

07 Counting

CS201 Discrete Mathematics

Instructor: Shan Chen

Counting

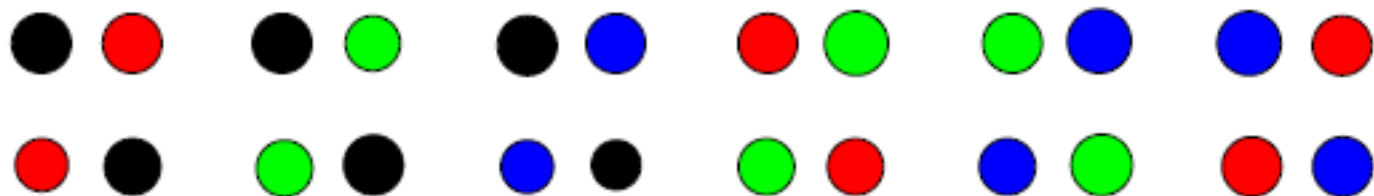
- **Combinatorics** is the study of arrangements of objects. It is an important part of discrete mathematics. The **counting of objects with certain properties** is an important part of combinatorics.

- Example: ● ● ● ●

- How many different ways to choose 2 balls out of 4 colored balls?



- What about when **order matters**?



Counting

- **Combinatorics** is the study of arrangements of objects. It is an important part of discrete mathematics. The **counting of objects with certain properties** is an important part of combinatorics.
- More examples:
 - the number of steps in a computer program
 - the number of passwords between **6 ~ 10** characters
 - the number of telephone numbers with **8** digits
- Counting may be **very hard** and not trivial.
 - usually can be simplified by **decomposing the problem**

The Counting Basics

The Sum Rule

- A count decomposes into a **set** of **independent** counts.
 - Elements of different counts are **alternatives**.
- Example: To travel from city A to B , you may **either** fly, **or** take a train, **or** take a bus. There are **12** different flights, **5** different trains and **10** different buses.
 - How many options do you have to travel from A to B ?
 $12 + 5 + 10 = 27$
- **The Sum Rule:** If a count of elements can be broken down into a **set of independent counts**, where the first count yields n_1 elements, the 2nd n_2 elements, and k -th n_k elements, then the total number of elements is $n = n_1 + n_2 + \cdots + n_k$.

The Product Rule

- A count decomposes into a **sequence** of **dependent** counts.
 - Each element in one count is **associated with all elements** of the next count.
- Example: In an auditorium, the seats are labeled by a letter **and** numbers between **1** and **50** (e.g., **A23**).
 - What is the total number of seats?
 $26 \times 50 = 1300$
- **The Product Rule:** If a count of elements can be broken down into a **sequence of dependent counts**, where the first count yields n_1 elements, the 2nd n_2 elements, and k -th n_k elements, then the total number of elements is $n = n_1 \times n_2 \times \cdots \times n_k$.

Other Rules

- **The Subtraction Rule:** If a task can be done in n_1 or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two ways.
 - E.g., $|A \cup B| = |A| + |B| - |A \cap B|$
- **The Division Rule:** If a task can be done using a procedure that can be carried out in n “fine-grained” ways, and every “giant” way w corresponds to exactly d of the n “fine-grained” ways, then there are n/d “giant” ways to do it.
 - E.g., how many kilobytes in one gigabyte? $10^9 / 10^3 = 10^6$

More Complex Counting

- Typically, a counting problem requires a **combination** of more than one rule.
- Example: Each password is **6 to 8 characters** long, where each character is a lowercase letter or a digit. Each password must contain **at least one digit**.
 - How many possible passwords are there?

