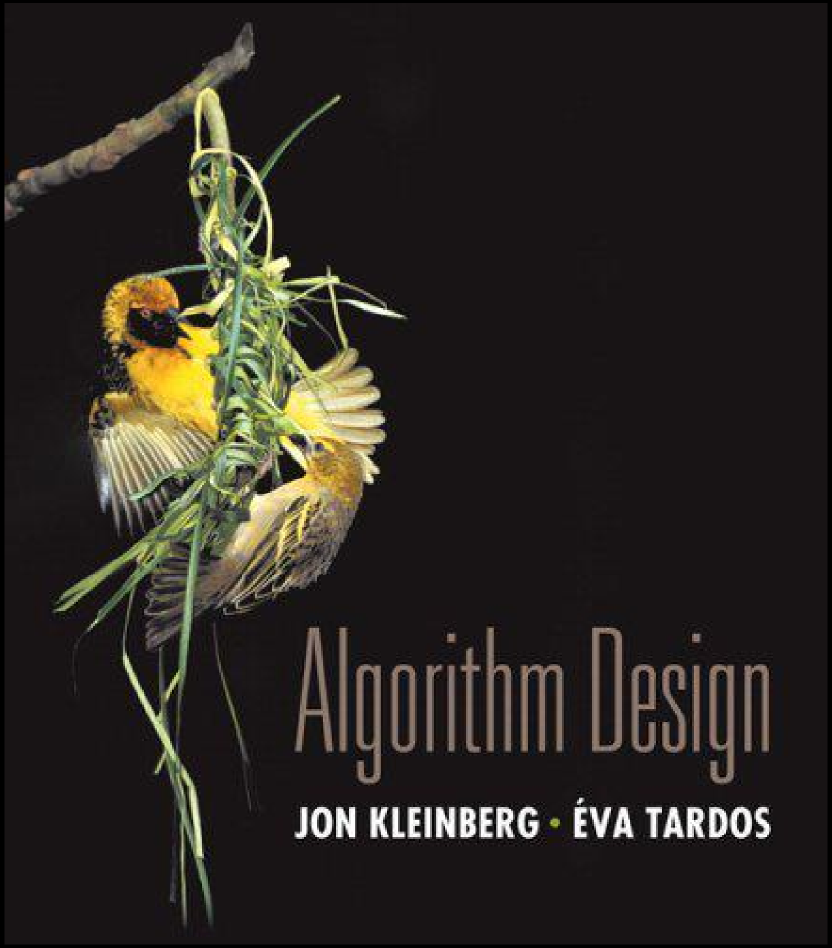


Chapter 4

Greedy Algorithms

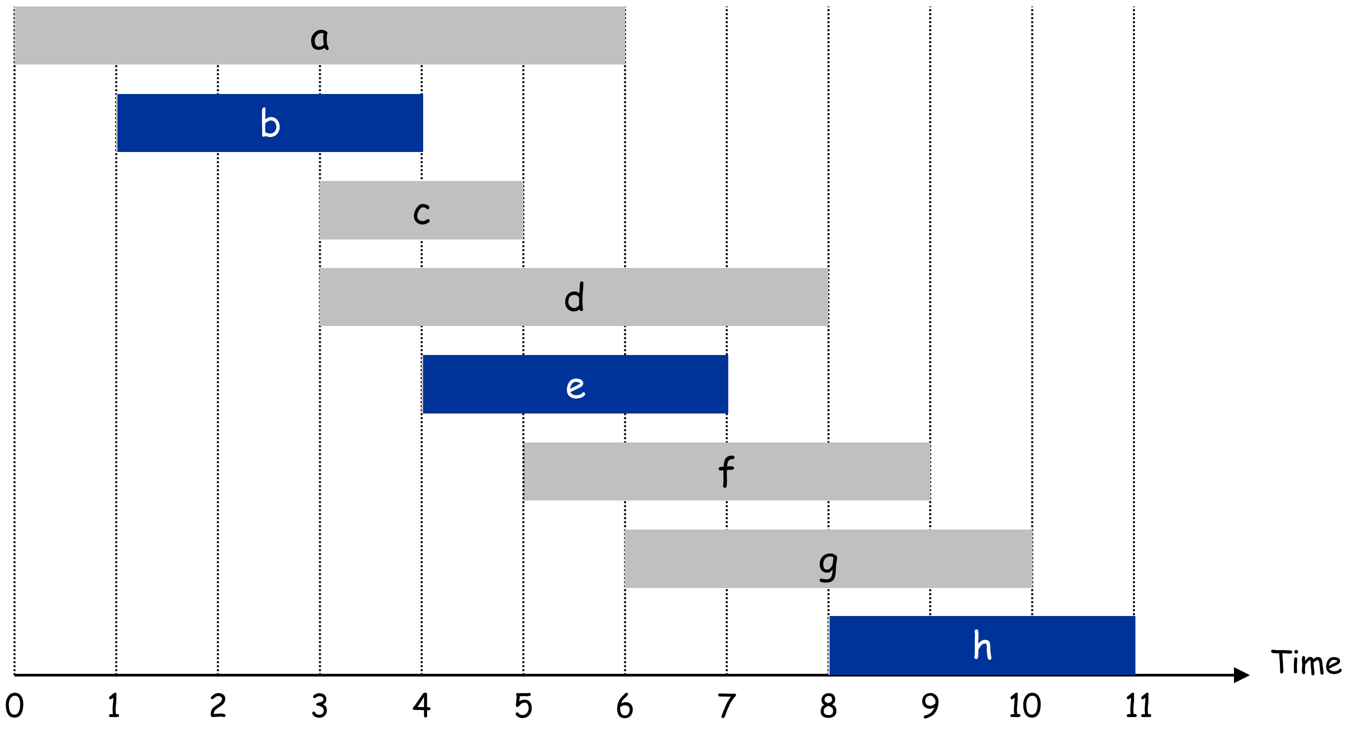
Slides by Kevin Wayne.

4.1.1 Interval Scheduling

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum cardinality subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of s_j.
- [Earliest finish time] Consider jobs in ascending order of f_j.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

  set of jobs selected
$A \leftarrow \phi$
for j = 1 to n {
   if (job j compatible with A)
      $A \leftarrow A \cup \{j\}$
}
return A
```

Implementation. O(n log n).

