by what we just did P is closed and as a closed subset of the compact set C it is also compact. This is a contradiction to the fact that P is infinite as explained above. $\square$

This lemma helps us prove the following characterisation of finite CW-complexes:

Lemma 1.1.21. *X is a finite CW-complex iff X compact.*

Proof. For the forward direction we know that $X = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I_n} \overline{e}_i^n$ which by assumption and lemma 1.1.6 is compact as a finite union of compact sets.

The backward direction follows from lemma 1.1.20 and corollary 1.1.11. $\square$

1.2. Constructions

In this section we will discuss how to get new CW-complexes from existing ones. We can start with some easy ones.

1.2.1. Skeletons as CW-complexes

The n-skeletons of a CW-complex X are again CW-complexes:

Lemma 1.2.1. *Let $-1 \le n \le \infty$. Then X_n is a CW-complex together with the cells $J_m := I_m$ for $m < n+1$ and $J_m = \varnothing$ otherwise.*

Proof. We need to verify the five conditions of definition 1.1.2. Conditions (i), (ii) and (iii) follow directly from X fulfilling these conditions and condition (v) is given by the definition of the n-skeleton. Thus we only need to worry about condition (iv), i.e. that X_n has weak topology. It follows easily from lemma 1.1.6 that for a set $A \subseteq X_n$ that is closed in X_n the intersection $A \cap \overline{e}_i^m$ is closed in X_n for every $m \in \mathbb{N}$ and $i \in J_m$. We can therefore directly consider the other direction. Let $A \subseteq X_n$ be a set such that for every $m \in \mathbb{N}$ and $i \in J_m$ the intersection $A \cap \overline{e}_i^m$ is closed in X_n. We need to show that A is closed in X_n. It suffices to show that A is closed in X. By lemma 1.1.13 we need to prove that for every $m \in \mathbb{N}$ and $i \in I_m$ either $A \cap \overline{e}_i^m$ or $A \cap e_i^m$ is closed. Let us start with the case $i \in J_m$. By assumption $A \cap \overline{e}_i^m$ is closed in X_n. The definition of the subspace topology tells us that there exists a closed set $C \subseteq X$ such that $C \cap X_n = A \cap \overline{e}_i^m$. But since X_n is closed by lemma 1.1.14 that means that $A \cap \overline{e}_i^m$ is also closed in X. So we are done for this case. For the case $i \notin J_m$ notice that by lemma 1.1.10 we get $A \cap e_i^m \subseteq X_n \cap e_i^m = (\bigcup_{l<n+1} \bigcup_{j\in I_l} e_j^l) \cap e_i^m = \varnothing$ since different open cells of X are disjoint. The empty set is obviously closed. $\square$

1.2.2. Disjoint union of CW-complexes

Additionally we can get a CW-complex by taking the disjoint union of two CW-complexes:

Lemma 1.2.2. *Let X and Y be two CW-complexes with indexing sets $(I_{1,n})_{n\in\mathbb{N}}$ and $(I_{2,n})_{n\in\mathbb{N}}$. Then $X \amalg Y$ is a CW-complex with indexing sets $J_n := I_{1,n} \cup I_{2,n}$.*

Proof. We need to show that this construction satisfies the conditions of definition 1.1.2. Conditions (i), (ii), (iii) and (v) follow directly from X and Y fulfilling these conditions. So we again only need to focus on condition (iv), i.e. the weak topology. The forward direction follows in the same way as in a lot of the other proofs. For the backwards direction take $A \subseteq X \amalg Y$ such that $A \cap \overline{e}_i^n$ is closed in $X \amalg Y$ for every $n \in \mathbb{N}$ and $i \in J_n$. We need to show that A is closed in $X \amalg Y$. by the definition of the disjoint union topology that is equivalent to $A \cap X$ being closed in X and $A \cap Y$ being closed in Y. We will show this for X. By the weak topology it is enough to show that $A \cap X \cap \overline{e}_i^n$ is closed in X for every $n \in \mathbb{N}$ and $i \in I_{1,n}$. But we have $A \cap X \cap \overline{e}_i^n = (A \cap \overline{e}_i^n) \cap X$ which is closed in X by assumption and the definition of the disjoint union topology. □

1.2.3. Subcomplexes

One important way to get a new CW-complex from an existing one is to consider subcomplexes which we will discuss in this section.

Let X be a CW-complex. A subcomplex of X is defined as follows:

Definition 1.2.3. A subcomplex of X is a set $E \subseteq X$ together with a set $J_n \subseteq I_n$ for every $n \in \mathbb{N}$ such that:

(i) E is closed.

(ii) $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} e_i^n = E$.

Note that here we want E to be the union of the open cells instead of the union of the closed cells as in definition 1.1.2. But we can prove the other version easily:

Lemma 1.2.4. *Let $E \subseteq X$ be a subcomplex. Then $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} \overline{e}_i^n = E$.*

Proof. Let $n \in \mathbb{N}$ and $i \in J_n$. It is enough to show that $\overline{e}_i^n \subseteq E$. By lemma 1.1.7 $\overline{e}_i^n = \overline{e_i^n}$. Since E is closed by property (i) $\overline{e}_i^n \subseteq E$ is equivalent to $e_i^n \subseteq E$ which is true by property (ii). □

Example 1.2.5. We have already proven that every n-skeleton is a subcomplex with lemma 1.1.14 and lemma 1.1.10. This section therefore provides us with an alternative way to show that n-skeletons are CW-complexes.