$$P = P_6 + P_7 + P_8$$

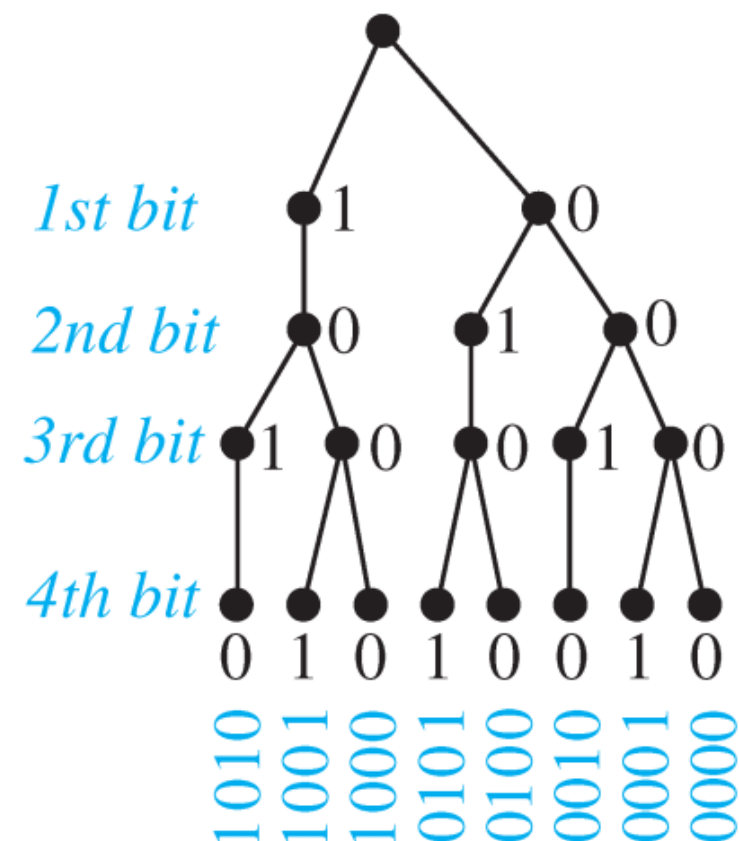
$$P_6 = 36^6 - 26^6$$

$$P_7 = 36^7 - 26^7$$

$$P_8 = 36^8 - 26^8$$

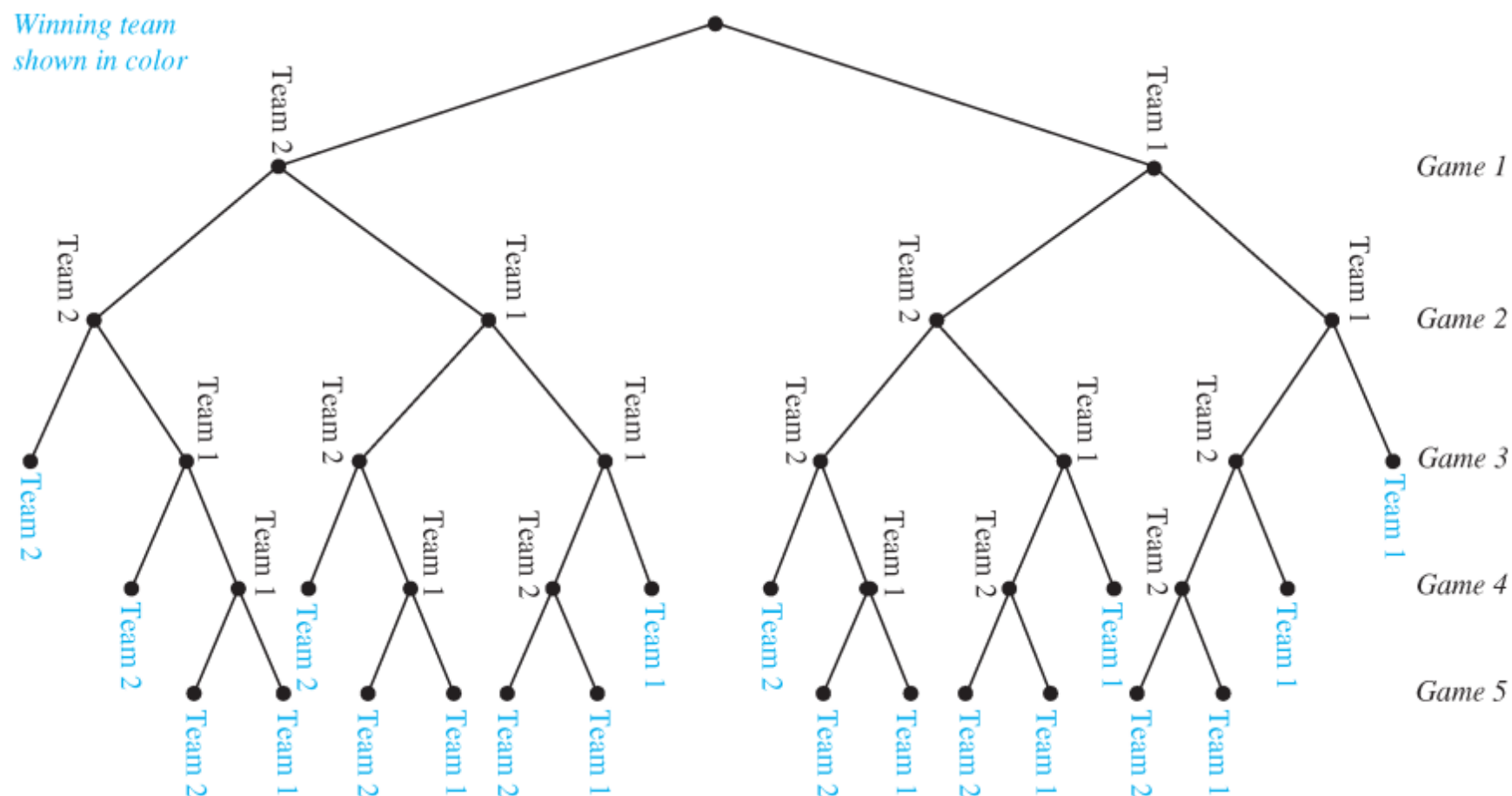
Tree Diagrams

- A **tree** is a structure that consists of a **root**, **branches** and **leaves**.
 - It can represent a counting problem and record the choices we made for alternatives, with the **possible outcomes on the leaves**.
- Example: What is the number of bit strings of length 4 that **do not have consecutive 1s**?



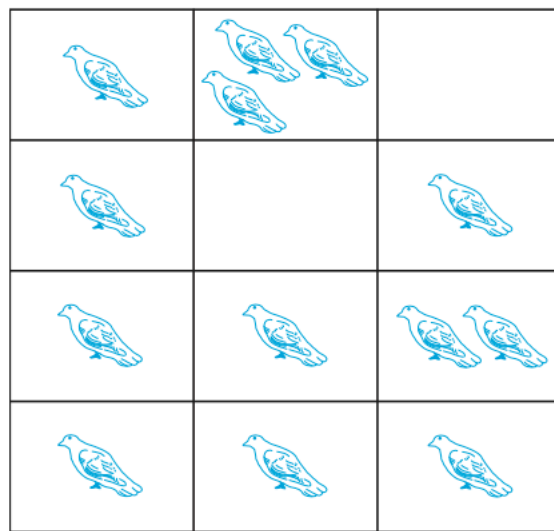
Tree Diagrams

- A **tree** is a structure that consists of a **root**, **branches** and **leaves**.
 - It can represent a counting problem and record the choices we made for alternatives, with the **possible outcomes on the leaves**.
- Example: The first team that **wins 3 out of 5 games** wins the playoff. In how many different ways can the playoff occur?

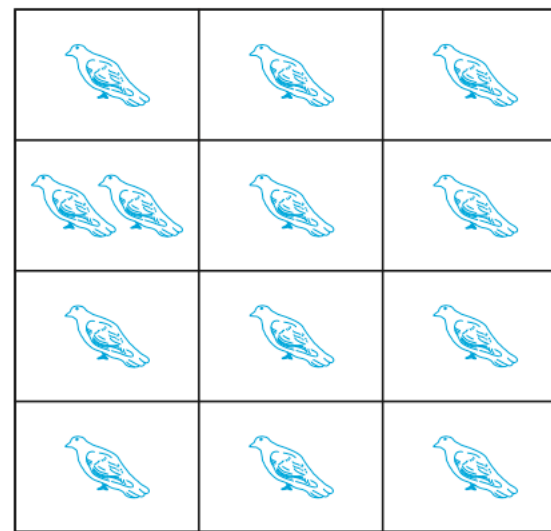


The Pigeonhole Principle

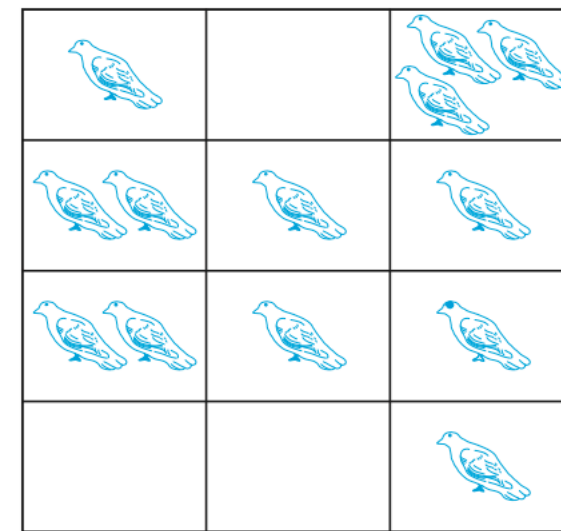
- If a flock of **13 pigeons** flies into a set of **12 pigeonholes** to roost, then **at least one** pigeonhole must have **at least two** pigeons in it.



