工程概率统计

Probability and Statistics for Engineering

第一章 概率论基础

Chapter 1 Basic Ideas in Probability

Chapter 1 Basic Ideas in Probability

1.1 Basic Concepts

- What's your understanding of probability?
- In daily life, people usually interpret probability as the measure of an individual's degree of belief in the statement that he or she is making.

Example 1.1

Probability in daily life:

- It is 40% probable that 吴承恩 actually wrote the novel *Journey to the West* (西游记).
- The probability that Oswald acted alone in assassinating Kennedy is 0.8.
- You are 90% sure that you will receive at least an A if you work hard in this course.

- Using the concept of probability in this way is acceptable in daily life, but in academic contexts, probability has a more rigorous and precise definition.
- Probability theory studies random events, but whether the novel *Journey to the West* was written by 吴承恩 is not random, it is just unknown.

1.1 Basic Concepts

- What exactly does "random" mean?
- The concept of "random" refers to events or outcomes that cannot be predicted with certainty, even though the set of possible outcomes is known.
- Therefore, randomness implies uncertainty, but it is not entirely uncertain.
- The essence of probability theory is to transform individual randomness into overall certainty.
- Based on the understanding of "random," we introduce basic concepts such as random experiments, sample space, and random events.
- With these basic concepts, we can describe and study random phenomena using mathematical methods.

1.1 Basic Concepts

Random Experiment

A random experiment (随机试验), often simply called an experiment, represents the realization or observation of a random phenomenon and has the following characteristics:

- it can be repeated under the same conditions;
- all possible outcomes are clearly known;
- exactly one of these possible outcomes occurs each time, but it cannot be determined in advance which outcome will occur.

Sample Space and Random Event

- Each possible fundamental outcome (基本结果) of a random experiment is called a sample point (样本点), usually denoted as ω.
- A set that includes all sample points of the experiment is called the sample space (样本空间), usually denoted as Ω.
- From a set theory perspective, a random event (随机事件), or event, is a subset of the sample space Ω, typically denoted by uppercase letters (e.g., A, B, C, etc.)
- If the outcome is a sample point in event A, then we say that event A happens.

1.1 Basic Concepts

Example 1.2

A person driving to work needs to pass through three traffic lights. At each traffic light, he either comes to a red light (denoted as 0) or a green light (denoted as 1). Then:

- The sample space is $\Omega = \{000, 001, 010, 100, 011, 101, 110, 111\}$;
- The event that "just one of the three traffic lights is red" is $A = \{011, 101, 110\} \subset \Omega$.

Example 1.3

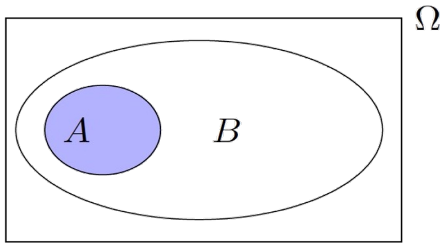
A couple decides to have children and plans to have children until a boy is born. Write B for boy and G for girl, then:

- The sample space is $\Omega = \{\mathrm{B}, \mathrm{GB}, \mathrm{GGB}, \mathrm{GGGB}, \dots\}$;
- This example shows that the sample space is not necessarily finite.

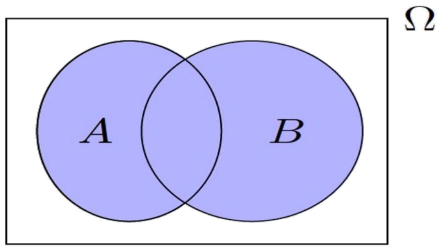
1.1 Basic Concepts

- From a set theory perspective, the relationships and operations of events become very clear:

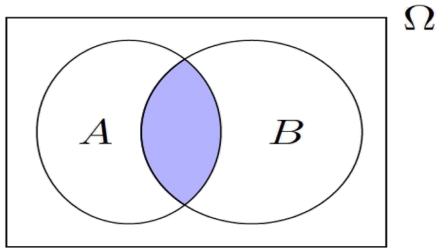
  - Inclusion $A \subset B$: All sample points in event A are also in B, i.e., B happens when A happens. (包含)



  - Sum/union $A \cup B (A+B)$: $= \{\omega | \omega \in A \text{ or } \omega \in B\}$, at least one of A and B happens. (和/并)

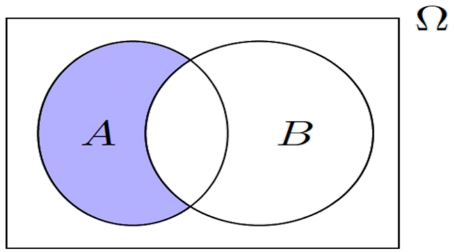


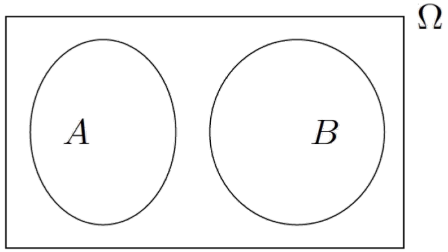
  - Product/intersection $A \cap B (AB)$: $= \{\omega | \omega \in A \text{ and } \omega \in B\}$, A and B both happen. (积/交)

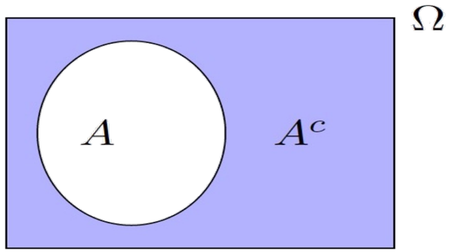


1.1 Basic Concepts

- From a set theory perspective, the relationships and operations of events become very clear：
  - Difference $A - B$ $(A\backslash B)$: $= \{\omega | \omega \in A \text{ and } \omega \notin B\}$, A happens and B does not happen. (差)
  - Mutually exclusive/disjoint $A \cap B = \emptyset$: A and B cannot happen at the same time. (互斥/互不相容)
  - Complement $A \cap B = \emptyset$ & $A \cup B = \Omega$: either A or B happens, denoted as $B = A^c$ or $\overline{A}$. (对立/互补)







1.1 Basic Concepts

- The operations of events obey certain rules similar to the rules of sets:
  - Communicative laws (交换律):

    $$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

  - Associative laws (结合律):

    $$A \cup (B \cup C) = (A \cup B) \cup C$$

    $$A \cap (B \cap C) = (A \cap B) \cap C$$

  - Distributive law (分配律):

