# DIGITAL LOGIC

## Chapter 3 part1:
## Gate-Level Minimization

2024 Fall



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# Today’s Agenda

- Recap
- Context
  - Gate level minimization using the Map Method
  - Product of sums simplification
  - Don’t Care Conditions
- Reading: Textbook, Chapter 3.1-3.5

# Recap

- Lecture2
  - Boolean Functions
    - binary variables
      - 0 and 1
    - Operators
      - AND, OR, NOT
    - 3 representations
      - Truth table
        - unique
      - Algebraic expression
      - Logic diagram
  - Basic Axiom and Theorems
    - One variable
      - Identity, Null Element, Idempotency, Involution, Complements
    - Multiple variables
      - Commutativity, Associativity, Distributivity, Absorption, Combining, Simplification, Consensus, Demorgan's
        - xy = yx
        - (xy)z = x(yz)
        - x(y + z) = xy + xz
        - (x + y)(x + z) = x + yz
        - (x + y)' = x'y'
  - Canonical and Standard forms
    - Canonical
      - Sum of Minterm (Product term with all variables)
      - Product of Maxterm (Sum term with all variables)
      - Mi = mi'
    - Standard
      - Sum of products (can have less variables)
      - Product of sums (can have less variables)

# Outline

- **Map Method Simplification**
- Product of sums simplification
- Don’t Care Conditions

# Boolean function simplification

- A function's truth-table representation is unique, while its algebraic expression is not unique.
- Complexity of digital circuit (gate count) $\propto$ complexity of algebraic expression (literal count)
  - F=x'y'z+x'yz+xy' (3 AND term, 8 literals)
  - F=x'z+xy' (2 AND terms, 4 literals)
- The simplest algebraic expression is one that has minimum number of terms with the smallest possible number of literals in each term
- Methods for gate-level minimization:
  - **Algebraic method**(逻辑代数): Boolean algebra (Last lecture)
  - **Karnaugh map**(卡诺图): the map method (This lecture)

# Karnaugh Map (K-map)

- An array of squares each representing one minterm to be minimized
- Each K-map defines a unique Boolean function
  - A Boolean function can be represented by a truth table, a Boolean expression, or a map
- K-map is a visual diagram of all possible ways a function may be expressed
- Used for manual minimization of Boolean functions

# Merging Minterms

- In function F, m1 and m3 in the truth table differ only in one position

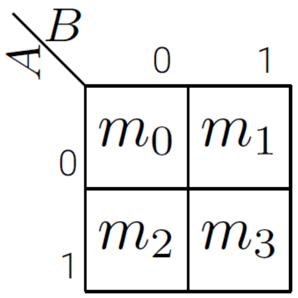
  001
  011 → 0?1

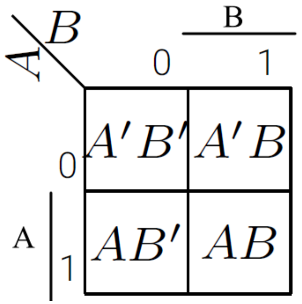
  - ? matches either 0 or 1
- The minterms in a function can be merged to form a simpler product term
  - 001 → x’y’z
  - 011 → x’yz
  - 0?1 → x’y’z+x’yz = x’z(y’+y) = x’z

| x | y | z | F |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

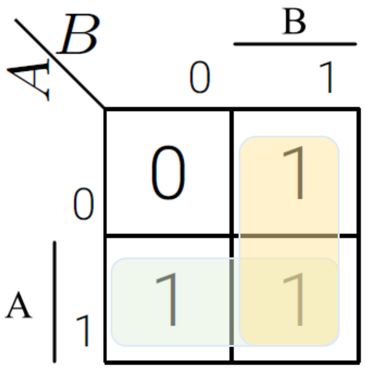
# Two-Variable Map

- A two-variable map
  - is A truth table in square diagram
  - 4 minterms: A’B’, A’B, AB’, AB
  - row 0 stands for A’; row 1 stands for A
  - column 0 stands for B'; column 1 stands for B




| | A | B | F |
|---|---|---|---|
| $m_0$ | 0 | 0 | 0 |
| $m_1$ | 0 | 1 | 1 |
| $m_2$ | 1 | 0 | 1 |
| $m_3$ | 1 | 1 | 1 |