(a)



(b)



(c)

- **The Pigeonhole Principle:** If k is a positive integer and **$k + 1$ or more objects** are placed into **k boxes**, then there is **at least one** box containing **two or more** of the objects.
 - proof by contradiction

The Generalized Pigeonhole Principle

- **The Generalized Pigeonhole Principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
 - proof by contradiction
- Example:
 - We had 99 registered students in the beginning of this semester. At least how many of them were born in the same month?
 $\lceil 99/12 \rceil = 9$
 - Now we have 96 students left. What about now?
 $\lceil 96/12 \rceil = 8$

Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange or select some distinct elements from a set.
- A **permutation** of a set of distinct objects is an **ordered arrangement** of these objects.
 - E.g., in how many ways can we select three students from a group of five students and arrange them to stand in line for a picture?
- A **combination** of a set of distinct objects is an **unordered selection** of these objects.
 - E.g., how many different committees of three students can be formed from a group of five students?

r -Permutations and r -Combinations

- An **ordered arrangement** of r distinct elements from a set of size n is called a **r -permutation**.
 - An n -permutation of a set of size n is simply called a **permutation**.
- Example: What are the 3-permutations of $\{1, 2, 3, 4\}$?
 - $L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}$
 - This type of “dictionary” ordering (where we treat numbers as letters) is called a **lexicographic ordering** and is used quite often.
- An **unordered selection** of r distinct elements from a set of size n is called a **r -combination**.
- Example: what are the 3-combinations of $\{1, 2, 3, 4\}$?
 - $L = \{123, 124, 134, 234\}$

Counting r -Permutations

- **Theorem:** Let n, r be integers and $0 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = n! / (n - r)!$$

r -permutations of a set with n distinct elements.

- Note that $P(n, 0) = 1$, i.e., there is one way to order 0 element.

- Proof:

- There are n choices for the **first** number.
For each way of choosing the **first** number, there are $n - 1$ choices for the **second** number.
...
For each way of choosing the first $r - 1$ numbers, there are $n - r + 1$ choices for the r -th number.
- Therefore, by the **product rule**, there are $n(n - 1)(n - 2) \cdots (n - r + 1)$ r -permutations, which is equal to $n! / (n - r)!$.

Counting r -Combinations

- **Theorem:** Let n, r be integers and $0 \leq r \leq n$, then there are

$$C(n, r) = P(n, r) / P(r, r) = n! / r! (n - r)!$$

r -combinations of a set with n distinct elements.

- Note that $C(n, 0) = 1$, i.e., there is one way to choose 0 element

- Proof:

- Since the order of elements in an r -combination does not matter and there are $P(r, r)$ different ways to order the r elements in an r -combination, each of the $C(n, r)$ r -combinations corresponds to exactly $P(r, r)$ r -permutations.
- Therefore, by the **division rule**, we have $C(n, r) = P(n, r) / P(r, r) = n! / r! (n - r)!$.

Exercise (5 mins)

- Answer the following questions:
 - How many different bit strings of length 7 are there?
 - How many different functions from a set with m elements to a set with n elements?
 - How many injective functions from a set with m elements to a set with n elements ($m \leq n$)?
 - How many onto functions from a set with m elements to a set with n elements ($m \geq n$)?

Exercise (5 mins)

○ Answer the following questions:

- How many different bit strings of length 7 are there?

$$2^7$$

- How many different functions from a set with m elements to a set with n elements?

$$n^m$$

- How many injective functions from a set with m elements to a set with n elements ($m \leq n$)?

$$P(n, m) = n(n - 1) \cdots (n - m + 1)$$

- How many onto functions from a set with m elements to a set with n elements ($m \geq n$)?

$$n^m - C(n, 1)(n - 1)^m + C(n, 2)(n - 2)^m - \cdots + (-1)^{n-1}C(n, n - 1)1^m$$

** to be proved soon in later sections*

The Birthday Problem

- **The Birthday Paradox:** Suppose that 23 students are in a room. What is the probability that at least two of them share a birthday?
 - It's greater than a half!
- Assume a year has 365 days and there are no twins in the room. Calculate the probability as follows:
 - Sample space: $|S| = 365^n$ * all cases occur equally likely
 - A_n : “for n students in a room ≥ 2 of them share a birthday”
 B_n : “for n students in a room none of them share a birthday”
 - $\#E$: number of cases favorable to event E
 - $\#A_n + \#B_n = |S| = 365^n$
 - $\#B_n = C(365, n) = 365 \times 364 \times \dots \times (365 - (n - 1))$
 - $\Pr[A_n] = \#A_n / |S| = 1 - \#B_n / |S|$ * classical probability

The Birthday Problem

- Probabilities of A_n and B_n : ($23 \approx \sqrt{2 \cdot 365}$)

n	A_n	B_n	n	A_n	B_n
1	0.00000000	1.00000000	16	0.28360400	0.71639599
2	0.00273972	0.99726027	17	0.31500766	0.68499233
3	0.00820416	0.99179583	18	0.34691141	0.65308858
4	0.01635591	0.98364408	19	0.37911852	0.62088147
5	0.02713557	0.97286442	20	0.41143838	0.58856161
6	0.04046248	0.95953751	21	0.44368833	0.55631166
7	0.05623570	0.94376429	22	0.47569530	0.52430469
8	0.07433529	0.92566470	23	0.50729723	0.49270276
9	0.09462383	0.90537616	24	0.53834425	0.46165574
10	0.11694817	0.88305182	25	0.56869970	0.43130029
11	0.14114137	0.85885862	26	0.59824082	0.40175917
12	0.16702478	0.83297521	27	0.62685928	0.37314071
13	0.19441027	0.80558972	28	0.65446147	0.34553852
14	0.22310251	0.77689748	29	0.68096853	0.31903146
15	0.25290131	0.74709868	30	0.70631624	0.29368375

The Birthday Attack

- In cryptography, the **birthday attack** is an attack that uses the probabilistic model shown in the birthday problem to reduce the complexity of **finding a collision for a hash function**.
 - Assume a hash function has **independent random outputs**.
 - Each hash output can be viewed as a student's birthday.
- Recall the birthday problem:
 - A_n : “for n students in a room ≥ 2 of them share a birthday”
 - B_n : “for n students in a room **none** of them share a birthday”

$$\Pr[B_n] = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right) = \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$$\Pr[A_n] = 1 - \Pr[B_n] = 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right) \quad p(n; H) := 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{H}\right)$$

The Birthday Attack

- In cryptography, the **birthday attack** is an attack that uses the probabilistic model shown in the birthday problem to reduce the complexity of **finding a collision for a hash function**.
- **Q:** What is the **smallest** number of inputs that we have to choose such that the probability of finding a hash collision is $\geq p$?
 - Let H be the number of possible hash outputs.
 - The **collision probability** when choosing n random hash outputs is

$$p(n; H) := 1 - \prod_{i=1}^{n-1} \left(1 - \frac{i}{H}\right)$$

Since $e^x = 1 + x + \frac{x^2}{2!} + \dots$, for $|x| \ll 1$, $e^x \approx 1 + x$

Thus, we have $e^{-i/H} \approx 1 - \frac{i}{H}$. $p(n; H) \approx 1 - e^{-n(n-1)/2H} \approx 1 - e^{-n^2/2H}$

- By **inverting** the above expression, we have the **smallest** number n :

$$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}.$$

Binomial Coefficients and Identities

Binomial Coefficients

- **Theorem:** For integers n and k with $0 \leq k \leq n$, the number of k -element subsets of an n -element set is

$$\binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}.$$

This is the number of k -combinations of a set with n elements.

