STA219 Assignment 6

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1. Standard normal distribution is symmetric, i.e. the probability only depends on the distance from the origin. Therefore, we can consider using the polar coordinate system to describe it.

   Let $R = \sqrt{-2\ln U_1}$, $\theta = 2\pi U_2$, we have $\theta \sim \mathrm{U}(0, 2\pi)$, $f_\theta(\omega) = \dfrac{1}{2\pi}$, and $Z_1 = R\cos\theta$, $Z_2 = R\sin\theta$.

   $\because P(R^2 \le t) = P(-2\ln U_1 \le t) = P(U_1 \ge e^{-\frac{t}{2}}) = 1 - e^{-\frac{t}{2}}$

   $\therefore R^2 \sim \mathrm{Exp}(\dfrac{1}{2})$.

   $\therefore F_R(r) = F_{R^2}(r^2) = 1 - e^{-\frac{r^2}{2}}$, $f_R(r) = \dfrac{dF_R(r)}{dr} = re^{-\frac{r^2}{2}}$.

   $\therefore R$ and θ are independent, $f_{R,\theta}(r, \omega) = f_R(r)f_\theta(\omega) = \dfrac{re^{-\frac{r^2}{2}}}{2\pi}$.

   $\because Z_1 = R\cos\theta$, $Z_2 = R\sin\theta$

   $\therefore R = Z_1^2 + Z_2^2$, $f_{Z_1,Z_2}(z_1, z_2) = \dfrac{1}{r}f_{R,\theta}(r, \omega) = \dfrac{e^{-\frac{z_1^2+z_2^2}{2}}}{2\pi} = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}} \cdot \dfrac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}}$.

   $\therefore Z_1$ and Z_2 are independent, and $f_{Z_1}(z_1) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}}$, $f_{Z_2}(z_2) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}}$.

   $\therefore P(Z_1 \le a, Z_2 \le b) = \displaystyle\int_\infty^b \int_\infty^a f_{Z_1,Z_2}(z_1, z_2)\, dz_1 dz_2 = \int_\infty^a f_{Z_1}(z_1)\, dz_1 \int_\infty^b f_{Z_2}(z_2)\, dz_2 = \Phi(a)\Phi(b)$.

   $\therefore Z_1$ and Z_2 are a pair of independent standard normal random variables.

2. (1) Sample mean: $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^n X_i = \dfrac{1}{10}(17.2 + 22.1 + 18.5 + 17.2 + 18.6 + 14.8 + 21.7 + 15.8 + 16.3 + 22.8) = 18.5$.

   Sample variance: $S^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^n (X_i - \overline{X})^2 = \dfrac{1}{9}[(17.2 - 18.5)^2 + (22.1 - 18.5)^2 + (18.5 - 18.5)^2 + (17.2 - 18.5)^2 +$

   $(18.6 - 18.5)^2 + (14.8 - 18.5)^2 + (21.7 - 18.5)^2 + (15.8 - 18.5)^2 + (16.3 - 18.5)^2 + (22.8 - 18.5)^2]$

   ≈ 7.88.

   Sample standard deviation: $S = \sqrt{S^2} \approx 2.81$.

   (2) Sample lower quartile: $Q_{0.25} = X_{(\lfloor 10\times 0.25+1\rfloor)} = X_{(3)} = 16.3$.

   Sample upper quartile: $Q_{0.75} = X_{(\lfloor 10\times 0.75+1\rfloor)} = X_{(8)} = 21.7$.

   Sample interquartile range: $Q_{0.75} - Q_{0.25} = 21.7 - 16.3 = 5.4$.

3. (1) $\because X \sim \mathrm{U}(0, \theta)$

   $\therefore \mathrm{E}(X) = \dfrac{\theta}{2}$, $\mathrm{Var}(X) = \dfrac{\theta^2}{12}$, $f_X(x) = \dfrac{1}{\theta}$, $F_X(x) = \dfrac{x}{\theta}$.