It’s ok for groups to overlap, if that makes them larger

$m_1$+$m_2$+$m_3$ = A‘B+AB' +AB = A+B

# Three-variable Map

- Minterms are arranged in the Gray-code sequence
- Any 2 (horizontally or vertically) adjacent squares differ by exactly 1 variable, which is complemented in one square and uncomplemented in the other.
- Any 2 minterms in adjacent squares that are ORed together will cause a removal of the different variable (adjacent applies not only the middle squares but also the boundary squares)

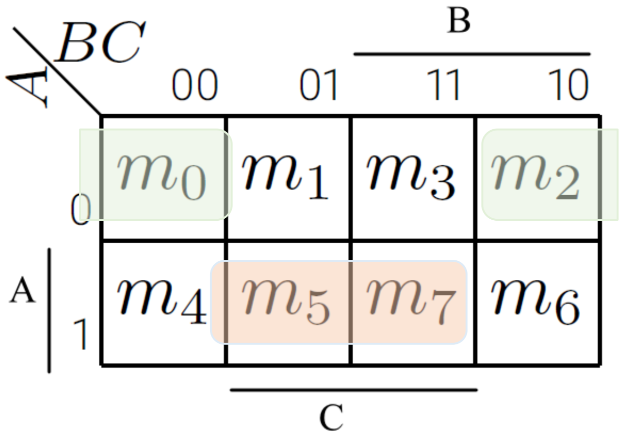
| A \ BC | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | $m_0$ | $m_1$ | $m_3$ | $m_2$ |
| 1 | $m_4$ | $m_5$ | $m_7$ | $m_6$ |

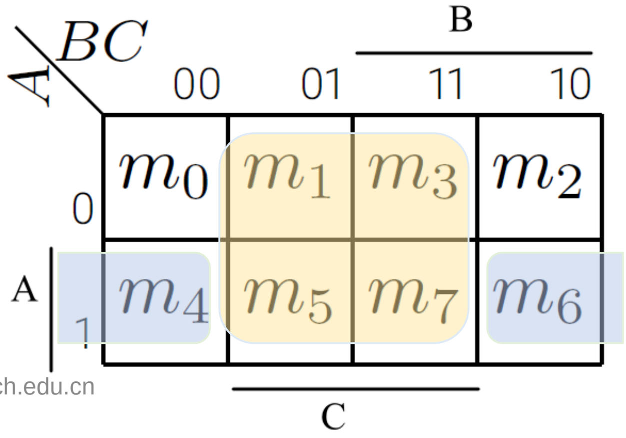
# Three-variable Map

- Example (adjacent squares)
- $m_5+m_7$ = AB'C+ABC = AC(B+B') = AC
- $m_0+m_2$ = A'B'C'+A'BC' = A'C'(B+B') = A'C'
- $m_4+m_6$ = AB'C' + ABC' = AC' (B'+B) = AC'
- $m_1+m_3+m_5+m_7$

  = A'B'C+A'BC+AB'C+ABC=A'C(B+B')+AC(B+B')

  = A'C+AC=C

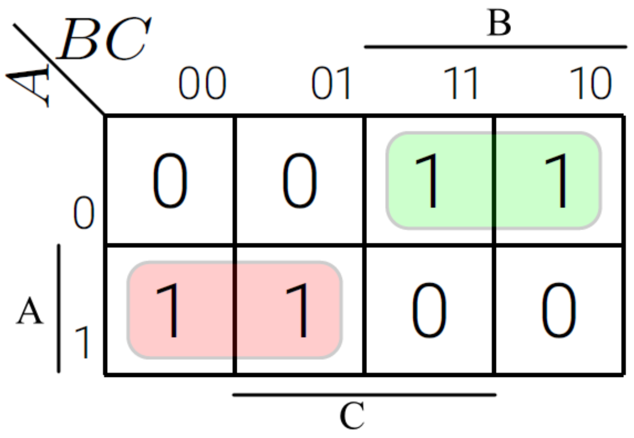


# Example

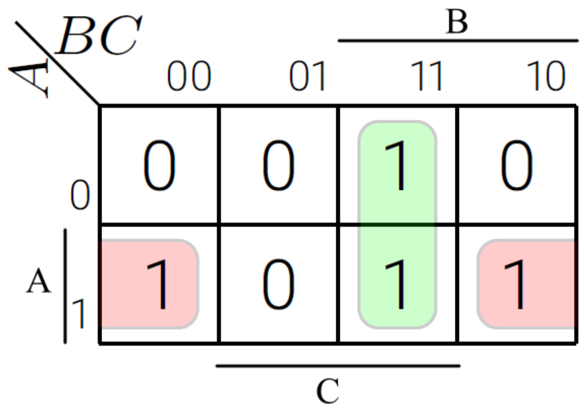
- Simplify the following Boolean functions.

