CS 241 - STAT 250

Lecture # 8

.4 (Continued) Expected Value

Moment Generating Function (MGF)

Def: The MGF of a RV X is defined as

$$M_X(s) = E[e^{sX}]$$

$$= \int_{-\infty}^{+\infty} e^{sx} f_X(x)\, dx$$

for $s \in [0, b)$

roperties of MGF

$$) = E\left[e^{sX}\right] = E\left[1 + sX + \frac{s^2X^2}{2} + \frac{s^3X^3}{3!} + \cdots\right]$$

$$= 1 + s\,E[X] + \frac{s^2}{2}\,E[X^2] + \cdots$$

$$= \sum_{k=0}^{+\infty} \frac{s^k}{k!}\,E[X^k]$$

Moments of RV X

$$M_X(s)\Big|_{s=0} = E\left[e^{0X}\right] = E(1) = 1$$

$$\frac{dM_X(s)}{ds}\Big|_{s=0} = E\left[x\,e^{sx}\right]\Big|_{s=0} = E(X) \leftarrow \text{First Moment of } X$$

$$\frac{d^2 \mathcal{M}_X(s)}{ds^2}\Bigg|_{s=0} = E\left[x^2 e^{sx}\right]\Bigg|_{s=0} = E\left[X^2\right] \leftarrow \text{2}^{\text{nd}} \text{ Moment of RV } X$$

$$\vdots$$

$$\boxed{E\left[X^n\right] = \frac{d^n \mathcal{M}_X(s)}{ds^n}\Bigg|_{s=0}}$$

Characteristic Function (CF)

<u>Example:</u>

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < +\infty$$

$\leftarrow$ Cauchy PDF

$$M_X(s) = E\left[e^{sX}\right]$$

$$= \int_{-\infty}^{+\infty} \frac{e^{sx}}{\pi(1+x^2)}\, dx \xrightarrow{\text{does not converge}} +\infty$$

$\rightarrow$ in this case MGF does not exist

- We can use the characteristic Function (CF)

defined as: $\Phi_X(\omega) = E\left[e^{j\omega X}\right]$ with $j^2 = -1$

$$= \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)\, dx$$

Properties of CF

$$|\Phi_X(w)| = |E[e^{jwX}]|$$

$$= \left| \int_{-\infty}^{+\infty} e^{jwx} f_X(x)\, dx \right|$$

$$\leq \int_{-\infty}^{+\infty} \underbrace{|e^{jwx}|}_{=1} \cdot \underbrace{|f_X(x)|}_{= f_X(x)}\, dx$$

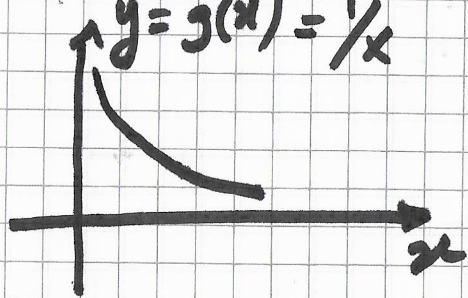
$$= \int_{-\infty}^{+\infty} f_X(x)\, dx = 1$$

$$E[X^n] = (-j)^n \left. \frac{d^n \Phi_X(w)}{dw^n} \right|_{w=0}$$

3.5.b Examples

1. RV X with PDF $f_X(x) = \begin{cases} 4x^3 & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Q: Find the PDF of $Y = 1/X$.



$R_Y = [1, +\infty[$

use method of transformation

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} \quad \text{with } x_1 = g^{-1}(y)$$

$g'(x) = -\frac{1}{x^2}$

$y = \frac{1}{x} = g(x)$

then $x = 1/y = g^{-1}(y)$

Thus

$$f_Y(y) = \frac{f_X(1/y)}{\left|-\frac{1}{x^2}\right|} = \frac{4/y^3}{y^2} = \frac{4}{y^5} \quad y \geq 1$$

0 otherwise

Thus

$$\boxed{f_Y(y) = \begin{cases} 4/y^5 & \text{for } y \geq 1 \\ 0 & \text{otherwise} \end{cases}}$$

