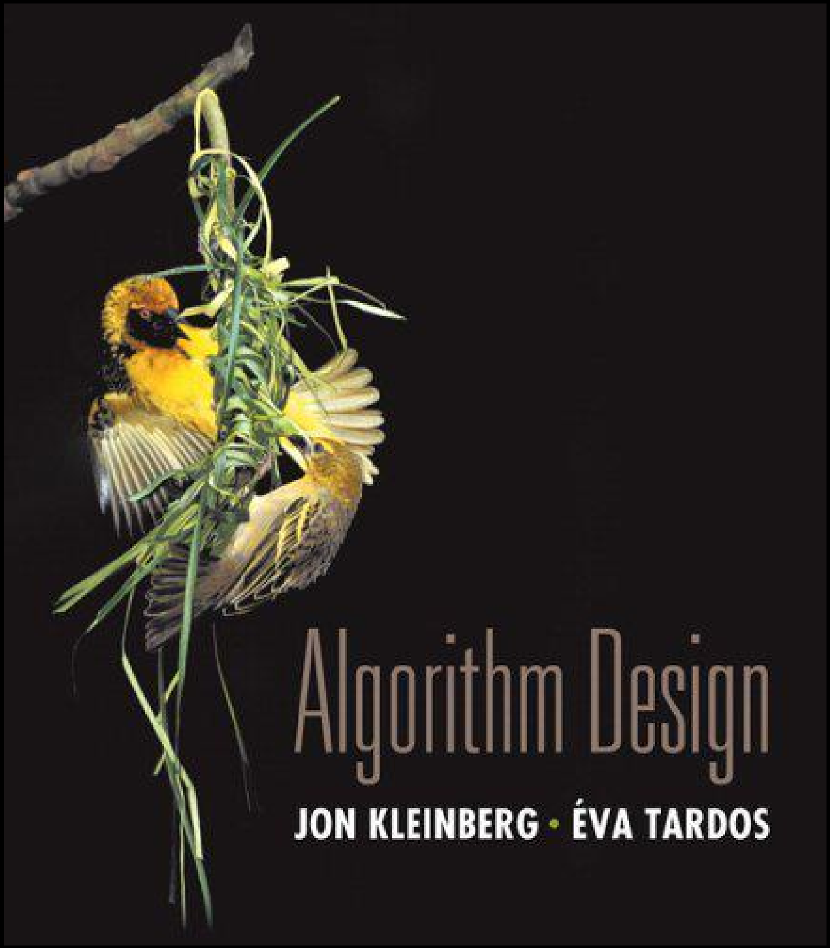


How to Multiply

integers, matrices, and polynomials

Slides by Kevin Wayne.

Complex Multiplication

Complex multiplication. $(a + bi)(c + di) = x + yi.$

Grade-school. $x = ac - bd, \ y = bc + ad.$

4 multiplications, 2 additions

Q. Is it possible to do with fewer multiplications?

Complex Multiplication

Complex multiplication. $(a + bi)\,(c + di) = x + yi.$

Grade-school. $x = ac - bd,\;\; y = bc + ad.$

4 multiplications, 2 additions

Q. Is it possible to do with fewer multiplications?

A. Yes. [Gauss] $x = ac - bd,\;\; y = (a + b)\,(c + d) - ac - bd.$

3 multiplications, 5 additions

Remark. Improvement if no hardware multiply.

Divide into more than 2 subproblems

What happens if the divide-and-conquer algorithms that create recursive calls on q sub-problems of size $n/2$ each with $q>2$?

If $T(n)$ obeys the following recurrence relation

$$T(n) \le qT(n/2) + cn$$

when $n>2$ *and* $T(2) \le c$.

$T(\cdot)$ *satisfying the above with* $q > 2$ *is bounded by* $O(n^{\log_2 q})$.

When $q=3$, $O(n^{\log_2 q}) = O(n^{1.585})$

When $q=4$, $O(n^{\log_2 q}) = O(n^2)$

For details, please read the Section 5.2 of the Textbook

5.5 Integer Multiplication

Integer Addition

Addition. Given two n-bit integers a and b, compute $a + b$.
Grade-school. $\Theta(n)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Remark. Grade-school addition algorithm is optimal.

Integer Multiplication

Multiplication. Given two n-bit integers a and b, compute $a \times b$.
Grade-school. $\Theta(n^2)$ bit operations.

```
                  1 1 0 1 0 1 0 1
              × 0 1 1 1 1 1 0 1
-------------------------------
                  1 1 0 1 0 1 0 1
                0 0 0 0 0 0 0 0 0
              1 1 0 1 0 1 0 1 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    0 0 0 0 0 0 0 0 0
-------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1
```

Q. Is grade-school multiplication algorithm optimal?

Divide-and-Conquer Multiplication: Warmup

To multiply two n-bit integers a and b:

- Multiply four $\frac{1}{2}n$-bit integers, recursively.
- Add and shift to obtain result.

$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= \left(2^{n/2} \cdot a_1 + a_0\right)\left(2^{n/2} \cdot b_1 + b_0\right) = 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0
\end{aligned}
$$

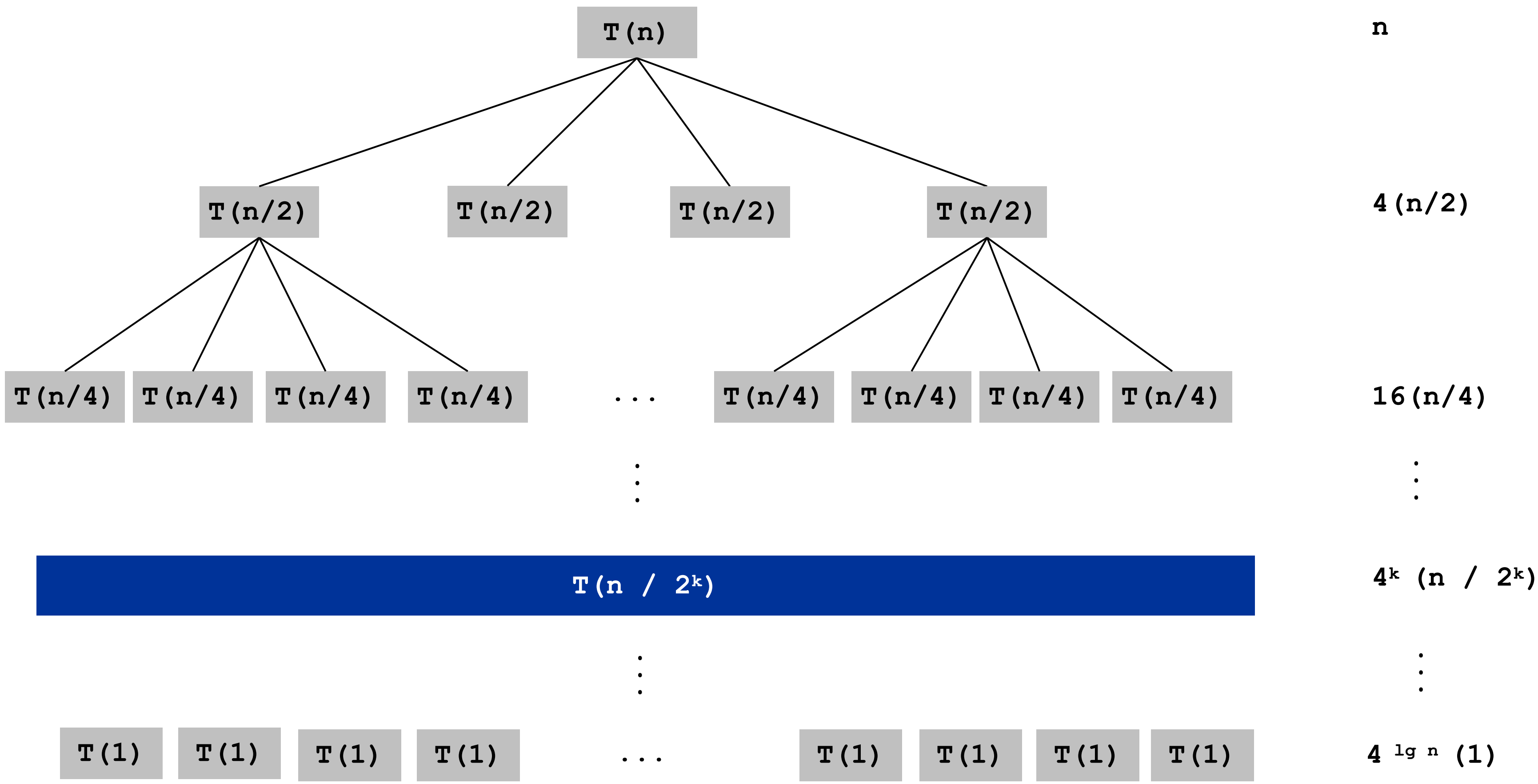
Ex. $\quad a = \underbrace{1000}_{a_1}\underbrace{1101}_{a_0} \qquad b = \underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$

