STA219: Probability and Statistics for Engineering

Assignment 6

Note: The assignment can be answered in Chinese or English, either is fine. Please provide derivation and computation details, not just the final answer. Please submit a PDF file on BB.

Part I: Calculations and derivations by hand

1. (15 points) Continued with Example 4.5 (3) in the slide of Chapter 4, prove that if $U_1 \sim \mathrm{U}[0,1]$ and $U_2 \sim \mathrm{U}[0,1]$ are independent, and let

$$\begin{cases} Z_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) \\ Z_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2) \end{cases},$$

then Z_1 and Z_2 are a pair of independent standard normal random variables.

(Hint: Show that $P(Z_1 \le a, Z_2 \le b) = \Phi(a)\Phi(b)$ for all a and b, which requires variable substitution in a double integral.)

2. (10 points) A network provider investigates the load of its network. The number of concurrent users is recorded at ten locations (thousands of people),

17.2 22.1 18.5 17.2 18.6 14.8 21.7 15.8 16.3 22.8

(1) Compute the sample mean, sample variance, and sample standard deviation of the number of concurrent users. (5 points)

(2) Compute the sample lower and upper quartile, and sample interquartile range. (5 points)

3. (20 points) Let X_1, X_2, X_3 be a simple random sample from the population $X \sim \mathrm{U}(0,\theta)$.

(1) Show that $\hat{\theta}_1 = \frac{4}{3}X_{(3)}$ and $\hat{\theta}_2 = 4X_{(1)}$ are both unbiased estimators of θ. (10 points)

(2) Which of these two estimators is more efficient? (10 points)

Part II: Implementations by Python

Note: Please provide both the results and the code, and present them in one PDF file.

4. (15 points) Continued with Problem 8 in HW5, generate 10000 numbers from the following distributions based on a uniform distribution random number generator in Python, plot the histogram of the generated numbers, and compare it with the theoretical PMF/PDF of the following distributions:

(1) Geometric(0.5); (5 points)

(2) Standard Cauchy distribution. (10 points)

5. (20 points) Apply the rejection sampling technique to sample from

$$f^*(x) = 0.6\exp\{-(x+5)^2/2\} + 0.4\exp\{-(x-1)^2/0.5\},\ x \in (-\infty, \infty).$$

(1) Choose an appropriate proposal distribution and plot it to show that it covers the target distribution. (5 points)

(2) Apply the rejection sampling method to generate 500,000 samples, plot the histogram of the generated samples, and compare it with the theoretical PDF $f(x)$ (normalized $f^*(x)$). (10 points)

(3) What is the acceptance proportion? (5 points)

6. (20 points) A forest consists of $1{,}000$ trees forming a perfect 20×50 rectangle as in the figure below. The northwestern (top-left) corner tree catches fire. Wind blows from the northwest, therefore trees can only catch fire from its buring left and above neighbors (火只会向东或向南蔓延). Assume the probability that any tree catches fire from its burning left neighbor is 0.8, and the probabilities to catch fire from trees immediately to the above is 0.3.

(1) Conduct a Monte Carlo study to estimate the probability that more than 30% of the forest will eventually be burning. With probability 0.95, your answer should differ from the true value by no more than 0.005. (10 points)

(You should first determine the value of n that satisfies the requirement, and then conduct a Monte Carlo study to estimate the probability.)

(2) Based on the same study, predict the total number of affected trees X. (5 points)

(3) What is the corresponding standard deviation of X. (5 points)