Solution of Problem Set 1

Problem 1:

(a) $[2, 8]$
(b) $[6, 7)$
(c) $(-\infty, 0) \cup (1, \infty)$
(d) $[7, 8]$

Problem 2:

a. $A \cup B = \{1, 2, 4, 5\}$.
b. $A \cap B = \{2\}$.
c. $\overline{A} = \{3, 4, 5, 6\}$ ($\overline{A}$ consists of elements that are in S but not in A).
d. $\overline{B} = \{1, 3, 6\}$.
e. We have

$$(A \cup B)^c = \{1, 2, 4, 5\}^c = \{3, 6\},$$

which is the same as

$$A^c \cap B^c = \{3, 4, 5, 6\} \cap \{1, 3, 6\} = \{3, 6\}.$$

f. We have

$$A \cap (B \cup C) = \{1, 2\} \cap \{1, 2, 4, 5, 6\} = \{1, 2\},$$

which is the same as

$$(A \cap B) \cup (A \cap C) = \{2\} \cup \{1\} = \{1, 2\}.$$

Problem 3:

$$\bigcup_{i=1}^{N} A_i = \{x : 0 \leq x \leq N\} = A_N$$

$$\bigcap_{i=1}^{N} A_i = \{x : 0 \leq x \leq N\} = A_1$$

No, since $A_1 \subset A_2 \subset .... \subset A_N$.

Problem 4:

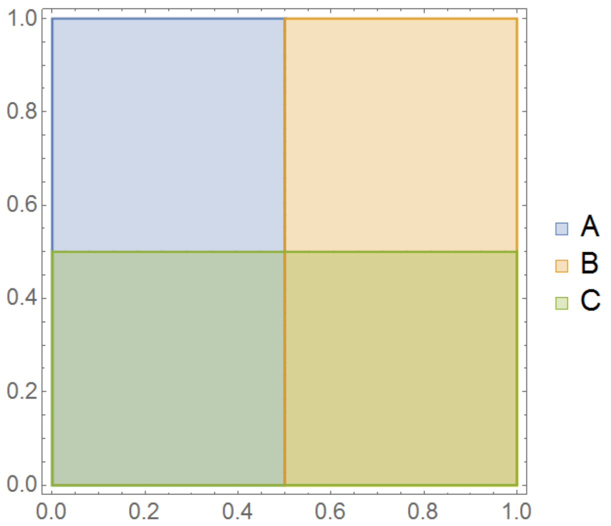
Remember that a partition of S is a collection of nonempty sets that are disjoint and their union is S. There are 5 possible partitions for $S = \{1, 2, 3\}$:

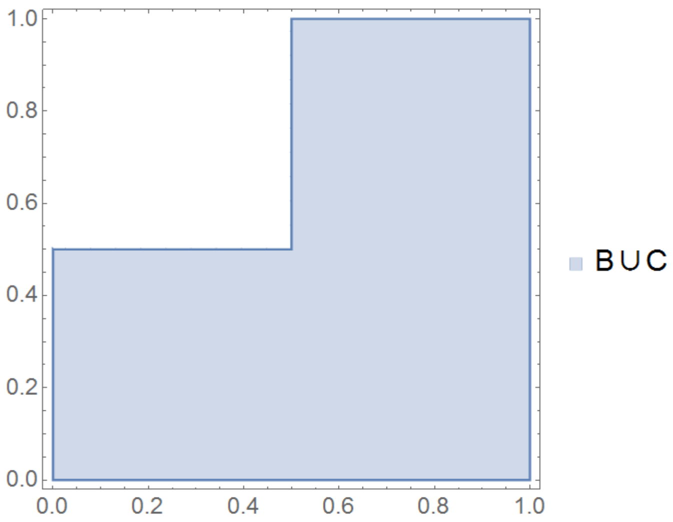
1. $\{1\}, \{2\}, \{3\}$;
2. $\{1, 2\}, \{3\}$;
3. $\{1, 3\}, \{2\}$;
4. $\{2, 3\}, \{1\}$;
5. $\{1, 2, 3\}$.

