Discrete Mathematics Assignment 2

12312706 Zhou Liangyu

Q1: (a) Yes. $\because A \cap B \neq \varnothing$

$\therefore \exists x\{x|x \in A \wedge x \in B\}$.

$\because A - B = \{y|y \in A \wedge y \notin B\}$

$\therefore A - B \subseteq A$.

$\because x \neq y$

$\therefore A - B \neq A$.

$\therefore A - B \subset A$.

(b) No. If $A = B$:

$\because A = B$

$\therefore A \subseteq B,\ |A \cup B| = |A|$, which contradicts to $|A \cup B| \geq 2|A|$.

(c) No. $\overline{A \cup B} = \{x|x \notin A, x \notin B\}$.

$\because \overline{A - B} = \{y|y \notin A \wedge y \in B\} + \{y|y \in A \wedge y \in B\} + \{y|y \notin A \wedge y \notin B\}$,

$\overline{B - A} = \{z|z \in A \wedge z \notin B\} + \{z|z \in A \wedge z \in B\} + \{z|z \notin A \wedge z \notin B\}$

$\therefore \overline{(A - B)} \cap \overline{(B - A)} = \{k|k \in A \wedge k \in B\}$.

$\therefore \overline{(A - B)} \cap \overline{(B - A)} \neq \overline{A \cup B}$.

Q2: (a) $\overline{A \cap (B \cup C)}$

$= \overline{A} \cup \overline{(B \cup C)}$ De Morgan's

$= \overline{A} \cup (\overline{B} \cap \overline{C}) = \overline{A} \cup \overline{(B \cup C)}$ De Morgan's

$= \overline{(C \cup B)} \cup \overline{A}$ Commutative

(b) $(A - B) \cap (B - A)$

$= \{x|x \in A \wedge x \notin B\} \cap \{x|x \in B \wedge x \notin A\}$ Definition of difference

$= (A \cap \overline{B}) \cap (B \cap \overline{A})$ Definition

$= (A \cap B) \cap (\overline{B} \cap \overline{A})$ Distributive

$= (A \cap B) \cap (\overline{A \cup B})$ De Morgan's

$= \varnothing$ Definition

Q3: (a) For any finite subset of A, according to the definition of countable, it is countable.

For any infinite subset of A, let B denote such a subset. Arrange the elements of A in a sequence $\{x_n\}$ of distinct terms: $x_1,\ x_2,\ \ldots,\ x_n$. Construct a sequence $\{n_k\}$ that n_k is the smallest positive integer such that $x_{n_k} \in B$. Suppose $n_1,\ n_2,\ \ldots,\ n_k\ (k \in \mathbb{Z}^+)$ have been chosen. Let n_{k+1} be the smallest positive integer such that $x_{n_{k+1}} \in B$.

Consider the function between $\mathbb{Z}^+$ and B: $g:\mathbb{Z}^+ \to B,\ g(k) = x_{n_k}$.

$\because$ $\{x_n\}$ are distinct terms.

$\therefore$ $\{n_k\}$ are distinct terms.

$\therefore$ g is injective.

$\because$ For all n_k, there exists a x_{n_k} in B corresponding to it.

$\therefore$ g is onto.

$\therefore$ g is bijective.

$\therefore$ B is countable.

(b) $\because$ $g : A \to B$ is onto.

$\therefore$ For every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.

For every element of B that have more than one element of A maps to it, delete all but one of the mappings to that element, so only one element of A maps to it. Thus there are only injective mappings left. Those elements of A that still have mappings form a subset of A, and this subset is countable according to (a). Let E denote this subset.

Since every element of B is mapped to a unique element of E, there exists a bijection $f : E \to B$.

$\because$ E is countable

$\therefore$ There exists a bijection $g : \mathbb{Z}^+ \to E$.

$\therefore$ $g \circ f : \mathbb{Z}^+ \to B$ is bijective.

$\therefore$ B is countable.

Q4: (a) Yes. $A \,\Delta\, (B \,\Delta\, C)$

$= A \,\Delta\, (\{x|x \in B \wedge x \notin C\} \cup \{x|x \in C \wedge x \notin B\})$

$= \{x|x \in A \wedge x \notin B \wedge x \notin C\} \cup \{x|x \notin A \wedge x \in B \wedge x \notin C\} \cup \{x|x \notin A \wedge x \in C \wedge x \notin B\})$

$= (\{x|x \in A \wedge x \notin B\} \cup \{x|x \in B \wedge x \notin A\}) \,\Delta\, C$

$= (A \,\Delta\, B) \,\Delta\, C.$

(b) $A \neq B$. Proof: If $C = \varnothing$:

$$A \Delta C = \{x|x \in A \wedge x \notin C\} \cup \{x|x \in C \wedge x \notin A\} = A,$$

$$B \Delta C = \{x|x \in B \wedge x \notin C\} \cup \{x|x \in C \wedge x \notin B\} = B.$$

but there is not any information about $A = B$ or not.

