

DIGITAL LOGIC

Lecture 2 Boolean Algebra

2024 Fall

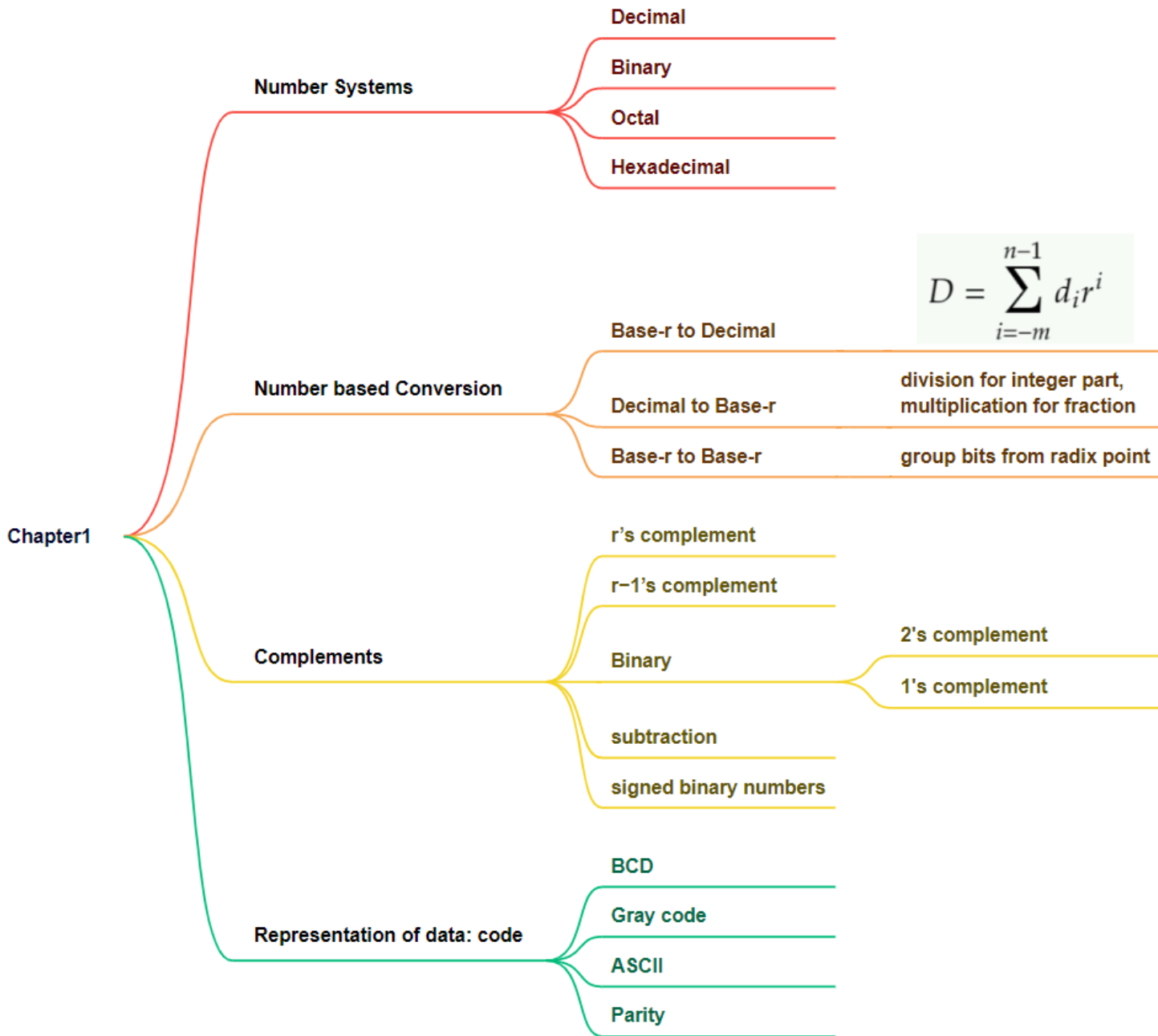
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Today's Agenda