- Properties:

- $\binom{n}{0} = \binom{n}{n} = 1$ * only one subset of size 0 and one of size n
- $\binom{n}{k} = \binom{n}{n-k}$ * obvious by definition
- $\sum_{k=0}^n \binom{n}{k} = 2^n$ * the number of subsets of an n -element set

Binomial Coefficients

- Each row starts and ends with a **1**

- $\binom{n}{0} = \binom{n}{n} = 1$

- Second half of each row is the **reverse** of the first half.

- $\binom{n}{k} = \binom{n}{n-k}$

- Sum on the **n** -th row is **2^n**

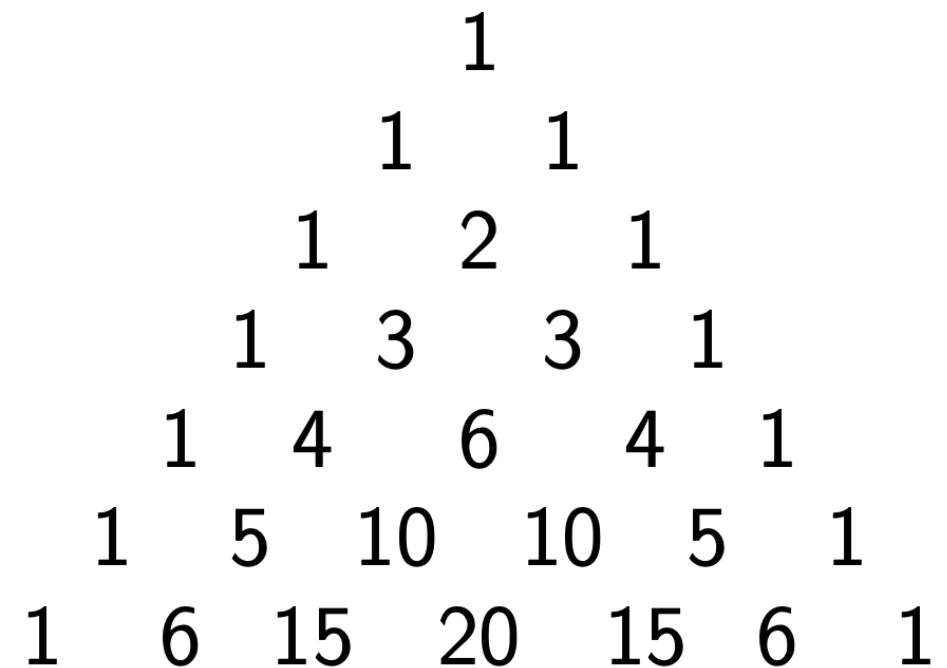
- $\sum_{k=0}^n \binom{n}{k} = 2^n$

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Pascal's Triangle

- Take the table and shift each row slightly such that middle element is in the center

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

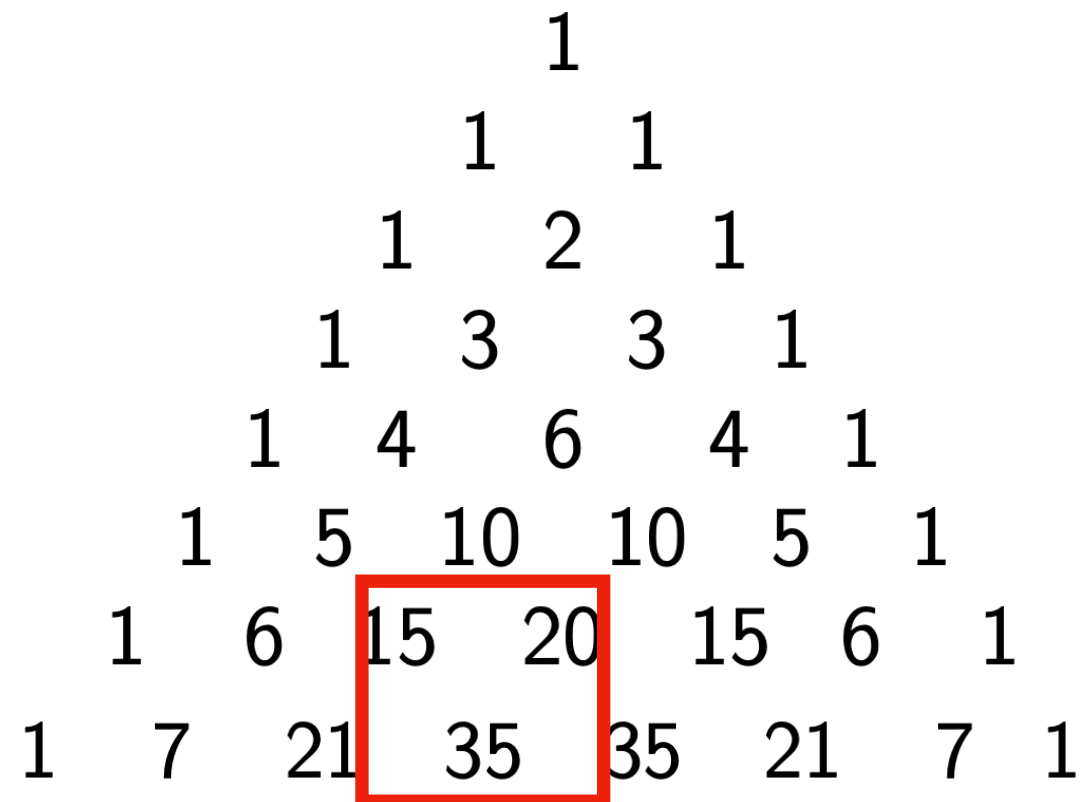


Pascal's Identity

- **Q:** What is the next row?
 - Each (non-1) entry in Pascal's triangle is the sum of the two entries directly above it.

- **Pascal's identity:**

- $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
- A purely algebraic proof (i.e., by manipulating formulas) is possible but complicated.
- We will apply a so-called combinatorial argument (or combinatorial proof).



A Combinatorial Proof

- **Pascal's identity:** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- A combinatorial proof:
 - Let S_1 be the set of all k -element subsets and x_n be the n -th element. Partition S_1 into S_2 and S_3 and apply the **sum rule**:
 - S_2 : the set of k -element subsets that **contain** x_n .
 - S_3 : the set of k -element subsets that **don't contain** x_n .
 - $|S_1| = \binom{n}{k}$: number of k -subsets of an n -element set
 - $|S_2| = \binom{n-1}{k-1}$: number of $(k-1)$ -subsets of an $(n-1)$ -element set
 - $|S_3| = \binom{n-1}{k}$: number of k -subsets of an $(n-1)$ -element set

Blaise Pascal

- French mathematician (1623~1662)
 - a founder of probability theory
 - inventor of one of the first mechanical calculating machines
 - Pascal programming language named in honor of him



The Binomial Theorem

- **The Binomial Theorem:** Let x and y be variables, and let n be a nonnegative integer. Then $\sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = (x + y)^n$.

** the proof is easy: how many ways one can derive the term $x^{n-k} y^k$?*

- Let $x = y = 1$. We have $\sum_{k=0}^n \binom{n}{k} = 2^n$.
- Let $x = 1, y = -1$. We have $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$.
- Let $y = 1$. We have $\sum_{k=0}^n \binom{n}{k} x^k = (1 + x)^n$.

