The product of CW-complexes

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Reminder: Definition of CW-complexes

Definition: CW-complex

Let X be a Hausdorff space. A CW-structure on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of maps $(Q_i^n\colon D_i^n \to X)_{n\geq 0, i\in I_n}$ s.t.

(i) $Q_i^n|_{\mathrm{int}(D_i^n)} : \mathrm{int}(D_i^n) \to Q_i^n(\mathrm{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\mathrm{int}(D_i^n))$ an n-cell (or a cell of dimension n).

(ii) For all $m, n \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$, $Q_i^n(\mathrm{int}(D_i^n))$ and $Q_j^m(\mathrm{int}(D_j^m))$ are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n\geq 0} \bigcup_{i\in I_n} e_i^n = X$.

Lean: Definition of CW-Complexes

Lean: CW-complex

```
class CWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = closedBall 0 1
  cont (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  cont_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n
    i).target
  pairwiseDisjoint :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1
    ni.2 '' ball 0 1)
  mapsto (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
   MapsTo (map n i) (sphere 0 1) (⋃ (m < n) (j ∈ I m), map m j ''
    closedBall 0 1)
  closed (A : Set X) (asubc : A ⊆ C) : IsClosed A ↔ ∀ n j, IsClosed
    (A ∩ map n j '' closedBall 0 1)
  union : ⋃ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C
```

K-ification

Definition: k-ification

Let X be a topological space. Then we can define another topological space X_c on the same set by setting:

$$A \subseteq X_c \text{ open } \iff A \cap C \text{ open in } C \text{ for every compact set } C \subseteq X.$$

We call X_c the k-ification of X.

Lean: k-ification

```
def kification (X : Type*) := X

instance instkification {X : Type*} [t : TopologicalSpace X] :
    TopologicalSpace (kification X) where
  IsOpen A := ∀ (B : t.Compacts), ∃ (C: t.Opens), A ∩ B.1 = C.1 ∩ B.1
  isOpen_univ := ...
  isOpen_inter := ...
  isOpen_sUnion := ...
```

Product of CW-complexes

Theorem (product of CW-complexes)

Let X, Y be CW-complexes with families of characteristic maps $(Q_i^n \colon D_i^n \to X)_{n,i}$ and $(P_j^m \colon D_j^m \to Y)_{m,j}$.
Then we get a CW-structure on $(X \times Y)_c$ with characteristic maps $(Q_i^n \times P_j^m \colon D_i^n \times D_j^m \to (X \times Y)_c)_{n,m,i,j}$.

Lean: product of CW-complexes

```
def Prodkification X Y := kification (X × Y)
infixr:35 " ×ₖ "  => Prodkification

instance CWComplex_product : CWComplex (X := X ×ₖ Y) (C ×ˢ D) where
  cell n := (Σ' (m : ℕ) (l : ℕ) (hml : m + l = n), cell C m × cell D l)
  map n i := match i with
    | ⟨m, l, hmln, j, k⟩ =>
      hmln ▸ Equiv.transPartialEquiv ((IsometryEquivFinMap m
    l).symm).toEquiv
      (PartialEquiv.prod (map m j) (map l k))
```

Why don't people care about the proof?

- Answer: Most important spaces are already k-spaces.

Lemma (Examples of k-spaces)

(i) Weakly locally compact spaces are k-spaces.

(ii) Sequential spaces (e.g. first countable spaces, metric spaces) are k-spaces.

Examples of spaces that are not k-spaces

(i) Every non-discrete anti-compact T_1 space (e.g. cocountable topology on uncountable set) is not a k-space.

(ii) The product of uncountably many copies of $\mathbb{R}$ is not a k-space.

Proof of continuity

Lemma (alternate definition of k-ification)

We have

$$A \subseteq X_c \text{ closed} \iff A \cap C \text{ closed in } C \text{ for every compact set } C \subseteq X.$$

Lemma

Let X and Y be topological spaces with X compact. Let $f: X \to Y$ be a continuous. Then $f: X \to Y_c$ is continuous.

Proof of weak topology

Lemma

Let X be a CW-complex and $C \subseteq X$ a compact set. Then C is disjoint with all but finitely many cells of X.

Lemma

$(X \times Y)_c$ has weak topology i.e. $A \subseteq (X \times Y)_c$ is closed iff $Q_i^n \times P_j^m(D^{n+m}) \cap A$ is closed for all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.