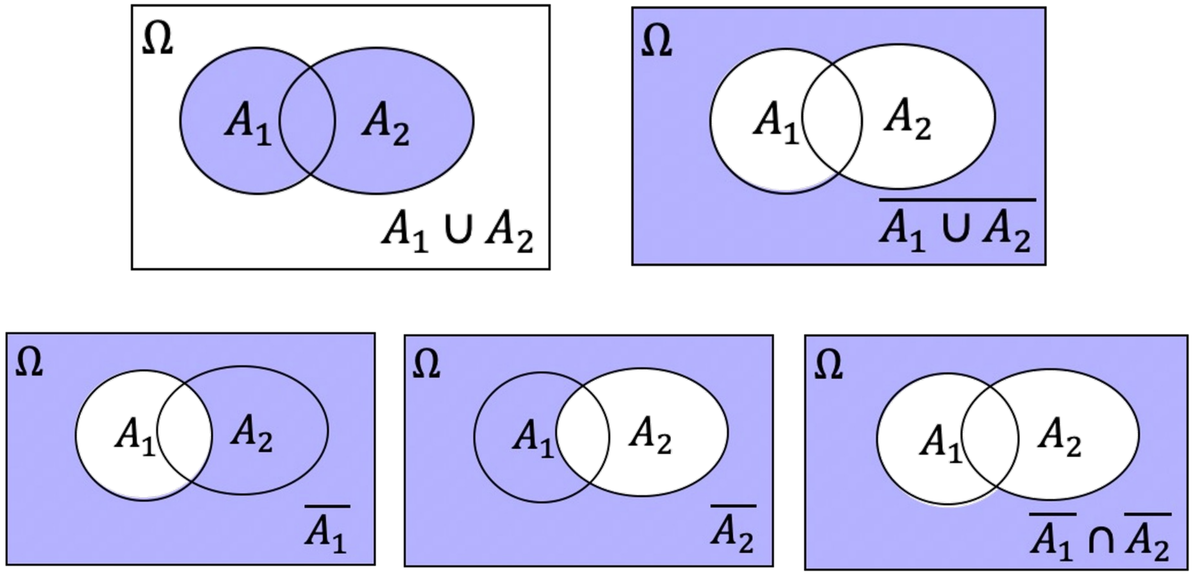
    $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

    $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

  - De Morgan's laws (德·摩根律):

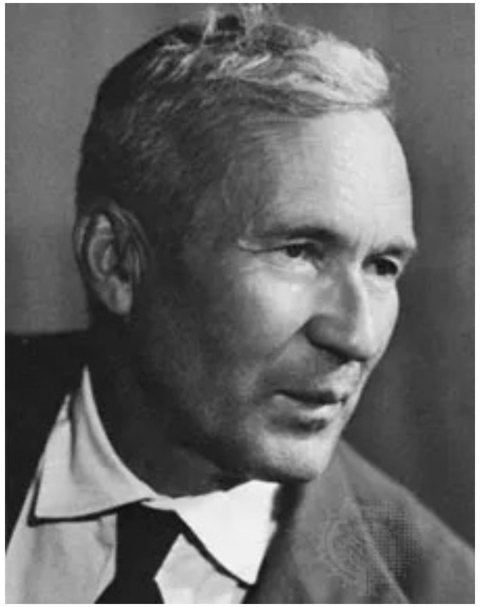
    $$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcap_{i=1}^{\infty} \overline{A_i}, \qquad \overline{\bigcap_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i}$$

A brief illustration of the De Morgan's laws



1.1 Basic Concepts

- Having defined the sample space and random event, we can now discuss the probability of events.
- It is agreed that probability is that it is a quantitative description of the likelihood of a random event occurring. But what's the formal mathematical definition of probability?



Andrey Kolmogorov
(1903-1987)

Probability

Probability measure (概率测度), or simply probability, is a real-valued function defined on subsets of the sample space Ω, satisfying the following three axioms:

- Non-negativity (非负性): for any event $A \subseteq \Omega$, we have $P(A) \geq 0$.
- Normalization (规范性): $P(\Omega) = 1$.
- Additivity (可加性): for any mutually exclusive events $A_1, A_2, \ldots$, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

1.1 Basic Concepts

- Following the three axioms, many useful rules for computing probability can be derived.

Properties of Probability

- $P(\emptyset) = 0$.

  Proof: Consider a sequence of events $A_1 = \Omega, A_2 = A_3 = \cdots = \emptyset$. Then, these events are mutually exclusive and $\Omega = A_1 \cup A_2 \cup \cdots$. Therefore, by the third axiom (i.e., additivity):

$$P(\Omega) = P(\Omega) + \sum_{i=1}^{\infty} P(\emptyset),$$

  which implies that $P(\emptyset) = 0$.

- Finite additivity (有限可加性): for any finite sequence of mutually exclusive events $A_1, A_2, \dots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

1.1 Basic Concepts

Properties of Probability

- The complement rule: $P(\overline{A}) = 1 - P(A)$.

  Proof: Let $A_1 = A$ and $A_2 = \overline{A}$, then by the finite additivity with $n = 2$,

$$1 = P(\Omega) = P(A \cup \overline{A}) = P(A) + P(\overline{A}) \Rightarrow P(\overline{A}) = 1 - P(A).$$

- The numeric bound: $0 \leq P(A) \leq 1$.
- Monotonicity (单调性): if $A \subseteq B$, then $P(A) \leq P(B)$ and $P(B - A) = P(B) - P(A)$.
- The addition law (加法定律): $P(A \cup B) = P(A) + P(B) - P(AB)$.
- The inclusion-exclusion principle (容斥原理): ($A_1, A_2, \dots, A_n$ not necessarily mutually exclusive)

$$P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i<j \leq n} P(A_i A_j) + \sum_{1 \leq i<j<k \leq n} P(A_i A_j A_k)$$

$$- \sum_{1 \leq i<j<k<l \leq n} P(A_i A_j A_k A_l) + \dots + (-1)^{n-1} P(A_1 A_2 \dots A_n)$$

+ for odd counts
- for even counts

1.1 Basic Concepts

Example 1.4

Let A, B, C be three random events, and $P(A) = P(B) = P(C) = 0.25$, $P(AB) = P(BC) = 0$, $P(AC) = 0.125$. Please find the probability that at least one of A, B, C happens.

Solution

By the monotonicity of probability, since $ABC \subseteq AB$, then $P(ABC) \leq P(AB) = 0$.

Following the non-negativity axiom, $P(ABC) = 0$.

The probability that at least one of A, B, C happens is expressed as $P(A \cup B \cup C)$. Finally, by the inclusion-exclusion principle:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

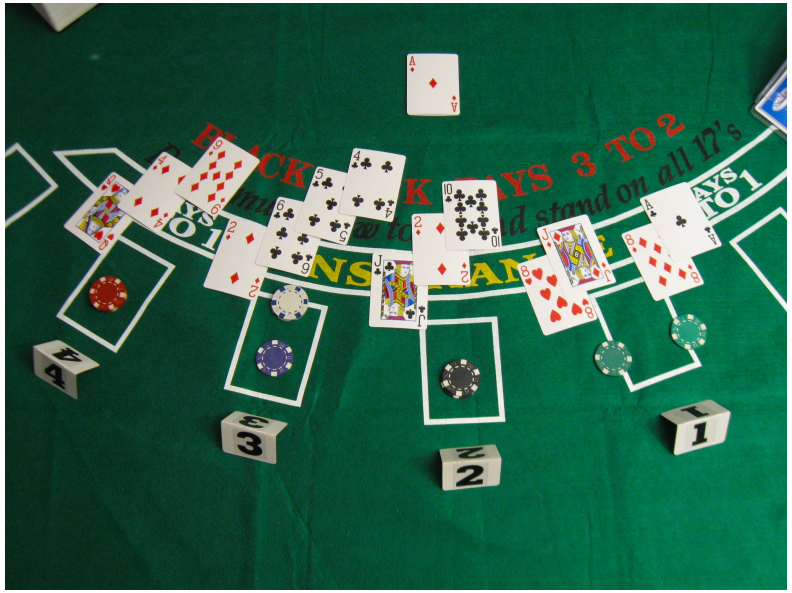
$$= 0.25 + 0.25 + 0.25 - 0 - 0 - 0.125 + 0 = 0.625.$$