F = A’BC+A’BC’+AB’C’+AB’C
= A'B + AB’




Green circle: A’BC+A’BC’ = A’B
Red circle: AB’C’+AB’C = AB’

F = A’BC+AB’C’+ABC’+ABC
= BC + AC’

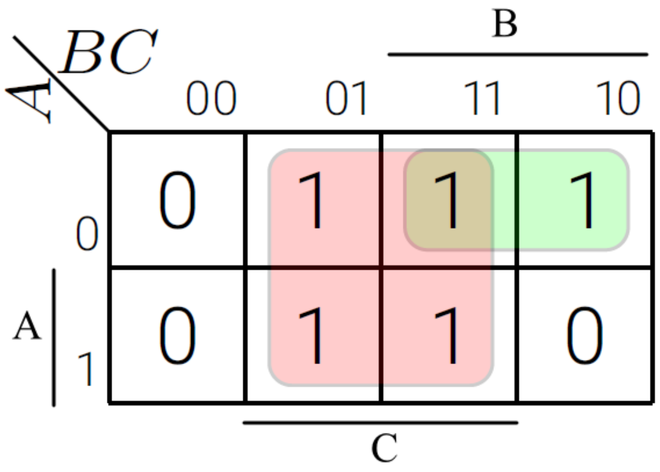



Green circle: A’BC + ABC = BC
Red circle: AB’C’ + ABC’ = AC’

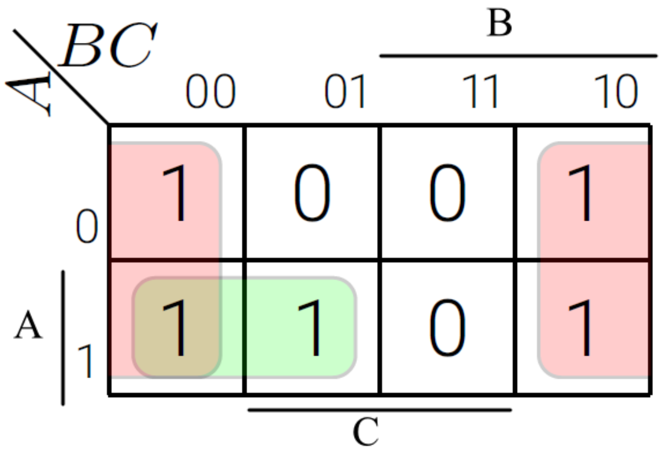
# Example

- Simplify the following Boolean functions.

$F = \sum(1, 2, 3, 5, 7) = C + A'B$




$F= \sum(0, 2, 4, 5, 6) = C' + AB'$




It's ok for groups to overlap,
if that makes them larger

# Exercise

- Simplify the following Boolean function.
  - F = A'C + A'B + AB'C + BC

    = ?

| | | | |
|---|---|---|---|
| | | | |

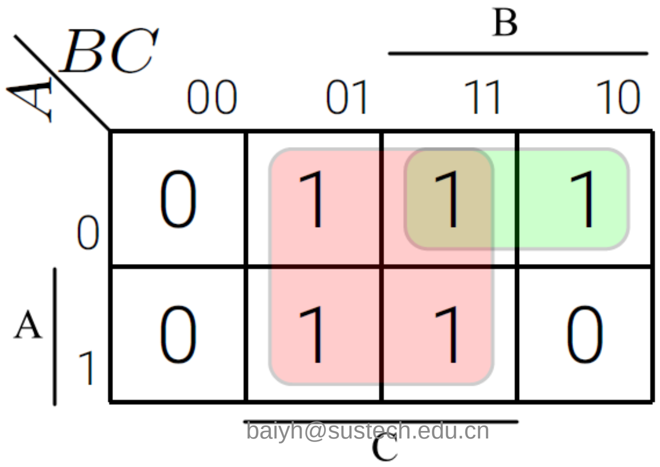
# Exercise

- Simplify the following Boolean function.
  - F = A'C + A'B + AB'C + BC = ?
- Solution:
  - Express it in sum of minterms.
  - Find the minimal sum of products expression.
  - F = A'C + A'B + AB'C + BC

