Discrete Mathematics Assignment 5

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1. (a) R_1 is irreflexive, symmetric and transitive.

   (b) R_2 is reflexive, symmetric and transitive.

   (c) R_3 is irreflexive, antisymmetric and transitive.

   (d) R_4 is irreflexive, antisymmetric and transitive.

   (e) R_5 is reflexive, antisymmetric and transitive.

2. (a) Counterexample: Consider a relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$ on set $A = \{1,2,3\}$.

   $\because (1,1),(2,2),(3,3) \in R$

   $\therefore$ R is reflexive.

   $\because$ For $(1,2)$ and $(2,3) \in R$, we have $(2,1)$ and $(3,2) \in R$

   $\therefore$ R is symmetric.

   $\because$ For $(1,2)$ and $(2,3) \in R$, we don't have $(1,3) \in R$

   $\therefore$ R is not transitive.

   $\therefore$ If R is reflexive and symmetric, then R has no need to be transitive.

   (b) $\because$ $R_1,\ R_2$ are reflexive

   $\therefore$ For all $a \in A$, $(a,a) \in R_1$ and $(a,a) \in R_2$.

   $\therefore$ For all $a \in A$, $(a,a) \in R_1 \cup R_2$.

   $\therefore$ R_2 is reflexive.

   (c) Counterexample: Consider relations $R_1 = \{(1,2),(2,3)\}$, $R_2 = \{(2,1),(3,2)\}$ on set $A = \{1,2,3\}$.

   $\because$ For elements m_{ij} of M_{R_1} and M_{R_2}, $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

   $\therefore$ R_1 and R_2 are both antisymmetric.

   $\because$ $R_1 \cup R_2 = \{(1,2),(2,3),(2,1),(3,2)\}$

   For $(1,2)$ and $(2,3) \in R_1 \cup R_2$, we have $(2,1)$ and $(3,2) \in R_1 \cup R_2$, but $1 \neq 2,\ 2 \neq 3$.

   $\therefore$ $R_1 \cup R_2$ is not antisymmetric.

   $\therefore$ If R_1, R_2 are antisymmetric, then $R_1 \cup R_2$ has no need to be antisymmetric.

3. (a) $\because$ $s_{C_2}(R) = \{a \in R \mid C_2(a) = T\}$

   $\therefore$ $s_{C_1}(s_{C_2}(R)) = \{a \in R \mid C_1(a) = T \wedge C_2(a) = T\} = s_{C_1 \wedge C_2}(R)$.

   (b) $\because$ $P_{i_1,i_2,\ldots,i_m}(R \cup S) = \{(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \mid (a_i, a_2, \ldots, a_m) \in R \cup S\}$,

   $P_{i_1,i_2,\ldots,i_m}(R) = \{(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \mid (a_i, a_2, \ldots, a_m) \in R\}$,

   $P_{i_1,i_2,\ldots,i_m}(S) = \{(a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \mid (a_i, a_2, \ldots, a_m) \in S\}$

   $\therefore$ $P_{i_1,i_2,\ldots,i_m}(R \cup S) = P_{i_1,i_2,\ldots,i_m}(R) \cup P_{i_1,i_2,\ldots,i_m}(S)$.

4. (a) Proof by induction:

   Basic step: $R^1 = R$ is symmetric.

   Inductive step: Assume that for an arbitrary positive integer k, R^k is symmetric.

   $\therefore$ For all $(a,b) \in R^k$, we have $(b,a) \in R^k$.

   $\because$ $R^{k+1} = R^k \circ R$

$\therefore$ For all $(a, c) \in R^{k+1}$, there exists $(a, b), (b, c) \in R^k$.

$\because R^k$ is symmetric

$\therefore (c, b), (b, a) \in R^k$.

$\therefore (c, a) \in R^{k+1}$.

$\therefore R^{k+1}$ is symmetric.

(b) Proof: Consider a pair $(a, b) \in R^*$.

$\because R^* = \bigcup_{k=1}^{\infty} R^k$

$\therefore$ There exists pairs $(a, m_1), (m_1, m_2), (m_2, m_3), \ldots, (m_{k-1}, b) \in R$.

$\because R$ is symmetric

$\therefore (b, m_{k-1}), \ldots, (m_2, m_1), (m_1, a) \in R$.

$\therefore (b, a) \in R^*$

$\therefore R^*$ is symmetric.

5. Consider a relation R. Let S be the symmetric closure of R, T be the transitive closure of R.

   Let T_S be the transitive closure of S, S_T be the symmetric closure of T.

   Consider pair $(a, b) \in S_T$, then we have $(a, b) \in T$ or $(b, a) \in T$, or both. We also have $(b, a) \in S_T$.

   If $(a, b) \in T$, then there exists pairs $(a, m_1), (m_1, m_2), (m_2, m_3), \ldots, (m_k, b) \in R$.

   $\therefore$ There exists $(b, m_k), \ldots, (m_2, m_1), (m_1, a) \in S$.

   $\therefore (a, b), (b, a) \in T_S$.

   If $(b, a) \in T$, then there exists pairs $(b, n_k), \ldots, (n_2, n_1), (n_1, a) \in R$.

   $\therefore$ There exists $(a, n_1), (n_1, n_2), (n_2, n_3), \ldots, (n_k, b) \in S$.

   $\therefore (b, a), (a, b) \in T_S$.

   If both, obviously $(a, b), (b, a) \in T_S$.

   $\therefore$ The transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

6. Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$, $v_5 = e$.

