Problem Set 3
Continuous and Mixed Random Variables

Date: Tuesday, June 25^{th}, 2019.

Problem 1:

Let X be a continuous random variable with the following PDF

$$f_X(x) = \begin{cases} ce^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

a. Find c.
b. Find the CDF of X, $F_X(x)$
c. Find $P(2 < X < 5)$
d. Find $\mathbb{E}(X)$.

Problem 2:

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq \frac{2}{3} | X > \frac{1}{3})$.

Problem 3:

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find $\mathbb{E}(X^n)$ for $n = 1, 2, 3, \ldots$
b. Find the variance of X.

Problem 4:

Let X be a uniform$(0, 1)$ random variable and let $Y = e^{-X}$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find $\mathbb{E}(Y)$

Problem 5:

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = X^2$.

a. Find the CDF of Y.
b. Find the PDF of Y.
c. Find $\mathbb{E}(Y)$.

Problem 6:

Let $X \sim$ Exponential(λ), and $Y = aX$, where a is a positive real number. Show that

$$Y \sim \text{Exponential}\left(\frac{\lambda}{a}\right)$$

Problem 7:

Let $X \sim$ Exponential(λ). Show that
a. $\mathbb{E}(X^n) = \frac{n}{\lambda}\mathbb{E}(X^{n-1})$ for $n = 1, 2, 3, \ldots$
b. $\mathbb{E}(X^n) = \frac{n!}{\lambda^n}$ for $n = 1, 2, 3, \ldots$

Problem 8:

Let $X \sim \mathcal{N}(3, 9)$.
a. Find $P(X > 0)$.
b. Find $P(-3 < X < 8)$.
c. Find $P(X > 5|X > 3)$.

Problem 9:

Let $X \sim \mathcal{N}(3, 9)$ and $Y = 5 - X$.
a. Find $P(X > 2)$.
b. Find $P(-1 < Y < 3)$.
c. Find $P(X > 4|Y < 2)$.

Problem 10:

Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 1 & x \geq 1 \\ \frac{1}{2} + \frac{x}{2} & 0 \leq x < 1 \\ 0 & x < 0 \end{cases}$$

(a) What kind of random variable is X: discrete, continuous, or mixed?
(b) Find the PDF of X, $f_X(x)$.
(c) Find $E(e^X)$.
(d) Find $P(X = 0|X \leq 0.5)$.

Problem 11:

Let $X \sim$ Exponential(2) and $Y = 2 + 3X$.
a. Find $P(X > 2)$.
b. Find $\mathbb{E}(Y)$ and Var(Y).
c. Find $P(X > 2|Y < 11)$.

Problem 12:

Let X be a $Uniform(2, 2)$ continuous random variable. We define $Y = g(X)$, where the function $g(x)$

is defined as

$$g(x) = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the CDF and PDF of Y.

Problem 13:

Let X be a random variable with the following CDF

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < \frac{1}{4} \\ x + \frac{1}{2} & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases}$$

a. Plot $F_X(x)$ and explain why X is a mixed random variable.
b. Find $P\left(X \leq \frac{1}{3}\right)$
c. Find $P\left(X \geq \frac{1}{4}\right)$
d. Write the CDF of X in the form of

$$F_X(x) = C(x) + D(x),$$

where $C(x)$ is a continuous function and $D(x)$ is in the form of a staircase function, i.e.

$$D(x) = \sum_k a_k u(x - x_k)$$

e. Find $c(x) = \frac{d}{dx}C(x)$.
f. Find $\mathbb{E}(X)$ using $\mathbb{E}(X) = \int_{-\infty}^{\infty} xc(x)dx + \sum_k x_k a_k$

Problem 14:

Let X be a random variable with the following CDF

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < \frac{1}{4} \\ x + \frac{1}{2} & \text{for } \frac{1}{4} \leq x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases}$$

a. Find the generalized PDF of X, $f_X(x)$.
b. Find $\mathbb{E}(X)$ using $f_X(x)$.
c. Find Var(X) using $f_X(x)$.

Problem 15:

Let X be a mixed random variable with the following generalized PDF

$$f_X(x) = \frac{1}{3}\delta(x+2) + \frac{1}{6}\delta(x-1) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

a. Find $P(X = 1)$ and $P(X = -2)$.
b. Find $P(X \geq 1)$.
c. Find $P(X = 1|X \geq 1)$

d. Find $\mathbb{E}(X)$ and $\text{Var}(X)$

Problem 16:
A company makes a certain device. We are interested in the lifetime of the device. It is estimated that around 2% of the devices are defective from the start so they have a lifetime of 0 years. If a device is not defective, then the lifetime of the device is exponentially distributed with a parameter $\lambda = 2$ years. Let X be the lifetime of a randomly chosen device.

a. Find the generalized PDF of X.
b. Find $P(X \geq 1)$.
c. Find $P(X > 2|X \geq 1)$
d. Find $\mathbb{E}(X)$ and $\text{Var}(X)$

Problem 17:
A continuous random variable is said to have a Laplace(μ, b) distribution if its PDF is given by

$$f_X(x) = \frac{1}{2b}\text{exp}\left(-\frac{|x-\mu|}{b}\right) = \begin{cases} \frac{1}{2b}\text{exp}\left(\frac{x-\mu}{b}\right) & \text{if } x < \mu \\ \frac{1}{2b}\text{exp}\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$$

where $\mu \in \mathbb{R}$ and $b > 0$.

