Discrete Mathematics Assignment 4

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1. (a) Let $A = \{P \mid P \text{ can be proved by weak induction}\}$, $B = \{P \mid P \text{ can be proved by strong induction}\}$.

$\because$ The inductive step in weak induction is "Prove $\forall k \in \mathbb{Z}^+,\ P(k) \to P(k+1)$ is true",

the inductive step in strong induction is "Prove $\forall k \in \mathbb{Z}^+,\ P(1) \land \cdots \land P(k) \to P(k+1)$ is true"

$\therefore$ If a statement P can be proved by strong induction, then it can also be proved by weak induction.

$\therefore A \subset B$.

$\therefore$ Strong induction implies weak induction.

Let P be a statement in B, then we know that when $P(1)$ is true, $P(1) \land \cdots \land P(k) \to P(k+1)$.

Let $Q(n) = P(1) \land \cdots \land P(n)$, then $Q(1) = P(1)$.

$\because P(1) \land \cdots \land P(k) \to P(k+1)$

$\therefore P(1) \land \cdots \land P(k) \to P(1) \land \cdots \land P(k) \land P(k+1)$, i.e. $Q(k) \to Q(k+1)$

$\therefore Q$ can be proved by weak induction.

$\because$ Whenever Q is true, P is true

$\therefore P$ can be proved by weak induction.

$\therefore B \subset A$.

$\therefore$ Weak induction implies strong induction.

$\therefore$ Weak induction and strong induction are equivalent.

(b) Let P be a statement that can be proved by weak induction,

S be the set that contains all $n \in \mathbb{N}$ such that $P(n)$ is false.

If weak induction doesn't implies the well-ordering principle, then S must be non-empty.

By well-ordering principle, let the smallest element in S be k.

$\because P(1)$ is true.

$\therefore k \neq 1$.

$\because k - 1 \in \mathbb{N}$, k is the smallest element in S

$\therefore k - 1 \notin S$, i.e. $P(k-1)$ is true.

$\because P(1)$ is true, by weak induction we can prove that $P(k)$ is true.

$\therefore k \notin S$, which contradicts to our assumption.

$\therefore$ If P can be proved by weak induction, then S must be empty.

$\therefore$ Weak induction implies the well-ordering principle.

2. Proof: Let $P(n)$ represent $(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$.

Base step: When $n = 1$, $LHS = A_1 \cap B$, $RHS = A_1 \cap B$. $\therefore P(1)$ is true.

Inductive step: Assume that for an arbitrary positive integer k, $P(k)$ is true.

$\therefore (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_k - B) = (A_1 \cap A_2 \cap \cdots \cap A_k) - B$.

$\therefore (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_k - B) \cap (A_{k+1} - B)$

$= ((A_1 \cap A_2 \cap \cdots \cap A_k) - B) \cap (A_{k+1} - B)$

$= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cap B^c) \cap (A_{k+1} \cap B^c)$

$= (A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cap B^c$

$= (A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) - B.$

$\therefore P(k+1)$ is true.

$\therefore$ By mathematical induction, we know that

$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$ holds for all positive integer n.

3. Proof: Let $P(n)$ represent that for any real number $h > -1$ and integer $n \geq 0$, we have $(1+h)^n \geq 1 + nh$.

   Base step: When $n = 0$, $(1+h)^0 = 1 + 0 \cdot h$. $P(1)$ is true.

   Inductive step: Assume that for an arbitrary positive integer k, $P(k)$ is true.

   $\therefore (1+h)^k \geq 1 + kh.$

   $\therefore (1+h)^{k+1} = (1+kh) \cdot (1+h) = 1 + (k+1)h + kh^2 \geq 1 + (k+1)h.$

   $\therefore P(k+1)$ is true.

   $\therefore$ By mathematical induction, we know that $(1+h)^n \geq 1 + nh$ holds for all positive integer n.

4. Proof: Let $P(n)$ represent "if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some integer $i \in 1, 2, \ldots, n$".

   Base step: When $n = 1$, the condition $p \mid a_1$ given in the problem directly implies $p \mid a_1$. $P(1)$ is true.