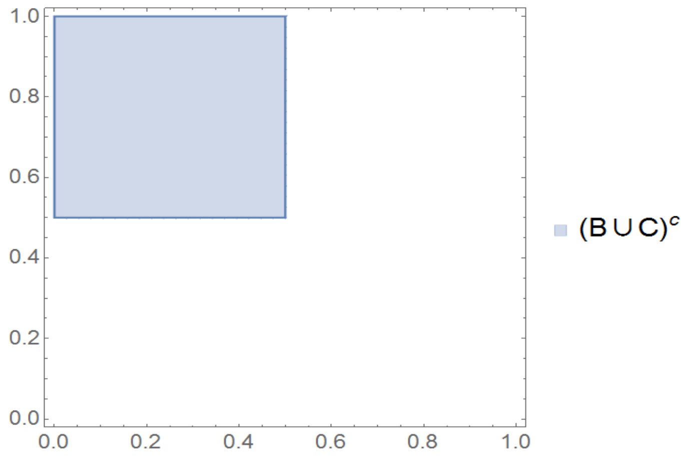
Problem 5:

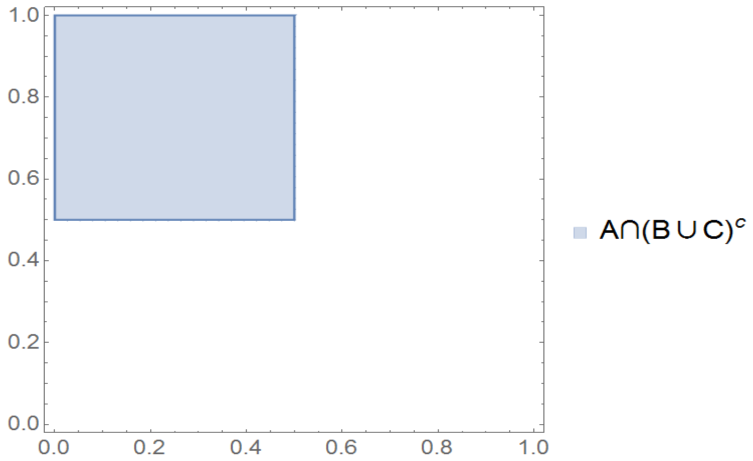
If $x \in A \cap B^c$, then $x \in A$ and $x \in B^c$.
If $x \in B^c$, then $x \notin B$. Therefore, we have $x \in A$ and $x \notin B$.
By definition, this means that $A \setminus B$. Thus, we get $A \cap B^c = A \setminus B$.

Problem 6:









Problem 7:

(a)

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

(b)

$$\begin{aligned} A \cup C &= \{1, 2, 3, 7, 8, 9, 10\} \\ B &= \{2, 3, \cdots, 7\} \\ \text{thus:} \quad (A \cup C) - B &= \{1, 8, 9, 10\} \end{aligned}$$

(c)

$$\begin{aligned} \bar{A} &= \{4, 5, \cdots, 10\} \\ B - C &= \{2, 3, 4, 5, 6\} \\ \text{thus:} \quad \bar{A} \cup (B - C) &= \{2, 3, \cdots, 10\} \end{aligned}$$

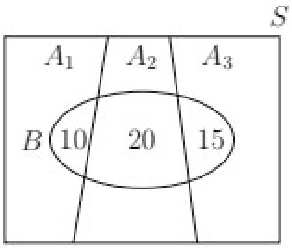
(d) No, since they are not disjoint. For example,

$$A \cap B = \{2, 3\} \neq \varnothing$$

Problem 8:

(a) A={(H,H),(H,T)}.
(b) B={(H,T),(T,H),(T,T)}.
(c) C={(H,T),(T,H)}.

Problem 9:



We see that in fact the sets $B \cap A_1$, $B \cap A_2$, and $B \cap A_3$ form a partition of B. Therefore

$$\begin{aligned}|B| &= |B \cap A_1| + |B \cap A_2| + |B \cap A_3| \\ &= 10 + 20 + 15 \\ &= 45\end{aligned}$$

Problem 10:

a)

$$\begin{aligned}P(a) + P(b) + P(d) &= 1 \\ P(a) &= 0.5 \\ P(d) &= 0.25\end{aligned}$$

Therefore $P(b) = 0.25$.

b)

$$\begin{aligned}P(\{b, d\}) &= P(b) + P(d) \\ &= 0.5\end{aligned}$$

Problem 11:

a)

