Solution of Problem Set 5

Problem 1:

This is, in fact, a hypergeometric distribution. First, note that we must have $X+Y=10$, so

$$\begin{aligned} R_{XY} &= \{(i,j) | i+j=10, i,j \in \mathbb{Z}, i,j \geq 0\} \\ &= \{(0,10),(1,9),(2,8),\ldots,(10,0)\}. \end{aligned}$$

Then, we can write

$$P_{XY}(i,j) = \begin{cases} \frac{\binom{40}{i}\binom{60}{j}}{\binom{100}{10}} & i+j=10, i,j \in \mathbb{Z}, i,j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem2:

To find the CDF of Z, we can write

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(\max(X,Y) \leq z) \\ &= P\Big((X \leq z) \text{ and } (Y \leq z)\Big) \\ &= P(X \leq z)P(Y \leq z) && (\text{ since } X \text{ and } Y \text{ are independent}) \\ &= F_X(z)F_Y(z). \end{aligned}$$

To find the CDF of W, we can write

$$\begin{aligned} F_W(w) &= P(W \leq w) \\ &= P(\min(X,Y) \leq w) \\ &= 1 - P(\min(X,Y) > w) \\ &= 1 - P\Big((X > w) \text{ and } (Y > w)\Big) \\ &= 1 - P(X > w)P(Y > w) && (\text{ since } X \text{ and } Y \text{ are independent}) \\ &= 1 - (1 - F_X(w))(1 - F_Y(w)) \\ &= F_X(w) + F_Y(w) - F_X(w)F_Y(w). \end{aligned}$$

Problem3:

a. The range of Z is given by

$$R_Z = \left\{\frac{m}{n} | m, n \in \mathbb{N}\right\},$$

which is the set of all positive rational numbers.

b. To find PMF of Z, let $m, n \in \mathbb{N}$ such that $(m, n) = 1$, where (m, n) is the largest divisor of m and n. Then

$$\begin{aligned}
P_Z\left(\frac{m}{n}\right) &= \sum_{k=1}^{\infty} P(X = mk, Y = nk) \\
&= \sum_{k=1}^{\infty} P(X = mk)P(Y = nk) \qquad \text{(since } X \text{ and } Y \text{are independent)} \\
&= \sum_{k=1}^{\infty} pq^{mk-1}pq^{nk-1} \qquad \text{(where } q = 1 - p\text{)} \\
&= p^2q^{-2} \sum_{k=1}^{\infty} q^{(m+n)k} \\
&= \frac{p^2q^{m+n-2}}{1 - q^{m+n}} \\
&= \frac{p^2(1-p)^{m+n-2}}{1-(1-p)^{m+n}}.
\end{aligned}$$

c. Find EZ: We can use LOTUS to find EZ. Let us first remember the following useful identities:

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \qquad \text{for } |x| < 1,$$

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{for } |x| < 1.$$

The first one is obtained by taking derivative of the geometric sum formula, and the second one is a Taylor series. Now, let's apply LOTUS.

$$\begin{aligned}
E\left[\frac{X}{Y}\right] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m}{n} P(X = m, Y = n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{m}{n} p^2q^{m-1}q^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} p^2q^{n-1} \sum_{m=1}^{\infty} mq^{m-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} p^2q^{n-1} \frac{1}{(1-q)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} q^{n-1} \\
&= \frac{1}{q} \sum_{n=1}^{\infty} \frac{q^n}{n} \\
&= \frac{1}{1-p} \ln\frac{1}{p}.
\end{aligned}$$

Problem 4:

We can write

$$f_{X,Y}(x,y) = f_X(x) f_Y(y),$$

where

$$f_X(x) = 2e^{-2x}u(x), \qquad f_Y(y) = 3e^{-3y}u(y).$$

Thus, X and Y are independent. Since X and Y are independent, we have $E[Y|X>2] = E[Y]$. Note that $Y \sim Exponential(3)$, thus $EY = \frac{1}{3}$. We have

$$\begin{aligned} P(X>Y) &= \int_0^\infty \int_y^\infty 6e^{-(2x+3y)}\,dxdy \\ &= \int_0^\infty 3e^{-5y}\,dy \\ &= \frac{3}{5}. \end{aligned}$$

Problem 5:

First note that, by the assumption

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2x} & -x \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1, -x \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$f_{XY}(x,y) = \begin{cases} 1 & |y| \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