$$
T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \quad \Rightarrow \quad T(n) = \Theta(n^2)
$$

Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\lg n} n\, 2^k = n\left(\frac{2^{1+\lg n} - 1}{2 - 1}\right) = 2n^2 - n$$



Karatsuba Multiplication

To multiply two n-bit integers a and b:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= 2^{n} \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^{n} \cdot \underset{1}{a_1 b_1} + 2^{n/2} \cdot \left(\underset{2}{(a_1 + a_0)(b_1 + b_0)} - \underset{1}{a_1 b_1} - \underset{3}{a_0 b_0} \right) + \underset{3}{a_0 b_0}
\end{aligned}
$$

Karatsuba Multiplication

To multiply two n-bit integers a and b:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
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ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^n \cdot \underset{1}{a_1 b_1} + 2^{n/2} \cdot \left(\underset{2}{(a_1 + a_0)(b_1 + b_0)} - \underset{1}{a_1 b_1} - \underset{3}{a_0 b_0}\right) + \underset{3}{a_0 b_0}
\end{aligned}
$$

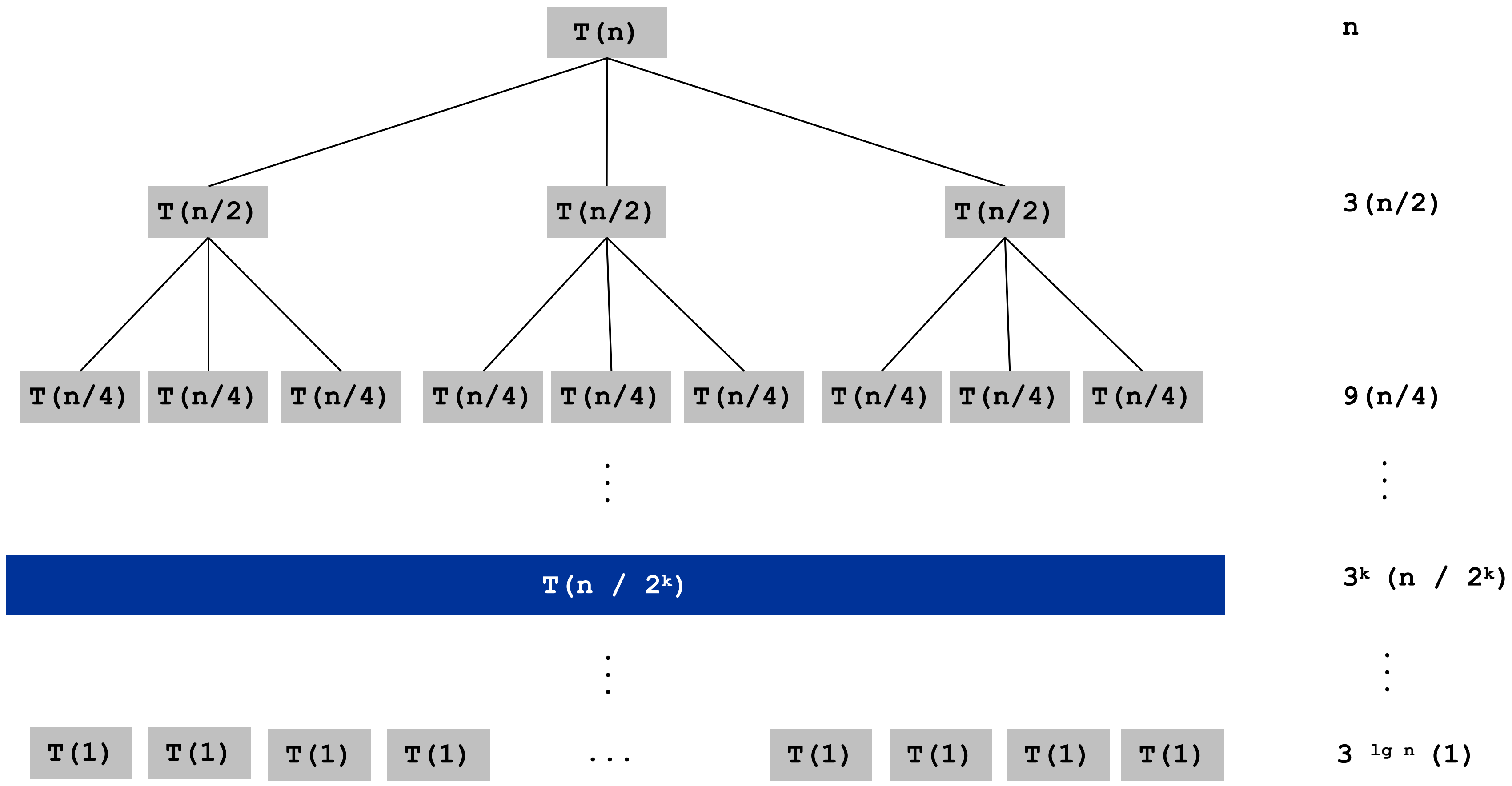
Theorem. [Karatsuba-Ofman 1962] Can multiply two n-bit integers in $O(n^{1.585})$ bit operations.

$$
T(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\lg 3}) = O(n^{1.585})
$$

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\lg n} n \left(\tfrac{3}{2}\right)^k = n \left(\frac{\left(\tfrac{3}{2}\right)^{1+\lg n} - 1}{\tfrac{3}{2} - 1} \right) = 3n^{\lg 3} - 2n$$