- Recap
- Context
 - Boolean Algebra (布尔代数)
 - Axioms (公理) and Theorems(定理)
 - Boolean Functions (布尔方程)
 - Canonical (范式) and Standard form(标准式)
- Reading: Textbook, Chapter 2



Recap



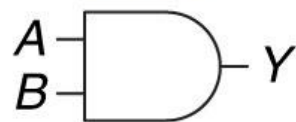
Outline

- **Axioms and Theorems of Boolean Algebra**
- Simplify Boolean Functions
- Canonical and Standard form
- Other Logic Operations

Binary Logic

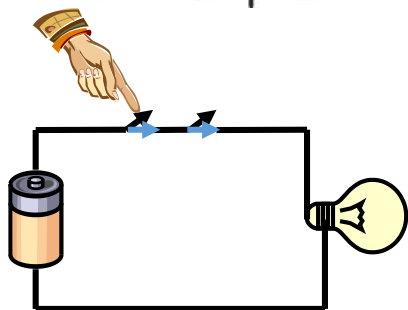
- Deal with Variables like A, B... taking two values:
 - '0', '1'; 'L', 'H'; 'T', 'F'

AND

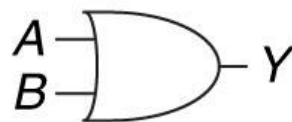


$$Y = AB$$

A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

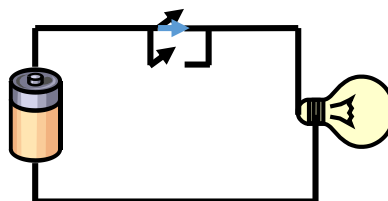


OR

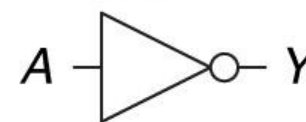


$$Y = A + B$$

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1



NOT

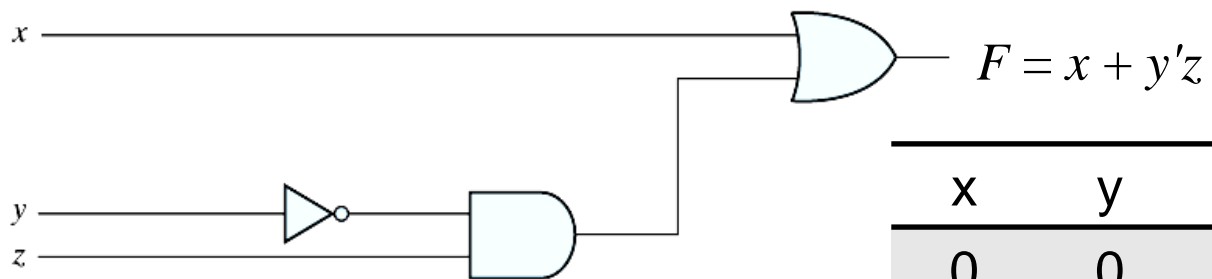


$$Y = A'$$

A	Y
0	1
1	0

Boolean Equation and Truth Table

- Boolean Equation: $F = x + y'z$
- Logic diagram:



x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

- if $x = y = 0, z = 1$
 - $F = 0 + 1 \cdot 1 = 1$
- Truth table (真值表)
 - The truth table of F has 2^n entries
(n = num of inputs)

Boolean Algebra

- Boolean algebra(逻辑代数), a deductive mathematical system developed by George Boole in 1854, deals with the rules by which logical operations are carried out.
- Boolean algebra is an algebraic structure defined by
 - a set of elements S : binary variables;
 - a set of binary operators: AND(\cdot), OR($+$) and NOT($'$);
 - and a number of Axioms/theorems.

