工程概率统计

Probability and Statistics for Engineering

第三章 联合分布

Chapter 3 Joint Distributions

Chapter 3 Joint Distributions

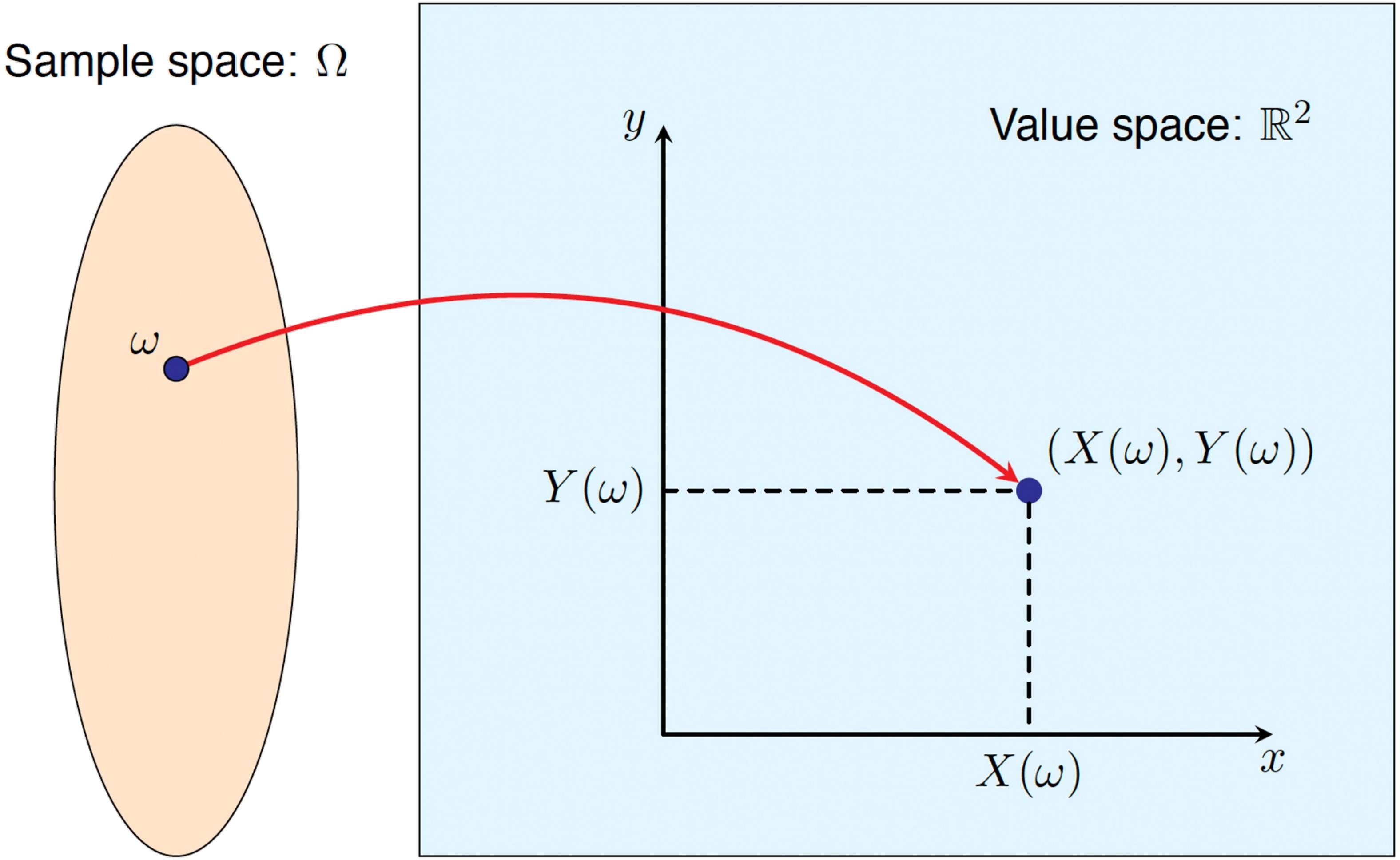
3.1 Random Vector and Joint Distribution

- We have been considering a single random variable each time, however, often we need to deal with several random variables. E.g.,
  - The height and weight of a randomly selected person.
  - The temperature, humidity, wind, precipitation of Singapore on a randomly selected day.
- Since different metrics of the same object are typically associated with each other, they should be considered simultaneously.
- Here we will talk about how to study two random variables X and Y simultaneously, all the concepts can be extended to n random variables $X_1, X_2, \dots, X_n$.

Random Vector

We say that (X, Y) is a **random vector (随机向量)** if $\omega \in \Omega \longmapsto (X(\omega), Y(\omega))$ is a function valued on $\mathbb{R}^2$. (X, Y) can also be called a two-dimensional random variable (二维随机变量).

3.1 Random Vector and Joint Distribution



3.1 Random Vector and Joint Distribution

- The major concepts that will be introduced for random vectors are:
  - (Joint) CDF/PMF/PDF;
  - Marginal CDF/PMF/PDF;
  - Conditional PMF/PDF.
- First, for a discrete/continuous random vector, we can describe its distribution using the (joint) CDF (cumulative distribution function).

Joint Cumulative Distribution Function

For a random vector (X, Y), either discrete or continuous, its cumulative distribution function (CDF, 累积分布函数) is defined as

$$F(x, y) = P(X \leq x, Y \leq y), \forall x, y \in \mathbb{R}.$$

$F(x, y)$ is also called the joint CDF (联合累积分布函数) of X and Y.

- You can treat $\{X \leq x\}$ as an event A, and $\{Y \leq y\}$ as an event B, then

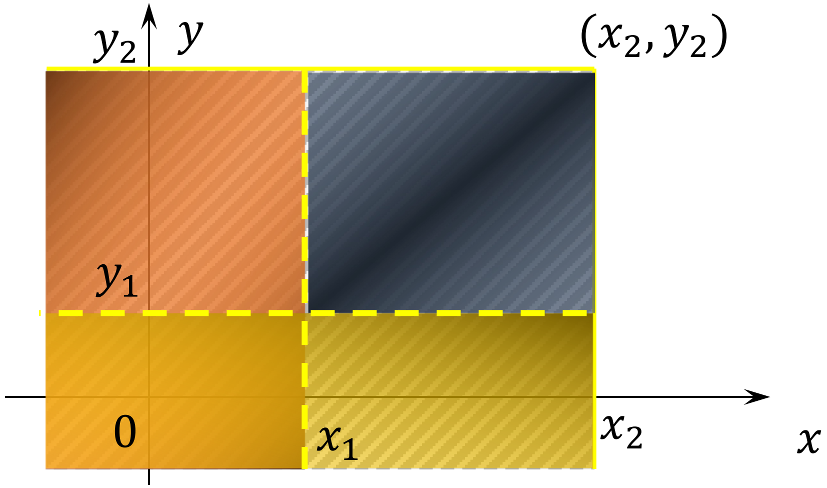
$$F(x, y) = P(A \cap B).$$

- It follows that for $\forall x, y \in \mathbb{R}$, we have $0 \leq F(x, y) \leq 1$ and

$$F(+\infty, +\infty) = 1, F(-\infty, -\infty) = 0,$$

$$F(-\infty, y) = 0, F(x, -\infty) = 0.$$

3.1 Random Vector and Joint Distribution



- The joint CDF can be used to compute $P(x_1 < X \le x_2, y_1 < Y \le y_2)$ for any $-\infty < x_1 < x_2 < \infty, -\infty < y_1 < y_2 < \infty$.

$$P\{x_1 < X \le x_2,\ \ y_1 < Y \le y_2\}$$
$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$
$$\ge 0.$$

- With the joint CDF, it is straight forward to define the marginal CDF.

