Discrete Mathematics Assignment 6

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1. (a) If every two distinct vertices are connected by an edge, the graph G, a complete graph, has the maximum number of edges. When there is no edges in G, G has the minimum number of edges.

   $\therefore$ The maximum number of edges in G is $\dfrac{n(n-1)}{2}$, the minimum number of edges in G is 0.

   (b) When a vertex is adjacent to all other vertices, it has the maximum number of degrees. When a vertex doesn't have any edges, it has the minimum number of degrees.

   $\therefore$ The maximum degrees of each vertex in G is $n-1$, and the minimum degrees of each vertex in G is 0.

   (c) Assume that G is not connected while the minimum degree of any vertex of G is greater than or equal to $\frac{n-1}{2}$, then it has at least 2 disconnected components G_1 and G_2.

   For an arbitrary vertex n_1 in G_1 and n_2 in G_2, we have $deg(n_1) \leq v_1 - 1$ and $deg(n_2) \leq v_2 - 1$.

   According to our assumption, $deg(n_1) \geq \frac{n-1}{2}$ and $deg(n_2) \geq \frac{n-1}{2}$. Therefore $\frac{n-1}{2} \leq v_1 - 1$ and $\frac{n-1}{2} \leq v_2 - 1$.

   $\therefore n-1 \leq v_1 + v_2 - 2$, i.e. $v_1 + v_2 \geq n+1$, which contradicts to $v_1 + v_2 = n$.

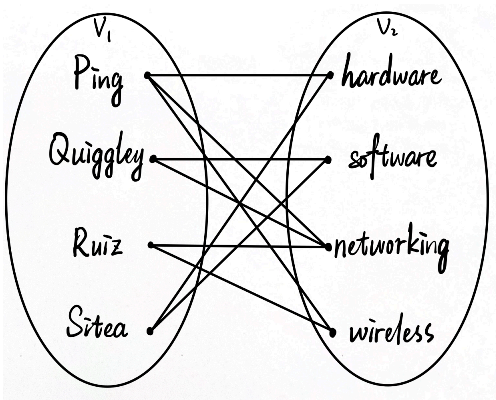
   $\therefore$ If the minimum degree of any vertex of G is greater than or equal to $\frac{n-1}{2}$, then G must be connected.

2. Since every vertices in K_n is adjacent to all other vertices, $\overline{K_n}$ has the same vertices as G, and no vertex in $\overline{K_n}$ is adjacent to other vertices.

   Since the edges between two vertices in $K_{m,n}$ if and only if one vertex is in the first subset and the other vertex is in the second subset, $\overline{K_{m,n}}$ has the same vertices as G, with an edge between two vertices if and only if both two vertex is in the first subset or the second subset.

   Since the edges in C_n are $\{v_1,\ v_2\}, \{v_2,\ v_3\}, \ldots, \{v_{n-1},\ v_n\}, \{v_n,\ v_1\}$, $\overline{C_n}$ has the same vertices as G, and every vertex in $\overline{C_n}$ is adjacent to every other vertex, except for the two vertices that are adjacent to it in the cycle C_n.

   Since two vertices in Q_n are adjacent if and only if they represent differ in exactly one bit position, $\overline{Q_n}$ has the same vertices as G, and two vertices in $\overline{Q_n}$ are adjacent if and only if they represent differ in more than one bit position.

3. (a) 

   (b) $\because$ Every employee is qualified to support more than one areas

   $\therefore$ For every vertex in V_1, it has edges that connect to more than one vertex in V_2.

   $\therefore$ For any subset A that only contains one vertex in V_1, it holds that $|N(A)| \geq |A|$. It is obvious that for any subset A that contains more than one vertex in V_1, the inequality still holds.

   $\therefore$ According to Hall's Marriage Theorem, there exists a complete matching from V_1 to V_2, i.e. there is an assignment of employees to support areas so that each employee is assigned one area to support.

   One possible assignment:

   Ping $\rightarrow$ Wireless, Quiggley $\rightarrow$ Software, Ruiz $\rightarrow$ Networking, Sitea $\rightarrow$ Hardware.

