Solution of Problem Set 3

Problem 1:

(a)

To find c

$$1 = \int_{-\infty}^{\infty} f_X(u)du = \int_0^{\infty} ce^{-4u}du$$
$$= \frac{c}{4}\Big[-e^{-4x}\Big]_0^{\infty} = \frac{c}{4}$$

Thus, we must have $c = 4$.

(b)

To find the CDF of X, we use $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, so for $x < 0$, we obtain $F_X(x) = 0$. For $x \geq 0$, we have

$$F_X(x) = \int_0^x 4e^{-4u}du = -\Big[e^{-4x}\Big]_0^x = 1 - e^{-4x}.$$

Thus,

$$F_X(x) = \begin{cases} 1 - e^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c)

We can find $P(2 < X < 5)$ using either the CDF or the PDF. If we use the CDF, we have

$$P(2 < X < 5) = F_X(5) - F_X(2) = \left[1 - e^{-20}\right] - \left[1 - e^{-8}\right] = e^{-8} - e^{-20}.$$

Equivalently, we can use the PDF. We have

$$P(2 < X < 5) = \int_2^5 f_X(t)dt =$$
$$\int_2^5 4e^{-4t}dt = e^{-8} - e^{-20}.$$

(d)

As we saw, the PDF of X is given by

$$f_X(x) = \begin{cases} 4e^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so to find its expected value, we can write:

$$
\begin{aligned}
EX &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_{0}^{\infty} x(4e^{-4x})dx \\
&= \Big[-xe^{-4x}\Big]_0^{\infty} + \int_0^{\infty} e^{-4x}dx \\
&= 0 + \Big[-\frac{1}{4}e^{-4x}\Big]_0^{\infty} = \frac{1}{4}
\end{aligned}
$$

Problem2:

We have

$$
\begin{aligned}
P(X \leq \tfrac{2}{3} | X > \tfrac{1}{3}) &= \frac{P(\frac{1}{3} < X \leq \frac{2}{3})}{P(X > \frac{1}{3})} \\
&= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 dx}{\int_{\frac{1}{3}}^{1} 4x^3 dx} \\
&= \frac{3}{16}.
\end{aligned}
$$

Problem3:

(a)

Using LOTUS we have

$$
\begin{aligned}
E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x)dx \\
&= \int_0^1 x^n(x^2 + \frac{2}{3})dx \\
&= \int_0^1 (x^{n+2} + \frac{2}{3}x^n)dx \\
&= \left[\frac{1}{n+3}x^{n+3} + \frac{2}{3(n+1)}x^{n+1}\right]_0^1 \\
&= \frac{1}{n+3} + \frac{2}{3(n+1)} \\
&= \frac{5n+9}{3(n+1)(n+3)}. \quad \text{where } n = 1, 2, 3, \cdots
\end{aligned}
$$

(b)

We know that

$$\mathrm{Var}(X) = EX^2 - (EX)^2.$$

So we need to find the values of EX and EX^2

$$E[X] = \frac{7}{12}$$

$$E[X^2] = \frac{19}{45}$$

Thus, we have

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{19}{45} - (\frac{7}{12})^2 = 0.0819.$$

Problem 4:

Solution: First, note that we already know the CDF and PDF of X. In particular,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

It is a good idea to think about the range of Y before finding the distribution. Since e^{-x} is a decreasing function of x and $R_X = [0, 1]$, we conclude that $R_Y = [e^{-1}, 1]$. So we immediately know that

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = 0, \qquad \text{for } y < e^{-1} \\ F_Y(y) &= P(Y \leq y) = 1, \qquad \text{for } y \geq 1. \end{aligned}$$

(a) To find $F_Y(y)$ for $y \in [e^{-1}, 1]$, we can write

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(e^{-X} \leq y) \\ &= P(-X \leq \ln y) \quad \text{(since } e^{-x} \text{ is a decreasing function)} \\ &= P(X \geq -\ln y) \\ &= 1 - P(X \leq -\ln y) \\ &= 1 - F_X(\ln \frac{1}{y}) = 1 + \ln(y) \end{aligned}$$

$$\text{(since } 0 \leq \ln \frac{1}{y} \leq 1 \text{ we can write } F_X(\ln \frac{1}{y}) = \ln \frac{1}{y} = -\ln(y))$$

To summarize

$$F_Y(y) = \begin{cases} 0 & \text{for } y < \frac{1}{e} \\ 1 + \ln(y) & \text{for } \frac{1}{e} \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

(b) The above CDF is a continuous function, so we can obtain the PDF of Y by taking its derivative. We have