CDF _ based approach

$$F_Y(y) = \text{Prob}[Y \leq y]$$

$$= \text{Prob}\left[\frac{1}{X} \leq y\right]$$

$$= \text{Prob}\left[X \geq 1/y\right]$$

$$= 1 - F_X(1/y)$$

$$F_X(x) = \int_0^x 4\, x_1^3 \, dx_1 = 4 \left.\frac{x_1^4}{4}\right|_0^x = x^4$$

$$f_Y(y) = -\left(-\frac{1}{y^2}\right) f_X\left(\frac{1}{y}\right) = \frac{1}{y^2} \cdot \frac{4}{y^3} = \frac{4}{y^5}.$$

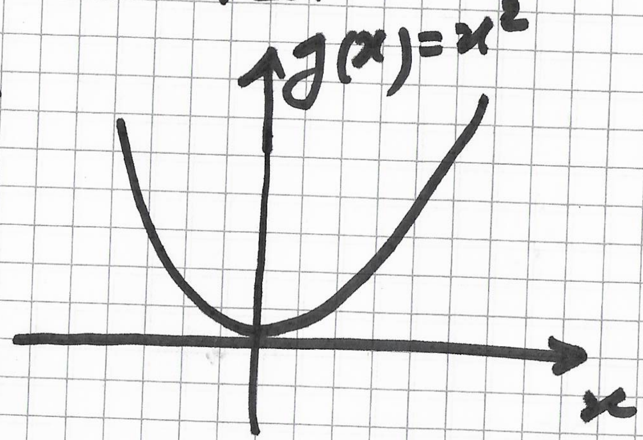
Example 2

RV X with PDF $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Let $Y = g(X) = X^2$

$\downarrow$

$g'(x) = 2x$



$R_Y = [0, +\infty[$

$x_1 = +\sqrt{y}$ or $x_2 = -\sqrt{y}$

Applying Theorem 2

$$f_Y(y) = \left| \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \right|$$

$$= \frac{f_X(+\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|}$$

$$= \frac{1}{2\sqrt{y}} \left(f_X(+\sqrt{y}) + f_X(-\sqrt{y}) \right)$$

$$= \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \right)$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \text{ for } y \geq 0 \quad = 0 \text{ otherwise}}$$

CDF approach.

$$F_Y(y) = \text{Prob}[Y \le y]$$

$$= \text{Prob}[X^2 \le y]$$

$$= \text{Prob}[-\sqrt{y} \le X \le +\sqrt{y}] \quad \text{for } y \ge 0$$

$$F_Y(y) = F_X(+\sqrt{y}) - F_X(-\sqrt{y})$$

derivative

Same as with Theorem 2

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(+\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{\ldots}\left(f_X(+\sqrt{y}) + f_X(-\sqrt{y})\right) \quad y \ge 0$$

3.6 Special Continuous Distributions

a- Uniform distribution

PDF $\quad f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$

CDF $\quad F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & \text{for } x \geq b \end{cases}$

- $E[X] = \dfrac{a+b}{2}$
- $Var[X] = \dfrac{(b-a)^2}{12}$

b_ Exponential Distribution

. $X \sim$ Exponential (λ) if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

. CDF of X $\quad F_X(x) = \int_0^x \lambda e^{-\lambda u} \, du$

$$= \cancel{\lambda} \frac{-1}{\cancel{\lambda}} e^{-\lambda u} \Big|_0^x$$

$$= -e^{-\lambda x} + 1 \implies$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwize} \end{cases}$$

- $E(X) = \int_0^{+\infty} x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$

- $Var(X) = E(X^2) - (\underbrace{E(X)}_{=1/\lambda})^2 = \dots = \frac{1}{\lambda^2}$.

→ Properties of Exponential RV

P1: Relation to Geometric RV

. The exponential distribution can be viewed as

The continuous version of the geometric RV

- Suppose that you toss a coin until you observe the first head and the RV X be the time you observe the first head

So if we let Δ be the time between two tosses and the proba of heads

$$p = \Delta . \lambda \quad \text{then}$$

$$\text{as} \quad \Delta \longrightarrow 0 , \quad X \xrightarrow{\approx} Exp(\lambda)$$

P2: Memoryless Property of Exponential RV

Theorem.