Chapter 1 Basic Ideas in Probability

- 1.1 Basic Concepts
- 1.2 Computing Probabilities
- 1.3 Conditional Probability and Independence

1.2 Computing Probabilities

- Probability is a precise description of the likelihood of a random event occurring. Is it necessary to precisely compute probability?
- In everyday life, it may not be necessary, as it is difficult for us to distinguish between events with probabilities of 0.3 and 0.4.
- In professional fields, however, precise probability measurement becomes very important. E.g.,
  - In a **casino**, through precise probability measurement and design, the house only needs to have a slightly higher probability of winning than the players to make a profit from an overall perspective.
  - **Insurance companies** also operate a probability-based business. They design and price insurance products by accurately calculating the probability of claims, ensuring that they make a profit overall.



1.2 Computing Probabilities

- The probability computations in early days were mostly based on a relatively simple model known as the classical model of probability (古典概型).

Classical Model of Probability

If a random experiment satisfies:

- there are only finite number of sample points in the sample space Ω, i.e.,
$$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\},$$
- each sample point is equally likely to occur, i.e.,
$$P(\{\omega_1\}) = P(\{\omega_2\}) = \cdots = P(\{\omega_n\}) = \frac{1}{n},$$

then the probability model of the experiment is called a classical model of probability.

Probability Computation Under the Classical Model of Probability

Assume that there are k sample points in event A, then the probability of event A is

$$P(A) = \frac{\text{Number of sample points in } A}{\text{Total number of sample points in } \Omega} = \frac{k}{n}.$$

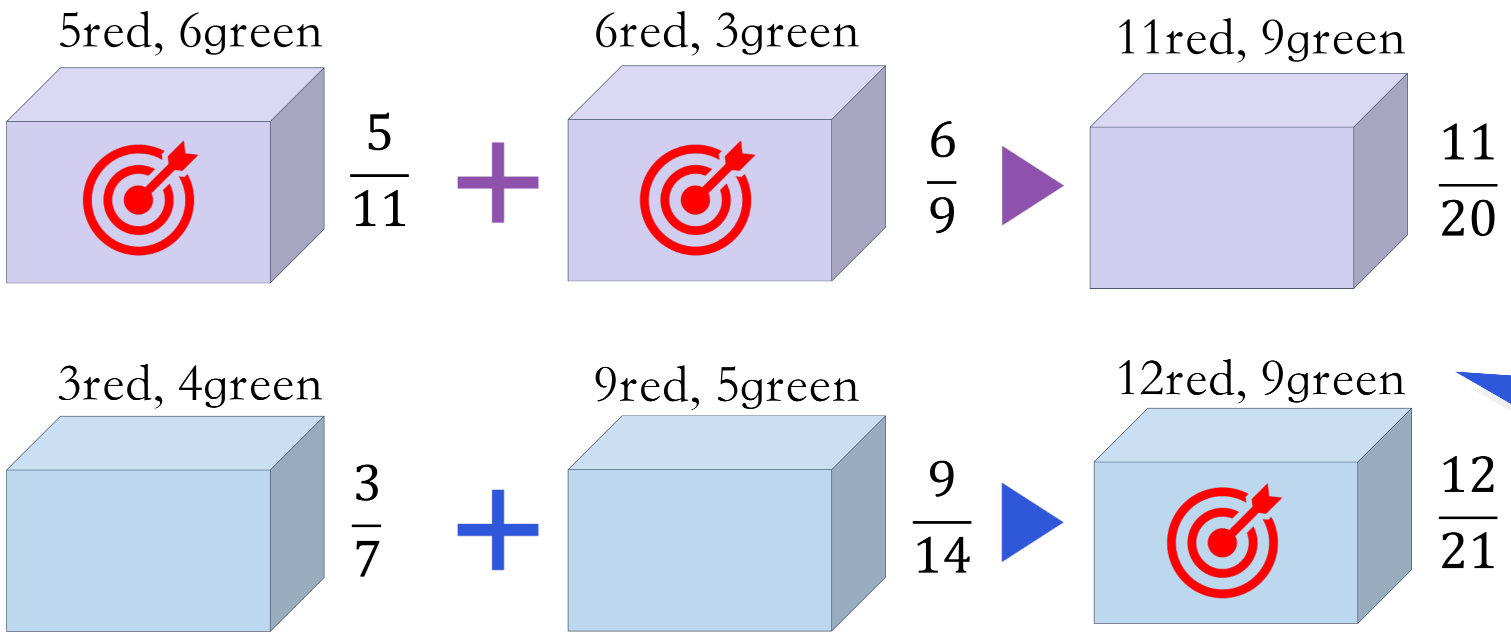
Hence, under the classical model of probability, probability computation only involve counting the number of sample points in the sample space and the event of interest.

- The probability of a random event is the proportion of that event, a subset of the sample space, within the sample space.

1.2 Computing Probabilities

Example 1.5

If two boxes each contain a certain number of red and green balls, and you are allowed to first choose a box and then randomly pick a ball from the chosen box. If a red ball is picked, you win a prize. Then which box should you choose from?



Simpson's Paradox
(辛普森悖论)

(加州大学伯克利分校在1970年代的性别歧视)

1.2 Computing Probabilities

- However, more often, the number of sample points is not easy to determine and requires more systematic counting methods.

- We introduce/review two commonly used counting methods: the addition and multiplication principles, and permutations and combinations.

Addition and Multiplication Principles

- **Addition principle (加法原理)**: If there are n types of methods to complete a task, with m_1 specific methods in the first type, m_2 specific methods in the second type, ..., and m_n specific methods in the nth type, then the total number of specific methods to complete this task is:

$$N = m_1 + m_2 + \cdots + m_n.$$

- **Multiplication principle (乘法原理)**: If there are n steps to complete a task, with m_1 possible methods for the first step， m_2 possible methods for the second step, $\cdots$, m_n methods for the nth step， then the total number of methods to complete this task is:

$$N = m_1 \times m_2 \times \cdots \times m_n.$$

1.2 Computing Probabilities

Permutation and Combination

- **Permutation (排列)**: If k elements are randomly selected **without replacement (无重复随机抽取)** from n distinct elements $(k \leq n)$ and placed in order, then the number of different permutations is

$$A_n^k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

- **Combination (组合)**: The number of combinations of k elements randomly selected **without replacement (无重复随机抽取)** from n distinct elements $(k \leq n)$, where the order does not matter, is given by

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

- **Think:** what's the permutation and combination under the case of **randomly selecting with replacement (有重复随机抽取)**?