Here are some alternative ways to define subcomplexes. These are taken from chapter 7.4 in [Jän01]. The proof that these three notions are equivalent can be found in there. We will just show the direction that is useful to us.

Lemma 1.2.6. *Let $E \subseteq X$ and $J_n \subseteq I_n$ for $n \in \mathbb{N}$ be such that*

(i) For every $n \in \mathbb{N}$ and $i \in I_n$ we have $\overline{e}_i^n \subseteq E$.

(ii) $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} e_i^n = E$.

Then E is a subcomplex of X.

Proof. Property (ii) in definition 1.2.3 is clear immediately. So we only need to show that E is closed. We apply lemma 1.1.13 which means we only need to show that for every $n \in \mathbb{N}$ and $i \in I_n$ either $E \cap \overline{e}_i^n$ or $E \cap e_i^n$ is closed. So let $n \in \mathbb{N}$ and $i \in I_n$. We differentiate the cases $i \in J_n$ and $i \notin J_n$. For the first one notice that by property (i) E can be expressed as a union of closed cells: $E = \bigcup_{m\in\mathbb{N}} \bigcup_{j\in J_n} e_j^m \subseteq \bigcup_{m\in\mathbb{N}} \bigcup_{j\in J_n} \overline{e}_j^m \subseteq E$. This gives us $E \cap \overline{e}_i^n = \overline{e}_i^n$ which is closed by lemma 1.1.6. Now for the case $i \notin J_n$ the disjointness of the open cells of X gives us that $E \cap e_i^n = (\bigcup_{m\in\mathbb{N}} \bigcup_{j\in J_n} e_j^m) \cap e_i^n = \varnothing$ which is obviously closed. $\square$

And here is a third way to express the property of being a subcomplex:

Lemma 1.2.7. *Let $E \subseteq X$ and $J_n \subseteq I_n$ for $n \in \mathbb{N}$ be such that*

(i) E is a CW-complex with respect to the cells determined by X and J_n.

(ii) $\bigcup_{n\in\mathbb{N}} \bigcup_{i\in J_n} e_i^n = E$.

Then E is a subcomplex of X.

Proof. We will show that this satisfies the properties of the construction above in lemma 1.2.6. Property (ii) is again immediate. Property (i) combined with the definition 1.1.2 of a CW-complex immediately gives us property (i) of lemma 1.2.6. $\square$

Now we can show that a subcomplex is indeed again a CW-complex:

Lemma 1.2.8. *Let $E \subseteq X$ together with $J_n \subseteq I_n$ for every $n \in \mathbb{N}$ be a subcomplex of the CW-complex X. Then E is again a CW-complex with respect to the cells determined by J_n and X.*

Proof. We show this by verifying the properties in the definition 1.1.2 of a CW-complex. Properties (i) and (ii) are immediate and we already covered property (v) in lemma 1.2.4.

Let us consider property (iii) i.e. closure finiteness. So let $n \in \mathbb{N}$ and $i \in J_n$. By closure finiteness of X we know that there is a finite set $E \subseteq \bigcup_{m<n} I_n$ such that $\partial e_i^n \subseteq \bigcup_{e\in E} e$. We define $E' := \{e_j^m \in E \mid j \in J_m\}$. We want to show that $\partial e_i^n \subseteq \bigcup_{e\in E'} e$. Take $x \in \partial e_i^n$. By $\partial e_i^n \subseteq \bigcup_{e\in E} e$ there is an $e_j^m \in E$ such that $x \in e_j^m$. It is obviously enough to show that $j \in J_m$. By lemma 1.2.4 we know that $x \in \partial e_i^n \subseteq \overline{e}_i^n \subseteq E$. But since $E = \bigcup_{m'\in\mathbb{N}} \bigcup_{j'\in J_{m'}} e_{j'}^{m'}$ there is $m' \in \mathbb{N}$ and $j' \in J_{m'}$ such that $x \in e_{j'}^{m'}$. We know that the open cells of X are disjoint which gives us $(m, j) = (m', j')$. That directly implies $j \in J_m$ which we wanted to show.

Lastly we need to show property (iv), i.e. that E has weak topology. Like in a lot of our other proofs $A \subseteq E$ being closed implies that $A \cap \overline{e}_i^n$ is closed for every $n \in \mathbb{N}$ and $i \in J_n$. So now take $A \subseteq E$ such that $A \cap \overline{e}_i^n$ is closed in E for every $n \in \mathbb{N}$ and $i \in J_n$. We need to show that A is closed in E. It is enough to show that A is closed in X. We apply lemma 1.1.13 which means we only need to show that for every $n \in \mathbb{N}$ and $j \in I_n$ either $A \cap \overline{e}_j^n$ or $A \cap e_j^n$ is closed. We look at two cases. Firstly consider $j \in J_n$. Then $A \cap \overline{e}_i^n$ is closed in E by assumption. By the definition of the subspace topology this means that there exists a closed set $B \subseteq X$ such that $A \cap \overline{e}_i^n = E \cap B$. But since E is closed by assumption (i) of definition 1.2.3 of a subcomplex that means that $A \cap \overline{e}_i^n$ is the intersection of two closed sets in X making it also closed. Now let us cover the case $j \notin J_n$. This gives us $A \cap e_j^n \subseteq E \cap e_j^n = (\bigcup_{m\in\mathbb{N}} \bigcup_{i\in J_m} e_i^m) \cap e_j^n = \varnothing$ where the last equality holds since the open cells of X are pairwise disjoint. Thus $A \cap e_j^n = \varnothing$ which is obviously closed. $\square$