(c) Let $A = [0, 1],\ B = (0, 1] \cup \{x|x = \frac{1}{n}, n \in \mathbb{N}\}$.

$\because$ $[0, 1]$ and $(0, 1]$ are uncountable

$\therefore$ A and B are uncountable.

$\therefore$ $A \,\Delta\, B = \{0\} \cup \{x|x = \frac{1}{n}, n \in \mathbb{N}\}$, and $A \,\Delta\, B$ is infinite and countable because there exists a bijection between $\mathbb{Z}^+$ and $A \,\Delta\, B : f(x) = \frac{1}{x}$.

Q5: To calculate $|A \cup B \cup C|$, we need to add $|A|+|B|+|C|$ first. Since $|A \cap B|$ is counted twice in $(|A|+|B|)$ part, $|B \cap C|$ is counted twice in $(|B|+|C|)$ part and $|A \cap C|$ is counted twice in $(|A|+|C|)$ part, we need to subtract $(|A \cap B|+|A \cap C|+|B \cap C|)$ once. However, $|A \cap B \cap C|$ is subtracted three times in the declined part, so we need to add $|A \cap B \cap C|$ once.

Q6: $\because |A| = |B|$

$\therefore$ There exists a bijection $f : A \to B$. $f(a) = b$ for $a \in A$ and $b \in B$.

$\because |C| = |D|$

$\therefore$ There exists a bijection $g : C \to D$. $g(c) = d$ for $c \in C$ and $d \in D$.

Construct a function $h(a, c) = (f(a), g(c))$.

$\because f(a)$ and $g(c)$ are bijective function.

$\therefore h(a, c) = (f(a), g(c)) = (b, d)$ is a bijective function.

$\therefore$ There exists a bijection $h : A \times C \to B \times D$.

$\therefore |A \times C| = |B \times D|$.

Q7: (a) f must be one-to-one.

$\because g : A \to B$ and $f : B \to C$

$\therefore f \circ g : A \to C$.

If f is not one-to-one, there exists $x, y \in B$ and $x \neq y$ implies $f(x) = f(y)$.

$\because g$ is one-to-one

$\therefore$ There exists $m, n \in A$ and $m \neq n$ such that $g(m) = x$ and $g(n) = y$, which implies $f \circ g\,(m) = f \circ g\,(n)$ and leads to a contradiction to "$f \circ g$ is one-to-one".

$\therefore f$ is one-to-one.

(b) g must be one-to-one.

Assume that g is not one-to-one, then there exists $m, n \in A$ and $m \neq n$ such that $g(m) = g(n)$.

$\because f$ is one-to-one

$\therefore f(g(m)) = f(g(n))$, i.e. $f \circ g(m) = f \circ g(n)$ implies $m = n$, which contradicts to "$f \circ g$ is one-to-one".

$\therefore g$ is one-to-one.

(c) g must be one-to-one.

Assume g isn't one-to-one.

$\therefore$ There exists $m \neq n$ such that $g(m) = g(n)$.

$\because f \circ g$ is one-to-one

$\therefore$ For $m \neq n$, $f \circ g(m) \neq f \circ g(n)$.

$\because g(m) = g(n)$

$\therefore f(g(m)) = f(g(n))$, i.e. $f \circ g(m) = f \circ g(n)$, which contradicts to $f \circ g(m) \neq f \circ g(n)$.

$\therefore g$ must be one-to-one.

(d) f must be onto.

$\because f \circ g$ is onto

$\therefore$ For every $c \in C$, there exists $a \in A$ such that $f \circ g(a) = c$.

$\because f \circ g(a) = f(g(a)), g(a) = b$

$\therefore f(b) = c$, i.e. for every $c \in C$, there exists $b \in B$ such that $f(b) = c$.

$\therefore f$ is onto.

(e) There is no need for g to be onto.

Consider the following example: $A = \{1,2\}, B = \{1,2,3\}, C = \{1,2\}$.

$$g(1) = 1,\ g(2) = 2.\ f(1) = 1,\ f(2) = 2,\ f(3) = 2.$$

$\therefore f \circ g(1) = 1, f \circ g(2) = 2$.

$\therefore f \circ g$ is onto, with g is not onto because it doesn't map any element in A to 3 in B.

Q8: Proof: $\because x = \lfloor x \rfloor + y$ for $0 \le y < 1$

$\therefore 3x = 3\lfloor x \rfloor + 3y$.

Case 1: $0 \le y < \frac{1}{3}$

we have $0 \le 3y < 1$.

$\lfloor 3x \rfloor = \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor \frac{1}{3} \rfloor = \lfloor x \rfloor$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor \frac{2}{3} \rfloor = \lfloor x \rfloor$

$\therefore \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Case 2: $\frac{1}{3} \le y < \frac{2}{3}$

we have $1 \le 3y < 2$.

$\lfloor 3x \rfloor = \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor + 1$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor \lfloor x \rfloor + y + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor \lfloor x \rfloor + y + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1$

$\therefore \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Case 3: $\frac{2}{3} \le y < 1$

we have $2 \le 3y < 3$.