Boolean Axioms and Theorems of One Variable

- **Axioms** and **theorems** to simplify Boolean equations
- **Duality** (对偶性) in Axioms and theorems:
 - Replace \cdot with $+$, 0 with 1 , the $=$ relation remains

	Theorem	Dual	Name
1	$x + 0 = x$	$x \cdot 1 = x$	Identity
2	$x + 1 = 1$	$x \cdot 0 = 0$	Null Element
3	$x + x = x$	$x \cdot x = x$	Idempotency
4	$(x')' = x$		Involution
5	$x + x' = 1$	$x \cdot x' = 0$	Complements

- Operator precedence
 - Parentheses $>$ NOT $>$ AND $>$ OR

Boolean Axioms and Theorems of Several Variables

- Dual: Replace \bullet with $+$, 0 with 1 , the $=$ relation remains

	Theorem	Dual	Name
6	$xy = yx$	$x + y = y + x$	Commutativity
7	$(xy)z = x(yz)$	$(x + y) + z = x + (y + z)$	Associativity
8	$x(y + z) = xy + xz$	$x + yz = (x + y)(x + z)$	Distributivity
9	$x + xy = x$	$x(x + y) = x$	Absorption
10	$xy + xy' = x$	$(x + y)(x + y') = x$	Combining
11	$(x + y')y = xy$	$xy' + y = x + y$	Simplification
12	$xy + x'z + yz$ $= xy + x'z$	$(x + y)(x' + z)(y + z)$ $= (x + y)(x' + z)$	Consensus
13	$(x + y)' = x'y'$	$(xy)' = x' + y'$	DeMorgan's law

Note: 8's Dual differs from traditional algebra: OR (+) distributes over AND (\bullet)

Proofs (1)

Algebraic method

- **Absorption**

- $x + xy = x$

- pf: $x + xy = x \cdot 1 + x \cdot y = x(1+y) = x$

- **Combining**

- $(x + y)(x + y') = x$

- pf: $(x + y)(x + y') = x + yy' = x + 0 = x$

- **Simplification**

- $xy' + y = x + y$

- pf: $xy' + y = xy' + (x+x')y = xy' + xy + x'y$
 $= (xy' + xy) + (x'y) = x(y'+y) + y(x+x') = x + y$

- **Consensus**

- $xy + x'z + yz = xy + x'z$

- pf: $xy + x'z + yz = xy + x'z + (x+x')yz$
 $= xy + x'z + xyz + x'yz$
 $= (xy + xyz) + (x'z + x'zy) = xy + x'z$

Proofs (2)

- **DeMorgan's Law**

- $(x + y)' = x'y'$

$$(xy)' = x' + y'$$

Truth table method

pf:

x	y	x'	y'	$(x+y)'$	$x'y'$	$x'+y'$	$(xy)'$
0	0	1	1	1	1	1	1
0	1	1	0	0	0	1	1
1	0	0	1	0	0	1	1
1	1	0	0	0	0	0	0

- **Associativity**

- $(xy)z = x(yz)$

- $(x + y) + z = x + (y + z)$

x	y	z	$(xy)z$	$x(yz)$	$(x+y)+z$	$x+(y+z)$
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	0	1	1
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	0	0	1	1
1	1	0	0	0	1	1
1	1	1	1	1	1	1

Outline

- Axioms and Theorems of Boolean Algebra
- **Simplify Boolean Functions**
- Canonical and Standard form
- Other Logic Operations

Boolean Functions

- A Boolean function from an algebraic expression can be realized to a logic diagram composed of logic gates.
 - Binary variables
 - operators OR, AND, NOT
 - Parentheses
- Terminology:
 - **Literal**: A variable or its complement
 - **Product term**: literals connected by •
 - **Sum term**: literals connected by +
- Example:
 - $A'B'C + A'BC + AB'$ has 8 literals, 3 product term
 - $(A+B'+C)(A'+C)$ has 5 literals, 2 sum term

