1) $T(n) = T(n-3) + 3\lg(n)$

upper bound: $T(n) \le cn\lg(n)$

$\le c(n-3)\lg(n-3) + 3\lg(n)$

$= cn\lg(n-3) - 3c\lg(n-3) + \lg(n)$

for $n \ge 4$ $3c\lg(n-3) \ge 3\lg(n/2)$

$\le cn\lg(n) - 3c\lg(n) + 3c + \lg(n)$

$-3c\lg(n) + 3c + \lg(n) \le 0$

$c \le \frac{1}{3}\lg(n)(c - \frac{1}{3})$

$\lg(n) \ge \frac{3c}{c-1}$

$c = 4$, $n = 16$

lowe bound: $T(n) = \Omega(n\lg n)$ $T(n) \ge cn\lg n + dn$

$T(n) \ge T(n-3) + 3\lg(n)$

$\ge c(n-3)\lg(n-3) + d(n-3) + 3\lg(n)$

$= cn\lg(n-3) - 3c\lg(n-3) + dn - 3d + 3\lg(n)$

for $n \ge 4$ $\lg(n-1) \ge \lg(n/2)$

$\ge cn\lg(n/2) - 3c\lg(n-3) + dn - 3d + 3\lg(n)$

$T = \Theta(n\lg n)$, $T = \Omega(n\lg n)$

$T(n) = n\lg(n)$

$= cn\lg(n) - cn - 3c\lg(n-3) + dn - 3d + 3\lg(n)$

$\ge cn\lg(n)$

$\lg(n-3)$

if $-cn - 3c\lg(n-3) + dn - 3d + \lg(n) \ge 0$

$(d-c)n \ge (3c-1)\lg(n-3) + 3d$

$n = 4$, $c = 2$, $d = 1$

2) $T(n) = 4T(\frac{n}{3}) + n^2$ guess function $T(n) = O(n^{\log_3 4})$

$T(n) \leq cn^{\log_3 4}$ $c > 0$

upper bound

$T(n) = 4(c(\frac{n}{3})^{\log_3 4}) + n^2$

$= \frac{4}{3} c\, n^{\log_3 4} + n^2$

$\leq cn^{\log_3 4}$ $c > 1$

lower bound $T(n) \geq (c\, n^{\log_3 4}) + dn$

$T(n) \geq 4(c(\frac{n}{3})^{\log_3 4} + d(\frac{n}{3})) + n^2$

$= \frac{4}{3} cn^{\log_3 4} + \frac{4}{3} dn + \frac{3}{4}n^2$

$cn^{\log_3 4} = dn + \frac{3}{4}n^2 \geq 0$

Master:

$n^{\log_b a} = n^{\log_3 4}$ $f(n) = n^2$

$4(\frac{n}{3})^{2.5} \leq c(n^{2.5})$

$\frac{4}{3^{2.5}} = 0.256 \leq c < 1$

$T(n) = \Theta n^{\log_3 4} + \Theta n^2 = \Theta(n^2)$

3) $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n \qquad T(n) \le cn$

$T(n) = c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n$

$= \frac{7cn}{8} + n$

$= n(\frac{7c}{8} + 1)$

$\le cn \quad \exists\ c \geq 8$

$T(n) \geq \Omega(n) \qquad T(n) = \Theta(n)$

4) $T(n) = 4T(\frac{n}{2}) + n \qquad T(n) \le cn^2 \quad c > 0$

upper

$T(n) \le 4(c(\frac{n}{2})^2) + n$

$\le cn^2 + n \qquad n \geq 1 \quad c > 0$

lower bund:

$T(n) \geq 4(c(\frac{n}{2})^2 + d(\frac{n}{2})) + n$

$\geq cn^2 + 2dn + n$

$n(2d+1) \geq 0$

$d \geq 0 \quad n \geq 1$

Master: Case 1

$f(n) = n,\ n^2$

$n^{2-\epsilon} \geq n$

$\epsilon = 1$

$T(n) = \Theta(n^2) + \Theta(O(n^1))$

$= \Theta(n^2)$

$T(n) = O(n^2) = \Omega(n^2) = \Theta(n^2)$

5) $T(n) = 27(\frac{n}{3}) + n^3$ $\quad T(n) \leq Cn^3$

upper $T(n) \leq 27(C(\frac{n}{3})^3) + n^3$

$\leq Cn^3 + n^3$ $\quad C \geq 1 \quad n \geq 1$

lower:

$T(n) \geq 27(C(\frac{n}{3})^3 + d(\frac{n}{3})) + n^3$

$\geq Cn^3 + 9dn - n^3$

$9dn + n^3 \geq 0$

$n \geq 1 \quad d \geq 1$

$T(n) = \Theta(n^3) = \Omega(n^3) = \Theta(n^3)$

Master Case: 2

$f(n) = n^3 \quad n^{\log_b a} = n^3$

$\Theta(n^3) = n^3 \lg' n$

Part 2

1) $T(n) = 3(\frac{n}{2}) + n^2$

$n^{\log_2 3} < f(n) = n^2$

Case 3:

$n^{\log_2 3 + \epsilon} \leq n^2 \quad \epsilon \geq 0.415$

$3(\frac{n}{2})^2 \leq C(n^2)$

$\frac{3}{4} \leq C \leq 1$

$T(n) = \Theta(n^2)$

2) $T(n) = 2^n T(\frac{n}{2}) + n^n$

$n^{\log_2(2^n)} = n^n$ $\quad f(n) = n^n$

Case 2:

$\Theta(n^n) = \Theta(n^n \lg n) + \Theta(n^n) = \Theta(n^n \lg n)$

3) $T(n) = 3T(\frac{n}{4}) + n\lg n$

$n^{\lg_4 3} \le 1$ $\quad f(n) = n\lg n$

Case 3:

$n^{\lg_4 3 - \epsilon} \le 1$

$\epsilon \ge 0.207$

$\frac{3}{4}(\frac{n}{4}) \le cn\lg n$

$c \ge \frac{3}{4}$

$\Theta = \Theta(n\lg n) + \Theta(n^{\lg_4 3}) = \Theta(n\lg n)$

4) $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n}$

$n^{\log_2 2} = n^1$ $\quad f(n) = \frac{n}{\log(n)}$

It can't determine by master method when $2 \le n \le 4$ $f(n) > n$ but when $n = 4$ they are equal, and $n \ge 4$ $f(n)$ greater than $n^{\log_b a}$

5) $T(n) = 0.5\,T(\frac{n}{2}) + \frac{1}{n}$

$n^{\log_2 0.5} = n^{-1} \quad f(n) = n^{-1}$

Case 2:

$\Theta(n^{0-1}) = \Theta(n^{-1} \lg^0 n)$

$T(n) = \Theta(n^{-1}) + (\Theta(n^{ba} \lg^1 n) = \Theta(n^{-1} \lg n)$