   $\because f_{\min}(x) = nf(x)[1 - F(x)]^{n-1} = \dfrac{3}{\theta}(1 - \dfrac{x}{\theta})^2 = \dfrac{2(\theta - x)^2}{\theta^3}$, $f_{\max}(x) = nf(x)F(x)^2 = \dfrac{3}{\theta}(\dfrac{x}{\theta})^2 = \dfrac{3x^2}{\theta^3}$.

   $\therefore \mathrm{E}(X_{(1)}) = \displaystyle\int_0^\theta x f_{\min}(x)\, dx = \int_0^\theta \frac{3x(\theta - x)^2}{\theta^3}\, dx = -\frac{1}{\theta^3}\int_0^\theta x\, d(\theta - x)^3 = -\frac{1}{\theta^3}[0 - \int_0^\theta (\theta - x)^3\, dx] = \frac{\theta}{4}$,

   $\mathrm{E}(X_{(3)}) = \displaystyle\int_0^\theta x f_{\max}(x)\, dx = \int_0^\theta \frac{3x^3}{\theta^3}\, dx = \frac{3x^4}{4\theta^3}\bigg|_0^\theta = \frac{3\theta}{4}$.

   $\therefore \mathrm{E}(\hat{\theta}_1) = \dfrac{4}{3}\mathrm{E}(X_{(3)}) = \theta$, $\mathrm{E}(\hat{\theta}_2) = 4\mathrm{E}(X_{(1)}) = \theta$.

   $\therefore \hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators of θ.

$$(2) \because \mathrm{E}(X_{(1)}^2) = \int_0^\theta x^2 f_{\min}(x)\,dx = \int_0^\theta \frac{3x^2(\theta-x)^2}{\theta^3}\,dx = \frac{1}{\theta^3}\int_0^\theta (\theta-x)^2\,dx^3 = \frac{1}{\theta^3}[0 - \int_0^\theta x^3\,d(\theta-x)^2] = \frac{\theta^2}{10},$$

$$\mathrm{E}(X_{(3)}^2) = \int_0^\theta x^2 f_{\max}(x)\,dx = \int_0^\theta \frac{3x^4}{\theta^3}\,dx = \frac{3x^5}{5\theta^3}\Big|_0^\theta = \frac{3\theta^2}{5}.$$

$$\therefore \mathrm{Var}(X_{(1)}) = \mathrm{E}(X_{(1)}^2) - \mathrm{E}(X_{(1)})^2 = \frac{\theta^2}{10} - \frac{\theta^2}{16} = \frac{3\theta^2}{80},\ \mathrm{Var}(X_{(3)}) = \mathrm{E}(X_{(3)}^2) - \mathrm{E}(X_{(3)})^2 = \frac{3\theta^2}{5} - \frac{9\theta^2}{16} = \frac{3\theta^2}{80}.$$

$$\therefore \mathrm{Var}(\hat\theta_1) = \frac{16}{9}\mathrm{Var}(X_{(3)}) = \frac{\theta^2}{15},\ \mathrm{Var}(\hat\theta_2) = 16\mathrm{Var}(X_{(1)}) = \frac{3\theta^2}{5}.$$