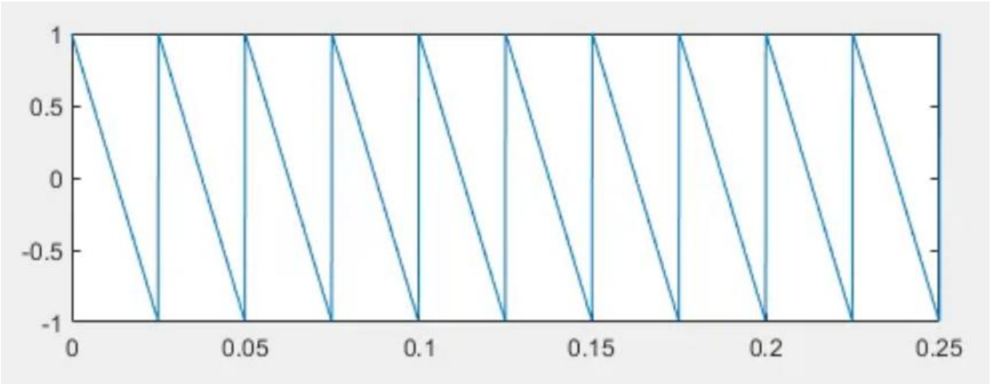


5.6 Convolution and FFT

Fourier Analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any periodic function can be expressed as the sum of a series of sinusoids.

sufficiently smooth



Sawtooth: $$y(t) = \frac{2}{\pi} \sum_{k=1}^{N} \frac{\sin kt}{k}$$

Fourier Analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any periodic function can be expressed as the sum of a series of sinusoids.

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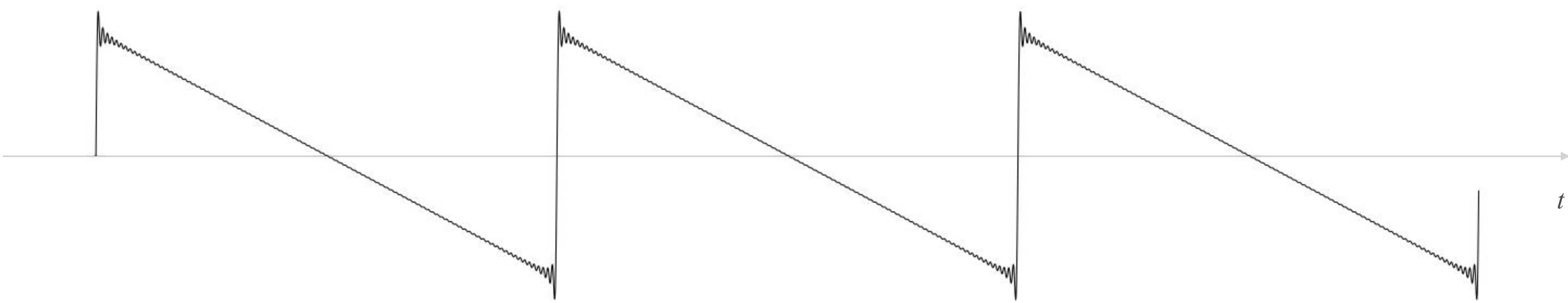


Sawtooth: $y(t) = \dfrac{2}{\pi} \sum_{k=1}^{N} \dfrac{\sin kt}{k}$ $N = 100$

Euler's Identity

Sinusoids. Sum of sine and cosines.

$$e^{jx} = \cos x + j \sin x$$
$$e^{-jx} = \cos x - j \sin x$$

- $e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$
- $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

Euler's identity

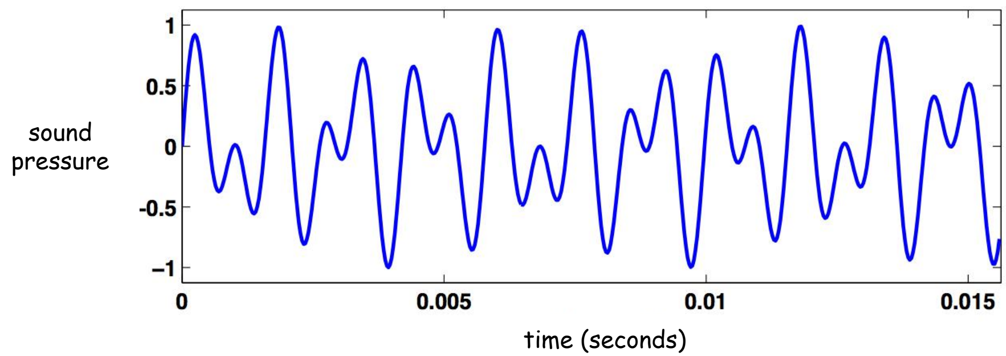
Sinusoids. Sum of complex exponentials.

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$
$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

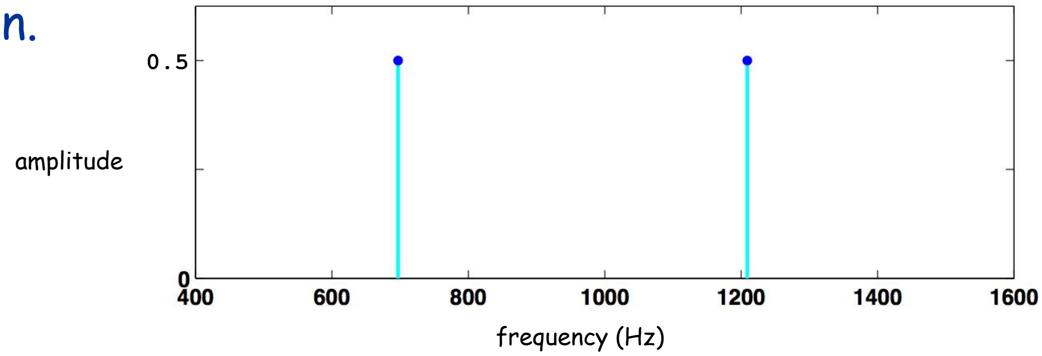
Time Domain vs. Frequency Domain

Signal. [touch tone button 1] $y(t) = \frac{1}{2}\sin(2\pi \cdot 697\ t) + \frac{1}{2}\sin(2\pi \cdot 1209\ t)$

Time domain.



Frequency domain.