1.2 Computing Probabilities

Example 1.6

Suppose a class of n students has been allocated $m\,(< n)$ concert tickets by the university. The teacher decides to distribute the tickets by drawing lots.

The teacher prepares a hat containing n slips of paper, with m slips marked with a "1" and the remaining slips marked with a "0".

The students take turns drawing slips from the hat, and those who draw a slip marked with a "1" will get a concert ticket.

If you are one of the students in the class and you really want to get a ticket, would you choose to draw early or late?

1.2 Computing Probabilities

Solution

- The essential question is: Suppose you are the lth to draw a slip, does the probability of you getting a ticket depend on l?
- To compute probability, we first need to figure out what's the sample space?
- Considering the ticket drawing of the class as a random experiment, then each possible outcome of the drawing process is a sample point (i.e., an n-digit number with m digits being 1 and the remaining being 0), the sample space is the set of all these sample points.
- Then, determine the number of sample points in the sample space: C_n^m.
- Let A denote the event that you get a ticket, then the number of sample points in event A is: C_{n-1}^{m-1}.
- Put these together, the probability of event A is

$$P(A) = \frac{\text{Number of sample points in } A}{\text{Total number of sample points in } \Omega} = \frac{C_{n-1}^{m-1}}{C_n^m} = \frac{m}{n}.$$

- Therefore, the probability of getting a ticket does not depend on the order in which you draw a slip!

1.2 Computing Probabilities

Example 1.7

(Birthday problem) There are n ($n < 365$) students in a class, what's the probability that at least two students have the same birthday? (leap years are not taken into account)

Solution

- Each sequence of n birthdays is a sample point, and the sample space is the set of all possible sequences of n birthdays.
- Then, the number of sample points in the sample space is: 365^n.
- Let A be the event that at least two students have the same birthday, then $\overline{A}$ is the event that every student has a different birthday.
- It is easier to determine the number of sample points in $\overline{A}$:

$$A_{365}^{n} = 365 \times 364 \times \cdots \times (365 - n + 1)$$

- Therefore, the probability of event A is computed as:

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{A_{365}^{n}}{365^n}.$$

1.2 Computing Probabilities

Solution

Results							
n	20	25	30	40	50	55	100
$P(A)$	0.41	0.57	0.71	0.89	0.97	0.99	0.9999997

- In a class of 50 students, the probability that at least two people share the same birthday is as high as 97%.
- In a class of 100 students, it is almost certain that at least two people will share the same birthday.
- Does this surprise you?

1.2 Computing Probabilities

Example 1.8

(Matching problem 配对问题)

- There are n students in a class with student IDs $1, 2, \dots, n$.
- Before the last Chinese New Year, everyone prepared a gift which was numbered with his/her student ID.
- Then all the gifts were put into a bag, and everyone randomly select a gift from the bag.
- What's the probability that at least one student get his/her own gift?

Solution

- Each permutation of $\{1, 2, \dots, n\}$ is a sample point, representing the gift numbers corresponding to students with IDs $1, 2, \dots, n$.
- So, the number of sample points in the sample space is: $A_n^n = n!$.
- Let A be the event that at least one student gets his/her own gift, then directly determining the number of sample points in A is not so straightforward.

1.2 Computing Probabilities

Solution

- The idea is to break down vague and complex problems into clear and simple ones.
- Let A_i be the event that the student with ID i get his/her own gift, then

$$P(A) = P(A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \cdots + (-1)^{n-1} P(A_1 A_2 \ldots A_n)$$

- The events A_i, $A_i A_j$, $A_1 A_2 \cdots A_n$ are very clear and their probability computation is simple:

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{C_n^1}, \quad P(A_i A_j) = \frac{(n-2)!}{n!} = \frac{1}{2!\, C_n^2}, \cdots$$

- Put these together, we have:

$$P(A) = \frac{1}{C_n^1} \times C_n^1 - \frac{1}{2!\, C_n^2} \times C_n^2 + \frac{1}{3!\, C_n^3} \times C_n^3 + \cdots + (-1)^{n-1} \frac{1}{n!}$$
$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{n-1} \frac{1}{n!} \approx 1 - e^{-1} \approx 0.632.$$

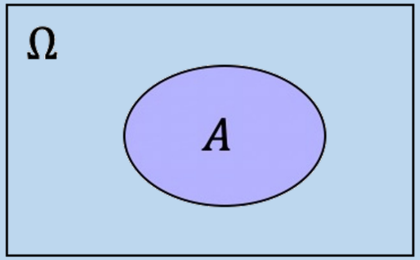
1.2 Computing Probabilities

- The classical model of probability assumes finite number of sample points and equal likelihood.
- Another model, called **the geometric model of probability (几何概型)**, is an extension of the classical model to an infinite number of sample points while maintaining equal likelihood.

Geometric Model of Probability and its Probability Computation

If a random experiment can be represented as randomly throwing a point onto a bounded region Ω, where the point is **equally likely** to land at any position within the region, then the probability model of the experiment is called a **geometric model of probability**. Let A be the event that the point lands at a subregion A of Ω, then the probability of event A is computed as

$$P(A) = \frac{\text{The length/area/volume of } A}{\text{Total length/area/volume of } \Omega}.$$



- No matter which model is used, probability is essentially the proportion of the random event, a subsect of the sample space, within the sample space.

1.2 Computing Probabilities

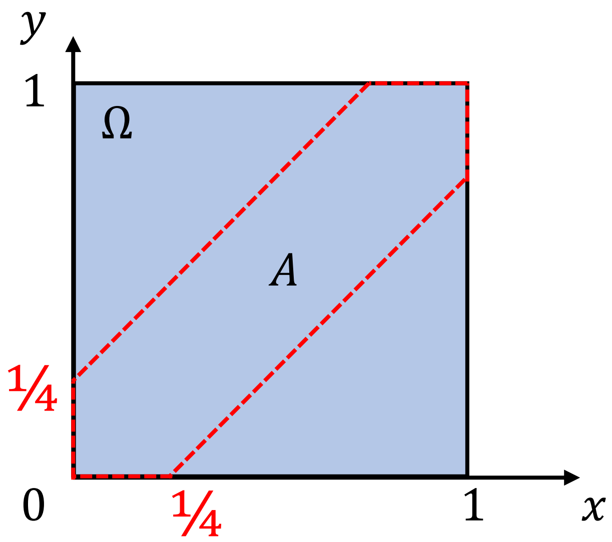
Example 1.9

- Romeo and Juliet plan to meet in an evening, and each will arrive at the garden between 7pm and 8pm, with all pairs of arrival time equally likely.
- The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived.
- What is the probability that they will meet?

Solution

- Let 7pm be the origin, x-axis and y-axis be the arrival time of Romeo and Juliet, respectively.
- Then the sample space is $\Omega = \{(x, y) \in [0,1]\times[0,1]\}$.
- For Romeo and Juliet to meet, we need $|x - y| \leq 1/4, 0 \leq x, y \leq 1$.

