04 Complexity of Algorithms

CS201 Discrete Mathematics

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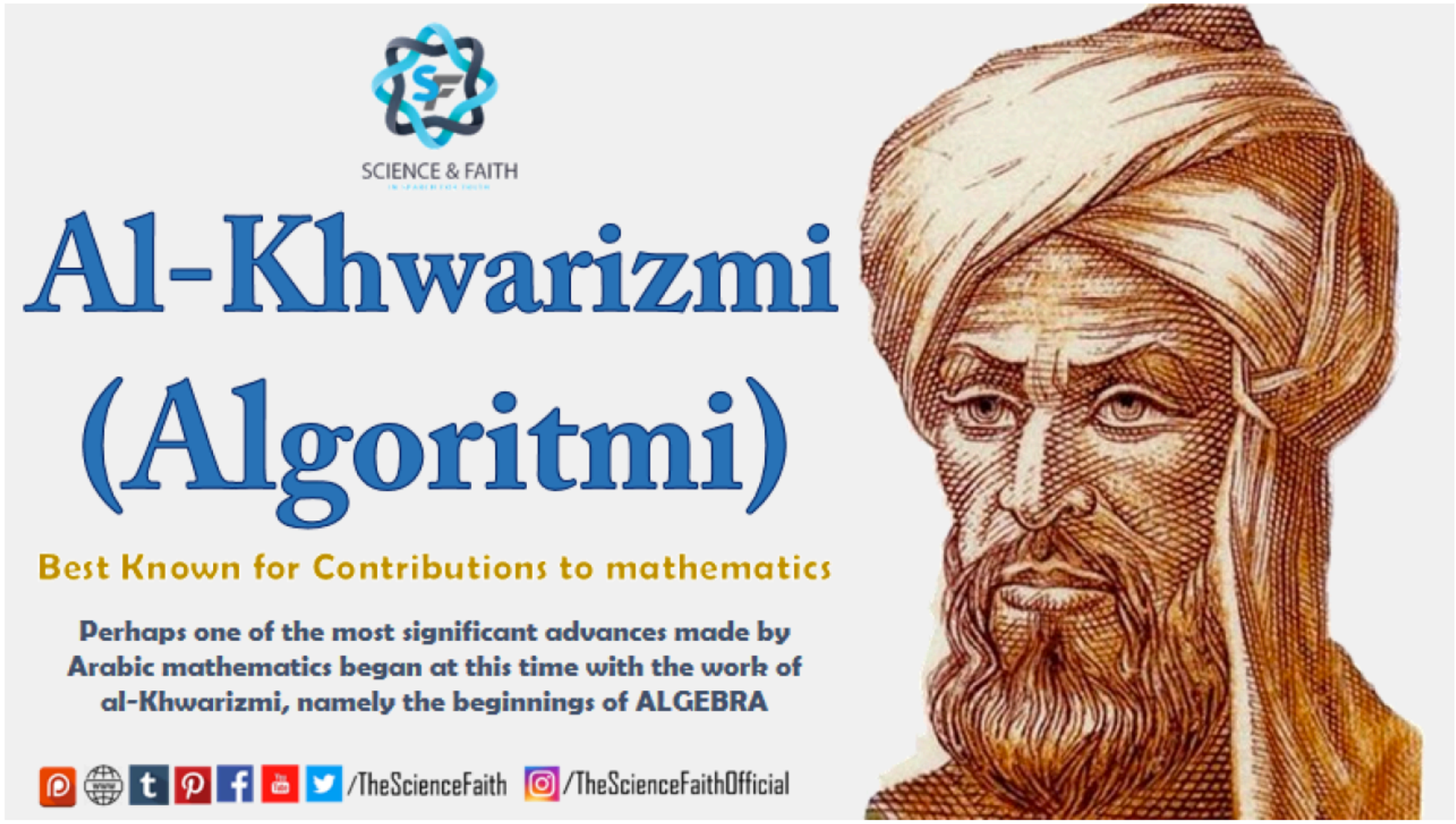
Algorithms

- An algorithm is a finite sequence of precise instructions for performing a computation or for solving a problem.



Al-Khwarizmi Persian polymath

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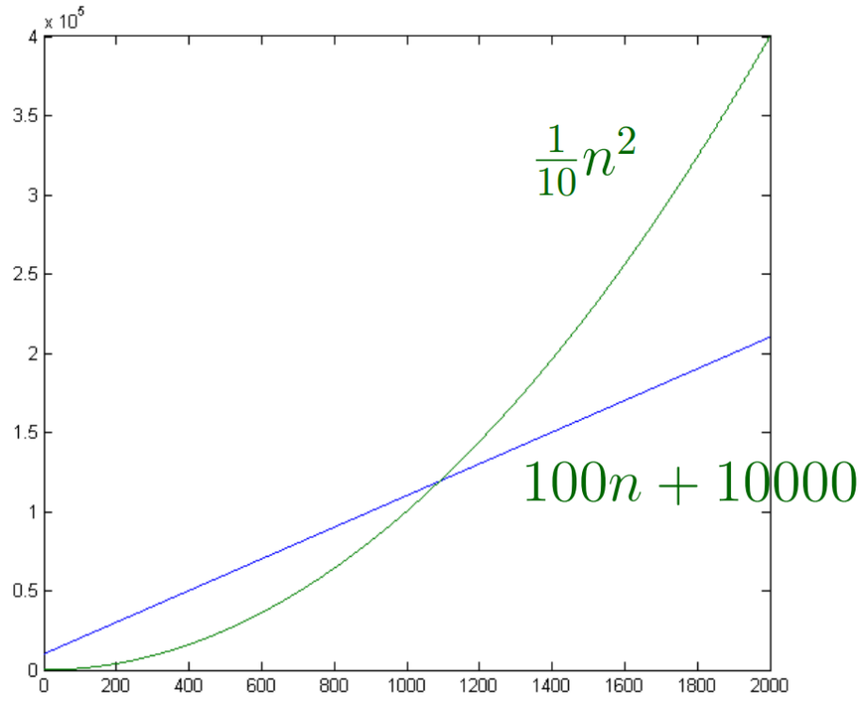
The Growth of Functions

Which Function is Larger?

- **Q:** Which function is "larger"? $n^2/10$ vs $100n + 10000$
- **A:** It depends on the value of n.