$\because \mathrm{Var}(\hat\theta_1) < \mathrm{Var}(\hat\theta_2)$

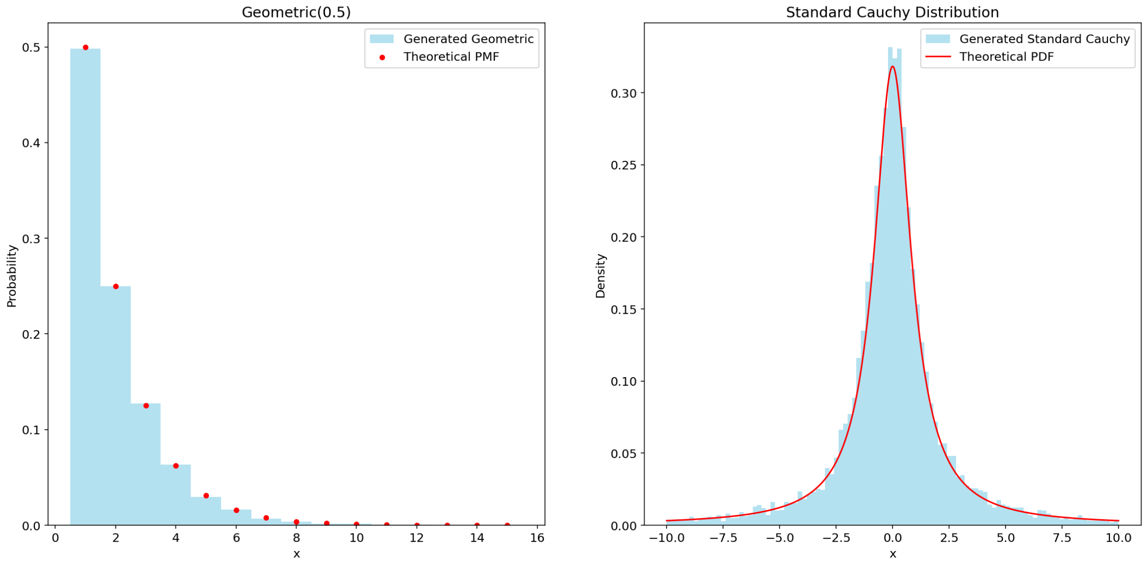
$\therefore \hat\theta_1$ is more efficient than $\hat\theta_2$.

4. Python Code:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import geom, cauchy

# Set seed
np.random.seed(20250510)

# Generate uniform samples
n_samples = 10000
unif_samples1 = np.random.uniform(0, 1, n_samples)
unif_samples2 = np.random.uniform(0, 1, n_samples)

# Transform
p_geom = 0.5
geo_samples = np.ceil(np.log(1 - unif_samples1) / np.log(1 - p_geom))
stdcauchy_samples = np.tan(np.pi * (unif_samples2 - 0.5))

# Theoretical PMF/PDF
geo_x = np.arange(min(geo_samples), max(geo_samples))
geo_pmf = geom.pmf(geo_x, p_geom)
stdcauchy_x = np.linspace(-10, 10, 1000)
stdcauchy_pdf = cauchy.pdf(stdcauchy_x)

# Adjust the plots
plt.figure(figsize=(19.5, 9), dpi=143.4)
plt.rcParams['font.size'] = 12
plt.tight_layout()

# Plot - Gerometric(0.5)
plt.subplot(1, 2, 1)
plt.hist(geo_samples, bins=np.arange(1, 17)-0.5, density=True, alpha=0.6, color='skyblue', label='Generated Geometric')
plt.scatter(geo_x, geo_pmf, color='red', s=20, marker='o', label='Theoretical PMF', zorder=2)
plt.xlabel('x')
plt.ylabel('Probability')
plt.title('Geometric(0.5)')
plt.legend()

# Plot - Standard Cauchy Distribution
plt.subplot(1, 2, 2)
plt.hist(stdcauchy_samples, bins=100, density=True, range=(-10, 10), alpha=0.6, color='skyblue', label='Generated Standard Cauchy')
plt.plot(stdcauchy_x, stdcauchy_pdf, 'r-', label='Theoretical PDF')
plt.xlabel('x')
```

```python
plt.ylabel('Density')
plt.title('Standard Cauchy Distribution')
plt.legend()

