Quiz 2: (50p max; 60p total = 10p each + 10p free)
Please write your answers in English and submit to Blackboard

1. Prove by induction that for $n \in \mathbb{Z}^+$:

   $$1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + n \cdot 2^{n-1} = (n-1) \cdot 2^n + 1$$

2. Solve the recurrence relation $T(n) = 4T(n/2) + n^2$ by iterating it, where n is a power of 2 and $T(1) = 1$.

3. Use a combinatorial proof to show the identity holds for $n \in \mathbb{Z}^+$:

   $$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4. How many onto functions from a set with 10 elements to a set with 4 elements? Briefly explain why this can be solved by the inclusion-exclusion principle. (Please write your answer with binomial coefficients. No need to calculate the exact number.)

5. Solve the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ $(n \geq 2)$ with $a_0 = 1$, $a_1 = 2$.

Solutions

- Q1. Proof by (mathematical) induction:
  - Let $P(n)$ be the predicate that the equation in the problem is true.
  - **Basis step:** $P(1)$ is true, because $1 \cdot 2^0 = 1 = (1-1) \cdot 2^1 + 1$.
  - **Inductive step:** From the inductive hypothesis, i.e., $P(k)$ is true for an arbitrary positive integer k, we need to show that $P(k+1)$ is true, i.e., $1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + (k+1) \cdot 2^{k+1-1} = (k+1-1) \cdot 2^{k+1} + 1$.

$$
\begin{aligned}
& 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + k \cdot 2^{k-1} + (k+1) \cdot 2^{k+1-1} \\
&= (k-1) \cdot 2^k + 1 + (k+1) \cdot 2^{k+1-1} = (k-1+k+1) \cdot 2^k + 1 \\
&= k \cdot 2^{k+1} + 1 = (k+1-1) \cdot 2^{k+1} + 1
\end{aligned}
$$

  - By mathematical induction, $P(n)$ is true for all positive integers n.

- Q2. Iterating the recurrence yields $T(n) = 4^{\log_2 n} T(1) + n^2 \log_2 n$.
  Note that $4^{\log_2 n} = n^2$ and $T(1) = 1$, we have $T(n) = n^2 (\log_2 n + 1)$.
  ** The students need to write out the iterating procedures.*

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Solutions

- Q3. To perform a combinatorial proof, we need to construct a combinatorial problem and count the result in two ways.
  - Let us choose *2* team leaders out of *2n* team members.
  - It is easy to see there are $\binom{2n}{2}$ ways to find these two leaders.
  - Alternatively, one can split them into two groups of *n* people each, then choose these two leaders. We can choose two leaders from the same group, which yields $2\binom{n}{2}$ ways as there are 2 groups, or choosing one leader from each group, which yields n^2 ways.
  - Together, the proof is concluded.
- Q4. Please refer to page 39 of slides 07. The answer is

$$4^{10} - \binom{4}{1} \cdot 3^{10} + \binom{4}{2} \cdot 2^{10} - \binom{4}{3} \cdot 1^{10}$$

* *explanation is required*

Solutions

- Q5. Solution:
  - The characteristic equation (CE) is

    $$r^2 - 6r + 9 = 0$$

  - The only root is 3. So, assume that

    $$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$$

  - By the two initial conditions, we have

    $$a_0 = \alpha_1 = 1$$

    $$a_1 = 3\alpha_1 + 3\alpha_2 = 2$$

  - We get $\alpha_1 = 1$ and $\alpha_2 = -1/3$. Therefore

    $$a_n = 3^n - 1/3 \cdot n \cdot 3^n = (1 - n/3)\, 3^n$$