Now let us look at some properties of subcomplexes:

Lemma 1.2.9. *A union of subcomplexes $(E_i)_{i\in\iota}$ of X with indexing sets $(I_{i,n})_{i\in\iota,n\in\mathbb{N}}$ is again a subcomplex of X with the indexing set $\bigcup_{i\in\iota} I_{i,n}$ for every $n \in \mathbb{N}$.*

Proof. We show that this construction satisfies the assumptions of lemma 1.2.6. Property (ii) follows easily from that the fact that each of the subcomplexes E_i is the union of its open cells. So let us look at property (i). Take $n \in \mathbb{N}$ and $j \in \bigcup_{i\in\iota} I_{i,n}$. Then there is a $i \in \iota$ such that $j \in I_{i,n}$. With lemma 1.2.4 we get $\overline{e}_j^n \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{j\in I_{i,n}} \overline{e}_j^n = E_i \subseteq \bigcup_{i\in\iota} E_i$ which means we are done. $\square$

Remark 1.2.10. We say a subcomplex is finite when it is finite as a CW-complex. It is easy to see that taking a finite union of finite subcomplexes of X yields again a finite subcomplex of X.

Here are two examples of finite subcomplexes that we will need:

Example 1.2.11.

(i) Let $i \in I_0$. Then $\overline{e}_i^0$ is a finite subcomplex of X with the indexing sets $J_0 = \{i\}$ and $J_n = \varnothing$ for $n > 0$.

(ii) Let E together with the indexing sets $(J_n)_{n\in\mathbb{N}}$ be a finite subcomplex of X and $n \in \mathbb{N}$ and $i \in I_n$ such that ∂e_i^n is included in a union of cells of E of dimension less than n. Then $E \cup e_i^n$ together with $J'_n = J_n \cup \{i\}$ and $J'_m = J'_m$ for $m \neq n$ is a finite subcomplex of X.

We will omit the proofs of these examples as they are quite direct to see.
This helps us get the following lemma:

Lemma 1.2.12. *Let $n \in \mathbb{N}$ and $i \in I_n$. Then there is a finite subcomplex of X such that i is among its cells.*

Proof. We show this by strong induction over n. The base case $n = 0$ is directly given by the first example in 1.2.11. For the induction step assume that the statement is true for all $m \leq n$. We now need to show that it then also holds for $n + 1$. By closure finiteness of X there is a finite set F of cells of X with dimension less than $n+1$ such that $\partial e_i^{n+1} \subseteq \bigcup_{e\in F} \overline{e}$. By induction each cell $e \in F$ is part of a finite subcomplex E_e of X. By lemma 1.2.9 and remark 1.2.10 $\bigcup_{e\in F} E_e$ is again a finite subcomplex of X. The second example in 1.2.11 now allows us to attach the cell e_i^{n+1} to this subcomplex yielding a finite subcomplex with e_i^{n+1} among its cells. $\square$

Corollary 1.2.13. *Every finite set of cells of X is contained in a finite subcomplex of X.*

Proof. Let F be the set of finite cells. By the above lemma 1.2.12 each cell $e \in F$ is contained in a finite subcomplex E_e. By lemma 1.2.9 and remark 1.2.10 $\bigcup_{e\in F} E_e$ is again a finite subcomplex of X and we obviously have $\bigcup_{e\in F} e \subseteq \bigcup_{e\in F} E_e$. $\square$

Corollary 1.2.14. *Let $C \subseteq X$ be compact. Then C is contained in a finite subcomplex of X.*

Proof. We know from lemma 1.1.20 and property (v) in definition 1.1.2 that C is contained in a finite union of cells of X. And now the above corollary tells us that these finite cells and therefore C is contained in a finite subcomplex of X. $\square$

1.2.4. Product of CW-complexes

In this subsection we will talk about the product of CW-complexes.

1.2.4.1. Counterexample

We will first show that the product of two CW-complexes is not necessarily a CW-complex with respect to the natural cell structure.

Remark 1.2.15. The statement that we would want but is unfortunately false is the following:

Let X, Y be CW-complexes with families of characteristic maps $(Q_i^n \colon D_i^n \to X)_{n\in\mathbb{N}, i\in I_n}$ and $(P_j^m \colon D_j^m \to Y)_{m\in\mathbb{N}, j\in J_n}$. Then we would want to get a CW-structure on $X \times Y$ with characteristic maps $(Q_i^n \times P_j^m \colon D_i^n \times D_j^m \to X \times Y)_{n,m\in\mathbb{N}, i\in I_n, j\in J_m}$. The indexing sets K_l are given by $K_l = \bigcup_{n+m=l} I_n \times J_m$ for every $l \in \mathbb{N}$.

We will discuss a counterexample first presented by Dowker in [Dow52].

We firstly define the two relevant spaces:

Definition 1.2.16. Let $X = \bigvee_{i\in\iota} A_i$ where A_i is the unit interval for every $i \in \iota$ and ι is the set of all infinite sequences in $\mathbb{N}$. X has a 0-cell at the base point of the wedge sum which we will label 0_X and assume to be the 0 of all of the intervals. The rest of the 0-cells are the 1's of the intervals. The 1-cells are the interiors of the intervals.

