Problem Set 2
Discrete Random Variables

Date: Monday 24^{th} June, 2019.

Problem 1:

Let X be a discrete random variable with the following PMF:

$$P_X(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0 \\ \frac{1}{3} & \text{for } x = 1 \\ \frac{1}{6} & \text{for } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find R_X, the range of the random variable X.
(b) Find $P(X \geq 1.5)$.
(c) Find $P(0 < X < 2)$.
(d) Find $P(X = 0 | X < 2)$.

Problem 2:

Alice, Bob, and Charlotte are looking for butterflies. They look in three separate parts of a field, so that their probabilities of success do not affect each other.

- Alice finds 1 butterfly with probability 17%, and otherwise does not find one.
- Bob finds 1 butterfly with probability 25%, and otherwise does not find one.
- Charlotte finds 1 butterfly with probability 45%, and otherwise does not find one.

Let X be the number of butterflies that they catch altogether. Find the PMF of X.

Problem 3:

Let X be the number of the cars being repaired at a repair shop. We have the following information:

- At any time, there are at most 3 cars being repaired.
- The probability of having 2 cars at the shop is the same as the probability of having one car.
- The probability of having no car at the shop is the same as the probability of having 3 cars.
- The probability of having 1 or 2 cars is half of the probability of having 0 or 3 cars.

Find the PMF of X.

Problem 4:

Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{8} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P(X \leq 2 \text{ and } Y \leq 2)$.
(b) Find $P(X > 2 \text{ or } Y > 2)$.
(c) Find $P(X > 2 | Y > 2)$.
(d) Find $P(X < Y)$.

Problem 5:

50 students live in a dormitory. The parking lot has the capacity for 30 cars. If each student has a car with probability 1/2 (independently from other students), what is the probability that there won't be enough parking spaces for all the cars?

Problem 6:

An urn consists of 20 red balls and 30 green balls. We choose 10 balls at random from the urn. The sampling is done without replacement (repetition not allowed).

(a) What is the probability that there will be exactly 4 red balls among the chosen balls?
(b) Given that there are at least 3 red balls among the chosen balls, what is the probability that there are exactly 4 red balls?

Problem 7:

The number of emails that I get in a weekday (Monday through Friday) can be modeled by a Poisson distribution with an average of 16 emails per minute. The number of emails that I receive on weekends (Saturday and Sunday) can be modeled by a Poisson distribution with an average of 1/30 emails per minute.

(a) What is the probability that I get no emails in an interval of length 4 hours on a Sunday?
(b) A random day is chosen (all days of the week are equally likely to be selected), and a random interval of length one hour is selected on the chosen day. It is observed that I did not receive any emails in that interval. What is the probability that the chosen day is a weekday?

Problem 8:

Let X be a discrete random variable with the following PMF:

$$P_X(k) = \begin{cases} 0.5 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.
(b) Find $Var(X)$ and $SD(X)$.
(c) If $Y = 2X$, find $E(Y)$.

Problem 9:

Suppose that $Y = -2X + 3$. If we know $E(Y) = 1$ and $E(Y^2) = 9$, find $E(X)$ and $Var(X)$.

Problem 10:

Let X be a random variable with mean $E(X) = \mu$. Define the function $f(\alpha)$ as

$$f(\alpha) = E[(X - \alpha)^2].$$

Find the value of α that minimizes f.

Problem 11:

An urn contains 9 balls, 4 of which are red and 5 of which are blue. We draw a ball out of the urn 10 times, taking care to replace the ball and shake up the urn between draws. Let X be the number of times that we draw a red ball.

(a) What kind of random variable is X and what are the parameters of the random variable?
(b) What are $E[X]$ and $Var(X)$?
(c) What is the probability that $X \leq 3$?

Problem 12:

Suppose that there are 4 red plates and 4 blue plates on a shelf. You randomly select 4 of them, with all choices equally likely. Let X denote the number of blue plates selected. Find $F_X(x)$, the CDF of X, for $x \in \{0, 1, 2, 3, 4\}$.

Problem 13:

Suppose that X has probability mass function $p_X(x) = \left(\frac{2}{7}\right)\left(\frac{5}{7}\right)^{x-1}$, for integer $x \geq 1$

(a) Find the probability $P(3 \leq X \leq 5)$.
(b) Find the probability $P(a \leq X \leq b)$, where $1 \leq a \leq b$.

Problem 14:

I roll a fair die repeatedly until a number larger than 4 is observed. If N is the total number of times that I roll the die, find $P(N = k)$, for $k = 1, 2, 3, \ldots$.

Problem 15:

Let X and Y be two independent random variables. Suppose that we know $Var(2XY) = 6$ and $Var(X + 2Y) = 9$. Find $Var(X)$ and $Var(Y)$.