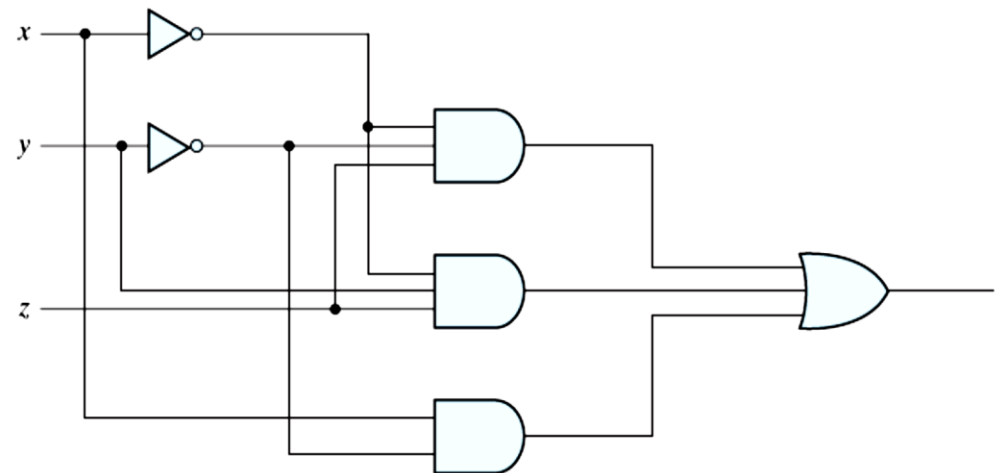
Boolean Functions

- Each Boolean function has
 - only one representation in truth table
 - but a variety of ways in algebraic form/gate implementation.
- Examples
 - $F_1 = x' y' z + x' y z + x y'$
 - $F_2 = x y' + x' z$
 - $F_1 = F_2$
 - Same truth table
 - Different algebraic expression

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0

Gate Implementation

- $F_1 = x'y'z + x'yz + xy'$
 - 8 literals
 - 3 terms (implementation with a gate)
- $F_2 = x'z + xy'$
 - 4 literals
 - 2 terms
 - **Simpler** circuit, more **economical**

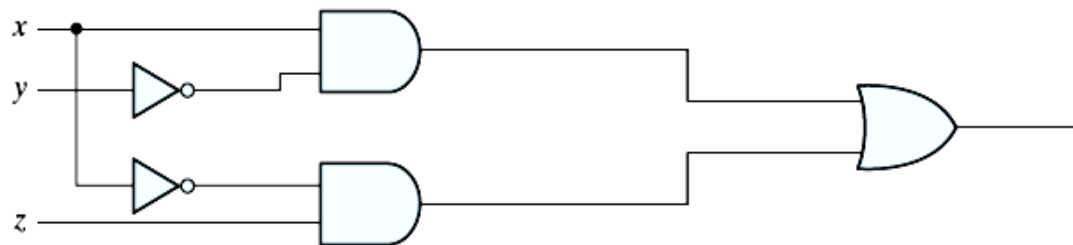


$$F_1 = x'y'z + x'yz + xy'$$

$$= x'z(y' + y) + xy'$$

$$= x'z + xy' = F_2$$

Distributivity
Complements



Algebraic Simplification

- Minimize the number of literals and terms for a simpler circuits (less expensive)
- Algebraic simplification can minimize literals and terms. However, no specific rules to guarantee the optimal results
- Usually not possible by hand for complex functions, use computer minimization program
- More advanced techniques in the next lectures (K-Map)
- Useful rules
 - Distributivity
 - Idempotency
 - Complements
 - DeMorgan's
 - etc

Example

- Examples:

$$\begin{aligned} F &= A'BC + A' \\ &= A'(BC + 1) && \text{Distributivity} \\ &= A' && \text{Null Element} \end{aligned}$$

$$\begin{aligned} F &= XYZ + XY'Z + XYZ' \\ &= XYZ + XY'Z + \textcolor{red}{XYZ} + XYZ' && \text{Idempotency} \\ &= XZ(Y + Y') + XY(Z + Z') && \text{Distributivity} \\ &= XZ + XY && \text{Complements} \\ &= X(Y + Z) && \text{Distributivity} \end{aligned}$$