a. If $X \sim$ Laplace$(0, 1)$, find $\mathbb{E}(X)$ and $\text{Var}(X)$.
b. If $X \sim$ Laplace$(0, 1)$ and $Y = bX + \mu$, show that $Y \sim$ Laplace(μ, b).
c. Let $Y \sim$ Laplace(μ, b), where $\mu \in \mathbb{R}$ and $b > 0$. Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

Problem 18:
Let $X \sim$ Laplace$(0, b)$, i.e.

$$f_X(x) = \frac{1}{2b}\text{exp}\left(-\frac{|x|}{b}\right)$$

where $b > 0$. Define $Y = |X|$. Show that $Y \sim$ Exponential $\left(\frac{1}{b}\right)$.

Problem 19:
A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Determine $E(X)$.

Problem 20:
A continuous random variable is said to have a **Rayleigh** distribution with parameter σ if its PDF is given by

$$f_X(x) = \frac{x}{\sigma^2}\text{exp}\left(-\frac{x^2}{2\sigma^2}\right)u(x) = \begin{cases} \frac{x}{\sigma^2}\text{exp}\left(-\frac{x^2}{2\sigma^2}\right) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $\sigma > 0$.

a. If $X \sim$ Rayleigh(σ), find $\mathbb{E}(X)$.
b. If $X \sim$ Rayleigh(σ), find the CDF of X, $F_X(x)$.
c. If $X \sim$ Exponential(1) and $Y = \sqrt{2\sigma^2 X}$, show that $Y \sim$ Rayleigh(σ).

Problem 21:

A continuous random variable is said to have a Pareto(x_m, α) distribution if its PDF is given by

$$f_X(x) = \begin{cases} \alpha \frac{x_m^\alpha}{x^{\alpha+1}} & \text{for } x \geq x_m \\ 0 & \text{for } x < x_m \end{cases}$$

where $x_m, \alpha > 0$. Let $X \sim$ Pareto(x_m, α).

a. Find the CDF of X, $F_X(x)$.

b. Find $P(X > 3x_m | X > 2x_m)$.

Problem 22:

Let $Z \sim \mathcal{N}(0, 1)$. If we define $X = e^{\sigma Z + \mu}$, then we say that X has a log-normal distribution with parameters μ and σ, and we write $X \sim$ LogNormal(μ, σ).

a. If $X \sim$ LogNormal(μ, σ), find the CDF of X in terms of the Φ function.

b. Find $\mathbb{E}(X)$ and Var(X).

Problem 23:

Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim$ Exponential(λ). Define

$$Y = X_1 + X_2 + \cdots + X_n$$

As we will see later, Y has a **Gamma** distribution with parameters n and λ, i.e., $Y \sim$ Gamma(n, λ). Using this, show that if $Y \sim$ Gamma(n, λ), then $\mathbb{E}(Y) = \frac{n}{\lambda}$ and Var$(Y) = \frac{n}{\lambda^2}$.

Problem 24:

Assume that the continuous RV X has the PDF $f(x) = ae^{-2a|x|}$. In what follows, we may need to use

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \; a \neq 0, \; n = 0, 1, \ldots.$$

(a) For what values of a is $f_X(x)$ a valid PDF?

(b) Find the mean of X.

(c) Find the variance of X.

(d) Deduce the standard deviation of X.

(e) Compute $P(-2 < X < 2)$.

Problem 25:

We define the function $f_X(\cdot)$ as follows:

$$f_X(x) = \begin{cases} Ke^{ax} & \text{if } x \leq 0, \\ Ke^{-bx} & \text{otherwise,} \end{cases}$$

where $a > 0$, $b > 0$, and K is a positive constant.

(a) Find K such that $f_X(\cdot)$ is a valid PDF.

(b) Find the corresponding CDF.

Problem 26:

Let $X \sim N(0, 1)$. Use a computer simulation to determine the number of outcomes out of 1000 that you would expect to occur within the interval $[1, 2]$.

Problem 27:

Use Python to solve this problem.

Total serum IgE (immunoglobulin E) concentration allergy tests allow for the measurement of the total IgE level in a serum sample. Elevated levels of IgE are associated with the presence of an allergy. The log concentration of IgE (in IU/ml) in a cohort of healthy subjects is distributed as a normal $N(9, (0.9)^2)$ random variable. What is the probability that in a randomly selected subject from the same cohort the log concentration will

(a) Exceed 10 IU/ml?

(b) Be between 8.1 and 9.9 IU/ml?

(c) Differ from the mean by no more than 1.8 IU/ml?

(d) Find the number x_0 such that the IgE log concentration in 90% of the subjects from the same cohort exceeds x_0.

Problem 28:

Use Python to solve this problem.

The time intervals between successive barges passing a certain point on a busy waterway have an exponential distribution with mean 8 minutes.

(a) Find the probability that the time interval between two successive barges is less than 5 minutes.

(b) Find a time interval t such that we can be 95% sure that the time interval between two successive barges will be greater than t.