AMCS 241 - STAT 250

Fall' 2018

$\boxed{\text{Lecture \# 7}}$

Chapter 3: Continuous RVs

$\boxed{3.1}$ Introduction

Recall:

- Discrete RVs can only take a countable number of possible outcomes
- Continuous RVs have a range in the form of:
  - interval on the real number line
  - Union of non-overlapping interval on real-line.

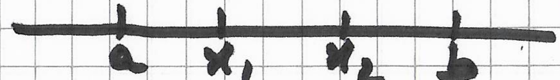
3.2 Example

→ Choose a real number uniformily distributed in the interval $[a, b]$ and call it X.

→ By <u>uniformly</u> at random, we mean that all intervals in $[a, b]$ that have the same length will have the same probability.

$$\rightarrow \quad P[X \in [a, b]] = 1$$

$$\rightarrow \quad a \leq x_1 \leq x_2 \leq b$$

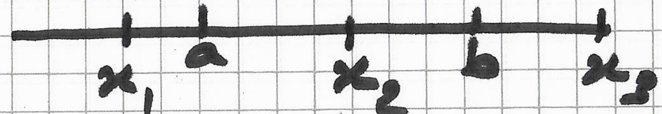


$$P[X \in [x_1, x_2]] = \frac{x_2 - x_1}{b - a}$$

• From the same old definition of CDF

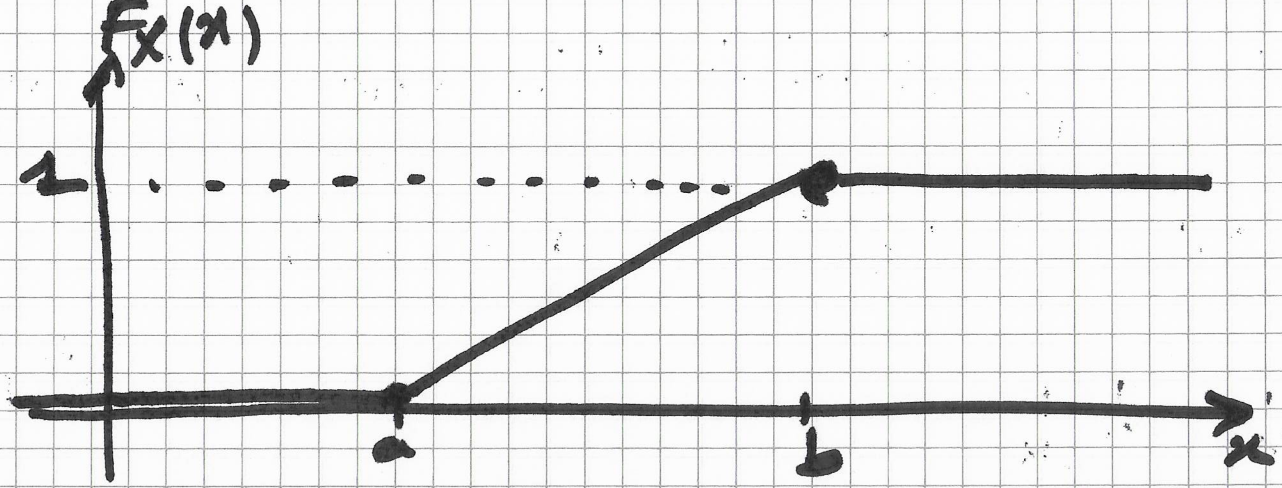
$$F_X(x) = \text{Prob}\,[X \leq x]$$ we can

write



$$F_X(x) = \begin{cases} 0 & \text{for} \quad x < a \\ \dfrac{x-a}{b-a} & \text{for} \quad a \leq x < b \\ 1 & \quad b \leq x \end{cases}$$

So the CDF of a continuous RV uniformily distributed over $[a, b]$ is given by



3.3 CDF and Probability Density Function (PDF)

$\circledast$ $\boxed{\text{CDF}}$ **Def:**

A RV X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

→ The CDF of a continuous function does not have jumps. (this is consistent with $P_X(x) = 0$)

$P[X = x] = 0$

⊛ $\boxed{\text{PDF}}$

- For continuous RVs, the CDF works but PMF does not since $P[X = x] = 0$

- Instead we introduce the PDF

- For continuous RV, we define the function

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P[x < X \leq x + \Delta]}{\Delta}$$