Exercise:

$$\begin{aligned} F &= A'B'C + A'BC + AB' \\ &= ? \\ &= ? \end{aligned}$$

Boolean Function complement

- The complement of any function F is F' , which can be obtained by DeMorgan's Theorem
 - Take the **dual** of expression, and then complement each literal in F
- Example: $F_3 = x'y'z + x'yz + xy'$
 - Step1, Dual: Replace \cdot with $+$, 0 with 1

$$x'y'z + x'yz + xy' \xrightarrow{\text{Dual}} (x'+y'+z)(x'+y+z)(x+y')$$

- Step2, complement each literal in F

$$\begin{aligned} F_3' &= (x'y'z + x'yz + xy')' \\ &= (x+y+z')(x+y'+z')(x'+y) \end{aligned} \quad \text{DeMorgan}$$

Pay attention! The dual is not duality!

$$x'y'z + x'yz + xy' \neq (x'+y'+z)(x'+y+z)(x+y')$$

Outline

- Axioms and Theorems of Boolean Algebra
- Simplify Boolean Functions
- **Canonical and Standard form**
- Other Logic Operations

Minterms and Maxterms

- Minterms and Maxterms
- A **minterm**(最小项): an AND term consists of all literals in their normal form or in their complement form.
 - For example, two binary variables x and y ,
 - $x'y'$, $x'y$, xy' , xy ($m_0 \sim m_3$)
 - n variables can be combined to form 2^n minterms
- A **maxterm**(最大项): an OR term
 - For example, two binary variables x and y ,
 - $x+y$, $x+y'$, $x'+y$, $x'+y'$ ($M_0 \sim M_3$)
 - 2^n maxterms
- Each maxterm is the complement of its corresponding minterm and vice versa. ($M_i = m_i'$)

Minterms and Maxterms

- Canonical forms
 - sum-of-minterms (som)
 - product-of-maxterms (pom)

Example: Minterms and maxterms for three binary variables

			Minterms		Maxterms	
<i>x</i>	<i>y</i>	<i>z</i>	Term	Designation	Term	Designation
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

Canonical Forms

- A Boolean function $F = xy + x'z$ can be expressed by

- a truth table

- either of the 2 canonical forms

- sum-of-minterms

- $F = x'y'z + x'yz + xyz' + xyz$
 $= m_1 + m_3 + m_6 + m_7 = \sum(1,3,6,7)$

- product-of-maxterms

- $F = (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')$
 $= M_0 \cdot M_2 \cdot M_4 \cdot M_5 = \prod(0,2,4,5)$

Why $F = \sum(1,3,6,7) = \prod(0,2,4,5)$?

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Conversion between som and pom

- To convert from one canonical(som: Sum of Minterms) to another(pom: Product of Maxterms), interchange \sum and \prod , and list the numbers that were excluded from the original form

$$F = \sum(1, 3, 6, 7) = m_1 + m_3 + m_6 + m_7$$

$$F' = \sum(0, 2, 4, 5) = m_0 + m_2 + m_4 + m_5$$

$$F = \sum(1, 3, 6, 7)$$

$$= (F')' = (m_0 + m_2 + m_4 + m_5)'$$

$$= m'_0 m'_2 m'_4 m'_5$$

$$= M_0 M_2 M_4 M_5$$

$$= \prod(0, 2, 4, 5)$$

(som)

(Convolution)

(DeMorgan's)

($M_i = m'_i$)

(pom)

x	y	z	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	0

Represent a Function in Canonical Forms

- Example: Express $F = A + B'C$ as a sum of minterms.
 - by truth table
 - or by expanding the missing variables in each term, using $1 = x + x'$, $0 = xx'$
- Hint: $xy = xy(z + z') = xyz + xyz'$

$$F = A + B'C$$

$$= A(B + B') + B'C$$

$$= AB + AB' + B'C$$

$$= \boxed{AB(C + C')} + AB'(C + C') + (A + A')B'C$$

$$= ABC + ABC' + AB'C + AB'C' + A'B'C$$

$$= m_1 + m_4 + m_5 + m_6 + m_7$$

$$= \sum(1, 4, 5, 6, 7)$$

Truth Table for $F = A + B'C$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Represent a Function in Canonical Forms