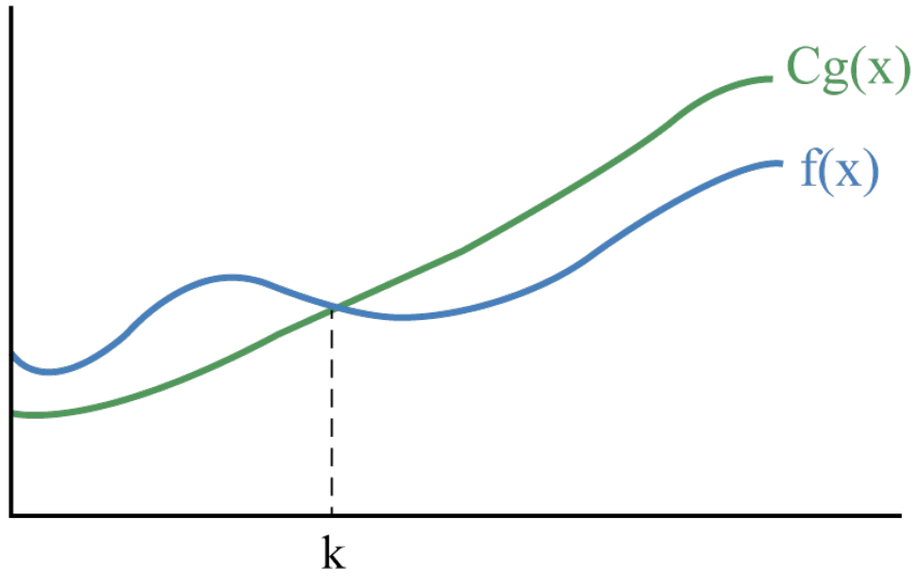


- In computer science, usually we are interested in what happens when the problem input size n gets big.
- When n is "large enough", $n^2/10$ gets bigger than $100n + 10000$ and stays bigger for larger n.

Big-*O* Notation

- **Definition:** Let *f* and *g* be functions from ***Z*** (or ***R***) to ***R***. We say that *f(x)* = *O(g(x))* (read as *f(x)* is big-oh of *g(x)*), if there exist positive constants *C* and *k* such that

$$|f(x)| \leq C|g(x)|, \text{ whenever } x > k.$$



https://calcworkshop.com/functions/big-o/

- Big-*O* gives an upper bound on the growth of a function. It tells us that a function grows at most as fast as the other function.

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Big-*O* Notation

- Example: $100n + 10000 = O(n^2/10)$
  - Let $k = 2000$, we can verify that $\forall n > k,\ 100n + 10000 < n^2/10$
  - Note that the opposite is not true: $n^2/10 \neq O(100n + 10000)$
- Some other $O(n^2)$ functions:
  - $4n^2$
  - $8n^2 + 2n - 3$
  - $n^2/5 + n^{1/2} - 10 \log n$
  - $n(n - 3)$

Big-*O* Estimates for Polynomials

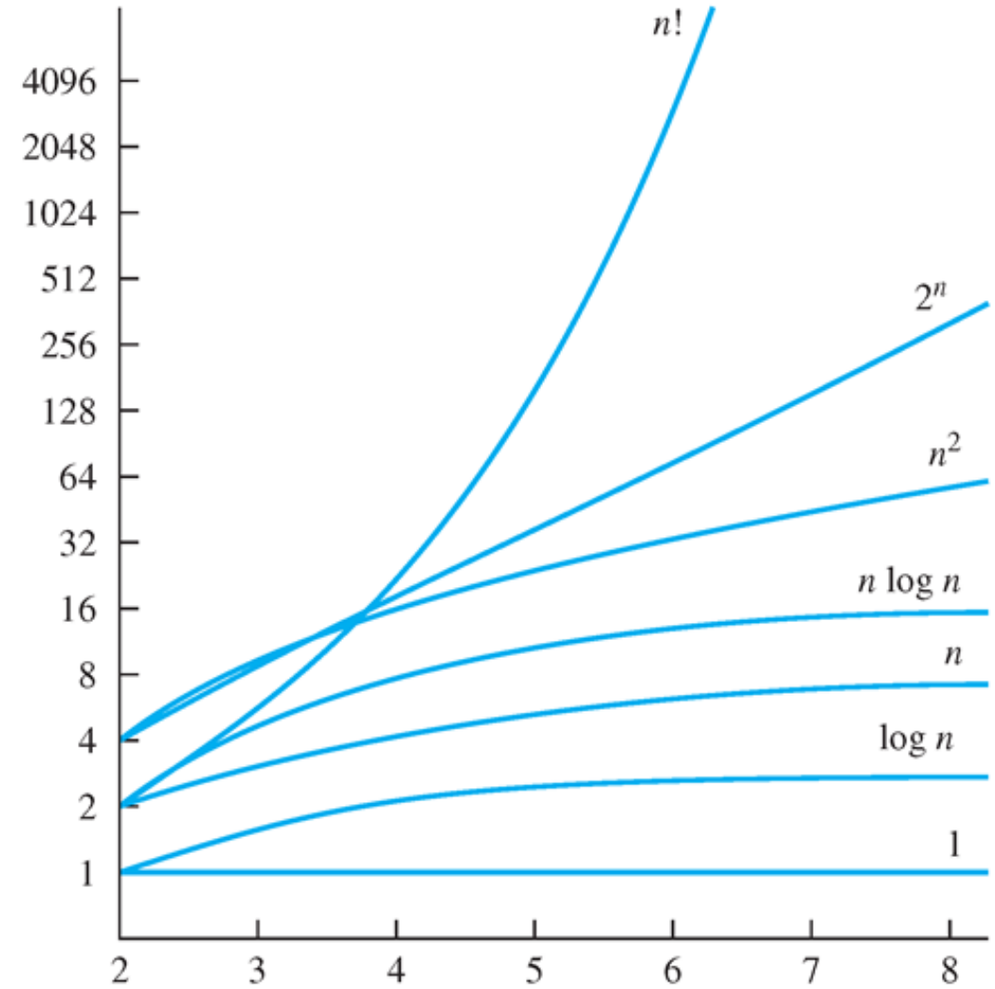
- **Theorem:** Let function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers. Then, $f(x) = O(x^n)$.
  - The leading term $a_n x^n$ of a polynomial dominates its growth.
- Proof:
  - Assuming $x > 1$, we have

$$\begin{aligned} |f(x)| &= |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\ &\le |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\ &= x^n (|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\ &\le x^n (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|) \end{aligned}$$

  - Choose $k = 1$ and $C = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$, then $|f(x)| \le C x^n$ whenever $x > k$.

