STA219 Assignment 1

1. Let A = computer have problems with MB, B = computer have problems with HD.

   $\because P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.3 - 0.15 = 0.55$

   $\therefore$ The probability that a 10-year old computer still has fully functioning MB and HD is $1 - 0.55 = 0.45$.

2. (1) Let A = a programmer knows Java, B = a programmer knows Python.

   Then $P(A) = 0.7$, $P(B) = 0.6$, $P(A \cap B) = 0.5$.

   $\therefore P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (0.7 + 0.6 - 0.5) = 0.2$.

   $\therefore$ The probability that he/she does not know Python and does not know Java is 0.2.

   (2) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.7 + 0.6 - 0.5 = 0.8$.

   $P(A \cup B) - P(B) = 0.8 - 0.6 = 0.2$.

   $\therefore$ The probability that he/she knows Java but not Python is 0.2.

   (3) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.5}{0.6} = 0.8333$.

   $\therefore$ The probability that he/she knows Java given that he/she knows Python is 0.8333.

3. If k elements are randomly selected with replacement from n distinct elements ($k \leq n$) and placed in order, there are n elements can be chosen in each selection. According to multiplication principle, the number of the permutations is given by n^k.

   If k elements are randomly selected with replacement from n distinct elements ($k \leq n$), where the order does not matter, what matters is the number of times each element is selected. Let the number of times the first element is selected be m_1, the second element be m_2, ..., the n-th element be m_n, then we have $m_1 + m_2 + \ldots + m_n = k$.

   To find non-negative solutions for this equation, we can imagine a box containing n balls. To divide these n balls into k parts (the number of balls in each part can be 0), we need to insert $k-1$ boards between the balls. Each board can be inserted at any position, which equals to choosing $k-1$ positions from $n+k-1$ positions to insert the boards, then the total number to choose is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$. Therefore, the number of the combinations is given by $\binom{n+k-1}{n}$.

4. (1) If we randomly pick $2k$ shoes from n pairs of shoes, the total number of ways to pick is $\binom{2n}{2k}$.

   If exactly k pairs are formed among the $2k$ shoes picked, the number of possible cases is $\binom{n}{k}$.

   $\therefore P(\text{Exactly k pairs are formed among the 2k shoes picked}) = \dfrac{\binom{n}{k}}{\binom{2n}{2k}}$.

   (2) If no pair is formed among the $2k$ shoes picked, the result is: Pick $2k$ pair of shoes from n pairs of shoes, then pick out either the left or right shoe from each pair. Therefore, the number of possible cases is $\binom{n}{2k} \cdot 2^{2k}$.

   $\therefore P(\text{No pair is formed among the 2k shoes picked}) = \dfrac{\binom{n}{2k} \cdot 2^{2k}}{\binom{2n}{2k}}$.

   (3) If exactly one pair is formed among the $2k$ shoes picked, the result is: Pick one pair of shoes from n pair of shoes, and pick $2k-2$ pair of shoes from $n-1$ pairs of shoes, then pick out either the left or right shoe from each pair. Therefore, the number of possible cases is $\binom{n}{1} \cdot \binom{n-1}{2k-2} \cdot 2^{2k-2}$.

$$\therefore P(\text{Exactly one pair is formed among the 2k shoes picked}) = \frac{\binom{n}{1}\cdot\binom{n-1}{2k-2}\cdot 2^{2k-2}}{\binom{2n}{2k}}.$$

5. Let A_i = The i-th couple is paired together, $1 \le i < j < k \le 4$.

   If exactly k couples are paired together, since the rest couples can be randomly paired, the permutation number of pairing is $(k-1)!$. The total permutation number of pairing is $A_4^4 = 4!$, therefore:

   $P(A_i) = \frac{3!}{4!} = \frac{1}{4}$, $P(A_i \cap A_j) = \frac{2!}{4!} = \frac{1}{12}$, $P(A_i \cap A_j \cap A_k) = \frac{1!}{4!} = \frac{1}{24}$, $P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{4!} = \frac{1}{24}$.

