STA219 Assignment 3

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1. (1) Let X represent the household income, then $X \sim N(900, 200^2)$, $\mu = 900$, $\sigma = 200$.

$$X = 600 \to Z = \frac{600 - 900}{200} = -1.5,\ X = 1200 \to Z = \frac{1200 - 900}{200} = 1.5.$$

$$\therefore P(600 \le X \le 1200) = P(-1.5 \le Z \le 1.5) = P(Z \le 1.5) - P(Z \ge -1.5) = \Phi(1.5) - (1 - \Phi(1.5))$$

$$= 0.9332 - (1 - 0.9332) = 0.8664.$$

$\therefore$ The proportion of "the middle class" is about 86.64%.

(2) $\because$ According to the Standard Normal Table, $P(Z < -1.88) = 0.0301$.

$\therefore X = \sigma Z + \mu = 200 * (-1.88) + 900 = 524$.

$\therefore$ Families whose income below 524 coins will receive food stamps.

2. $\because \Delta = b^2 - 4ac = 4^2 - 4 \cdot X = 16 - 4X$.

$\therefore \Delta = 16 - 4X < 0 \Leftrightarrow X > 4$.

$\because P(\Delta < 0) = P(X > 4) = 0.5$.

$\therefore \mu = 4$.

3. Let X represent the English score of one student.

$\because$ According to the Standard Normal Table, $P(Z > 2.00) = P(Z < -2.00) = 0.0228 \approx 0.023$.

$$\therefore \sigma = \frac{X - \mu}{Z} = \frac{96 - 72}{2} = 12.$$

$$\because X = 60 \to Z = \frac{60 - 72}{12} = -1,\ X = 84 \to Z = \frac{84 - 72}{12} = 1$$

$$\therefore P(60 \le X \le 84) = P(-1 \le Z \le 1) = P(Z \le 1) - P(Z \ge -1) = \Phi(1) - (1 - \Phi(1))$$

$$= 0.8413 - (1 - 0.8413) = 0.6826.$$

$\therefore$ The probability that the score is between 60 and 84 points is 68.26%.

4. Let D represent the diameter and S represent the area, then $D \sim \text{Uniform}(a, b)$ and $S = \dfrac{\pi D^2}{4}$.

$$\because E(D) = \frac{a + b}{2},\ E(D^2) = \frac{a^2 + ab + b^2}{3}$$

$$\therefore E(S) = \frac{\pi}{4} E(D^2) = \frac{\pi}{4} \cdot \frac{a^2 + ab + b^2}{3} = \frac{\pi(a^2 + ab + b^2)}{12}.$$

5. $\because \Phi(Z)$ is the CDF of Z

$\therefore$ It's a continuous and non-decreasing function, i.e. its inverse function $\Phi^{-1}(Z)$ exists.

$\therefore$ Define $Y = \Phi(Z)$, then $Y \sim \text{Uniform}(0, 1)$.

$$\therefore E(\Phi(Z)) = \frac{1}{2},\ \text{Var}(\Phi(Z)) = \frac{1}{12}.$$

6. (1) $\because X \sim N(0, 1)$

$$\therefore \phi(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

$\because Y = g(X) = |X|$ is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, 0)$.

$\therefore$ For any $y \in (0, \infty)$, we have $F_Y(y) = P(Y_1 \le y) = P(|X| \le y) = P(-y \le X \le y) = \Phi(y) - \Phi(-y)$.

$$\therefore f_Y(y) = F_Y'(y) = \phi(y) + \phi(-y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-y)^2}{2}} = \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

$$\therefore f_Y(y) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, & y \ge 0 \\ 0, & \text{otherwise} \end{cases}.$$

(2) $\because Y = g(X) = 2X^2 + 1$ is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, 0)$.

$\therefore$ For any $y \in [1, \infty)$, we have $F_Y(y) = P(2X^2 + 1 \le y) = P(-\sqrt{\frac{y-1}{2}} \le X \le \sqrt{\frac{y-1}{2}}) = \Phi(\sqrt{\frac{y-1}{2}}) - \Phi(-\sqrt{\frac{y-1}{2}})$.

