CS201: Discrete Mathematics (Fall 2024)
Written Assignment #5 - Solutions
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 13 (please submit to Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**15p**) Let S be the set of all strings (including the empty string) of English letters. Determine whether the following relations defined on S are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*. No proof is required.

(a) (**3p**) $R_1 = \{(a, b)|a$ and b have no letters in common$\}$

(b) (**3p**) $R_2 = \{(a, b)|a$ and b are of the same length$\}$

(c) (**3p**) $R_3 = \{(a, b)|a$ is shorter than $b\}$

(d) (**3p**) $R_4 = \{(a, b)|a$ and b have exactly one letter in common$\}$

(e) (**3p**) $R_5 = \{(a, b)|a$ contains b as a substring$\}$

Solution:

(a) symmetric. (not irreflexive due to empty string)

(b) reflexive, symmetric, transitive.

(c) irreflexive, antisymmetric, transitive.

(d) symmetric.

(e) reflexive, antisymmetric, transitive.

□

Q.2 (**15p**) Consider relations on a set A. **Prove or disprove** the following statements:

(a) (**5p**) If R is reflexive and symmetric, then R is transitive.

(b) (**5p**) If R_1, R_2 are reflexive, then $R_1 \cup R_2$ is reflexive.

(c) (**5p**) If R_1, R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Solution:

(a) False. Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}.$$

Then R is reflexive and symmetric, but not transitive, e.g., $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$.

(b) True. By definition, for any $a \in A$, we have $(a, a) \in R_1$ and $(a, a) \in R_2$, so $(a, a) \in R_1 \cup R_2$.

(c) False. Counterexample: Consider $A = \{1, 2\}$ and $R_1 = \{(1, 2)\}$, $R_2 = \{(2, 1)\}$. Then R_1, R_2 are antisymmetric, but it is obvious that $R_1 \cup R_2 = \{(1, 2), (2, 1)\}$ is not antisymmetric.

□

Q.3 (**10p**) Prove the following statements about n-ary relations:

(a) (**5p**) If C_1 and C_2 are conditions that elements of the n-ary relation $R : A_1, \ldots, A_n$ may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.

(b) (**5p**) If R and S are n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cup S) = P_{i_1,i_2,\ldots,i_m}(R) \cup P_{i_1,i_2,\ldots,i_m}(S)$.

Solution:

(a) By definition, we have $s_{C_1 \wedge C_2}(R) = \{a \in R \mid (C_1 \wedge C_2)(a) = T\}$. Furthermore, we have $s_{C_1}(s_{C_2}(R)) = s_{C_2}(R) \cap \{a \in R \mid C_1(a) = T\} = \{a \in R \mid C_1(a) = T\} \cap \{a \in R \mid C_2(a) = T\} = \{a \in R \mid C_2(a) = T \wedge C_1(a) = T\} = \{a \in R \mid C_1(a) \wedge C_2(a) = T\} = s_{C_1 \wedge C_2}(R)$.

(b) By definition of the projection operator, both sides of this equation pick out the m-tuples consisting of the i_1-th, i_2-th, …, i_m-th components of n-tuples in R or S.

□

Q.4 (**10p**) Suppose that a relation R on a set A is symmetric.

(a) (**7p**) Prove that, for any positive integer $n \geq 1$, R^n is symmetric.

(b) (**3p**) Prove that R^* is symmetric.

Solution:

(a) Let $P(n)$ denote the statement "R^n is symmetric". We prove it by strong induction.

Basis step: It is known that $P(1)$ is true, i.e., R is symmetric. We show that $P(2)$ is also true. By definition, if $(a, b) \in R^2$, then there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R$. Since R is symmetric, we have $(c, a) \in R$ and $(b, c) \in R$. By the definition of relation composition, this implies $(b, a) \in R^2$ and hence $P(2)$ is true.

Inductive step: Assume that $P(j)$ is true for all $1 \leq j \leq k$ where $k \geq 2$, i.e., R^j is symmetric. We show that $R^{k+1} = R^k \circ R$ is also symmetric. If $(a, b) \in R^{k+1}$, by definition there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R^k$. By inductive hypothesis, we know $(c, a) \in R$ and $(b, c) \in R^k$. Furthermore, since $R^k = R^{k-1} \circ R$, there exists $d \in A$ such that $(b, d) \in R$ and $(d, c) \in R^{k-1}$. Therefore, we have $(d, a) \in R^k$. Note that $(b, d) \in R$, we get $(b, a) \in R^{k+1}$.

By strong induction, we have that R^n is symmetric for all $n \geq 1$.

(b) Recall that $R^* = \bigcup_{n=1}^{\infty} R^n$. If $(a, b) \in R^*$, then there exists some positive integer n such that $(a, b) \in R^n$. In (a) we already proved that R^n is symmetric, so $(b, a) \in R^n$, and hence $(b, a) \in R^*$. By definition, R^* is symmetric.