Lemma 1.2.17. *X together with the described cell-structure is a CW-complex.*

Proof. Firstly note that the wedge sum is defined to be $\bigvee_{i\in\iota} A_i := \coprod_{i\in\iota} A_i/\sim$ where $\sim$ is the equivalence relation identifying all the 0's of the intervals. It is easy to see from the definition that the wedge sum of Hausdorff spaces is again a Hausdorff space. We now need to verify the five conditions of definition 1.1.2. They are all relatively self-evident except for condition (iv) which says that X needs to have weak topology. The forward direction follows in the same way as always. For the backward direction take a set $C \subseteq X$ such that $C \cap \overline{e}_i^n$ is closed for all the closed cells of X. Note that the only relevant information this gives us is that $C \cap A_i$ is closed in X for every $i \in \iota$. We need to show that C is closed in X. By the quotient topology C is closed in X if its preimage $q^{-1}(C)$ under the quotient map $q\colon \coprod_{i\in\iota} A_i \to \coprod_{i\in\iota} A_i/\sim$ is closed in the disjoint union. But by the disjoint union topology $q^{-1}(C)$ is closed iff $q^{-1}(C) \cap A_i = C \cap A_i$ is closed in A_i for every $i \in \iota$ which is true by assumption. $\square$

Definition 1.2.18. Let $Y = \bigvee_{j\in\mathbb{N}} B_k$ where B_k is the unit interval for every $j \in \mathbb{N}$. X has a 0-cell at the base point of the wedge sum which we will label 0_Y and assume to be the 0 of all of the intervals. The rest of the 0-cells are the 1's of the intervals. The 1-cells are the interiors of the intervals.

Lemma 1.2.19. *Y together with the described cell-structure is a CW-complex.*

Proof. Completely analogue to the proof of lemma 1.2.17. □

Lemma 1.2.20. *The space $X \times Y$ is not a CW-complex with respect to the cell-structure proposed in 1.2.15.*

Proof. We show that $X \times Y$ does not have weak topology by finding a set that by the weak topology should be closed in $X \times Y$ but is not. For $i \in \iota$ and $j \in \mathbb{N}$ we define $p_{i,j} = (1/i_j, 1/i_j) \in A_i \times B_j$ where i_j is the j'th element of the sequence i. Set $P \coloneqq \{p_{i,j} \mid i \in \iota, j \in \mathbb{N}\}$.

Let us first show that P would be closed if $X \times Y$ had weak topology. We need to show that its intersection with every closed cell of $X \times Y$ is closed. The closed cells of $X \times Y$ are the following: The 0-cells are products of 0-cells i.e. singletons of the form (x, y) where $x \in \{1_i \mid i \in \iota\} \cup \{0_X\}$ and $y \in \{1_j \mid j \in \mathbb{N}\} \cup \{0_Y\}$. The intersection of P with any closed 0-cell is empty and therefore closed. The 1-cells are product of 0-cells with 1-cells. The two different options are $A_i \times \{x\}$ with $i \in \iota$ and $x \in \{1_i \mid i \in \iota\} \cup \{0_X\}$ and $\{y\} \times B_j$ with $y \in \{1_j \mid j \in \mathbb{N}\} \cup \{0_Y\}$ and $j \in \mathbb{N}$. The intersection of P with any closed 1-cell is thus also empty and closed. So lastly let us consider the 2-cells. They are of the form $A_i \times B_j$ with $i \in \iota$ and $j \in \mathbb{N}$. For the intersection we get $P \cap (A_i \times B_j) = p_{i,j}$ which is closed. Therefore P would be closed in the weak topology.

Now we prove that P is not closed in $X \times Y$. We show that the complement P^c of P in $X \times Y$ is not open by showing that every open neighbourhood of $(0_X, 0_Y)$ contains a point of P. A base for the product topology is given by $\{U \times V \mid U \subset X$ is open in X, $V \subseteq Y$ is open in Y$\}$. It is easy to see that it suffices to prove our desired property for the base. Now lets examine what open neighbourhoods of 0_X in X look like. By the definition of the wedge sum an open neighbourhood of 0_X is of the form $\bigvee_{i\in\iota} U_i$ where U_i is an open neighbourhood of 0 in A_i for $i \in \iota$. For each of the U_i's there is an $x_i > 0$ such that $[0, x_i) \subseteq U_i$. It is therefore enough to show our claim for these sets. Arguing in the same manner for Y allows us to reduce our aim to the set $\{(\bigvee_{i\in\iota}[0, x_i)) \times (\bigvee_{j\in\mathbb{N}}[0, y_j)) \mid x_i > 0$ for all $i \in \iota, y_j > 0$ for all $j \in \mathbb{N}\}$. Picking such an open neighbourhood $(\bigvee_{i\in\iota}[0, x_i)) \times (\bigvee_{j\in\mathbb{N}}[0, y_j))$ we need to find a p in P such that p is in that neighbourhood. We pick an $i' \in \iota$ such that for every $j \in \mathbb{N}$ we have $i'_j > \max(j, 1/y_j)$. Then we pick $j' \in \mathbb{N}$ such that $j' > 1/x_{i'}$. That gives us $1/i'_{j'} < 1/j' < x_{j'}$ and $1/i'_{j'} < y_{j'}$ which means that $p_{i',j'} = (1/i'_{j'}, 1/i'_{j'}) \in [0, x_{i'}) \times [0, y_{j'}) \subseteq (\bigvee_{i\in\iota}[0, x_i)) \times (\bigvee_{j\in\mathbb{N}}[0, y_j))$. Thus P is not closed. □

1.2.4.2. K-spaces and the k-ification

Before we can move on to discuss the product of CW-complexes we need to discuss its topology. Therefore we will now study k-spaces and the k-ification.

