Formalisation of CW-complexes

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1. Introduction

Here you have your introduction. This template is mainly based on Felix Boes Master thesis you can find it here https://github.com/felixboes/masters_thesis/tree/master

1.1. First section

Congratulations you have created your first section

1.1.1. Subsections

If you need to divide your thesis use subsections.
You can iterate this with subsub-sections if you want. If you want to do this you have to change the depth of the subsub-sections in the main.
If you want to create a subsection, which does not appear in the contents table use "subsection*"

1.2. How to cite

In this section we will discuss how to cite.
Just add your reference in masterthesis_your_name_bibliography.bib
Then use this line [Abh05] or [Dis06]

2. One

In this chapter we will learn some useful tools.

2.1. How to cross reference

In this section we will learn to cross reference.
For this just use command 1.1. You previously need to create a label.
You can also reference equations in this way

$$\langle v, \mathrm{Re} Av \rangle = \langle v, U^* DUv \rangle = \langle Uv, DUv \rangle \geq \lambda_1 \|Uv\|^2 = \lambda_1 \|v\|^2 \tag{2.1}$$

Lemma 2.1.1. *Let $A \in \mathbb{C}^{M\times M}$ be diagonally dominant then A is invertible.*

This equations can be accsessed by 2.1 and 2.1.1

2.2. How to create Index

In this section we will learn to add elements to the index.
Just use the command as in the example.

Definition 2.2.1. A vectorspace is...

2.3. How to create symbol index

In this section we will learn to add elements to the symbol index.
Just use the command as in the example.

3. Definition

In this chapter we will introduce CW-complexes and prove basic facts about them.

3.1. Definition of a CW-complexes

The modern definition of a CW-complex is the following:

Definition 3.1.1. Let X be a topological space. A CW-complex on X is a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ such that

(i) For every $n \geq 0$ there is a pushout of topological spaces

$$\begin{tikzcd} \coprod_{i \in I_n} S_i^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d, hook, "\coprod_{i \in I_n} j_i"'] & X_{n-1} \arrow[d] \\ \coprod_{i \in I_n} D_i^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"'] & X_n \end{tikzcd}$$

where I_n is any indexing set and $j_i \colon S_i^{n-1} \to D_i^n$ is the usual inclusion for every $i \in I_n$.

(ii) We have $X = \bigcup_{n \geq 0} X_n$.

(iii) X has weak topology, i.e. $A \subseteq X$ is open $\iff$ $A \cap X_n$ is open in X_n for every n.

X_n is called the n-skeleton. An element $e^n \in \pi_0(X_n \setminus X_{n-1})$ is called an (open)n-cell. Q_i^n is called a characteristic map.

In this thesis we will however focus on the historical definition of CW-complexes first presented by Whitehead in [Whi18].

Definition 3.1.2. Let X be a Hausdorff space. A CW-structure on X consists of a family of indexing sets $(I_n)_{n \in \mathbb{N}}$ and a family of maps $(Q_i^n \colon D_i^n \to X)_{n \geq 0, i \in I_n}$ s.t.

(i) $Q_i^n|_{\mathrm{int}(D_i^n)} : \mathrm{int}(D_i^n) \to Q_i^n(\mathrm{int}(D_i^n))$ is a homeomorphism. We call $e_i^n \coloneqq Q_i^n(\mathrm{int}(D_i^n))$ an (open) n-cell (or a cell of dimension n).

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n, i) \neq (m, j)$ the cells $Q_i^n(\mathrm{int}(D_i^n))$ and $Q_j^m(\mathrm{int}(D_j^m))$ are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n \geq 0} \bigcup_{i \in I_n} e_i^n = X$.

We call Q_i^n a characteristic map and $\overline{e}_i^n := Q_i^n(D_i^n)$ a closed n-cell for any i and n.

4. Product

In this chapter we will talk about the product.

4.1. K-spaces and the k-ification

Before we can move on to discuss the product of CW-complexes we need to discuss its topology. Therefore we will study k-spaces and the k-ification in this section.

A k-space or also called a compactly generated space is defined for our purposes as follows. Note that we mean quasi-compactness when talking about compactness.