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{y} & \text{for } e^{-1} \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) To find the EY, we can directly apply LOTUS,

$$\begin{aligned} E[Y] = E[e^{-X}] &= \int_{-\infty}^{\infty} e^{-x} f_X(x)dx \\ &= \int_0^1 e^{-x}dx \\ &= 1 - e^{-1}. \end{aligned}$$

For this problem, we could also find EY using the PDF of Y,

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y)dy \\ &= \int_{e^{-1}}^{1} y \frac{1}{y} dy \\ &= 1 - e^{-1}. \end{aligned}$$

Problem 5:

(a) First, we note that $R_Y = [0, 4]$. As usual, we start with the CDF. For $y \in [0, 4]$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(0 \leq X \leq \sqrt{y}) \quad \text{since } x \text{ is not negative} \\ &= \int_0^{\sqrt{y}} \frac{5}{32} x^4 dx \\ &= \frac{1}{32} (\sqrt{y})^5 \\ &= \frac{1}{32} y^2 \sqrt{y} \end{aligned}$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{1}{32} y^2 \sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 1 & \text{for } y > 4. \end{cases}$$

(b)

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{5}{64} y \sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(c)

To find the EY, we can directly apply LOTUS,

$$\begin{aligned} E[Y] = E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^2 x^2 \cdot \frac{5}{32} x^4 dx \\ &= \int_0^2 \frac{5}{32} x^6 dx \\ &= \frac{5}{32} \times \frac{1}{7} \times 2^7 = \frac{20}{7}. \end{aligned}$$

Problem6:

Solution: Since $X \sim Exponential(\lambda)$, we have $F_X(x) = \big(1 - e^{-\lambda x}\big) u(x)$. To show $Y \sim Exponential(\frac{\lambda}{a})$, it suffices to show that

$$F_Y(y) = \big(1 - e^{-\frac{\lambda}{a} y}\big) u(y).$$

We have

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P\left(aX \leq y\right) \\
&= P(X \leq \frac{y}{a}) \\
&= \left(1 - e^{-\lambda \frac{y}{a}}\right) u(\frac{y}{a}) \\
&= \left(1 - e^{-\frac{\lambda}{a} y}\right) u(y).
\end{aligned}$$

Problem7:

Solution:

(a) We use integration by part (choosing $u = x^n$ and $v = -e^{-\lambda x}$)

$$\begin{aligned}
EX^n &= \int_0^\infty x^n \lambda e^{-\lambda x} \, dx \\
&= \left[-x^n e^{-\lambda x}\right]_0^\infty + n \int_0^\infty x^{n-1} e^{-\lambda x} \, dx \\
&= 0 + \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} \, dx \\
&= \frac{n}{\lambda} EX^{n-1}.
\end{aligned}$$

(b) We can prove this by induction using part (a). Note that for $n = 1$, we have

$$EX = \frac{1}{\lambda} = \frac{1!}{\lambda^1}.$$

Now, if we have $EX^n = \frac{n!}{\lambda^n}$, we can write

$$\begin{aligned}
EX^{n+1} &= \frac{n+1}{\lambda} EX^n \\
&= \frac{n+1}{\lambda} \cdot \frac{n!}{\lambda^n} \\
&= \frac{(n+1)!}{\lambda^{n+1}}.
\end{aligned}$$

Problem8:

(a) Find $P(X > 0)$:

$$\begin{aligned} P(X > 0) &= 1 - P(X \leq 0) \\ &= 1 - F_X(0) \\ &= 1 - \Phi\left(\frac{0-3}{3}\right) \\ &= 1 - \Phi(-1) = \Phi(1) \qquad \big(\text{since } \Phi(-x) = 1 - \Phi(x)\big) \end{aligned}$$

(b) Find $P(-3 < X < 8)$:

$$\begin{aligned} P(-3 < X < 8) &= F_X(8) - F_X(-3) \\ &= \Phi\left(\frac{8-3}{3}\right) - \Phi\left(\frac{(-3)-3}{3}\right) \\ &= \Phi(\frac{5}{3}) - \Phi(-2) \\ &= \Phi(\frac{5}{3}) + \Phi(2) - 1 \qquad \big(\text{since } \Phi(-x) = 1 - \Phi(x)\big) \end{aligned}$$

(c) Find $P(X > 5|X > 3)$:

$$\begin{aligned} P(X > 5|X > 3) &= \frac{P(X > 5, X > 3)}{P(X > 3)} \\ &= \frac{P(X > 5)}{P(X > 3)} \\ &= \frac{1 - \Phi\left(\frac{5-3}{3}\right)}{1 - \Phi\left(\frac{3-3}{3}\right)} \\ &= \frac{1 - \Phi(\frac{2}{3})}{1 - \Phi(0)} \\ &= 2 \times (1 - \Phi(\frac{2}{3})) \end{aligned}$$

