HPI Hasso Plattner Institut
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Summer Term 2017

Homework 3 – *Nature Inspired Algorithms*

https://hpi.de/friedrich/teaching/ss17/natinsalg.html

The goal of this homework is to review some of the basic concepts in the theoretical analysis of nature-inspired algorithms. The lecture slides for lectures 3 and 4 (posted to the Moodle) should be a helpful resource for this assignment. You will also need the concept of *conditional expectation*, which is outlined in the gray box below.

The homework is submitted on Moodle (https://hpi.de/friedrich/moodle/) by uploading a PDF file with your solutions. You are welcome to write your solutions out by hand and scan them.

> Given two events A and B from a probability space with $\Pr(B) > 0$, the conditional probability of A given B is defined as $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B)$. Roughly speaking, the *conditional expectation*[a] of a random variable is the expectation *given* that a certain event has occured. For example, let X be the random variable that corresponds to the value that comes up on the toss of a six-sided die. The expectation of X is
>
> $$E(X) = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega) = \sum_{i=1}^{6} i/6 = \frac{21}{6} = 3.5.$$
>
> Let $\mathcal{E}$ be the event that the die comes up with an even number. The expectation of X *conditioned* on $\mathcal{E}$ is
>
> $$\begin{aligned} E(X \mid \mathcal{E}) &= \sum_{\omega \in \Omega} X(\omega) \Pr(\omega \mid \mathcal{E}) = \sum_{\omega \in \Omega} X(\omega) \frac{\Pr(\{\omega\} \cap \mathcal{E})}{\Pr(\mathcal{E})} \\ &= \frac{(0 + 2/6 + 0 + 4/6 + 0 + 6/6)}{1/2} = 4. \end{aligned}$$
>
> We used the fact that $\Pr(\emptyset) = 0$.
>
> [a] http://en.wikipedia.org/wiki/Conditional_expectation

Theorem 1 (Law of Total Expectation) *Let $\mathcal{E}$ and $\mathcal{F}$ be mutually disjoint events and $\mathcal{E} \cup \mathcal{F} = \Omega$. Denote as $E(T \mid \mathcal{E})$ the expectation of T conditioned on event $\mathcal{E}$. Then $E(T) = E(T \mid \mathcal{E}) \Pr(\mathcal{E}) + E(T \mid \mathcal{F}) \Pr(\mathcal{F})$.*

Earlier in the semester, I claimed that there are functions on which the runtime of the (1+1) EA can be worse than the worst-case runtime of blind RandomSearch.

Define the function $g\colon \{0,1\}^n \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} 2 & \text{if } |x|_1 = n, \\ 0 & \text{if } 3n/4 \le |x|_1 < n, \\ 1 & \text{otherwise;} \end{cases}$$

Exercise 4 *Answer the following.*

(a) What is the expected runtime of RandomSearch *to maximize g?*

(b) Let $\mathcal{E}$ be the event that the initial solution of the (1+1) EA has at most $n/4$ zero bits. Derive an upper bound on $\Pr(\mathcal{E})$.
Hint: use one of the methods we discussed in Lecture 4.

(c) Suppose $\mathcal{E}$ has **not** *occurred. Give a lower bound on the expected time for the (1+1) EA to first generate the all-ones string under this condition.*

(d) Conclude that the expected runtime to maximize g for the (1+1) EA is worse than the runtime of RandomSearch *on g.*
Hint: use the law in Theorem 1.