STA219 Assignment 5

12312706 Zhou Liangyu

1. Let X_i be the working time of the i-th component, then $X_i \sim \text{Exp}(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}$, $\text{E}(X) = \dfrac{1}{\lambda}$, $\text{Var}(X) = \dfrac{1}{\lambda^2}$.

   Since the device functions normally only when all 3 components are working properly, $T = \min(T_1, T_2, T_3)$.

   $\because X_i \overset{\text{i.i.d.}}{\sim} \text{Exp}(\lambda)$

   $\therefore f_T(t) = nf_X(t)(1 - F_X(t))^{n-1} = 3\lambda e^{-\lambda t}(1 - (1 - \lambda e^{-\lambda t}))^2 = 3\lambda e^{-3\lambda t}$.

2. (1) $X - Y \sim \text{N}(\mu_X - \mu_Y, \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2) = \text{N}(0, 2 - 2\rho)$.

   $\therefore X - Y$ follows normal distribution, $\mu_{X-Y} = 0$, $\sigma_{X-Y}^2 = 2 - 2\rho$.

   $$\therefore f_{X-Y}(z) = \frac{1}{\sqrt{2\pi\sigma_{X-Y}^2}} e^{-\dfrac{(z - \mu_{X-Y})^2}{2\sigma_{X-Y}^2}} = \frac{1}{2\sqrt{\pi(1-\rho)}} e^{-\dfrac{z^2}{4 - 4\rho}}.$$

   (2) $\text{Cov}(X - Y, XY) = \text{Cov}(X, XY) - \text{Cov}(Y, XY)$

   $= (\text{E}(X^2Y) - \text{E}(X)\text{E}(XY)) - (\text{E}(XY^2) - \text{E}(Y)\text{E}(XY))$

   $= \text{E}(X^2Y) - \text{E}(XY^2)$.

   According to the symmetry of X and Y, $\text{E}(X^2Y) = \text{E}(XY^2)$.

   $\therefore \text{Cov}(X - Y, XY) = 0$.

   $$\therefore \text{Cor}(X - Y, XY) = \frac{\text{Cov}(X - Y, XY)}{\sqrt{\text{Var(X-Y)Var(XY)}}} = 0.$$

3. (1) $$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{y^2}{2\sigma_y^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{1}{\sqrt{1-\rho^2}})^2(\frac{x}{\sigma_x} - \rho\frac{y}{\sigma_y})^2}\, dx.$$

   Let $t = \dfrac{1}{\sqrt{1-\rho^2}}(\dfrac{x}{\sigma_x} - \rho\dfrac{y}{\sigma_y})$, then $dx = \sigma_x\sqrt{1-\rho^2}\, dt$.

   $$\therefore f_Y(y) = \frac{1}{2\pi\sigma_y} e^{-\frac{y^2}{2\sigma_y^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt.$$

   $\because$ According to the normalization of standard normal distribution, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = 1$

   $$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{y^2}{2\sigma_y^2}}.$$

   (2) $$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

   $$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2}(\frac{1}{\sqrt{1-\rho^2}})^2(\frac{x}{\sigma_x} - \rho\frac{y}{\sigma_y})^2 - \frac{y^2}{2\sigma_y^2}} \cdot \sqrt{2\pi}\sigma_y e^{-\frac{y^2}{2\sigma_y^2}}$$

   $$= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2}(\frac{1}{\sqrt{1-\rho^2}})^2(\frac{x}{\sigma_x} - \rho\frac{y}{\sigma_y})^2}$$

   $$= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_x^2}(x - \rho\frac{\sigma_x y}{\sigma_y})^2}.$$

   $$\therefore X|Y = y \sim \text{N}(\rho\frac{\sigma_X y}{\sigma_Y}, \sigma_X^2(1 - \rho^2)).$$

   $$\therefore E(X|Y = y) = \rho\frac{\sigma_X y}{\sigma_Y},\ Var(X|Y = y) = \sigma_X^2(1 - \rho^2).$$