Problem 9:

(a) Find $P(X > 2)$: We have $\mu_X = 3$ and $\sigma_X = 3$. Thus,

$$P(X > 2) = 1 - \Phi\left(\frac{2-3}{3}\right)$$
$$= 1 - \Phi(\frac{-1}{3}) = \Phi(\frac{1}{3})$$

(b) Find $P(-1 < Y < 3)$: Since $Y = 5 - X$, we have $Y \sim N(2, 9)$. Therefore,

$$P(-1 < Y < 3) = \Phi\left(\frac{3-2}{3}\right) - \Phi\left(\frac{(-1)-2}{3}\right)$$
$$= \Phi(\frac{1}{3}) - \Phi(-1).$$

(c) Find $P(X > 4|Y < 2)$:

$$\begin{aligned} P(X > 4|Y < 2) &= P(X > 4|5 - X < 2) \\ &= P(X > 4|X > 3) \\ &= \frac{P(X > 4, X > 3)}{P(X > 3)} \\ &= \frac{P(X > 4)}{P(X > 3)} \\ &= \frac{1 - \Phi(\frac{4-3}{3})}{1 - \Phi(\frac{3-3}{3})} \\ &= \frac{1 - \Phi(\frac{1}{3})}{1 - \Phi(0)} \\ &= 2(1 - \Phi(\frac{1}{3})) \end{aligned}$$

Problem 10:

a. What kind of random variable is X: discrete, continuous, or mixed? We note that the CDF has a discontinuity at $x = 0$, and it is continuous at other points. Since $F_X(x)$ is not flat in other locations, we conclude X is a mixed random variable. Indeed, we can write

$$F_X(x) = \frac{1}{2}u(x) + \frac{1}{2}F_Y(x),$$

where Y is a $Uniform(0, 1)$ random variable. If we use the interpretation of Problem 1, we can say the following. We toss a fair coin. If it lands heads then $X = 0$, otherwise X is obtained according the a $Uniform(0, 1)$ distribution.

b. Find the PDF of X, $f_X(x)$: By differentiating the CDF, we obtain

$$f_X(x) = \frac{1}{2}\delta(x) + \frac{1}{2}f_Y(x),$$

where $f_Y(x)$ is the PDF of $Uniform(0, 1)$, i.e.,

$$f_Y(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

c. Find $E(e^X)$: We can use LOTUS to write

$$\begin{aligned} E(e^X) &= \int_{-\infty}^{\infty} e^x f_X(x)dx \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^x \delta(x)dx + \tfrac{1}{2}\int_{-\infty}^{\infty} e^x f_Y(x)dx \\ &= \tfrac{1}{2}e^0 + \tfrac{1}{2}\int_0^1 e^x dx \\ &= \tfrac{1}{2} + \tfrac{1}{2}(e-1) \\ &= \tfrac{1}{2}e. \end{aligned}$$

Here is another way to think about this part: similar to part (c) of Problem 1, we can write

$$\begin{aligned} E(e^X) &= \tfrac{1}{2} \times e^0 + \tfrac{1}{2}E[e^Y] \\ &= \tfrac{1}{2} + \tfrac{1}{2}\int_0^1 e^y dy \\ &= \tfrac{1}{2}e. \end{aligned}$$

d. Find $P(X=0|X \leq 0.5)$: We have

$$\begin{aligned} P(X=0|X \leq 0.5) &= \frac{P(X=0, X\leq 0.5)}{P(X\leq 0.5)} \\ &= \frac{P(X=0)}{P(X\leq 0.5)} \\ &= \frac{0.5}{\int_0^{0.5} f_X(x)dx} \\ &= \frac{0.5}{0.75} = \frac{2}{3}. \end{aligned}$$

Problem 11:

(a) Find $P(X > 2)$:

$$\begin{aligned} P(X>2) &= 1 - P(X \leq 2) \\ &= 1 - F_X(2) = 1 - (1 - e^{-4}) = e^{-4} \end{aligned}$$

(b) Find EY:

Since $Y = 2 + 3X$,

we have $EY = 2 + 3EX = 2 + 3 \times \frac{1}{2} = \frac{7}{2}$.