Marginal CDF

Let $F(x, y)$ be the joint CDF of (X, Y), $F_X(x)$ be the CDF of X without considering Y, and $F_Y(y)$ be the CDF of Y without considering X, then for $\forall x, y \in \mathbb{R}$:

$$F_X(x) = F(x, \infty), F_Y(y) = F(\infty, y).$$

F_X (F_Y) is also called the marginal CDF (边缘累积分布函数) of X (Y).

Note: While the joint CDF uniquely determines the marginal CDFs, the reverse is not true.

3.1 Random Vector and Joint Distribution

Example 3.1

- Suppose that the joint CDF of random vector (X, Y) is (a, b, c are constants)

$$F(x,y) = a\left(b + \arctan\frac{x}{2}\right)\left(c + \arctan\frac{y}{2}\right), -\infty < x, y < \infty.$$

- 1. Determine the value of a, b, c. 2. Calculate $P(-2 < X \le 2, -2 < Y \le 2)$.
- 3. Obtain the marginal CDFs of X and Y.

Solution

- 1. By the basic properties of the joint CDF, we have:

$$F(+\infty,+\infty) = a\left(b+\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right) = 1,\ F(-\infty,+\infty) = a\left(b-\frac{\pi}{2}\right)\left(c+\frac{\pi}{2}\right) = 0,$$

$$F(+\infty,-\infty) = a\left(b+\frac{\pi}{2}\right)\left(c-\frac{\pi}{2}\right) = 0. \quad \Rightarrow b = \frac{\pi}{2}, c = \frac{\pi}{2}, a = \frac{1}{\pi^2}.$$

- Therefore, the joint CDF is

$$F(x,y) = \frac{1}{\pi^2}\left(\frac{\pi}{2} + \arctan\frac{x}{2}\right)\left(\frac{\pi}{2} + \arctan\frac{y}{2}\right).$$

Solution

- 2. By the definition and property of the joint CDF, we have:

$$P(-2 < X \leq 2, -2 < Y \leq 2) = F(2,2) - F(2,-2) - F(-2,2) + F(-2,-2)$$

$$= \frac{1}{\pi^2}\left[\left(\frac{\pi}{2} + \arctan(1)\right)^2 - 2\left(\frac{\pi}{2} + \arctan(1)\right)\left(\frac{\pi}{2} + \arctan(-1)\right) + \left(\frac{\pi}{2} + \arctan(-1)\right)^2\right]$$

$$= \frac{1}{\pi^2}\left[\left(\frac{\pi}{2} + \arctan(1)\right) - \left(\frac{\pi}{2} + \arctan(-1)\right)\right]^2 = \frac{1}{\pi^2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]^2 = \frac{1}{4}.$$

- 3. By the definition of the marginal distribution, we have:

$$F_X(x) = F(x, +\infty) = \frac{1}{\pi^2}\left(\frac{\pi}{2} + \arctan\frac{x}{2}\right)\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{x}{2},$$

$$F_Y(y) = F(+\infty, y) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{y}{2}.$$

3.1 Random Vector and Joint Distribution

- Then for discrete random vectors, we introduce the joint and marginal PMF.

Joint and Marginal PMF for Discrete Random Vector

- For a random vector (X, Y), let $S_X = \{x_1, x_2, \ldots\}$ and $S_Y = \{y_1, y_2, \ldots\}$ be the support of X and Y, respectively. Then the joint PMF (联合概率质量函数) of (X, Y) is defined as

$$p(x_i, y_j) = P(X = x_i, Y = y_j) \triangleq p_{ij}, i, j = 1, 2, \ldots.$$

- The marginal PMF (边缘概率质量函数) of X is the PMF of X without considering Y:

$$p_X(x_i) = P(X = x_i) = \sum_{j=1}^{\infty} p_{ij} \triangleq p_{i\cdot}, i = 1, 2, \cdots.$$

- Similarly, the marginal PMF of Y is

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{\infty} p_{ij} \triangleq p_{\cdot j}, j = 1, 2, \ldots.$$

- The joint PMF satisfies:
- Non-negativity: $p_{ij} \geq 0, i, j = 1, 2, \ldots;$
- Normalization: $\sum_i \sum_j p_{ij} = 1.$

- The joint PMF is typically displayed in a tabular format:

$X \backslash Y$	y_1	y_2	$\cdots$	y_j	$\cdots$
x_1	p_{11}	p_{12}	$\cdots$	p_{1j}	$\cdots$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

3.1 Random Vector and Joint Distribution

Example 3.2

- Two dice are tossed independently. Let X be the smaller number of points and Y be the larger number of points. If both dice show the same number, say, z points, then $X = Y = z$.
- 1. Find the joint PMF of (X, Y); 2. Find the marginal PMF of X.

Solution

- 1. Since the two dice are tossed independently, it is not difficult to obtain the joint PMF of (X, Y):

$X\backslash Y$	1	2	3	4	5	6
1	1/36	1/18	1/18	1/18	1/18	1/18
2	0	1/36	1/18	1/18	1/18	1/18
3	0	0	1/36	1/18	1/18	1/18
4	0	0	0	1/36	1/18	1/18
5	0	0	0	0	1/36	1/18
6	0	0	0	0	0	1/36

- 2. With the joint PMF of (X, Y), the marginal PMF of X can be directly obtained as:

Value	1	2	3	4	5	6
Prob.	$\frac{11}{36}$	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{1}{12}$	$\frac{1}{36}$

3.1 Random Vector and Joint Distribution

- Then for continuous random vectors, we introduce the joint PDF.

Joint PDF for Continuous Random Vector

- (X,Y) is said to be a continuous random vector if there exists a non-negative function $f(x,y)$, defined for all $(x,y)\in\mathbb{R}^2$, satisfies that for any $D\subset\mathbb{R}^2$,

$$P\big((X,Y)\in D\big)=\iint_{(x,y)\in D} f(x,y)\,dxdy\,.$$

$f(x,y)$ is called the joint PDF (联合概率密度函数) of (X,Y).

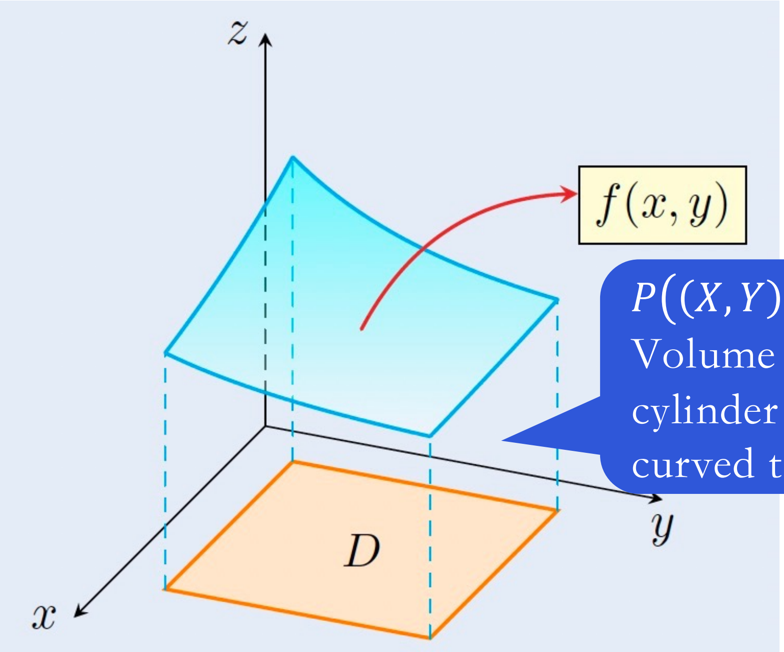
- Particularly, we have the joint CDF of (X,Y) to be