□

Q.5 (**5p**) Prove that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution: Suppose that (a, b) is in the symmetric closure of the transitive closure of an arbitrary relation R. It is sufficient to show that (a, b) is also in the transitive closure of the symmetric closure of R. We know that at least one of (a, b) and (b, a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward. Therefore, (a, b) is in the transitive closure of the symmetric closure of R.

□

Q.6 (**10p**) Use the Floyd-Warshall algorithm to find the transitive closures of the relation $R = \{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$ on set $\{a, b, c, d, e\}$.

Solution: Since R is defined on a set consisting of 5 elements, we only need to compute W_5.

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

□

Q.7 (**10p**) Consider the relation $R = \{(x, y) \mid x - y \in \mathbf{Z}\}$.

(a) (**7p**) Prove that R is an equivalence relation on the set of real numbers $\mathbf{R}$.

(b) (**3p**) Describe what elements the following equivalence classes consist of: $[1]$, $[\frac{1}{2}]$, and $[\pi]$.

Solution:

(a) We prove that the given relation R is an equivalence relation:

- *Reflexive:* For all $a \in \mathbf{Z}$, $a - a \in \mathbf{Z}$, so $(a, a) \in R$.
- *Symmetric:* If $(a, b) \in R$, then by definition $a - b \in \mathbf{Z}$. It is clear that $b - a = -(a - b) \in \mathbf{Z}$ and hence $(b, a) \in R$.
- *Transitive:* If $(a, b) \in R$ and $(b, c) \in R$, then we have $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. Therefore, we have $a - c = a - b + b - c \in \mathbf{Z}$ and hence $(a, c) \in R$.

(b) By definition, we have $[1] = \mathbf{Z}$, $[\frac{1}{2}] = \{\frac{1}{2} + n \mid n \in \mathbf{Z}\}$, and $[\pi] = \{\pi + n \mid n \in \mathbf{Z}\}$.

□

Q.8 (**10p**) For any functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$. We say f is *dominated* by g, denoted by $f \preceq g$, if and only if $\forall x \in \mathbf{R}, f(x) \leq g(x)$ holds. Prove or disprove the following:

(a) (**7p**) The relation $\preceq$ is a partial ordering.

(b) (**3p**) The relation $\preceq$ is a total ordering.

Solution:

(a) True. We can prove it as follows:

Reflexive: For all $x \in \mathbf{R}$, $f(x) \le f(x)$, so $f \preceq f$.

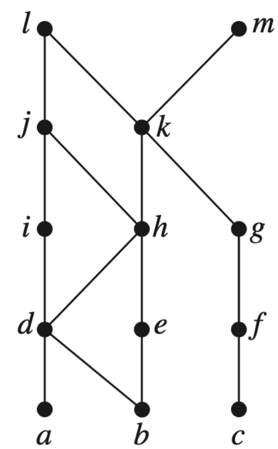
Antisymmetric: If $f \preceq g$ and $g \preceq f$, then for all $x \in \mathbf{R}$ we have $f(x) \le g(x) \le f(x)$ and hence $f(x) = g(x)$, i.e., $f = g$.

Transitive: If $f \preceq g$ and $g \preceq h$, then for all $x \in \mathbf{R}$ we have $f(x) \le g(x) \le h(x)$ and hence $f(x) \le h(x)$, i.e., $f \preceq h$.

(b) False. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\le -1 = g(1)$ and $g(-1) = 1 \not\le -1 = f(-1)$. So it is not the case that for all x, $f(x) \le g(x)$, and it is not the case that for all x, $g(x) \le f(x)$. That is, these two functions are not comparable.

$\square$

Q.9 (**20p**) Answer questions about the partial order represented by the Hasse diagram below:



(a) (**3p**) Find the maximal elements.

(b) (**3p**) Find the minimal elements.

(c) (**2p**) Is there a greatest element?

(d) (**2p**) Is there a least element?

(e) (**3p**) Find all upper bounds of $\{a, e, f\}$.

(f) (**2p**) Find the least upper bound of $\{a, e, f\}$, if it exists.

(g) (**3p**) Find all lower bounds of $\{h, i, j\}$.

(h) (**2p**) Find the greatest lower bound of $\{h, i, j\}$, if it exists.

Solution:

(a) l, m

(b) a, b, c

(c) No

(d) No

(e) k, l, m

(f) k

(g) d, a, b

(h) d

$\square$

Q.10 (**5p**) Topological sorting. Find **all** compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, |)$.

Solution: It is not hard to draw a Hasse diagram for the given poset (omitted here). Then, one can follow the topological algorithm learned in class and list all the compatible total orderings as follows: $\{2, 3, 4, 6, 12\}$, $\{2, 3, 6, 4, 12\}$, $\{2, 4, 3, 6, 12\}$, $\{3, 2, 4, 6, 12\}$, $\{3, 2, 6, 4, 12\}$.

$\square$