We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

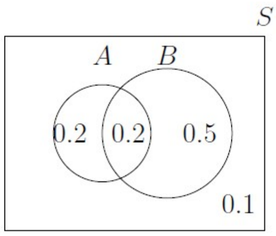
Thus,

$$0.9 = 0.7 + 0.4 - P(A \cap B)$$

which results in:

$$P(A \cap B) = 0.2.$$

A Venn diagram is useful here:



b)

$$\begin{aligned} P(A^c \cap B) &= P(B - A) \\ &= P(B) - P(B \cap A) \\ &= 0.7 - 0.2 \\ &= 0.5 \end{aligned}$$

c)

$$P(A - B) = 0.2$$

d)

$$\begin{aligned} P(A^c - B) &= P(A^c \cap B^c) \\ &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - 0.9 \\ &= 0.1 \end{aligned}$$

e)

$$\begin{aligned} P(A^c \cup B) &= P(A^c) + P(B) - P(A^c \cap B) \\ &= 1 - P(A) + P(B) - P(B - A) \\ &= 1 - 0.4 + 0.7 - 0.5 \\ &= 0.8 \end{aligned}$$

Problem 12:

The sample space has 36 elements.
$S = \{(1, 1),(1, 2), \cdots ,(1, 6), (2, 1),(2, 2), \cdots ,(2, 6), \ldots (6, 1),(6, 2), \cdots ,(6, 6)\}$

(a) The event $X_2 = 4$ can be represented by the set. $A = \{(1, 4),(2, 4),(3, 4),(4, 4), (5, 4),(6, 4)\}$.
Thus, $P(A) = |A|/|S| = 6/36 = 1/6$

(b) B = {(x1, x2)|x1 + x2 = 7} = {(1, 6),(2, 5),(3, 4),(4, 3),(5, 2),(6, 1)}
Therefore P(B) = |B|/|S| = 6/36 = 1/6.

(c) C = {(X1, X2)|X1 6= 2, X2 ≥ 4} = {(1, 4),(1, 5),(1, 6), (3, 4),(3, 5),(3, 6), (4, 4),(4, 5),(4, 6), (5, 4),(5, 5),(5, 6), (6, 4),(6, 5),(6, 6)}
Therefore |C| = 15.
Which results in: P(C) = |C|/|S| =15/36 = 5/12.

Problem 13:

Solution: The total number of ways to take out 8 balls is the number of unordered samples of size 8, without replacement, from 30, i.e., $\binom{30}{8}$. This is the denominator in the probability we need to compute. For the numerator, we need to count those samples that consist of exactly 3 red and 5 blue balls. These samples can be obtained in two stages, by first picking 3 out of the 10 red balls ($\binom{10}{3}$ ways to do this), and then picking 5 out of the 20 blue balls ($\binom{20}{5}$ ways to do this). The total number of such samples is therefore $\binom{10}{3}\binom{20}{5}$, and the probability sought is $\binom{10}{3}\binom{20}{5}/\binom{30}{8} (= 0.31\ldots)$.

Problem 14:

(a) Let A be the event that there are exactly 4 black cell phones among the 10 chosen cell phones. Then:

$$P(A) = \frac{|A|}{|S|}$$

$$|S| = \binom{50}{10}$$

$$|A| = \binom{20}{4}\binom{30}{6}$$

Thus:

$$P(A) = \frac{\binom{20}{4}\binom{30}{6}}{\binom{50}{10}}.$$

(b) Let B be the event that there are less than 3 black cell phones among the chosen phones. Then:

$$\begin{aligned} P(B) &= P(\text{"0 black phones" or "1 black phones" or "2 black phones"}) \\ &= \frac{\binom{20}{0}\binom{30}{10} + \binom{20}{1}\binom{30}{9} + \binom{20}{2}\binom{30}{8}}{\binom{50}{10}} \end{aligned} \tag{4}$$

Problem 15:

a)

$$
\begin{aligned}
P(A|B) &= \frac{P(A\cap B)}{P(B)} \\
&= \frac{0.2}{0.35} \\
&= \frac{4}{7}
\end{aligned}
$$

b)

$$
\begin{aligned}
P(C|B) &= \frac{P(C\cap B)}{P(B)} \\
&= \frac{0.15}{0.35} \\
&= \frac{3}{7}
\end{aligned}
$$

c)

$$
\begin{aligned}
P(B|A\cup C) &= \frac{P(B\cap (A\cup C))}{P(A\cup C)} \\
&= \frac{0.1+0.1+0.05}{0.2+0.1+0.1+0.1+0.5+0.05} \\
&= \frac{0.25}{0.7} \\
&= \frac{5}{14}
\end{aligned}
$$

d)

$$
\begin{aligned}
P(B|A,C) &= \frac{P(B\cap A\cap C)}{P(A\cap C)} \\
&= \frac{0.1}{0.2} \\
&= \frac{1}{2}
\end{aligned}
$$