Some Big-*O* Estimates

- $c = O(1)$ *for constant c*
- $cn = O(n)$ *for constant c*
- $1 + 2 + \cdots + n = O(n^2)$
- $n! = O(n^n)$
- $\log n! = O(n \log n)$
- $\log_a n = O(n)$ *for* $a > 0$
- $n^a = O(n^b)$ *for* $0 \le a \le b$
- $n^a = O(2^n)$



Combination of Functions

- **Theorem:** If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then
  $(f_1 + f_2)(x) = O(max(|g_1(x)|, |g_2(x)|))$

- Proof:
  - By definition, there exist positive constants C_1, C_2, k_1, k_2 such that
    $|f_1(x)| \leq C_1|g_1(x)|$ *when* $x > k_1$
    $|f_2(x)| \leq C_2|g_2(x)|$ *when* $x > k_2$
  - Let $g(x) = max(|g_1(x)|, |g_2(x)|)$, when $x > max(k_1, k_2)$ we have

$$\begin{aligned} |(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \\ &\leq C_1|g_1(x)| + C_2|g_2(x)| \leq C_1|g(x)| + C_2|g(x)| \\ &= (C_1 + C_2)|g(x)| \end{aligned}$$

  - The proof is concluded with $C = C_1 + C_2$ and $k = max(k_1, k_2)$.

Combination of Functions

- **Theorem:** If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then
  $(f_1f_2)(x) = O(g_1g_2(x))$

- Proof: *very similar to the previous theorem*
  - By definition, there exist positive constants C_1, C_2, k_1, k_2 such that
    $|f_1(x)| \leq C_1|g_1(x)|$ *when* $x > k_1$
    $|f_2(x)| \leq C_2|g_2(x)|$ *when* $x > k_2$
  - Let $g(x) = g_1g_2(x)$, when $x > max(k_1, k_2)$ we have

$$\begin{aligned} |(f_1f_2)(x)| &= |f_1(x)f_2(x)| = |f_1(x)||f_2(x)| \\ &\leq C_1|g_1(x)|C_2|g_2(x)| = C_1C_2|g_1(x)g_2(x)| \\ &= C_1C_2|g(x)| \end{aligned}$$

  - The proof is concluded with $C = C_1C_2$ and $k = max(k_1, k_2)$.

Exercise *(3 mins)*

- Sort the following functions by order of growth:
  - $f_1(n) = (1.5)^n$
  - $f_2(n) = 8n^3 + 17n^2 + 111$
  - $f_3(n) = (\log n)^2$
  - $f_4(n) = 2^n$
  - $f_5(n) = \log(\log n)$
  - $f_6(n) = n^2(\log n)^3$
  - $f_7(n) = 2^n(n^2 + 1)$
  - $f_8(n) = 8n^3 + n(\log n)^2$
  - $f_9(n) = 100000$
  - $f_{10}(n) = n!$

Exercise *(3 mins)*

- Sort the following functions by order of growth:
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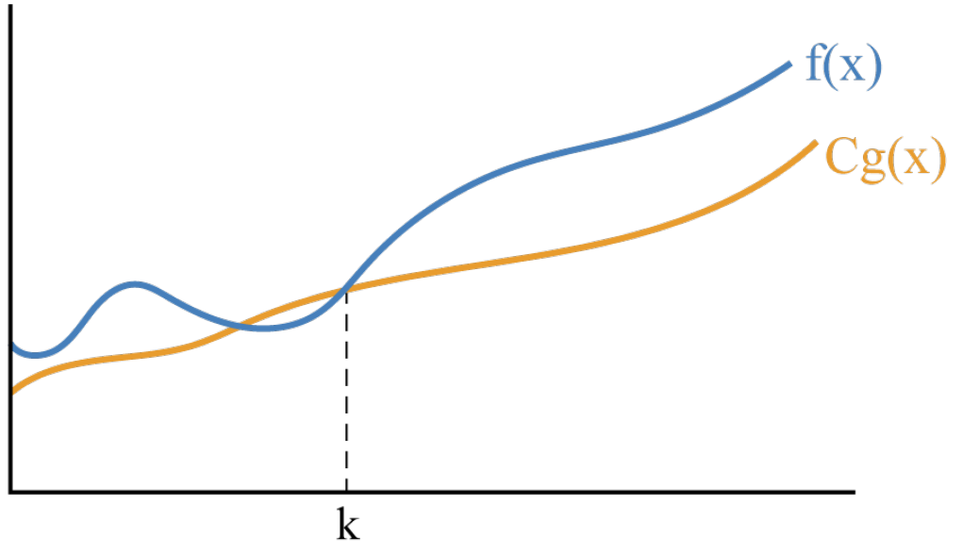
  - $f_7(n) = 2^n(n^2 + 1)$
  - $f_8(n) = 8n^3 + n(\log n)^2$
  - $f_9(n) = 100000$
  - $f_{10}(n) = n!$
- Solution:
  - $f_9 < f_5 < f_3 < f_6 < f_8 < f_2 < f_1 < f_4 < f_7 < f_{10}$

Big-Ω Notation

- **Definition:** Let f and g be functions from $\boldsymbol{Z}$ (or $\boldsymbol{R}$) to $\boldsymbol{R}$. We say that $f(x) = \Omega(g(x))$ (read as $f(x)$ is big-omega of $g(x)$), if there exist positive constants C and k such that