Definition 4.1.1. Let X be a topological space. We call X a k-space if

$$A \subseteq X \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

There are a lot of different definitions in the literature. The most popular ones all agree on Hausdorff spaces. An overview of these different notions can be found on Wikipedia [Wik24].

It will also be helpful to characterise k-spaces via closed sets:

Lemma 4.1.2. *Let X be a topological space. X is a k-space iff*

$$A \subseteq X \textit{ is closed} \iff \textit{for all compact sets } C \subseteq X \textit{ the intersection } A \cap C \textit{ is closed in } C.$$

Proof. We only show that forward direction as the backward direction follows in the same way. Of the equivalence that we now need to show the forward direction is trivial. Thus let $A \subseteq X$ be a set such that for all compact sets $C \subseteq X$ the intersection $A \cap C$ is closed in C. It is enough to show that A^c is open. By definition of the k-space that is the case if for every compact set $C \subseteq X$ the intersection $A^c \cap C$ is open in C. Take any compact $C \subseteq X$. By assumption $A \cap C$ is closed in C. Since $A \cap C$ is the complement of $A^c \cap C$ in C, this immediately gives us that $A^c \cap C$ is open in C. □

We also define a way to make any topological space into a k-space which we call the k-ification:

Definition 4.1.3. Let X be a topological space. We can define another topological space X_c on the same set by setting

$$A \subseteq X_c \text{ is open} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is open in } C.$$

We call X_c the k-ification of X.

It is easy to see that this gives us a finer topology:

Lemma 4.1.4. *$A \subseteq X$ is open $\implies$ $A \subseteq X_c$ is open.*

Again it it useful to characterise the closed sets in the k-ification:

Lemma 4.1.5. *$A \subseteq X_c$ is closed $\iff$ $A \cap C$ is closed in C for all compact sets $C \subseteq X$.*

Proof. Completely analogue to the proof of lemma 4.1.2. ☐

To show that the k-ification actually fulfils its purpose of turning any space into a k-space, we first need the following lemma:

Lemma 4.1.6. *$A \subseteq X$ is compact $\iff$ $A \subseteq X_c$ is compact.*

Proof. For the backward direction notice that lemma 4.1.4 is another way of stating that the map $\mathrm{id}\colon X_c \to X$ is continuous. As the image of a compact set under a continuous map, that makes $A \subseteq X$ compact.

For the forward direction take $A \subseteq X$ compact. To show that $A \subseteq X_c$ is compact, take an open cover $(U_i)_{i\in\iota}$ of A in X_c. For every $i \in \iota$ there is by definition of the k-ification an open set $V_i \subseteq X$ such that $V_i \cap A = U_i \cap A$. $(V_i)_{i\in\iota}$ is an (open) cover of A in X:

$$A = A \cap \bigcup_{i\in\iota} U_i = \bigcup_{i\in\iota}(A \cap U_i) = \bigcup_{i\in\iota}(A \cap V_i) = A \cap \bigcup_{i\in\iota} V_i \subseteq \bigcup_{i\in\iota} V_i.$$

Thus there is a finite subcover $(V_i)_{i\in\iota'}$ of A in X. $(U_i)_{i\in\iota'}$ is now a finite subcover of A in X_c:

$$A = A \cap \bigcup_{i\in\iota'} V_i = \bigcup_{i\in\iota'}(A \cap V_i) = \bigcup_{i\in\iota'}(A \cap U_i) = A \cap \bigcup_{i\in\iota'} U_i \subseteq \bigcup_{i\in\iota'} U_i.$$

☐

Now we are ready to move on to the promised lemma:

Lemma 4.1.7. *X_c is a k-space for every topological space X.*

Proof. We need to show that a set $A \subseteq X_c$ is open iff $A \cap C$ is open in C for every compact set $C \subseteq X_c$. The forward direction is again trivial.

