STA219 Assignment 2

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1. (1) Fischer will win the game if he is the first player to win a game with less than 10 successive draws. Therefore, he will experience n consecutive draws before winning this match, where $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

$$\therefore P(\text{Fischer wins the match}) = \sum_{k=0}^{9} (0.3)^k \cdot 0.4 = \frac{1-(0.3)^{10}}{1-0.3} \cdot 0.4 \approx 0.5714.$$

(2) For a match with a clear winner, suppose that the duration of the match is n $(n < 10)$. The players will experience $n-1$ successive draws, and then the winner will win the last game, where $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

If the duration of the match is 10, the players will experience at least 9 successive draws.

Let X represent the duration of the match, then $X \sim \text{Geometric}(p)$ where $p = 0.4 + 0.3 = 0.7$, and the PMF of X is given by

$$p(x) = \begin{cases} 0.7 \cdot 0.3^{x-1}, & x = 1, 2, \ldots, 9 \\ 0.3^9, & x = 10 \end{cases}.$$

2. (1) Let X represent the number of trials is needed to open the door.

$$P(X=1) = \binom{5}{1} = \frac{1}{5},\ P(X=2) = \binom{5}{4}\binom{4}{1} = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5},$$

$$P(X=3) = \binom{5}{4}\binom{4}{3}\binom{3}{1} = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{5},\ P(X=4) = \binom{5}{4}\binom{4}{3}\binom{3}{2}\binom{2}{1} = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{5},$$

$$P(X=5) = 1 - P(X=1) - P(X=2) - P(X=3) - P(X=4) = \frac{1}{5}.$$

$\therefore$ The PMF of the number of trials is given by $p(x) = \dfrac{1}{5}$.

(2) $X \sim \text{Geometric}(p)$, $p = 0.2$.

$\therefore$ The PMF of the number of trials is given by $p(x) = p(1-p)^{x-1} = 0.2 \cdot 0.8^{x-1},\ x = 1, 2, \ldots$.

3. $P(X=0) = \binom{10}{8} = \dfrac{4}{5},\ P(X=1) = \binom{10}{2}\binom{9}{8} = \dfrac{2}{10} \cdot \dfrac{8}{9} = \dfrac{8}{45},\ P(X=2) = \binom{10}{2}\binom{9}{1} = \dfrac{2}{10} \cdot \dfrac{1}{9} = \dfrac{1}{45}.$

$$\mathrm{E}(X) = \sum_{k=0}^{2} x_k p_k = 0 \cdot \frac{4}{5} + 1 \cdot \frac{8}{45} + 2 \cdot \frac{1}{45} = \frac{2}{9}.$$

$$\mathrm{Var}(X) = \sum_{k=0}^{2} (x_k - \mathrm{E}(X))^2 p_k = (0 - \frac{2}{9})^2 \cdot \frac{4}{5} + (1 - \frac{2}{9})^2 \cdot \frac{8}{45} + (2 - \frac{2}{9})^2 \cdot \frac{1}{45} = \frac{88}{405} \approx 0.2173.$$

4. According to normalization of PDF, $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_0^1 (ax + bx^2)dx = \frac{1}{2}ax^2 + \frac{1}{3}bx^3\Big|_0^1 = 1.$

$$\therefore \frac{1}{2}a + \frac{1}{3}b = 1.$$

$$\because \mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 (ax^2 + bx^3)dx = \frac{1}{3}ax^3 + \frac{1}{4}bx^4\Big|_0^1 = \frac{2}{3}$$

$$\therefore \frac{1}{3}a + \frac{1}{4}b = \frac{2}{3}.$$

$$\therefore a = 2,\ b = 0.$$

$$\therefore \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathrm{E}(X))^2 f(x)dx = \int_0^1 (x - \frac{2}{3})^2 \cdot 2x\ dx = \frac{1}{2}x^4 - \frac{8}{9}x^3 + \frac{4}{9}x^2\Big|_0^1 = \frac{1}{18}.$$

5. $\because p_x(k) = \dfrac{\lambda^k}{k!}e^{-\lambda}$

$\therefore \dfrac{p_x(k+1)}{p_x(k)} = \dfrac{\frac{\lambda^{k+1}}{(k+1)!}e^{-\lambda}}{\frac{\lambda^k}{k!}e^{-\lambda}} = \dfrac{\lambda}{k+1}.$

$\therefore$ When k does not reach the largest integer not exceeding λ, $\dfrac{p_x(k+1)}{p_x(k)} > 1$, $p_x(k)$ increases monotonically;

when k reaches the largest integer not exceeding λ, $\dfrac{p_x(k+1)}{p_x(k)} \leq 1$, and after that point, $\dfrac{p_x(k+1)}{p_x(k)} < 1$, $p_x(k)$ decreases monotonically with k.

6. For the insurance company to **make no profit** in this life insurance, the minimum number of deaths is $\dfrac{2500 \times 12}{2000} = 15.$

$X \sim \text{Binomial}(2500, 0.002).$

Since $n = 2500 > 100$ and $p = 0.002 < 0.05$, we apply $\text{Poisson}(5)$ to approximate $\text{Binomial}(2500, 0.002)$.

$P(X \leq 15) \approx \displaystyle\sum_{k=0}^{15} \frac{5^k}{k!}e^{-5} \approx 0.999931.$

$\therefore P(X > 15) = 1 - P(X \leq 15) = 1 - 0.999931 = 0.000069 = 0.0069\%.$

$\therefore$ The probability that the insurance company loses money in this life insurance is 0.0069%.

7. (1) The expected time between jobs is $\dfrac{60}{3} = 20$ minutes.

(2) Since the jobs are sent to the printer in constant rate, the number of jobs sent to the printer per hour follows $\text{Poisson}(3)$.

Let X be the time until the next job is sent(in hour), $t = 5$ minutes $= \dfrac{1}{12}$ hour, then $X \sim \text{Exp}(3)$.

$\therefore P(X \leq t) = 1 - e^{-3 \cdot \frac{1}{12}} = 1 - e^{-\frac{1}{4}} \approx 0.2212.$

$\therefore$ The probability that the next job is sent within 5 minutes is 0.2212.

8. Considered that the exponential distribution often arises as the distribution of the amount of time until some specific event occurs, we divide the time interval $[0, t]$ into n subintervals, ensuring that the probability of the event occurring is equal within each subinterval. Therefore, we can apply the geometric distribution to describe it.

For $X \sim \text{Exp}(\lambda)$, in each subinterval, the probability of the event occurring is given by $p = \dfrac{\lambda}{n}$.

Let N represent the number of subintervals before the event occurs, then $N \sim \text{Geometric}(\dfrac{\lambda}{n})$.

$\therefore \quad P(X = t) = P(N = tn) = \dfrac{\lambda}{n}(1 - \dfrac{\lambda}{n})^{tn-1}$, which is similar to the PDF of $X \sim \text{Exp}(\lambda)$, and also $P(X > t) = P(N > tn) = (1 - p)^{tn}$.

Let $m = \dfrac{1}{n}$, then $\displaystyle\lim_{n \to \infty} P(X > t) = \lim_{n \to \infty} P(N > tn) = \lim_{m \to 0}(1 - \lambda m)^{\frac{t}{m}} = e^{-\lambda t}$, which is same as the result calculated using the CDF of $X \sim \text{Exp}(\lambda)$.

Thus, the exponential distribution is the geometric distribution in continuous situation, i.e. the number of discrete experiments approaches infinity.