Jacobian matrix and Hessian matrix

The defination of Jacobian and Hessian matrix is simple and obvious, in which the function is a vector function rather than numeric function.
Let's talk about Hessian matrix more about the details and properties.
Once Hessian matrix is real and symmetric, which requires that anywhere that the second partial derivatives are continuous. which impies that $\mathbf{H}_{ij} = \mathbf{H}_{ji}$

$$\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_i \partial \mathbf{x}_j} = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_j \partial \mathbf{x}_j}$$

We can decompose it into a set of real eigenvalues and an orthogonal basis of eigenvectors, i.e., $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ where $\mathbf{u}_i$ represent an eigenvector, each pair of them is linearly independent.

The second derivative in a specific direction represented by a unit vector $\mathbf{d}$ is given by $\mathbf{d}^T\mathbf{H}\mathbf{d}$. If $\mathbf{d}$ is an eigenvector, then the second derivative is given by the corresponding eigenvalue,