$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,dudv\,.$$

- It follows that

$$f(x,y)=\frac{\partial^2}{\partial x\partial y}F(x,y).$$

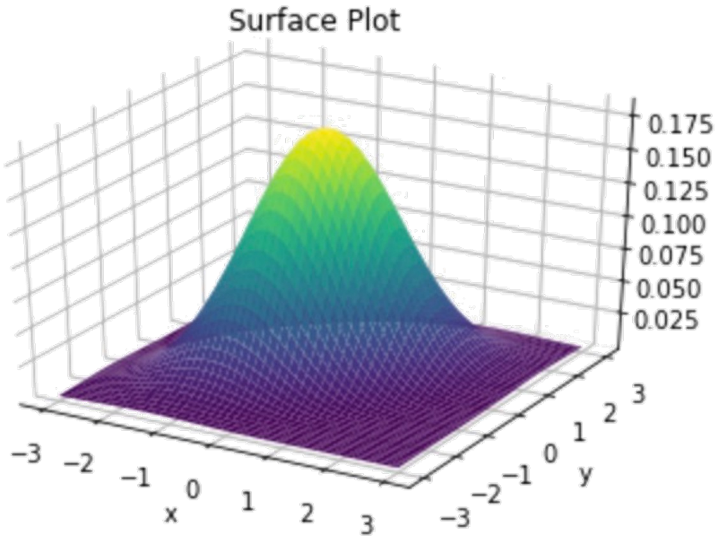
- The joint PDF satisfies:
- Non-negativity: $f(x,y)\geq 0, \forall (x,y)\in\mathbb{R}$;
- Normalization: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy=1$.

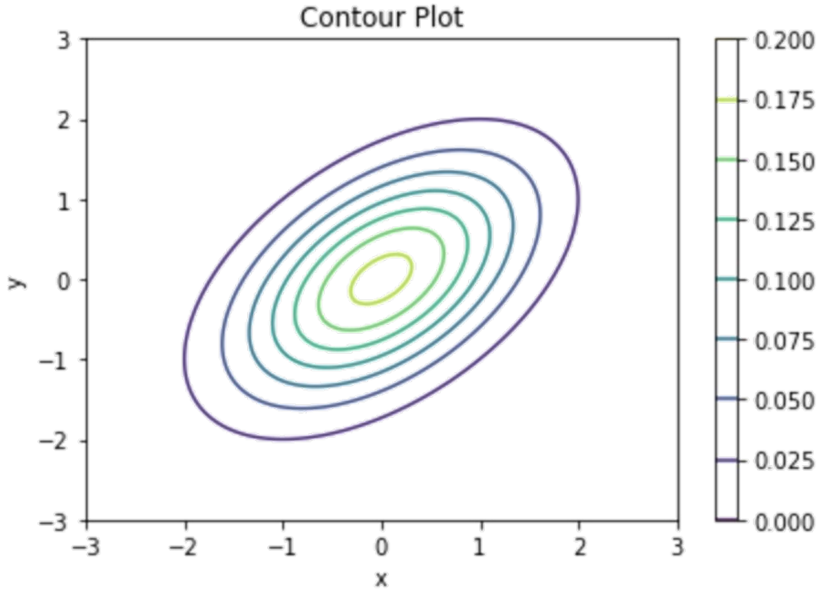


$P\big((X,Y)\in D\big)=$ Volume of the cylinder with a curved top

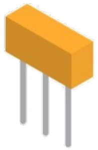
3.1 Random Vector and Joint Distribution

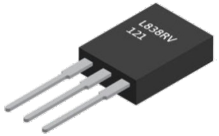
- Similar to the PDF of a single r.v., the joint PDF $f(x,y) \neq P(X=x, Y=y)$. Instead, it reflects the degree to which the probability is concentrated around (x,y).
- The joint PDF are typically visualized with the **surface plot (曲面图)** or the **contour plot (等高线图)** which help in intuitively understanding the distribution and relationship between X and Y.
  - **Surface plot**: a 3D plot where the height represents the value of the joint PDF at each point (x,y).
  - **Contour plot**: a 2D plot showing level curves where the joint PDF has constant values.





3.1 Random Vector and Joint Distribution



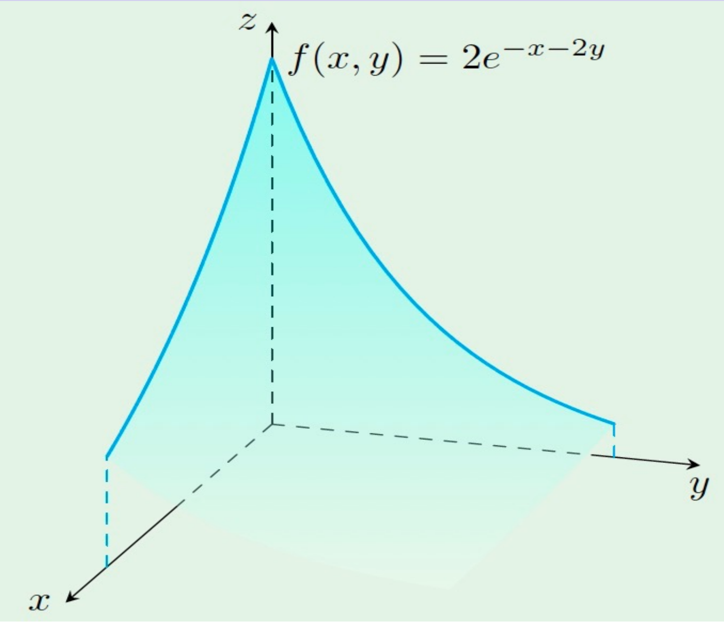


Example 3.3

- The lifetime (in years) of two electronic components of a randomly selected machine is denoted by r.v.s X and Y, which has a joint PDF

$$f(x,y)=\begin{cases}2e^{-x-2y}, & 0<x,y<\infty\\ 0, & \text{otherwise}\end{cases}.$$

- Compute: 1. $P(X<1, Y<1)$; 2. $P(X<Y)$.



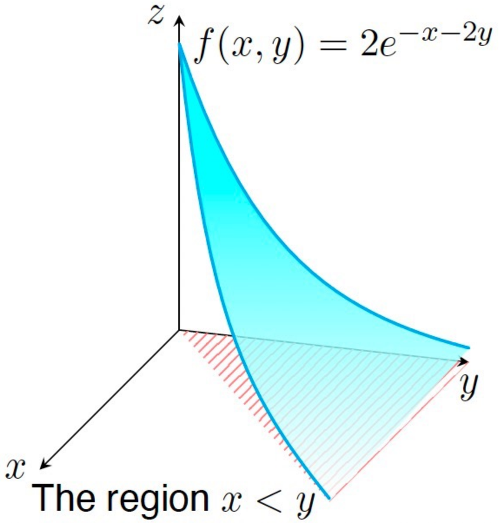
Solution

- 1. By the definition of the joint PDF of (X, Y), we have:

$$P(X<1, Y<1) = \int_{-\infty}^{1}\left(\int_{-\infty}^{1} f(x,y)\,dx\right)dy = \int_{0}^{1}\left(\int_{0}^{1} e^{-x}dx\right)2e^{-2y}dy$$
$$= (1-e^{-1})\int_{0}^{1} 2e^{-2y}dy = (1-e^{-1})(1-e^{-2}) \approx 0.5466.$$

- 2. Again, by the joint PMF of (X, Y):

$$P(X<Y) = \iint_{(x,y):x<y} f(x,y)dxdy = \int_{0}^{\infty}\left(\int_{0}^{y} 2e^{-x-2y}dx\right)dy$$
$$= \int_{0}^{\infty} 2e^{-2y}(1-e^{-y})dy = \int_{0}^{\infty} 2e^{-2y}dy - \int_{0}^{\infty} 2e^{-3y}dy$$
$$= 1-\frac{2}{3} = \frac{1}{3}.$$



3.1 Random Vector and Joint Distribution

- With the joint PDF, the marginal PDF can be determined.
- The marginal PDF mirrors the definition of the marginal PMF for the discrete case, except with sums replaced by integrals and the joint PMF replaced by the joint PDF.