$\mathrm{Var}(Y) = \mathrm{Var}(2+3X) = 9 \times \mathrm{Var}(X) = 9 \times \frac{1}{4} = \frac{9}{4}$

(c) Find $P(X > 2|Y < 11)$:

$$\begin{aligned} P(X>2|Y<11) &= P(X>2|2+3X<11) \\ &= P(X>2|X<3) \\ &= \frac{P(X>2, X<3)}{P(X<3)} \\ &= \frac{P(2<X<3)}{P(X<3)} \\ &= \frac{e^{-4} - e^{-6}}{1 - e^{-6}} \end{aligned}$$

Problem 12:

Note that $R_Y = [0,1]$. Therefore,

$$F_Y(y) = 0, \quad \text{for} \quad y < 0,$$
$$F_Y(y) = 1, \quad \text{for} \quad y \geq 1.$$

We also note that

$$P(Y = 0) = P(X < 0) = \frac{1}{2},$$
$$P(Y = 1) = P(X > 1) = \frac{1}{4}.$$

Also for $0 < y < 1$,

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y) = \frac{y+2}{4}.$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 1 & y \geq 1 \\ \frac{y+2}{4} & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

In particular, we note that there are two jumps in the CDF, one at $y = 0$ and another at $y = 1$. We can find the generalized PDF of Y by differentiating $F_Y(y)$:

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{4}\delta(y-1) + \frac{1}{4}(u(y) - u(y-1)).$$

Problem 13:

(a)

X is a mixed random variable because CDF is not a continuous function nor in the form of a staircase function.

(b)

$$P(X \leq \frac{1}{3}) = F_X(\frac{1}{3}) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

(c)

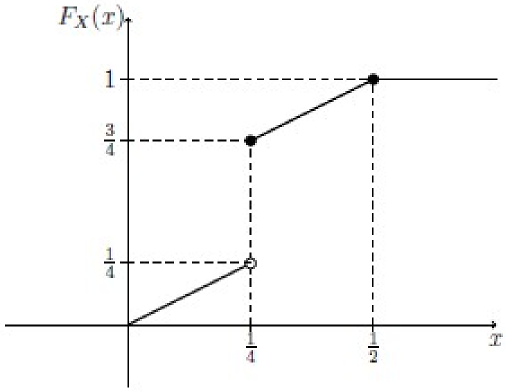


Figure 1: CDF of the Mixed random variable

$$
\begin{aligned}
P(X \geq \frac{1}{4}) &= 1 - P(X < \frac{1}{4}) \\
&= 1 - P(X \leq \frac{1}{4}) + P(X = \frac{1}{4}) \\
&= 1 - F_X(\frac{1}{4}) + \frac{1}{2} = 1 - \frac{3}{4} + \frac{1}{2} = \frac{3}{4}
\end{aligned}
$$

(d)

We can write:

$F_X(x) = C(x) + D(x)$

where

$$
C(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \text{for } x \geq \frac{1}{2} \end{cases}
$$

and

$$
D(x) = \begin{cases} 0 & \text{for } x < \frac{1}{4} \\ \frac{1}{2} & \text{for } x \geq \frac{1}{4} \end{cases}
$$

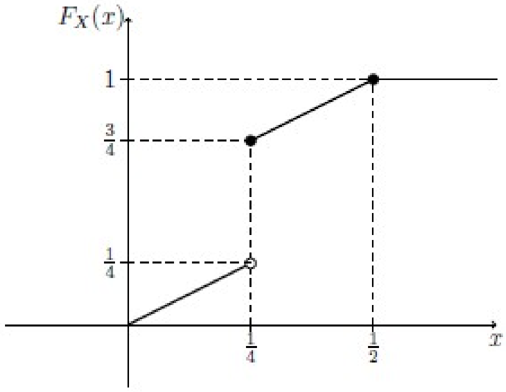


Figure 1: CDF of the Mixed random variable

$$\begin{aligned}
P(X \geq \frac{1}{4}) &= 1 - P(X < \frac{1}{4}) \\
&= 1 - P(X \leq \frac{1}{4}) + P(X = \frac{1}{4}) \\
&= 1 - F_X(\frac{1}{4}) + \frac{1}{2} = 1 - \frac{3}{4} + \frac{1}{2} = \frac{3}{4}
\end{aligned}$$

(d)

We can write:

$F_X(x) = C(x) + D(x)$

where

$$C(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \text{for } x \geq \frac{1}{2} \end{cases}$$

and

$$D(x) = \begin{cases} 0 & \text{for } x < \frac{1}{4} \\ \frac{1}{2} & \text{for } x \geq \frac{1}{4} \end{cases}$$

Problem 14:

(a)

We can find $f_X(x)$ by differentiating $F_X(x)$. We must pay special attention to the jumps in $F_X(x)$:

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ 0 & \text{else} \end{cases} + \frac{1}{2}\delta(x - \frac{1}{4})$$

Thus, we can write:

$f_X(x) = \frac{1}{2}\delta(x - \frac{1}{4}) + (u(x) - u(x - \frac{1}{2}))$

(b)

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} \frac{1}{2} x \delta(x - \frac{1}{4}) dx + \int_0^{\frac{1}{2}} x dx \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{8} = \frac{1}{4} \end{aligned}$$

