Problem Set 1
Basic Concepts and Combinatorics

Date: Sunday 23^{rd} June, 2019.

Problem 1:

When working with real numbers, our universal set is $\mathbb{R}$. Find each of the following sets.
(a) $[6,8] \cup [2,7)$.
(b) $[6,8] \cap [2,7)$.
(c) $[0,1]^c$.
(d) $[6,8] - (2,7)$.

Problem 2:

If the universal set is given by $S = \{1,2,3,4,5,6\}$, and $A = \{1,2\}$, $B = \{2,4,5\}$, $C = \{1,5,6\}$ are three sets, find the following sets:
(a) $A \cup B$.
(b) $A \cap B$.
(c) $\overline{A}$.
(d) $\overline{B}$.
(e) Check De Morgan's law by finding $(A \cup B)^c$ and $A^c \cap B^c$.
(f) Check the distributive law by finding $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

Problem 3:

Given the sets $A_i = \{x : 0 \leq x \leq i\}$ for $i = 1, 2, \ldots, N$, find $\bigcup_{i=1}^{N} A_i$ and $\bigcap_{i=1}^{N} A_i$. Are the A_i's disjoint?

Problem 4:

Let $S = \{1,2,3\}$. Write all the possible partitions of S.

Problem 5:

Prove that if $x \in A \cap B^c$, then $x \in A \setminus B$.

Problem 6:

The sets A, B ,C are subsets of $\mathcal{S} = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. They are defined as

$$A = \{(x,y) : x \leq \frac{1}{2}, 0 \leq y \leq 1\}$$
$$B = \{(x,y) : x \geq \frac{1}{2}, 0 \leq y \leq 1\}$$
$$C = \{(x,y) : 0 \leq x \leq 1, y \leq \frac{1}{2}\}$$

Explicitly determine the set $A \cap (B \cup C)^c$ by drawing a picture of it as well as pictures of all the individual sets. For simplicity you can ignore the edges of the sets in drawing any diagrams. Can you represent the resultant set using only unions and complements?

Problem 7:

Suppose that the universal set S is defined as $S = \{1, 2, \ldots, 10\}$ and $A = \{1, 2, 3\}$, $B = \{X \in S : 2 \leq X \leq 7\}$, and $C = \{7, 8, 9, 10\}$.

(a) Find $A \cup B$.
(b) Find $(A \cup C) - B$.
(c) Find $\bar{A} \cup (B - C)$.
(d) Do A, B, and C form a partition of S?

Problem 8:

A coin is tossed twice. Let S be the set of all possible pairs that can be observed, i.e., $S = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$. Write the following sets by listing their elements.

(a) A: The first coin toss results in head.
(b) B: At least one tail is observed.
(c) C: The two coin tosses result in different outcomes.

Problem 9:

Suppose that A_1, A_2, A_3 form a partition of the universal set S. Let B be an arbitrary set. Assume that we know that

$$|B \cap A_1| = 10,$$
$$|B \cap A_2| = 20,$$
$$|B \cap A_3| = 15.$$

Find $|B|$.

Problem 10:

Two teams A and B play a soccer match, and we are interested in the winner. The sample space can be defined as $S = \{a, b, d\}$, where a shows the outcome that A wins, b shows the outcome that B wins, and d shows the outcome that they draw. Suppose we know that:

- the probability that A wins is $P(a) = P(\{a\}) = 0.5$,
- the probability of a draw is $P(d) = P(\{d\}) = 0.25$.

(a) Find the probability that B wins.
(b) Find the probability that B wins or a draw occurs.

Problem 11:

Let A and B be two events such that $P(A) = 0.4$, $P(B) = 0.7$, and $P(A \cup B) = 0.9$.

(a) Find $P(A \cap B)$.
(b) Find $P(A^c \cap B)$.
(c) Find $P(A - B)$.
(d) Find $P(A^c - B)$.
(e) Find $P(A^c \cup B)$.

Problem 12:

I roll a fair die twice and obtain two numbers: X_1 = result of the first roll, X_2 = result of the second roll.

(a) Find the probability that $X_2 = 4$.
(b) Find the probability that $X_1 + X_2 = 7$.
(c) Find the probability that $X_1 \neq 2$ and $X_2 \geq 4$.

Problem 13:

An urn (box) contains 30 balls, of which 10 are red and the other 20 blue. Suppose you take out 8 balls from this urn, without replacement. What is the probability that among the 8 balls in this sample exactly 3 are red and 5 are blue?

