CS201: Discrete Mathematics (Fall 2024)
Written Assignment #1 - Solutions
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Oct 11 (please submit to Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**5p**) Consider the following propositions:

p : You get an A on the final.

q : You do all the assignments.

r : You get an A in this course.

Translate the following statements to formulas using p, q, r and logical connectives.

(a) (**1p**) You get an A either in this course or on the final.

(b) (**1p**) To get an A in this course, it is necessary for you to do all the assignments.

(c) (**1p**) You do all the assignments, but you don't get an A on the final; nevertheless, you get an A in this course.

(d) (**1p**) If you don't get an A in this course, then you don't get an A on the final or don't do all the assignments.

(e) (**1p**) You get an A in this course if and only if you do all the assignments and get an A on the final.

Solution:

(a) $r \oplus p$

(b) $r \rightarrow q$

(c) $q \wedge \neg p \wedge r$

(d) $\neg r \rightarrow (\neg p \vee \neg q)$

(e) $r \leftrightarrow (q \wedge p)$

□

Q.2 (**10p**) Construct a truth table for each of the following compound propositions:

(a) (**1p**) $p \oplus \neg p$

(b) (**2p**) $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

(c) (**2p**) $(p \oplus q) \rightarrow (p \vee \neg q)$

(d) (**5p**) $(p \rightarrow \neg q) \leftrightarrow (r \rightarrow (p \vee \neg q))$

Solution:

(a)

p	$\neg p$	$(p \oplus \neg p)$
T	F	T
F	T	T

(b)

p	q	$(p \to q) \wedge (\neg p \leftrightarrow q)$
T	T	F
T	F	F
F	T	T
F	F	F

(c)

p	q	$(p \oplus q) \to (p \vee \neg q)$
T	T	T
T	F	T
F	T	F
F	F	T

(d)

p	q	r	$p \to \neg q$	$(p \vee \neg q)$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

□

Q.3 (**15p**) Use logical equivalences to prove the following statements. Please write out the names of laws used at each step (see the lecture slides for examples).

(a) (**4p**) $\neg(p \to q) \to p$ is a tautology.

(b) (**4p**) $(p \wedge \neg q) \to r$ and $p \to (q \vee r)$ are equivalent.

(c) (**7p**) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
& \neg(p \to q) \to p \\
\equiv\ & \neg\neg(p \to q) \vee p \quad \text{Useful} \\
\equiv\ & (p \to q) \vee p \quad \text{Double negation} \\
\equiv\ & (\neg p \vee q) \vee p \quad \text{Useful} \\
\equiv\ & (\neg p \vee p) \vee q \quad \text{Commutative} \\
\equiv\ & T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(b) We have

$$\begin{aligned}
&(p \wedge \neg q) \to r \\
&\equiv \neg(p \wedge \neg q) \vee r \quad \text{Useful} \\
&\equiv (\neg p \vee q) \vee r \quad \text{De Morgan's} \\
&\equiv \neg p \vee (q \vee r) \quad \text{Associative} \\
&\equiv p \to (q \vee r) \quad \text{Useful}
\end{aligned}$$

Therefore, they are equivalent.

(c) We have

$$\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan's} \\
&\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Negation} \\
&\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv T \vee \neg r \quad \text{Negation} \\
&\equiv T \quad \text{Identity}
\end{aligned}$$

Therefore, it is a tautology.

□

Q.4 (**10p**) Determine whether or not the following pairs of statements are logically equivalent, and explain your answer. (Truth tables are not necessary if your explanation is clear.)

(a) (**2p**) $p \oplus q$ and $\neg p \vee \neg q$

(b) (**2p**) $\neg q \wedge (p \leftrightarrow q)$ and $\neg p$

(c) (**3p**) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(d) (**3p**) $(p \to q) \to r$ and $p \to (q \to r)$

Solution:

(a) The combined truth table is:

p	q	$p \oplus q$	$\neg p \vee \neg q$
F	F	F	T
F	T	T	T
T	F	T	T
T	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(b) The combined truth table is:

p	q	$\neg q$	$p \leftrightarrow q$	$\neg q \wedge (p \leftrightarrow q)$	$\neg p$
F	F	T	T	T	T
F	T	F	F	F	T
T	F	T	F	F	F
T	T	F	T	F	F

By comparing the last two columns, we have that they are not equivalent.

(c) The second statement is false only when p is true and $q \vee r$ is false, which means both q and r are false.

The first statement if false only when both $p \to q$ and $p \to r$ are false. This only happens when p is true, and both q and r are false.

Thus, these two statements are logically equivalent.

(d) These two statements are not logically equivalent. It suffices to give a counterexample. When p, q and r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

□

Q.5 (**5p**) Determine for which values of p, q, r the statement $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true and for which values it is false, and explain your reasoning. Do not use a truth table.

Solution: The statement is false when p, q, r have the same truth value and is true otherwise. The explanation is as follows. The first clause is true if and only if at least one of p, q and r is true. The second clause is true if and only if at least one of the three variables is false. Therefore the entire statement is true if and only if there is at least one T and one F among the truth values of the variables, in other words, that they don't all have the same truth value.