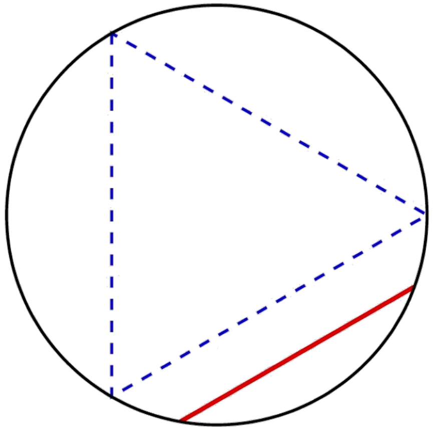
$$P(A) = \frac{\text{The area of } A}{\text{Total area of } \Omega} = 1 - \frac{3}{4}\times\frac{3}{4} = \frac{7}{16}.$$



Example 1.10

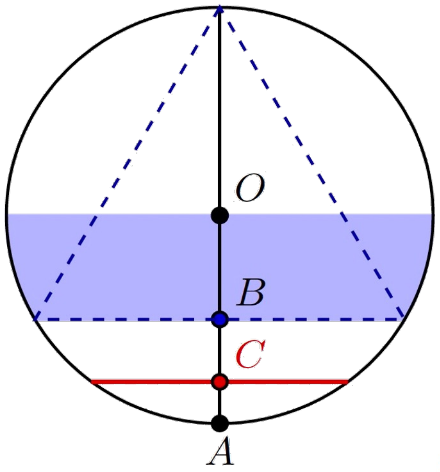
Bertrand's paradox (贝特朗悖论)

Consider a circle with radius (半径) 1. What is the probability that a randomly draw chord (弦) of the circle is longer than the side of the inscribed equilateral triangle (内切等边三角形) of the circle?



Solution 1

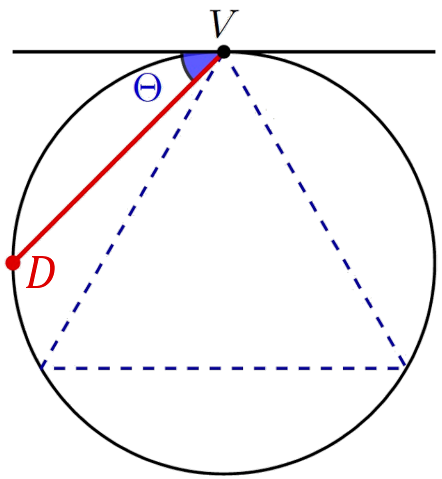
- Take a radius of the circle OA, and randomly choose a point C on the radius (all points being equally likely), then draw a chord through C orthogonal to OA.
- By elementary geometry, OA intersects the triangle at the midpoint of OA, say B.
- Then the sample space is all points on OA
- For the chord to be longer than the side of the triangle, C must fall on OB.
- Therefore, the probability is $\dfrac{\text{The length of } OB}{\text{The length of } OA} = \dfrac{1}{2}$.



1.2 Computing Probabilities

Solution 2

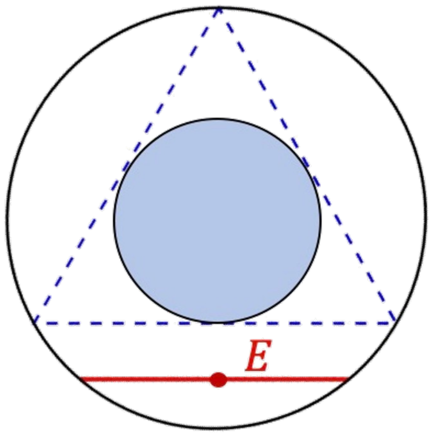
- Take a point on the circle, e.g., the vertex V, draw the tangent (切线) to the circle through V.
- Then randomly choose another point D on the circle and draw a chord by connecting V and D that forms a random angle Θ with the tangent.
- Then the sample space is $\Omega = \{\Theta \in [0, \pi]\}$, and for the chord to be longer than the side of the inscribed equilateral triangle, we need $\pi/3 < \Theta < 2\pi/3$.
- Therefore, the probability is $\dfrac{\pi/3}{\pi} = \dfrac{1}{3}$.



1.2 Computing Probabilities

Solution 3

- Randomly choose a point E within the circle and draw a chord with E as the midpoint.
- For the chord to be longer than the side of the inscribed equilateral triangle, E must fall within the inscribed circle (内切圆) of the triangle, whose radius is 1/2.
- Then the sample space is the original circle, and the event of interest is is the inscribed circle.
- Therefore, the probability is $\dfrac{\pi/4}{\pi} = \dfrac{1}{4}$.



1.2 Computing Probabilities

Discussion

- Which solution is correct?
- The three solutions are all correct!
- The reason why the same question have three different answers is that the "randomly draw chord" is not clearly defined in the problem.
- Different assumptions of equal likelihood lead to different sample spaces:
  - In Solution 1, the midpoint of the chord is assumed to be equally likely chosen on a radius, so the sample space consists of all points on the radius.
  - In Solution 2, the other endpoint of the chord is assumed to be equally likely chosen on the circle, so the sample space consists of all points on the circle.
  - In Solution 3, the midpoint of the chord is assumed to be equally likely chosen within the circle, so the sample space consists of all points within the circle.
- Therefore, when computing probability, it is crucial to clearly define the sample space first.

Chapter 1 Basic Ideas in Probability

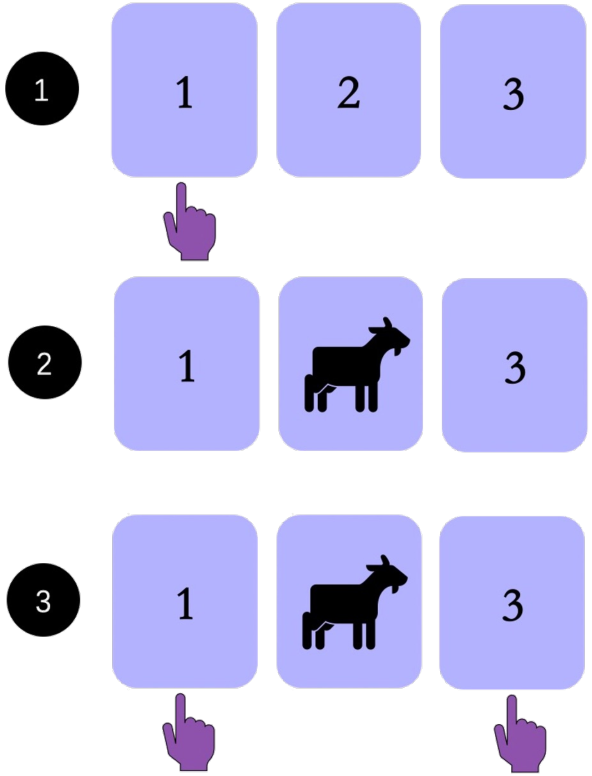
- 1.1 Basic Concepts
- 1.2 Computing Probabilities
- 1.3 Conditional Probability and Independence

1.3 Conditional Probability and Independence

- Let’s first look at a problem that once sparked widespread discussion in the United States—the **Monty Hall problem (三门问题)**.