$\therefore f_Y(y) = F_Y'(y) = \frac{1}{2\sqrt{2(y-1)}}(\phi(\sqrt{\frac{y-1}{2}}) + \phi(-\sqrt{\frac{y-1}{2}})) = \frac{1}{2\sqrt{2(y-1)}}(\frac{1}{\sqrt{2\pi}}e^{-\frac{y-1}{4}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{y-1}{4}}) = \frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}}$.

$$\therefore f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{\pi(y-1)}} e^{-\frac{y-1}{4}}, & y \ge 1 \\ 0, & \text{otherwise} \end{cases}.$$

7. $\because X \sim \text{Exp}(2)$

$\therefore \phi(X) = 2e^{-2x}$ when $x \ge 0$.

$\therefore \Phi(X) = \displaystyle\int_0^x 2e^{-2u} du = 1 - e^{-2x}$.

For $Y = e^{-2X}$:

For any $y \in (0,1)$, we have $F_Y(y) = P(Y \le y) = P(e^{-2X} \le y) = P(X \ge -\frac{1}{2}\ln y) = 1 - \Phi(-\frac{1}{2}\ln y) = 1 - (1 - e^{-2\cdot -\frac{1}{2}\ln y}) = y$.

$\therefore Y = 1 - e^{-2X}$ follow the uniform distribution on $(0,1)$.

For $Y = 1 - e^{-2X}$:

For any $y \in (0,1)$, we have $F_Y(y) = P(1 - e^{-2X} \le y) = P(X \le -\frac{1}{2}\ln(1-y)) = \Phi(-\frac{1}{2}\ln(1-y)) = 1 - e^{-2\cdot -\frac{1}{2}\ln(1-y)} = y$.

$\therefore Y = 1 - e^{-2X}$ follow the uniform distribution on $(0,1)$.

8. (1) According to the normalization, $\displaystyle\int_0^\infty \int_0^\infty ke^{-(3x+4y)} dxdy = \int_0^\infty -\frac{k}{3}e^{-4y} dy = \frac{k}{12} = 1$.

$\therefore k = 12$.

$\therefore$ The joint CDF of (X, Y) when $0 < x, y < \infty$ is given by

$$F(x,y) = \int_0^y \int_0^x 12e^{-(3u+4v)} dudv = 12 \cdot \frac{1-e^{-3x}}{3} \cdot \frac{1-e^{-4y}}{4} = (1-e^{-3x})(1-e^{-4y}).$$

(2) $\because X > 0, Y > 0, X + Y < 1$

$\therefore X < 1 - Y, Y < 1$.

$$\begin{aligned}\therefore P(X+Y<1) &= \int_0^1 \int_0^{1-y} 12e^{-(3x+4y)} dxdy \\ &= \int_0^1 4(1 - e^{-3(1-y)})e^{-4y} dy \\ &= 4(\int_0^1 e^{-4y} dy - \int_0^1 e^{-3-y} dy) \\ &= 4[\frac{1-e^{-4}}{4} - (e^{-3} - e^{-4})] \\ &= -4e^{-3} + 3e^{-4} + 1.\end{aligned}$$

9. For $0 < x < y < \infty$, $f_X(x) = \displaystyle\int_x^\infty e^{-y} dy = e^{-x}$, $f_Y(y) = \displaystyle\int_0^y e^{-y} dx = ye^{-y}$.

10. $\because f_{X|Y}(x|y) \triangleq \dfrac{f(x,y)}{f_Y(y)}$

$\therefore$ For $0 < x < y < 1$, $f(x,y) = f_Y(y) f_{X|Y}(x|y) = 5y^4 \dfrac{3x^2}{y^3} = 15x^2 y$.

$\because 0 < x < y < 1$

$\therefore X > 0.5$ only occurs when $y > 0.5$.

$\therefore P(X > 0.5) = \displaystyle\int_{0.5}^1 \int_{0.5}^y 15x^2 y dxdy = \int_{0.5}^1 (5y^4 - \frac{5}{8}y) dy = (y^5 - \frac{5}{16}y^2)\Big|_{0.5}^1 = (1 - \frac{5}{16}) - (\frac{1}{32} - \frac{5}{64}) = \frac{47}{64}$.