4. The number of edges in a complete graph is $\frac{v(v-1)}{2}$.

$\because$ G and $\overline{G}$ are isomorphic.

$\therefore$ They have the same number of edges.

$\therefore$ The number of edges in G = The number of edges in $\overline{G} = \frac{v(v-1)}{4}$.

$\because$ The number of edges is an integer.

$\therefore v(v-1) = 4k, k = 0, 1, 2, \ldots$

$\therefore v(v-1) \equiv 0 \pmod 4$.

$\therefore v \equiv 0 \pmod 4$ or $v - 1 \equiv 0 \pmod 4$.

$\therefore v \equiv 0$ or $1 \pmod 4$.

5. (a) $\because$ The vertices in the graph is connected by edges, and the length of the edges is non-zero.

$\therefore dist(u, v) \geq 0$.

$\because$ "only if" part: If there is a cycle that begins and ends at the same vertex, the cycle won't be the shortest path from u to u, because we needn't to traverse any path to arrive u from u, i.e. the shortest path length is 0.

"if" part: If $dist(u, v) = 0$, it means that no edges are traversed, which can only happen if u and v are the same vertex, i.e. $u = v$.

$\therefore dist(u, v) = 0$ if and only if $u = v$.

(b) If there exists a shortest path from u to v, we can obtain a shortest path from v to u by reversing the path, and they share the same length.

$\therefore dist(u, v) = dist(v, u)$.

(c) If the vertex w is on the shortest path from u to v, it is obviously $dist(u, v) = dist(u, w) + dist(w, v)$.

If w is not on the shortest path from u to v, since the shortest path from u to v must have length less than or equal to the length of any other path from u to v, $dist(u, v) \leq dist(u, w) + dist(w, v)$.

$\therefore dist(u, v) \leq dist(u, w) + dist(w, v)$.

6. (a) Since one player can't compete against himself, a tournament directed graph cannot have cycles of length 1. If there exists a cycle of length 2, it means that $x \rightarrow y$ and $y \rightarrow x$, i.e. x beats y and y beats x. It's impossible because there can only be one winner in their game, thus a tournament directed graph cannot have cycles of length 2.

(b) Since $x \rightarrow y$ and $y \rightarrow x$ cannot appear at the same time, a tournament graph is always antisymmetric.

Since $x \rightarrow x$ is impossible, a tournament graph is never reflexive and always irreflexive.

If $x \rightarrow y$ and $y \rightarrow z$, it is uncertain whether $x \rightarrow z$ because the winner of the game between x and z is independent with the winner of the game between x and y and the game between y and z. Thus a tournament graph is sometimes transitive.

(c) "only if" part: If a tournament directed graph represents a strict total ordering, the relation is transitive, i.e. if $x \rightarrow y$ and $y \rightarrow z$, then $x \rightarrow z$ rather than $z \rightarrow x$. It proves that there are no cycles of length 3.

"if" part: If there are no cycles of length 3, for any three distinct players x, y, and z, there can be no cyclic relationships between them, i.e. if $x \rightarrow y$ and $y \rightarrow z$, then $z \rightarrow x$ is impossible, thus the relation is transitive. Since every game has a winner, every two players are comparable. In (a), we have already proven that the relation is irreflexive and antisymmetric. Thus a tournament directed graph represents a strict total ordering.

7. Assume that an edge e that is not contained in a circuit is not a *cut edge*. Let the two endpoints of e be u and v.

Since it is not a *cut edge*, its removal doesn't influence the connectivity of the graph, thus there must exists at least one path from u to v.

It means that in the original graph, there exists at least 2 paths from u to v, and they form a circuit.

However, e is one of those paths, which contradicts to the assumption that e is not contained in a circuit.

Therefore, if e is not contained in a circuit, it must be a *cut edge*.

8. Consider a connected regular simple graph G of degree k.

   Each vertex in G is adjacent to k other vertices. Since one edge has two endpoints, each edge in G shares one of its two endpoints with $2(k-1)$ other edges, which leads to an vertex in $L(G)$ of degree $2(k-1)$.