(c)

$$\begin{aligned} EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2} x^2 \delta(x - \frac{1}{4}) + x^2(u(x) - u(x - \frac{1}{2})) \right) dx \\ &= \frac{1}{2} \times (\frac{1}{4})^2 + \int_0^{\frac{1}{2}} x^2 dx \\ &= \frac{1}{32} + \frac{1}{3} \times \frac{1}{8} = \frac{7}{96}. \end{aligned}$$

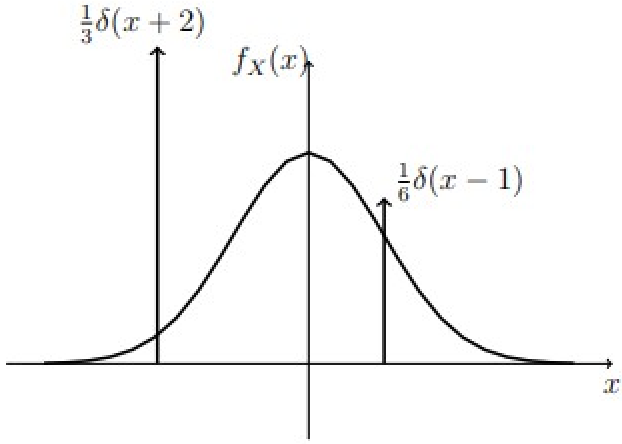
Thus,

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 \\ &= \frac{7}{96} - \left(\frac{1}{4}\right)^2 \\ &= \frac{1}{96} \end{aligned}$$

Problem 15:

Note that $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the PDF of a standard normal random variable.

So we can plot the PDF of X as follows:



(a)
$$P(X=1)=\frac{1}{6} \qquad P(X=-2)=\frac{1}{3}$$

(b)
$$\begin{aligned} P(X \geq 1) &= P(X=1)+\int_1^\infty \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{6}+\frac{1}{2}\left[1-\phi\left(\frac{1-0}{1}\right)\right] \\ &= \frac{1}{6}+\frac{1}{2}\left[1-\phi(1)\right] \\ &= \frac{1}{6}+\frac{1}{2}\phi(-1) \end{aligned}$$

(c)
$$\begin{aligned} P(X=1|X \geq 1) &= \frac{P(X=1 \text{ and } X \geq 1)}{P(X \geq 1)} \\ &= \frac{P(X=1)}{P(X \geq 1)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{2}\phi(-1)} \end{aligned}$$

(d)
$$EX = \frac{1}{6}\cdot 1 + \frac{1}{3}\cdot(-2) + \frac{1}{2}EZ \quad \text{where } Z \sim N(0,1)$$

Thus,

$$EX = \frac{1}{6} - \frac{2}{3} + 0 = -\frac{1}{2}$$

$$\begin{aligned}
EX^2 &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{3}x^2\delta(x+2) + \frac{1}{6}x^2\delta(x-1) + \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}x^2 e^{-\frac{x^2}{2}}\right)dx \\
&= \frac{1}{3}\cdot(-2)^2 + \frac{1}{6}\cdot 1^2 + \frac{1}{2}EZ^2 \quad \text{where } Z \sim N(0,1) \\
&= \frac{4}{3} + \frac{1}{6} + \frac{1}{2} = 2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= EX^2 - (EX)^2 \\
&= 2 - \left(\frac{1}{2}\right)^2 \\
&= \frac{7}{4}
\end{aligned}$$

Problem 16:

(a)
$$f_X(x) = \frac{2}{100}\delta(x) + \frac{98}{100}f_Z(x) \quad \text{where } f_Z(x) \text{ is the} \sim Exponential(\lambda = 2) \text{ PDF}$$

Thus:

$$\begin{aligned} f_X(x) &= \frac{1}{50}\delta(x) + \frac{49}{50} \cdot 2e^{-2x}u(x) \\ &= \frac{1}{50}\delta(x) + \frac{49}{25} \cdot e^{-2x}u(x) \end{aligned}$$

(b)
$$P(X \geq 1) = \int_1^\infty f_X(x)dx = \frac{49}{50}\int_1^\infty f_Z(x)dx = \frac{49}{50}e^{-2}$$

(c)
$$\begin{aligned} P(X > 2 | X \geq 1) &= \frac{P(X > 2 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X > 2)}{P(X \geq 1)} \\ &= \frac{\frac{49}{50}e^{-2\times 2}}{\frac{49}{50}e^{-2\times 1}} = e^{-2} \end{aligned}$$