Marginal PDF for Continuous Random Vector

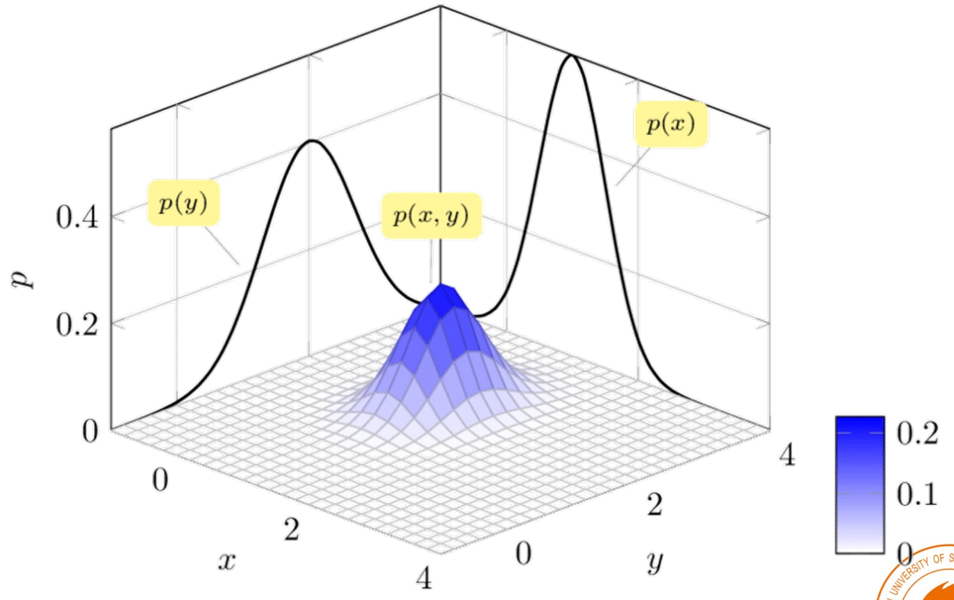
- (X, Y) a continuous random vector with joint PDF $f(x, y)$, then the **marginal PDF (边缘概率密度函数)** of X, i.e., the PDF of X without considering Y is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy .$$

- Likewise, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

- Note that a joint PDF uniquely defines the marginal PDFs, however, the reverse is not true.



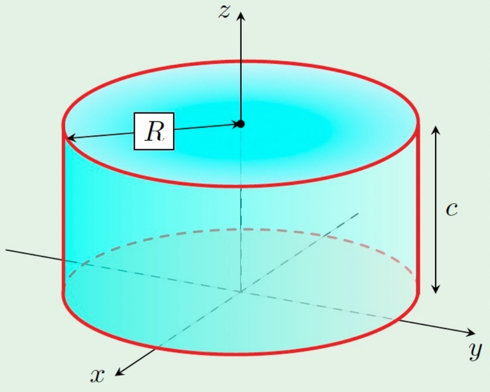
3.1 Random Vector and Joint Distribution

Example 3.4

- Suppose that the joint PDF of a random vector (X, Y) is given by

$$f(x,y) = \begin{cases} c, \text{if } x^2 + y^2 \le R^2 \\ 0, \qquad \text{otherwise} \end{cases}$$

- 1. Determine the constant c;
- 2. Find the marginal PDF of X.



Solution

- 1. By the normalization property of the joint PDF:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1,$$

- it follows that

$$1 = \iint_{(x,y):x^2+y^2 \le R^2} c\, dxdy \Longrightarrow c = \frac{1}{\pi R^2}.$$

- 2. With the joint PDF of (X, Y), the marginal PDF of X can be directly obtained as:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} c dy = \frac{2\sqrt{R^2 - x^2}}{\pi R^2},$$

- if $|x| \le R$; and $f_X(x) = 0$, otherwise.

3.1 Random Vector and Joint Distribution

- Finally, we talk about the conditional PMF/PDF.

Conditional PMF/PDF

- For a **discrete** random vector (X, Y) with joint PMF $p(x, y)$, the conditional PMF (条件概率质量函数) of X given $Y = y$ is defined as

$$p_{X|Y}(x|y) \triangleq P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)},$$

for all values of y such that $p_Y(y) > 0$.

- For a **continuous** random vector (X, Y) with joint PDF $f(x, y)$, the conditional PDF (条件概率密度函数) of X given $Y = y$ is defined as

$$f_{X|Y}(x|y) \triangleq \frac{f(x, y)}{f_Y(y)},$$

for all values of y such that $f_Y(y) > 0$.

- The conditional PDF mirrors the definition of the conditional PMF for the discrete case, except with the joint/marginal PMF replaced by the joint/marginal PDF.

- The conditional PMF/PDFs also satisfy the non-negativity and normalization properties.

- Question: for the continuous case, $P(Y = y) = 0$, so that $P(X = x|Y = y)$ or $P(X \leq x|Y = y)$ is not defined. Then how to understand the conditional PDF of X given $Y = y$?

3.1 Random Vector and Joint Distribution

- Here we talk about how to understand the conditional PDF of X given $Y = y$.
- Conditioning on $Y = y$ can be understood as conditioning on $\{y \le Y \le y+\varepsilon\}$ where $\varepsilon \to 0$.
- Consider the conditional CDF:

$$P\{X \le x | y < Y \le y+\varepsilon\} = \frac{P\{X \le x, y < Y \le y+\varepsilon\}}{P\{y < Y \le y+\varepsilon\}} = \frac{\int_{-\infty}^{x}\int_{y}^{y+\varepsilon} f(u,v)dvdu}{\int_{y}^{y+\varepsilon} f_Y(y)dy}$$

$$= \frac{\varepsilon\int_{-\infty}^{x} f(u,y_\varepsilon)du}{\varepsilon f_Y(\tilde{y}_\varepsilon)} \to \int_{-\infty}^{x} \frac{f(u,y)}{f_Y(y)} du \quad (\varepsilon \to 0).$$

By the mean value theorem of integrals (积分中值定理)

The conditional PDF

- By the definition of the conditional PDF, we have

$$f(x,y) = f_{X|Y}(x|y) f_Y(y).$$

- Take the integration w.r.t. y on both sides, the marginal distribution of X can be expressed as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)dy$$

The law of total probability under the continuous case

3.1 Random Vector and Joint Distribution

Example 3.5

- For a randomly selected person in an automobile accident, let X be his/her extent of injury and Y be the type of safety restraint he/she was wearing at the time of the accident. The joint PMF of X and Y is

$X \backslash Y$	1 (None)	2 (Belt Only)	3 (Belt and Harness)	$p_X(x)$
1 (None)	0.065	0.075	0.060	0.20
2 (Minor)	0.165	0.160	0.125	0.45
3 (Major)	0.145	0.10	0.055	0.30
4 (Death)	0.025	0.015	0.010	0.05
$p_Y(y)$	0.40	0.35	0.25	1.00

- 1. What is the PMF of extent of injury for a randomly selected person with no restraint?
- 2. What is the PMF of extent of injury for a randomly selected person with belt and harness?