- Why the name “**binomial coefficients**”?
 - Because those numbers occur as **coefficients** in the expansion of powers of **binomial expressions** such as $(x + y)^n$.

Trinomial Coefficients

- **Q:** What is the coefficient of $x^{k_1} y^{k_2} z^{k_3}$ in $(x + y + z)^n$?
- **A:** If we have k_1 red labels, k_2 blue labels, and $k_3 = n - k_1 - k_2$ purple labels, then in how many different ways can we apply these labels to n objects?
 - How many ways to choose the k_1 red items? How many ways to choose the k_2 blue items from the remaining $n - k_1$ items? Finally, the remaining $k_3 = n - k_1 - k_2$ items get labelled purple.

$$\binom{n}{k_1} \binom{n - k_1}{k_2} = \frac{n!}{k_1!(n - k_1)!} \frac{(n - k_1)!}{(k_2)!(n - k_1 - k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$

- If $k_1 + k_2 + k_3 = n$, we call $n!/(k_1! k_2! k_3!)$ a trinomial coefficient and denote it as $\binom{n}{k_1 k_2 k_3}$.

Inclusion-Exclusion

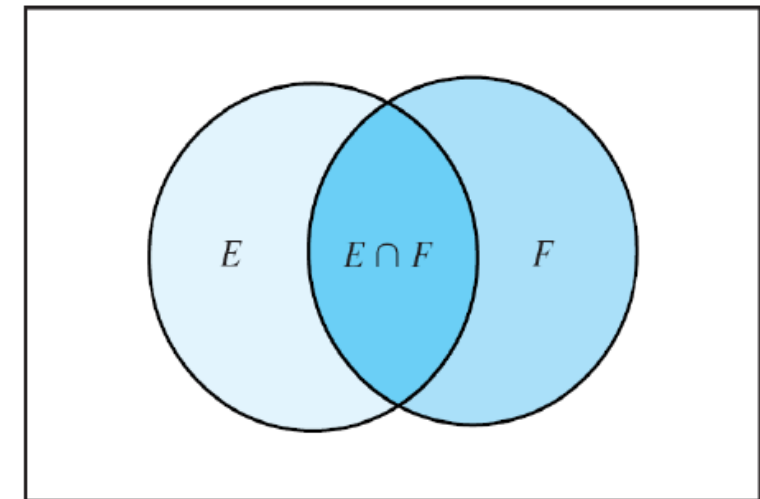
The Inclusion-Exclusion Principle

- The principle is used in counts where the **decomposition yields two dependent counting tasks** with **overlapping** elements
 - If we use the **sum rule**, some elements would be counted **twice**.
- The principle uses the **subtraction rule** to correct for the **overlapping** elements after the sum.
 - two-set case: $|A \cup B| = |A| + |B| - |A \cap B|$
- Example: How many bit strings of length 8 that **start with a '1' bit** or **end with the two bits '00'**?
 - It is easy to count bit strings starting with **'1'**: 2^7
 - It is easy to count bit strings ending with **'00'**: 2^6
 - **Deduct** the **overcounted** number of strings starting with **'1'** and ending with **'00'**: 2^5

The Inclusion-Exclusion Principle

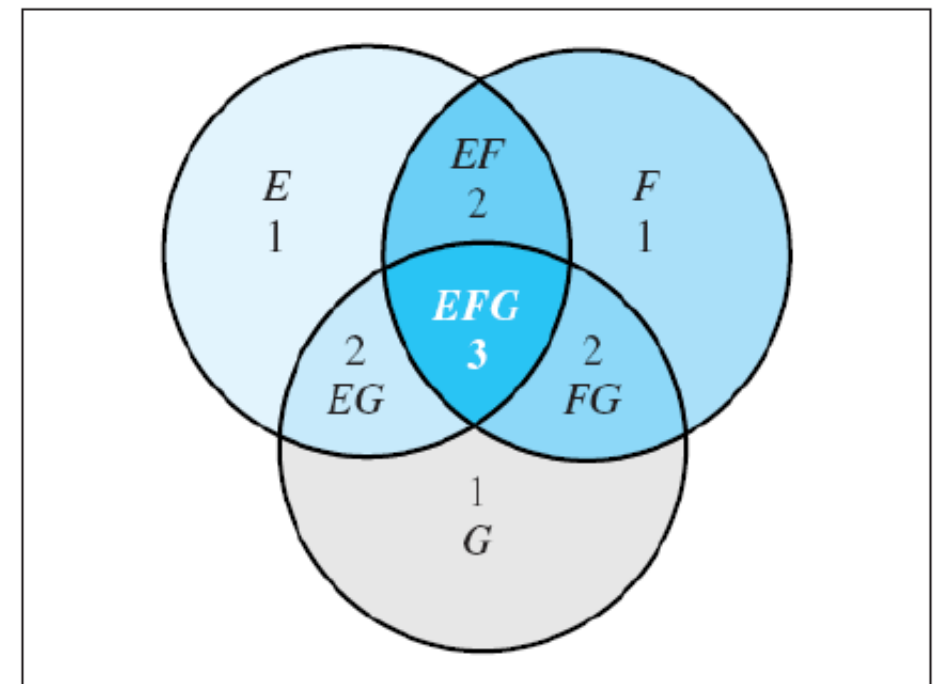
- Two sets:

- $|E \cup F| = |E| + |F| - |E \cap F|$
- $E \cap F$ got counted twice and deducted once



- Three sets:

- $|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|$
- $E \cap F, E \cap G, F \cap G$ got counted twice and then deducted once
- $E \cap F \cap G$ got counted three times then deducted three times and finally got counted once



The Inclusion-Exclusion Principle

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- Proof by induction:

- **Basis step:** obviously true for $n = 1, 2$
- **Inductive step:** By the two-set case of this principle, i.e., thinking of $\cup_{i=1}^{n-1} E_i$ and E_n as two sets, we have

$$|\cup_{i=1}^n E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |(\cup_{i=1}^{n-1} E_i) \cap E_n|$$

Let $G_i = E_i \cap E_n$, by the **distributive law**, we have

$$|(\cup_{i=1}^{n-1} E_i) \cap E_n| = |\cup_{i=1}^{n-1} (E_i \cap E_n)| = |\cup_{i=1}^{n-1} G_i|$$

Therefore, $|\cup_{i=1}^n E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |\cup_{i=1}^{n-1} G_i|$

The Inclusion-Exclusion Principle

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- Proof by induction:

- **Inductive step:** By the two-set case and $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^n E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |\cup_{i=1}^{n-1} G_i|$$

By inductive hypothesis, we have:

$$\begin{aligned} |\cup_{i=1}^{n-1} E_i| &= \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{0 \leq i_1 < i_2 < \dots < i_k \leq n-1} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}| \\ - |\cup_{i=1}^{n-1} G_i| &= \sum_{k=1}^{n-1} - (-1)^{k+1} \sum_{0 \leq i_1 < i_2 < \dots < i_k \leq n-1} |G_{i_1} \cap G_{i_2} \cap \dots \cap G_{i_k}| \end{aligned}$$

By definition, $|G_{i_1} \cap G_{i_2} \cap \dots \cap G_{i_k}| = |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} \cap E_n|$.