   $\therefore$ Each vertex in $L(G)$ has $2(k-1)$ degree.

   Each edge in G shares one of its two endpoints with other edges, which means that the corresponding vertex in $L(G)$ is adjacent to other vertices in $L(G)$.

   $\therefore L(G)$ is connected.

   $\therefore L(G)$ has an Euler circuit.

9. Base case: In a 2-cube Q_2, 00 and 01, 01 and 11, 11 and 10, 10 and 00 are adjacent. Thus there exists a Hamiltonian circuit in Q_2: $00 \to 01 \to 11 \to 10 \to 00$.

   Inductive step: Assume that an arbitrary n-cube Q_n has a Hamiltonian circuit.

   Construct a $n+1$-cube: Let $Q_{0,n}$ and $Q_{1,n}$ be two copies of Q_n. Rename every vertex s in $Q_{0,n}$ by $0s$ and every vertex s in $Q_{1,n}$ by $1s$. Adding edges to the pairs $(0s, 1s)$, we can get a $n+1$-cube Q_{n+1}.

   To find a Hamiltonian circuit, we can traverse the Hamiltonian circuit in $Q_{0,n}$, and at the end vertex $0s$ of that circuit, we cross the edge $(0s, 1s)$ we added above to arrive $Q_{1,n}$. Then we traverse the Hamiltonian circuit in $Q_{1,n}$ from the vertex $1s$. After arriving the end vertex of that circuit, the path we just took forms a Hamiltonian circuit. Therefore, Q_{n+1}

   $\therefore$ By induction, it follows that Q_n has a Hamiltonian circuit for all $n \geq 2$.

10. Assume that G is planar, then $e_G \leq 3v - 6 = 3 \cdot 11 - 6 = 27$.

    $\because e_G + e_{\overline{G}} = \frac{v(v-1)}{2} = \frac{11 \cdot 10}{2} = 55$.

    $\therefore$ If $e_G > 27$, then G is non planar.

    If $e_G \leq 27$, then $e_{\overline{G}} \geq 28$, which means that $e_{\overline{G}}$ is non planar.

    $\therefore G$ or $\overline{G}$ is nonplanar.

11. Since there is no simple circuits of length 4 or less, and each region shares one edge at most, we have $5r \leq 2e$.

    $\because r = e - v + 2$

    $\therefore e - v + 2 \leq \frac{2}{5}e$.

    $\therefore \frac{3}{5}e \leq v - 2$.

    $\therefore e \leq \frac{5}{3}v - \frac{10}{3}$.

12. Consider an arbitrary student among the 17 students, according to Pigeonhole Principle, he will discuss the same problem with at least $\lceil \frac{16}{3} \rceil = 6$ other students.

    Consider the 6 other students:

    If any pair of the 6 students discuss the same problem with the student we mentioned at the beginning, then we got 3 students who all pairwise discuss the same problem.

    If the problems discussed among these 6 students are all different from the student we mentioned at the beginning:

    Consider an arbitrary student among the 6 students, according to Pigeonhole Principle, he will discuss the same problem with at least $\lceil \frac{5}{2} \rceil = 3$ other students. If any pair of the 3 other students discuss the same problem with he, then we got 3 students who all pairwise discuss the same problem. If not, the 3 students only have one choice of problem to discuss, and we also got 3 students who all pairwise discuss the same problem.

    Therefore, there are at least 3 students who all pairwise discuss the same problem.

13. (a) Let F_n be the n-th Fibonacci number.

    The leaves of T_n are the leaves of T_{n-1} and T_{n-2} since the root does not count as a leaf. Thus, the number of leaves L_n is given by: $L_n = L_{n-1} + L_{n-2}$. Since $L_1 = 1$ and $L_2 = 1$, we have $L_n = F_n$.

Since the Fibonacci tree T_n is constructed by add T_{n-1} as the left child and T_{n-2} as right child of a root node, the Fibonacci tree is a full binary tree. According to the principle of full binary tree the number of internal nodes is one less than the number of leaves. Thus, the number of internal nodes is given by $I_n = L_n - 1 = F_n - 1$.