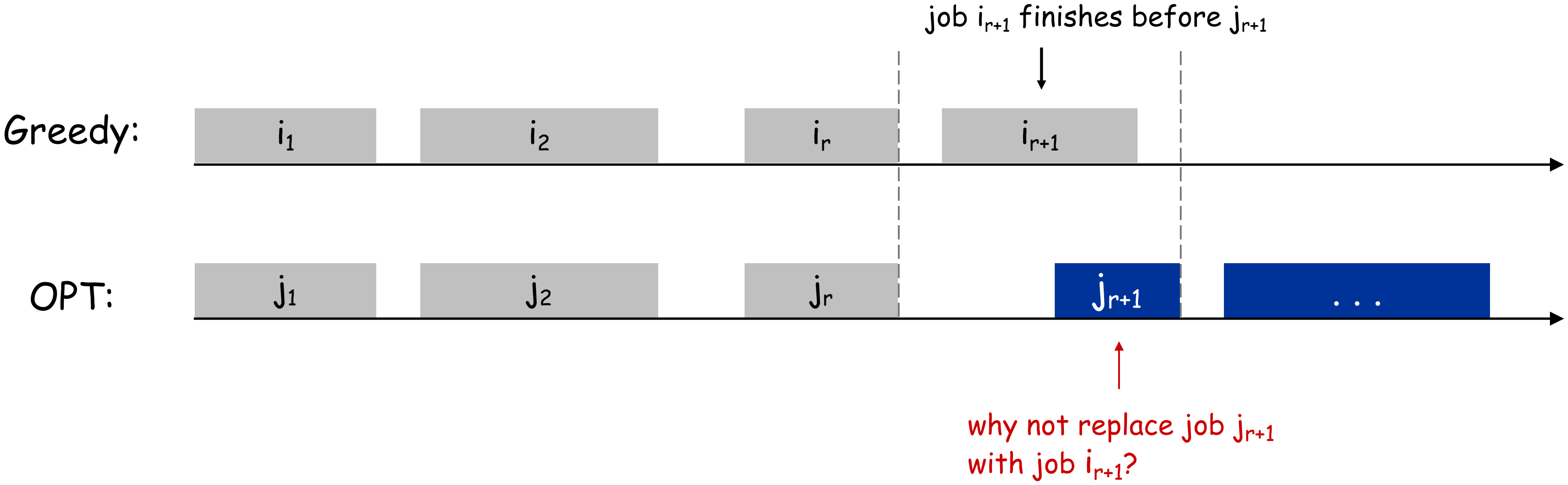
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \ge f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



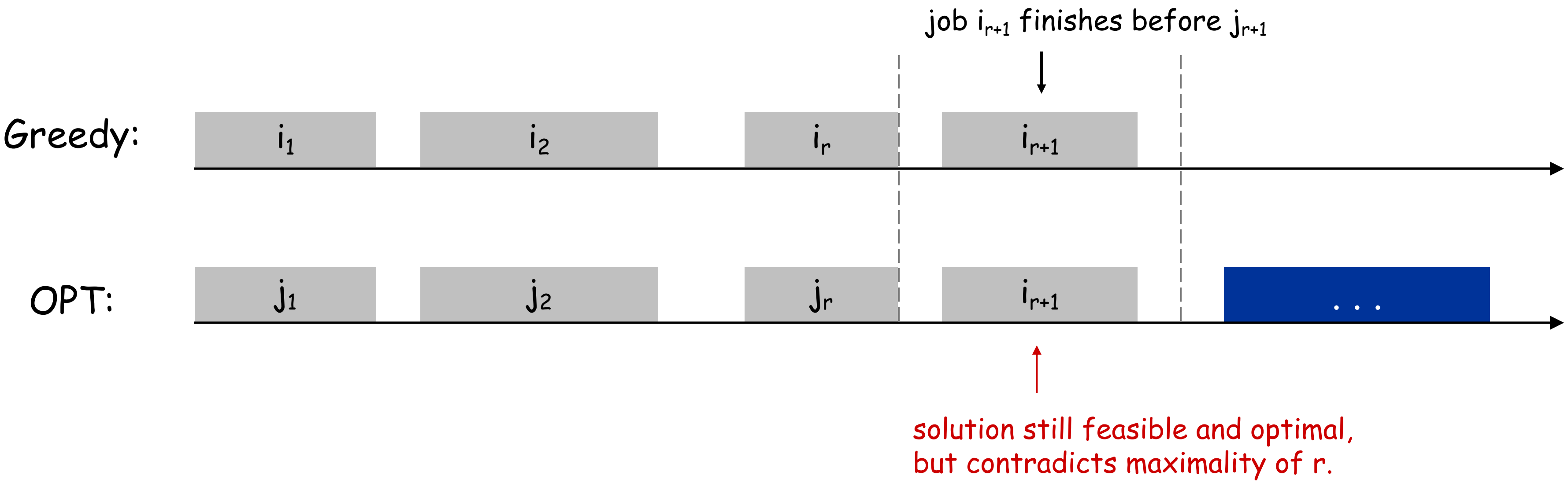
why not replace job j_{r+1} with job i_{r+1}?

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



solution still feasible and optimal, but contradicts maximality of r.

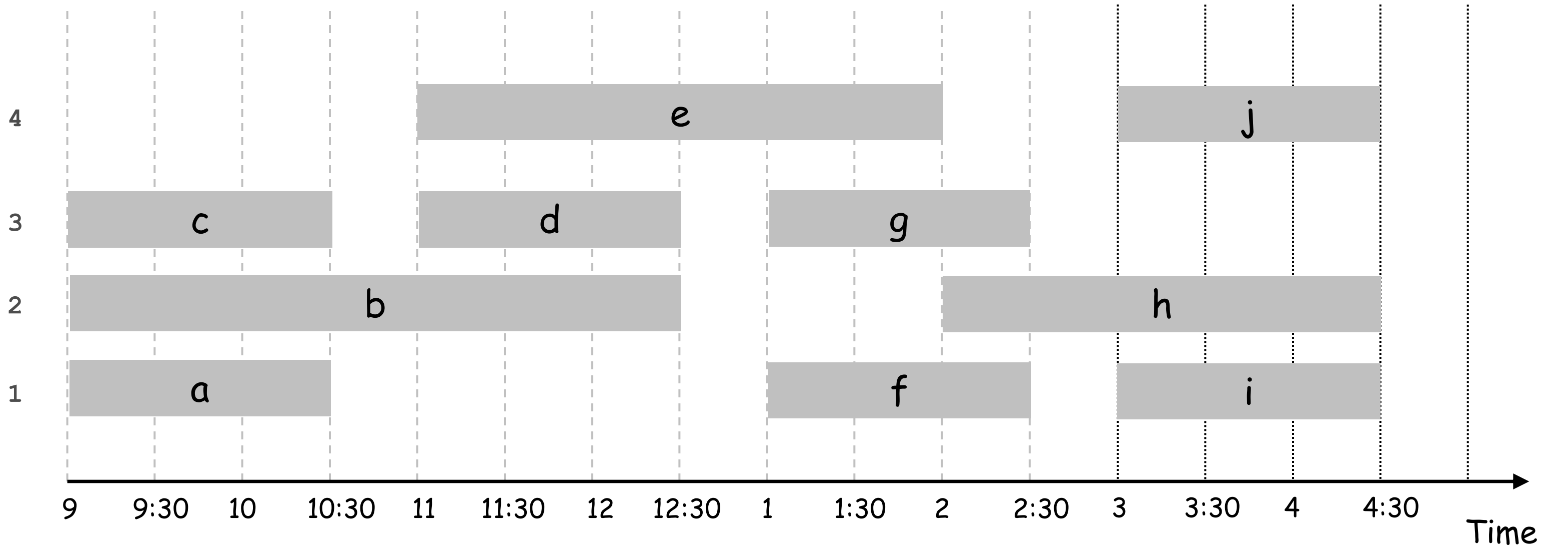
4.1.2 Interval Partitioning

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

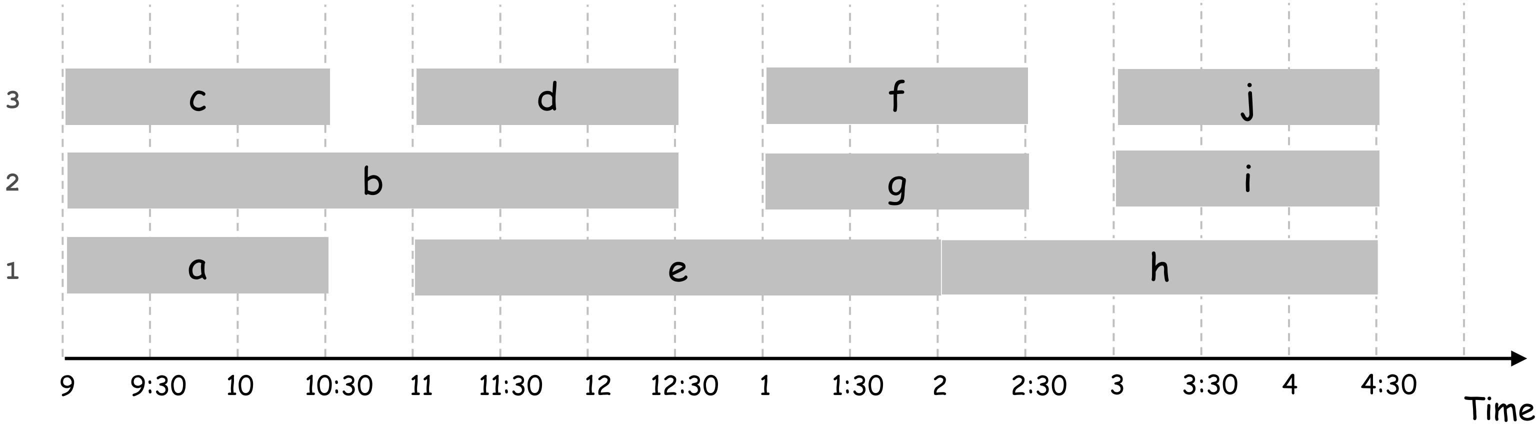


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

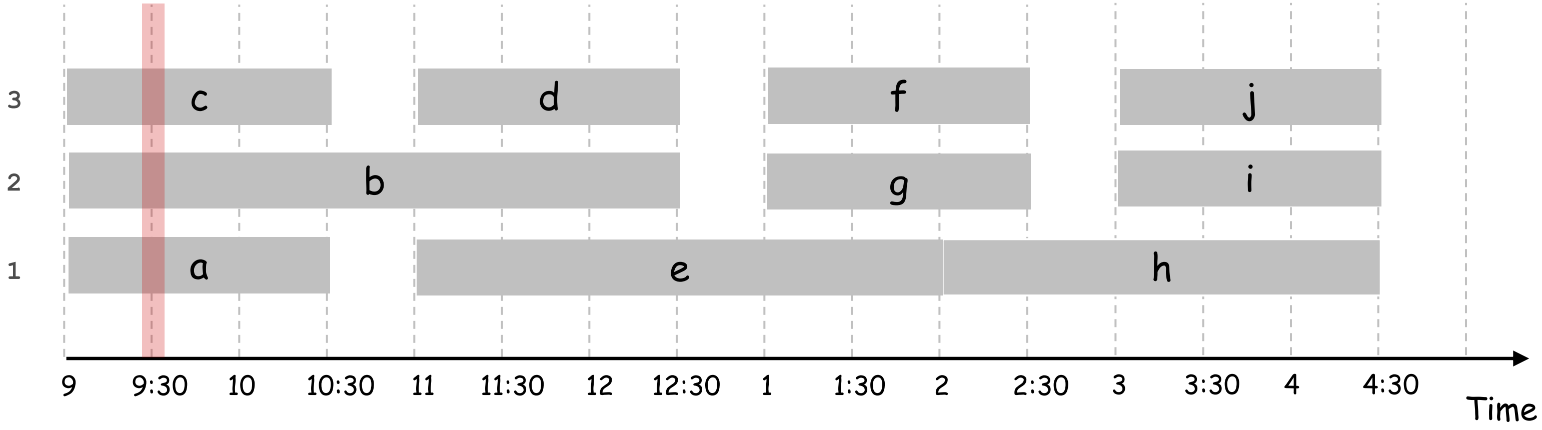
Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \leq s_2 \leq ... \leq s_n$.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. O(n log n).

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- These d jobs (one from each classroom) each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- If d >depth, then there are d lectures overlapping which is impossible, $\Rightarrow$ d must be $\leq depth$
- Key observation $\Rightarrow$ all schedules use $\geq$ depth classrooms. ▪

4.2 Scheduling to Minimize Lateness

Scheduling to Minimizing Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, \; f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time t_j.

- [Earliest deadline first] Consider jobs in ascending order of deadline d_j.