Reference: Cleve Moler, Numerical Computing with MATLAB

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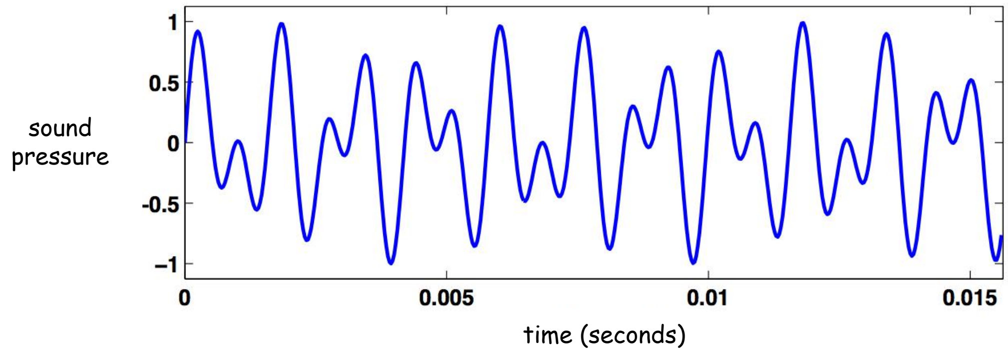
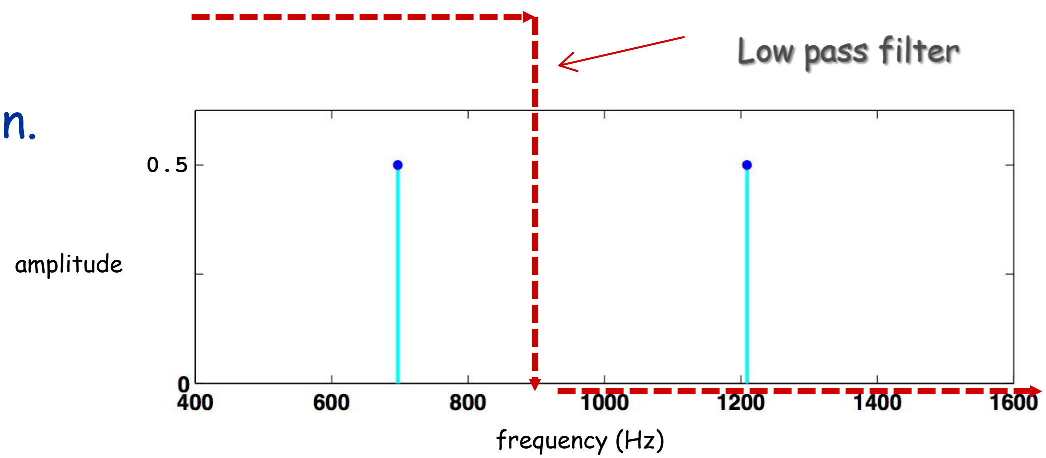
Time domain.



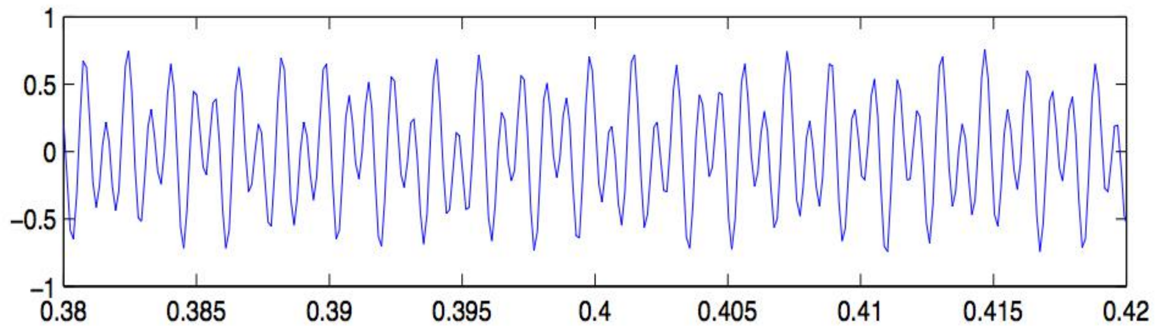
Frequency domain.



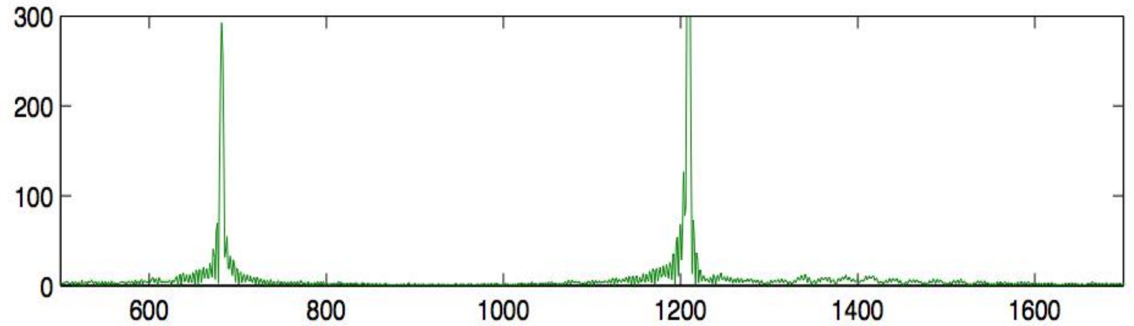
Reference: Cleve Moler, Numerical Computing with MATLAB

Time Domain vs. Frequency Domain

Signal. [recording, 8192 samples per second]



Magnitude of discrete Fourier transform.



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Time Domain vs. Frequency Domain

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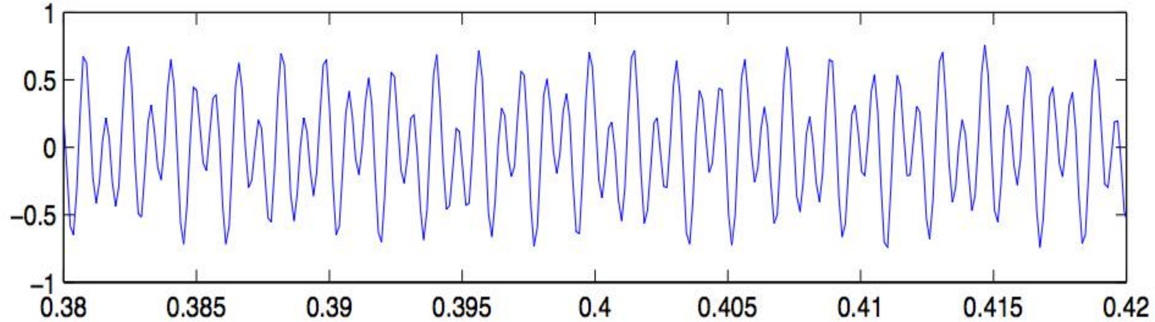
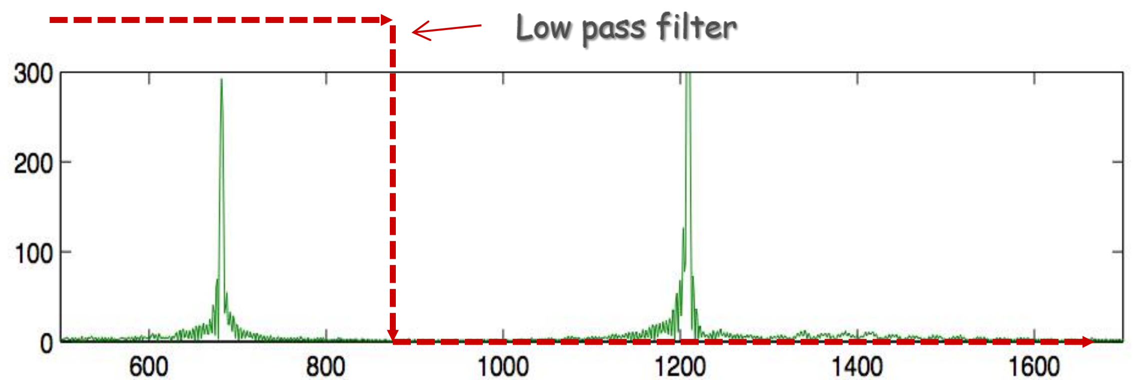


Magnitude of discrete Fourier transform.



Reference: Cleve Moler, Numerical Computing with MATLAB

Fast Fourier Transform

FFT. Fast way to convert between time-domain and frequency-domain.