A k-space or also called a compactly generated space is defined for our purposes as follows. Note that we mean quasi-compactness when talking about compactness.

Definition 1.2.21. Let X be a topological space. We call X a *k-space* if

$$A \subseteq X \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

There are a lot of different definitions in the literature. The most popular ones all agree on Hausdorff spaces. An overview of these different notions can be found on Wikipedia [Wik24].

It will also be helpful to characterise k-spaces via closed sets:

Lemma 1.2.22. *Let X be a topological space. X is a k-space iff*

$$A \subseteq X \textit{ is closed} \iff \textit{for all compact sets } C \subseteq X \textit{ the intersection } A \cap C \textit{ is closed in } C.$$

Proof. We only show that forward direction as the backward direction follows in the same way. Of the equivalence that we now need to show the forward direction is trivial. Thus let $A \subseteq X$ be a set such that for all compact sets $C \subseteq X$ the intersection $A \cap C$ is closed in C. It is enough to show that A^c is open. By definition of the k-space that is the case if for every compact set $C \subseteq X$ the intersection $A^c \cap C$ is open in C. Take any compact $C \subseteq X$. By assumption $A \cap C$ is closed in C. Since $A \cap C$ is the complement of $A^c \cap C$ in C, this immediately gives us that $A^c \cap C$ is open in C. ☐

We also define a way to make any topological space into a k-space which we call the k-ification:

Definition 1.2.23. Let X be a topological space. We can define another topological space X_c on the same set by setting

$$A \subseteq X_c \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

We call X_c the *k-ification* of X.

It is easy to see that this gives us a finer topology:

Lemma 1.2.24. *$A \subseteq X$ is open $\implies$ $A \subseteq X_c$ is open.*

Again it it useful to characterise the closed sets in the k-ification:

Lemma 1.2.25. *$A \subseteq X_c$ is closed $\iff$ $A \cap C$ is closed in C for all compact sets $C \subseteq X$.*

Proof. Completely analogue to the proof of lemma 1.2.22. ☐

To show that the k-ification actually fulfils its purpose of turning any space into a k-space, we first need the following lemma:

Lemma 1.2.26. *$A \subseteq X$ is compact $\iff$ $A \subseteq X_c$ is compact.*

Proof. For the backward direction notice that lemma 1.2.24 is another way of stating that the map $\mathrm{id}\colon X_c \to X$ is continuous. As the image of a compact set under a continuous map, that makes $A \subseteq X$ compact.

For the forward direction take $A \subseteq X$ compact. To show that $A \subseteq X_c$ is compact, take an open cover $(U_i)_{i\in\iota}$ of A in X_c. For every $i \in \iota$ there is by definition of the k-ification an open set $V_i \subseteq X$ such that $V_i \cap A = U_i \cap A$. $(V_i)_{i\in\iota}$ is an (open) cover of A in X:

$$A = A \cap \bigcup_{i\in\iota} U_i = \bigcup_{i\in\iota}(A \cap U_i) = \bigcup_{i\in\iota}(A \cap V_i) = A \cap \bigcup_{i\in\iota} V_i \subseteq \bigcup_{i\in\iota} V_i.$$

Thus there is a finite subcover $(V_i)_{i\in\iota'}$ of A in X. $(U_i)_{i\in\iota'}$ is now a finite subcover of A in X_c:

$$A = A \cap \bigcup_{i\in\iota'} V_i = \bigcup_{i\in\iota'}(A \cap V_i) = \bigcup_{i\in\iota'}(A \cap U_i) = A \cap \bigcup_{i\in\iota'} U_i \subseteq \bigcup_{i\in\iota'} U_i.$$

□

Now we are ready to move on to the promised lemma:

Lemma 1.2.27. *X_c is a k-space for every topological space X.*

Proof. We need to show that a set $A \subseteq X_c$ is open iff $A\cap C$ is open in C for every compact set $C \subseteq X_c$. The forward direction is again trivial.

For the backward direction take a set $A \subseteq X_c$ such that for every compact set $C \subseteq X_c$ the intersection $A\cap C$ is open in C. By the definition of the k-ification it is enough to show that for every compact set $C \subseteq X$ the intersection $A \cap C$ is open in C. So let $C \subseteq X$ be a compact set. By 1.2.26 C is also compact in X. By assumption this means that $A \cap C$ is open in $C \subseteq X_c$ (in the subspace topology of the k-ification). Thus there is an open set $B \subset X_c$ such that $A \cap C = B \cap C$. By the definition of the k-ification $B \cap C$ is open in $C \subseteq X$. That means there is an open set $E \subseteq X$ such that $B \cap C = E \cap C$. But that now gives us $A\cap C = B\cap C = E\cap C$ with which we can conclude that $A\cap C$ is open in $C \subseteq X$ (in the subspace topology of the original topology of X). □

If we already have a k-space, then the k-ification just maintains the topology of our space:

Lemma 1.2.28. *Let X be a k-space. Then the topologies of X and X_c coincide.*

Proof. Notice that the characterisation of open sets in X and X_c respectively agree in this setting. □

Corollary 1.2.29. *The k-ification is idempotent.*

Now we will characterise continuous maps to and from the k-ification. Going from the k-ification is not a big issue:

Lemma 1.2.30. *Let $f: X \to Y$ be a continuous map of topological spaces. Then $f: X_c \to Y$ is continuous.*

