STA219 Assignment 4

12312706 Zhou Liangyu

1. According to the joint PMF of (X, Y), we have:

   Normalization: $a + \frac{1}{9} + c + \frac{1}{9} + b + \frac{1}{3} = 1$, i.e. $a + b + c = \frac{4}{9}$.

   Marginal PMF: $p_X(x_1) = a + \frac{1}{9} + c,\ p_X(x_2) = \frac{1}{9} + b + \frac{1}{3},$

   $p_Y(y_1) = a + \frac{1}{9},\ p_Y(y_2) = \frac{1}{9} + b,\ p_Y(y_3) = c + \frac{1}{3}.$

   $\because p_X(x_1)p_Y(y_1) = (a + \frac{1}{9} + c)(a + \frac{1}{9}) = a,\ p_X(x_2)p_Y(y_2) = (\frac{1}{9} + b + \frac{1}{3})(\frac{1}{9} + b) = b,\ p_X(x_1)p_Y(y_3) = (a + \frac{1}{9} + c)(c + \frac{1}{3}) = c$

   $\therefore a = \frac{1}{18},\ b = \frac{2}{9},\ c = \frac{1}{6}.$

2. $\because X \sim \mathrm{Poisson}(\lambda),\ Y \sim \mathrm{Poisson}(\lambda)$

   $\therefore \mathrm{Var}(X) = \mathrm{Var}(Y) = \lambda.$

   $$\begin{aligned}\therefore \mathrm{Cov}(U, V) &= \mathrm{Cov}(2X + Y, 2X - Y) \\ &= \mathrm{Cov}(2X + Y, 2X) - \mathrm{Cov}(2X + Y, Y) \\ &= \mathrm{Cov}(2X, 2X) + \mathrm{Cov}(Y, 2X) - \mathrm{Cov}(2X, Y) - \mathrm{Cov}(Y, Y) \\ &= 4\mathrm{Cov}(X, X) - \mathrm{Cov}(Y, Y) \\ &= 4\mathrm{Var}(X) - Var(Y) \\ &= 3\lambda.\end{aligned}$$

   $\because \mathrm{Var}(U) = \mathrm{Var}(V) = 4\mathrm{Var}(X) + \mathrm{Var}(Y) = 5\lambda$

   $\therefore \mathrm{Cor}(U, V) = \dfrac{\mathrm{Cov}(U, V)}{\sqrt{\mathrm{Var}(U)\mathrm{Var}(V)}} = \dfrac{3\lambda}{5\lambda} = \dfrac{3}{5}.$

3. (1) $\mathrm{E}(X) = \int_0^1 \int_{-x}^{x} x f(x, y)\, dydx = \int_0^1 \int_{-x}^{x} x\, dydx = \frac{2}{3}$. $\mathrm{E}(Y) = \int_0^1 \int_{-x}^{x} y f(x, y)\, dydx = \int_0^1 \int_{-x}^{x} y\, dydx = 0.$

   $\because \mathrm{E}(XY) = \int_0^1 \int_{-x}^{x} xy f(x, y)\, dydx = \int_0^1 \int_{-x}^{x} xy\, dydx = 0$

   $\therefore \mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 0 - \frac{2}{3} \cdot 0 = 0.$

   (2) $f_X(x) = \int_{-x}^{x} f(x, y)\, dy = \int_{-x}^{x} 1\, dy = 2x,\ f_Y(y) = \int_0^1 f(x, y)\, dx = \int_{|y|}^{1} 1\, dx = 1 - |y|.$

   $\because f_X(x) f_Y(y) = 2x(1 - |y|) \neq 1 = f(x, y)$

   $\therefore X$ and Y are not independent.

4. (1) $\because f_Y(y) = \int_0^1 f(x, y)\, dx = \int_0^1 (x + y)\, dx = \frac{1}{2} + y.$

   $\therefore f_{X|Y}(x, y) = \dfrac{f(x, y)}{f_Y(y)} = \dfrac{x + y}{\frac{1}{2} + y}.$

   $\therefore \mathrm{E}(X|Y = y) = \int_0^1 x f_{X|Y}(x, y)\, dx = \int_0^1 \dfrac{x(x + y)}{\frac{1}{2} + y} = \dfrac{\frac{1}{3} + \frac{1}{2}y}{\frac{1}{2} + y} = \dfrac{2 + 3y}{3 + 6y}.$

   (2) $\mathrm{E}(X) = \int_0^1 \mathrm{E}(X|Y = y) \cdot f_Y(y)\, dy = \int_0^1 (\frac{1}{3} + \frac{1}{2}y)\, dy = \frac{5}{12}.$

5. $\because X \sim \mathrm{U}(0, 1),\ Y \sim \mathrm{U}(0, 1)$

   $\therefore f_X(x) = f_Y(y) = 1,\ 0 \leq x, y \leq 1,\ 0 \leq t \leq 2.$

   To ensure $f_Y(t - x) > 0$, it's necessary that $0 \leq t - x \leq 1$, i.e. $t - 1 \leq x \leq t$.