Alternate viewpoint. Fast way to multiply and evaluate polynomials.

we take this approach

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{m-1}x^{m-1}$$
$$B(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1}$$
$$C(x) = A(x)B(x) = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k + \cdots + c_{n+m-2}x^{n+m-2}$$
$$c_k = \sum_{(i,j):i+j=k} a_ib_j$$

If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. -Numerical Recipes

Fast Fourier Transform: Applications

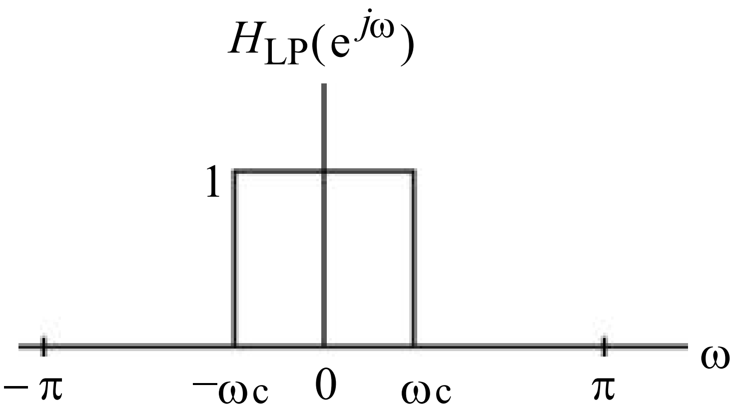
Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry…
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Shor's quantum factoring algorithm.
- …

> The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. -Charles van Loan

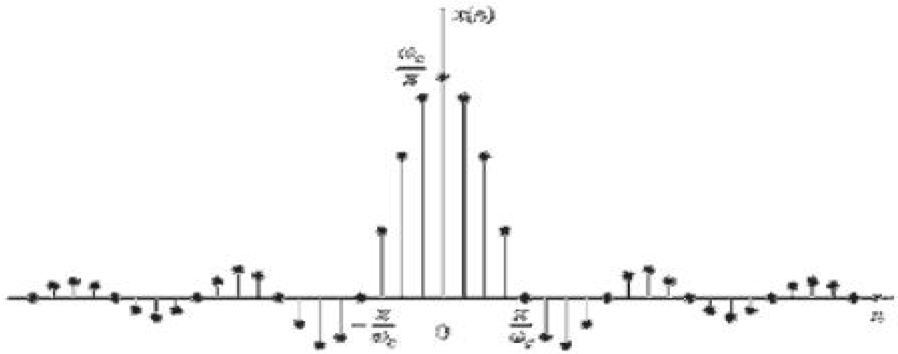
Digital filtering

Ideal low-pass filter



impulse response

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \leq n \leq \infty$$



Signal to be processed

x[n] → **Digital Filtering** h[n] (impulse response) → Processed signals

y[n]

Y(w) = X(w) H(w) →

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

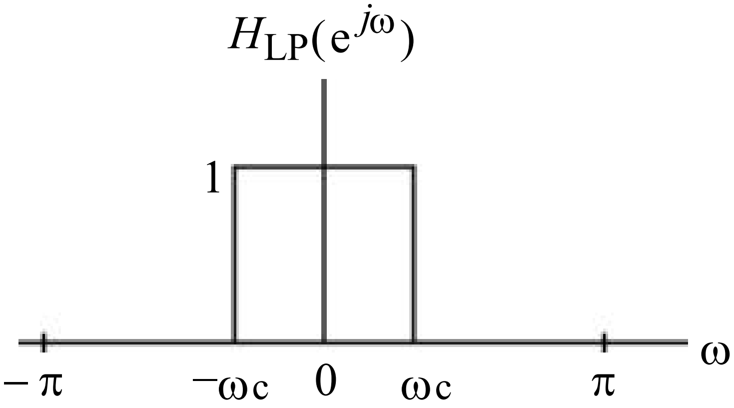
Convolution

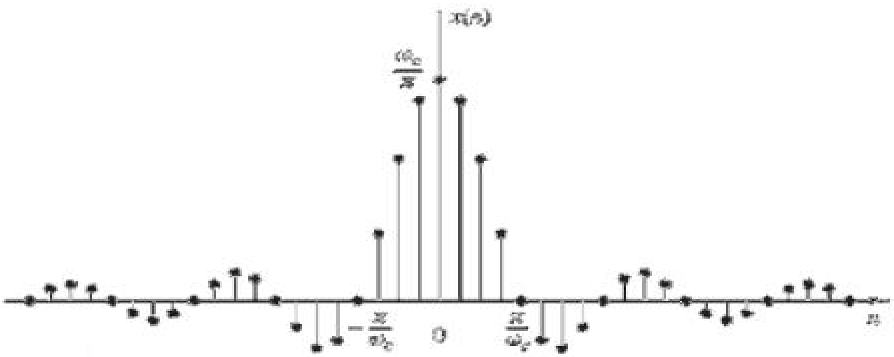
Digital filtering

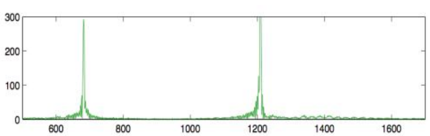
Ideal low-pass filter

impulse response

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$

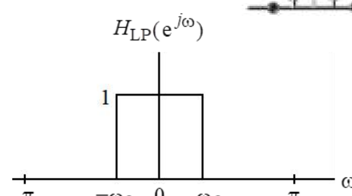






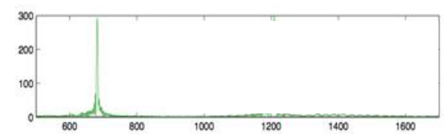
Signal to be processed

x[n]



Digital Filtering

h[n] (impulse response)



Processed signals

y[n]

Y(w) = X(w) H(w) $\longrightarrow$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