Adding the internal nodes and the leaves together, we can get the number of vertices $v_n = 2F_n - 1$.

(b) According to the construction of Fibonacci tree T_n, for $n \geq 2$, each time n increases by 1, its height also increases by 1. Since when $n = 1$ or 2 the height is 0, the height H_n is given by $H_n = n - 2$.

14.

$3\ 2\ \times\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ \times\ -$

$\downarrow\ 3 \times 2 = 6$

$6\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ \times\ -$

$\downarrow\ 6^2 = 36$

$36\ 5\ 3\ -\ 8\ 4\ /\ \times\ -$

$\downarrow\ 5-3=2$

$36\ 2\ 8\ 4\ /\ \times\ -$

$\downarrow\ 8/4=2$

$36\ 2\ 2\ \times\ -$

$\downarrow\ 2\times 2=4$

$36\ 4\ -$

$\downarrow\ 36-4=32$

32

15. (a) We start from node a, and initial the value $L(a)$ to $L(a) = 0$.

Update the values of node b and d: $L(b) = 5$, $L(d) = 2$, and add a to set S.

Then we choose node d, update the values of node e, g and h: $L(e) = 2 + 7 = 9$, $L(g) = 2 + 6 = 8$, $L(h) = 2 + 8 = 10$, and add d to set S.

Then we choose node b, update the values of node c and f: $L(c) = 5 + 4 = 9$, $L(f) = 5 + 6 = 11$, and add b to set S.

Then we choose node g, there is no need to update values, and add g to set S.

Then we choose node c, there is no need to update values, and add c to set S.

Then we choose node e, update the values of node f: $L(f) = 9 + 1 = 10$, and add e to set S.

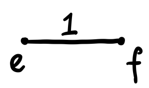
Then we choose node f, update the values of node i: $L(i) = 10 + 4 = 14$, and add f to set S.

Then we choose node h, update the values of node i: $L(i) = 10 + 2 = 12$, and add h to set S.

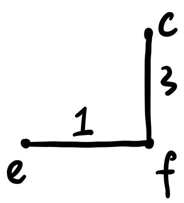
Then we choose node i, there is no need to update values, and add i to set S.

$\therefore$ The shortest path from a to f is $a \to b \to e \to f$., and its length is $L(f) = 10$.

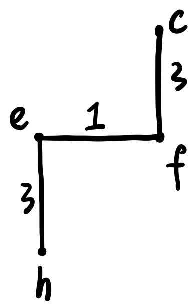
(b) We start from the edge $\{e, f\} = 1$, and add it to MST.



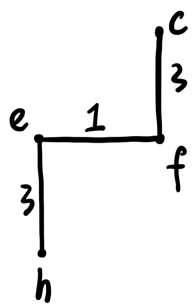
Then we choose the edge $\{f, c\} = 3$, and add it to MST.



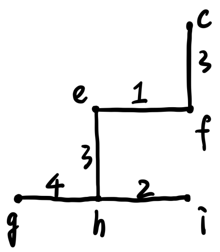
Then we choose the edge $\{e, h\} = 3$, and add it to MST.



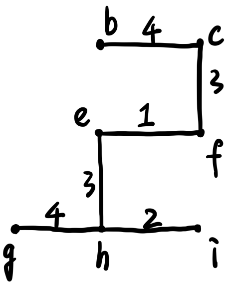
Then we choose the edge $\{h, i\} = 2$, and add it to MST.



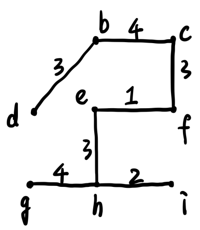
Then we choose the edge $\{g, h\} = 4$, and add it to MST.



Then we choose the edge $\{b, c\} = 4$, and add it to MST.



Then we choose the edge $\{b, d\} = 3$, and add it to MST.



Then we choose the edge $\{a, d\} = 2$, and add it to MST.