Problem Set 5
Bivariate Discrete and Continuous Random Variables

Date: Thurssday June 27^{th}, 2019.

Problem 1:
I have a bag containing 40 blue marbles and 60 red marbles. I choose 10 marbles (without replacement) at random. Let X be the number of blue marbles and Y be the number of red marbles. Find the joint PMF of X and Y.

Problem 2:
Let X and Y be two independent discrete random variables with CDFs F_X and F_Y. Define $Z = \max(X, Y)$ and $W = \min(X, Y)$. Find the CDFs of Z and W.

Problem 3:
Let $X, Y \sim$ Geometric(p) be independent, and let $Z = \frac{X}{Y}$.
(a) Find the range of Z.
(b) Find the PMF of Z.
(c) Find $E[Z]$.

Problem 4:
Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x, y) = 6\exp\left(-(2x + 3y)\right),\ x, y \geq 0.$$

(a) Are X and Y independent?
(b) Find $E\left[Y|X > 2\right]$.
(c) Find $P\left(X > Y\right)$

Problem 5:
Let X be continuous random variable with PDF

$$f_X(x) = 2x,\ \ 0 \leq z \leq 1.$$

We know that given $X = x$,, the random variable Y is uniformly distributed on $[-x, x]$.
(a) Find the joint PDF $f_{X,Y}(x, y)$.
(b) Find $f_Y(y)$.
(c) Find $P\left(|Y| < X^3\right)$.

Problem 6:
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x, y) = 6xy,\ \ 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}$$

(a) Show $R_{X,Y}$ in the $x - y$ plane.
(b) Find $f_X(x)$ and $f_Y(y)$.
(c) Are X and Y independent?
(d) Find the conditional PDF of X given $Y = y$, $f_{X|Y}(x|y)$.

(e) Find $E[X|Y=y]$, $0 \leq y \leq 1$.
(f) Find $var[X|Y=y]$, $0 \leq y \leq 1$.

Problem 7:

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = 2, \ y+x \leq 1, x > 0, y > 0.$$

Find $cov(X,Y)$ and $\rho(X,Y)$.

Problem 8:

I roll a fair die n times. Let X be the number of 1's that I observe and let Y be the number of 2's that I observe. Find $cov(X,Y)$ and $\rho(X,Y)$. Hint: One way to solve this problem is to look at $var(X+Y)$.

Problem 9:

I have a coin with $P(H) = p$. I toss the coin repeatedly until I observe two consecutive heads. Let X be the total number of coin tosses. Find $E(X)$.

Problem 10:

Consider two random variables X and Y with joint PMF given in the following table

	$Y=2$	$Y=4$	$Y=5$
$X=1$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$
$X=2$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{8}$
$X=3$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$

(a) Find $P(X \leq 2, Y \leq 4)$.
(b) Find the marginal PMFs of X and Y.
(c) Find $P(Y=2|X=1)$.
(d) Are X and Y independent?

Problem 11:

Let X and Y be two random variables, and let U, and V be the normalized versions of X and Y as defined

$$U = \frac{X - EX}{\sigma_X}, \quad V = \frac{Y - EY}{\sigma_Y}.$$

Use the fact that $Var(U+V) \geq 0$ to show that $|\rho(X,Y)| \leq 1$.

Problem 12:

Let X and Y be jointly (bivariate) normal, with $Var(X) = Var(Y)$. Show that the two random variables $X+Y$ and $X-Y$ are independent.

Problem 13:

Let X and Y be jointly normal random variables with parameters $\mu_X = 0$, $\sigma_X^2 = 1$, $\mu_Y = -1$, $\sigma_Y^2 = 4$, and $\rho = \frac{1}{2}$.

(a) Find $P(X+Y>0)$.
(b) Find the constant a if we know $aX+Y$ and $X+2Y$ are independent.

Problem 14:
Let X and Y be two independent $Uniform(0,1)$ random variables. Let also $Z = max(X,Y)$ and $W = min(X,Y)$. $FindCov(Z,W)$.

Problem 15:
Consider two random variables X and Y with joint PMF given by

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}, \text{for } k,l = 1,2,3,...$$

(a) Show that X and Y are independent and find the marginal PMFs of X and Y.
(b) Find $P(X^2+Y^2 \leq 10)$.

Problem 16:
Let X and Y be two independent random variables with PMFs

$$P_X(k) = P_Y(k) = \begin{cases} \frac{1}{5} & \text{for } x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

Define $Z = X - Y$. Find the PMF of Z.

Problem 17:
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2} & 0 \leq x, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c.
(b) Find $P(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.
(c) Find $P(0 \leq X \leq 1)$.

Problem 18:
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} e^{-xy} & 1 \leq x \leq e, y > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PDFs, $f_X(x)$ and $f_Y(y)$.
(b) Write an integral to compute $P(0 \leq Y \leq 1, 1 \leq X \leq \sqrt{e})$.

Problem 19:
Let X and Y be two jointly continuous random variables with joint CDF

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-2y} + e^{-(x+2y)} & x,y > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint PDF, $f_{XY}(x,y)$. Are X and Y independent?
(b) Find $P(X < 2Y)$.

Problem 20:
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x^2 + \frac{1}{3}y & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq y \leq 1$, find the following:
(a) The conditional PDF of X given $Y = y$.
(b) $P(X > 0|Y = y)$. Does this value depend on y?
(c) Are X and Y independent?

Problem 21:
Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}x^2 + \frac{2}{3}y & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $E[Y|X = 0]$ and $Var(Y|X = 0)$.

Problem 22:
Let X and Y be two independent $Uniform(0, 2)$ random variables. Find $P(XY < 1)$.

Problem 23:
Suppose $X \sim Exponential(1)$ and given $X = x$, Y is a uniform random variable in $[0, x]$, i.e.,

$$Y|X = x \quad \sim \quad Uniform(0, x),$$

or equivalently

$$Y|X \quad \sim \quad Uniform(0, X),$$

(a) Find $E[Y]$.
(b) Find $Var(Y)$.

Problem 24:
Let X and Y be two independent $Uniform(0, 1)$ random variables. Find
(a) $E[XY]$.
(b) $E[e^{XY}]$.
(c) $E[X^2 + Y^2 + XY]$.
(d) $E[Ye^{XY}]$.

Problem 25:
Let X and Y be two random variables. Suppose that $\sigma_X^2 = 4$, and $\sigma_Y^2 = 9$. If we know that the two random variables $Z = 2XY$ and $W = X + Y$ are independent, find $Cov(X, Y)$ and $\rho(X, Y)$.