   Inductive step: Assume that for an arbitrary positive integer k, $P(k)$ is true.

   $\therefore$ If $p \mid a_1 a_2 \cdots a_k$, then $p \mid a_i$ for some integer $i \in 1, 2, \ldots, k$.

   Suppose that $p \mid a_1 a_2 \cdots a_k a_{k+1}$.

   $\therefore p \mid a_1 a_2 \cdots a_k$ or $p \mid a_{k+1}$.

   If $p \mid a_1 a_2 \cdots a_k$, then we know that $p \mid a_i$ for some integer $i \in 1, 2, \ldots, k$,

   which also satisfies $p \mid a_i$ for some integer $i \in 1, 2, \ldots, k+1$.

   If $p \mid a_{k+1}$, then we are directly done.

   $\therefore P(k+1)$ is true.

   $\therefore$ By mathematical induction, we know that "if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some integer $i \in 1, 2, \ldots, n$" holds for all positive integer n.

5. (a) $\because$ Four 3-cent stamps form a 12-cent stamp. $\therefore P(12)$ is true.

   $\because$ Two 3-cent stamps and one 7-cent stamp form a 13-cent stamp. $\therefore P(13)$ is true.

   $\because$ Two 7-cent stamps form a 14-cent stamp. $\therefore P(14)$ is true.

   (b) The inductive hypothesis of the proof is "for an arbitrary positive integer k, Postage of k cents can be formed using just 3-cent stamps and 7-cent stamps is true".

   (c) We need to prove $P(k+1)$ is true in the inductive step.

   (d) Since $k+1 \geq 15$, $k-2 \geq 12$, by the inductive hypothesis, we know that $P(k)$, $P(k-1)$, $P(k-2)$ are true.

   $\because k+1 = (k-2) + 3$

   $\therefore$ By adding one 3-cent stamp to the $k-2$ situation, we can get a postage of $k+1$ cents.

   $\therefore P(k+1)$ is true.

   (e) In (a), we have proved $P(12)$, $P(13)$ and $P(14)$ is true by completing the base step.

   For $n \geq 15$ situations, we have proved $P(n)$ is true by completing the inductive step.

   Therefore, we can show that this statement is true whenever $n \geq 12$.

6. Let the elements a_i in the ordered array A be numbered from 1 to n. x is the target number.

   $binarySearch(A, x, l, r)$

   if $l > r$ **then return** 0

$m := \lfloor (l+r)/2 \rfloor$

if $a_m = x$ **then return** m

else if $a_m > x$ **then return** $binarySearch(A, x, l, m-1)$

else $a_m < x$ **then return** $binarySearch(A, x, m+1, r)$

7. $\because 1 \le a < 2$

$\therefore (\frac{a}{2})^k \le 1,\ k \ge 0.\ a^{log_2 n} = n^{log_2 a} < n.$

$\therefore \sum_{i=0}^{m} (\frac{a}{2})^i = 1 + \frac{a}{2} + \frac{a^2}{2^2} + ... + \frac{a^m}{2^m} = \Theta(1).$

$T(n) = aT(n/2) + n$

$= a^2 T(n/2^2) + (a/2+1)n$

$= a^3 T(n/2^3) + (a^2/2^2 + a/2 + 1)n$

$= ...$

$= a^{log_2 n} T(n/2^{log_2 n}) + n \sum_{i=0}^{log_2 n - 1} (\frac{a}{2})^i$

$= n^{log_2 a} T(1) + n \cdot \Theta(1).$

$\therefore T(n) = \Theta(n).$

8. The number of all bit strings of length 8 is: $2^8 = 256$.

0000 can be placed at 5 positions in the string: $1-4, 2-5, 3-6, 4-7, 5-8$.

The rest 4 positions can be any numbers, and there are $2^4 = 16$ possible combinations for the four numbers.

$\therefore$ The number of bit strings containing 4 consecutive 0s is $5 \cdot 16 = 80$.

Similarly, the number of bit strings containing 4 consecutive 1s is 80.

