

CS201: Discrete Mathematics (Fall 2024)
Written Assignment #5
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 13 (please submit to Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (15p) Let S be the set of all strings (including the empty string) of English letters. Determine whether the following relations are *reflexive*, *irreflexive*, *symmetric*, *antisymmetric*, and/or *transitive*. No proof is required.

- (a) (3p) $R_1 = \{(a, b) | a \text{ and } b \text{ have no letters in common}\}$
- (b) (3p) $R_2 = \{(a, b) | a \text{ and } b \text{ are of the same length}\}$
- (c) (3p) $R_3 = \{(a, b) | a \text{ is shorter than } b\}$
- (d) (3p) $R_4 = \{(a, b) | a \text{ and } b \text{ have exactly one letter in common}\}$
- (e) (3p) $R_5 = \{(a, b) | a \text{ contains } b \text{ as a substring}\}$

Q.2 (15p) Consider relations on a set A . **Prove or disprove** the following statements:

- (a) (5p) If R is reflexive and symmetric, then R is transitive.
- (b) (5p) If R_1, R_2 are reflexive, then $R_1 \cup R_2$ is reflexive.
- (c) (5p) If R_1, R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Q.3 (10p) Prove the following statements about n -ary relations:

- (a) (5p) If C_1 and C_2 are conditions that elements of the n -ary relation $R : A_1, \dots, A_n$ may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.
- (b) (5p) If R and S are n -ary relations, then $P_{i_1, i_2, \dots, i_m}(R \cup S) = P_{i_1, i_2, \dots, i_m}(R) \cup P_{i_1, i_2, \dots, i_m}(S)$.

Q.4 (10p) Suppose that a relation R on a set A is symmetric.

- (a) (7p) Prove that, for any positive integer $n \geq 1$, R^n is symmetric.
- (b) (3p) Prove that R^* is symmetric.

Q.5 (5p) Prove that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Q.6 (10p) Use the Floyd-Warshall algorithm to find the transitive closures of the relation $R = \{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$ on set $\{a, b, c, d, e\}$.

Q.7 (10p) Consider the relation $R = \{(x, y) | x - y \in \mathbf{Z}\}$.

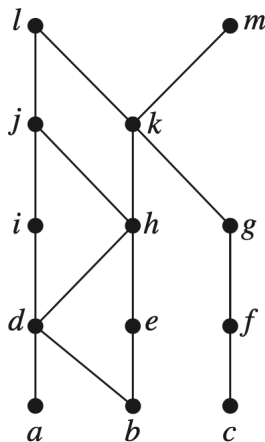
- (a) (7p) Prove that R is an equivalence relation on the set of real numbers \mathbf{R} .
- (b) (3p) Describe what elements the following equivalence classes consist of: $[1]$, $[\frac{1}{2}]$, and $[\pi]$.

Q.8 (10p) For any functions $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$. We say f is *dominated* by g , denoted by $f \preceq g$, if and only if $\forall x \in \mathbf{R}, f(x) \leq g(x)$ holds. Prove or disprove the following:

(a) **(7p)** The relation \preceq is a partial ordering.

(b) **(3p)** The relation \preceq is a total ordering.

Q.9 **(20p)** Answer questions about the partial order represented by the Hasse diagram below:



(a) **(3p)** Find the maximal elements.

(b) **(3p)** Find the minimal elements.

(c) **(2p)** Is there a greatest element?

(d) **(2p)** Is there a least element?

(e) **(3p)** Find all upper bounds of $\{a, e, f\}$.

(f) **(2p)** Find the least upper bound of $\{a, e, f\}$, if it exists.

(g) **(3p)** Find all lower bounds of $\{h, i, j\}$.

(h) **(2p)** Find the greatest lower bound of $\{h, i, j\}$, if it exists.

Q.10 **(5p)** Topological sorting. Find **all** compatible total orderings for the poset $(\{2, 3, 4, 6, 12\}, |)$.