Problem 16:

Let $X \sim Geometric(\frac{1}{3})$, and let $Y = |X - 5|$. Find the range and PMF of Y.

Problem 17:

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} \frac{1}{21} & \text{for } k \in \{-10, -9, \cdots, -1, 0, 1, \cdots, 9, 10\} \\ 0 & \text{otherwise} \end{cases}$$

The random variable $Y = g(X)$ is defined as

$$Y = g(X) = \begin{cases} 0 & \text{if } X \leq 0 \\ X & \text{if } 0 < X \leq 5 \\ 5 & \text{otherwise} \end{cases}$$

Find the PMF of Y.

Problem 18:

The median of a random variable X is defined as any number m that satisfies both of the following conditions:

$$P(X \geq m) \geq \frac{1}{2} \quad \text{and} \quad P(X \leq m) \geq \frac{1}{2}$$

Note that the median of X is not necessarily unique. Find the median of X if

(a) The PMF of X is given by

$$P_X(k) = \begin{cases} 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) X is the result of a rolling of a fair die.

Problem 19:

You just rented a large house and the realtor gave you 5 keys, one for each of the 5 doors of the house. Unfortunately, all keys look identical. so to open the front door, you try them at random. Find the PMF of the number of trials you will need to open the door, under the following alternative assumptions:

(a) after an unsuccessful trial. you mark the corresponding key, so that you never try it again.

(b) at each trial you are equally likely to choose any key.

Problem 20:

A city's temperature is modeled as a random variable with mean and standard deviation both equal to 10 degrees Celsius. A day is described as "normal" if the temperature during that day ranges within one standard deviation from the mean. If X is the temperature in Celsius, the temperature in Fahrenheit is $Y = 32 + \frac{9}{5}X$. What would be the temperature range for a normal day if temperature were expressed in degrees Fahrenheit?

Problem 21:

Two coins are simultaneously tossed until one of them comes up a head and the other a tail. The first coin comes up a head with probability p and the second with probability q. All tosses are assumed independent.

(a) Find the PMF, the expected value, and the variance of the number of tosses.

(b) What is the probability that the last toss of the first coin is a head?

Problem 22:

Suppose that a drawer contains 8 marbles: 2 are red, 2 are blue, 2 are green, and 2 are yellow. The marbles are rolling around in a drawer, so that all possibilities are equally likely when they are drawn. Alice chooses 2 marbles without replacement, and then Bob also chooses 2 marbles without replacement. Let Y denote the number of red marbles that Alice gets, and let X denote the number of red marbles that Bob gets.

(a) Find probability mass function for Y , i.e., for the number of red marbles that Alice gets, i.e., find $p_Y(y)$ for $y = 0, 1, 2$.

(b) Find $E(Y)$.

(c) Is $E(X)$ the same as $E(Y)$? i.e., is the expected number of marbles that Bob gets the same or different? Why or why not?

Problem 23:

At a party a large barrel is filled with 99 gag gifts and 1 diamond ring, all enclosed in identical boxes. Each person at the party is given a chance to pick a box from the barrel, open the box to see if the diamond is inside, and if not, to close the box and return it to the barrel. What is the probability that at least 19 persons will choose gag gifts before the diamond ring is selected?

Problem 24:

Catherine watches raindrops hit the window. The number of raindrops that fall in a fixed period of time is Poisson with an average of 6 per minute.

(a) What is the probability that exactly 5 raindrops fall during the next one minute?
(b) What is the probability that no raindrops fall during the next one minute?
(c) What is the probability that 4 or more raindrops fall during the next one minute?

Problem 25:

In the Mathematics library, suppose that the number of errors in a book chapter (selected at random) is modelled by a Poisson random variable, with mean 2.8.

(a) Find the probability that such a randomly chosen book chapter has at most 4 errors.
(b) Find the conditional probability that such a randomly chosen book chapter has at most 3 errors, given that it has at most 5 errors.

Problem 26:

If X is a geometric random variable with $p = 0.25$, what is the probability that $X >= 4$? Verify your result by performing a computer simulation.

Problem 27:

Estimate the PMF for a geometric random variable with $p = 0.25$ for $k = 1, 2, \dots, 20$ using a computer simulation and compare it to the true PMF.

Problem 28:

If X is a discrete random variable has the PMF $p_X[k] = \frac{1}{4}$ for $k = -1$ and $p_X[k] = \frac{3}{4}$ for $k = 1$, find the mean and variance. Perform a computer simulation to estimate its mean and variance.

Problem 29:

Suppose that 10% of a population are left-handed ($p = 0.1$).

(a) We want to know, out of a random sample of 10 people, what is the probability of 3 of these 10 people being left handed?
(b) What is the probability of 2 or fewer people being left-handed from a selection of 10 people?