First, note that $R_Y = [-1,1]$. To find $f_Y(y)$, we can write

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_{|y|}^{1} 1dx \\ &= 1 - |y|. \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} 1-|y| & |y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To find $P(|Y| < X^3)$, we can use the law of total probability (Equation 5.16):

$$\begin{aligned} P(|Y| < X^3) &= \int_0^1 P(|Y| < X^3 | X = x) f_X(x) dx \\ &= \int_0^1 P(|Y| < x^3 | X = x) 2x dx \\ &= \int_0^1 \left(\frac{2x^3}{2x}\right) 2x dx \qquad \text{since } Y|X = x \sim Uniform(-x, x) \\ &= \frac{1}{2}. \end{aligned}$$

Problem 6:

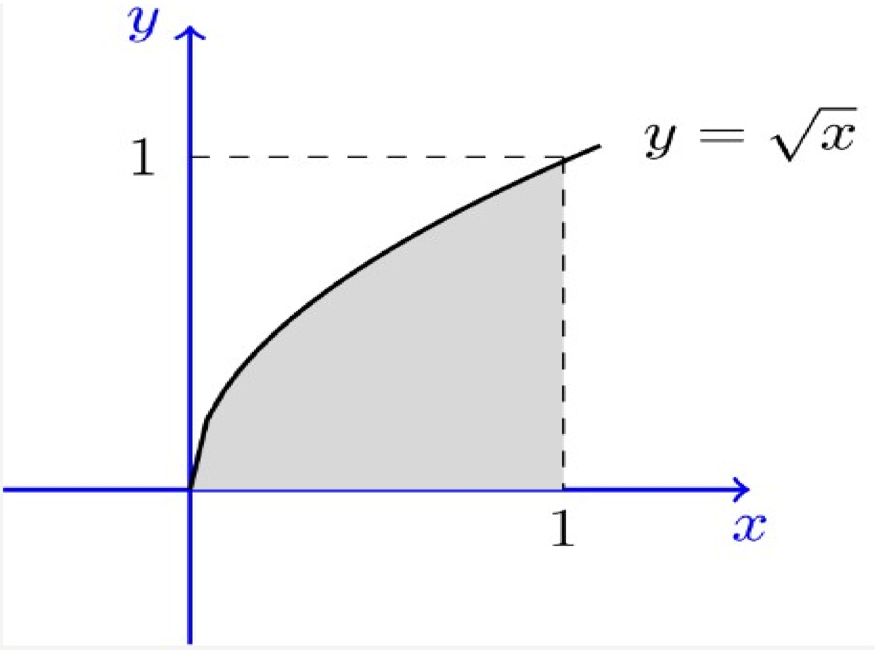

Figure 5.9: The figure shows R_{XY} for Solved Problem 4.

First, note that $R_X = R_Y = [0, 1]$. To find $f_X(x)$ for $0 \le x \le 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) \ dy \\ &= \int_0^{\sqrt{x}} 6xy \ dy \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

To find $f_Y(y)$ **for** $0 \leq y \leq 1$, **we can write**

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y) \; dx \\ &= \int_{y^2}^{1} 6xy \; dx \\ &= 3y(1-y^4). \end{aligned}$$

$$f_Y(y) = \begin{cases} 3y(1-y^4) & 0 \leq y \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

X **and** Y **are not independent, since** $f_{XY}(x,y) \neq f_x(x)f_Y(y)$. **We have**

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{2x}{1-y^4} & y^2 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We have

$$\begin{aligned} E[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y) \; dx \\ &= \int_{y^2}^{1} x \frac{2x}{1-y^4} \; dx \\ &= \frac{2(1-y^6)}{3(1-y^4)}. \end{aligned}$$

We have

$$\begin{aligned} E[X^2|Y=y] &= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y) \; dx \\ &= \int_{y^2}^{1} x^2 \frac{2x}{1-y^4} \; dx \\ &= \frac{1-y^8}{2(1-y^4)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}(X|Y=y) &= E[X^2|Y=y] - (E[X|Y=y])^2 \\ &= \frac{1-y^8}{2(1-y^4)} - \left(\frac{2(1-y^6)}{3(1-y^4)}\right)^2. \end{aligned}$$

Problem 7:

For $0 \leq x \leq 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^{1-x} 2dy \\ &= 2(1-x). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$\begin{aligned} EX &= \int_0^1 2x(1-x)dx \\ &= \frac{1}{3} = EY, \end{aligned}$$

$$\begin{aligned} EX^2 &= \int_0^1 2x^2(1-x)dx \\ &= \frac{1}{6} = EY^2. \end{aligned}$$

Thus,

$$Var(X) = Var(Y) = \frac{1}{18}.$$

We also have

$$\begin{aligned} EXY &= \int_0^1 \int_0^{1-x} 2xydydx \\ &= \int_0^1 x(1-x)^2 dx \\ &= \frac{1}{12}. \end{aligned}$$

Now, we can find $\mathrm{Cov}(X,Y)$ and $\rho(X,Y)$:

$$\begin{aligned}\mathrm{Cov}(X,Y) &= EXY - EXEY \\ &= \frac{1}{12} - \left(\frac{1}{3}\right)^2 \\ &= -\frac{1}{36},\end{aligned}$$

$$\begin{aligned}\rho(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{Var(X)Var(Y)}} \\ &= -\frac{1}{2}.\end{aligned}$$

Problem 8:

Note that you can look at this as a binomial experiment. In particular, we can say that X and Y are $Binomial(n, \frac{1}{6})$. Also, $X+Y$ is $Binomial(n, \frac{2}{6})$. Remember the variance of a $Binomial(n,p)$ random variable is $np(1-p)$. Thus, we can write

$$\begin{aligned}n\frac{2}{6}.\frac{4}{6} &= Var(X+Y) \\ &= Var(X) + Var(Y) + 2\mathrm{Cov}(X,Y) \\ &= n\frac{1}{6}.\frac{5}{6} + n\frac{1}{6}.\frac{5}{6} + 2\mathrm{Cov}(X,Y).\end{aligned}$$

Thus,

$$\mathrm{Cov}(X,Y) = -\frac{n}{36}.$$

And,

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{Var(X)Var(Y)}} = -\frac{1}{5}.$$

Problem 9:

Let $\mu = EX$. We first condition on the result of the first coin toss. Specifically,

$$\begin{aligned}\mu = EX &= E[X|H]P(H) + E[X|T]P(T)\\ &= E[X|H]p + (1+\mu)(1-p).\end{aligned}$$

In this equation, $E[X|T] = 1 + EX$, because the tosses are independent, so if the first toss is tails, it is like starting over on the second toss. Thus,

$$p\mu = pE[X|H] + (1-p) \tag{5.14}$$

We still need to find $E[X|H]$ so we condition on the second coin toss

$$\begin{aligned}E[X|H] &= E[X|HH]P + E[X|HT](1-p)\\ &= 2p + (2+\mu)(1-p)\\ &= 2 + (1-p)\mu.\end{aligned}$$

Here, $E[X|HT] = 2 + EX$ because, if the first two tosses are HT, we have wasted two coin tosses and we start over at the third toss. By letting $E[X|H] = 2 + (1-p)\mu$ in Equation 5.14, we obtain

$$\mu = EX = \frac{1+p}{p^2}.$$

Problem10:

a. To find $P(X \le 2, Y \le 4)$, we can write

$$\begin{aligned}P(X \le 2, Y \le 4) &= P_{XY}(1,2) + P_{XY}(1,4) + P_{XY}(2,2) + P_{XY}(2,4)\\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{3}{8}.\end{aligned}$$

b. Note from the table that

$$R_X = \{1,2,3\} \text{ and } R_Y = \{2,4,5\}.$$

Now we can use Equation 5.1 to find the marginal PMFs:

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{11}{24} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 2 \\ \frac{1}{4} & y = 4 \\ \frac{1}{4} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

c. Using the formula for conditional probability, we have

$$\begin{aligned} P(Y=2|X=1) &= \frac{P(X=1,Y=2)}{P(X=1)} \\ &= \frac{P_{XY}(1,2)}{P_X(1)} \\ &= \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}. \end{aligned}$$

d. Are X and Y independent? To check whether X and Y are independent, we need to check that $P(X=x_i, Y=y_j) = P(X=x_i)P(Y=y_j)$, for all $x_i \in R_X$ and all $y_j \in R_Y$. Looking at the table and the results from previous parts, we find

$$P(X=2,Y=2) = \frac{1}{6} \neq P(X=2)P(Y=2) = \frac{3}{16}.$$

Thus, we conclude that X and Y are not independent.