Example 1.11

- The problem originates from the American TV show *Let's Make a Deal*, named after the show's host, Monty Hall.
- The show came into the public eye in 1975. Contestants would see three closed doors, with a car behind one of them and a goat behind each of the other two. If they successfully pick the door with the car, they will win the car.
- You are first asked to choose one of the doors, say Door 1.
- Then, the host (who knows where the car is) opens one of the remaining two doors, revealing a goat, say behind Door 2.
- Finally, the host gives you a chance to change your choice—either stick with your original choice (Door 1) or switch to Door 3.
- What’s your choice?



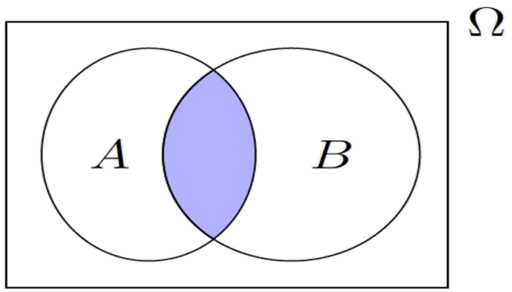
1.3 Conditional Probability and Independence

- After learning about conditional probability, you will find a logically rigorous way to solve the Monty Hall problem.
- Conditional probability refers to the idea that the probability of a random event will change given the occurrence of another event.

Conditional Probability

Let A and B be two random events and $P(B) > 0$. Then the conditional probability (条件概率) of event A given that event B occurs is defined as

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}.$$

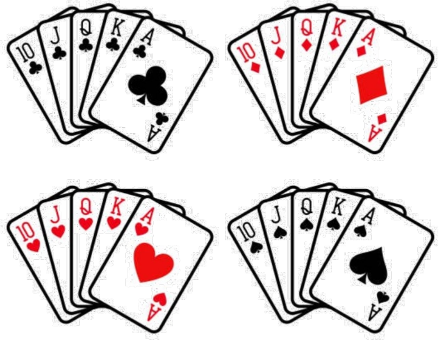


- The intuitive understanding of $P(A|B)$ is that the occurrence of event B provides new information that updates our belief on the likelihood of event A.
- The idea behind this definition is that if event B has occurred, the sample space becomes B instead of Ω.

1.3 Conditional Probability and Independence

Example 1.12

- You are playing a poker game where you are dealt 5 cards face down.
- A royal flush (皇家同花顺) is a hand of AKQJ10 all in one suit.
- 1. What is the probability that you are dealt a royal flush?
- 2. If one of the cards that you are dealt lands face up, showing the Ace of spades (黑桃A), what is the probability now?



Solution

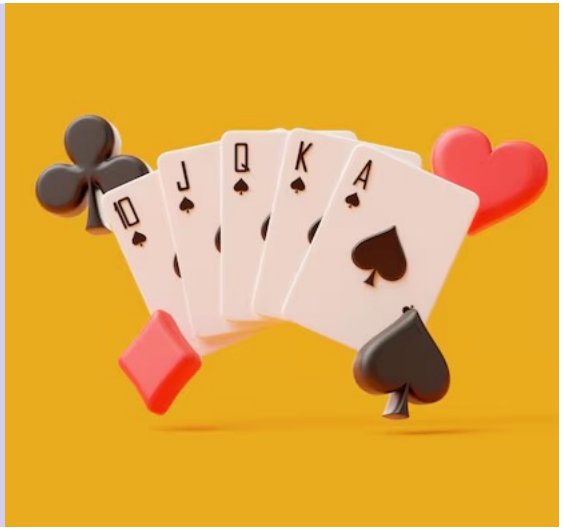
- Let A be the event of a royal flush, then the number of sample points in A is 4 (one for each suit).
- The sample space is all possible combinations of 5 cards with total number of sample points C_{52}^5.
- So, the probability of A is:

$$P(A) = \frac{4}{C_{52}^5} = \frac{1}{649{,}740}.$$

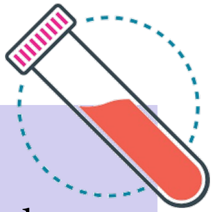
Solution

- Let B be the event that one of the 5 cards is the Ace of spades, then the number of sample points in B is C_{51}^{4}.
- $AB = A \cap B$ is the event of a royal flush of spades, so the number of sample points in AB is 1. Consequently, the probability of A given B is

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1}{C_{51}^{4}} = \frac{1}{249{,}900}.$$



1.3 Conditional Probability and Independence



Example 1.13

- You have a blood test for a rare disease that occurs by chance in 1 person in 100,000.
- If you have the disease, the test will report that you do with probability 0.95 (and that you do not with probability 0.05).
- If you do not have the disease, the test will report a false positive with probability 0.001.
- If the test says you have the disease, what is the probability that you actually have the disease?

Thinking

- First, we express the given conditions in the example using probability language.
- Let A be the event that the test reports a positive result for a randomly chosen person, B be the event that a randomly chosen person has the disease, then:

$$P(A|B) = 0.95, P\left(A|\overline{B}\right) = 0.001, P(B) = 0.00001.$$

- The probability to be computed is expressed as $P(B|A) = P(AB)/P(A)$.
- How to compute $P(AB)$ and $P(A)$ based on the given conditions?

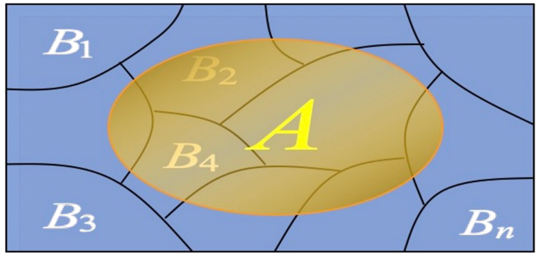
1.3 Conditional Probability and Independence

- To solve the problem, we introduce the multiplication law (乘法定律, used to compute $P(AB)$) and the law of total probability (全概率定律, used to compute $P(A)$).

Multiplication Law

Let A and B be two random events and $P(B) > 0$. Then

$$P(AB) = P(A|B)P(B).$$



Law of Total Probability

Let A and B be two random events, then (assume that $P(A|B) = 0$ if $P(B) = 0$)

$$P(A) = P(AB) + P(A\overline{B}) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B}).$$

More generally, if $B_1, B_2, \ldots, B_n$ are n mutually exclusive random events, and $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$, then we call $B_1, B_2, \ldots, B_n$ a **partition (分割/分划)** of the sample space Ω, and

$$\begin{aligned} P(A) &= P(A \cap \Omega) = P(AB_1 \cup AB_2 \cup \cdots \cup AB_n) = P(AB_1) + P(AB_2) + \cdots + P(AB_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n). \end{aligned}$$

1.3 Conditional Probability and Independence

- Like the addition law which has a general version extended to multiple events, i.e., the inclusion-exclusion principle, the multiplication law also has a general version: $(P(A_1A_2\cdots A_{n-1})>0)$

$$\begin{aligned}P(A_1A_2\cdots A_n) &= P(A_n|A_1A_2\cdots A_{n-1})P(A_1A_2\cdots A_{n-1}) = \cdots \\ &= P(A_n|A_1A_2\cdots A_{n-1})P(A_{n-1}|A_1A_2\cdots A_{n-2})\cdots P(A_2|A_1)P(A_1),\end{aligned}$$

- which is called the chain rule (链式法则) for random events.