    = A'C(B+B') + A'B(C+C') + AB'C + (A+A')BC

    = A'BC + A'B'C + ~~A'BC~~ + A'BC' + AB'C + ABC + ~~A'BC~~

    = ∑(1, 2, 3, 5, 7) = C + A'B

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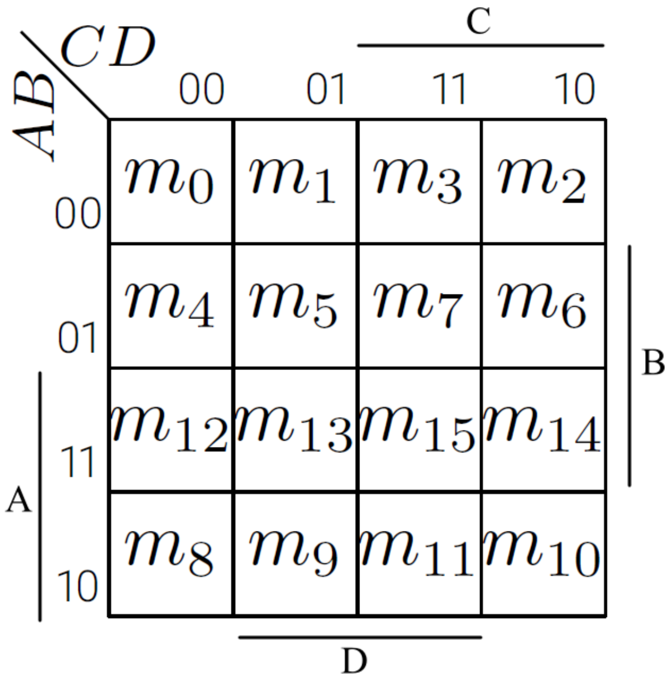
# Four-Variable Map

- The map
  - 16 minterms
  - Combinations of 2, 4, 8, and 16 adjacent squares



# Implicants

- Implicant: any product term that implies the function
  - A product term that makes a function to be true

| | minterm | implicant | |
|---|---|---|---|
| $m_1$ | √ | √ | 1-minterm |
| $m_2$ | √ | X | 0-minterm |
| $x'z$ | X | √ | 1 product term |

| x | y | z | F | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $m_0$ |
| 0 | 0 | 1 | 1 | $m_1$ |
| 0 | 1 | 0 | 0 | $m_2$ |
| 0 | 1 | 1 | 1 | $m_3$ |
| 1 | 0 | 0 | 1 | $m_4$ |
| 1 | 0 | 1 | 1 | $m_5$ |
| 1 | 1 | 0 | 0 | $m_6$ |
| 1 | 1 | 1 | 0 | $m_7$ |

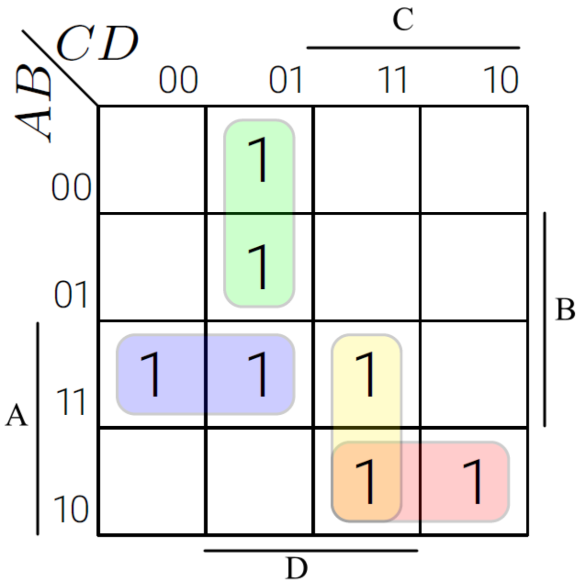
- Prime implicant (PI) (质蕴含)
  - A 1-product term obtained by combining the maximum possible number of adjacent squares in the map.
- Essential prime implicant (EPI) (基本质蕴含)
  - If a minterm in a square is covered by only one prime implicant

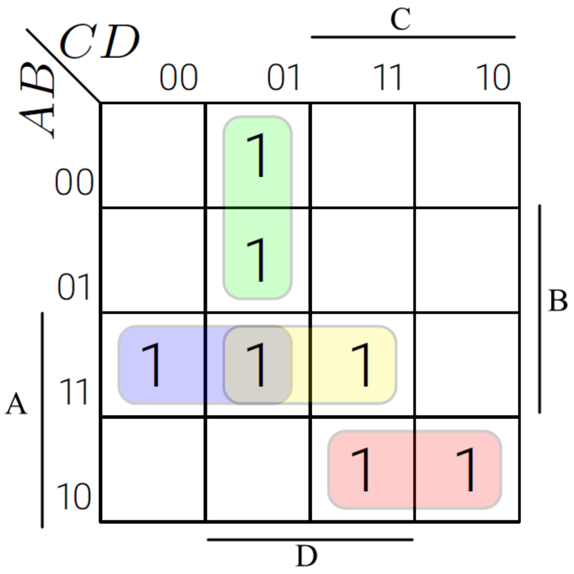
# Tips for simplification

- Simplification Steps:
  - Determine all essential prime implicants.
  - Find other prime implicants that cover remaining minterms.
  - Logical sum all prime implicants.
- Tips:
  - Minimize the number of groups
  - Maximize the group size
  - It’s ok for groups to overlap, if that makes them larger