$$|f(x)| \geq C|g(x)|, \text{ whenever } x > k.$$

  - Note: $f(x) = \Omega(g(x))$ if and only if $g(x) = O(f(x))$

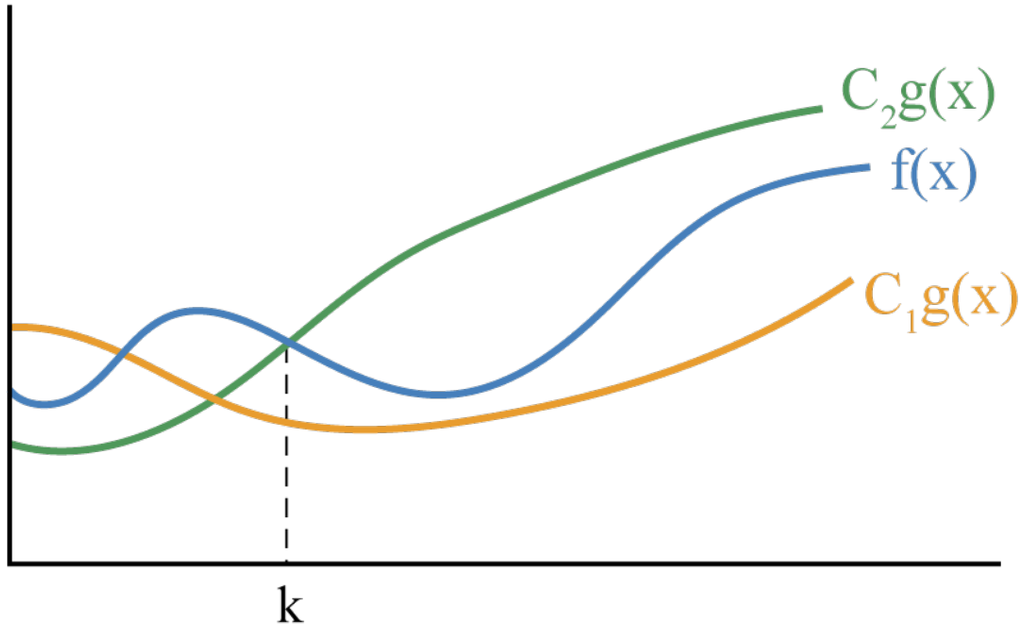


https://calcworkshop.com/functions/big-o/

- Big-Ω gives a lower bound on the growth of a function. It tells us that a function grows at least as fast as the other function.

Big-Θ Notation

- **Definition:** Let *f* and *g* be functions from ***Z*** (or ***R***) to ***R***. We say that *f(x) = Θ(g(x))* (read as *f(x)* is big-theta of *g(x)*), if they have the same order of growth: *f(x) = O(g(x))* and *f(x) = Ω(g(x))*.



https://calcworkshop.com/functions/big-o/

- Note: *f(x) = Θ(g(x))* is equivalent to *g(x) = Θ(f(x))*

Exercise *(2 mins)*

- True or False?
  - $3n^2 + 4n = \Theta(n)$?
  - $3n^2 + 4n = \Theta(n^2)$?
  - $3n^2 + 4n = \Theta(n^3)$?
  - $n/5 + 10n \log n = \Theta(n^2)$?
  - $n^2/5 + 10n \log n = \Theta(n \log n)$?
  - $n^2/5 + 10n \log n = \Theta(n^2)$?

Exercise *(2 mins)*

- True or False?
  - $3n^2 + 4n = \Theta(n)$? — *False*, but $\Omega(n)$
  - $3n^2 + 4n = \Theta(n^2)$? — *True*
  - $3n^2 + 4n = \Theta(n^3)$? — *False*, but $O(n^3)$
  - $n/5 + 10n \log n = \Theta(n^2)$? — *False*, but $O(n^2)$
  - $n^2/5 + 10n \log n = \Theta(n \log n)$? — *False*, but $\Omega(n \log n)$
  - $n^2/5 + 10n \log n = \Theta(n^2)$? — *True*

Complexity of Algorithms

Problems and Algorithms

- **Computational problem:** a task solved by a computer, which formally is a set of problem instances together with a (perhaps empty) set of solutions for every instance.
  - An instance is just a specific problem input, not the problem itself.
- Example:
  - Computational problem: integer factorization
  - A problem instance: factoring the integer *12*
  - Solution to the above instance: $12 = 3 \times 4$
- **Algorithm:** a finite sequence of precise instructions for performing a computation or for solving a problem.
- We say an algorithm solves the problem if it halts (ends) with the correct solution as output for every input problem instance.

Algorithms: Example

- **Computational problem:** a task solved by a computer.
- **Algorithm:** a finite sequence of precise instructions for performing a computation or for solving a problem.
- We say an algorithm solves the problem if it halts (ends) with the correct solution as output for every input problem instance.
- Example: algorithm for solving the sum of $a_1, a_2, \ldots, a_n$
  - Step 1: set $S = 0$
  - Step 2: for $i = 1$ to n, $S := S + a_i$ (i.e., assign S the value $S + a_i$)
  - Step 3: output S

** problem instance example: < 8, 3, 6, 7, 1, 2, 9 > (here n = 7)*

Time and Space Complexity

- **Time complexity:** the number of machine operations (addition, multiplication, assignment, etc.) required by an algorithm
- **Space complexity:** the amount of memory used by an algorithm
- Example: algorithm for solving the sum of $a_1, a_2, \ldots, a_n$

  - Step 1: set $S = 0$
  - Step 2: for $i = 1$ to n, $S := S + a_i$ (i.e., assign S the value $S + a_i$)
  - Step 3: output S
  - **time complexity:** $O(n)$ ** we often ignore operations on iterator i*

    Step 2 takes n operations (in-place additions). Steps 1 and 3 each take 1 operation. Altogether this algorithm takes $n + 2$ operations.
  - **space complexity:** $O(n)$

    The input numbers take $O(n)$ memory and S, i take $O(1)$ memory.

Example: Horner's Method

- Example: consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$
  - **direct computation:** *3* additions and *6* multiplications
  - **better solution:** evaluate $f(x) = 1 + x(2 + x(3 + 4x))$ instead, which takes *3* additions and *3* multiplications

- **Polynomial evaluation:** $f(x) = a_0 + a_1x + \cdots + a_nx^n$

- **Horner's method:** $f(x) = a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + xa_n) \cdots))$
  - Step 1: set $S = a_n$
  - Step 2: for $i = 1$ to n, $S := a_{n-i} + xS$
  - Step 3: output S
  - **time complexity:** $O(n)$

    Steps 1 and 3 each take *1* operation. Step 2 takes *3n* operations: *n* multiplications, *n* additions, *n* assignments.

Another Example

- Determine the **time complexity** of the following algorithm:

```
for i := 1 to n
    for j := 1 to n
        a := 2 * n + i * j;
    end for
end for
```

- Solution:
  - In each iteration, computing the value of a takes 4 operations (2 multiplications, 1 addition and 1 assignment). There are n^2 iterations in two loops. So it takes $n^2 \times 4 = 4n^2$ operations. The **time complexity** of this algorithm is $O(n^2)$.
  - Note: can compute $2 \times n$ only once but still needs $O(n^2)$ complexity.