- Example: Express $F = xy + x'z$ as a product of maxterms.
 - by truth table
 - First convert to product of sum form, then expand, using $1=x+x'$, $0=xx'$
- Hints: $x + y = (x + y + zz') = (x+y+z)(x+y+z')$

$$F = xy + x'z$$

$$= (xy + x')(xy + z)$$

$$= (x+x')(y+x')(x+z)(y+z)$$

$$= (x'+y)(x+z)(y+z)$$

$$= (x'+y+zz')(x+z+yy')(y+z+xx')$$

$$= (x'+y+z)(x'+y+z')(x+z+y)(x+z+y')(y+z+x)(y+z+x')$$

$$= (x+y+z)(x+y'+z)(x'+y+z)(x'+y+z')$$

$$= M_0 M_2 M_4 M_5$$

$$= \prod(0, 2, 4, 5)$$

$$x + yz = (x + y)(x + z)$$

Distributivity

Tips: You can also use
DeMorgan's Law
(Involution first)

Exercise

- How to convert $f = x + y'z$ into canonical form?

$$f = x + y'z$$
$$= ?$$

Standard Forms

- Canonical forms are very seldom the ones with the least number of literals.
- Standard forms: the terms that form the function may have fewer literals than the minterms.
 - Sum of products(sop): $F_1 = y' + xy + x'yz'$
 - Product of sums(pos): $F_2 = x(y' + z)(x' + y + z')$
 - $F_3 = A'B'CD + ABC'D'$
- Standard forms are not unique!

Outline

- Axioms and Theorems of Boolean Algebra
- Simplify Boolean Functions
- Canonical and Standard form
- **Other Logic Operations**

Other Logic Operations

- 2^n rows in the truth table of n binary variables.
- 2^{2^n} functions for n binary variables.
- 16 functions of two binary variables.

Truth Tables for the 16 Functions of Two Binary Variables

x	y	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

- All the new symbols except for the exclusive-OR symbol are not in common use by digital designers.

Boolean Expressions

- When the three operators AND, OR, and NOT are applied on two variables A and B, they form 16 Boolean functions:

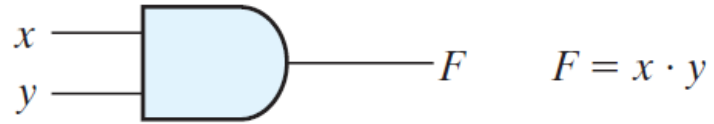
Boolean Functions	Operator Symbol	Name	Comments
$F_0 = 0$		Null	Binary constant 0
$F_1 = xy$	$x \cdot y$	AND	x and y
$F_2 = xy'$	x/y	Inhibition	x, but not y
$F_3 = x$		Transfer	x
$F_4 = x'y$	y/x	Inhibition	y, but not x
$F_5 = y$		Transfer	y
$F_6 = xy' + x'y$	$x \oplus y$	Exclusive-OR	x or y, but not both
$F_7 = x + y$	$x + y$	OR	x or y
$F_8 = (x + y)'$	$x \downarrow y$	NOR	Not-OR
$F_9 = xy + x'y'$	$(x \oplus y)'$	Equivalence	x equals y
$F_{10} = y'$	y'	Complement	Not y
$F_{11} = x + y'$	$x \subset y$	Implication	If y, then x
$F_{12} = x'$	x'	Complement	Not x
$F_{13} = x' + y$	$x \supset y$	Implication	If x, then y
$F_{14} = (xy)'$	$x \uparrow y$	NAND	Not-AND
$F_{15} = 1$		Identity	Binary constant 1

Digital Logic Gates

- Consider the 16 functions in previous Table
 - Two are equal to a constant (F_0 and F_{15}).
 - Four are repeated twice (F_4 , F_5 , F_{10} and F_{11}).
 - Inhibition (F_2) and implication (F_{13}) are not commutative or associative.
 - The other eight are used as standard gates:
 - complement (F_{12})
 - transfer (F_3)
 - AND (F_1)
 - OR (F_7)
 - NAND (F_{14})
 - NOR (F_8)
 - XOR (F_6)
 - equivalence (XNOR) (F_9)
 - Complement: inverter.
 - Transfer: buffer (increasing drive strength).
 - Equivalence: XNOR.