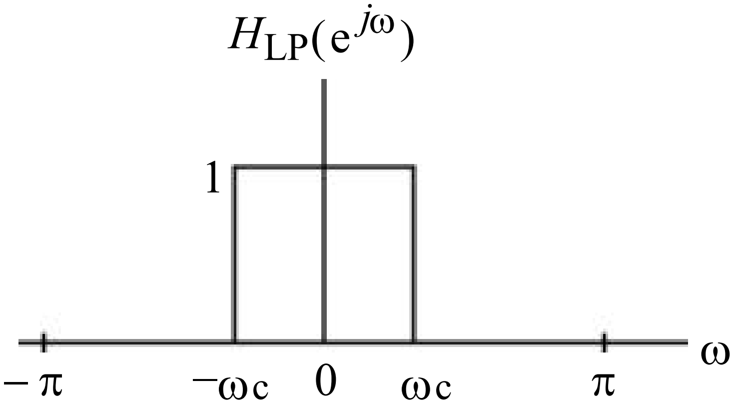
Convolution

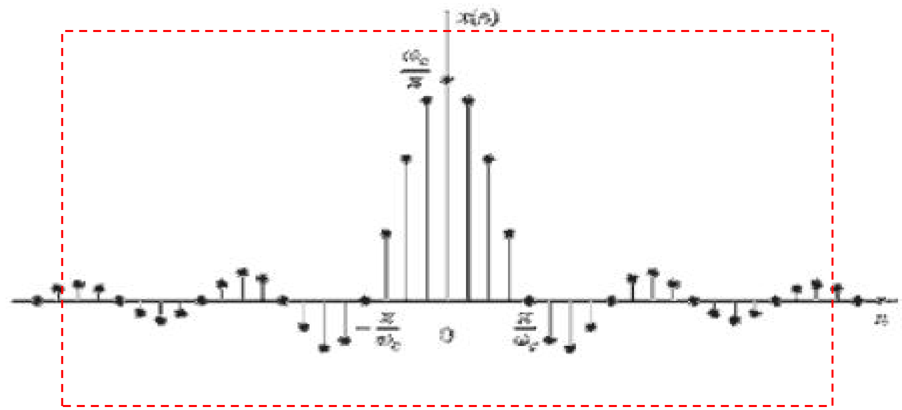
Digital filtering

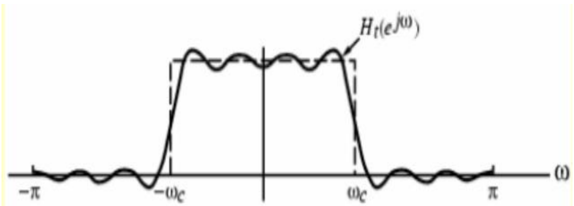
Ideal low-pass filter

impulse response

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$







Signal to be processed

x[n]

Digital Filtering

h[n] (impulse response)

Processed signals

y[n]

Truncate to limited number

Y(w) = X(w) H(w)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Convolution

Fast Fourier Transform: Brief History

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

Importance not fully realized until advent of digital computers.

Fourier Series (FS)

- Fourier's original work: A periodic function can be represented as a finite, weighted sum of sinusoids that are integer multiples of the fundamental frequency Ω_0 of the signal. These frequencies are said to be harmonically related, or simply harmonics.

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Continuous Time Fourier Transform (CTFT)

- Extension of Fourier series to non-periodic functions: Any continuous aperiodic function can be represented as an infinite sum (integral) of sinusoids. The sinusoids are no longer integer multiples of a specific frequency.

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Discrete Time Fourier Transform (DTFT)

- Extension of FT to discrete sequences. Any discrete function can also be represented as an infinite sum (integral) of sinusoids. While time domain is discretized, frequency domain is still continuous.

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- Because DTFT is defined as an infinite sum, the frequency representation is not discrete. An extension to DTFT is DFT, where the frequency variable is also discretized.

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Fast Fourier Transform (FFT)

- Mathematically identical to DFT, however a significantly more efficient implementation. FFT is what signal processing made possible today!

Fourier Series (FS)

Continuous vs Discrete

- Fourier's original work: A periodic function can be represented as a finite, weighted sum of sinusoids that are integer multiples of the fundamental frequency Ω_0 of the signal. These frequencies are said to be harmonically related, or simply harmonics.

Continuous Time Fourier Transform (CTFT)

Continuous vs Continuous

- Extension of Fourier series to non-periodic functions: Any continuous aperiodic function can be represented as an infinite sum (integral) of sinusoids. The sinusoids are no longer integer multiples of a specific frequency.

Discrete Time Fourier Transform (DTFT)

Discrete vs Continuous

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Discrete vs Discrete

- Because DTFT is defined as an infinite sum, the frequency representation is not discrete. An extension to DTFT is DFT, where the frequency variable is also discretized.

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Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_{n-1} + b_{n-1}) x^{n-1}$$

Evaluate. $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

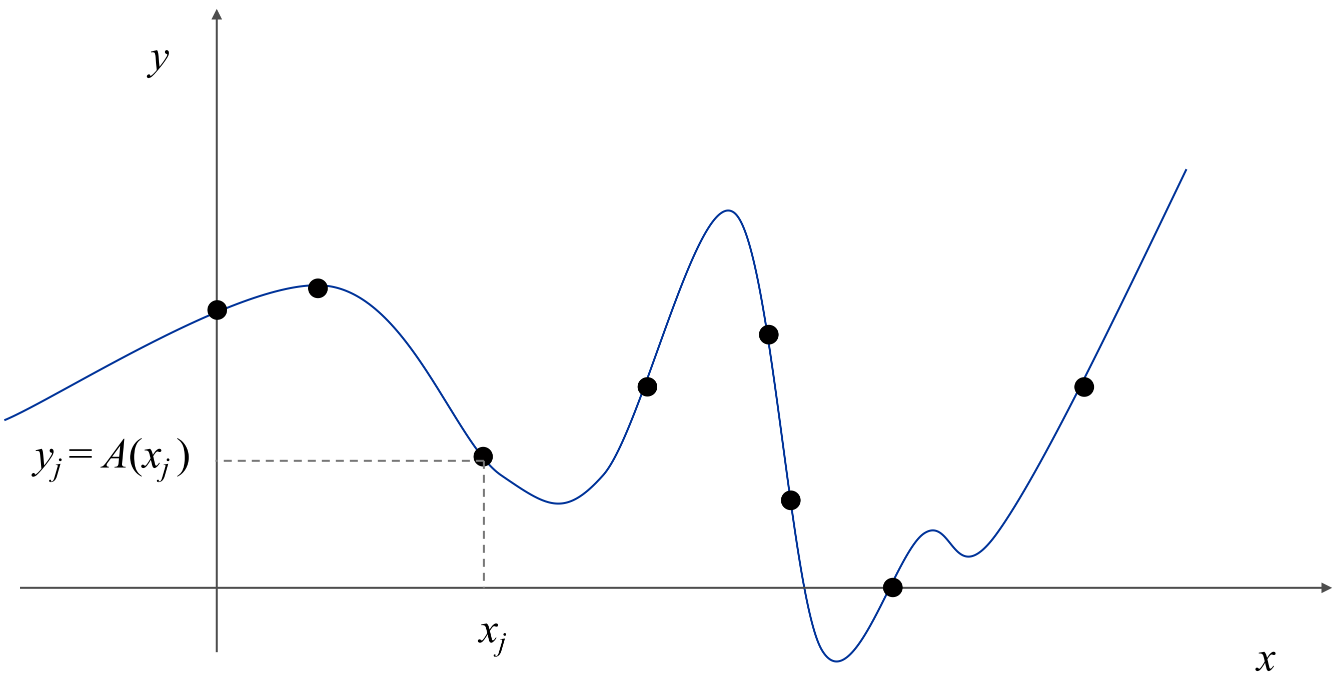
Multiply (convolve). $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \;\; \text{where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

Fundamental theorem of algebra. [Gauss, PhD thesis] A degree n polynomial with complex coefficients has exactly n complex roots.

Corollary. A degree n-1 polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): \ (x_0, y_0), \ \ldots, (x_{n\text{-}1}, y_{n-1})$$
$$B(x): \ (x_0, \ z_0), \ \ldots, (x_{n\text{-}1}, z_{n-1})$$

Add. $O(n)$ arithmetic operations.

$$A(x)+B(x): \ \ (x_0, y_0+z_0), \ldots, (x_{n\text{-}1}, y_{n-1}+z_{n-1})$$

Multiply (convolve). $O(n)$, but need $2n$-1 points.

$$A(x) \times B(x): \ \ (x_0, y_0 \times z_0), \ldots, (x_{2n\text{-}1}, y_{2n-1} \times z_{2n-1})$$

Evaluate. $O(n^2)$ using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Goal. Efficient conversion between two representations $\Rightarrow$ all ops fast.

$$a_0, a_1, \ldots, a_{n-1} \rightleftarrows (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Converting Between Two Representations: Brute Force

Coefficient $\Rightarrow$ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Running time. $O(n^2)$ for matrix-vector multiply (or n Horner's).