4. (1) $\because X|Y = y \sim \text{N}(\mu_x + \dfrac{\rho\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1 - \rho^2))$

$$\therefore X|Y=6800 \sim \mathrm{N}(4500+\frac{0.65\times 1500}{2000}\times(6800-5500), 1500^2\times(1-0.65^2)) = \mathrm{N}(5133.75, 1299375).$$

$$\therefore P(X<6800|Y=6800) = P(\frac{X-\mu_{X|Y}}{\sigma_{X|Y}} < \frac{6800-5133.75}{\sqrt{1299375}}) \approx \Phi(1.46) = 0.9279.$$

(2) $P(X>Y \mid X+Y=12000) = P(X-Y>0 \mid X+Y=12000)$. Let $W_1 = X-Y$ and $W_2 = X+Y$.

$$\because \boldsymbol{\mu} = \begin{pmatrix}\mu_X\\ \mu_Y\end{pmatrix} = \begin{pmatrix}4500\\ 5500\end{pmatrix},\ \boldsymbol{\Sigma} = \begin{pmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{pmatrix} = \begin{pmatrix}1500^2 & 1950000\\ 1950000 & 2000^2\end{pmatrix}.$$

$$\therefore \begin{pmatrix}X-Y\\ X+Y\end{pmatrix} = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix} \sim \mathrm{N}(\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}4500\\ 5500\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1500^2 & 1950000\\ 1950000 & 2000^2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}^\top) = \mathrm{N}(\begin{pmatrix}-1000\\ 10000\end{pmatrix}, \begin{pmatrix}2350000 & -1750000\\ -1750000 & 10150000\end{pmatrix}).$$

$$\because \rho_{12} = \frac{-1750000}{\sqrt{2350000\times 10150000}} = -0.35832,\ \sigma_{W_1} = \sqrt{2350000} \approx 1532.97,\ \sigma_{W_2} = \sqrt{10150000} \approx 3185.91$$

$$\therefore W_1|W_2=12000 \sim \mathrm{N}(-1000+\frac{-0.35832\times 1532.97}{3185.91}\times 2000, 2350000\times(1-0.35832^2)) = \mathrm{N}(-1344.83, 1431.18^2).$$

$$\therefore P(W_1>0|W_2=12000) = P(\frac{W_1-\mu_{W_1|W_2}}{\sigma_{W_1|W_2}} > \frac{1344.83}{1431.18}) \approx 1-\Phi(0.94) = 0.1736.$$