For the backward direction take a set $A \subseteq X_c$ such that for every compact set $C \subseteq X_c$ the intersection $A \cap C$ is open in C. By the definition of the k-ification it is enough to show that for every compact set $C \subseteq X$ the intersection $A \cap C$ is open in C. So let $C \subseteq X$ be a compact set. By 4.1.6 C is also compact in X. By assumption this means that $A \cap C$ is open in $C \subseteq X_c$ (in the subspace topology of the k-ification). Thus there is an open set $B \subset X_c$ such that $A \cap C = B \cap C$. By the definition of the k-ification $B \cap C$ is open in $C \subseteq X$. That means there is an open set $E \subseteq X$ such that $B \cap C = E \cap C$. But that now gives us $A \cap C = B \cap C = E \cap C$ with which we can conclude that $A \cap C$ is open in $C \subseteq X$ (in the subspace topology of the original topology of X). ☐

If we already have a k-space, then the k-ification just maintains the topology of our space:

Lemma 4.1.8. *Let X be a k-space. Then the topologies of X and X_c coincide.*

Proof. Notice that the characterisation of open sets in X and X_c respectively agree in this setting. ☐

Corollary 4.1.9. *The k-ification is idempotent.*

Now we will characterise continuous maps to and from the k-ification. Going from the k-ification is not a big issue:

Lemma 4.1.10. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y$ is continuous.*

Proof. This follows easily from lemma 4.1.4. □

More interesting questions are when a map to the k-ification or a map from a k-ification to a k-ification is continuous. The following two lemmas and proofs that answer these questions are based on lemma 46.4 of [Mun14]. The next lemma is the first step towards the answer:

Lemma 4.1.11. *Let X be a compact space and $f\colon X \to Y$ be a continuous map. Then $f\colon X \to Y_c$ is continuous.*

Proof. We want to show that for every closed $A \subseteq Y_c$ the preimage $f^{-1}(A)$ is closed in X. Take any closed set $A \subseteq Y_c$. We know by lemma 4.1.5 that $A \cap C$ is closed in C for every compact $C \subseteq Y$. As the image of a compact set $f(X)$ is compact. Thus $A \cap f(X)$ is closed in $f(X) \subseteq Y$. By the definition of the subspace topology there is a closed set $B \subseteq Y$ such that $A \cap f(X) = B \cap f(x)$. Now we have

$$f^{-1}(A) = f^{-1}(A \cap f(X)) = f^{-1}(B \cap f(X)) = f^{-1}(B)$$

which is closed as the preimage of a closed set under a continuous map. □

Now this helps us get the following lemma:

Lemma 4.1.12. *Let $f\colon X \to Y$ be a map of topological spaces such that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y$ is continuous. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. The last lemma together with our assumption tells us that for every compact $C \subseteq X$ the restriction $f|_C : C \to Y_c$ is continuous. To show the claim take any open $A \subseteq Y_c$. We need to show that $f^{-1}(A) \subseteq X_c$ is open. By definition of the k-ification this set is open if for all compact sets $C \subseteq X$ the intersection $f^{-1}(A) \cap C$ is open in C. Take any compact set $C \subseteq X$. As noted above we now know that $f|_C : C \to Y_c$ is continuous. Or in other words we know that for every open $B \subseteq Y_c$ there is an open set $E \subseteq X$ such that $f^{-1}(B) \cap C = E \cap C$. Applying this to the set $A \subseteq Y_c$ gives us an open set $E \subseteq X$ such that $f^{-1}(A) \cap C = E \cap C$. But that is just another way of stating that $f^{-1}(A)$ is open in $C \subseteq X$. □

That yields the following corollary:

Corollary 4.1.13. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f\colon X_c \to Y_c$ is continuous.*

Proof. This situation trivially fulfils the conditions of the previous lemma. □

If you look at the discussion of the product of CW-complexes in some topology books, for example [Hat01] and [Lüc05], you will notice that the k-ification rarely gets discussed in detail. One possible reason for this is that most common spaces that you encounter are already k-spaces. Lemma 4.1.8 then allows you to ignore the k-ification entirely. We will therefore discuss in the remainder of this section which spaces are k-spaces and which are not. The first example are weakly locally compact spaces.

Definition 4.1.14. Let X be a topological space. We call X weakly locally compact if every point $x \in X$ has some compact neighbourhood.