# Example

- Simplify the following Boolean functions
- F = $m_1+m_5+m_{10}+m_{11}+m_{12}+m_{13}+m_{15}$
    - start with EPIs
    - Green circle: A′B′C′D + A′BC′D = A′C′D
    - Purple circle: ABC′D′ + ABC′D = ABC′
    - Red circle: …
    - …
  - F = A′C′D + ABC′ + ACD + AB′C
- This reduced expression is not a unique one
  - If pairs are formed in different ways, the simplified expression will be different.
  - F = A′C′D + ABC′ + ABD + AB′C

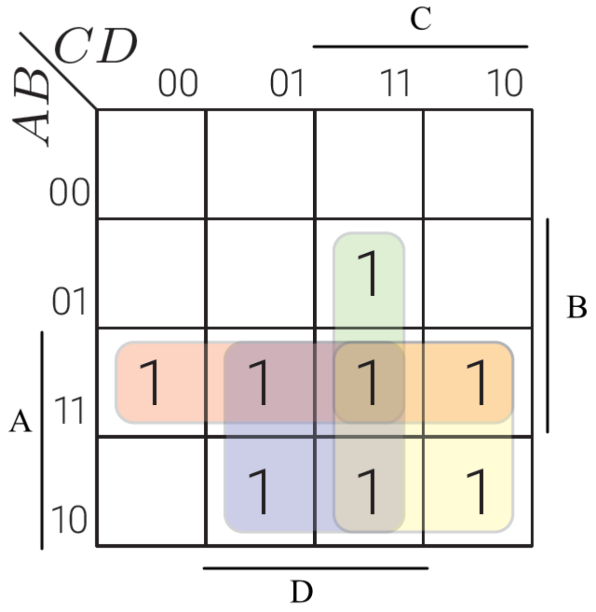


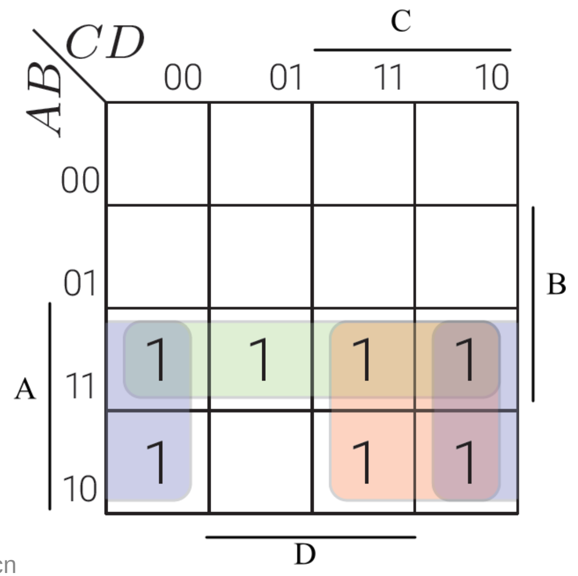
# Example

- Simplify the following Boolean functions.

$F = \sum(7, 9, 10, 11, 12, 13, 14, 15)$
$= AB + AC + AD + BCD$




$F(A,B,C,D) = ABCD + AB'C'D' + AB'C + AB$
$= ABCD + AB'C'D' + AB'C(D + D')$
$+AB(C + C')(D + D')$
$= \ldots$
$= \sum(8, 10, 11, 12, 13, 14, 15)$
$=AB + AC + AD'$

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# Exercise

- Simplify the following Boolean functions.