# Save the plot for assignment
plt.savefig('Geometric_StdCauchy_Generate.png')

# Show
plt.show()
```

Plots:

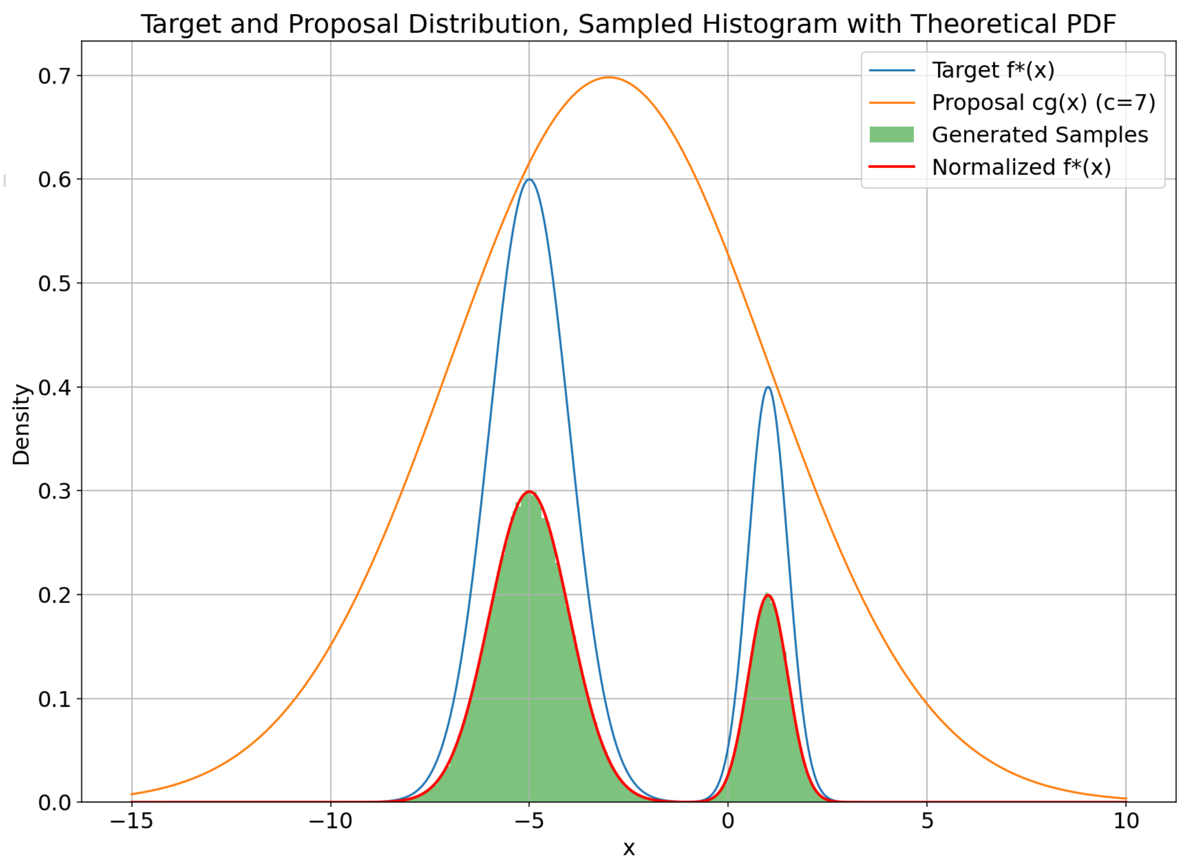


5. (1) (2) (3) Proposal distribution: $N(-3, 16)$. Acceptance proportion: 28.64%.

See the following Python code and results:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
from scipy.integrate import quad
from scipy.optimize import minimize_scalar

# Reject sampling
def reject_sampling(n_samples, mu, sigma, c):
    samples = []
    accepted = 0
    total_trials = 0

    while accepted < n_samples:
        batch_size = 100000
        x_prop = np.random.normal(loc=mu, scale=sigma, size=batch_size)
        u = np.random.rand(batch_size)

        g_x = norm.pdf(x_prop, loc=mu, scale=sigma)
        accept_prob = fs(x_prop) / (c * g_x)

        accept_mask = u <= accept_prob
        accepted_samples = x_prop[accept_mask]

        samples.extend(accepted_samples)
```

```
        accepted += len(accepted_samples)
        total_trials += batch_size

    accept_rate = accepted / total_trials
    return np.array(samples[:n_samples]), accept_rate

fs = lambda x: 0.6 * np.exp(-((x + 5) ** 2) / 2) + 0.4 * np.exp(-((x - 1) ** 2) / 0.5)
c1, _ = quad(fs, -np.inf, np.inf)
mu = -3
sigma = 4
g = lambda x: norm.pdf(x, loc=mu, scale=sigma)
c = 7  # by trial and error

n_samples = 500000
samples, accept_rate = reject_sampling(n_samples, mu, sigma, c)
x = np.linspace(-15, 10, 1000)

# Set plot parameters
plt.figure(figsize=(14.3, 10), dpi=143.4)
plt.rcParams['font.size'] = 16
plt.tight_layout()

# Plot to show that it covers the target distribution, the histogram of the generated samples, and compare it with the theoretical PDF
plt.plot(x, fs(x), label='Target f*(x)')
plt.plot(x, c*g(x), label=f'Proposal cg(x) (c={c})')
plt.hist(samples, bins=200, density=True, alpha=0.6, label='Generated Samples')
plt.plot(x, fs(x)/c1, 'r-', label='Normalized f*(x)', linewidth=2)
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()
plt.title("Target and Proposal Distribution, Sampled Histogram with Theoretical PDF")
plt.grid(True)
plt.savefig('RejectSampling.png')
plt.show()

print(f"Acceptance Rate = {accept_rate:.2%}")
```