□

Q.6 (**5p**) Prove that if $p \to q$, $\neg p \to \neg r$, $s \vee r$, then $q \vee s$. Please write out the names of inference rules used at each step (see the lecture slides for examples).

Solution:

Step	**Reason**
(1) $p \to q$	Premise
(2) $\neg q \to \neg p$	Contrapositive of (1)
(3) $\neg p \to \neg r$	Premise
(4) $\neg q \to \neg r$	Hypothetical syllogism from (2) (3)
(5) $q \vee \neg r$	Useful of (4)
(6) $s \vee r$	Premise
(7) $q \vee s$	Resolution from (5) (6)

□

Q.7 (**7p**) Prove that if $p \wedge q$, $q \to \neg(p \wedge r)$, $s \to r$, then $\neg s$. Please write out the names of inference rules used at each step (see the lecture slides for examples).

Solution:

Step	**Reason**
(1) $p \wedge q$	Premise
(2) q	Simplication of (1)
(3) $q \to \neg(p \wedge r)$	Premise
(4) $\neg(p \wedge r)$	Modens ponens from (2) (3)
(5) $\neg p \vee \neg r$	De Morgan's of (4)
(6) p	Simplication of (1)
(7) $\neg r$	Disjunctive syllogism from (5) (6)
(8) $s \to r$	Premise
(9) $\neg s$	Modus tollens from (7) (8)

□

Q.8 (**5p**) Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at SUSTech.

(a) (**1p**) There is a student who either can speak Russian or knows C++.

(b) (**1p**) There is a student who can speak Russian but who doesn't know C++.

(c) (**1p**) Every student can speak Russian and knows C++.

(d) (**1p**) No student can speak Russian or knows C++.

(e) (**1p**) If a student can speak Russian then he/she does not know C++.

Solution:

(a) $\exists x(P(x) \oplus Q(x))$

(b) $\exists x(P(x) \wedge \neg Q(x))$

(c) $\forall x(P(x) \wedge Q(x))$

(d) $\forall x \neg(P(x) \vee Q(x))$

(e) $\forall x(P(x) \to \neg Q(x))$

□

Q.9 (**8p**) Let $L(x, y)$ be the statement "x loves y", where the domain for both x and y consists of all people in the world. Translate the following statements to quantified formulas. (Hint: you can use $=, \neq$ to connect variables.)

(a) (**1p**) Everybody loves somebody.

(b) (**2p**) There is someone who loves only himself or herself but no other person.

(c) (**5p**) There are exactly two people whom Lynn loves.

Solution:

(a) $\forall x \exists y\ L(x, y)$

(b) $\exists x \forall y(L(x,y) \leftrightarrow x = y)$ **or** $\exists x(L(x,x) \land \forall y(L(x,y) \rightarrow x = y))$

(c) $\exists x \exists y(x \neq y \land L(Lynn,x) \land L(Lynn,y) \land (\forall z(L(Lynn,z) \rightarrow (z = x \lor z = y)))$ **or** $\exists x \exists y(x \neq y \land (\forall z(L(Lynn,z) \leftrightarrow (z = x \lor z = y)))$

$\square$

Q.10 (**8p**) Express the negations of each of the following statements such that all negation symbols *immediately precede* predicates.

(a) (**2p**) $\exists z \forall y \forall x T(x,y,z)$

(b) (**3p**) $\exists x \exists y P(x,y) \land \forall x \forall y Q(x,y)$

(c) (**3p**) $\forall x \exists y(P(x,y) \rightarrow Q(x,y))$

Solution:

(a)

$$\begin{aligned}
\neg \exists z \forall y \forall x T(x,y,z) &\equiv \forall z \neg \forall y \forall x T(x,y,z) \\
&\equiv \forall z \exists y \neg \forall x T(x,y,z) \\
&\equiv \forall z \exists y \exists x \neg T(x,y,z)
\end{aligned}$$

(b)

$$\begin{aligned}
\neg(\exists x \exists y P(x,y) \land \forall x \forall y Q(x,y)) &\equiv \neg \exists x \exists y P(x,y) \lor \neg \forall x \forall y Q(x,y) \\
&\equiv \forall x \neg \exists y P(x,y) \lor \exists x \neg \forall y Q(x,y) \\
&\equiv \forall x \forall y \neg P(x,y) \lor \exists x \exists y \neg Q(x,y)
\end{aligned}$$

(c)

$$\begin{aligned}
\neg \forall x \exists y(P(x,y) \rightarrow Q(x,y)) &\equiv \exists x \neg \exists y(P(x,y) \rightarrow Q(x,y)) \\
&\equiv \exists x \forall y \neg(\neg P(x,y) \lor Q(x,y)) \\
&\equiv \exists x \forall y(P(x,y) \land \neg Q(x,y)).
\end{aligned}$$

$\square$

Q.11 (**10p**) Consider this argument: "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners." Answer the following questions.