Solution of Example 1.13

- By the multiplication law and the law of total probability:

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|\overline{B})P(\overline{B})} = \frac{0.95\times 0.00001}{0.95\times 0.00001 + 0.001\times 0.99999} \approx 0.0094.$$

- Even if the test accuracy is very high, the probability that a positive result is correct is very small.
- Does this contradict your experience or intuition? Why does this happen?
- The reason is that incidence of the disease in the population is very low.

1.3 Conditional Probability and Independence

- Actually, a famous theorem has been used in the solution of the blood test example, which is the Bayes' theorem (贝叶斯定理).

Bayes' Theorem / Bayes' Rule

Let $B_1, B_2, \ldots, B_n$ be random events and $B_1, B_2, \ldots, B_n$ is a partition of the sample space. Then for any event A such that $P(A) > 0$, we have

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}.$$

- $P(B_i|A)$ is the probability of B_i given that A occurs, also termed the **posterior probability (后验概率)** of B_i.
- $P(B_i)$ is called the **prior probability (先验概率)** or marginal probability (边缘概率) of B_i, which refers to the probability value in the absence of any other prior information.

Thomas Bayes 1702-1761
British Mathematician

- The Bayes' rule is widely used across various fields, especially in scenarios involving uncertainty and decision-making under incomplete information.

1.3 Conditional Probability and Independence

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Is the Bayes' rule widely used?

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Yes, Bayes' rule is widely used across various fields, especially in scenarios involving uncertainty and decision-making under incomplete information. Here are some key applications:

1. **Medical Diagnosis:** In healthcare, Bayes' theorem helps calculate the probability that a patient has a particular disease given a positive test result, considering both the test's reliability and the disease's prevalence.

2. **Machine Learning and AI:** Many algorithms, like the Naive Bayes classifier, rely on Bayes' rule to make predictions. It’s used in spam detection, sentiment analysis, and document classification.

3. **Finance:** Bayes' theorem is used in risk assessment, portfolio management, and market predictions, updating investment strategies as new information becomes available.

4. **Forensics and Legal Reasoning:** In court cases, Bayes' theorem can assist in evaluating evidence, determining the likelihood of guilt based on the available data.

5. **Cognitive Science:** Bayes' rule models human reasoning and decision-making, particularly how people update their beliefs in light of new evidence.

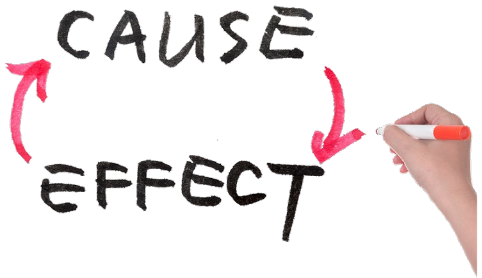
Overall, Bayes' rule is a powerful tool wherever probabilistic reasoning and inference are needed, from science and engineering to everyday decision-making.

1.3 Conditional Probability and Independence

- A real-world example of the application of Bayes' rule – the Clinical Decision Support System
  - $B_1, B_2, \dots, B_n$ represent various diseases, and A represents a certain symptom or indicator (such as the level of transaminase in the blood 血液中的转氨酶含量).
  - The prior probability $P(B_i)$ can be determined using statistical methods.
  - $P(A|B_i)$ can be determined using medical knowledge.
  - By applying the Bayes' rule, we can calculate the posterior probability $P(B_i|A)$.
  - The diseases corresponding to the larger $P(B_i|A)$ can be provided to the doctor for further clinical diagnosis.

1.3 Conditional Probability and Independence

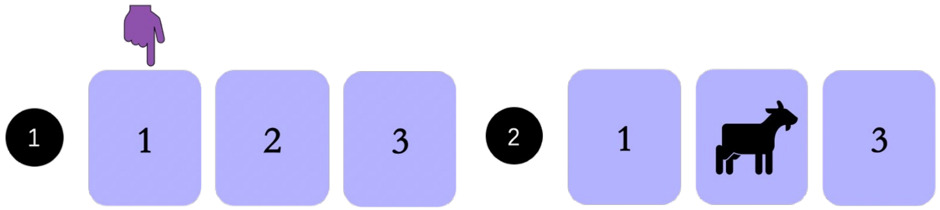
- How to understand the Bayes' rule?
  - In reality, we can view event A as the outcome (结果) and events $B_1, B_2, \ldots, B_n$ as the various causes (原因) leading to this outcome.
  - The law of total probability infers the probability of the outcome A occurring based on the different causes—it's a **reasoning process from cause to effect**.



  - However, there is another important scenario in our daily lives: we observe a certain phenomenon and then reason backward to determine the probabilities of various causes that led to it. Simply put, it is **reasoning from effect to cause**.
  - The conditional probability $P(B_i|A)$ obtained by the Bayes' rule helps us infer the likelihood that a specific cause B_i led to the observed outcome A, supporting our subsequent decision-making.
  - The posterior probability $P(B_i|A)$ is a revision of the prior probability $P(B_i)$ after acquiring new information.

1.3 Conditional Probability and Independence

- Lastly, let's solve the Monty Hall problem.



Solution of Example 1.11

- Without loss of generality, suppose that you chose Door 1 and the host opened Door 2.
- Let B_1, B_2, B_3 be event that there is a car behind Door 1, 2, 3 and A be the event that the host opens Door 2. Then:

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}, P(A|B_1) = \frac{1}{2}, P(A|B_2) = 0, P(A|B_3) = 1.$$

- The probability of interest is $P(B_1|A)$ and $P(B_3|A)$. By the law of total probability, we have

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) = \frac{1}{2}\times\frac{1}{3} + 1\times\frac{1}{3} = \frac{1}{2}.$$

- Finally, by the Bayes' rule:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{1/2 * 1/3}{1/2} = \frac{1}{3}, \quad P(B_3|A) = \frac{P(A|B_3)P(B_3)}{P(A)} = \frac{1 * 1/3}{1/2} = \frac{2}{3}.$$

- Therefore, you should switch to Door 3!

1.3 Conditional Probability and Independence

- Finally, let's discuss the independence (独立性) between events. You might have learned about some definitions of independence before, like:
  - The independence between two events means that the occurrence of one event does not affect the probability of another event.
  - If $P(AB) = P(A)P(B)$, then we claim that events A and B are independent.
- However, the first is not a mathematically rigorous definition, the second seems difficult to understand intuitively.
- With the help of conditional probability, the independence between events can be easily and clearly defined.
- "The occurrence of A does not affect the probability of B, and vice versa" can be expressed as

$$P(A|B) = P(A), P(B|A) = P(B).$$

- By the multiplication law, we therefore have

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) = P(A)P(B).$$

Example 1.14

Let A and B be two random events, what is the relationship between the following two statements?