3.1 Random Vector and Joint Distribution

Solution

- 1. By the definition of conditional PMF, we have the conditional distribution of X given $Y = 1$ is

Value	1 (None)	2 (Minor)	3 (Major)	4 (Death)
Prob.	$\frac{0.065}{0.4} = 0.1625$	$\frac{0.165}{0.4} = 0.4125$	$\frac{0.145}{0.4} = 0.3625$	$\frac{0.025}{0.4} = 0.0625$

- 2. Similarly, the conditional distribution of X given $Y = 3$ is

Value	1 (None)	2 (Minor)	3 (Major)	4 (Death)
Prob.	$\frac{0.06}{0.25} = 0.24$	$\frac{0.125}{0.25} = 0.50$	$\frac{0.055}{0.25} = 0.22$	$\frac{0.01}{0.25} = 0.04$

- With belt and harness, the probability of a major injury or death is 16.5% lower than the case without any safety restraint.

3.1 Random Vector and Joint Distribution

Example 3.4 (Continued)

- Determine the conditional PDF of X given $Y = y$ (where $|y| \leq R$).

Solution

- Similar with 2, we obtain the marginal PDF of Y:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \begin{cases} \dfrac{2\sqrt{R^2 - y^2}}{\pi R^2}, & \text{if } |y| \leq R \\ 0, & \text{otherwise} \end{cases}.$$

- Then, by the definition, the conditional PDF of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} \dfrac{1}{2\sqrt{R^2 - y^2}}, & \text{if } |x| \leq \sqrt{R^2 - y^2} \\ 0, & \text{otherwise} \end{cases}.$$

- This suggest that given $Y = y$, X follows $\text{Uniform}[-\sqrt{R^2 - y^2}, \sqrt{R^2 - y^2}]$.
- Since $f_{X|Y}(x|y) \neq f_X(x)$, we say that X is not independent of Y.

Chapter 3 Joint Distributions

3.2 Relationship Between Two Random Variables

- In the end of Section 3.1, we mentioned the concept of independence between random variables.
- Recall the independence between random events, the independence between random variables can be defined similarly: the value of Y does not affect the distribution of X or vice versa.
- For example, for the continuous case:

$$f_{X|Y}(x|y) = f_X(x) \Longrightarrow f(x,y) = f_{X|Y}(x|y)f_Y(y) = f_X(x)f_Y(y),$$

or

$$f_{Y|X}(y|x) = f_Y(y) \Longrightarrow f(x,y) = f_{Y|X}(y|x)f_X(x) = f_X(x)f_Y(y).$$

By $f(x,y) = f_X(x)f_Y(y)$, we have

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)du\,dv = \int_{-\infty}^{x} f_X(u)du \int_{-\infty}^{y} f_Y(v)dv = F_X(x)F_Y(y).$$

3.2 Relationship Between Two Random Variables

Independence of Random Variables

- Let $F(x_1, x_2, \dots x_n)$ be the joint CDF of $(X_1, X_2, \dots, X_n)$, $F_{X_i}(x_i)$ be the marginal CDF of X_i, then if for $\forall x_1, x_2, \dots, x_n \in \mathbb{R}$ we have

$$F(x_1, x_2, \dots, x_n) = F_{X_1}(x_1) F_{X_2}(x_2) \cdots F_{X_n}(x_n),$$

  then we say that random variables $X_1, X_2, \cdots, X_n$ are **(mutually) independent (相互独立)**.

- For discrete random variables $X_1, X_2, \cdots, X_n$, if they are independent, then the PMF satisfies

$$p(x_1, x_2, \cdots, x_n) = P(X_1 = x_1, \dots, X_n = x_n) = p_{X_1}(x_1) p_{X_2}(x_2) \cdots p_{X_n}(x_n).$$

- For continuous random variables $X_1, X_2, \cdots, X_n$, if they are independent, then the PDF satisfies

$$f(x_1, x_2, \cdots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

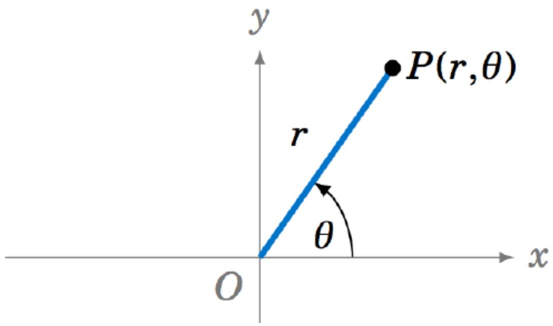
- **Question:** the definition of independence between three or more random events require multiple equations, why the independence between three or more random variables only require one?

3.2 Relationship Between Two Random Variables

Example 3.6

- The PDF of a standard normal random variable Z is $f(z) = ce^{-z^2/2}$, $-\infty < z < \infty$.
- We already know that $c = 1/\sqrt{2\pi}$, however, how is this value obtained?
- Surprisingly, the easiest way to determine c is to define two independent standard normal random variables and use the fact that their joint PDF must integrate to 1.

3.2 Relationship Between Two Random Variables



Solution

- Let random variables $X \sim \mathrm{N}(0,1)$, $Y \sim \mathrm{N}(0,1)$, X and Y are independent. Then the joint PDF of X and Y is

$$f(x,y) = ce^{-\frac{x^2}{2}} \cdot ce^{-\frac{y^2}{2}} = c^2 e^{-\frac{x^2+y^2}{2}}.$$

- Since the joint PDF must integrate to 1, we have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2 e^{-\frac{x^2+y^2}{2}} dx\, dy.$$

- Surprisingly, this double integral can be evaluated even though the single integral could not.
- To evaluate the double integral, we convert to polar coordinates (极坐标), using the substitutions $x = r\sin\theta$, $y = r\cos\theta$ and $dxdy = rdrd\theta$:

$$c^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dx\, dy = c^2\int_{0}^{2\pi}\int_{0}^{\infty} e^{-\frac{r^2}{2}} rdr\, d\theta = c^2\int_{0}^{2\pi} 1 d\theta = 2\pi c^2 \Longrightarrow c = \frac{1}{\sqrt{2\pi}}.$$

3.2 Relationship Between Two Random Variables

Example 3.7

- Suppose that the number of people who enter a shopping mall on a randomly selected weekday follows a Poisson distribution with parameter λ.
- If each person who enters the shopping mall is a male with probability 0.2 and a female with probability 0.8.
- Show that the number of males and females entering the shopping mall are independent Poisson random variables with parameters 0.2λ and 0.8λ, respectively.

Solution

- Let random variables X_1 and X_2 be the number of males and females that enter the shopping mall. By the definition of independence, we need to show that for $\forall i_1, i_2 = 0, 1, \ldots,$

$$P(X_1 = i_1, X_2 = i_2) = P(X_1 = i_1)P(X_2 = i_2).$$

3.2 Relationship Between Two Random Variables

Solution

- Let $Y=X_1+X_2$ be the total number of people that enter the shopping mall, then $Y\sim\text{Poisson}(\lambda)$, i.e.,

$$P(Y=i_1+i_2)=e^{-\lambda}\frac{\lambda^{i_1+i_2}}{(i_1+i_2)!}.$$

- Given that $Y=i_1+i_2$, it follows that $X_1\sim\text{Binomial}(i_1+i_2,0.2)$, so that

$$P(X_1=i_1|Y=i_1+i_2)=\binom{i_1+i_2}{i_1}0.2^{i_1}0.8^{i_2}=\frac{(i_1+i_2)!}{i_1!\,i_2!}0.2^{i_1}0.8^{i_2}.$$

- Therefore,

$$\begin{aligned}P(X_1=i_1,X_2=i_2)&=P(X_1=i_1,X_2=i_2|Y=i_1+i_2)P(Y=i_1+i_2)\\&=\frac{(i_1+i_2)!}{i_1!\,i_2!}0.2^{i_1}0.8^{i_2}\times e^{-\lambda}\frac{\lambda^{i_1+i_2}}{(i_1+i_2)!}=\left(e^{-0.2\lambda}\frac{(0.2\lambda)^{i_1}}{i_1!}\right)\left(e^{-0.8\lambda}\frac{(0.8\lambda)^{i_2}}{i_2!}\right).\end{aligned}$$

- With the joint PMF, it is not difficult to determine the marginal PMFs, i.e., $X_1\sim\text{Poisson}(0.2\lambda)$ and $X_2\sim\text{Poisson}(0.8\lambda)$, and the independence between X_1 and X_2 is proved.