(a) (**3p**) Define the predicates and translate each sentence to a quantified formula.

(b) (**7p**) Show the formal proof steps and explain which rule of inference is used for each step.

Solution:

(a) Let $s(x)$ be "x is a movie produced by Sayles," let $c(x)$ be "x is a movie about coal miners", and let $w(x)$ be "movie x is wonderful", where the universe for x is all movies. We are given premises $\forall x(s(x) \rightarrow w(x))$ and $\exists x(s(x) \land c(x))$, and we want to conclude $\exists x(c(x) \land w(x))$.

(b) The proof steps and rules of inferences are shown as follows:

Step	Reason
(1) $\exists x(s(x) \wedge c(x))$	Premise
(2) $s(y) \wedge c(y)$	Existential instantiation using (1)
(3) $s(y)$	Simplification of (2)
(4) $\forall x(s(x) \rightarrow w(x))$	Premise
(5) $s(y) \rightarrow w(y)$	Universal instantiation of (4)
(6) $w(y)$	Modus ponens using (3) and (5)
(7) $c(y)$	Simplification of (2)
(8) $w(y) \wedge c(y)$	Conjunction using (6) and (7)
(9) $\exists x(c(x) \wedge w(x))$	Existential generalization using (8)

□

Q.12 (**6p**) Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

□

Q.13 (**6p**) Prove that there is an irrational number between every two distinct rational numbers. (Hint: you can use the in-class learned fact that $\sqrt{2}$ is irrational.)

Solution: By finding a common denominator, we can assume the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \leq c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, which is wrong.

□

Q.14 (**10p**) Please read the following description carefully and answer the questions. In an auction, an auctioneer is responsible for selling a product, and bidders bid for the product. The winner of the auction gets the product and pays for it. We consider the so-called *second-price sealed-bid auction* as follows:

- There is one product to be sold.
- There are N bidders denoted by $B_1, B_2, \ldots, B_N$. Each bidder B_n ($n = 1, 2, \ldots, N$) has a valuation of v_n over the product.
- Every bidder submits his or her bid in a sealed envelope, so other bidders do not know his or her bid. Each bidder B_n submits a bid of b_n, which may or may not equal to v_n.
- After receiving the bids from all bidders, the auctioneer announces the winner and payment. The winner is the bidder who submits the highest bid. The payment of this winner is the second highest bid. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 4$, $b_3 = 5$. Then, the winner is bidder 3, and the payment is the second highest bid 4.

- If multiple bidders have the same bid, then they draw a lottery. Each of them has equally probability of winning. In this case, the payment is equal to their bids. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 5$, $b_3 = 5$. The winner is either 2 or 3 with equal probability. The payment is 5.
- After the auction, the payoffs of the bidders are as follows:
  - If bidder B_n wins, his or her payoff is equal to its valuation v_n minus the payment.
  - If bidder B_n loses, his or her payoff is 0. (Think why?)

Now, suppose you are a bidder in this auction, e.g., B_n, and you do not know other bidders' valuations and bids. You have your valuation v_n over the product and your goal is to choose your bid b_n to maximize your payoff. Prove that submitting a bid $b_n = v_n$ is your best strategy, i.e., it will always lead to a payoff that is greater than or equal to payoffs in other cases where $b_n \neq v_n$.

Note: the above second-price auction is widely used, due to the property that bidders are willing to submit their valuations as their bids.

(Hint: use proof by cases; consider the highest bid of the others, and compare it with your valuation v_n; enumerate all possibilities.)

Solution: Let b^* be the highest bid of all bidders except bidder B_n's bid b_n. Note that if B_n wins, then b^* is his or her payment.

We prove the statement by cases. There are three cases:

- $v_n < b^*$:
  - Submitting $b_n = v_n$: loses; payoff is zero.
  - Submitting $b_n < b^*$: loses; payoff is zero.
  - Submitting $b_n = b^*$: may win; if wins, payoff is negative, as $v_n - b^* < 0$; if loses, payoff is zero.
  - Submitting $b_n > b^*$: wins; payoff is negative, as $v_n - b^* < 0$.
- $v_n = b^*$:
  - Submitting $b_n = v_n$: may win; if wins, payoff is $v_n - b^* = 0$; if loses, payoff is zero.
  - Submitting $b_n > v_n$: wins; payoff is $v_n - b^* = 0$.
  - Submitting $b_n < v_n$: loses; payoff is zero.
- $v_n > b^*$:
  - Submitting $b_n = v_n$: wins; payoff is $v_n - b^* > 0$.
  - Submitting $b_n > b^*$: wins; payoff is still $v_n - b^* > 0$, same as above.
  - Submitting $b_n = b^*$: may win; if wins, payoff is $v_n - b^* > 0$, same as above; if loses, payoff is zero.
  - Submitting $b_n < b^*$: loses; payoff is zero.

For all these three cases, submitting bid $b_n = v_n$ will always lead to a payoff that is no smaller than the payoffs when submitting bid $b_n \neq v_n$.

□