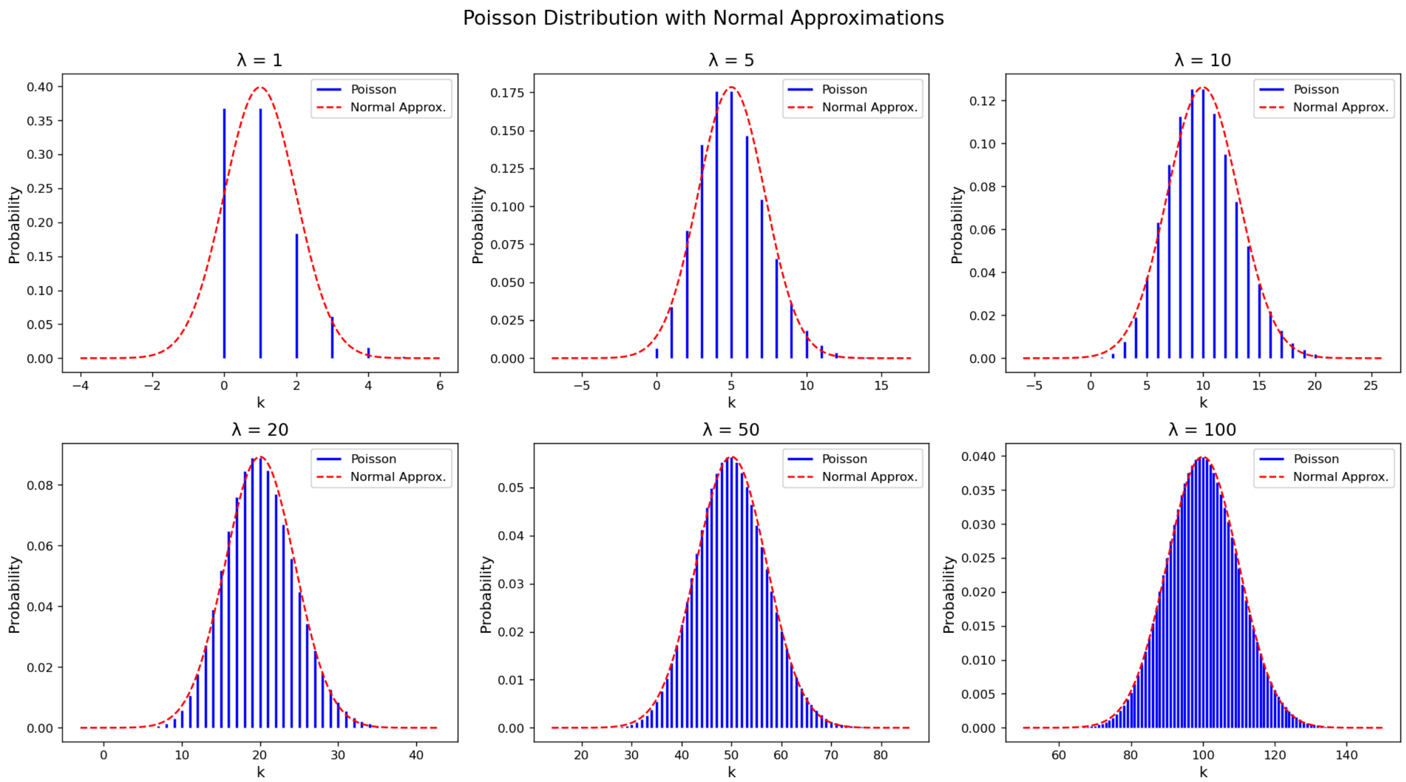
5. (1) Consider a r.v. $X \sim \text{Poisson}(\lambda)$ and its standardized version $Z = \frac{X-\lambda}{\lambda}$:

According to Central Limit Theorem, when $\lambda \to \infty$, Z converges in distribution to $\mathrm{N}(0,1)$, i.e. $X \sim \mathrm{N}(\lambda, \lambda)$.

(2) Python code and plots:

```python
import math
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import poisson, norm

# Parameters
lambdas = [1, 5, 10, 20, 50, 100]

# Create subplots
fig, axes = plt.subplots(2, 3, figsize=(16, 9), dpi=120)
fig.suptitle('Poisson Distribution with Normal Approximations', fontsize=16)

# Loop through each p value
for idx, (l, ax) in enumerate(zip(lambdas, axes.flat)):

    # Calculate mean and standard deviation for Normal approximation
    mean = l
    std = np.sqrt(l)

    # Define the x range
    lower = math.floor(mean - 5 * std)
    upper = math.ceil(mean + 5 * std)
    x_discrete = np.arange(lower, upper)
    x_continuous = np.arange(lower, upper, 0.01)

    # Poisson Distribution (λ = np)
    poisson_dist = poisson.pmf(x_discrete, mean)

    # Normal Approximation (mean = np, std = sqrt(np(1-p)))
    normal_dist = norm.pdf(x_continuous, mean, std)

    # Plot the distributions
    ax.vlines(x_discrete, ymin=0, ymax=poisson_dist, label='Poisson', color='blue', linewidth=2)
    ax.plot(x_continuous, normal_dist, '--', label='Normal Approx.', color='red')

    # Set the title and labels
```

```python
    ax.set_title(f'λ = {l}', fontsize=14)
    ax.set_xlabel('k', fontsize=12)
    ax.set_ylabel('Probability', fontsize=12)

    # Add a legend
    ax.legend()

# Adjust layout for better spacing
plt.tight_layout()
plt.subplots_adjust(top=0.9)

# Show the plot
plt.show()
```



It's obvious that when λ is small($\lambda = 1,\ 5$), the normal distribution is significantly different from the Poisson distribution, and the approximation effect is not good; when λ is large($\lambda \geq 20$), the shape of the normal distribution is close to the Poisson distribution, and the approximation works well.