$F(A,B,C,D)$
$= \sum(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$
$= ?$

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$F(A,B,C,D)$
$= A'B'C' + B'CD' + A'BCD' + AB'C'$
$= ?$

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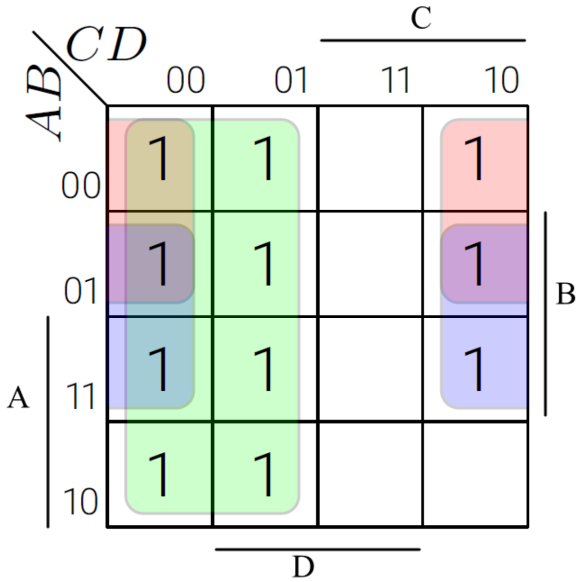
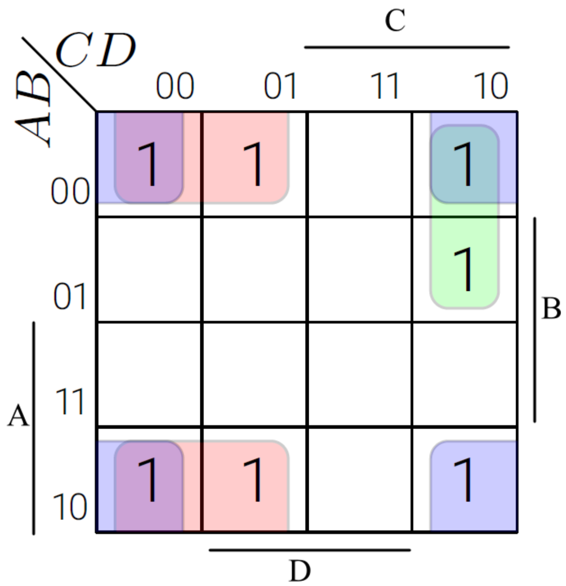
# Exercise

- Simplify the following Boolean functions.

F(A,B,C,D)
= ∑(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)
= C’ + A’D’ + BD’




F = A′B′C′ + B′CD′ + A′BCD′ + AB′C′
= A′B′C′(D + D′) + B′CD′(A + A′) +
A′BCD′+ AB′C′(D + D′)
= A′B′C′D + A′B′C′D′+ AB′CD′ + A′B′CD′+
A′BCD′+ AB′C′D + AB′C′D′
= ∑(0, 1, 2, 6, 8, 9, 10)
= B′C′+ B′D′+ A′CD′

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# K-map Summary

- Any $2^k$ adjacent squares, k=0,1,...,n, in an n-variable map represent an area that gives a product term of n-k literals

| K | # of adjacent squares | # of literals left in a term in an n-variable map | | |
|---|---|---|---|---|
| | | n=2 | n=3 | n=4 |
| 0 | 1 | 2 | 3 | 4 |
| 1 | 2 | 1 | 2 | 3 |
| 2 | 4 | 0 | 1 | 2 |
| 3 | 8 | | 0 | 1 |
| 4 | 16 | | | 0 |

- Five-Variable Map
  - Map for more than four variables becomes complicated
  - Five-variable map: two four-variable map (one on the top of the other), contains $2^5$ or 32 cells.

# Outline

- Map Method Simplification
- **Product of sums simplification**
- Don’t Care Conditions

# Product of Sums Simplification

- Previous Examples are Sum of Product Simplification
  - E.g. F = AB + A'D + AB'C (Product of sum form)
- How to find Product of Sum simplification
  - E.g. F = (A+B)(B+C')  (Sum of Product form)
- POS simplification Steps
  - Simplified F'  in the form of sum of products
    - Group adjacent 0-minterms squares together
  - Apply DeMorgan's theorem F = (F')'
  - F': sum of products → F: product of sums