Exercise *(2 mins)*

- Determine the **time complexity** of the following algorithm:

$$
\begin{aligned}
&S := 0 \\
&\text{for } i := 1 \text{ to } n \\
&\quad \text{for } j := 1 \text{ to } i \\
&\quad\quad S := S + i * j; \\
&\quad \text{end for} \\
&\text{end for}
\end{aligned}
$$

Exercise *(2 mins)*

- Determine the **time complexity** of the following algorithm:

$S := 0$
for $i := 1$ to n
  for $j := 1$ to i
    $S := S + i * j;$
  end for
end for

- Solution:
  - The first *S* assignment takes *1* operation. Computing *S* in each iteration takes *2* operations (1 multiplication and 1 in-place addition). There are $1 + 2 + \ldots + n = n(n + 1)/2$ iterations in two loops. Together it takes $1 + n(n + 1)/2 \times 2 = n^2 + n + 1$ operations.

    The **time complexity** of this algorithm is $O(n^2)$.

Types of Complexity Analysis

- Example: Insertion Sort

Input: $A[1 \ldots n]$ is an array of numbers

for $j := 2$ to n

  $key = A[j]$;

  $i = j - 1$;

  while $i \geq 1$ and $A[i] > key$ do

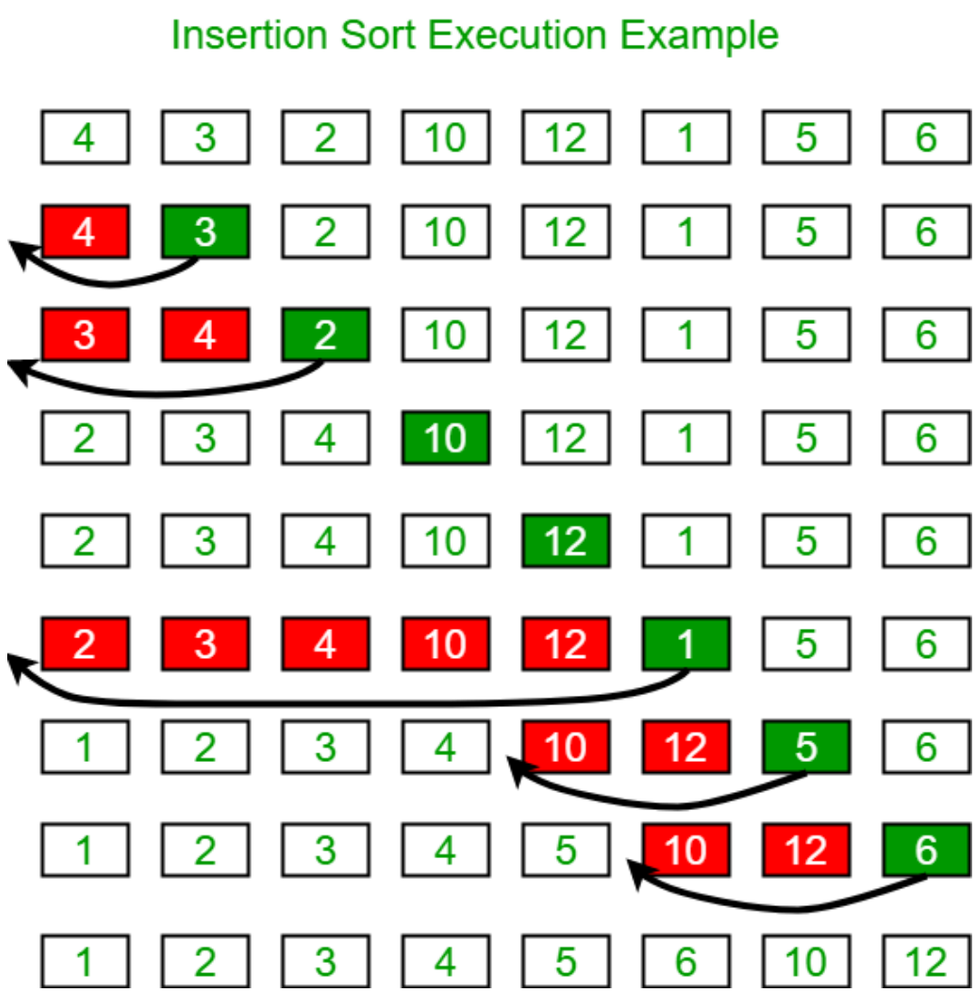
    $A[i + 1] = A[i]$;

    $i - -$;

  end while

  $A[i + 1] = key$;

end for



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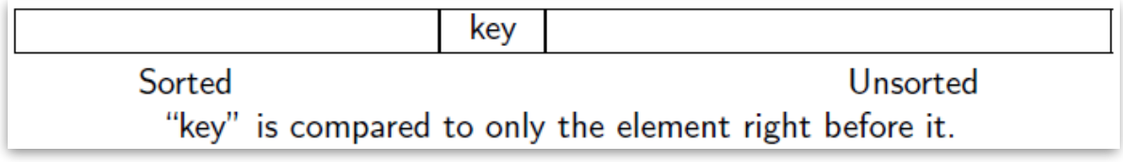
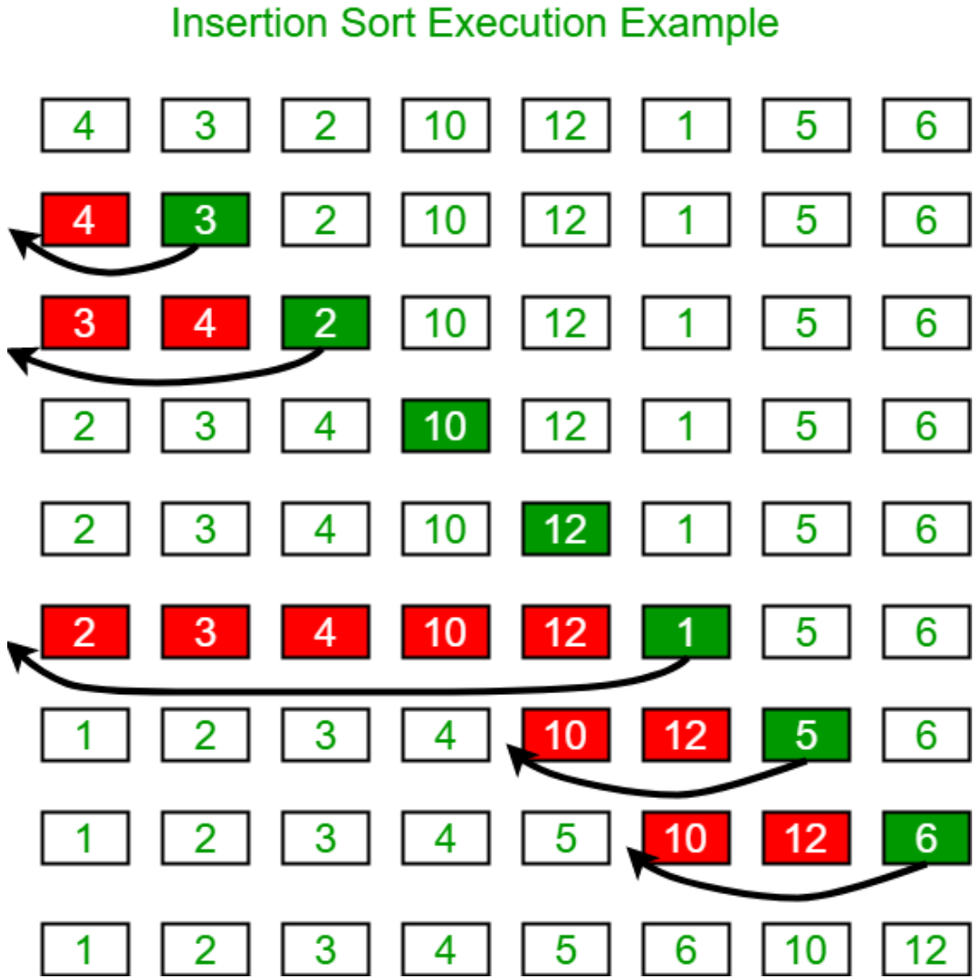
Complexity Analysis: Type I