This property is in some sources just called locally compact. The following proof is from Lemma 46.3 in [Mun14].

Lemma 4.1.15. *Weakly locally compact spaces are k-spaces.*

Proof. Let X be a weakly locally compact space. Let $A \subseteq X$. We need to show that A is open iff $A \cap C$ is open in C for every compact set C. The forward direction is trivial. So assume that that for every compact set C the intersection $A \cap C$ is open in C. A is open if it is a neighbourhood of every point $x \in A$. So fix any $x \in A$. Since X is weakly locally compact, x has a compact neighbourhood C. By definition of neighbourhoods there is an open set $U \subseteq C$ such that $x \in U$ and we need to find an open set $V \subseteq A$ such that $x \in V$. We show that $A \cap U$ fulfils these conditions. It is obvious that $A \cap U \subseteq A$ and $x \in A \cap U$. So it is left to show that $A \cap U$ is open. By assumption $A \cap C$ is open in C. That means that there is an open set B such that $A \cap C = B \cap C$. This now gives us

$$A \cap U = A \cap C \cap U = B \cap C \cap U = B \cap U$$

which is open as the intersection of two open sets. □

Another big class of spaces which are k-spaces are sequential spaces.

Definition 4.1.16. A set A in a topological space X is sequentially closed if for every convergent sequence contained in A its limit point is also in A. The sequential closure of a set A in X is defined as $\mathrm{scl}(A) = \{x \in X \mid \text{there is a sequence } (a_n)_{n\in\mathbb{N}} \subseteq A \text{ such that} (a_n)_{n\in\mathbb{N}} \text{ converges to } x\}$. A sequential space is a space in which all sequentially closed sets are closed.

We will need the following characterisation of sequentially closed sets:

Lemma 4.1.17. *A set $A \subseteq X$ is sequentially closed iff $A = scl(A)$.*

Proof. This is easy to see from the definitions. □

The following proof is based on [Sco16] and Lemma 46.3 in [Mun14].

Lemma 4.1.18. *Sequential Spaces are k-spaces.*

Proof. Let X be a Sequential Space. By lemma 4.1.2 it is enough to show that

$$A \subseteq X \text{ is closed} \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is closed in } C.$$

The forward direction is trivial. Let A be a set such that $A \cap C$ is closed in C for every compact set C. Since X is a sequential space it is enough to show that A is sequentially closed or by the previous lemma $A = \mathrm{scl}(A)$. The inclusion $A \subseteq \mathrm{scl}(A)$ is obvious. For the backward inclusion take $x \in \mathrm{scl}(A)$. We need to show that $x \in A$. By definition there is a sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$ that converges to x. It is well known (and can be shown directly from the definition of compactness) that the set $\{a_n \mid n \in \mathbb{N}\} \cup x$ is compact as the set of terms of a sequence together with the limit point of that sequence. By assumption that gives us that $A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x)$ is closed in $\{a_n \mid n \in \mathbb{N}\} \cup x$. In other words there is a closed set B such that

$$A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x).$$

With that we get

$$x \in A \iff x \in A \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) = B \cap (\{a_n \mid n \in \mathbb{N}\} \cup x) \iff x \in B$$

and for all $n \in \mathbb{N}$ we get $a_n \in B$ in the exact same way. Thus $(a_n)_{n\in\mathbb{N}} \subseteq B$. Since B is in particular sequentially closed this gives us $x \in B$ which is enough by the above equivalence. ☐

In particular sequential spaces include metric spaces:

Lemma 4.1.19. *Metric spaces are sequential spaces.*

Proof. Let X be a metric space and A be a sequentially closed set. We need to show that that A is closed which is equivalent to A^c being open. Assume towards a contradiction that A^c is not open. Then there is a point $x \in A^c$ such that for every $n \in \mathbb{N}$ the open ball $\mathrm{B}_{1/n}(x)$ contains a point $x_n \in A$. But then we have a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ that converges to $x \in A^c$. Thus A is not sequentially closed. Contradiction. ☐