6. (1) Let X and Y be the number of heads and tails in n tosses, then $X \sim \text{Binomial}(n, \dfrac{1}{2})$, $\text{E}(X) = \dfrac{n}{2}$, $\text{Var}(X) = \dfrac{n}{4}$.

   $\text{E}(X - Y) = \text{E}(2X - n) = 2\text{E}(X) - n = 0$, $\text{Var}(X - Y) = \text{Var}(2X - n) = 4\text{Var}(X) = n$.

   (2) Carl's opinion is correct. According to the Law of Large Numbers, as $n \to \infty$, $\dfrac{X}{n} \to \dfrac{1}{2}$.

   Based on the calculation above, although $\text{E}(X - Y) = 0$, since $\text{Var}(X - Y) = n$, $|X - Y|$ may increase as n increases. For example, when $n = 10^6$, the largest value of $|X - Y|$ may be 10^3. Therefore, the number of heads will not be close to the number of tails.

7. According to the Law of Large Numbers, $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables with expectation μ and variance σ^2, then Y_n converges in probability to $\text{E}(X_i)$ as $n \to \infty$.

   When n is large, we can apply CLT, and $Z_n = \dfrac{Y_n - n \cdot \dfrac{\mu}{n}}{\sqrt{n \cdot \dfrac{\sigma^2}{n^2}}}$ converges in distribution to a standard normal random variable.

   $\therefore Y_n \sim \text{N}(\mu, \dfrac{\sigma^2}{n})$, as $n \to \infty$, since $\text{Var}(Y_n) = \dfrac{\sigma^2}{n} \to 0$, Y_n converges to μ.

(1) When $X \sim \text{Poisson}(3)$, $\mu = 3$.

$\therefore Y_n$ converges to 3.

(2) When $X \sim \text{U}[-1,3]$, $\mu = 1$.

$\therefore Y_n$ converges to 1.

(3) When $X \sim \text{Exp}(5)$, $\mu = \frac{1}{5}$.

$\therefore Y_n$ converges to $\frac{1}{5}$.

8. (1) Consider the CDF of geometric distribution: $F(k) = \sum_{i=1}^{k} p(1-p)^{i-1} = p \cdot \frac{1-(1-p)^k}{1-(1-p)} = 1-(1-p)^k,\ k = 1, 2, \cdots$.

Let $U \sim \text{Uniform}(0,1)$. Divide the interval (0,1) into subintervals: $I_1 = (0, F(1)],\ I_2 = (F(1), F(2)], \cdots$.

For $u_1, u_2, \ldots, u_n$ from a uniform distribution random number generator, we have to find k_i s.t. $u_i \in (F(k_i - 1), F(k_i)]$.

$\therefore F(k_i) \geq u_i \Longrightarrow 1-(1-p)^{k_i} \geq u_i \Longrightarrow (1-p)^{k_i} \leq 1-u_i \Longrightarrow k_i \geq \frac{ln(1-u_i)}{ln(1-p)}$.

Since k_i is an integer, $k_i = \left\lceil \frac{ln(1-u_i)}{ln(1-p)} \right\rceil$, and k_i can be considered number generated from $\text{Geometric}(p)$.

(2) Consider the CDF of Cauchy distribution: $F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+u^2)}\, du = \frac{\arctan x}{\pi} + \frac{1}{2}$. We have $F^{-1}(x) = \tan(\pi(x - \frac{1}{2}))$.

According to the inverse transformation sampling, let $U \sim \text{Uniform}(0,1)$, for $u_1, u_2, \ldots, u_n$, define $x_i = \tan(\pi(x - \frac{1}{2}))$.

Then $x_1, x_2, \ldots, x_n$ can be considered numbers generated from standard Cauchy distribution.