Problem 16:

(a) Let A be the event that there is exactly 1 ace among the 5 chosen cards. In a shuffled card there are 4 aces and the other 48 cards are non-aces. Thus:

$$P(A) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$$

(b) Let B be the event that there is at least 1 ace among the 5 chosen cards. We can solve this problem in an easy way by calculating the $P(B^c)$. (B^c is the event in which no ace exists among the 5 chosen cards.)

$$\begin{aligned} P(B) &= 1 - P(\text{"no aces"}) \\ &= 1 - P(B^c) && (5) \\ &= 1 - \frac{\binom{48}{5}}{\binom{52}{5}} && (6) \end{aligned}$$

Problem 17:

Let A be the event that the deck contains exactly two aces and B the event that it contains at least one ace.

We use the formula for the conditional probability:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} = \frac{P(A)}{1 - P(B^c)} && (7) \\ & && (8) \end{aligned}$$

$$P(A) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

$$P(B^c) = \frac{\binom{48}{5}}{\binom{52}{5}}$$

By substituting $P(A)$ and $P(B^c)$ to the equation of $P(A|B)$, we have:

$$\begin{aligned} P(A|B) &= \frac{\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}}{1 - \frac{\binom{48}{5}}{\binom{52}{5}}} \\ &= \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5} - \binom{48}{5}} && (9) \end{aligned}$$

Problem 18:

A and B have 26 cards, which are 7 spades and 19 non-spades. We know there are 13 spades in a shuffled deck. Therefore, C and D have 6 spades and 20 non-spades. Thus, we can restate the problem as follows:

13 cards are chosen randomly from 26 cards (6 spades and 20 non-spades). What is the probability of choosing exactly 4 aces?

Let A be the event that 4 aces are among the 13 chosen cards.

$$P(A) = \frac{\binom{6}{4}\binom{20}{9}}{\binom{26}{13}}$$

Problem 19:

We have the following information:

Bag 1: 10 blue marbles.

Bag 2: 15 blue marbles.

Let A be the event that exactly 2 red marbles among the 5 chosen marbles exist. Let B_1 be the event that Bag 1 has been chosen. Let B_2 be the event that Bag 2 has been chosen.

We want to calculate $P(B_1|A)$. We use Bayes' rule:

$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \end{aligned} \tag{24}$$

In Bag 1, there are 10 blue and 6 red marbles. By substituting the values:

$$P(A|B_1) = \frac{\binom{6}{2}\binom{10}{3}}{\binom{16}{5}}$$

and

$$P(A|B_2) = \frac{\binom{6}{2}\binom{15}{3}}{\binom{21}{5}}$$

Thus:

$$\begin{aligned} P(B_1|A) &= \frac{\frac{\binom{6}{2}\binom{10}{3}}{\binom{16}{5}}}{\frac{\binom{6}{2}\binom{10}{3}}{\binom{16}{5}} + \frac{\binom{6}{2}\binom{15}{3}}{\binom{21}{5}}} \\ &= \frac{\binom{21}{5}\binom{10}{3}}{\binom{21}{5}\binom{10}{3} + \binom{15}{3}\binom{16}{5}} \end{aligned} \tag{25}$$

Problem 20:

This is another typical problem for which the law of total probability is useful. Let C_1 be the event that you choose a regular coin, and let C_2 be the event that you choose the two-headed coin. Note that C_1 and C_2 form a partition of the sample space. We already know that

$$P(H|C_1)=0.5,$$
$$P(H|C_2)=1.$$

(a) Thus, we can use the law of total probability to write

$$\begin{aligned} P(H) &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) \\ &= \frac{1}{2}.\frac{2}{3} + 1.\frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

(b) Now, for the second part of the problem, we are interested in $P(C_2|H)$. We use Bayes' rule

$$\begin{aligned} P(C_2|H) &= \frac{P(H|C_2)P(C_2)}{P(H)} \\ &= \frac{1.\frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2}. \end{aligned}$$

Problem 21:

(a) $P(HHH)=P(H)\cdot P(H)\cdot P(H)=0.5^3=1/8$.