(d)
$$\begin{aligned} EX &= \frac{1}{50} \cdot 0 + \frac{49}{50} \cdot EY \quad \text{where } Y \sim Exponential(\lambda = 2) \\ &= \frac{49}{50} \cdot \frac{1}{2} = 0.49 \end{aligned}$$

$$\text{Var}(X) = EX^2 - (EX)^2 = EX^2 - (0.49)^2$$

$$- (0.49)^2$$

$$\begin{aligned} EX^2 &= \frac{1}{50} \cdot 0 + \frac{49}{50} \cdot EY^2 \\ &= \frac{49}{50}\left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\right) \quad \text{where } \lambda = 2 \\ &= \frac{49}{50}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{49}{50} \end{aligned}$$

Thus:

$$\text{Var}(X) = \left(\frac{1}{2} \cdot \frac{49}{50}\right) - (0.49)^2 = 0.2499$$

Problem 17:

(a) $X \sim Laplace(0,1)$, so:

$$f_X(x) = \frac{1}{2}e^{-|x|} = \begin{cases} \frac{1}{2}e^{x} & \text{for } x < 0 \\ \frac{1}{2}e^{-x} & \text{for } x \geq 0 \end{cases}$$

Thus:

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} x f_X(x)dx = \frac{1}{2}\int_{-\infty}^{0} xe^{x}dx + \frac{1}{2}\int_{0}^{\infty} xe^{-x}dx \\ &= -\frac{1}{2}\int_{0}^{\infty} ye^{-y}dy + \frac{1}{2}\int_{0}^{\infty} xe^{-x}dx = 0 \quad (\text{let } y = -x) \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 = EX^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}dx = \int_{0}^{\infty} x^2 e^{-x}dx = 2 \end{aligned}$$

(b)

$Y = g(X)$ where $g(X) = bX + \mu$, $g'(X) = b$

$$f_Y(y) = \frac{f_X(\frac{y-\mu}{b})}{b} = \frac{1}{2b}\exp(-|\frac{y-\mu}{b}|)$$

Thus:

$Y \sim Laplace(\mu, b)$.

You can also show this by starting from the CDF:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(bX + \mu \leq y) \\ &= P(X \leq \frac{y-\mu}{b}) \\ &= F_X(\frac{y-\mu}{b}). \end{aligned}$$

Thus

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{f_X(\frac{y-\mu}{b})}{b} = \frac{1}{2b}\exp(-|\frac{y-\mu}{b}|). \end{aligned}$$

(c)

We can write $Y = bX + \mu$, where $X \sim Laplace(0,1)$

Thus by part (a), $EX = 0$ and $\text{Var}(X) = 2$

$$\begin{aligned} EY &= bEX + \mu = \mu \\ \text{Var}(Y) &= b^2\text{Var}(X) = 2b^2 \end{aligned}$$

Problem 18:

$X \sim Laplace(0, b)$, so:

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y)$$

Thus $F_Y(y) = 0$ for $y < 0$.
For $y \geq 0$, we have:

$$\begin{aligned} F_Y(y) &= P(|X| \leq y) = P(-y \leq X \leq y) \\ &= F_X(y) - F_X(-y) \end{aligned}$$

We can find the CDF of X as:

$$F_X(x) = \begin{cases} \frac{1}{2}e^{(\frac{x}{b})} & \text{for } x < 0 \\ 1 - \frac{1}{2}e^{-(\frac{x}{b})} & \text{for } x \geq 0 \end{cases}$$

Thus for $y \geq 0$:

$$\begin{aligned} F_Y(y) &= F_X(y) - F_X(-y) \\ &= 1 - \frac{1}{2}e^{-(\frac{y}{b})} - \frac{1}{2}e^{(\frac{-y}{b})} \\ &= 1 - e^{-(\frac{y}{b})} \end{aligned}$$

Thus:

$$F_Y(y) = \begin{cases} 1 - e^{-(\frac{y}{b})} & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude $Y \sim Exponential(\frac{1}{b})$

Problem19:

Solution: Since T is exponentially distributed with mean 3, the density of T is $f(t) = (1/3)e^{-t/3}$ for $t > 0$. Since $X = \max(T, 2)$, we have $X = 2$ if $0 \leq T \leq 2$ and $X = T$ if $2 < T < \infty$.

Thus,

$$\begin{aligned} E(X) &= \int_0^2 2\frac{1}{3}e^{-t/3}dt + \int_2^\infty t \cdot \frac{1}{3}e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) - te^{-t/3}\Big|_2^\infty + \int_2^\infty e^{-t/3}dt \\ &= 2(1 - e^{-2/3}) + 2e^{-t/3} + 3e^{-2/3} = 2 + 3e^{-2/3} \end{aligned}$$

Problem20:

(a)

$$\begin{aligned} EX(x) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{x^2}{\sigma^2} e^{-x^2/2\sigma^2} dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{\sigma^2} e^{-x^2/2\sigma^2} dx = \frac{\sqrt{2\pi}}{2\sigma} \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx \\ &= \frac{\sqrt{2\pi}}{2\sigma} EZ^2 \qquad \text{where } Z \sim N(0,\sigma) \\ &= \frac{\sqrt{2\pi}}{2\sigma} \sigma^2 = \sqrt{\frac{\pi}{2}} \sigma \end{aligned}$$

(b)

$F_X(x) = 0$ for $x < 0$.

