Quiz 1: (50p max; 60p total = 10p each + 10p free)
Please write your answers in English and submit to Blackboard

1. Determine if each statement is a tautology: * *answer Yes or No*
   (1) $(\neg p \vee r) \wedge (p \vee q) \rightarrow (q \vee r)$
   (2) $\neg\forall x \exists y\, (xy \neq 1) \leftrightarrow \exists x \forall y\, (xy = 1)$

2. Prove that the premises $p \rightarrow \neg q$, $\neg p \rightarrow s$, q imply conclusion s.

3. Prove $(B - A) \cap (C - A) = (B \cap C) - A$ with set identities.

4. Prove that "$|x + y| \leq |x| + |y|$ holds for any real numbers x, y".

5. Prove that "if A, B, C, D are (probably infinite) sets such that $|A| \leq |B|$ and $|C| = |D|$, then $|A \times C| \leq |B \times D|$".

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Solutions

- Q1. (1) Yes, resolution. (2) Yes, negation of nested quantifiers.
- Q2. Contrapositive of $p \to \neg q$ is $q \to \neg p$; Hypothetical syllogism of $q \to \neg p$ and $\neg p \to s$ is $q \to s$; Modus ponens of $q \to s$ and q is the conclusion s.
- Q3. Proof with set identities (no need to write out their names):

$$
\begin{aligned}
&(B - A) \cap (C - A) \\
&= (B \cap \bar{A}) \cap (C \cap \bar{A}) && \textit{* Definition} \\
&= (B \cap C) \cap (\bar{A} \cap \bar{A}) && \textit{* Communative} \\
&= (B \cap C) \cap \bar{A} && \textit{* Idempotent} \\
&= (B \cup C) - A && \textit{* Definition}
\end{aligned}
$$

- Q4. Use proof by cases, e.g., two cases $x \geq 0$, $x < 0$ and two sub-cases for each case $x + y \geq 0$, $x + y < 0$.

Solutions

- Q5. (key points: Cartesian product + injective/bijective functions)
  - By definition

    $|A| \leq |B|$ means that there is a injective function $f: A \to B$

    $|C| = |D|$ means that there is a bijective function $g: C \to D$
  - Then, by definition we need to show there is a injective function from $A \times C$ to $B \times D$. It suffices to show function (f, g) is injective, i.e., for any $(a, c), (a', c') \in A \times C$ such that $(a, c) \neq (a', c')$ we have $(f(a), g(c)) \neq (f(a'), g(c'))$.
  - Note that $(a, c) \neq (a', c')$ implies $a \neq a'$ or $c \neq c'$. The above holds because both f and g are injective:

    f injective: for any $a, a' \in A$ such that $a \neq a'$, we have $f(a) \neq f(a')$

    g injective: for any $c, c' \in B$ such that $c \neq c'$, we have $g(c) \neq g(c')$