The Inclusion-Exclusion Principle

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- Proof by induction:

- **Inductive step:** By the two-set case and $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^n E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |\cup_{i=1}^{n-1} G_i|$$

By inductive hypothesis, we have:

$$\begin{aligned} |\cup_{i=1}^{n-1} E_i| &= \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{0 \leq i_1 < i_2 < \dots < i_k < n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}| \\ - |\cup_{i=1}^{n-1} G_i| &= \sum_{k=1}^{n-1} (-1)^{(k+1)+1} \sum_{0 \leq i_1 < i_2 < \dots < i_k < n} |E_{i_1} \cap \dots \cap E_{i_k} \cap E_n| \end{aligned}$$

The 1st big sum captures all $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$ without E_n .

The 2nd big sum captures all $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} \cap E_n|$ for $k \geq 1$.

The Inclusion-Exclusion Principle

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- Proof by induction:

- **Inductive step:** By the two-set case and $G_i = E_i \cap E_n$, we have:

$$|\cup_{i=1}^n E_i| = |\cup_{i=1}^{n-1} E_i| + |E_n| - |\cup_{i=1}^{n-1} G_i|$$

By **inductive hypothesis**, we have:

The 1st big sum captures all $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$ **without E_n** .

The 2nd big sum captures all $|E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} \cap E_n|$ for $k \geq 1$.

The above two disjoint big sums together capture all possible combinations of $|E_{i_1} \cap \dots \cap E_{i_k}|$ except $|E_n|$. ** why?*

Therefore, the principle holds for n sets.

Example: Counting Onto Functions

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- We can use this principle to find the number of onto functions:
 - Let A, B be two sets with $|A| = m$ and $|B| = n$.
 - $\#(a)$: the number of onto functions from A to B
 - $\#(b)$: the number of non-onto functions from A to B , i.e., functions with at least one element of B having no preimage
 - Since there are n^m functions from A to B , we have $\#(a) + \#(b) = n^m$. So, in order to find $\#(a)$, we only need to calculate $\#(b)$.
 - E_i : set of functions such that the i -th element of B has no preimage

$$\#(b) = |\cup_{i=1}^n E_i|$$

Example: Counting Onto Functions

- **The Principle of Inclusion-Exclusion:** Let E_1, E_2, \dots, E_n be finite sets, then

$$|\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- We can use this principle to find the number of onto functions:
 - E_j : set of functions such that the j -th element of B has no preimage
 - By the **principle of inclusion-exclusion**,

$$\#(b) = |\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|$$

- Consider any index list i_1, i_2, \dots, i_k . * *how many such lists of size k ?*
Functions in $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}$ do not map to these k elements of B ,
so there are $(n - k)^m$ such functions in total. Therefore,

$$\#(b) = |\cup_{i=1}^n E_i| = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n - k)^m$$

Solving Linear Recurrence Relations

Linear Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

- **linear:** it is a linear combination of previous terms
- **homogeneous:** all terms are multiples of a_j s
- **degree k :** a_n is expressed by the previous k terms
- **constant coefficients:** coefficients are constants
- By induction, such a recurrence relation is uniquely determined by this recurrence relation, and k initial conditions a_0, a_1, \dots, a_{k-1} .

Linear Recurrence Relations

- **Definition:** A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

- Examples: are these linear homogeneous recurrence relations?
 - $P_n = \pi P_{n-1}$ *Yes, of degree 1*
 - $f_n = f_{n-1} + f_{n-2}$ *Yes, of degree 2*
 - $a_n = a_{n-1} + a_{n-2} \cdot a_{n-2}$ *No, not linear*
 - $H_n = 2H_{n-1} + 1$ *No, not homogeneous*
 - $B_n = nB_{n-1}$ *Yes, but coefficients are not constants*

Solving Recurrences of Degree 2

- Consider an arbitrary linear homogeneous relation of degree 2 with constant coefficients:

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$$

The **characteristic equation (CE)** is:

$$r^2 - c_1 \cdot r - c_2 = 0$$

- Theorem:** If this CE has two **distinct** roots r_1, r_2 , then a sequence $\{a_n\}$ is a solution of the recurrence relation **if and only if**

$$a_n = a_1 \cdot r_1^n + a_2 \cdot r_2^n \text{ for } n \geq 0,$$

where a_1, a_2 are constants.

- see the textbook for the proof (“if” part is easy but “only if” is tricky)*

Example

○ Solve the recurrence: $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2$, $a_1 = 1$.

○ Solution:

- The characteristic equation (CE) is

$$r^2 - 7r + 10 = 0$$

- Two roots are 2 and 5. So, assume that

$$a_n = a_1 \cdot 2^n + a_2 \cdot 5^n$$

- By the two initial conditions, we have

$$a_0 = a_1 + a_2 = 2$$

$$a_1 = 2a_1 + 5a_2 = 1$$

- We get $a_1 = 3$ and $a_2 = -1$. Therefore

$$a_n = 3 \cdot 2^n - 5^n$$

Exercise (3 mins)

- What is the closed-form expression of Fibonacci sequence F_n ?
 - $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

○ Solve the recurrence: $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2, a_1 = 1$.

○ Solution:

- The characteristic equation (CE) is

$$r^2 - 7r + 10 = 0$$

- Two roots are 2 and 5. So, assume that

$$a_n = a_1 \cdot 2^n + a_2 \cdot 5^n$$

- By the two initial conditions, we have

$$a_0 = a_1 + a_2 = 2$$

$$a_1 = 2a_1 + 5a_2 = 1$$

- We get $a_1 = 3$ and $a_2 = -1$. Therefore

$$a_n = 3 \cdot 2^n - 5^n$$

Exercise (3 mins)

- What is the closed-form expression of Fibonacci sequence F_n ?
 - $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

- Solution:

- Consider the characteristic equation (CE): $x^2 - x + 1 = 0$. There are two different roots

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}$$

- Then F_n is of the form $\alpha_1 \cdot \phi^n + \alpha_2 \cdot \psi^n$.
- By $F_0 = 0$ and $F_1 = 1$, we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1\phi + \alpha_2\psi = 1$, leading to $\alpha_1 = 1/\sqrt{5}, \alpha_2 = -1/\sqrt{5}$. Therefore,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Solving Recurrences of Degree k

- Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$$

The characteristic equation (CE) is:

$$r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} - \cdots - c_k = 0$$

- E.g., $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with CE: $r^3 - 2r^2 - 5r + 6 = 0$
- Theorem:** If this CE has k distinct roots r_1, r_2, \dots, r_k , then the solutions to the above recurrence is of the form
$$a_n = a_1 \cdot r_1^n + a_2 \cdot r_2^n + \cdots + a_k \cdot r_k^n$$
for $n \geq 0$, where a_1, a_2, \dots, a_k are constants.
- the proof is left as an exercise*

The Case of Degenerate Roots

- **Theorem:** If the CE $r^2 - c_1 \cdot r - c_2 = 0$ has **only one** root r_0 (i.e., with multiplicity 2), then

$$a_n = (a_1 + a_2 \cdot n) r_0^n = a_1 \cdot r_0^n + a_2 \cdot n \cdot r_0^n \text{ for } n \geq 0,$$

where a_1, a_2 are constants.