Proof. This follows easily from lemma 1.2.24. □

More interesting questions are when a map to the k-ification or a map from a k-ification to a k-ification is continuous. The following two lemmas and proofs that answer these questions are based on lemma 46.4 of [Mun14]. The next lemma is the first step towards the answer:

Lemma 1.2.31. *Let X be a compact space and $f: X \to Y$ be a continuous map. Then $f: X \to Y_c$ is continuous.*

Proof. We want to show that for every closed $A \subseteq Y_c$ the preimage $f^{-1}(A)$ is closed in X. Take any closed set $A \subseteq Y_c$. We know by lemma 1.2.25 that $A \cap C$ is closed in C for every compact $C \subseteq Y$. As the image of a compact set $f(X)$ is compact. Thus $A \cap f(X)$ is closed in $f(X) \subseteq Y$. By the definition of the subspace topology there is a closed set $B \subseteq Y$ such that $A \cap f(X) = B \cap f(x)$. Now we have

$$f^{-1}(A) = f^{-1}(A \cap f(X)) = f^{-1}(B \cap f(X)) = f^{-1}(B)$$

which is closed as the preimage of a closed set under a continuous map. □

Now this helps us get the following lemma:

Lemma 1.2.32. *Let $f\colon X \to Y$ be a map of topological spaces such that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y$ is continuous. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. The last lemma together with our assumption tells us that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y_c$ is continuous. To show the claim take any open $A \subseteq Y_c$. We need to show that $f^{-1}(A) \subseteq X_c$ is open. By definition of the k-ification this set is open if for all compact sets $C \subseteq X$ the intersection $f^{-1}(A) \cap C$ is open in C. Take any compact set $C \subseteq X$. As noted above we now know that $f|_C : C \to Y_c$ is continuous. Or in other words we know that for every open $B \subseteq Y_c$ there is an open set $E \subseteq X$ such that $f^{-1}(B) \cap C = E \cap C$. Applying this to the set $A \subseteq Y_c$ gives us an open set $E \subseteq X$ such that $f^{-1}(A) \cap C = E \cap C$. But that is just another way of stating that $f^{-1}(A)$ is open in $C \subseteq X$. □

That yields the following corollary:

Corollary 1.2.33. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. This situation trivially fulfils the conditions of the previous lemma. □

If you look at the discussion of the product of CW-complexes in some topology books, for example [Hat01] and [Lüc05], you will notice that the k-ification rarely gets discussed in detail. One possible reason for this is that most common spaces that you encounter are already k-spaces. Lemma 1.2.28 then allows you to ignore the k-ification entirely. We will therefore discuss in the remainder of this section which spaces are k-spaces and which are not. The first example are weakly locally compact spaces.

Definition 1.2.34. Let X be a topological space. We call X *weakly locally compact* if every point $x \in X$ has some compact neighbourhood.

This property is in some sources just called locally compact. The following proof is from Lemma 46.3 in [Mun14].

Lemma 1.2.35. *Weakly locally compact spaces are k-spaces.*

Proof. Let X be a weakly locally compact space. Let $A \subseteq X$. We need to show that A is open iff $A \cap C$ is open in C for every compact set C. The forward direction is trivial. So assume that that for every compact set C the intersection $A \cap C$ is open in C. A is open if it is a neighbourhood of every point $x \in A$. So fix any $x \in A$. Since X is weakly locally compact, x has a compact neighbourhood C. By definition of neighbourhoods there is an open set $U \subseteq C$ such that $x \in U$ and we need to find an open set $V \subseteq A$ such that $x \in V$. We show that $A \cap U$ fulfils these conditions. It is obvious that $A \cap U \subseteq A$ and $x \in A \cap U$. So it is left to show that $A \cap U$ is open. By assumption $A \cap C$ is open in C. That means that there is an open set B such that $A \cap C = B \cap C$. This now gives us

$$A \cap U = A \cap C \cap U = B \cap C \cap U = B \cap U$$

which is open as the intersection of two open sets. □

Another big class of spaces which are k-spaces are sequential spaces.

Definition 1.2.36. A set A in a topological space X is *sequentially closed* if for every convergent sequence contained in A its limit point is also in A. The *sequential closure* of a set A in X is defined as $\mathrm{scl}(A) = \{x \in X \mid \text{there is a sequence } (a_n)_{n\in\mathbb{N}} \subseteq A \text{ such that} (a_n)_{n\in\mathbb{N}} \text{ converges to } x\}$. A *sequential space* is a space in which all sequentially closed sets are closed.

We will need the following characterisation of sequentially closed sets:

Lemma 1.2.37. *A set $A \subseteq X$ is sequentially closed iff $A = scl(A)$.*

Proof. This is easy to see from the definitions. □

The following proof is based on [Sco16] and Lemma 46.3 in [Mun14].