- **Best-case complexity:**
  - constraints on the input instance rather than the input size
  - resulting in the fastest possible running time for the given size
- Example: Insertion Sort
  - A[1] ≤ A[2] ≤ A[3] ≤ ⋯ ≤ A[n]
  - **time complexity:** *Θ(n)*

    n – 1 comparisons





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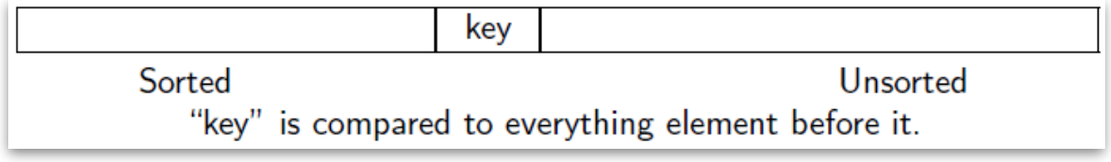
Complexity Analysis: Type II

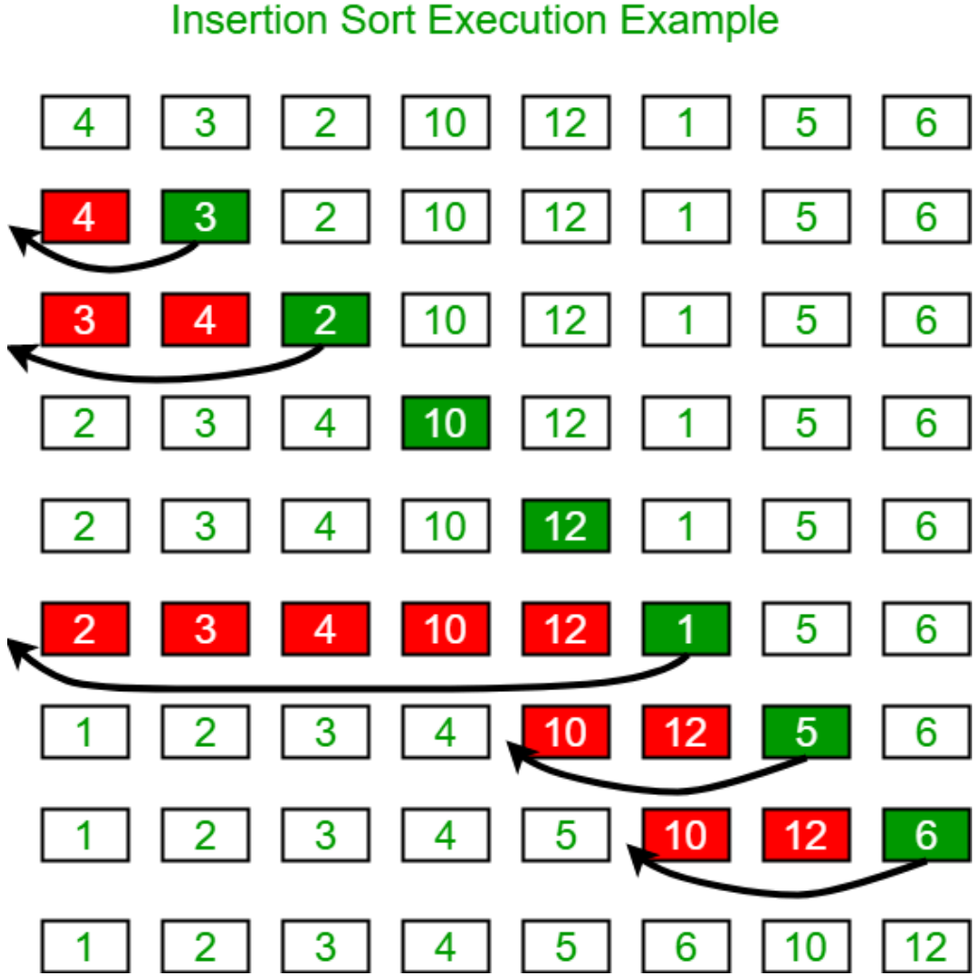
- **Worst-case complexity:**
  - constraints on the input instance rather than the input size
  - resulting in the slowest possible running time for the given size



- Example: Insertion Sort
  - A[1] ≥ A[2] ≥ A[3] ≥ ⋯ ≥ A[n]
  - **time complexity:** $\Theta(n^2)$

$$\sum_{j=2}^{n} j - 1 = \frac{n(n-1)}{2}$$

comparisons and swaps



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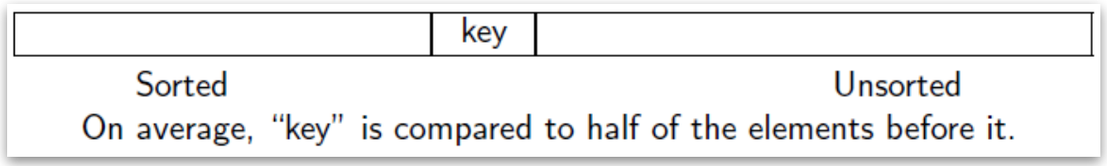
Complexity Analysis: Type III