Code running result:

```
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.65%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.62%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.64%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.64%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.64%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u 'c:\Users\Chaos Zhou\PyCharmProjects\STA219\Assignment6_P5.py'
Acceptance Rate = 28.60%
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219>
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```

6. (1) Let $\hat{p}_n$ be the frequency of that more than 30% of the forest will eventually be burning.

By the CLT, we have $\hat{p}_n \overset{\text{approx.}}{\sim} \mathrm{N}(p, \dfrac{p(1-p)}{n})$.

$\therefore P(|\hat{p}_n - p| \leq 0.005) \approx 2\Phi(\dfrac{0.005\sqrt{n}}{\sqrt{p(1-p)}}) - 1 \geq 0.95 \Rightarrow \dfrac{0.005\sqrt{n}}{\sqrt{p(1-p)}} \geq \Phi^{-1}(0.975) \approx 1.96.$

$\therefore n \geq \dfrac{1.96^2 p(1-p)}{0.005^2} \geq \dfrac{1.96^2 \cdot 0.25}{0.005^2} = 38416.$

Python code:

```python
import random as rd
from collections import deque
import numpy as np

def forest_mc():
    rows, cols = 20, 50
    forest = [[False] * cols for _ in range(rows)]
```

```python
    forest[0][0] = True
    q = deque([(0, 0)])
    burning_tree_count = 1
    while q:
        i, j = q.popleft()
        # right
        if j + 1 < cols and not forest[i][j + 1]:
            if rd.random() < 0.8:
                forest[i][j + 1] = True
                burning_tree_count += 1
                q.append((i, j + 1))
        # down
        if i + 1 < rows and not forest[i + 1][j]:
            if rd.random() < 0.3:
                forest[i + 1][j] = True
                burning_tree_count += 1
                q.append((i + 1, j))
    return burning_tree_count

n = 38416
samples = []
count = 0
for _ in range(n):
    x = forest_mc()
    samples.append(x)
    if x > 300:
        count += 1
prob = count / n
mean_x = np.mean(samples)
std_x = np.std(samples, ddof=1)

print(f"(1) The probability of X > 30%: {prob:.4f}")
print(f"(2) Mean of X: {mean_x:.2f}")
print(f"(3) Standard deviation of X: {std_x:.2f}")
```

The probability that more than 30% of the forest will eventually be burning: 0.0035.

(2) The total number of affected trees X: 43.

(3) The corresponding standard deviation of X: 62.99.

Code running result:

```
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u "c:\Users\Chaos Zhou\PyCharmProjects\STA219\a.py"
(1) The probability of X > 30%: 0.0035
(2) Mean of X: 43
(3) Standard deviation of X: 62.89
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u "c:\Users\Chaos Zhou\PyCharmProjects\STA219\a.py"
(1) The probability of X > 30%: 0.0036
(2) Mean of X: 44
(3) Standard deviation of X: 63.24
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u "c:\Users\Chaos Zhou\PyCharmProjects\STA219\a.py"
(1) The probability of X > 30%: 0.0037
(2) Mean of X: 43
(3) Standard deviation of X: 63.11
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219> python -u "c:\Users\Chaos Zhou\PyCharmProjects\STA219\a.py"
(1) The probability of X > 30%: 0.0034
(2) Mean of X: 43
(3) Standard deviation of X: 62.70
PS C:\Users\Chaos Zhou\PyCharmProjects\STA219>
行 46, 列 35   空格: 4   UTF-8   CRLF   {} Python   3.12.7
```