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	10	1
d_j	10	2

counterexample

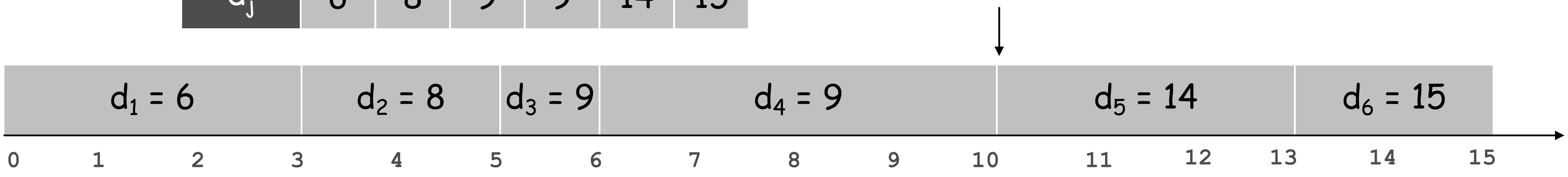
Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

Rename all jobs as 1, 2, …, n according to their deadlines

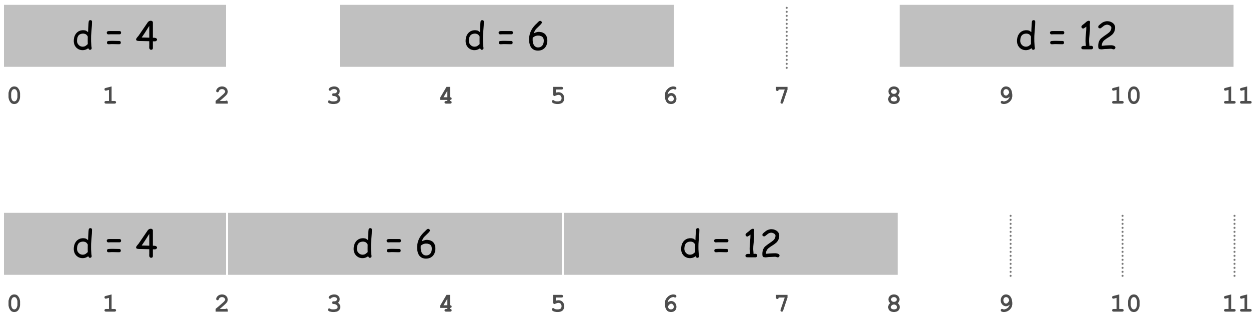
```
Sort n jobs by deadline so that $d_1 \le d_2 \le … \le d_n$

$t \leftarrow 0$
for j = 1 to n
    Assign job j to interval $[t, t + t_j]$
    $s_j \leftarrow t$, $f_j \leftarrow t + t_j$
    $t \leftarrow t + t_j$
output intervals $[s_j, f_j]$
```

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15


max lateness = 1

Minimizing Lateness: No Idle Time

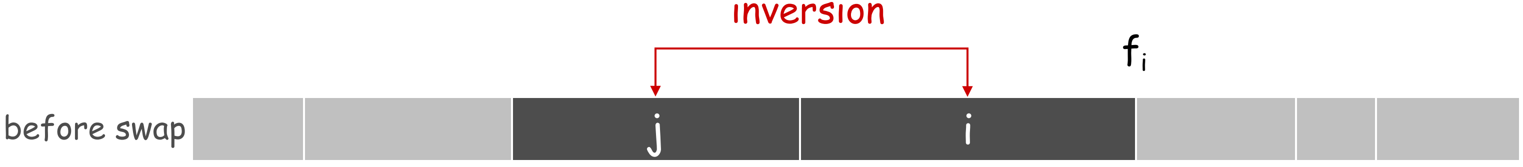
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



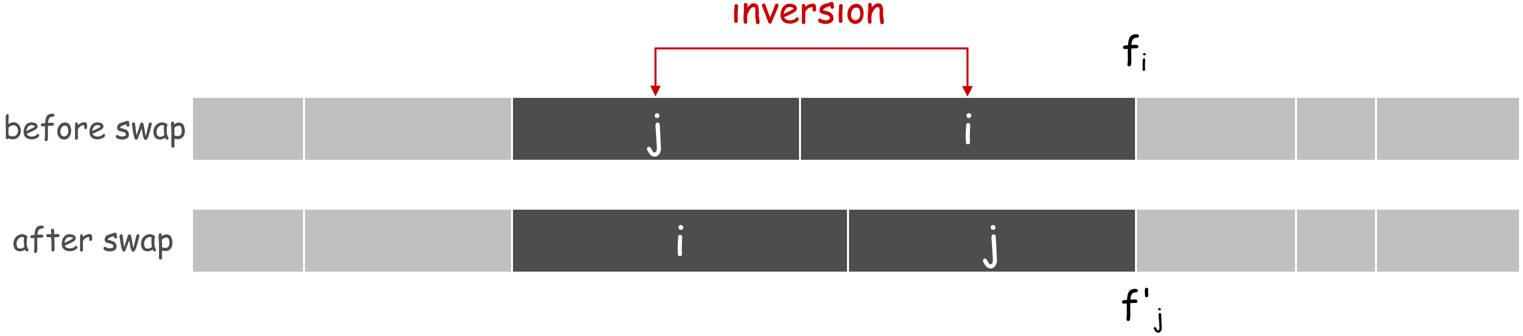
[as before, we assume jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$]

Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Minimizing Lateness: Inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.



Claim. Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is late:

$$
\begin{aligned}
\ell'_j &= f'_j - d_j && \text{(definition)} \\
&= f_i - d_j && (j \text{ finishes at time } f_i) \\
&\leq f_i - d_i && (i < j) \\
&\leq \ell_i && \text{(definition)}
\end{aligned}
$$

Minimizing Lateness: Analysis of Greedy Algorithm

Theorem. Greedy schedule S is optimal.

Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume S* has no idle time.
- If S* has no inversions, then S = S*. ← Greedy schedule has no inversions
- If S* has an inversion, let i-j be an adjacent inversion.
  - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of S* ▪

Greedy Analysis Strategies

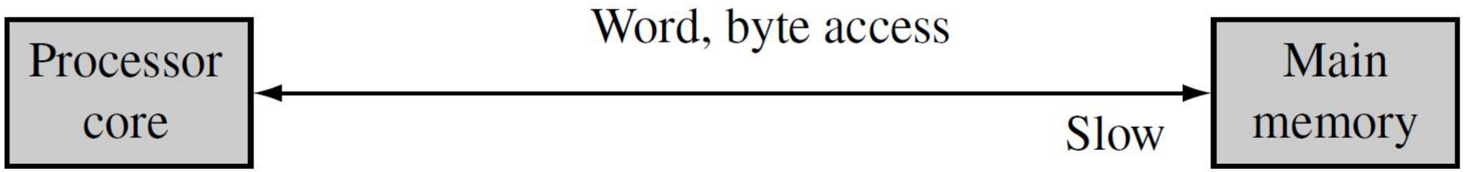
Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's. (Interval scheduling)

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound. (Interval Partitioning)

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality. (Scheduling to Minimize Lateness)

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, …

4.3 Optimal Caching



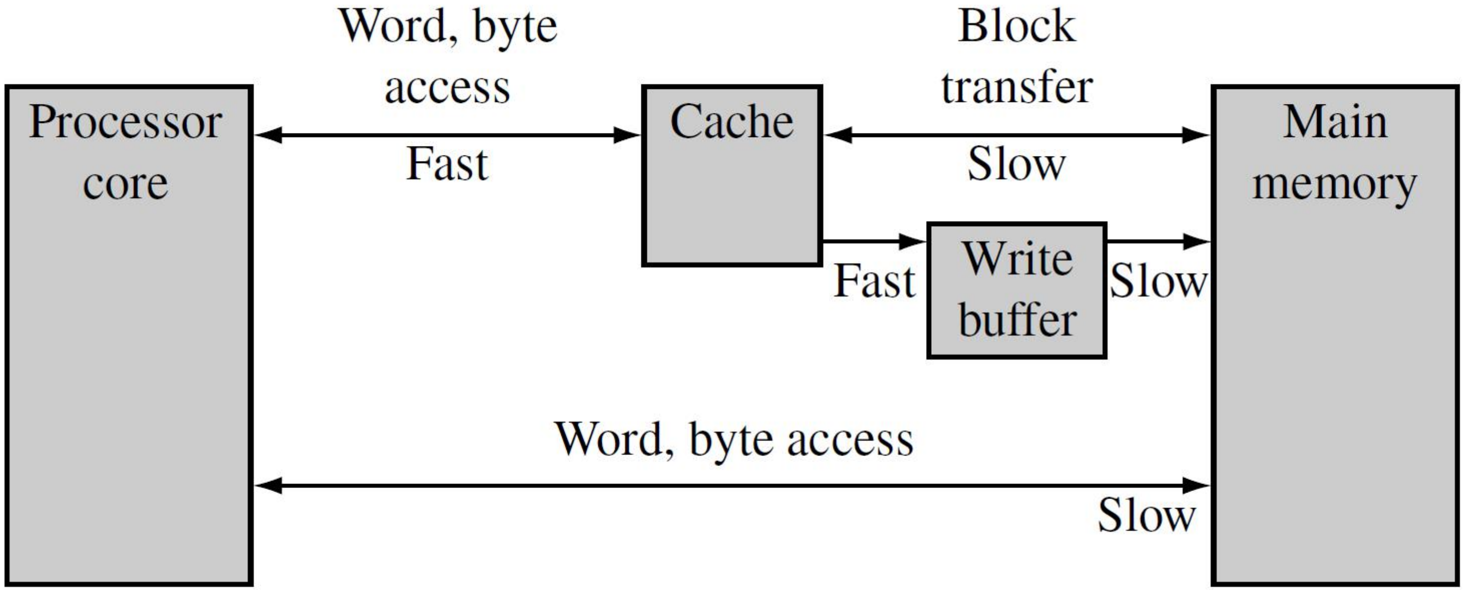
Processor core

Word, byte access

Slow

Main memory

Noncached system



Processor core

Word, byte access

Fast

Cache

Block transfer

Slow

Main memory

Fast

Write buffer

Slow

Word, byte access

Slow

Cached system

Optimal Offline Caching

Caching.

- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

to *evict* some other piece of data that is currently in the cache to make room for d_i .

Goal. *Eviction schedule* that minimizes number of *cache misses*.

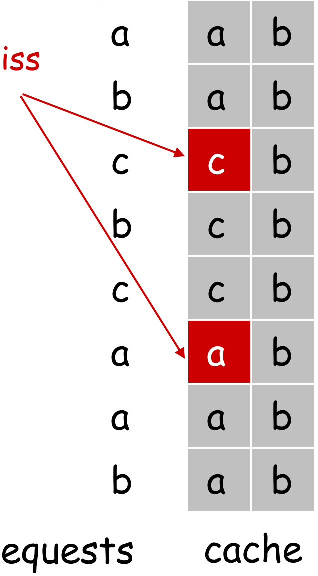
specifying which items should be evicted from the cache at which points in the sequence

Ex: k = 2, initial cache = ab,

requests: a, b, c, b, c, a, a, b.

Optimal eviction schedule: 2 cache misses.

red = cache miss

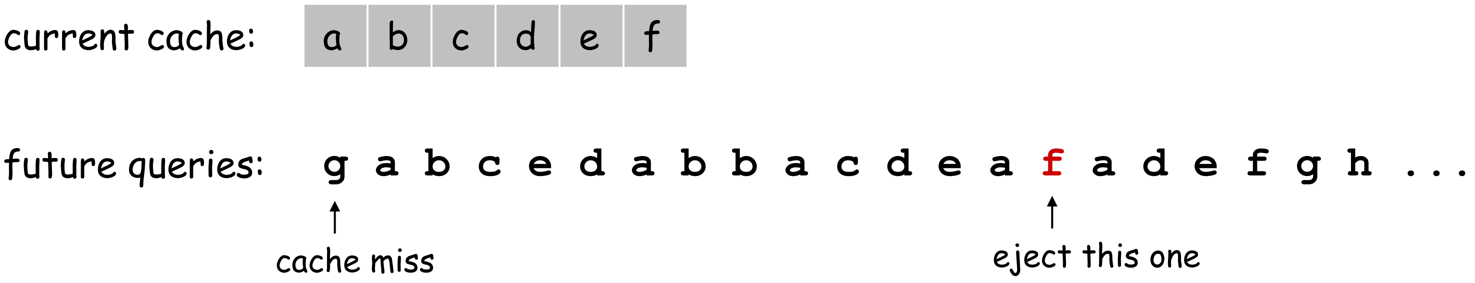


a	a	b
b	a	b
c	🟥 c	b
b	c	b
c	c	b
a	🟥 a	b
a	a	b
b	a	b

requests cache

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.



current cache: a b c d e f

future queries: `g a b c e d a b b a c d e a f a d e f g h ...`

Theorem. [Bellady, 1960s] FF is an optimal eviction schedule.

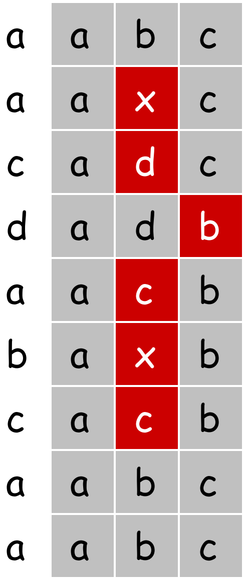
Pf. Algorithm and theorem are intuitive; proof is subtle.

Reduced Eviction Schedules

Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

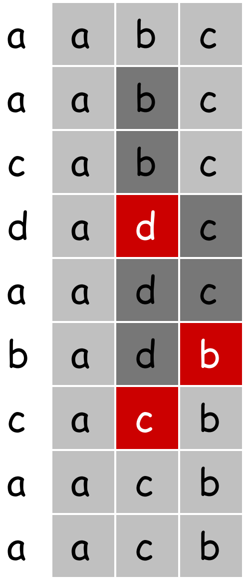
Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.

to evict some other piece of data that is currently in the cache to make room for data d_i .