Converting Between Two Representations: Brute Force

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix is invertible iff x_i distinct

Running time. $O(n^3)$ for Gaussian elimination.

or $O(n^{2.376})$ via fast matrix multiplication

Decimation in frequency. Break up polynomial into low and high powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{low}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$
- $A_{high}(x) = a_4 + a_5 x + a_6 x^2 + a_7 x^3.$
- $A(x) = A_{low}(x) + x^4 A_{high}(x).$

Decimation in time. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2).$

Coefficient to Point-Value Representation: Intuition

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$.
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2)$.

Intuition. Choose two points to be ± 1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1)$.
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1)$.

Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient $\Rightarrow$ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd powers.

- $A(x) \quad = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose four complex points to be ± 1, $\pm i$.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1).$
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1).$

Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 points.

Discrete Fourier Transform

Coefficient $\Rightarrow$ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea. Choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

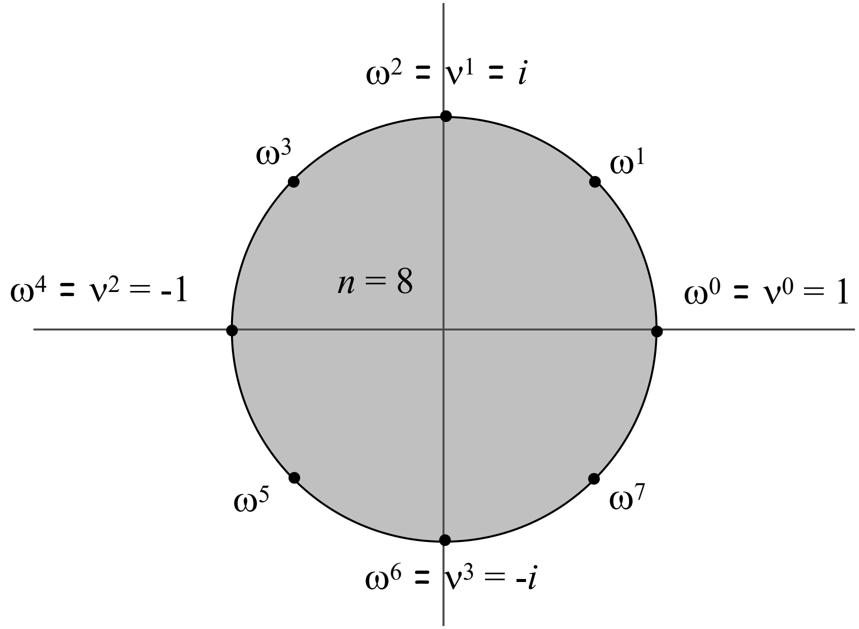
$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{\text{DFT}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}}_{\text{Fourier matrix } F_n} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Roots of Unity

Def. An n^{th} **root of unity** is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.
Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = \omega^2 = e^{4\pi i / n}$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + ... + a_{n-1}\, x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break up polynomial into even and odd powers.

- $A_{even}(x) \;=\; a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2}\, x^{n/2\,-\,1}$.
- $A_{odd}(x) \;=\; a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1}\, x^{n/2\,-\,1}$.
- $A(x) \;= A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

2T(n/2)

Combine.

$\nu^k = (\omega^k)^2$

- $A(\omega^k) \;= A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+\frac{1}{2}n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\nu^k = (\omega^{k+\frac{1}{2}n})^2$ $\quad$ $\omega^{k+\frac{1}{2}n} = -\omega^k$

O(n)

FFT Algorithm

```
fft(n, a₀,a₁,…,aₙ₋₁) {
   if (n == 1) return a₀

   (e₀,e₁,…,eₙ/₂₋₁) ← FFT(n/2, a₀,a₂,a₄,…,aₙ₋₂)
   (d₀,d₁,…,dₙ/₂₋₁) ← FFT(n/2, a₁,a₃,a₅,…,aₙ₋₁)

   for k = 0 to n/2 - 1 {
      ωᵏ ← e^(2πik/n)
      yₖ     ← eₖ + ωᵏ dₖ
      yₖ₊ₙ/₂ ← eₖ - ωᵏ dₖ
   }

   return (y₀,y₁,…,yₙ₋₁)
}
```

FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in $O(n \log n)$ steps.

assumes n is a power of 2

Running time.

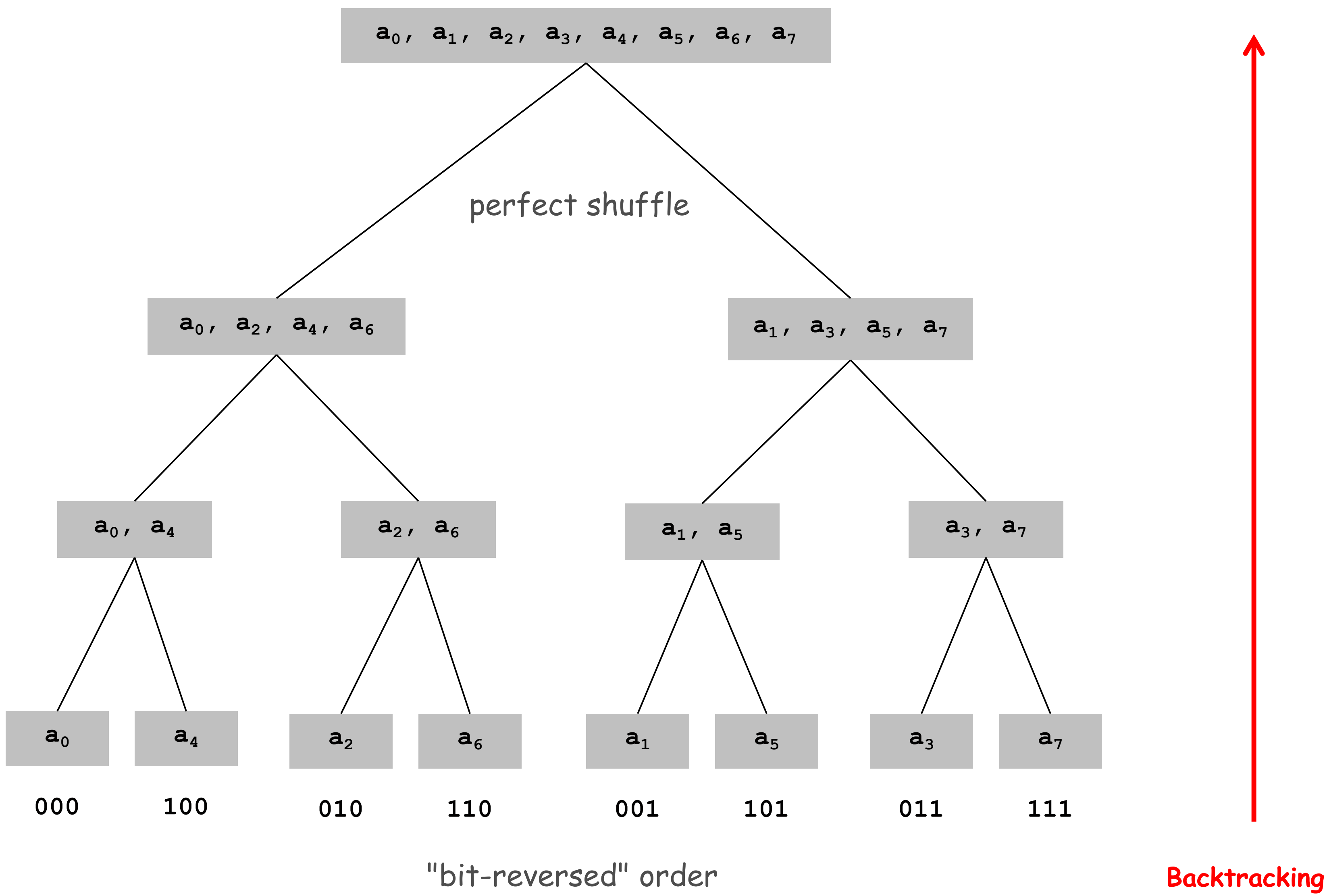
$$T(n) = 2T(n/2) + \Theta(n) \;\Rightarrow\; T(n) = \Theta(n \log n)$$