$\because$ Bit strings containing both 4 consecutive 0s and 4 consecutive 1s are 00001111 and 11110000, and they are counted twice in the calculation above.

$\therefore$ The number of bit strings of length 8 that contain either 4 consecutive 0s or 4 consecutive 1s is $80 + 80 - 2 = 158$.

9. (a) Choose the three-card rank: $\binom{13}{1}$. Choose 3 cards: $\binom{4}{3}$.

Choose the two-card rank: $\binom{12}{1}$. Choose 2 cards: $\binom{4}{2}$.

$\therefore$ The number of full houses is $\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}$.

(b) Choose 2 two-card ranks: $\binom{13}{2}$. Choose 2 cards for each rank: $\binom{4}{2}^2$.

Choose the third rank: $\binom{11}{1}$. Choose 1 card for the rank: $\binom{4}{1}$.

$\therefore$ The number of two pairs is $\binom{13}{2} \cdot \binom{4}{2}^2 \cdot \binom{11}{1} \cdot \binom{4}{1}$.

(c) Choose a suit for the flush: $\binom{4}{1}$. Choose 5 cards in the suit: $\binom{13}{5}$.

$\therefore$ The number of flushes is $\binom{4}{1} \cdot \binom{13}{5}$.

(d) According to the problem description, there are 10 possible combinations of cards to form a straight.

Choose a combination: $\binom{10}{1}$. Choose a suit for each card in the straight: $\binom{4}{1}^5$.

$\therefore$ The number of straights: $\binom{10}{1} \cdot \binom{4}{1}^5$.

(e) Choose a rank: $\binom{4}{1}$. Choose 4 cards in the rank: $\binom{13}{4}$.

Choose another rank: $\binom{3}{1}$. Choose 1 card in the rank: $\binom{13}{1}$.

$\therefore$ The number of quads is $\binom{4}{1} \cdot \binom{13}{4} \cdot \binom{3}{1} \cdot \binom{13}{1}$.

10. $\because$ For all $0 \le k \le n$ the combinations $\binom{n}{k}$ are integers

$\therefore \binom{2020}{1010}$ is divisible by some integers.

$\because 2022 = 2 \cdot 1011$, 2 and 1011 are coprime

$\therefore$ Decompose the problem into two subproblems: check if the binomial coefficient can is divisible by 2 and 1011.

The power of 2 in 2020! $= \lfloor \frac{2020}{2^1} \rfloor + \lfloor \frac{2020}{2^2} \rfloor + \lfloor \frac{2020}{2^3} \rfloor + \ldots + \lfloor \frac{2020}{2^{10}} \rfloor$

$$= 1010 + 505 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2013.$$

The power of 2 in $1010! \cdot 1010! = 2 \cdot (\lfloor \frac{1010}{2^1} \rfloor + \lfloor \frac{1010}{2^2} \rfloor + \lfloor \frac{1010}{2^3} \rfloor + \ldots + \lfloor \frac{1010}{2^9} \rfloor)$

$$= 2 \cdot (505 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1) = 2006.$$

$\because 2013 - 2006 = 7 > 0$

$\therefore \binom{2020}{1010} = \dfrac{2020!}{1010! \cdot 1010!}$ can is divisible by 2.

The power of 1011 in 2020! $= \lfloor \frac{2020}{1011^1} \rfloor = 1.$

The power of 1011 in $1010! \cdot 1010! = 2 \cdot \lfloor \frac{1010}{1011} \rfloor = 0.$

$\because 1 - 0 = 1 > 0$

$\therefore \binom{2020}{1010}$ can is divisible by 1011.

$\therefore \binom{2020}{1010}$ can is divisible by $2 \cdot 1011 = 2022$.

11. $LHS = \sum_{k=0}^{r} \binom{n+k}{k} = \sum_{k=0}^{r} \binom{n+k}{n}$, $RHS = \binom{n+r+1}{r} = \binom{n+r+1}{n+1}$.

The right hand side asks how many ways to pick $n+1$ elements from a set with size of $n+r+1$, and the answer is $\binom{n+r+1}{n+1}$.