(b) To find the probability of exactly one heads, we can write

$$\begin{aligned} P(\text{One heads}) &= P(HTT \cup THT \cup TTH) \\ &= P(HTT)+P(THT)+P(TTH) \\ &= 1/8 + 1/8 + 1/8 \\ &= 3/8. \end{aligned}$$

(c)

Let A_1 be the event that you observe at least one heads, and A_2 be the event that you observe at least two heads. Then

$$A1=S-\{TTT\}, \text{ and } P(A_1)=7/8;$$

$$A2=\{HHT,HTH,THH,HHH\}, \text{ and } P(A_2)=4/8.$$

Thus, we can write

$$P(A_2|A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{P(A_2)}{P(A_1)} = \frac{4}{8} \times \frac{8}{7} = \frac{4}{7}$$

Problem 22:

We know

$$P(A) = 0.7$$
$$P(B) = 0.4$$
$$P(A \cap B) = 0.2$$

Therefore:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = \frac{1}{2}$$

Problem 23:

We are given that people in the protest are either civilians or undercover police with the following probabilities:

- P(Person is a civilian) = P(C) = 0.98
- P(Person is an undercover police) = P(UP) = 0.02
- P(Violent|C) = 0.01,
- P(Peaceful|C) = 0.99,
- P(Violent|UP) = 0.5,
- P(Peaceful|UP) = 0.5.

You spot a person advocating violence in the crowd. We want to find the probability that this person is an undercover police officer, i.e., P(UP|Violent). Using Bayes' Rule we have

P(UP|Violent) = P(UP∩Violent)/P(Violent) = P(Violent|UP)×P(UP)/P(Violent) (*)

By the total probability rule we get,

P(Violent) = P(Violent|UP)×P(UP)+P(Violent|C)×P(C)
= 0.5×0.02+0.01×0.98 = 0.0198

Plugging Eq (**) and other given values to Eq (*), we get

P(UP|Violent) = 0.5×0.02/0.0198 = 50/99.

Problem 24:

(a) $$\binom{100}{10}$$

(b) $$\frac{\binom{40}{5}\binom{60}{5}}{\binom{100}{10}}$$

Problem 25:

(a)

$$\frac{\binom{6}{1}\binom{4}{2}}{\binom{10}{3}} = \frac{6 \times 6}{120} = \frac{3}{10}$$

(b)

$$\frac{\binom{6}{4}}{\binom{10}{3}} = \frac{15}{120} = \frac{1}{6}$$

(c)

$$\frac{\binom{4}{3}}{\binom{10}{3}} = \frac{4}{120} = \frac{1}{30}$$

Problem 26:

```
import numpy as np
import random
import pandas as pd

n_iter=100000
c = np.zeros(n_iter)
for i in range(n_iter):
    b=2
    w=3
    while (w>0):
        if (random.random() < b/5):
            b=b-1
            w=w+1
        else:
            w=w-1
            b=b+1
        c[i]=c[i]+1
unique_values = np.unique(c, return_counts=True)
tab = pd.DataFrame()
tab['Value'] = unique_values[0]
tab['count'] = unique_values[1]
tab['Percent'] = unique_values[1]/np.sum(unique_values[1])*100
print(tab.head(10))
```

Problem 27:

```
import random

n_iter = 1000 # number of trials
count = 0 # number of even number obtained
minimum = 1
maximum = 6
for i in range(n_iter):
   outcome = random.randint(minimum, maximum)
   if outcome == 2 or outcome == 4 or outcome == 6:
      count = count +1

probability_estimate = count/n_iter
print(probability_estimate)
```

Problem 28:

```
count = 0 # count of the desired event
E = [] # store all possibilities of the desired event
Ns = 6*5 # sampling without replacement
for i in range(1,7):
   for j in range(1,7):
      if (i <= 4 and j >=5) or (i >=5 and j <=4):
         E.append([i,j])
         count = count + 1

probability = count/Ns
print(probability)
```

Problem 29:

```
import random
n_iter = 10000
count = 0
for i in range(n_iter):
   x = [] # store the result of each trial
   for j in range(5):
      if random.random() <= 0.5:
         x.append(1) # success
      else:
         x.append(0) # not success
   if sum(x) == 3:
      count = count + 1

probability_estimate = count/n_iter
print(probability_estimate)
```