# Example

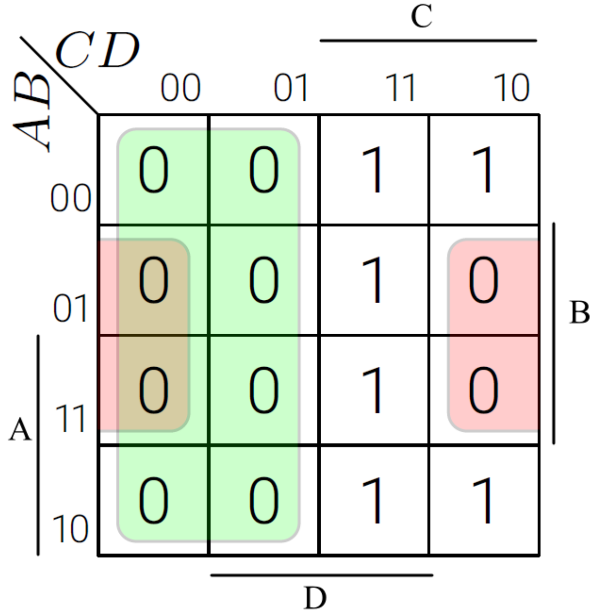
- Simplify the Boolean function into product of sums form:
  - F(A,B,C,D) = ∑(2, 3, 7, 10, 11, 15)
- Solution
  - Step1: group the 0-minterms to find F complement

    F’ = ∑(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)

    = C’ + BD’ (Group 0 minterms)
  - Step2: find the complement again to get original F

    F = (F’)’ = (C’ + BD’)’

    = C(B’+D) (DeMorgan’s)



# Exercise

- simplify F(A, B, C, D) = $\sum(0, 1, 2, 5, 8, 9, 10)$ into
  - sum-of-products form
    - F = ?
  - product-of-sums form
    - F = ?

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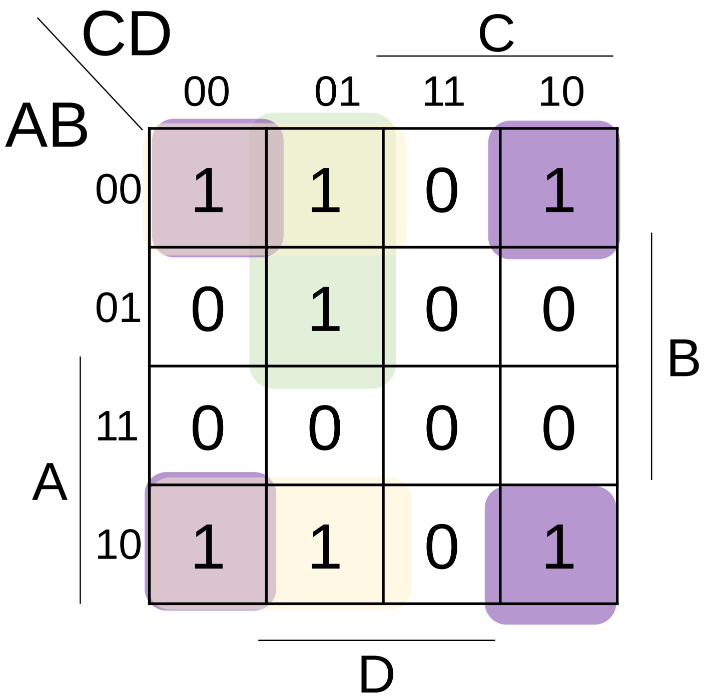
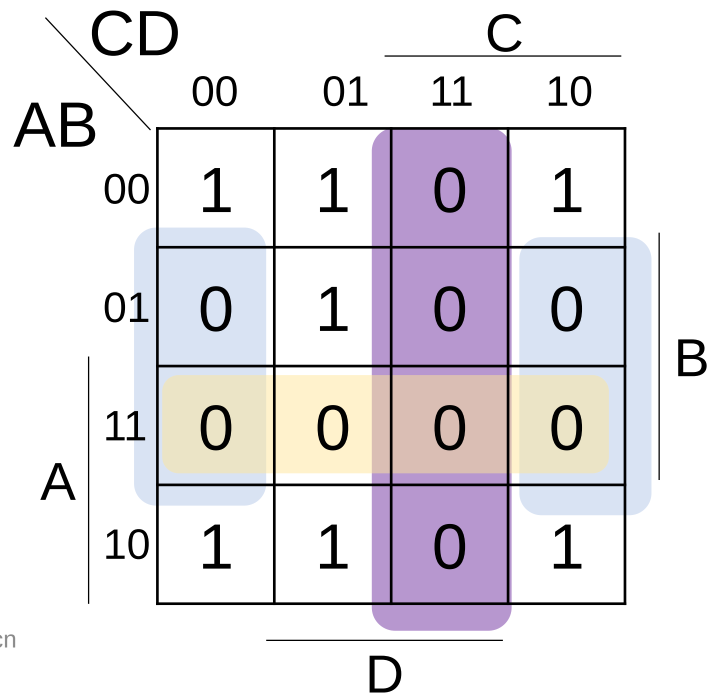
# Exercise