$$a_0, a_1, \ldots, a_{n-1} \quad \underset{???}{\overset{O(n \log n)}{\rightleftarrows}} \quad (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

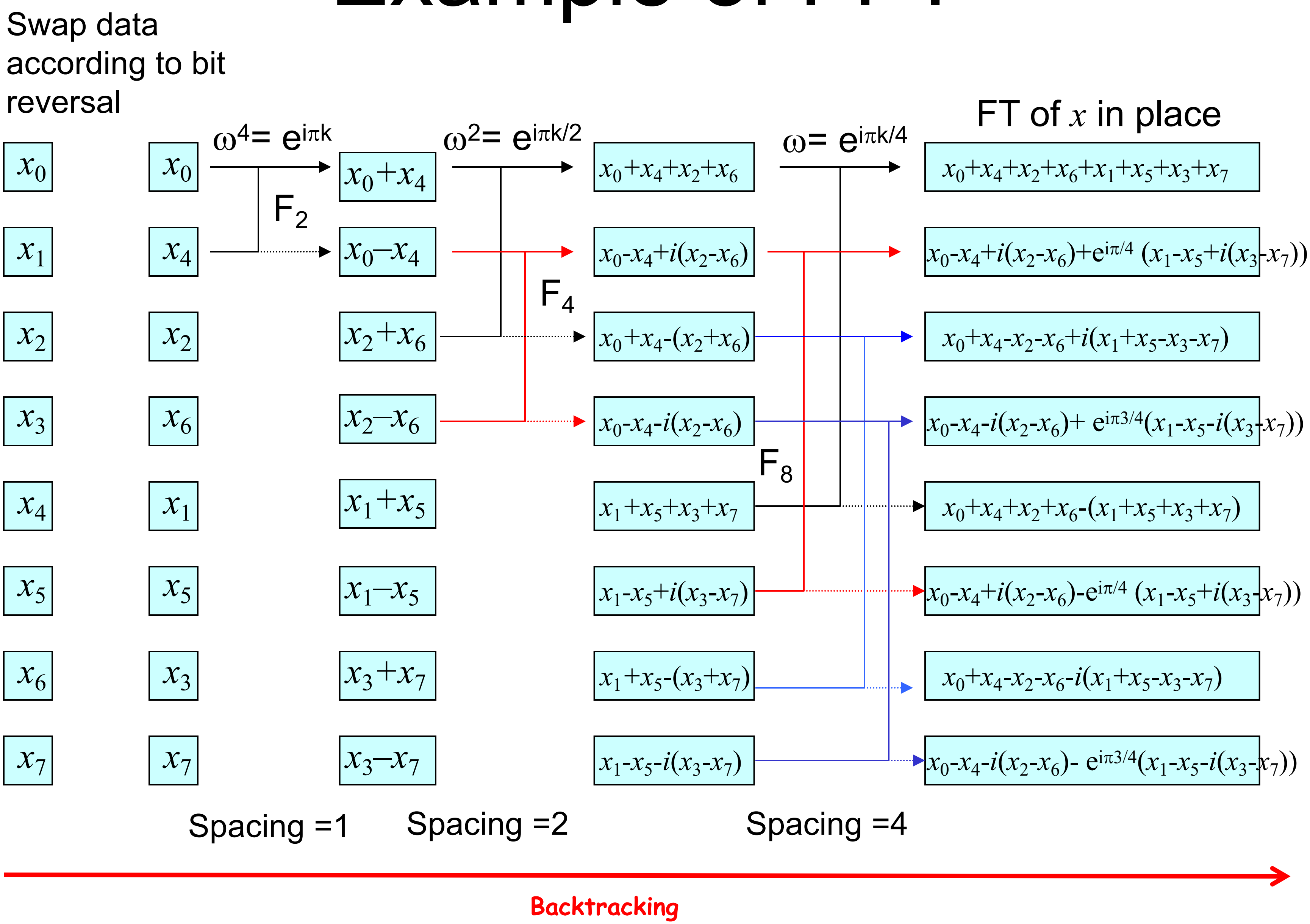
coefficient representation

point-value representation

Recursion Tree



Example of FFT



Inverse Discrete Fourier Transform

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Inverse DFT

Claim. Inverse of Fourier matrix F_n is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

$\frac{1}{\sqrt{n}} F_n$ is unitary

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n\, G_n\right)_{k\,k'} = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{kj}\,\omega^{-j\,k'} = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{(k-k')\,j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ series sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ series sums to 0. ▪

Inverse FFT: Algorithm

```
ifft(n, a₀,a₁,…,aₙ₋₁) {
   if (n == 1) return a₀

   (e₀,e₁,…,eₙ/₂₋₁) ← FFT(n/2, a₀,a₂,a₄,…,aₙ₋₂)
   (d₀,d₁,…,dₙ/₂₋₁) ← FFT(n/2, a₁,a₃,a₅,…,aₙ₋₁)

   for k = 0 to n/2 - 1 {
      ωᵏ ← e^(-2πik/n)
      yₖ₊ₙ/₂ ← (eₖ + ωᵏ dₖ) / n
      yₖ₊ₙ/₂ ← (eₖ - ωᵏ dₖ) / n
   }

   return (y₀,y₁,…,yₙ₋₁)
}
```

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in $O(n \log n)$ steps.

assumes n is a power of 2

$$a_0, a_1, \ldots, a_{n\text{-}1} \underset{O(n \log n)}{\overset{O(n \log n)}{\rightleftarrows}} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in $O(n \log n)$ steps.

pad with 0s to make n a power of 2

coefficient representation

$a_0, a_1, \ldots, a_{n-1}$
$b_0, b_1, \ldots, b_{n-1}$

2 FFTs $O(n \log n)$

$A(\omega^0), \ldots, A(\omega^{2n-1})$
$B(\omega^0), \ldots, B(\omega^{2n-1})$

point-value multiplication $O(n)$

$C(\omega^0), \ldots, C(\omega^{2n-1})$

inverse FFT $O(n \log n)$

coefficient representation

$c_0, c_1, \ldots, c_{2n-2}$