   $\because R = \{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$

   $\therefore w_{12} = w_{13} = w_{15} = w_{21} = w_{23} = w_{31} = w_{32} = w_{41} = w_{54} = 1$.

$$\therefore W_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   In the first iteration:

   $\because w_{21} = 1, w_{31} = 1, w_{41} = 1, w_{12} = 1, w_{13} = 1, w_{15} = 1$.

   $\therefore w_{22} = w_{25} = w_{33} = w_{35} = w_{42} = w_{43} = w_{45} = 1$.

$$\therefore W_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

   In the second iteration:

$\because w_{12} = 1, w_{21} = 1$

$\therefore w_{11} = 1.$

$$\therefore W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In the third iteration:

$$\therefore W_3 = W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In the 4th iteration:

$\because w_{54} = 1, w_{41} = 1, w_{42} = 1, w_{43} = 1, w_{45} = 1$

$\therefore w_{51} = w_{52} = w_{53} = w_{55} = 1.$

$$\therefore W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

In the 5th iteration:

$\because w_{15} = 1, w_{25} = 1, w_{35} = 1, w_{45} = 1, w_{55} = 1, w_{51} = 1, w_{52} = 1, w_{53} = 1, w_{54} = 1.$

$\therefore w_{14} = w_{24} = w_{34} = w_{44} = 1.$

$$\therefore W_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

W_5 is the transitive closure of R.

7. (a) **Reflexive**: For every $x \in \mathbb{R}$, we have $(x, x) = x - x = 0 \in \mathbb{Z}$

$\therefore R$ is reflexive.

Symmetric: Assume that for $x, y \in \mathbb{R}$, we have $(x, y) = x - y \in \mathbb{Z}$.

$\because (y, x) = y - x = -(x - y) \in \mathbb{Z}$

$\therefore R$ is symmetric.

Transitive: Assume that for $x, y, z \in \mathbb{R}$, we have $(x, y) = x - y \in \mathbb{Z}$, $(y, z) = y - z \in \mathbb{Z}$.

$\because (x, z) = x - z = (x - y) + (y - z),$

The sum of two integers is still integer.

$\therefore (x, z) = x - z \in \mathbb{Z}.$

$\therefore R$ is transitive.

(b) $\because [1] = \{x \in \mathbb{R} \mid 1 - x \in \mathbb{Z}\}$

$\therefore x = 1 + n, n \in \mathbb{Z}.$

$\therefore [1] = \{0, 1, -1, 2, -2, \dots\}$

$\because [\frac{1}{2}] = \{y \in \mathbb{R} \mid \frac{1}{2} - y \in \mathbb{Z}\}$

$\therefore y = \frac{1}{2} + n, n \in \mathbb{Z}.$

$\therefore [\frac{1}{2}] = \{\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, \ldots\}.$

$\because [\pi] = \{z \in \mathbb{R} \mid \pi - z \in \mathbb{Z}\}$

$\therefore z = \pi + n, n \in \mathbb{Z}.$

$\therefore [\pi] = \{\pi, \pi + 1, \pi - 1, \ldots\}.$

8. (a) **Reflexive**: $\because$ For any function f, $\forall x \in \mathbb{R}$, $f(x) = f(x)$, i.e. $f(x) \leq f(x)$, $f \preceq f$

   $\therefore$ The relation $\preceq$ is reflexive.

   Antisymmetric: For any function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$:

   If $f \preceq g$ and $g \preceq f$, i.e. $f(x) \leq g(x)$, $g(x) \leq f(x)$, then we have $f(x) = g(x)$.

   $\therefore$ The relation $\preceq$ is antisymmetric.

   Transitive: For three functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$:

   If $f \preceq g$ and $g \preceq h$, i.e. $f(x) \leq g(x)$, $g(x) \leq h(x)$, then we have $f(x) \leq h(x)$.

   $\therefore$ The relation $\preceq$ is transitive.

   (b) Counterexample: Consider two functions $f(x) = x : \mathbb{R} \to \mathbb{R}$ and $g(x) = -x : \mathbb{R} \to \mathbb{R}$.

   When $x \geq 0$, $f(x) \geq g(x)$; when $x < 0$, $f(x) < g(x)$.

   $\therefore f \preccurlyeq g$ or $g \preccurlyeq f$ not always holds.

   $\therefore$ The relation $\preceq$ is not a total ordering.

9. (a) Maximal elements: l, m.

   (b) Minimal elements: a, b, c.

   (c) $\because$ There are 2 maximal elements

   $\therefore$ The greatest element does not exist.

   (d) $\because$ There are 3 minimal elements

   $\therefore$ The least element does not exist.

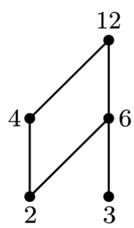
   (e) The upper bounds of $\{a, e, f\}$ are k, l, m.

   (f) The least upper bound of $\{a, e, f\}$ is k.

   (g) The lower bounds of $\{h, i, j\}$ are a, b, d.

   (h) The greatest lower bound of $\{h, i, j\}$ is d.

10. The Hasse Diagram for the poset $(\{2, 3, 4, 6, 12\}, \mid)$:



$\therefore$ The compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, \mid)$ are:

$$2 \prec 3 \prec 6 \prec 4 \prec 12$$
$$2 \prec 3 \prec 4 \prec 6 \prec 12$$
$$2 \prec 4 \prec 3 \prec 6 \prec 12$$
$$3 \prec 2 \prec 4 \prec 6 \prec 12$$
$$3 \prec 2 \prec 6 \prec 4 \prec 12$$