a	a	b	c
a	a	x	c
c	a	d	c
d	a	d	b
a	a	c	b
b	a	x	b
c	a	c	b
a	a	b	c
a	a	b	c

an unreduced schedule



a	a	b	c
a	a	b	c
c	a	b	c
d	a	d	c
a	a	d	c
b	a	d	b
c	a	c	b
a	a	c	b
a	a	c	b

a reduced schedule

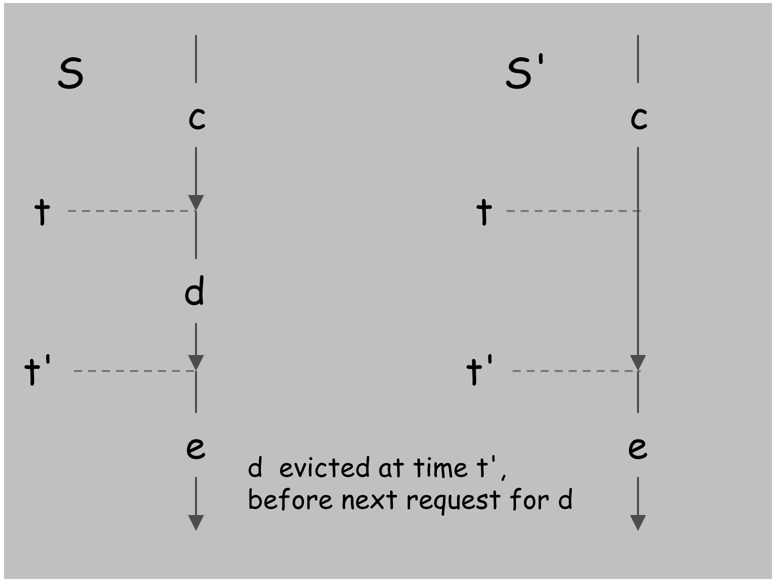
Reduced Eviction Schedules

Claim. Given any unreduced schedule S, can transform it into a reduced schedule S' with no more cache misses.

Pf. (by induction on number of unreduced items)

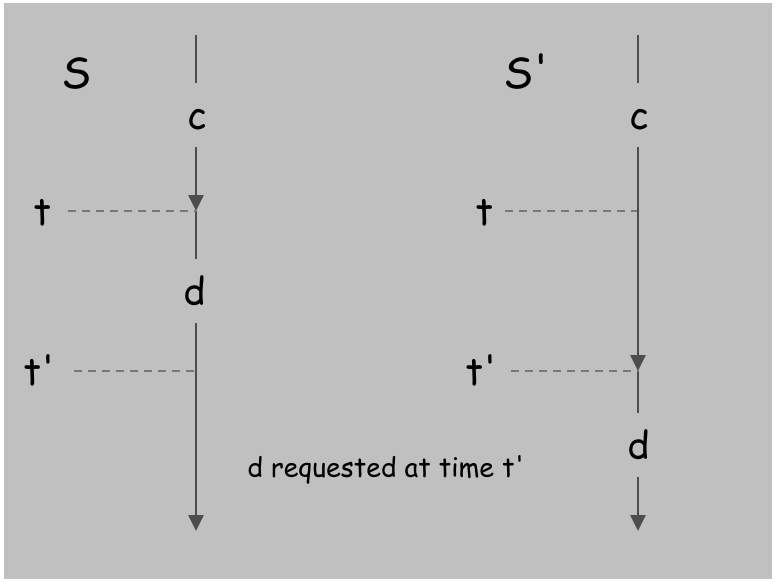
doesn't enter cache at requested time

- Suppose S brings d into the cache at time t, without a request.
- Let c be the item S evicts when it brings d into the cache.
- Case 1: d evicted at time t', before next request for d.
- Case 2: d requested at time t' before d is evicted. ▪



d evicted at time t', before next request for d

Case 1



d requested at time t'

Case 2

Farthest-In-Future: Analysis

Theorem. FF is an optimal eviction algorithm.
Pf. (by induction on number or requests j)

> Invariant: There exists an optimal reduced schedule S that makes the same eviction schedule as S_{FF} through the first j+1 requests.

Let S be an optimal reduced schedule that satisfies invariant through j requests. We produce S' that satisfies invariant through the first j+1 requests.

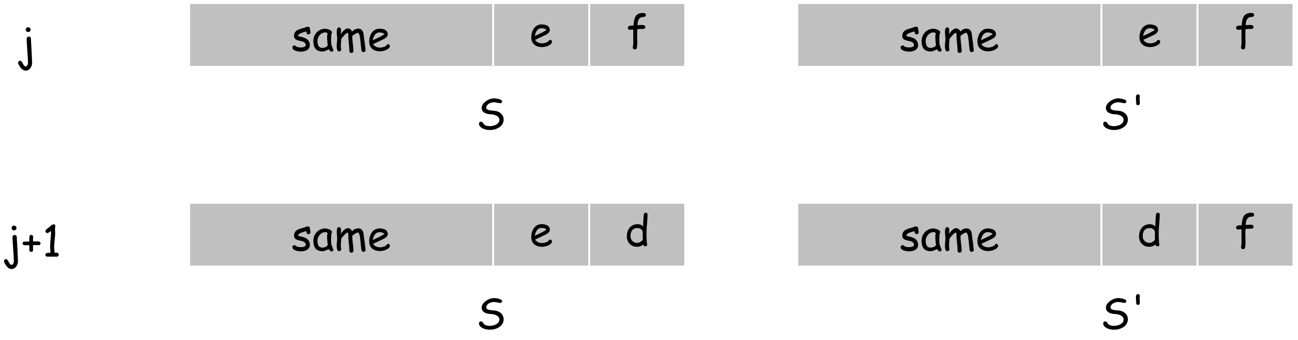
- Consider $(j+1)^{st}$ request $d = d_{j+1}$.
- Since S and S_{FF} have agreed up until now, they have the same cache contents before request j+1.
- Case 1: (d is already in the cache). S' = S satisfies invariant.
- Case 2: (d is not in the cache and S and S_{FF} evict the same element). S' = S satisfies invariant.

Farthest-In-Future: *Analysis*

Pf. (continued)

e is far away than f

- Case 3: (d is not in the cache; S_{FF} evicts e; S evicts f $\neq$ e).
  - begin construction of S' from S by evicting e instead of f

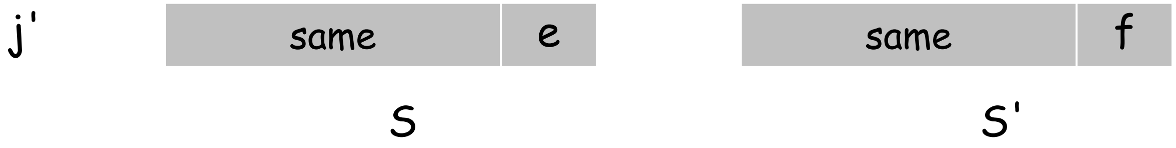


  - now S' agrees with S_{FF} on first j+1 requests; we show that having element f in cache is no worse than having element e

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)



- Case 3a: g = e. Can't happen with Farthest-In-Future since there must be a request for f before e.