Consider $X \sim Exp(\lambda)$ with $\lambda > 0$

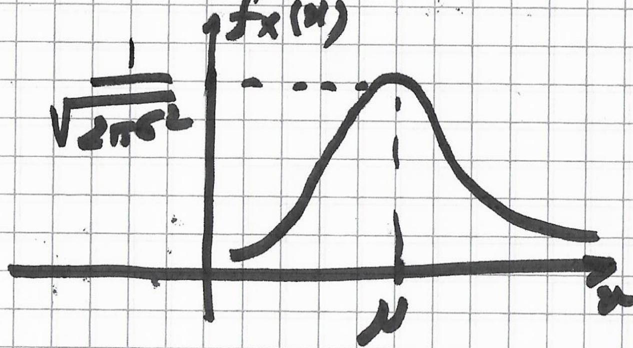
Then

$$P\left[X > x + a \,/\, X > a \right] = \text{Prob}\left[X > x \right]$$

for all a and $x \geq 0$

C- Gaussian (Normal) Distribution

$X \sim \mathcal{N}(\mu, \sigma^2)$ if



PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$-\infty < x < +\infty$$

Standard Gaussian $\mathcal{N}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

CDF of X = $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$

$\downarrow$

$\mathcal{N}(0,1)$

$$= \phi(x)$$

Matlab command
normcdf

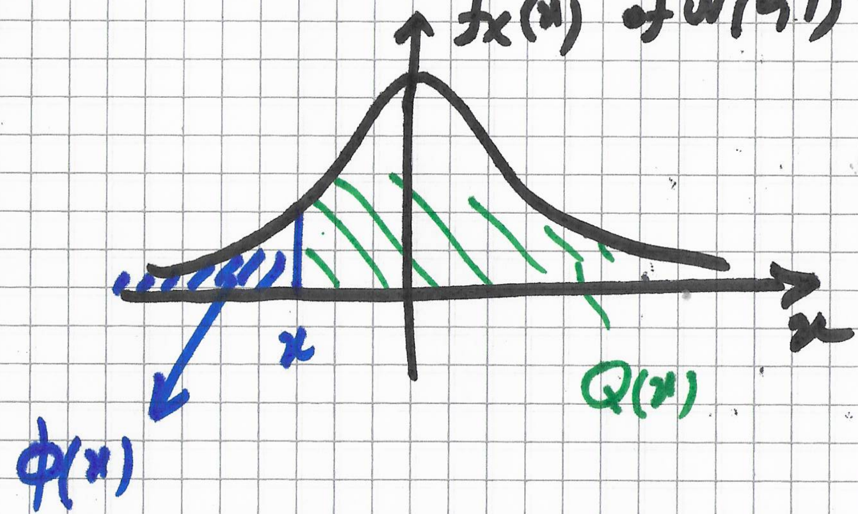
CDF of a standard Gaussian

For $X \sim \mathcal{N}(\mu, \sigma^2)$

$$F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

You can also use the Complementary CDF of a standard Gaussian RV

$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$



$$\Phi(x) + Q(x) = 1 .$$

Properties of $\phi(x)$

1. $\lim\limits_{x \to -\infty} \phi(x) = 0$

2. $\lim\limits_{x \to +\infty} \phi(x) = 1$

3. $\phi(0) = 1/2$

4. $\phi(-x) = \cancel{\phi(x)} = 1 - \phi(x)$ for all $x \in \mathbb{R}$.

5. Useful bounds

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2} \leq \underset{\underset{Q(x)}{\parallel}}{1 - \phi(x)} \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

6- If $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and

$$Y = aX + b$$

Then

$$Y \sim \mathcal{N}(\underbrace{a\mu_x + b}_{\mu_y}, \underbrace{a^2\sigma_x^2}_{\sigma_y^2})$$