3.2 Relationship Between Two Random Variables

- If X and Y are independent random variables, then, for any functions g and h, we have

$$\mathrm{E}[g(X)h(Y)] = \mathrm{E}[g(X)]\mathrm{E}[h(Y)],$$
$$\mathrm{Var}[g(X) \pm h(Y)] = \mathrm{Var}[g(X)] + \mathrm{Var}[h(Y)].$$

Proof: Without loss of generality, show the case for the continuous case.

Suppose that X and Y have joint density $f(x,y)$, then

$$\mathrm{E}[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dx\,dy$$
$$= \int_{-\infty}^{\infty} g(x)f_X(x)dx \int_{-\infty}^{\infty} h(y)f_Y(y)dy = \mathrm{E}[g(X)]\mathrm{E}[h(Y)].$$

Let $\mathrm{E}[g(X)] = a$ and $\mathrm{E}[h(Y)] = b$, then

$$\mathrm{Var}[g(X) + h(Y)] = \mathrm{E}[(g(X) + h(Y) - a - b)^2]$$
$$= \mathrm{E}[(g(X) - a)^2] + \mathrm{E}[(h(Y) - b)^2] + 2\mathrm{E}[(g(X) - a)(h(Y) - b)]$$
$$= \mathrm{Var}[g(X)] + \mathrm{Var}[h(Y)].$$

$= \mathrm{E}[g(X) - a]\mathrm{E}[h(Y) - b] = 0$

3.2 Relationship Between Two Random Variables

- Special case: $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ and $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ if X and Y are independent.
- What if X and Y are not independent?
- Think about the difference between $\mathrm{Var}(X+Y)$ and $\mathrm{Var}(X) + \mathrm{Var}(Y)$:

$$\mathrm{Var}(X+Y) - \mathrm{Var}(X) - \mathrm{Var}(Y) = 2\mathrm{E}\big[\big(X - \mathrm{E}(X)\big)\big(Y - \mathrm{E}(Y)\big)\big] = 2[\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)].$$

- So, if $\mathrm{E}\big[\big(X - \mathrm{E}(X)\big)\big(Y - \mathrm{E}(Y)\big)\big] \neq 0$, X and Y cannot be independent.
- Therefore, $\mathrm{E}\big[\big(X - \mathrm{E}(X)\big)\big(Y - \mathrm{E}(Y)\big)\big]$ can be used to measure the relationship between X and Y.

Covariance

- The **covariance (协方差)** between X and Y, denoted by $\mathrm{Cov}(X,Y)$, is defined by

$$\mathrm{Cov}(X,Y) \triangleq \mathrm{E}\big[\big(X - \mathrm{E}(X)\big)\big(Y - \mathrm{E}(Y)\big)\big] = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$$

- If X and Y are independent, then $\mathrm{Cov}(X,Y) = 0$. However, if $\mathrm{Cov}(X,Y) = 0$, X and Y may not be independent, we can only say that X and Y are **uncorrelated (不相关的)**.

3.2 Relationship Between Two Random Variables

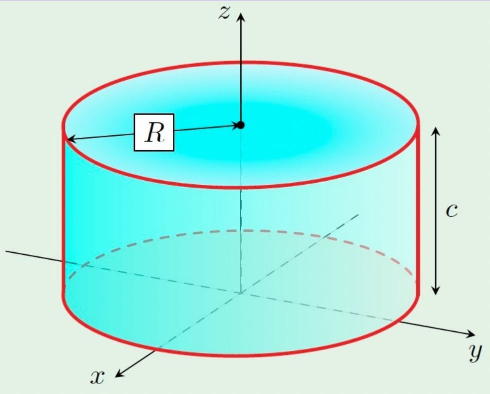
Example 3.4 (Continued)

- We already know that X and Y are not independent, however, show that $\mathrm{Cov}(X,Y)=0$.

Proof

- Since the marginal PDFs of X and Y are both even functions (偶函数), it follows directly that
$$\mathrm{E}(X)=\mathrm{E}(Y)=0.$$
- Then, we compute $\mathrm{E}(XY)$:

$$\mathrm{E}(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dx\,dy=\iint_{\{x^2+y^2\le R^2\}}cxydxdy$$
$$=c\int_{-R}^{R}\left(\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}}xdx\right)ydy=0.$$



- Therefore, $\mathrm{Cov}(X,Y)=\mathrm{E}(XY)-\mathrm{E}(X)\mathrm{E}(Y)=0$.

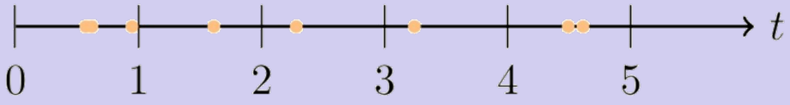
3.2 Relationship Between Two Random Variables

- Note that $\text{Cov}(X, Y)$ is positive when X and Y tend to vary in the same direction and negative when they tend to vary in the opposite direction.
- The covariance has the following properties: (a, b, c are constants)
  - Covariance-variance relationship: $\text{Cov}(X, X) = \text{Var}(X)$.
  $$\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\text{Cov}(X, Y).$$
  - Symmetry: $\text{Cov}(X, Y) = \text{Cov}(Y, X)$.
  - Constants cannot covary: $\text{Cov}(X, c) = 0$.
  - Pulling out constants: $\text{Cov}(aX, bY) = ab\text{Cov}(X, Y)$.
  - Distributive property: $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$.
  - Bilinear property:
  $$\text{Cov}(a_1X_1 + a_2X_2 + \cdots a_nX_n, b_1Y_1 + b_2Y_2 + \cdots + b_mY_m) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_ib_j\text{Cov}(X_i, Y_j).$$

3.2 Relationship Between Two Random Variables

Example 3.8

- A Geiger counter (盖革计数器) is a device used for detecting and measuring ionizing radiation (电离辐射).
- Each time it detects a radioactive particle(放射性粒子), it makes a clicking sound. E.g., the orange points below indicates the times at which the Geiger counter detect a particle.



- Suppose that in a city, radioactive particles reach a Geiger counter according to a Poisson process at a rate of $\lambda = 0.8$ particles per second.
- The time that the first particle is detected and the time that the second particle is detected are denoted by X and Y, respectively.
- Calculate the covariance between X and Y.

3.2 Relationship Between Two Random Variables

Solution

- Let $Z = Y - X$, then Z represents the time interval between the arrival of the first and the second particle.
- By our previous knowledge, we know that $X \sim \text{Exp}(\lambda)$, $Z \sim \text{Exp}(\lambda)$, and X and Z are independent.
- Therefore, we have

$$\mathrm{E}(X) = \mathrm{E}(Z) = \frac{1}{\lambda},\ \mathrm{Var}(X) = \mathrm{Var}(Z) = \frac{1}{\lambda^2},\ \text{and}\ \mathrm{Cov}(X, Z) = 0.$$

- Then,

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, X + Z) = \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Z) = \mathrm{Var}(X) = \frac{1}{\lambda^2} = 1.5625.$$

- $\mathrm{Cov}(X, Y) > 0$ is consistent with our intuition: the longer it takes for the first arrival to happen, the longer we will have to wait for the second arrival, since the second arrival has to happen after the first.

