CS201: Discrete Mathematics (Fall 2024)
Written Assignment #6
(100 points maximum but 110 points in total)
Deadline: 11:59pm on Dec 27 (please submit to Blackboard)
PLAGIARISM WILL BE PUNISHED SEVERELY

Q.1 (**10p**) Let G be a *simple* graph with n vertices.

(a) (**3p**) What is the *maximum* and *minimum* numbers of edges G can have? Explain.

(b) (**3p**) What is the *maximum* and *minimum* degrees each vertex in G can have? Explain.

(c) (**4p**) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Q.2 (**8p**) The *complementary graph* $\overline{G}$ of a simple graph G is a simple graph that has the same vertices as G, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of the following graphs: $\overline{K_n}$, $\overline{K_{m,n}}$, $\overline{C_n}$, $\overline{Q_n}$.

Q.3 (**8p**) Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

(a) (**3p**) Use a bipartite graph to model the four employees and their qualifications.

(b) (**5p**) Use Hall's marriage theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support. If such an assignment exists, find one.

Q.4 (**5p**) A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Q.5 (**9p**) Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimum* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(a) (**3p**) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(b) (**3p**) $dist(u, v) = dist(v, u)$.

(c) (**3p**) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Q.6 (**10p**) In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner — there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ if and only if x beats y in their game.

(a) (**3p**) Explain why a tournament directed graph cannot have cycles of length 1 or 2.

(b) (**3p**) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) (**4p**) A relation is called a *strict total ordering* if it is irreflexive, antisymmetric, transitive, and every two distinct elements in the set are comparable. Show that a tournament directed graph represents a strict total ordering if and only if there are no cycles of length 3.

Q.7 (**5p**) Let G be a connected simple graph. Show that if an edge in a connected graph is not contained in any simple circuit, then this edge is a *cut edge*.

Q.8 (**5p**) Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding distinct edges of G share a common endpoint. Prove that if a connected simple graph G is regular (i.e., all vertices have the same degree), then $L(G)$ has an Euler circuit.

Q.9 (**5p**) Prove that every n-cube Q_n ($n > 1$) has a Hamilton circuit. (Hint: Q_{n+1} can be constructed from two copies of Q_n.)

Q.10 (**5p**) Show that if G is a simple graph with at least 11 vertices, then G or $\overline{G}$ is nonplanar.

Q.11 (**5p**) Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - 10/3$ if $v \geq 4$.

Q.12 (**10p**) There are 17 students who communicate with each other discussing problems in discrete mathematics. There are only 3 possible problems, and each pair of students discuss one of these 3 problems. Prove that there are at least 3 students who all pairwise discuss the same problem.

Q.13 (**8p**) The *rooted Fibonacci trees* T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n \geq 3$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. Answer the following questions: (you can write your answers in terms of the Fibonacci numbers f_n)

(a) (**6p**) How many vertices, leaves, and internal vertices does T_n ($n \geq 1$) have? Explain.

(b) (**2p**) What is the height of T_n? Explain.

Q.14 (**5p**) Calculate the value of the postfix expression "$3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$" and show your computation steps.

Q.15 (**12p**) Consider the graph shown below and answer the following questions:

(a) (**6p**) Use Dijkstra's algorithm to find a shortest path and its length from a to f. Show your iteration steps.

(b) (**6p**) Use Prim's algorithm to find the minimum spanning tree of the following weighted connected undirected graph. For every spanning step, draw the corresponding subtree.