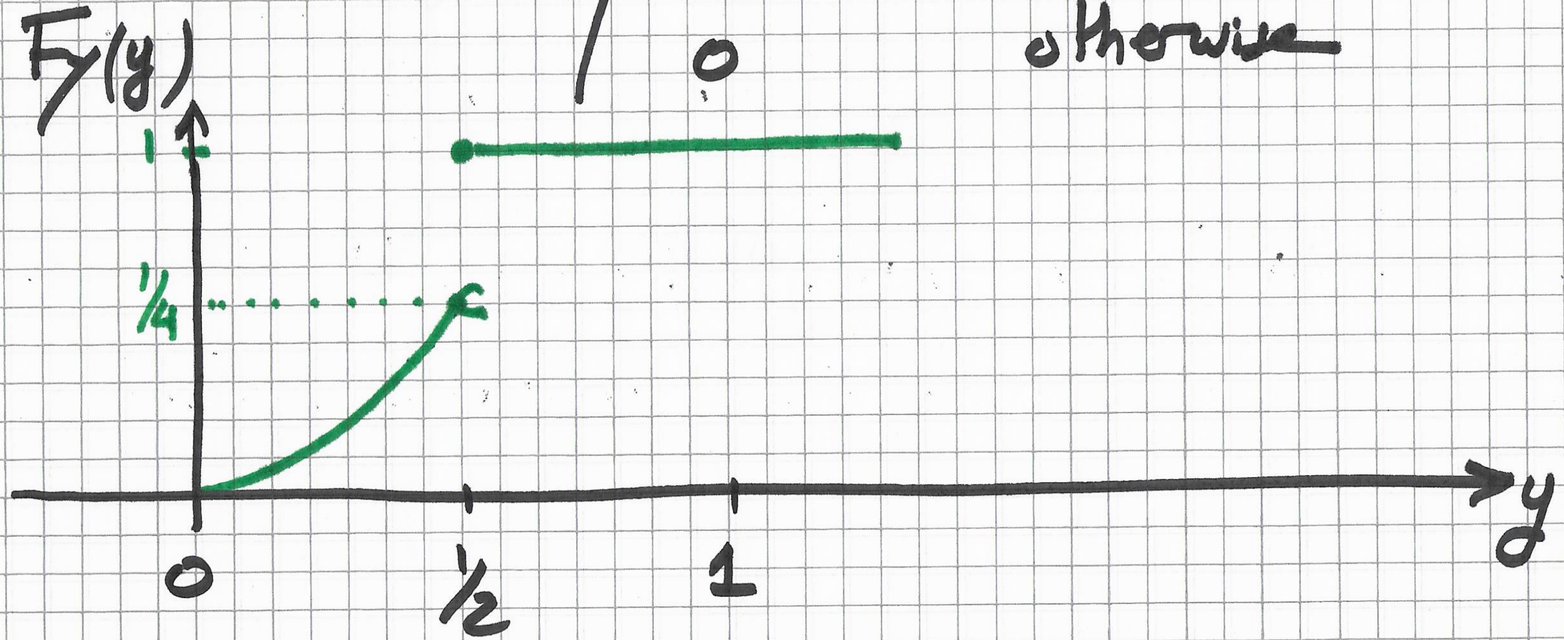
3.7 Mixed RVs

Def: A Mixed RV has a discrete part and Continuous part.

* **Example:**

Consider a mixed RV Y with CDF $F_Y(y)$ given by

$$F_Y(y) = \begin{cases} 1 & y \geq \frac{1}{2} \\ y^2 & 0 \leq y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



~~Note~~

- Note that the CDF of Y is not continuous as there is a jump at $y = \frac{1}{2}$

- Note also that we do not have the staircase form $\Rightarrow$ Y is not a discrete RV

$\rightarrow$ More generally the CDF of Y has a continuous part and discrete part

— So we can write

$$F_Y(y) = C(y) + D(y)$$

with

$$C(y) = \begin{cases} 1/4 & y \geq \frac{1}{2} \\ y^2 & 0 \leq y < 1/2 \\ 0 & y < 0 \end{cases}$$

$$D(y) = \begin{cases} 3/4 & y \geq 1/2 \\ 0 & y < 1/2 \end{cases}$$

* Using the delta function ~~$\$$~~ in the CDF expressions of mixed RVs (and discrete RV)

$$f_X(x) = \sum_k a_k\, \delta(x - x_k) + g(x)$$

where $a_k = \text{Prob}[X = x_k]$ and

$g(x) \geq 0$ that does not contain any delta functions. So we have

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = \sum_k a_k + \int_{-\infty}^{+\infty} g(x)\, dx = 1$$

→ As a special case, if X is discrete RV with range $R_X = \{x_1, x_2, \ldots\}$ and PMF $P_X(x_k)$ we can define its PDF as

$$f_X(x) = \sum_{x \in R_X} P_X(x_k)\, \delta(x - x_k)$$