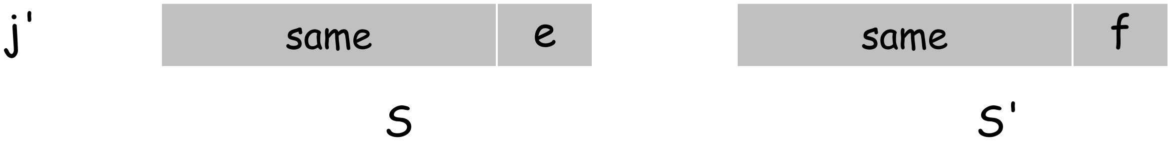
- Case 3b: g = f. Element f can't be in cache of S, so let e' be the element that S evicts.
  - if e' = e, S' accesses f from cache; now S and S' have same cache
  - if e' ≠ e, S' evicts e' and brings e into the cache; now S and S' have the same cache

Note: S' is no longer reduced, but can be transformed into a reduced schedule that agrees with S_{FF} through step j+1

Farthest-In-Future: Analysis

Let j' be the first time after j+1 that S and S' take a different action, and let g be item requested at time j'.

must involve e or f (or both)


j'

same | e

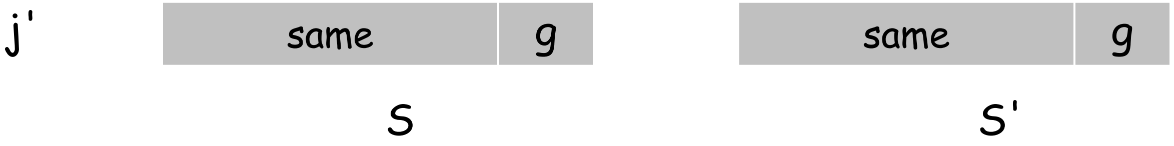
S

same | f

S'

otherwise S' would take the same action

- Case 3c: $g \neq e, f$. S must evict e.
  Make S' evict f; now S and S' have the same cache. ▪


j'

same | g

S

same | g

S'

Hence, in all these cases, we have a new reduced schedule S' that agrees with S_{FF} through the first $j + 1$ items and incurs no more misses than S does.

Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

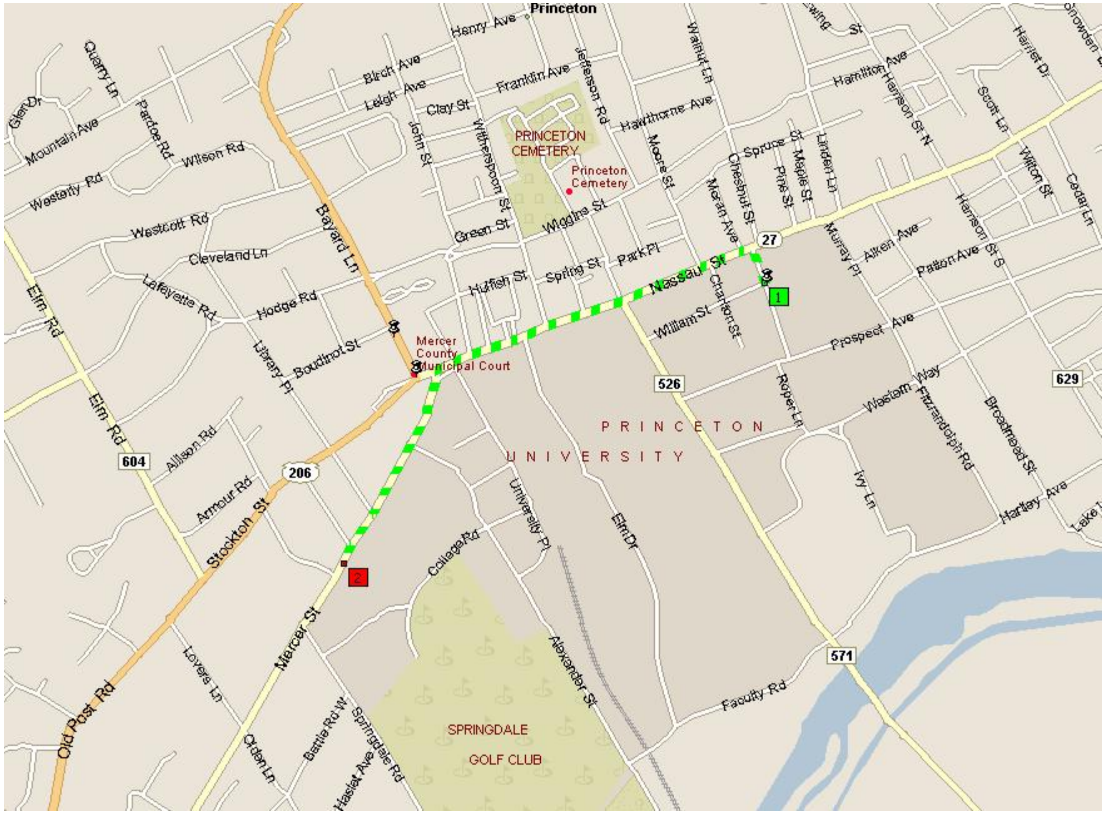
Least-Recently-Used (LRU). Evict page whose most recent access was earliest.

↑ FF with direction of time reversed!

Theorem. FF is an optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.