For $x > 0$:

$$\begin{aligned} F_X(x) &= \int_0^x \frac{u}{\sigma^2} e^{-u^2/2\sigma^2} du \\ &= \int_0^{\frac{x^2}{2\sigma^2}} e^{-t} dt = 1 - e^{-\frac{x^2}{2\sigma^2}} \end{aligned}$$

(c)

$Y = \sqrt{2\sigma^2 X}$ and $X \sim Exponential(1)$.

So, we know:

$$\begin{aligned} f_X(x) &= e^{-x} u(x) \\ F_X(x) &= (1 - e^{-x}) u(x) \end{aligned}$$

So, we conclude $R_Y = [0, \infty)$

For $y < 0$, we conclude that $F_Y(y) = 0$

For $y > 0$:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\sqrt{2\sigma^2 X} \leq y) = P(2\sigma^2 X \leq y^2) \\ &= P(X \leq \frac{y^2}{2\sigma^2}) = (1 - e^{-\frac{y^2}{2\sigma^2}}) u(x) \end{aligned}$$

which is the CDF of the Rayleigh distribution.

So, $Y \sim Rayleigh(\sigma)$

Problem21:

(a)

$$f_X(x) = \begin{cases} \alpha \frac{x_m^\alpha}{x^{\alpha+1}} & \text{for } x \geq x_m, \\ 0 & \text{for } x < x_m. \end{cases}$$

Note that $R_X = [x_m, \infty)$,

Thus, $F_X(x) = 0$ for $x < x_m$

For $x \geq x_m$:

$$\begin{aligned} F_X(x) &= \int_{x_m}^x \alpha \frac{x_m^\alpha}{x^{\alpha+1}} dx \\ &= \left[-\frac{x_m^\alpha}{x^\alpha} \right]_{x_m}^x = 1 - \left(\frac{x_m}{x}\right)^\alpha \end{aligned}$$

Thus:

$$F_X(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\alpha & \text{for } x \geq x_m \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} P(X > 3x_m | X > 2x_m) &= \frac{P(X > 3x_m \text{ and } X > 2x_m)}{P(X > 2x_m)} \\ &= \frac{P(X > 3x_m)}{P(X > 2x_m)} = \frac{\left(\frac{x_m}{3x_m}\right)^\alpha}{\left(\frac{x_m}{2x_m}\right)^\alpha} = \left(\frac{2}{3}\right)^\alpha \end{aligned}$$

Problem22:

$Z \sim N(0,1)$, $X = e^{\sigma Z+\mu}$

(a)

First note that $R_X = (0, \infty)$

For $x > 0$:

$$\begin{aligned}F_X(x) &= P(X < x)\\ &= P(e^{\sigma Z+\mu} < x) = P(\sigma Z + \mu < \ln x)\\ &= P(Z < \frac{\ln x - \mu}{\sigma}) = \phi(\frac{\ln x - \mu}{\sigma})\end{aligned}$$

(b)

$X = e^{\sigma Z+\mu}$

$$\begin{aligned}EX &= E[e^{\sigma Z+\mu}] = e^{\mu}E[e^{\sigma Z}]\\ &= e^{\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sigma u}e^{-\frac{u^2}{2}}du\\ &= e^{\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(u-\sigma)^2}e^{\frac{\sigma^2}{2}}du\\ &= e^{\mu+\frac{\sigma^2}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u-\sigma)^2}du\\ &= e^{\mu+\frac{\sigma^2}{2}} \qquad \text{because it is the integral of } N(\sigma,1) = 1\end{aligned}$$

$$EX^2 = E[e^{2\sigma Z+2\mu}] = e^{2\mu}e^{\frac{(2\sigma)^2}{2}} = e^{2\mu+2\sigma^2} \quad \text{similar to above}$$

Thus:

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

Problem 23:

$$Y = X_1 + X_2 + \cdots + X_n.$$

where $X_i \sim Exponential(\lambda)$

Thus:

$$\begin{aligned}EY &= EX_1 + EX_2 + \cdots + EX_n\\ &= \frac{1}{\lambda} + \frac{1}{\lambda} + \cdots + \frac{1}{\lambda} \quad \text{since } X_i \sim Exponential(\lambda)\\ &= \frac{n}{\lambda}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n) \quad \text{since } X_i\text{'s are independent}\\ &= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^2}\\ &= \frac{n}{\lambda^2}\end{aligned}$$