We can divide the process into r different cases:

We number the elements in the set from 1 to $n+r+1$.

Assume that the number of the rest n elements should be larger than the first element.

First we should pick one element from the first $r+1$ elements.

If the number of the first element is 1, then we have to pick n elements from the rest $n+r$ elements, i.e. $\binom{n+r}{n}$.

If the number of the first element is 2, then we have to pick n elements from the rest $n+r-1$ elements, i.e. $\binom{n+r-1}{n}$.

Therefore if the number of the first element is k, the rest of the n committee members can be chosen from the remaining $n+r+1-k$ persons in exactly $\binom{n+r+1-k}{n}$ ways,

This applies to $k = 1, 2, \ldots, r+1$.

Add these combinations together, we can get $\binom{n+r}{n} + \binom{n+r-1}{n} + \ldots + \binom{n+2}{n} + \binom{n+1}{n} = \sum_{k=0}^{r} \binom{n+k}{n} = \sum_{k=0}^{r} \binom{n+k}{k}$.

$\therefore \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{n+1} = \binom{n+r+1}{r}$.

12. $\because a_n = 3a_{n-2} + 2a_{n-3}$

$\therefore$ The characteristic equation is $r^3 - 3r - 2 = 0$, i.e. $(r+1)^2(r-2) = 0$.

$\therefore$ The CE has 2 distinct roots: $r_1 = -1$ and $r_2 = 2$. $m_1 = 2, m_2 = 1$.

$\therefore a_n = r_1^n(\alpha_{1,0}n^0 + \alpha_{1,1}n) + r_2^n \alpha_{2,0} n^0 = (-1)^n(\alpha_{1,0} + \alpha_{1,1}n) + 2^n \alpha_{2,0}$.

$\because a_0 = \alpha_{1,0} + \alpha_{2,0} = 1,$

$a_1 = (-1)(\alpha_{1,0} + \alpha_{1,1}) + 2\alpha_{2,0} = -5,$

$a_2 = \alpha_{1,0} + 2\alpha_{1,1} + 2^2 \alpha_{2,0} = 0$

$\therefore \alpha_{1,0} = 2,\ \alpha_{1,1} = 1,\ \alpha_{2,0} = -1.$

$\therefore a_n = (-1)^n(2+n) - 2^n.$

13. (a) The characteristic equation of the associated linear homogeneous recurrence relation is $r - 2 = 0$. The root is 2.

Assume $a_n = p(n) = An^2 + Bn + C$ is a particular solution to the original recurrence relation,

then $An^2 + Bn + C = 2(A(n-1)^2 + B(n-1) + C) + n^2$.

$\therefore (A+1)n^2 + (-4A+B)n + 2A - 2B + C = 0.$

$\therefore A+1=0,\ -4A+B=0,\ 2A-2B+C=0.$

$\therefore A=-1,\ B=-4,\ C=-6.$

$\therefore$ All of the solutions are of the form $a_n = \alpha_1 \cdot 2^n - n^2 - 4n - 6$.

(b) $\because$ The initial condition $a_1 = \alpha_1 \cdot 2 - 11 = 2$.

$\therefore \alpha_1 = \frac{13}{2}$.

$\therefore$ The solution of the recurrence relation is $a_n = 13 \cdot 2^{n-1} - n^2 - 4n - 6$.

14. Let $G(x)$ be the generating function of a_n for $n \geq 0$, we have $G(x) - a_0 = G(x) = 4xG(x) + \dfrac{x}{1-8x}$.

$$\begin{aligned}
\therefore G(x) &= \frac{x}{(1-4x)(1-8x)}. \\
&= \frac{1}{4}\left(\frac{1}{1-8x} - \frac{1}{1-4x}\right) \\
&= \frac{1}{4}\left(\sum_{n=0}^{\infty} 8^n x^n - \sum_{n=0}^{\infty} 4^n x^n\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}(8^n - 4^n)x^n.
\end{aligned}$$

$$\therefore a_n = \frac{1}{4}(8^n - 4^n).$$