- A and B are independent.
- A and B are mutually exclusive.

Solution

- A and B are independent $\Longrightarrow P(AB) = P(A)P(B)$.
- A and B are mutually exclusive $\Longrightarrow P(AB) = 0$.
- Therefore, if $P(A) > 0$ and $P(B) > 0$, then "A and B are independent" and "A and B are mutually exclusive" cannot happen at the same time.
- "A and B are mutually exclusive" suggests that A would never occur if B occurs, i.e., the occurrence of B provides new information on the occurrence of A, so the two events are not independent.

1.3 Conditional Probability and Independence

- Let's look at the formal definition of independence.

Independence

Let $A, B, C, A_1, A_2, \dots, A_n$ all denote random events.

- A and B are said to be independent if $P(AB) = P(A)P(B)$.
- A, B, C are said to be (mutually) independent (相互独立) if:

$$P(AB) = P(A)P(B), P(AC) = P(A)P(C), P(BC) = P(B)P(C),$$
$$P(ABC) = P(A)P(B)P(C).$$

- $A_1, A_2, \dots, A_n$ are said to be (mutually) independent (相互独立) if for every subset of the events, $A_{i_1}, A_{i_2}, \dots, A_{i_k}$ $(1 \le i_1 < \cdots < i_k \le n, k = 2, \dots, n)$, we have

$$P\left(A_{i_1} A_{i_2} \dots A_{i_k}\right) = P\left(A_{i_1}\right) P\left(A_{i_2}\right) \cdots P\left(A_{i_k}\right).$$

- The conditions in the definition above can all be derived using conditional probability.
  - You might naturally wonder why the independence of multiple events requires so many conditions. Aren't these conditions redundant?
  - For example, doesn't pairwise independence imply that all three events are mutually independent?

1.3 Conditional Probability and Independence

Example 1.15

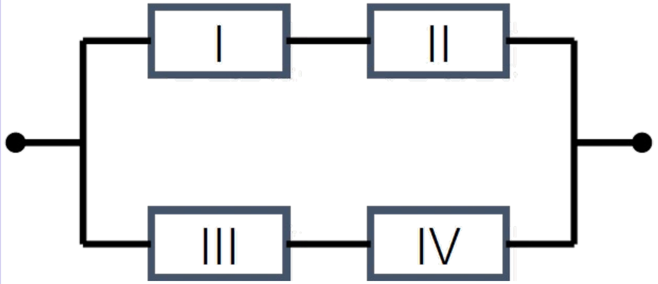
- Draw three cards from a properly shuffled standard deck, with replacement and reshuffling (i.e., draw a card, make a note, return to deck, shuffle, draw the next).
- Let A be the event that "card 1 and 2 have the same suit", B be the event that "card 2 and 3 have the same suit", C be the event that "card 1 and 3 have the same suit".
- Are the three events mutually independent?

Solution

- It is not difficult to obtain that $P(A) = P(B) = P(C) = 1/4$.
- Moreover, events AB, AC, BC, ABC all refer to the event that "card 1, 2, and 3 all have the same suit", so $P(AB) = P(AC) = P(BC) = P(ABC) = 1/16$.
- Therefore, we have $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$, $P(BC) = P(B)P(C)$, suggesting that the three events are pairwise independent.
- However, since $P(ABC) \neq P(A)P(B)P(C)$, the three events are not mutually independent.

Example 1.16

- A system has 4 components as shown in the picture.
- Assume the probability of each component working properly is p and the 4 components are independent of each other.
- What is the probability that the system works properly?



Solution

- Let A be the event that the system works properly and A_i be the event that component i works properly ($i = 1, 2, 3, 4$). Then $P(A_1) = P(A_2) = P(A_3) = P(A_4) = p$.
- From the structure of the system, it is not difficult to see $A = (A_1A_2) \cup (A_3A_4)$.
- Therefore, we have the reliability of the system to be

$$P(A) = P(A_1A_2 \cup A_3A_4) = P(A_1A_2) + P(A_3A_4) - P(A_1A_2A_3A_4)$$

By independence

$$= P(A_1)P(A_2) + P(A_3)P(A_4) - P(A_1)P(A_2)P(A_3)P(A_4) = p^2(2 - p^2).$$

1.3 Conditional Probability and Independence

- Sometimes, events may not be independent directly, but they are independent conditioned on the occurrence of a specific event. This is the concept of conditional independence (条件独立性).

Conditional Independence

Let $A_1, A_2, \ldots, A_n, B$ all denote random events.

- $A_1, A_2, \ldots, A_n$ are said to be conditionally independent conditioned on B if for every subset of the events, $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n, k = 2, \ldots, n)$, we have

$$P\left(A_{i_1}A_{i_2} \ldots A_{i_k}|B\right) = P\left(A_{i_1}|B\right)P\left(A_{i_2}|B\right) \cdots P\left(A_{i_k}|B\right).$$

- Conditional independence is a foundational concept widely used in various fields, enabling efficient computation and reasoning about complex systems.
- Some key areas where conditional independence is essential include graphical models (e.g., Bayesian networks), hidden Markov models, structural causal models in causal inference, topic models in natural language processing (e.g., Latent Dirichlet Allocation), etc.

1.3 Conditional Probability and Independence

Example 1.17

- Naïve Bayes is a simple yet powerful probabilistic machine learning algorithm used for classification tasks, particularly in text classification, such as spam filtering, sentiment analysis, etc.
- It is based on the Bayes' theorem with an assumption of conditional independence between the features that are used to perform classification.
- For example, in spam filtering, suppose that there are N words in an email, W_i denotes the event that the ith word is in the email ($i = 1, 2, \ldots, N$), and S denotes the event that the email is a spam, then

$$P(S|W_1 \cap W_2 \cap \cdots \cap W_N) = \frac{P(W_1 W_2 \cdots W_N|S)P(S)}{P(W_1 W_2 \cdots W_N|S)P(S) + P\left(W_1 W_2 \cdots W_N\middle|\bar{S}\right)P(\bar{S})}.$$

- By assuming conditional independence between the words appearing in an email given whether the email is a spam or not, we have

$$P(W_1 W_2 \cdots W_N|S) = P(W_1|S)P(W_2|S) \cdots P(W_N|S),$$

- which simplifies the computation greatly.
- However, the conditional independence assumption may not be realistic, this is why it is called "naïve".

1.3 Conditional Probability and Independence

- Finally, let's discuss the independence in reality.
  - In mathematics, it is not difficult to determine if events are independent since it is clearly defined.
  - However, in real life, it may not be easy to discern whether events are independent.
  - For example, we have heard of the famous butterfly effect, where "the flap of a butterfly's wings in South America could ultimately cause a tornado in Texas."
  - So, something seemingly unrelated could bring about a significant change.
  - Many instances of independence in real life are assumptions we make to greatly simplify problems, i.e., independence is often just a mathematical model we use to describe random events.

Example 1.18

Please compare the Taobao shopping system and the 12306 ticketing system from the perspective of system development difficulty.

End of Chapter 1