Problem24:

(1) . $f_X(x) \geq 0 \Rightarrow a\, e^{-2a|x|} \geq 0 \Rightarrow a \geq 0$

. $\int_{-\infty}^{+\infty} f(x)\, dx = 1 \Leftrightarrow \int_{-\infty}^{+\infty} a\, e^{-2a|x|}\, dx = 1$

$$2a \int_0^{+\infty} e^{-2ax} = 1 \underset{a \neq 0}{\Leftrightarrow} 2a \frac{-1}{2a} e^{-2ax}\Big|_0^{+\infty} = 1$$

So to be a valid PDF $a > 0$.

(2) $$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \underbrace{\underbrace{x}_{\text{odd functi}} \times \underbrace{a\, e^{-2a|x|}}_{\text{even function of } x}}_{\text{odd function}}\, dx \Rightarrow \boxed{E[X] = 0}$$

(4) $$\text{std}[X] = \sqrt{\text{Var}[X]} = \sigma_X = \frac{1}{\sqrt{2}\, a} \Rightarrow \boxed{\sigma_X = \frac{1}{\sqrt{2}\, a}}$$

(5) $$\Pr[-2 < X < 2] = \int_{-2}^{2} f_X(x)\, dx = \int_{-2}^{2} a\, e^{-2a|x|}\, dx$$

$$= 2a \int_0^2 e^{-2ax}\, dx = 2a \frac{1}{-2a} e^{-2ax}\Big|_0^{+2}$$

$$= -1\left[e^{-4a} - e^{0}\right] = 1 - e^{-4a}$$

$$\boxed{\Pr[-2 < X < 2] = 1 - e^{-4a}}$$

Problem25:

a) f_X valid $\Rightarrow \int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

$= \int_{-\infty}^{0} k e^{ax}\,dx + \int_{0}^{+\infty} k\, e^{-bx}\,dx$

$= k\left(\int_{-\infty}^{0} e^{ax}\,dx + \int_{0}^{+\infty} e^{-bx}\,dx\right)$

$= k\left(\left[\frac{1}{a}e^{ax}\right]_{-\infty}^{0} + \left[\frac{-1}{b}e^{-bx}\right]_{0}^{+\infty}\right)$

$= k\left(\frac{1}{a} + \frac{1}{b}\right) = 1$

$\Rightarrow \quad k = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = \frac{ab}{a+b}$

b) $F_X(x) = P(X \leq x)$

if $x \leq 0$

$F_X(x) = \int_{-\infty}^{x} k\, e^{ax}\,dx$

$= \frac{k}{a}\left[e^{ax}\right]_{-\infty}^{x} = \frac{k}{a} e^{ax}$

if $x \geq 0$

$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$= \int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{x} f_X(x)\,dx$

$= \int_{-\infty}^{0} k e^{ax}\,dx + \int_{0}^{x} k\, e^{-bx}\,dx$

$= \frac{k}{a} + \left[\frac{-k}{b}e^{-bx}\right]_{0}^{x}$

$= \frac{k}{a} + \frac{k}{b} - \frac{k}{b}e^{-bx} = k\left(\frac{1}{a} + \frac{1}{b} - \frac{e^{-bx}}{b}\right)$

$$\Rightarrow F_X(x) = \begin{cases} \frac{k}{a}e^{ax} = \frac{b}{a+b}e^{ax} & \text{if } x \leq 0 \\ k\left(\frac{1}{a} + \frac{1}{b} - \frac{e^{-bx}}{b}\right) = 1 - \frac{a}{a+b}e^{-bx} & \text{if } x \geq 0 \end{cases}$$

Problem 26:

```
import numpy as np

M = 1000
x = np.random.randn(M)
count = 0
for i in range(M):
   if x[i] >= 1 and x[i] <= 2:
      count = count + 1

print(count)
```

Problem 27:

```
from scipy import stats

X = stats.norm(9, 0.9) # define the normal RV

# (a)
1-X.cdf(10) # 0.1333

# (b)
X.cdf(9.9)-X.cdf(8.1) #0.6827

# (c)
X.cdf(9 + 1.8)-X.cdf(9 - 1.8) # 0.9545

# (d)
X.isf(q=0.9) #7.8466
```

Problem 28:

```
from scipy import stats

X = stats.expon(scale = 8)  # define the exponential RV (scale = 1 / lambda)

# (a)
X.cdf(5) # 0.4647

# (b)
X.isf(q=0.95) #0.410 minutes = 24.6 seconds
```