4.4 Shortest Paths in a Graph



shortest path from Princeton CS department to Einstein's house

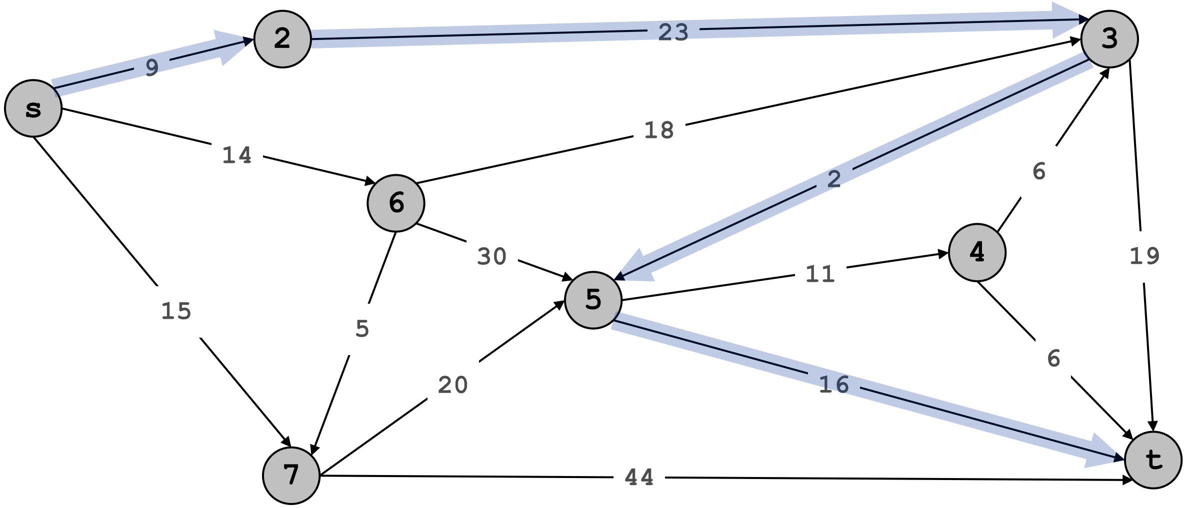
Shortest Path Problem

Shortest path network.

- Directed graph $G = (V, E)$.
- Source s, destination t.
- Length ℓ_e = length (cost) of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 50.

Dijkstra's Algorithm

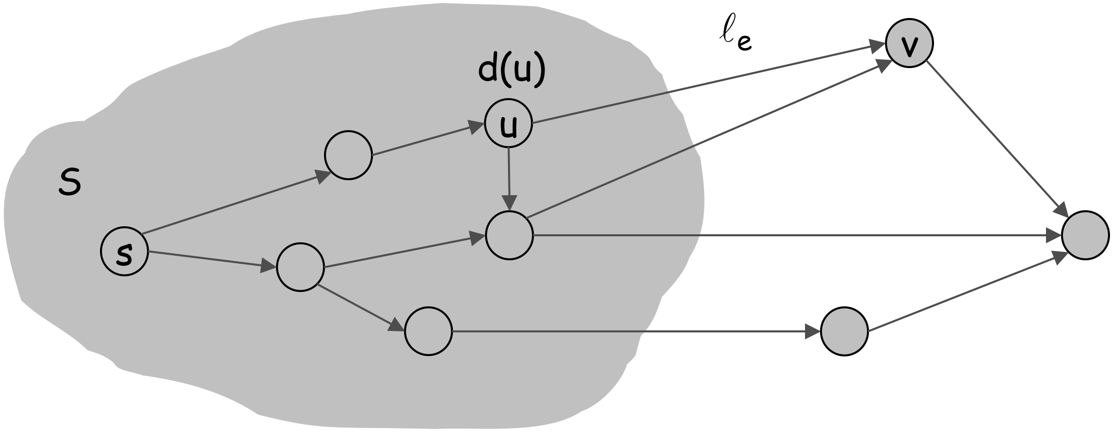
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



Dijkstra's Algorithm

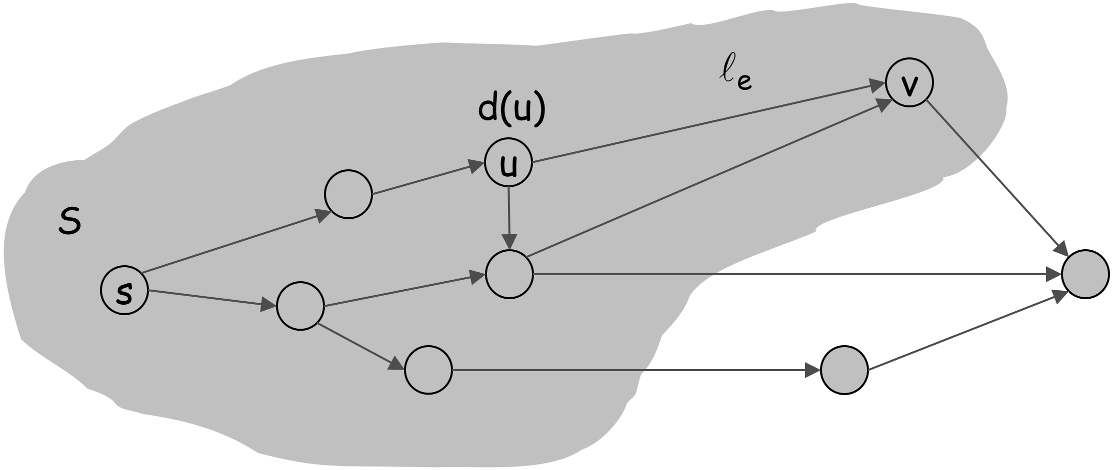
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e ,$$

shortest path to some u in explored part, followed by a single edge (u, v)

add v to S, and set d(v) = π(v).



Dijkstra's Algorithm: Proof of Correctness

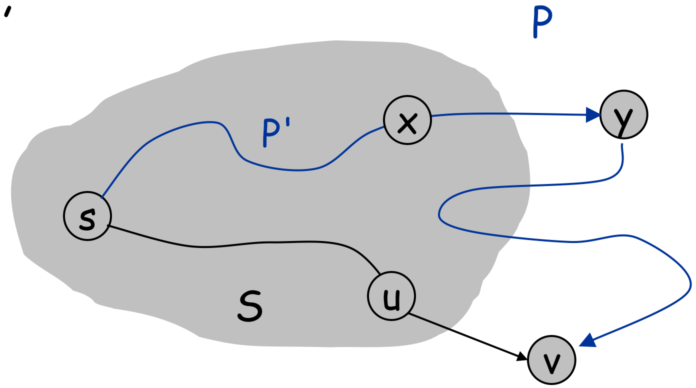
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s-u path.

Pf. (by induction on $|S|$)

Base case: $|S| = 1$ is trivial.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S, and let u-v be the chosen edge.
- The shortest s-u path plus (u, v) is an s-v path of length $\pi(v)$.
- Consider any other s-v path P. We'll see that it's no shorter than $\pi(v)$.
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S.



$$\underset{\substack{\uparrow\\ \text{nonnegative}\\ \text{weights}}}{\ell(P) \geq} \ell(P') + \ell(x,y) \underset{\substack{\uparrow\\ \text{inductive}\\ \text{hypothesis}}}{\geq} d(x) + \ell(x, y) \underset{\substack{\uparrow\\ \text{defn of } \pi(y)}}{\geq} \pi(y) \underset{\substack{\uparrow\\ \text{Dijkstra chose v}\\ \text{instead of y}}}{\geq} \pi(v)$$

Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e = (u,v)\,:\, u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update

$$\pi(w) = \min \{\ \pi(w),\ \ \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.