Corollary 4.1.20. *Metric spaces are k-spaces.*

Lastly we will discuss spaces that are not k-spaces:

Lemma 4.1.21. *Let X be an anti-compact T_1 space. Then X_c has discrete topology.*

Proof. Let $A \subseteq X_c$ be any set. We need to show that it is open. By the definition of the k-ification it is enough to show that $A \cap C$ is open in C for every compact set $C \subseteq X$. Since X is anti-compact C is finite. And by T_1 every finite set has discrete topology. Thus $A \cap C$ is open in C and X_c has discrete topology. ☐

Corollary 4.1.22. *Let X be a non-discrete anti-compact T_1 space. Then X is not a k-space.*

Proof. This follows easily from the previous lemma and lemma 4.1.8. ☐

That leads us to our first concrete example of a space that is not a k-space:

Example 4.1.23. Let X be any uncountable set. Equip X with the cocountable topology, i.e. let a set $A \subseteq X$ be open iff $A = \varnothing$ or A^c is countable. Then X is not a k-space.

Proof. It is easy to see by going through the axioms that the cocountable topology is indeed a topology. We will now show that this space satisfies the conditions of the previous corollary. X is clearly non-discrete. To see that X is a T_1 space take two distinct points a and b. Now let A be the set $X \setminus \{b\}$. This set is open since $\{b\}$ is countable and it obviously does not contain b. We lastly need to show that that X is anti-compact. To do that take any set $A \subseteq X$. Pick an (if possible infinite) countable subset $B \subseteq A$. Now for every $b \in B$ define $U_b = (X \setminus B) \cup \{b\}$. Since $U_b^c = B \setminus \{b\}$ is countable U_b is open for every $b \in B$. It is also easy to see that $A \subseteq \bigcup_{b \in B} U_b$. Thus $(U_b)_{b \in B}$ is an open cover of A. But since for every $b \in B$ there is no $b' \in B$ with $b \neq b'$ and $b \in U_{b'}$, $(U_b)_{b \in B}$ cannot have a proper subcover. Therefore A can only be compact if all these possible covers are already finite. That can only be the case if B and with that A are finite. □

Other examples can be found on π-base [PiB24].

4.2. The product of CW-complexes

We can now move on to discuss the product. We want to proof the following theorem:

Theorem 4.2.1.

Lemma 4.2.2. *$(X \times Y)_c$ has weak topology, i.e. $A \subseteq (X \times Y)_c$ is closed iff $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed for all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.*

Proof.

"⇒" Since D^{n+m} is compact, its image is compact and therefore closed. As the intersection of two closed sets $(Q_i^n \times P_j^m)(D^{n+m}) \cap A$ is closed as well.

"⇐" We know by definition of the k-ification that A is closed if for every compact set $C \subseteq X \times Y$ $A \cap C$ is closed in C. Take such a compact set C. The projections $\mathrm{pr}_1(C)$ and $\mathrm{pr}_2(C)$ are compact as images of a compact set. By ? there are finite sets $E \subseteq \{e_i^n \mid n \in \mathbb{N}, i \in I_n\}$ and $F \subseteq \{f_j^m \mid m \in \mathbb{N}, j \in J_m\}$ s.t $\mathrm{pr}_1(C) \subseteq \bigcup_{e \in E} e$ and $\mathrm{pr}_2(C) \subseteq \bigcup_{f \in F} f$. Thus

$$C \subseteq \mathrm{pr}_1(C) \times \mathrm{pr}_2(C) \subseteq \bigcup_{e \in E} e \times \bigcup_{f \in F} f = \bigcup_{e \in E} \bigcup_{f \in F} e \times f.$$

So C is included in a finite union of cells of $(X \times Y)_c$. Therefore

$$A \cap C = A \cap \left(\bigcup_{e \in E} \bigcup_{f \in F} e \times f \right) \cap C = \left(\bigcup_{e \in E} \bigcup_{f \in F} A \cap (e \times f) \right) \cap C$$

is closed since by assumption $A \cap (e \times f)$ is closed for every e and f and the union is finite. Thus $A \cap C$ is in particular closed in C. □

Appendix

A. a

B. b

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