Lemma 1.2.38. *Sequential Spaces are k-spaces.*

Proof. Let X be a Sequential Space. By lemma 1.2.22 it is enough to show that

$$A \subseteq X \text{ is closed} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is closed in } C.$$

The forward direction is trivial. Let A be a set such that $A \cap C$ is closed in C for every compact set C. Since X is a sequential space it is enough to show that A is sequentially closed or by the previous lemma $A = \mathrm{scl}(A)$. The inclusion $A \subseteq \mathrm{scl}(A)$ is obvious. For the backward inclusion take $x \in \mathrm{scl}(A)$. We need to show that $x \in A$. By definition there is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ that converges to x. It is well known (and can be shown directly from the definition of compactness) that the set $\{a_n \mid n \in \mathbb{N}\} \cup x$ is compact as the set of terms of a sequence together with the limit point of that sequence. By assumption that gives us that $A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x)$ is closed in $\{a_n \mid n \in \mathbb{N}\} \cup x$. In other words there is a closed set B such that

$$A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x).$$

With that we get

$$x \in A \iff x \in A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) \iff x \in B$$

and for all $n \in \mathbb{N}$ we get $a_n \in B$ in the exact same way. Thus $(a_n)_{n\in\mathbb{N}} \subseteq B$. Since B is in particular sequentially closed this gives us $x \in B$ which is enough by the above equivalence. □

In particular sequential spaces include metric spaces:

Lemma 1.2.39. *Metric spaces are sequential spaces.*

Proof. Let X be a metric space and A be a sequentially closed set. We need to show that that A is closed which is equivalent to A^c being open. Assume towards a contradiction that A^c is not open. Then there is a point $x \in A^c$ such that for every $n \in \mathbb{N}$ the open ball $\mathrm{B}_{1/n}(x)$ contains a point $x_n \in A$. But then we have a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ that converges to $x \in A^c$. Thus A is not sequentially closed. Contradiction. □

Corollary 1.2.40. *Metric spaces are k-spaces.*

Lastly we will discuss spaces that are not k-spaces:

Lemma 1.2.41. *Let X be an anti-compact T_1 space. Then X_c has discrete topology.*

Proof. Let $A \subseteq X_c$ be any set. We need to show that it is open. By the definition of the k-ification it is enough to show that $A \cap C$ is open in C for every compact set $C \subseteq X$. Since X is anti-compact C is finite. And by T_1 every finite set has discrete topology. Thus $A \cap C$ is open in C and X_c has discrete topology. □

Corollary 1.2.42. *Let X be a non-discrete anti-compact T_1 space. Then X is not a k-space.*

Proof. This follows easily from the previous lemma and lemma 1.2.28. □

That leads us to our first concrete example of a space that is not a k-space:

Example 1.2.43. Let X be any uncountable set. Equip X with the cocountable topology, i.e. let a set $A \subseteq X$ be open iff $A = \varnothing$ or A^c is countable. Then X is not a k-space.

Proof. It is easy to see by going through the axioms that the cocountable topology is indeed a topology. We will now show that this space satisfies the conditions of the previous corollary. X is clearly non-discrete. To see that X is a T_1 space take two distinct points a and b. Now let A be the set $X \setminus \{b\}$. This set is open since $\{b\}$ is countable and it obviously does not contain b. We lastly need to show that that X is anti-compact. To do that take any set $A \subseteq X$. Pick an (if possible infinite) countable subset $B \subseteq A$. Now for every $b \in B$ define $U_b = (X \setminus B) \cup \{b\}$. Since $U_b^c = B \setminus \{b\}$ is countable U_b is open for every $b \in B$. It is also easy to see that $A \subseteq \bigcup_{b\in B} U_b$. Thus $(U_b)_{b\in B}$ is an open cover of A. But since for every $b \in B$ there is no $b' \in B$ with $b \neq b'$ and $b \in U_{b'}$, $(U_b)_{b\in B}$ cannot have a proper subcover. Therefore A can only be compact if all these possible covers are already finite. That can only be the case if B and with that A are finite. □

Other examples can be found on π-base [PiB24].

1.2.4.3. Constructing the product

We can now move on to discuss the correct version of remark 1.2.15. For the rest of the section let X, Y be CW-complexes with families of characteristic maps $(Q_i^n\colon D_i^n \to X)_{n\in\mathbb{N},i\in I_n}$ and $(P_j^m\colon D_j^m \to Y)_{m\in\mathbb{N},j\in J_n}$. We will write the cells of X as e_i^n and the cells of Y as f_j^m. We want to show:

Theorem 1.2.44. *There is a CW-structure on $(X\times Y)_c$ with characteristic maps $(Q_i^n \times P_j^m\colon D_i^n \times D_j^m \to (X\times Y)_c)_{n,m\in\mathbb{N},i\in I_n,j\in J_m}$. The indexing sets $(K_l)_{l\in\mathbb{N}}$ are given by $K_l = \bigcup_{n+m=l} I_n \times J_m$ for every $l\in\mathbb{N}$ and the cells are therefore of the form $e_i^n \times f_j^m$ for $n,m\in\mathbb{N}$, $i\in I_n$ and $j\in J_m$.*

We will split the proof up into lemmas to have a better overview of the proof. Let us first show that the issue that occurred with the counterexample in an earlier section in lemma 1.2.20 works out here:

Lemma 1.2.45. *$(X\times Y)_c$ has weak topology, i.e. $A\subseteq (X\times Y)_c$ is closed iff $\overline{e}_i^n \times \overline{f}_j^m \cap A$ is closed for all $n,m\in\mathbb{N}$, $i\in I_n$ and $j\in J_m$.*

Proof. The forward direction follows from the fact that the product of closed sets is closed in the product topology and from lemma 1.2.24 that tells us that the k-ification is finer than the product topology.