- *the proof is left as an exercise*

- **Theorem:** Suppose the CE $r^k - c_1 \cdot r^{k-1} - \dots - c_k = 0$ has t **distinct** roots r_1, \dots, r_t with **multiplicities** 重数 m_1, \dots, m_t respectively, where $m_i \geq 1$ and $m_1 + \dots + m_t = k$. Then

$$a_n = \sum_{i=1}^t \left(\sum_{j=0}^{m_i-1} \alpha_{i,j} n^j \right) r_i^n \text{ for } n \geq 0,$$

where $\alpha_{i,j}$ are constants.

- *the proof is left as an exercise*

Exercise (3 mins)

- Solve the recurrence: $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1, a_1 = 0$.

○ **Theorem:** If the CE $r^2 - c_1 \cdot r - c_2 = 0$ has **only one** root r_0 (i.e., with multiplicity 2), then

$$a_n = (a_1 + a_2 \cdot n) r_0^n = a_1 \cdot r_0^n + a_2 \cdot n \cdot r_0^n \text{ for } n \geq 0,$$

where a_1, a_2 are constants.

Exercise (3 mins)

○ Solve the recurrence: $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1, a_1 = 0$.

○ Solution:

- The characteristic equation (CE) is

$$r^2 - 4r + 4 = 0$$

- The only root is 2. So, assume that

$$a_n = a_1 \cdot 2^n + a_2 \cdot n \cdot 2^n$$

- By the two initial conditions, we have

$$a_0 = a_1 = 1$$

$$a_1 = 2a_1 + 2a_2 = 0$$

- We get $a_1 = 1$ and $a_2 = -1$. Therefore

$$a_n = 2^n - n \cdot 2^n = (1 - n) 2^n$$

Solving Nonhomogeneous Recurrences

- **Theorem:** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} + F(n)$$

Then all its solutions are of the form

$$a_n = p(n) + h(n)$$

where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$$

- *the proof is left as an exercise*

Example

- Solve the recurrence: $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$.
- Solution:
 - The characteristic equation (CE) of the associated linear homogeneous recurrence relation is $r - 3 = 0$.
 - The root is 3. So, assume $a_n = p(n)$ is a particular solution to the original recurrence relation, then all of its solutions are of the form

$$a_n = a_1 \cdot 3^n + p(n)$$

- It is natural to try a degree-1 polynomial, i.e., $p(n) = cn + d$. Then,
 $cn + d = 3(c(n-1) + d) + 2n$, i.e., $(2c + 2)n + 2d - 3c = 0$
- We get $c = -1$ and $d = -3/2$. Therefore, $a_n = a_1 \cdot 3^n - n - 3/2$.
- By the initial condition $a_1 = 3a_1 - 1 - 3/2 = 3$, we get $a_1 = 11/6$.

Generating Functions

Generating Functions

- **Definition:** The **generating function** for the sequence $\{a_k\}$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

- We use generating functions to **characterize** sequences:

- $\sum_{k=0}^{\infty} 3x^k$: generating function for the sequence $\{a_k\}$ with $a_k = 3$
- $\sum_{k=0}^{\infty} 2^k x^k$: generating function for the sequence $\{a_k\}$ with $a_k = 2^k$

Generating Functions

- **Definition:** The **generating function** for the sequence $\{a_k\}$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k$$

- We use generating functions to **characterize** sequences:

- $\sum_{k=0}^{\infty} 3x^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n 3x^k = 3 \lim_{n \rightarrow \infty} \frac{1 - x^{n+1}}{1 - x} = \frac{3}{1 - x}$ for $|x| < 1$
- $\sum_{k=0}^{\infty} 2^k x^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n (2x)^k = \lim_{n \rightarrow \infty} \frac{1 - (2x)^{n+1}}{1 - 2x} = \frac{1}{1 - 2x}$ for $|2x| < 1$

Operations of Generating Functions

- **Theorem:** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k a_j b_{k-j} \right) x^k$$

** the proof can be found in a calculus course*

- Example: $f(x) = g(x) = \sum_{k=0}^{\infty} a^k x^k$
 - We know $G(x) = \sum_{k=0}^{\infty} a^k x^k = 1/(1 - ax)$ for $|ax| < 1$
 - Therefore,

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k a^j a^{k-j} \right) x^k = \sum_{k=0}^{\infty} (k+1) a^k x^k = \frac{1}{(1 - ax)^2}$$

The Case of Finite Sequences

- **Definition:** The generating function for the finite sequence a_0, a_1, \dots, a_n of real numbers is a polynomial of degree n

$$G(x) = a_0 + a_1x + \dots + a_nx^n$$

- A finite sequence a_0, a_1, \dots, a_n can be easily extended to infinity by setting $a_{n+1} = a_{n+2} = \dots = 0$.
- Example: What is the generating function for the finite sequence a_0, a_1, \dots, a_n , with $a_k = C(n, k)$?

$$G(x) = C(n,0) + C(n,1)x + \dots + C(n,n)x^n = (1+x)^n$$

Useful Generating Functions

$$(1 + x)^n = \sum_{k=0}^n C(n, k)x^k$$

$$(1 + ax)^n = \sum_{k=0}^n C(n, k)a^k x^k$$

$$(1 + x^r)^n = \sum_{k=0}^n C(n, k)x^{rk}$$

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$$

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2 x^2 + \dots$$

$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \dots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \dots$$

Generating Functions for Counting

- **Example 1:** Find the number of integer solutions of

$$x_1 + x_2 + x_3 = 17$$

such that $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, $4 \leq x_3 \leq 7$.

- Solution: Using generating functions, the above number equals the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

The answer is 3. ($4 + 6 + 7$, $5 + 5 + 7$, $5 + 6 + 6$)

Exercise (3 mins)

- In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two and no more than four cookies?

- **Example 1:** Find the number of integer solutions of

$$x_1 + x_2 + x_3 = 17$$

such that $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, $4 \leq x_3 \leq 7$.