When $0 \le t < 1, t-1 < 0$, the PMF of T is given by $f_T(t) = \int_0^t f_X(x) f_Y(t-x)\, dx = \int_0^t 1\, dx = t$.

When $1 \le t \le 2, t-1 > 0$, the PMF of T is given by $f_T(t) = \int_{t-1}^1 f_X(x) f_Y(t-x)\, dx = \int_{t-1}^1 1\, dx = 2-t$.

$$\therefore f_T(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

6. Let $S_n = X_1 + X_2 + X_3$, then $\mathrm{E}(S_n) = \mu_1 + \mu_2 + \mu_3 = 170$, $\mathrm{Var}(S_n) = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 36$, $S_n \sim \mathrm{N}(170, 36)$.

   $\therefore P(S_n \le 180) = P(\frac{S_n - \mu}{\sigma} \le \frac{180-170}{6}) = P(Z \le \frac{5}{3}) \approx \Phi(1.67) = 0.9525$.

7. Let X_i represent the lifespan of the i-th bulb and $S_n = X_1 + X_2 + \cdots + X_{40}$, then $X_i \sim \mathrm{Exp}(\frac{1}{25})$. $E(X_i) = 25$, $\mathrm{Var}(X_i) = 25^2$.

   Since $n = 40 \ge 30$, the CLT applies to S_n.

   $\therefore P(S_n > 900) = 1 - P(S_n \le 900) = 1 - P(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \le \frac{900 - 40 \cdot 25}{\sqrt{40} \cdot 25} = 1 - P(Z \le -0.63) \approx 1 - \Phi(-0.63) = 0.7357$.

8. Let X_i represent the occupancy of the i-th room, then $X_i \sim \mathrm{Bernoulli}(0.8)$.

   $\because np = 500 \cdot 0.8 = 400 \ge 0.5,\ n(1-p) = 500 \cdot 0.2 = 100 \ge 0.5$

   $\therefore S_n \overset{\text{approx.}}{\sim} N(np, np(1-p)) = N(400, 80)$.

   $\because P(S_n \le k) = P(\frac{S_n - \mu}{\sigma} \le \frac{k-400}{\sqrt{80}}) = P(Z \le \frac{k-400}{\sqrt{80}}) = \Phi(\frac{k-400}{\sqrt{80}}) = 0.99,\ \Phi(2.33) = 0.9901$

   $\therefore \frac{k-400}{\sqrt{80}} = 2.33$, i.e. $k = 2.33 \cdot \sqrt{80} + 400 = 420.84 \approx 421$.

   $\therefore 421 \cdot 2 = 842$ kW of power is needed to ensure a 99% probability of having enough power for the air conditioners.

9. $\because X \sim \mathrm{Exp}(\lambda),\ Y \sim \mathrm{Exp}(\mu),\ X$ and Y are independent

   $\therefore f_X(x) = \lambda e^{-\lambda x}$, $f_Y(y) = \mu e^{-\mu y}$, $f(x,y) = f_X(x) f_Y(y) = \lambda\mu e^{-\lambda x - \mu y}$ for $x, y \ge 0$.

   $\therefore P(X \le Y) = \int_0^\infty \int_0^y \lambda\mu e^{-\lambda x - \mu y}\, dx dy = \int_0^\infty \mu e^{-\mu y}(1 - e^{-\lambda y})\, dy = \frac{\lambda}{\mu + \lambda}$.

   $\therefore P(X > Y) = 1 - P(X \le Y) = 1 - \frac{\lambda}{\mu+\lambda} = \frac{\mu}{\mu+\lambda}$.

$$\therefore f_Z(z) = \begin{cases} \frac{\lambda}{\mu+\lambda}, z = 1 \\ \frac{\mu}{\mu+\lambda}, z = 0 \end{cases}.$$

10. $\because X$ and Y are i.i.d. random variables that follow geometric distribution

    $\therefore F_X(k) = F_Y(k) = 1 - (1-p)^k,\ F_Z(k) = F_X(k) F_Y(k) = (1 - (1-p)^k)^2$.

    $\therefore f_Z(k) = F_Z(k) - F_Z(k-1) = (1-(1-p)^k)^2 - (1-(1-p)^{k-1})^2$

    $= (2 - (1-p)^k - (1-p)^{k-1})((1-p)^{k-1} - (1-p)^k)$

    $= (2 + (p-2)(1-p)^{k-1}) p (1-p)^{k-1}$

    $= 2p(1-p)^{k-1} + p(p-2)(1-p)^{2k-2}$.