Summary of Logic Gates

AND



x	y	F
0	0	0
0	1	0
1	0	0
1	1	1

OR



x	y	F
0	0	0
0	1	1
1	0	1
1	1	1

Inverter



x	F
0	1
1	0

Buffer



x	F
0	0
1	1

Summary of Logic Gates

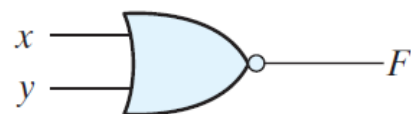
NAND



$$F = (xy)'$$

x	y	F
0	0	1
0	1	1
1	0	1
1	1	0

NOR



$$F = (x + y)'$$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	0

Exclusive-OR
(XOR)



$$F = xy' + x'y$$

$$= x \oplus y$$

x	y	F
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive-NOR
or
equivalence



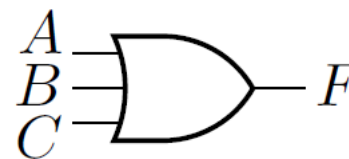
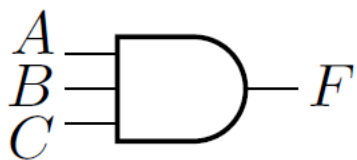
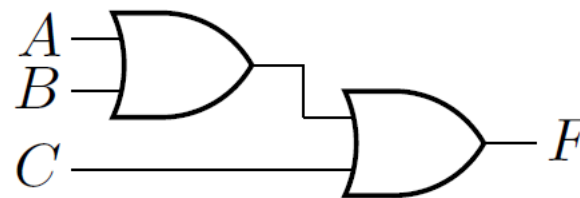
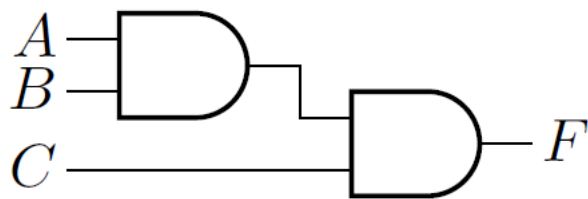
$$F = xy + x'y'$$

$$= (x \oplus y)'$$

x	y	F
0	0	1
0	1	0
1	0	0
1	1	1

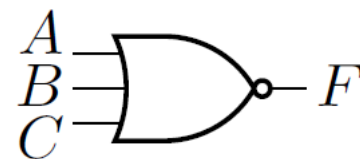
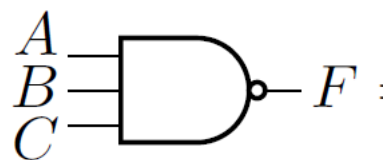
Multiple Inputs

- Extension to multiple inputs
 - A gate can be extended to multiple inputs.
 - AND and OR are commutative and associative.
 - $F = ABC = (AB)C$
 - $F = A + B + C = (A + B) + C$



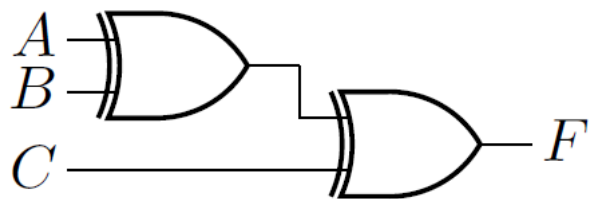
Multiple Inputs

- NAND and NOR are commutative **but not associative**
 - $((AB)'C)' \neq (A(BC)')'$: does not follow associativity.
 - $((A + B)' + C)' \neq (A + (B + C)')'$: does not follow associativity.



Multiple Inputs

- The XOR gates and equivalence gates both possess **commutative and associative properties**.
 - Gate output is low when even numbers of 1's are applied to the inputs, and when the number of 1's is odd the output is logic 1.
 - Multiple-input exclusive-OR and equivalence gates are uncommon in practice.



$$\begin{matrix} A \\ B \\ C \end{matrix} \text{ XOR } F = A \oplus B \oplus C$$