   $\therefore$ According to the inclusion-exclusion principle, $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$

   $-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4)$

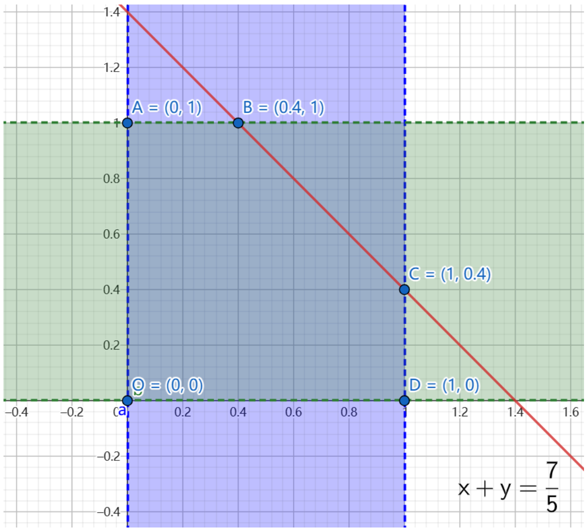
   $+P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_4)$

   $= 4\cdot\frac{1}{4} - 6\cdot\frac{1}{12} + 4\cdot\frac{1}{24} - \frac{1}{24} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{5}{8}$.

   $\therefore$ The probability that at least one couple is paired together is $\frac{5}{8}$.

6. Let the two numbers be x and y, then $0 \le x, y \le 1$.

   If $x + y < \frac{7}{5}$, it represents the area of the pentagon OABCD in the figure.



   $\therefore$ The probability is given by $\frac{1 - 0.6 \times 0.6 \times 0.5}{1} = 0.82$.

7. Let A = the rare disease occurs, B = the test says you have the disease, then:

   $P(A) = \frac{1}{1000} = 0.001$, $P(B|A) = 0.95$, $P(B|\overline{A}) = 0.001$.

   $\therefore P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\overline{A})P(B|\overline{A})} = \frac{0.001\cdot 0.95}{0.001\cdot 0.95 + 0.999\cdot 0.001} = 0.4874$.

   Compared to the probability in Example 1.13, the probability is much higher.

8. The first opinion is correct, because the gender of the two children are independent.

   Let A = meet a boy at first, $B_1 = \{GG\}$, $B_2 = \{BB\}$, $B_3 = \{BG\}$, $B_4 = \{GB\}$.

   $B_1,\ B_2,\ B_3$ and B_4 are mutually exclusive, $B_1 \cap B_2 \cap B_3 \cap B_4 = \Omega$, $P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$.

   $\because$ There is no boy in GG family, in BB family there are two boys, there is only one boy in BG and GB family

   $\therefore P(A|B_1) = 0$, $P(A|B_2) = 1$, $P(A|B_3) = \frac{1}{2}$, $P(A|B_4) = \frac{1}{2}$.

   $\therefore P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) + P(B_4)P(A|B_4)$

$$= \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}$$
$$= \frac{1}{2}.$$

$\because$ Only BB suggests that the other child is also a boy

$\therefore P(\text{the other child is also a boy}) = P(B_2|A) = \dfrac{P(B_2)P(A|B_2)}{P(A)} = \dfrac{\frac{1}{4} \cdot 1}{\frac{1}{2}} = \dfrac{1}{2}.$

$\therefore$ The second opinion is not correct, because it does not take the probability of encountering boys in different families into account.

9. According to the figure, when one of component 1, 2, 3 and one of component 4, 5 works well, the system works properly. Let A = one of component 1, 2, 3 works, B = one of component 4, 5 works, then

$$P(A) = 1 - 0.3 \cdot 0.3 \cdot 0.3 = 0.973,\ P(B) = 1 - 0.3 \cdot 0.3 = 0.91$$

$\because$ A and B are independent

$\therefore P(\text{the system works properly}) = P(A \cap B) = P(A) \cdot P(B) = 0.973 \cdot 0.91 = 0.88543.$

$\therefore$ The probability that the system works properly is 0.88543.