- Solution: Using generating functions, the above number equals the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

The answer is 3. (4 + 6 + 7, 5 + 5 + 7, 5 + 6 + 6)

Exercise (3 mins)

- In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two and no more than four cookies?
- Solution: Find the number of integer solutions of

$$x_1 + x_2 + x_3 = 8$$

such that $2 \leq x_i \leq 4$. Using generating functions, the number of solutions equals the coefficient of x^8 in the expansion of

$$(x^2 + x^3 + x^4)^3$$

The answer is 6. $(2 + 2 + 4, 2 + 3 + 3, 2 + 4 + 2, 3 + 2 + 3, 3 + 3 + 2, 4 + 2 + 2)$

Generating Functions for Counting

- **Example 2:** In how many ways can r dollars be paid by coins that are worth 1 dollar, 2 dollar and 5 dollar each?
- Solution: Using generating functions, the number of ways equals the coefficient of x^r in the expansion of the following functions:

- If order of coins does not matter:

$$(1 + x + x^2 + x^3 + \dots) (1 + x^2 + x^4 + x^6 + \dots) (1 + x^5 + x^{10} + x^{15} + \dots)$$

** the number of non-negative integer solutions of $x_1 + x_2 + x_3 = r$ such that x_2 is a multiple of 2 and x_3 is a multiple of 5*

- If order matters:

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \dots = 1 / (1 - x - x^2 - x^5)$$

** the number of ways using k coins for $k = 0, 1, 2, \dots$*

Generating Functions for Counting

- **Example 3:** Use **generating functions** to find the number of **k -combinations** of a set with n elements.
- Solution: Each of the n elements in the set contributes one term $(1 + x)$ to the generating function $(1 + x)^n = \sum_{k=0}^n a_k x^k$. Then, by the **binomial theorem** * *can be proved by induction*, we have

$$a_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Generating Functions for Counting

- **Example 4:** How many **multisets** of size **17** from the set $\{a, b, c\}$?
 - A **multiset** of size r is an r -combination with repetition allowed.
- Solution: This is **equivalent** to find the number of nonnegative integer solutions to

$$x_1 + x_2 + x_3 = 17$$

- This equals the number of ways to split **17** balls into **3** groups. Imagine **19** empty boxes are aligned in a line, then one just needs to choose **2** empty boxes and put balls in the remaining **17** boxes, then the balls are split into **3** groups. Therefore, the answer is $C(19, 2) = C(19, 17) = C(3 + 17 - 1, 17)$
- This is equal to the coefficient of x^{17} in the generating function:
$$(1 + x + x^2 + \dots)^3 = 1/(1 - x)^3 = \sum_{k=0}^{\infty} C(3 + k - 1, k)x^k$$

Useful Generating Functions (more)

- Hinted by **Example 4**:

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)x^k$$

$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)(-1)^k x^k$$

$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1, k)a^k x^k$$

- Taylor series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Solving Recurrences with $G(x)$

○ **Example 5:** Solve $a_k = 3a_{k-1}$ ($k \geq 1$) with $a_0 = 2$.

○ Solution:

- Let $G(x)$ be the generating function of $\{a_k\}$, we have

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k = a_0 = 2 \end{aligned}$$

- Then, we know

$$G(x) = \frac{2}{1-3x} = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

- Therefore, the solution is $a_k = 2 \cdot 3^k$ for $k \geq 0$.

Exercise (5 mins)

- Solve $a_k = 5a_{k-1} - 6a_{k-2}$ ($k \geq 2$) with $a_0 = 6$, $a_1 = 30$.

- **Example 5:** Solve $a_k = 3a_{k-1}$ ($k \geq 1$) with $a_0 = 2$.

- Solution:

- Let $G(x)$ be the generating function of $\{a_k\}$, we have

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1} x^k \\ &= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k = a_0 = 2 \end{aligned}$$

- Then, we know

$$G(x) = \frac{2}{1-3x} = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

- Therefore, the solution is $a_k = 2 \cdot 3^k$ for $k \geq 0$.

Exercise (5 mins)

○ Solve $a_k = 5a_{k-1} - 6a_{k-2}$ ($k \geq 2$) with $a_0 = 6$, $a_1 = 30$.

○ Solution:

- Let $G(x)$ be the generating function of $\{a_k\}$, we have

$$\begin{aligned} G(x) - 5xG(x) + 6x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 5a_{k-1} x^k + \sum_{k=2}^{\infty} 6a_{k-2} x^k \\ &= a_0 + a_1 x - 5a_0 x + \sum_{k=2}^{\infty} (a_k - 5a_{k-1} + 6a_{k-2}) x^k \\ &= a_0 + (a_1 - 5a_0)x = 6 + (30 - 5 \cdot 6)x = 6 \end{aligned}$$

- Then, we know

$$G(x) = \frac{6}{(1-2x)(1-3x)} = 6 \left(\frac{3}{1-3x} - \frac{2}{1-2x} \right) = \sum_{k=0}^{\infty} (18 \cdot 3^k - 12 \cdot 2^k) x^k$$

- Therefore, the solution is $a_k = 18 \cdot 3^k - 12 \cdot 2^k$ for $k \geq 0$.

Solving Recurrences with $G(x)$

◦ **Example 6:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ ($n \geq 2$) with $a_1 = 9$.

◦ Solution:

- Let $a_0 = 1$, then we have $a_1 = 8a_0 + 10^0 = 9$, so the recurrence relation now holds for $n \geq 1$. * $G(x)$ with n starting from 0 is simpler
- Let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$\begin{aligned} G(x) - a_0 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1} x^n + 10^{n-1} x^n) \\ &= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1} \\ &= 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1 - 10x) \end{aligned}$$

Solving Recurrences with $G(x)$

- **Example 6:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ ($n \geq 2$) with $a_1 = 9$.
- Solution:
 - Let $a_0 = 1$, then we have $a_1 = 8a_0 + 10^0 = 9$, so the recurrence relation now holds for $n \geq 1$. * $G(x)$ with n starting from 0 is simpler
 - Let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$G(x) - a_0 = G(x) - 1 = 8xG(x) + x/(1 - 10x)$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}$$

Solving Recurrences with $G(x)$

◦ **Example 6:** Solve $a_n = 8a_{n-1} + 10^{n-1}$ ($n \geq 2$) with $a_1 = 9$.

◦ Solution:

- Let $a_0 = 1$, then we have $a_1 = 8a_0 + 10^0 = 9$, so the recurrence relation now holds for $n \geq 1$. * $G(x)$ with n starting from 0 is simpler
- Let $G(x)$ be the generating function of $\{a_n\}$ for $n \geq 0$, we have

$$\begin{aligned} G(x) &= \frac{1 - 9x}{(1 - 8x)(1 - 10x)} = \frac{1}{2} \left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x} \right) \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n \right) = \sum_{n=0}^{\infty} \frac{1}{2} (8^n + 10^n) x^n \end{aligned}$$

Therefore, the solution is $a_n = \frac{1}{2}(8^n + 10^n)$ for $n \geq 1$.

Summary

- Using **generating functions** to solve recurrence relations:
 - **Step 1:** Based on the given recurrence and its initial conditions, find the closed-form expression of its generating function $G(x)$.

$$\text{E.g., } G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}$$

- **Step 2:** Rewrite $G(x)$ as the summation of an infinite (or finite) series.
** this step may be tricky*

$$\text{E.g., } G(x) = \frac{?}{1 - 8x} + \frac{?}{1 - 10x} = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

08 Relations

To be continued...

Quiz 2 and Assignment 4

- Quiz 2 will take place in class on Nov 29 and it captures materials from 06 Induction and Recursion to 07 Counting.
- Again, the quiz is open-book:
 - 3~6 questions in ~20 minutes
 - Must attend the quizzes in person and write in English.
 - Please bring several pieces of paper to write your answers on.
 - No electronic device is allowed during the quiz.
 - After quiz, please take photos of your quiz answers and submit to Blackboard (you will have ~5 minutes after quiz to do this).
- Deadline for Assignment 4: Nov 29