$\lfloor 3x \rfloor = \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor + 2$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor \lfloor x \rfloor + y + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor + 1$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor \lfloor x \rfloor + y + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1$

$\therefore \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

$\therefore \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor.$

Q9: Assume that $n^2 \leq k < (n+1)^2$, then $n \leq \lfloor \sqrt{k} \rfloor < n+1$.

$\because (n+1)^2 - n^2 = 2n+1$

$\therefore$ There are $2n+1$ integers k in the interval $[n^2,\ (n+1)^2)$. And they contributes $n(2n+1)$ to the summation.

Let $n = \lfloor \sqrt{m} \rfloor - 1$, if $m \geq (n+1)^2$, from $k = n+1$ to $k = m$, there are $m - (n+1)^2 + 1$ integers $n+1$.

$$\therefore \sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \sum_{k=1}^{n} k(2k+1) + (m - (n+1)^2 + 1)(n+1)$$

$$= 2(1^2 + 2^2 + \ldots\ldots + n^2) + (1 + 2 + \ldots\ldots + n) + (m - (n+1)^2 + 1)(n+1)$$

$$= \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + (m - (n+1)^2 + 1)(n+1)$$

$$= \frac{n(n+1)(4n+5)}{6} + (m - (n+1)^2 + 1)(n+1).$$

Q10: Proof: Construct two one-to-one functions: $f: (0,1) \to [0,2],\ f(x) = 2x.\ \ g: [0,2] \to (0,1),\ g(x) = \frac{1}{3}(x+1).$

$\therefore |(0,1)| \leq |[0,2]|,\ |[0,2]| \leq |(0,1)|.$

$\therefore |(0,1)| = |[0,2]|.$

Q11: Since the hotel has infinitely many rooms, we can move the guests to the rooms which has a number twice their current room number, i.e. the guest in room 1 moves to room 2, the guest in room 2 moves to room 4, an so on. It acts as the function $f(x) = 2x$. This action will make the rooms which have odd numbers empty. And the new guests can move to those rooms. Just like the function $f(x) = 2x - 1$. Because there are infinitely many natural numbers, there are infinitely many empty rooms for the new guests. So this action works for countably many new guests.

Q12: (a) Every program can be seem as a finite string constructed from the finite alphabet that consists of every characters that may appear in a program. Define an order for those characters. Enumerate every string s, if it's a valid program, add it to the set S. After enumeration, S represents the set of all computer programs in all existing programming languages, and the finite strings in S can be listed in a sequence of the length of the strings. This implies a bijection from Z^+ to the set of all computer programs in all existing programming languages.

(b) Assume that the set of all functions from Z^+ to the set of digits $\{0,1,...,9\}$ is countable, represented by S. Then every function f in S can be expressed in a sequence: $f_n = (f_n(1), f_n(2), \ldots, f_n(n))$, and $f_n(n)$ is a digit in $\{0,1,...,9\}$. Construct a function g that is not included in S:

$$g(n) = \begin{cases} f_n(n) + 1 & \text{if } f_n(n) \neq 9 \\ 0 & \text{if } f_n(n) = 9 \end{cases}$$

g is a function from Z^+ to the set of digits $\{0,1,...,9\}$, but $g(i) \neq f_i(i)$, which leads to a contradiction. Thus the set of all functions from Z^+ to the set of digits $\{0,1,...,9\}$ is uncountable.

Q13: $\because log_a n = \frac{log_2 n}{log_2 a},\ a > 1$

$\therefore$ Let $\frac{1}{log_2 a} = c$, then $log_a n = c \cdot log_2 n,\ c > 0.$

$\therefore \exists\, c_1 > 0$ such that $\forall\, n > 1$, $|log_a n| \leq c_1 \cdot |log_2 n|$. $\exists\, c_2 > 0$ such that $\forall\, n > 1$, $|log_a n| \geq c_2 \cdot |log_2 n|$.

$\therefore log_a n = O(log_2 n).\ log_a n = \Omega(log_2 n).$

$\therefore \Theta(log_a n) = \Theta(log_2 n).$

Q14: (a) $\because$ The while-loop halves the search space each time until the size of search space reaches 1.

$\therefore$ The number of steps is $log_2 n$.

$\therefore$ The time complexity is $\Theta(log\, n)$.

(b) $\because$ The while-loop in Algorithm 1 stops until the size of search space reaches 1.

$\therefore$ We can check if the middle element of the array matches the target before continuing the loop.

In best-case, the target is the middle element of the original array, and the time complexity is $\Theta(1)$.

In worst-case, the search has to continue through multiple splits, the time complexity remains $\Theta(log\, n)$.

(c) $\because$ In Algorithm 1, the variables i, j and only need a constant amount of space to store values.

$\therefore$ The space complexity is $\Theta(1)$.

(d) The input of binary search problem is an integer array of length n. According to the definition of fixed-length encoding and binary representation of integers on computers, every integer needs $\lceil log_2 n \rceil$ bits to represent itself. So the input size of the above binary search problem is $\Theta(n \cdot log_2 n)$.