Problem 11:

We have

$$\begin{aligned} \mathrm{Var}(U+V) &= \mathrm{Var}(U) + \mathrm{Var}(V) + 2\mathrm{Cov}(U,V) \\ &= 1+1+2\rho_{XY}. \end{aligned}$$

Since $\mathrm{Var}(U+V) \geq 0$, we conclude $\rho(X,Y) \geq -1$. Also, from this we conclude that

$$\rho(-X,Y) \geq -1.$$

But $\rho(-X,Y) = -\rho(X,Y)$, so we conclude $\rho(X,Y) \leq 1$.

Problem 12:

Note that since X and Y are jointly normal, we conclude that the random variables $X+Y$ and $X-Y$ are also jointly normal. We have

$$\begin{aligned} \mathrm{Cov}(X+Y,X-Y) &= \mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y) \\ &= Var(X) - Var(Y) \\ &= 0. \end{aligned}$$

Since $X+Y$ and $X-Y$ are jointly normal and uncorrelated, they are independent.

Problem 13:

1. Since X and Y are jointly normal, the random variable $U = X + Y$ is normal. We have

$$EU = EX + EY = -1,$$

$$\begin{aligned} Var(U) &= Var(X) + Var(Y) + 2\mathrm{Cov}(X,Y) \\ &= 1 + 4 + 2\sigma_X\sigma_Y\rho(X,Y) \\ &= 5 - 2 \times 1 \times 2 \times \frac{1}{2} \\ &= 3. \end{aligned}$$

Thus, $U \sim N(-1,3)$. Therefore,

$$P(U > 0) = 1 - \Phi\left(\frac{0-(-1)}{\sqrt{3}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{3}}\right) = 0.2819$$

2. Note that $aX + Y$ and $X + 2Y$ are jointly normal. Thus, for them, independence is equivalent to having $\mathrm{Cov}(aX+Y, X+2Y) = 0$. Also, note that $\mathrm{Cov}(X,Y) = \sigma_X\sigma_Y\rho(X,Y) = -1$. We have

$$\begin{aligned} \mathrm{Cov}(aX+Y, X+2Y) &= a\mathrm{Cov}(X,X) + 2a\mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) + 2\mathrm{Cov}(Y,Y) \\ &= a - (2a+1) + 8 \\ &= -a + 7. \end{aligned}$$

Thus, $a = 7$.

Problem 14:

It is useful to find the distributions of Z and W. To find the CDF of Z, we can write

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(\max(X,Y) \leq z) \\ &= P\Big((X \leq z) \text{ and } (Y \leq z)\Big) \\ &= P(X \leq z)P(Y \leq z) && \text{(since } X \text{ and } Y \text{ are independent)} \\ &= F_X(z)F_Y(z). \end{aligned}$$

Thus, we conclude

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ z^2 & 0 \leq z \leq 1 \\ 1 & z > 1 \end{cases}$$

Therefore,

$$f_Z(z) = \begin{cases} 2z & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

From this we obtain $EZ = \frac{2}{3}$. Note that we can find EW as follows

$$\begin{aligned} 1 = E[X+Y] &= E[Z+W] \\ &= EZ + EW \\ &= \frac{2}{3} + EW. \end{aligned}$$

Thus, $EW = \frac{1}{3}$.

$$
\begin{aligned}
\mathrm{Cov}(Z,W) &= E[ZW] - EZEW \\
&= E[XY] - EZEW \\
&= E[X]E[Y] - E[Z]E[W] \qquad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= \frac{1}{2}\cdot\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{3} \\
&= \frac{1}{36}.
\end{aligned}
$$

Note that $\mathrm{Cov}(Z,W) > 0$ as we expect intuitively.