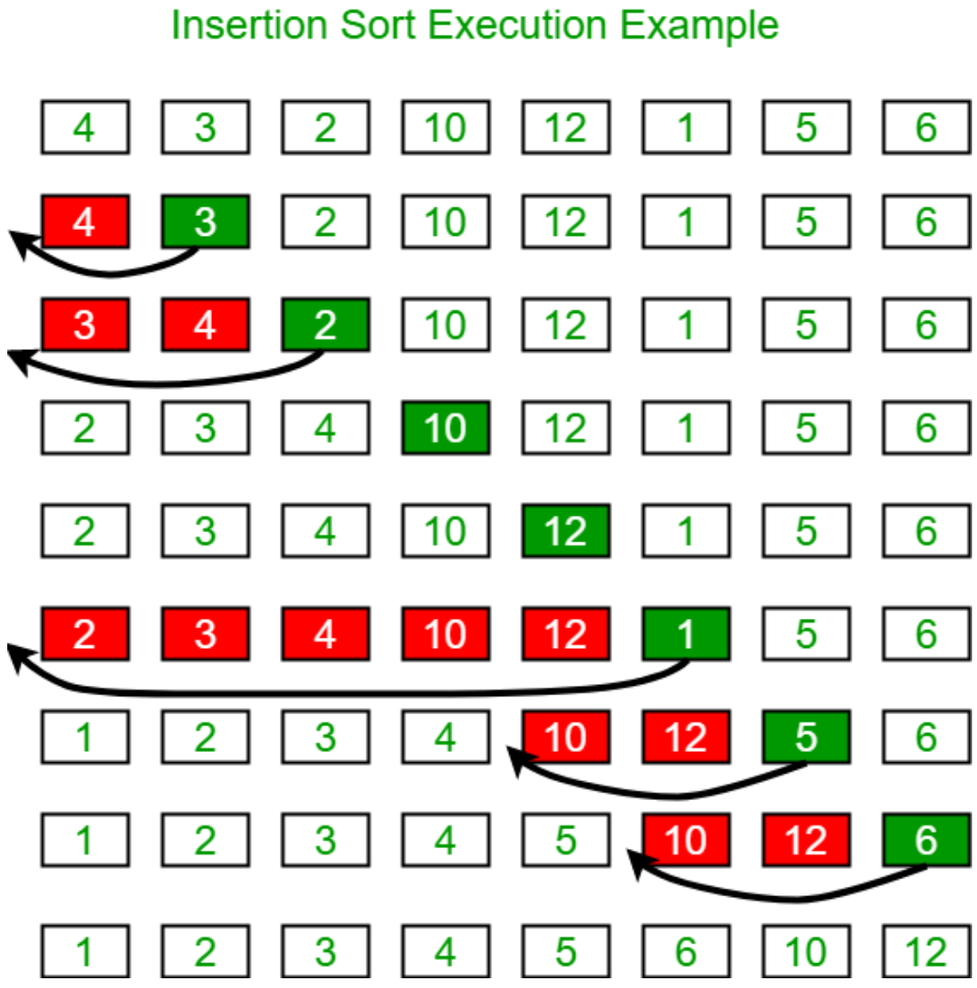
- **Average-case complexity:**
  - constraints on the input instance rather than the input size
  - average running time over all possible inputs for the given size (usually involving probability distribution on input instances)



- Example: Insertion Sort
  - uniformly random distribution
  - **time complexity:** $\Theta(n^2)$

$$\sum_{j=2}^{n} \frac{j-1}{2} = \frac{n(n-1)}{4}$$

comparisons and swaps



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Thoughts on Algorithm Design

- Algorithm design is mainly about designing algorithms that have small Big-*O* running time.
- Being able to design good algorithms lets us identify the hard parts of the problem and handle them effectively and efficiently.
- Too often, programmers try to solve problems using brute force techniques and end up with slow and complicated code!
- A few hours of abstract thought devoted to algorithm design could have speeded up and simplified the solution substantially!

Complexity of Problems

Dealing with Hard Problems

- What would you do if you cannot find an efficient algorithm for a given problem?

Blame yourself

Prove that no such algorithm exists

Dealing with Hard Problems

- Showing that a problem has efficient algorithms is relatively easy:
  - All we have to do is to demonstrate an efficient algorithm.
- Proving that no efficient algorithm exists for a particular problem is quite difficult:
  - How can we prove the non-existence of something?
- We will now learn about ***NP***-complete problems, which provide us with a way to approach the above question.

Introduction to *NP*-Complete

- ***NP*-complete problems:** a very large class of problems (> 3000) which is not known to have any "efficient" solutions.
  - Researchers have spent innumerable man-years trying to find efficient solutions to ***NP***-complete problems but failed.
  - So, ***NP***-complete problems are very likely to be hard.
- It is known that if any one of the ***NP***-complete problems has an efficient solution then all of the ***NP***-complete problems have efficient solutions.
- What we can do: prove that a hard problem is ***NP***-complete.
  - This shows no one can find an efficient solution so far.
- Next, we show how to define such complexity classes formally.

Problem Input Size Matters

- **COMPOSITE:** given a positive integer n, are there integers $d, k \geq 2$ such that $n = d \cdot k$?

- The naive algorithm for determining whether n is composite is to enumerate d from 2 to $n - 1$ to see if any of them divides n.
  - This takes $\Theta(n)$ division operations, which may seem very efficient (linear). However, it is problematic to treat the value of n as the input size of the problem, because integer n is typically processed as a binary string of length $\Theta(\log_2 n)$ rather than $\Theta(n)$. An efficient algorithm should have time complexity "close" to its input size $\Theta(\log_2 n)$ rather than the input value n.

    E.g., $n \times n$ needs only $O((\log_2 n)^2)$ bit operations ** shown in later sections*
  - The input size of COMPOSITE is $L = \log_2 n$. Then, the time complexity is actually $\Theta(n) = \Theta(2^L)$, i.e., exponential in the input size L so impractical.
    ** Note that integer division n/d also takes $O((\log_2 n)^2)$ bit operations as shown in later sections, but here we ignore it for simplicity.*

- **Takeaway:** we should use the input size to measure complexity.