Moving on to the backward direction we know by lemma 1.2.28 that the k-ification is a k-space and by lemma 1.2.25 that A is closed if for every compact set $C\subseteq (X\times Y)_c$ $A\cap C$ is closed in C. Take such a compact set C. The projections $\mathrm{pr}_1(C)$ and $\mathrm{pr}_2(C)$ are compact as images of a compact set. By lemma 1.1.20 there are finite sets $E\subseteq \{e_i^n \mid n\in\mathbb{N}, i\in I_n\}$ and $F\subseteq \{f_j^m \mid m\in\mathbb{N}, j\in J_m\}$ s.t $\mathrm{pr}_1(C)\subseteq \bigcup_{e\in E} e$ and $\mathrm{pr}_2(C)\subseteq \bigcup_{f\in F} f$. Thus

$$C\subseteq \mathrm{pr}_1(C)\times \mathrm{pr}_2(C) \subseteq \bigcup_{e\in E} e \times \bigcup_{f\in F} f = \bigcup_{e\in E}\bigcup_{f\in F} e\times f.$$

So C is included in a finite union of cells of $(X\times Y)_c$. Therefore

$$A\cap C = A\cap \left(\bigcup_{e\in E}\bigcup_{f\in F} e\times f\right)\cap C = \left(\bigcup_{e\in E}\bigcup_{f\in F} A\cap (e\times f)\right)\cap C$$

is closed since by assumption $A\cap(e\times f)$ is closed for every e and f and the union is finite. Thus $A\cap C$ is in particular closed in C. □

Before we can discuss closure finiteness we need to think about what the frontiers of cells look like in $(X\times Y)_c$:

Lemma 1.2.46. *Let $e_i^n\times f_j^m$ for $n,m\in\mathbb{N}$, $i\in I_n$ and $j\in J_m$. The frontier of that cell is $\partial e_i^n \times \overline{f}_j^m \cup \overline{e}_i^n \times \partial f_j^m$.*

Proof. The definition of the frontier gives us:

$$\begin{aligned}(Q_i^n\times P_j^m)(\partial D^{n+m}) &= (Q_i^n\times P_j^m)(\partial D^n\times D^m \cup D^n\times \partial D^m)\\ &= (Q_i^n\times P_j^m)(\partial D^n\times D^m)\cup (Q_i^n\times P_j^m)(D^n\times\partial D^m)\\ &= Q_i^n(\partial D^n)\times P_j^m(D^m)\cup Q_i^n(D^n)\times P_j^m(\partial D^m)\\ &= \partial e_i^n\times \overline{f}_j^m \cup \overline{e}_i^n\times \partial f_j^m.\end{aligned}$$

The equality $\partial D^{n+m} = \partial D^n \times D^m \cup D^n \times \partial D^m$ is true in any metric space and can be verified explicitly. $\square$

Using this we get closure finiteness:

Lemma 1.2.47. *$(X \times Y)_c$ has closure finiteness, i.e. each frontier of a cell is contained in a finite union of closed cells of a lower dimension.*

Proof. By the above lemma we need to verify that for all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$ the set $\partial e_i^n \times \overline{f}_j^m \cup \overline{e}_i^n \times \partial f_j^m$ is contained in a finite union of closed cells of $(X \times Y)_c$ of dimension less than $n+m$. We can show this separately for $\partial e_i^n \times \overline{f}_j^m$ and $\overline{e}_i^n \times \partial f_j^m$. We will do the proof for the the former as both proofs work in the same way. Since X fulfils closure finiteness there is a finite set E of cells of X of dimension less than n such that $\partial e_i^n \subset \bigcup_{e \in E} \overline{e}$. But that gives us $\partial e_i^n \times \overline{f}_j^m \subseteq \bigcup_{e \in E} \overline{e} \times \overline{f}_j^m$ which is a finite union of closed cells of $(X \times Y)_c$ of dimension less than $n+m$. $\square$

Now we can proof the desired theorem:

Proof of Theorem 1.2.44. It is well known and easy to see explicitly that the product of two Hausdorff spaces is again Hausdorff. Now we can go through the five conditions of definition 1.1.2.

Property (i) is given by the fact that the product of bijective maps is again bijection and continuity in both directions follows from lemma 1.2.30 and lemma 1.2.31.

For property (ii) pick any $n, m, n', m' \in \mathbb{N}$, $i \in I_n$, $j \in J_m$, $i' \in I_{n'}$ and $j' \in J_{m'}$ such that $(n+m, i, j) \neq (n'+m', i', j')$ then either $(n, i) \neq (n', i')$ or $(m, j) \neq (m', j')$. Thus

$$\overline{e}_i^n \times \overline{f}_j^m \cap \overline{e}_{i'}^{n'} \times \overline{f}_{j'}^{m'} = \left(\overline{e}_i^n \cap \overline{e}_{i'}^{n'}\right) \times \left(\overline{f}_j^m \cap \overline{f}_{j'}^{m'}\right) = \varnothing$$

since X and Y themselves fulfil property (ii).

We already covered property (iii) in lemma 1.2.47 and property (iv) in lemma 1.2.45. Property (v) is immediate. $\square$

Appendix

A. a

B. b

Symbol Index

D^n	The closed unit disk in $\mathbb{R}^n$, i.e. $D^n \coloneqq \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.
S^n	The boundary of the unit disk in $\mathbb{R}^n$, i.e. $S^n \coloneqq \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.
∂e^n	The frontier of an n-cell, i.e. $\partial e^n \coloneqq Q^n(\partial D^n)$. See definition 1.1.2.
$\overline{e}^n$	A closed n-cell, i.e. $\overline{e}^n \coloneqq Q^n(D^n)$. See definition 1.1.2.
e^n	An (open) n-cell, i.e. $e^n \coloneqq Q^n(\mathrm{int}(D^n))$. See definition 1.1.2.

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