Problem15:

$$
\begin{aligned}
R_X &= \{1,2,3,\cdots\} \\
R_Y &= \{1,2,3,\cdots\}
\end{aligned}
$$

$$P_{XY}(k,l) = \frac{1}{2^{k+l}}.$$

$$
\begin{aligned}
P_X(k) &= \sum_{l \in R_Y} P(X=k, Y=l) = \sum_{l=1}^{\infty} \frac{1}{2^{k+l}} \\
&= \frac{1}{2^k} \sum_{l=1}^{\infty} \frac{1}{2^l} = \frac{1}{2^k}.
\end{aligned}
$$

$$
\begin{aligned}
P_Y(l) &= \sum_{k \in R_X} P(X=k, Y=l) = \sum_{k=1}^{\infty} \frac{1}{2^{k+l}} \\
&= \frac{1}{2^l} \sum_{k=1}^{\infty} \frac{1}{2^l} = \frac{1}{2^l}.
\end{aligned}
$$

By calculating the marginal PMFs we observe that $P_{XY}(k,l) = P_X(k) \cdot P_Y(l)$ for all $k \in R_X$ and $l \in R_Y$ $(k,l = 1,2,3,\cdots)$. So, these two variables are independent.

(b)

There are different cases in which $X^2 + Y^2 \leq 10$:

$$\begin{aligned} P(X^2+Y^2 \leq 10) &= P(X=1,Y=1) + P(X=1,Y=2) + P(X=2,Y=1) \\ &+ P(X=2,Y=2) + P(X=1,Y=3) + P(X=3,Y=1) \\ &= \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} = \frac{11}{16}. \end{aligned}$$

Problem16:

To find PMF of Z, let $i,j \in \{1,2,3,4,5\}$. Then ,

$$\begin{aligned} P_Z(Z=i-j) &= \sum_{i,j\in\{1,2,3,4,5\}} P(X=i, Y=j) \\ &= \sum_{i,j\in\{1,2,3,4,5\}} P(X=i)P(Y=j) \quad \text{(since } X \text{ and } Y \text{ are independent)} \end{aligned}$$

$$P_Z(z) = \begin{cases} \frac{1}{25} & \text{for } z=-4 \\ \frac{2}{25} & \text{for } z=-3 \\ \frac{3}{25} & \text{for } z=-2 \\ \frac{4}{25} & \text{for } z=-1 \\ \frac{5}{25} & \text{for } z=0 \\ \frac{4}{25} & \text{for } z=1 \\ \frac{3}{25} & \text{for } z=2 \\ \frac{2}{25} & \text{for } z=3 \\ \frac{1}{25} & \text{for } z=4 \\ 0 & \text{otherwise} \end{cases}$$

Problem 17:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

$$= \int_{y=0}^{1}\int_{x=0}^{\infty}\left(\frac{1}{2}e^{-x} + \frac{cy}{(1+x)^2}\right)dxdy$$

$$= \int_{0}^{1}\left[-\frac{1}{2}e^{-x} - \frac{cy}{(1+x)}\right]_0^{\infty} dy$$

$$= \int_{0}^{1}(\frac{1}{2} + cy)dy$$

$$= \left[\frac{1}{2}y + \frac{1}{2}cy^2\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2}c$$

Thus, $c = 1$.

(b)

$$P(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$$

$$= \int_{y=0}^{\frac{1}{2}}\int_{x=0}^{1} \frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}dxdy$$

$$= \int_{0}^{\frac{1}{2}}\left[-\frac{1}{2}e^{-x} - \frac{y}{1+x}\right]_0^1 dy$$

$$= \int_{0}^{\frac{1}{2}}\left[(\frac{1}{2} + y) - (\frac{1}{2}e^{-1} + \frac{y}{2})\right] dy$$

$$= \frac{5}{16} - \frac{1}{4e}$$

(c)

$$P(0 \leq X \leq 1) = \int_{y=0}^{1}\int_{x=0}^{1}\left(\frac{1}{2}e^{-x} + \frac{y}{(1+x)^2}\right)dxdy$$

$$= \frac{3}{4} - \frac{1}{2e}$$

Problem 18:

for $1 < x < e$:

$$f_X(x) = \int_{0}^{\infty} e^{-xy}dy$$

$$= -\frac{1}{x}e^{-xy}\Big|_0^{\infty}$$

$$= \frac{1}{x}$$

$$f_X(x) = \begin{cases} \frac{1}{x} & 1 \leq x \leq e \\ 0 & \text{otherwise} \end{cases}$$

for $0 < y$

$$
\begin{aligned}
f_Y(y) &= \int_1^e e^{-xy} dx \\
&= \frac{1}{y}(e^{-y} - e^{-ey})
\end{aligned}
$$

Thus,

$$
f_Y(y) = \begin{cases} \frac{1}{y}(e^{-y} - e^{-ey}) & y > 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