- simplify F(A, B, C, D) = ∑(0, 1, 2, 5, 8, 9, 10) into
  - sum-of-products form
    - F = B'D' + B'C' + A'C'D (Group 1-minterms)
  - product-of-sums form
    - F' = AB+CD+BD' (Group 0-minterms)
    - F = (A'+B')(C'+D')(B'+D) (DeMorgan's)



# Outline

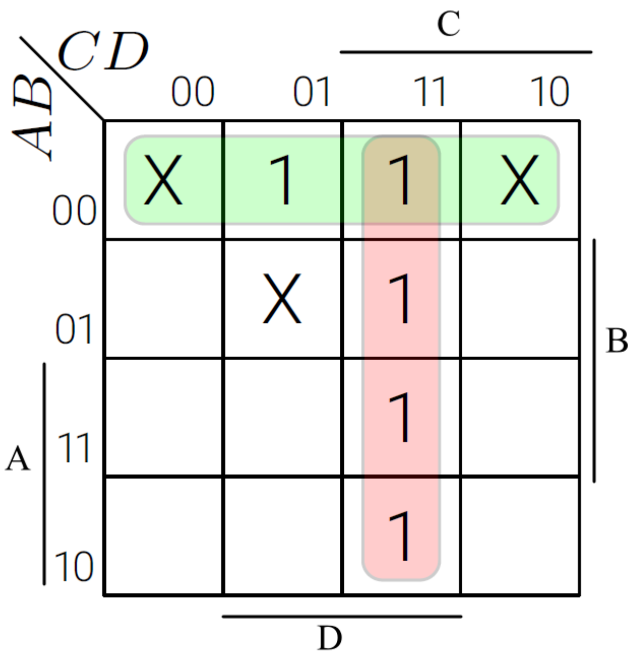
- Map Method Simplification
- Product of sums simplification
- **Don’t Care Conditions**

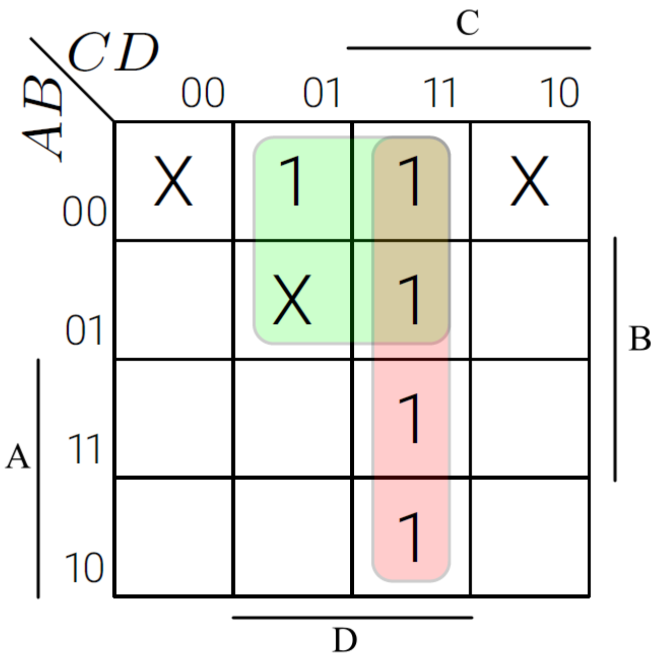
# Don’t care conditions

- Incompletely specified functions
  - Functions that have unspecified outputs for some input combinations
  - E.g. output are unspecified for 1010 to 1111 in 4-bit BCD code
- Don’t-care conditions
  - Unspecified minterms of a function, don’t-cares, Xs
  - Can be used on a map to provide further simplifications of the Boolean expression
  - Each X can be assigned an arbitrary value, 0 or 1, to help simplification procedure

# Example

- Simplify F(A, B, C, D) = ∑(1, 3, 7, 11, 15) with don't-care conditions d(A, B, C, D) = ∑(0, 2, 5).
  - F = A'B' + CD
  - or F = A'D + CD
  - Just make sure all 1 minterms are circled, thus both simplifications are acceptable



# Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A,B,C,D) = \Sigma(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$

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# Exercise

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function F(A,B,C,D) = Σ(0, 2, 3, 6, 7) + d(8, 10, 11, 15)
- F = A'C + B'D'

| AB \ CD | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 1 | 1 |
| 01 | 0 | 0 | 1 | 1 |
| 11 | 0 | 0 | X | 0 |
| 10 | X | 0 | X | X |