3.2 Relationship Between Two Random Variables

- While the covariance measures the relationship between two random variables, its value depends on the unit/scale on which we measure the random variables.
  - E.g., let X (in m) and Y be the height and weight (in kg) of a randomly selected person, and $\tilde{X} = 100X$ (i..e., $\tilde{X}$ is the height measured in cm) then $\mathrm{Cov}(\tilde{X}, Y) = 100\mathrm{Cov}(X, Y)$.
  - Therefore, a larger covariance does not necessarily suggest a stronger relationship.
- To make the measure comparable, we need to remove the impact of unit/scale.

Correlation Coefficient

- The **correlation coefficient (相关系数)** between X and Y, denoted by $\mathrm{Cor}(X, Y)$ or ρ_{XY}, is defined by

$$\rho_{XY} = \mathrm{Cor}(X, Y) \triangleq \mathrm{E}\left[\frac{(X - \mathrm{E}(X))}{\mathrm{SD}(X)} \cdot \frac{(Y - \mathrm{E}(Y))}{\mathrm{SD}(Y)}\right] = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

- ρ_{XY} is a normalized version of the covariance, which is a dimensionless quantity (无量纲数值).
- **Question**: what kind of relationship is ρ_{XY} measuring?

Example 3.9

- Calculate ρ_{XY} if the joint PDF of X and Y is $(0 < c \leq 1)$

$$f(x,y) = \frac{1}{2\pi c}\exp\left\{-\frac{x^2 - 2\sqrt{1-c^2}xy + y^2}{2c^2}\right\}, -\infty < x, y < \infty.$$

Solution

Consider the normal PDF

- Consider the marginal PDF of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\exp\left\{-\frac{\left(y-\sqrt{1-c^2}x\right)^2}{2c^2} - \frac{c^2x^2}{2c^2}\right\}dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

- So $X \sim \mathrm{N}(0,1)$ and similarly, we obtain $Y \sim \mathrm{N}(0,1)$. It follows that $\mathrm{E}(X) = \mathrm{E}(Y) = 0$, $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$.

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dx\,dy = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cv\left(cu + \sqrt{1-c^2}v\right)e^{-(u^2+v^2)/2}du\,dv$$

$$u = \frac{y - \sqrt{1-c^2}x}{c}, v = x$$

3.2 Relationship Between Two Random Variables

Solution

- Continued with the previous derivation

$$\mathrm{Cov}(X,Y)=\mathrm{E}(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dx\,dy=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(cuv+\sqrt{1-c^2}v^2\right)e^{-(u^2+v^2)/2}du\,dv$$

$$=\frac{c}{2\pi}\int_{-\infty}^{\infty} ue^{-u^2/2}du\int_{-\infty}^{\infty} ve^{-v^2/2}\,dv+\frac{\sqrt{1-c^2}}{2\pi}\int_{-\infty}^{\infty} e^{-u^2/2}du\int_{-\infty}^{\infty} v^2e^{-v^2/2}\,dv$$

$$=0+\frac{\sqrt{1-c^2}}{2\pi}\cdot\sqrt{2\pi}\cdot\sqrt{2\pi}=\sqrt{1-c^2}.$$

- Therefore, $\rho_{XY}=\mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}=\sqrt{1-c^2}$.
- This is an example showing that the marginal PDFs cannot uniquely determine the joint PDF.

3.2 Relationship Between Two Random Variables

- ρ_{XY} actually measure the direction and strength of **the linear relationship** between X and Y.

Proof: Consider to use a linear function of X to approximate Y, i.e., $\hat{Y} = a + bX$.

Then, the mean squared error (MSE, 均方误差) of the approximation is

$$\begin{aligned}\mathrm{MSE} &= \mathrm{E}\left[\left(Y-\hat{Y}\right)^2\right] = \mathrm{E}[(Y-a-bX)^2] \\ &= \mathrm{E}(Y^2) + b^2\mathrm{E}(X^2) + a^2 - 2b\mathrm{E}(XY) + 2ab\mathrm{E}(X) - 2a\mathrm{E}(Y).\end{aligned}$$

Next, we would like to minimize the MSE w.r.t. a and b.

$$\begin{cases} \dfrac{\partial \mathrm{MSE}}{\partial a} = 2a + 2b\mathrm{E}(X) - 2\mathrm{E}(Y) = 0 \\ \dfrac{\partial \mathrm{MSE}}{\partial b} = 2b\mathrm{E}(X^2) - 2\mathrm{E}(XY) + 2a\mathrm{E}(X) = 0 \end{cases} \Rightarrow \begin{cases} b_0 = \dfrac{\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)}{\mathrm{E}(X^2) - [\mathrm{E}(X)]^2} = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \\ a_0 = \mathrm{E}(Y) - b_0\mathrm{E}(X) = \mathrm{E}(Y) - \mathrm{E}(X)\dfrac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \end{cases}$$

Therefore,

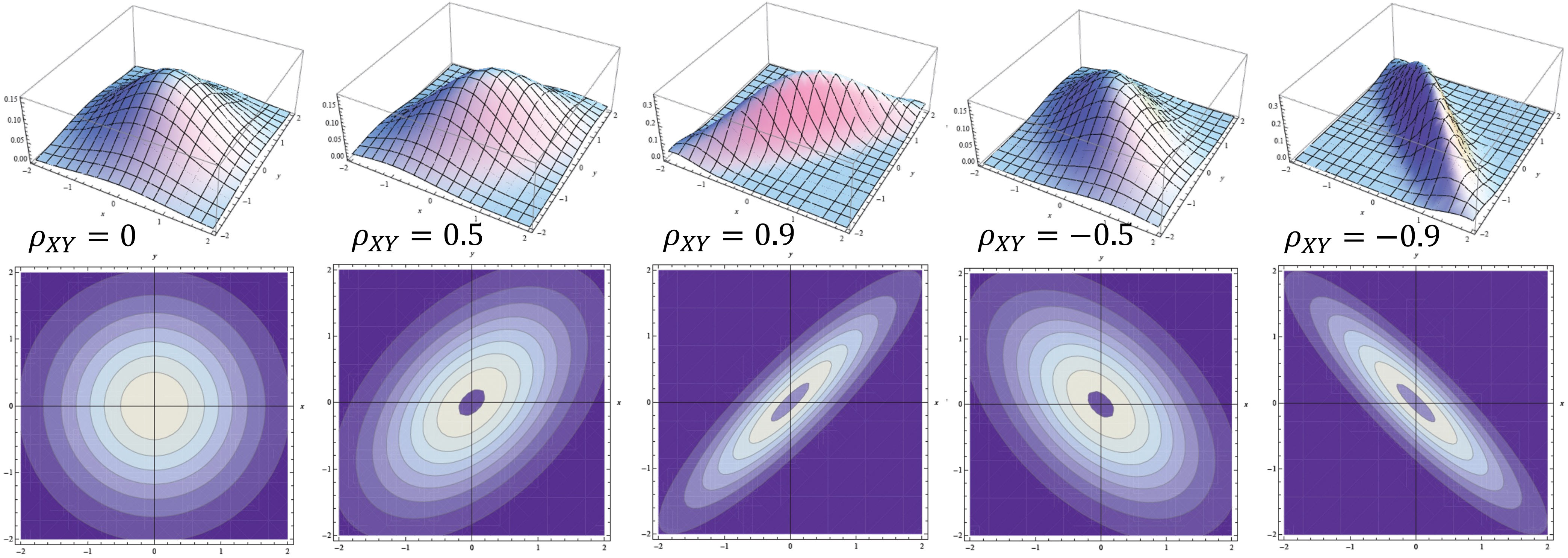
$$\begin{aligned}\min_{a,b} \mathrm{MSE} &= \mathrm{E}[(Y - a_0 - b_0X)^2] = \mathrm{E}[(Y - \mathrm{E}(Y) + b_0\mathrm{E}(X) - b_0X)^2] \\ &= \mathrm{Var}(Y) + b_0^2\mathrm{Var}(X) - 2b_0\mathrm{Cov}(X,Y) = \mathrm{Var}(Y)\left[1 - \frac{[\mathrm{Cov}(X,Y)]^2}{\mathrm{Var}(X)\mathrm{Var}(Y)}\right] = \mathrm{Var}(Y)(1-\rho_{XY}^2).\end{aligned}$$

3.2 Relationship Between Two Random Variables

- Since $\min_{a,b} \text{MSE} = \text{Var}(Y)(1-\rho_{XY}^2) \geq 0$, so $\rho_{XY}^2 \leq 1 \Rightarrow -1 \leq \rho_{XY} \leq 1$.