(b)

$$
\begin{aligned}
P(0 \le Y \le 1, 1 \le X \le \sqrt{e}) &= \int_{x=1}^{\sqrt{e}} \int_{y=0}^{1} e^{-xy} dy dx \\
&= \frac{1}{2} - \int_1^{\sqrt{e}} \frac{1}{x} e^{-x} dx
\end{aligned}
$$

Problem 19:

(a) Note that we can write $F_{XY}(x, y)$ as

$$
\begin{aligned}
F_{XY}(x, y) &= \left(1 - e^{-x}\right) u(x)(1 - e^{-2y})u(y) \\
&= (\text{a function of } \quad x) \cdot (\text{a function of } \quad y) \\
&= F_X(x) \cdot F_Y(y)
\end{aligned}
$$

i.e. X and Y are independent.

$$
F_X(x) = (1 - e^{(-x)})u(x)
$$

Thus $X \sim Exponential(1)$. So, we have $f_X(x) = e^{-x}u(x)$. Similarly, $f_Y(y) = 2e^{-2y}u(y)$ which results in:

$$
f_{XY}(x, y) = 2e^{(-x+2y)}u(x)u(y)
$$

(b)

$$
\begin{aligned}
P(X < 2Y) &= \int_{y=0}^{\infty} \int_{x=0}^{2y} 2e^{-(x+2y)} dx dy \\
&= \frac{1}{2}
\end{aligned}
$$

Problem 20:

(a) Let us first find $f_Y(y)$:

$$\begin{aligned} f_Y(y) &= \int_{-1}^{+1} (x^2 + \frac{1}{3}y)dx = [\frac{1}{3}x^3 + \frac{1}{3}yx]_{-1}^{+1} \\ &= \frac{2}{3}y + \frac{2}{3} \quad \text{for} \quad 0 \leq y \leq 1 \end{aligned}$$

Thus, for $0 \leq y \leq 1$, we obtain:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{x^2 + \frac{1}{3}y}{\frac{2}{3}y + \frac{2}{3}} = \frac{3x^2 + y}{2y + 2} \quad \text{for} \quad -1 \leq x \leq 1$$

For $0 \leq y \leq 1$:

$$f_{X|Y}(x|y) = \begin{cases} \frac{3x^2+y}{2y+2} & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$\begin{aligned} P(X > 0|Y = y) &= \int_0^1 f_{X|Y}(x|y)dx = \int_0^1 \frac{3x^2 + y}{2y + 2}dx \\ &= \frac{1}{2y+2}\int_0^1 (3x^2 + y)dx \\ &= \frac{1}{2y+2}\left[(x^3 + yx)\right]_0^1 = \frac{y+1}{2(y+1)} = \frac{1}{2} \end{aligned}$$

Thus it does not depend on y.

(c) X and Y are not independent. Since $f_{X|Y}(x|y)$ depends on y.

Problem 21:

Let us first find $f_{Y|X}(y|x)$. To do so, we need $f_X(x)$:

$$\begin{aligned} f_X(x) &= \int_0^1 (\frac{1}{2}x^2 + \frac{2}{3}y)dy = [\frac{1}{2}x^2y + \frac{1}{3}y^2]_0^1 \\ &= \frac{1}{2}x^2 + \frac{1}{3} \quad \text{for} \quad -1 \leq x \leq +1 \end{aligned}$$

Thus:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\frac{1}{2}x^2 + \frac{2}{3}y}{\frac{1}{2}x^2 + \frac{1}{3}}$$

Therefore:

$$f_{Y|X}(y|0) = \frac{\frac{2}{3}y}{\frac{1}{3}} = 2y \quad \text{for} \quad 0 \leq y \leq 1$$

Thus:

$$E[Y|X = 0] = \int_0^1 y f_{Y|X}(y|0)dy = \int_0^1 2y^2 dy = \frac{2}{3}$$

$$E[Y^2|X = 0] = \int_0^1 y^2 f_{Y|X}(y|0)dy = \int_0^1 2y^3 dy = \frac{1}{2}$$

Therefore:

$$\text{Var}(Y|X = 0) = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Problem 22:

We have:

$$\begin{aligned}
P(XY<1) &= \int_0^2 P(XY<1|Y=y)f_Y(y)dy \quad \text{Law of total prob} \\
&= \int_0^2 P(XY<1|Y=y)\frac{1}{2}dy \quad \text{Since } Y \sim Uniform(0,2) \\
&= \frac{1}{2}\int_0^2 P(X<\frac{1}{y})dy \quad X \text{ and } Y \text{ indep} \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} 1dy + \frac{1}{2}\int_{\frac{1}{2}}^2 \frac{1}{2y}dy \quad X \sim Uniform(0,2) \\
&= \frac{1}{4}[1+\ln 4] \approx 0.597
\end{aligned}$$

Problem 23:

Remember that if $Y \sim Uniform(a,b)$, then $EY = \frac{a+b}{2}$ and $\mathrm{Var}(Y) = \frac{(b-a)^2}{12}$

(a)

Using the law of total expectation:

$$\begin{aligned}
E[Y] &= \int_0^\infty E[Y|X=x]f_X(x)dx \\
&= \int_0^\infty E[Y|X=x]e^{-x}dx \quad \text{Since } Y|X \sim Uniform(0,X) \\
&= \int_0^\infty \frac{x}{2}e^{-x}dx = \frac{1}{2}[\int_0^\infty xe^{-x}dx] \\
&= \frac{1}{2}\cdot 1 = \frac{1}{2}
\end{aligned}$$

(b)

$$EY^2 = \int_0^\infty E[Y^2|X=x]f_X(x)dx = \int_0^\infty E[Y^2|X=x]e^{-x}dx \quad \text{Law of total expectation}$$

$Y|X \sim Uniform(0,X)$

$$\begin{aligned}
E[Y^2|X=x] &= \mathrm{Var}(Y|X=x) + (E[Y|X=x])^2 \\
&= \frac{x^2}{12} + \frac{x^2}{4} = \frac{x^2}{3} \\
EY^2 &= \int_0^\infty \frac{x^2}{3}e^{-x}dx = \frac{1}{3}\int_0^\infty x^2e^{-x}dx \\
&= \frac{1}{3}EW^2 = \frac{1}{3}[\mathrm{Var}(w) + (Ew)^2] = \frac{1}{3}(1+1) = \frac{2}{3} \quad \text{where } w \sim Exponential(1)
\end{aligned}$$

Therefore:

$$EY^2 = \frac{2}{3} \quad \mathrm{Var}(Y) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Problem 24:

(a)
$$\begin{aligned} E[XY] &= E[X]\cdot E[Y] \quad \text{Since } X \text{and } Y \text{ are indep} \\ &= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \end{aligned}$$

(b)

$$\begin{aligned} E[e^{X+Y}] &= E[e^X\cdot e^Y] = E[e^X]E[e^Y] \\ E[e^X] = E[e^Y] &= \int_0^1 e^x\cdot 1 \; dx = e-1 \end{aligned}$$

Therefore:

$$E[e^{X+Y}] = (e-1)(e-1) = (e-1)^2$$

(c)

$$\begin{aligned} E[X^2+Y^2+XY] &= E[X^2]+E[Y^2]+E[XY] \quad \text{linearity of expectation} \\ &= 2EX^2 + EXEY \end{aligned}$$

$$EX^2 = \int_0^1 x^2\cdot 1dx = \frac{1}{3}$$

Therefore:

$$E[X^2+Y^2+XY] = \frac{2}{3}+\frac{1}{4} = \frac{11}{12}$$

(d)

$$\begin{aligned} E[Ye^{XY}] &= \int_0^1\int_0^1 ye^{xy}dxdy \quad \text{LOTUS} \\ &= \int_0^1 [e^{xy}]_0^1 dy = \int_0^1 [e^y-1]dy = e-2 \end{aligned}$$

Problem 25:

Z and W are independent, thus $\text{Cov}(Z,W)=0$. Therefore:

$$\begin{aligned} 0 &= \text{Cov}(Z,W) = \text{Cov}(2X-Y, X+Y) \\ &= 2\cdot\text{Var}(X) + 2\cdot\text{Cov}(X,Y) - \text{Cov}(Y,X) - \text{Var}(Y) \\ &= 2\times 4 + \text{Cov}(X,Y) - 9 \end{aligned}$$

Therefore:

$$\begin{aligned} \text{Cov}(X,Y) &= 1 \\ \rho(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} \\ &= \frac{1}{2\times 3} = \frac{1}{6} \end{aligned}$$