Problem 14:

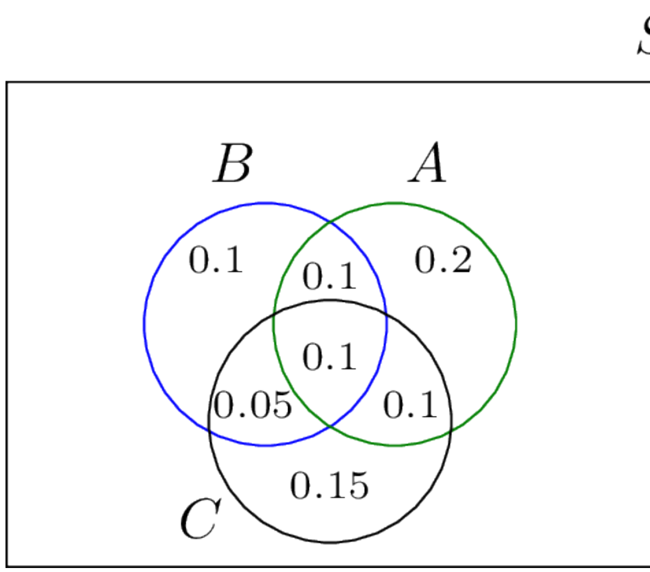
There are 20 black cell phones and 30 white cell phones in a store. An employee takes 10 phones at random. Find the probability that

(a) there will be exactly 4 black cell phones among the chosen phones,
(b) there will be less than 3 black cell phones among the chosen phones.

Does this data suggest that "getting an A" and "living on campus" are dependent or independent?

Problem 15:

Let A, B, and C be three events with probabilities given below:



(a) Find $P(A|B)$
(b) Find $P(C|B)$
(c) Find $P(B|A \cup C)$
(d) Find $P(B|A, C) = P(B|A \cap C)$

Problem 16:

Five cards are dealt from a shuffled deck. What is the probability that the dealt hand contains

(a) exactly one ace,
(b) at least one ace?

Problem 17:

Five cards are dealt from a shuffled deck. What is the probability that the dealt hand contains exactly two aces, given that we know it contains at least one ace?

Problem 18:

The 52 cards in a shuffled deck are dealt equally among four players, call them A, B, C, and D. If A and B have exactly 7 spades, what is the probability that C has exactly 4 spades?

Problem 19:

I have two bags. Bag 1 contains 10 blue marbles, while Bag 2 contains 15 blue marbles. I pick one of the bags at random, and throw 6 red marbles in it. Then I shake the bag and choose 5 marbles (without replacement) at random from the bag. If there are exactly 2 red marbles among the 5 chosen marbles, what is the probability that I have chosen Bag 1?

Problem 20:

A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$),

(a) You pick a coin at random and toss it. What is the probability that it lands heads up?
(b) You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

Problem 21:

You toss a fair coin three times. We assume that the coin tosses are independent.

(a) What is the probability of three heads, HHH?
(b) What is the probability that you observe exactly one heads?
(c) Given that you have observed at least one heads, what is the probability that you observe at least two heads?

Problem 22:

Suppose that, of all the customers at a coffee shop,

- 70% purchase a cup of coffee
- 40% purchase a piece of cake
- 20% purchase both a cup of coffee and a piece of cake

Given that a randomly chosen customer has purchased a piece of cake, what is the probability that he/she has also purchased a cup of coffee.

Problem 23:

In a demonstration/protest march, there are 2 kinds of people: civilians and undercover police. You know that 99% of the civilians are peaceful, and 1% advocate violence. You also know that 50% of the undercover police are agent provocateurs (who advocate violence) and 50% of them are peaceful. 2% of the protestors are undercover police. Given that you see a person in the demonstration advocating violence, what is the probability that this person is undercover police?

Problem 24:

Assume a committee of 10 has to be selected from a group of 100 people, of which 40 are men and 60 are women.

(a) How many ways are there to choose such a committee?
(b) What is the probability that a randomly selected committee of 10 consists of exactly 5 men and 5 women?

Problem 25:

In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

(a) one is boy and two are girls.
(b) all are boys.
(c) all are girls.

Problem 26:

A jar contains three white and two black balls. Each time, you pick at random one ball from the jar. If it is a white ball, a black ball is inserted instead; otherwise, a white ball is inserted instead. You continue until all balls in the jar are black. What is the probability that you need 17 picks to achieve this? Use Python to solve this problem.

Problem 27:

Use a computer simulation to simulate the tossing of a fair die. Based on the simulation what is the probability of obtaining an even number?

Problem 28:

An urn contains 4 red balls and 2 black balls. Two balls are chosen at random and without replacement. What is the probability of obtaining one red ball and one black ball in any order? Verify your results by enumerating all possibilities using a computer evaluation.

Problem 29:

A sequence of independent sub-experiments is conducted. Each sub-experiment has the outcomes "success", "failure", or "don't know". If $P(\text{success}) = \frac{1}{2}$ and $P(\text{failure}) = \frac{1}{4}$, what is the probability of 3 successes in 5 trials?