Note that $P[x < X \leq x + \Delta] =$

$$F_X(x+\Delta) - F_X(x)$$

Thus we get

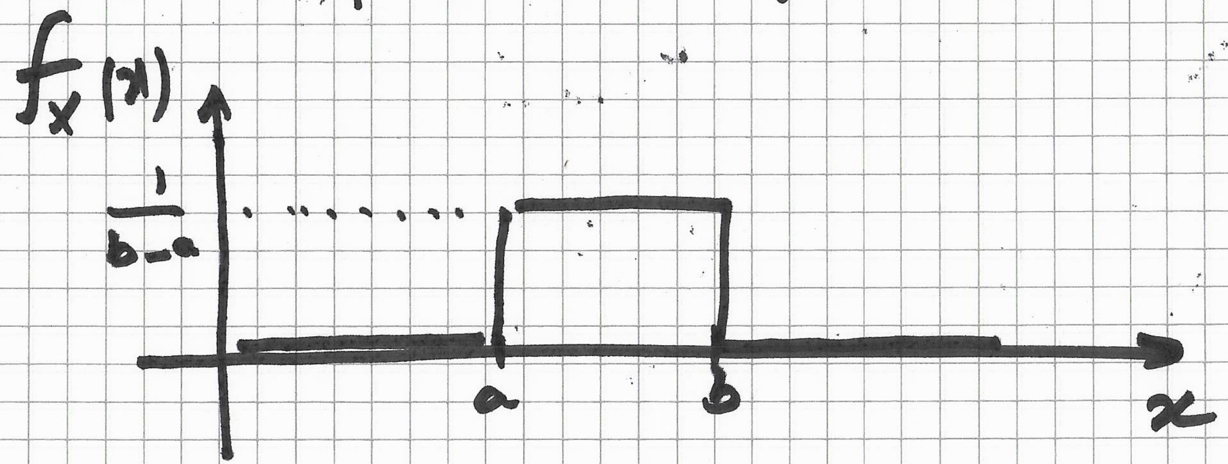
$$f_X(x) = \lim_{\Delta \to 0^+} \frac{F_X(x+\Delta) - F_X(x)}{\Delta}$$

$$= \frac{dF_X(x)}{dx} = F_X'(x)$$

$$\Rightarrow \boxed{f_X(x) = \frac{dF_X(x)}{dx}}$$

Going back to uniform RV example

$$f_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & \text{for } a \le x < b \\ 0 & \text{for } x \geq b \end{cases}$$



* Properties of PDF

. Consider a continuous RV with PDF $f_X(x)$, we have:

1- $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

2- $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

3- $P[a < x \leq b] = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$

4- More generally for a set A

$$P[X \in A] = \int_A f_X(x)\, dx$$

5- Range of X is defined as

$$R_X = \{ x \;/\; f_X(x) > 0 \}$$

3.4 Expected Value and Variance

Def $\mu_X = E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$

More generally $E[g(X)] = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx$

$$\sigma_X^2 = Var[X] = E[(X - \underbrace{\mu_X}_{E[X]})^2] = E[X^2] - (E[X])^2$$

- Moments . centered around mean $\longrightarrow E[(X-\mu_X)^n]$.

- Moments centered around zero:

$$E[X^n].$$

Some properties

- $E[aX+b] = a\,E(X) + b \quad (a,b) \in \mathbb{R}$
- $E[X_1 + X_2 + \dots + X_n] = E(X_1) + E(X_2) + \dots + E(X_n)$
- $Var[aX+b] = a^2\, Var[X]$

→ Define a **Standard RV**

For any RV X, we can define/introduce ~~the~~ its standarized RV

$$Z = \frac{X - \mu_X}{\sigma_X}$$

$$E[Z] = E\left[\frac{1}{\sigma_X} X - \frac{\mu_X}{\sigma_X}\right] = \frac{\overset{\mu_X}{\overset{\shortparallel}{E(X)}}}{\sigma_X} - \frac{\mu_X}{\sigma_X} = 0$$

$$Var[Z] = Var\left(\frac{1}{\sigma_X} X - \frac{\mu_X}{\sigma_X}\right] = \frac{1}{\sigma_X^2} \overset{\sigma_X^2}{\overset{\shortparallel}{Var(X)}} = 1$$

Z is a standard RV with mean 0 and variance (standard Deviation) 1.

3.5 Function of Continuous RV

→ If X is a continuous RV, then $Y = g(X)$ is also a RV.

→ Typically, we start by finding the CDF of Y ⇒ we differentiate CDF to deduce the PDF of Y.

a- Method of Transformations

<u>Theorem 1</u>: Suppose that X is a continuous RV and $g: \mathbb{R} \to \mathbb{R}$ is a stricly monotonic differentiable function. Let $Y = g(X)$

Then the PDF of Y is given by

$$f_Y(y) = \begin{cases} \dfrac{f_X(x_1)}{|g'(x_1)|} = f_X(x_1) \cdot \left|\dfrac{dx_1}{dy}\right| \\ \qquad \text{with } g(x_1) = y \\ = 0 \quad \text{if } g(x) = y \text{ does not have a solution.} \end{cases}$$

→ We can extend the previous Theorem to a more general case for g not monotonic.

we can divide g into finite number of monotonic differentiable fcts

Theorem 2

Consider a continuous RV X with domain R_x. Let $Y = g(X)$. Suppose that we can partition R_x into a finite number of intervals such that $g(x)$ is strictly monotone and differentiable on each partition. Then the PDF of Y is given by

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} = \sum_{i=1}^{n} f_X(x_i) \left| \frac{dx_i}{dy} \right|$$

with $x_1, x_2, \ldots, x_n$ are real solutions for $g(x) = y$.