The Input Size of Problems

- Complexity of a problem is measured in terms of its input size.
  - The input size of a problem is the number of bits needed to encode the input of the problem.
- The optimal input size, determined by an optimal encoding method, is hard to compute in most cases.
- For most problems, it is sufficient to choose some natural and simple encoding method and use its encoded input size.

Problem Input Size: Examples

- Example 1: COMPOSITE
  - What is the input size of this problem?

    Any integer $n \geq 1$ can be represented as a binary string $a_0 a_1 \cdots a_L$ of length $\lceil \log_2 (n + 1) \rceil$. Therefore, a natural measure of the input size is $\lceil \log_2 (n + 1) \rceil$ (or $\Theta(\log_2 n)$ for simplicity).

- Example 2: sorting n integers $a_1, \ldots, a_n$
  - What is the input size of this problem?

    Fixed-length encoding: all input numbers share the same length. We write every input integer a_i as a binary string of the same length $m = \lceil \log_2 \max(|a_i| + 1) \rceil + 1$ (with an extra bit for +/– sign).

    This natural encoding gives the input size $n \cdot m$.

Decision and Optimization Problems

- **Decision problem:** a problem that has a yes or no answer.
  - E.g., "Given $n > 0$, is integer m such that $m^m < n$?"
- **Optimization problem:** a problem that asks for some answer that maximizes or minimizes a particular objective function.
  - E.g., "Given $n > 0$, what is the largest integer m such that $m^m < n$?"

- Given an algorithm for solving the optimization problem, solving the corresponding decision problem is usually trivial.
  - Contrapositive: if we prove that a given decision problem is hard to solve efficiently, then the corresponding optimization problem must be (at least as) hard.
- The other direction (decision → optimization) often works too.
  - E.g., use binary search to find the max m in the above examples

Complexity Classes

- **Computational complexity theory** is a field that deals with:
  - classification of certain "decision problems" into several classes:
    the class of "easy" problems
    the class of "hard" problems
    the class of "hardest" problems
    …
  - relations among the above classes
  - properties of problems in the above classes
- How to classify decision problems?
  - Use polynomial-time algorithms (often called efficient algorithms)

Polynomial-Time Algorithms

- **Polynomial-time algorithm:** an algorithm that runs in time $O(n^c)$, where $c > 0$ is a constant number independent of n, and n is the input size of the problem that the algorithm solves.
  - E.g., popular sorting algorithms are polynomial-time algorithms.

- Our expectations for efficient algorithms:
  - When the input size of an efficient algorithm expands from n to n^c (for any constant $c > 0$), the algorithm should still be efficient.
  - An algorithm that is composed by several efficient algorithms should still be efficient.

- These somehow explain why people chose polynomial-time to define efficient algorithms, because the common operations (e.g., addition, subtraction, multiplication, composition, etc.) are closed for polynomials.

Non-Polynomial-Time Algorithms

- **Non-polynomial-time algorithm:** an algorithm of which the running time is not $O(n^c)$ for any constant $c > 0$.
  - E.g., the brute-force algorithm for solving COMPOSITE
- Non-polynomial-time algorithms are usually impractical.
  - E.g., exponential time 2^n for $n = 100$ takes billions of years!!!
- **Caveat:** even polynomial-time algorithms could be impractical.
  - E.g., a $\Theta(n^{20})$ algorithm may not be very practical for $n = 100$.

Tractable Problems and Class *P*

- **Tractable problem:** a problem that is solvable in polynomial time (or the problem is in polynomial time). That is, there exists a polynomial-time algorithm that solves the problem.

- **Class *P*** consists of all decision problems that are solvable in polynomial time. That is, there is a polynomial-time algorithm that decides if any given input instance is a yes-input or a no-input.
  - E.g., PRIMES (determining whether a number is prime) is in ***P***.

- How to prove that a decision problem is in ***P***?
  - Find a polynomial-time algorithm * *relatively easy*

- How to prove that a decision problem is not in ***P***?
  - Prove that there exists no polynomial-time algorithm for solving this problem * *much much harder*

Certificates and Class *NP*

- A decision problem is usually formulated as: "Is there an object satisfying some conditions?"

- A certificate/proof/witness for a yes-input is a specific object that is used to verify/prove/show that this input is indeed a yes-input.
  - E.g., the COMPOSITE problem can be formulated as:
    "Is there an integer d $(1 < d < n)$ such that d divides n?"
    A certificate for a composite number n (a yes-input of COMPOSITE) can be one of such integer factors d.

- **Class *NP*** *(Nondeterministic Polynomial-time)* consists of all decision problems for which there is a polynomial-time algorithm V such that, an input is a yes-input if and only if there is a certificate with which V can verify the input is indeed a yes-input.
  - E.g., COMPOSITE is in ***NP*** because a certificate d always exists for a composite number n and whether d divides n can be verified in polynomial time: $O((\log_2 n)^2)$ bit operations.

P = NP?

- Whether ***P*** = ***NP*** is one of the most important problems in CS.
- It is not hard to see that ***P*** ⊆ ***NP***. ** why?*
- Intuitively, ***NP*** ⊆ ***P*** is doubtful.
  - Just being able to verify a certificate in polynomial time does not necessarily mean we can tell whether an input is a yes-input or a no-input in polynomial time, e.g., certificates may be hard to find.
  - So far, we are still far from solving it and do not know the answer. However, the search for such a solution has provided us with deep insights into what distinguishes “easy” problems from “hard” ones.

NP-Complete and *NP*-Hard

- ***NP*-complete:** consists of the hardest problems in ***NP***.
  - ***NP***-complete problems are reducible to each other, i.e., they are equivalently hard.

    If solving problem A can be transformed into solving problem B, we say A reduces to B. This also means B is at least as hard as A.

- ***NP*-hard:** consists of decision problems that are at least as hard as those in ***NP***-complete.
  - Note: some ***NP***-hard problems may not even belong to ***NP***.

05 Number Theory and Cryptography

To be continued…

Midterm Exam (Early Notice)

- Midterm exam will take place in class (16:20~18:10) on Nov 1 and it captures materials from 02 Logic and Proofs to 05 Number Theory and Cryptography.
  - Midterm exam is closed-book.
  - If your student ID **< 12311200**, please go to classroom **107**.
  - If your student ID **> 12311200**, please go to classroom **108**.