Not necessarily independent!

  - $0 < \rho_{XY} \leq 1$: positively correlated; $-1 \leq \rho_{XY} < 0$: negatively correlated; $\rho_{XY} = 0$: uncorrelated.
  - When $|\rho_{XY}|$ is closer to 1, the mean squared error is smaller, i.e., the relationship between X and Y is closer to linear. Specifically, if $\rho_{XY} = \pm 1$, X and Y have **an almost perfect linear relationship**.



$\rho_{XY} = 0$ $\rho_{XY} = 0.5$ $\rho_{XY} = 0.9$ $\rho_{XY} = -0.5$ $\rho_{XY} = -0.9$

3.2 Relationship Between Two Random Variables

Example 3.10

- We would like to invest \$10,000 into shares of companies XX and YY.
- Shares of XX cost \$20 per share and the market analysis shows that the expected return is \$1 per share, with a standard deviation of \$0.5.
- Shares of YY cost \$50 per share, with an expected return of \$2.5 and a SD of \$1.
- What is the optimal portfolio (资产组合) consisting of shares of XX and YY, given their correlation coefficient ρ? (Note: number of shares can be any non-negative real value)

Solution

- Suppose that c dollars are invested into XX and $(10{,}000 - c)$ dollars into YY, the resulting return is R_c.
- Let r.v.s X, Y denote the return per share of XX and YY, respectively. First, consider the expected return:

$$\mathrm{E}(R_c) = \mathrm{E}\left(X\times\frac{c}{20} + Y\times\frac{10000-c}{50}\right) = 1\times\frac{c}{20} + 2.5\times\frac{10000-c}{50} = \$500.$$

- Therefore, the expected return does not vary with c.

3.2 Relationship Between Two Random Variables

Solution

- Next, consider the variance of the return

$$
\begin{aligned}
\mathrm{Var}(R_c) &= \mathrm{Var}\left(X\times\frac{c}{20}+Y\times\frac{10000-c}{50}\right)\\
&= \left(\frac{c}{20}\right)^2\mathrm{Var}(X)+\left(\frac{10000-c}{50}\right)^2\mathrm{Var}(Y)+2\left(\frac{c}{20}\right)\left(\frac{10000-c}{50}\right)\mathrm{Cov}(X,Y)\\
&= \left(\frac{c}{20}\right)^2 0.5^2+\left(\frac{10000-c}{50}\right)^2 1^2+2\left(\frac{c}{20}\right)\left(\frac{10000-c}{50}\right)\rho\times 0.5\times 1\\
&= \left(\frac{41-40\rho}{40000}\right)c^2-(8-10\rho)c+40000.
\end{aligned}
$$

- Minimizing $\mathrm{Var}(R_c)$ w.r.t. c, we have:
  - If $\rho \geq 0.8$, $c = 0$, i.e., all $10000 are invested into YY. In this case, $\mathrm{Var}(R_c) = 40000$.
  - If $\rho < 0.8$, $c = 40000(4-5\rho)/(41-40\rho)$. In this case

Perfect risk hedging!

$$\mathrm{Var}(R_c) = 40000 - 40000\times\frac{(4-5\rho)^2}{41-40\rho},$$ specifically, when $\rho = -1$, $\mathrm{Var}(R_c) = 0$.

3.2 Relationship Between Two Random Variables

- The conditional expectation (条件期望) of a random variable X given another random variable Y is the expected value of X when Y is known, denoted as $E(X|Y)$.

Conditional Expectation

- For a **discrete** random vector (X, Y), the conditional expectation of X given $Y = y$ is

$$E(X \mid Y = y) \triangleq \sum_{k=1}^{\infty} x_k \cdot P(X = x_k | Y = y).$$

- For a **continuous** random vector (X, Y), the conditional expectation of X given $Y = y$ is

$$E(X \mid Y = y) \triangleq \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx,$$

- where $f_{X|Y}(x|y)$ is the conditional PDF of X given $Y = y$.

- $E(X|Y = y)$ is a function of y, and we can denote $g(y) \triangleq E(X|Y = y)$.
- Then $E(X|Y) = g(Y)$, which is a function of Y, not a constant. It is itself a **random variable**.
- If X and Y are independent, then $E(X|Y) = E(X)$.

3.2 Relationship Between Two Random Variables

- The Law of Total Expectation (全期望公式) ***connects*** the **unconditional expectation** of X to its **conditional expectation** given Y.

Law of Total Expectation

- If (X, Y) is a random vector, and $E(X)$ exists, then
$$E(X) = E(E(X \mid Y)).$$
- Specifically, if (X, Y) are **discrete** random vector, then
$$E(X) = \sum_j E(X|Y = y_j) \cdot P(Y = y_j).$$
- If (X, Y) are **continuous** random vector, then
$$E(X) = \int_{-\infty}^{\infty} E(X|Y = y) \cdot f_Y(y)\, dy.$$

- $E(X)$ can be computed by first finding the conditional expectation $E(X|Y = y)$, and then taking the expectation of this conditional expectation over all possible values of Y.
- Useful when it is easier to compute $E(X|Y = y)$ than $E(X)$ directly. The whole sample space is split into several subspaces $\{Y = y\}$.

3.2 Relationship Between Two Random Variables

Example 3.11

- A miner (矿工) is trapped in a mine (矿井) with three doors.
- The first door leads to a tunnel (隧道) that takes 3 hours to reach a safe area.
- The second door leads to a tunnel that takes 5 hours to return to the original location.
- The third door leads to a tunnel that takes 7 hours to also return to the original location.
- Assume the miner always chooses one of the three doors with equal probability. What is the average time it will take for him to reach the safe area?

Solution

- Let X be the time in hours it takes for the miner to reach the safe area. The possible values of X are:

$$3, 5+3, 7+3, 5+5+3, 5+7+3, 7+7+3, \cdots$$

- It is difficult to write out the probability distribution of X, so we cannot directly compute $E(X)$.
- Instead, let Y represent the first door chosen, where $\{Y=j\}$ means the miner chooses the j-th door. Then,

$$P(Y=1)=P(Y=2)=P(Y=3)=\frac{1}{3}.$$

3.2 Relationship Between Two Random Variables

Solution

- If the first door is chosen, it takes 3 hours to reach the safe area, so
$$E(X \mid Y=1)=3.$$
- If the second door is chosen, it takes 5 hours to return to the original location, so
$$E(X \mid Y=2)=5+E(X).$$
- If the third door is chosen, it takes 7 hours to return to the original location, so
$$E(X \mid Y=3)=7+E(X).$$
- Thus, by the Law of Total Expectation,
$$\begin{aligned} E(X) &= \sum_j E(X|Y=y_j)\cdot P(Y=y_j) \\ &= \frac{1}{3}[3+5+E(X)+7+E(X)]=5+\frac{2}{3}E(X), \end{aligned}$$
- Solving the equation, we get $E(X)=15$.
- Therefore, the miner will take an average of 15 hours to reach the safe area.