B样条

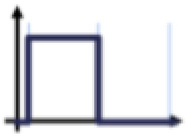
1. B样条基函数

1.1. De Boor递推

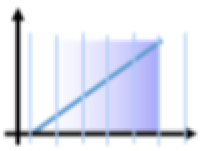
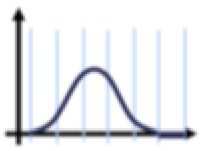
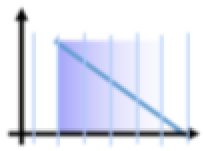
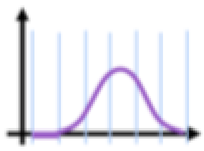
1.1.1. 单位情况

k阶（$k-1$度）单位B样条基函数表示为：

$$N_i^1(t)=\begin{cases}1, & i\le t<i+1\\ 0, & \text{otherwise}\end{cases}$$



$$\begin{aligned}N_i^k(t)&=\frac{t-i}{(i+k-1)-i}N_i^{k-1}(t)+\frac{(i+k)-t}{(i+k)-(i+1)}N_{i+1}^{k-1}(t)\\&=\frac{t-i}{k-1}N_i^{k-1}(t)+\frac{i+k-t}{k-1}N_{i+1}^{k-1}(t)\end{aligned}$$

1.1.2. 一般情况

- 给定：结序列$t_0<t_1<\cdots<t_n<\cdots<t_{n+k}$

  （$(t_0,t_1,\cdots,t_{n+k})$称为结向量）
- 归一化的k阶（$k-1$度）单位B样条基函数$N_{i,k}$定义为：

$$N_i^1(t)=\begin{cases}1, & i\le t<i+1\\ 0, & \text{otherwise}\end{cases}$$
$$N_{i,k}(t)=\frac{t-t_i}{t_{i+k-1}-t_i}N_{i,k-1}(t)+\frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}N_{i+1,k-1}(t)$$

  其中，$k>1$且$i=0,\cdots,n$

1.2. 核心思想

- 设计基函数$\boldsymbol{b}(t)$
- 性质：
  - $\boldsymbol{b}(t)$为C^2连续
  - $\boldsymbol{b}(t)$是三次分段多项式
  - $\boldsymbol{b}(t)$具有局部控制性质
  - 叠加位移的$\boldsymbol{b}(t+i)$组成一个整体的划分
  - 对所有的t，有$\boldsymbol{b}(t)\ge 0$
- 简而言之
  - 基函数中具有所有所需的性质
  - 基函数的线性组合也将有这些性质

1.3. 基函数性质

- 对$t_i < t < t_{i+k}$，有$N_{i,k}(t) > 0$
- 对$t_0 < t < t_i$或$t_{i+k} < t < t_{n+k}$，有$N_{i,k}(t) = 0$
- 对$t_{k-1} \leq t \leq t_{n+1}$，有$\sum_{i=1}^{n} N_{i,k}(t) = 1$
- 对于$t_i \leq t_j \leq t_{i+k}$，基函数$N_{i,k}(t)$在结点$t_j$处有$C^{k-2}$连续性
- 区间$[t_i, t_{i+k}]$称为$N_{i,k}$的支撑（support）

2. B样条曲线

2.1. B样条曲线简介

- 给定$n+1$个控制点$\boldsymbol{d}_0, \cdots, \boldsymbol{d}_n \in \mathbb{R}^3$，结向量$T = (t_0, \cdots, t_n, \cdots, t_{n+k})$
- k阶B样条曲线$\boldsymbol{x}(t)$定义为：

$$\boldsymbol{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \boldsymbol{d}_i$$

- 点$\boldsymbol{d}_i$称为de Boor points

2.1.1. 重复结向量

- B样条曲线允许：$T = (t_0, \cdots, t_n, \cdots, t_{n+k})$，$t_0 \le t_1 \le \cdots \le t_{n+k}$
- 只要不超过k个结重合，B样条函数$N_{i,k}(i = 0, \cdots, n)$的递归定义依然有效
- 多结重合的效果：
  - 设：$t_0 = t_1 = \cdots = t_{k-1}$
  - 且$t_{n+1} = t_{n+2} = \cdots = t_{n+k}$

  则将插值$\boldsymbol{d}_0$和$\boldsymbol{d}_n$

2.2. B样条的性质

2.2.1. B样条函数 vs Bernstein多项式

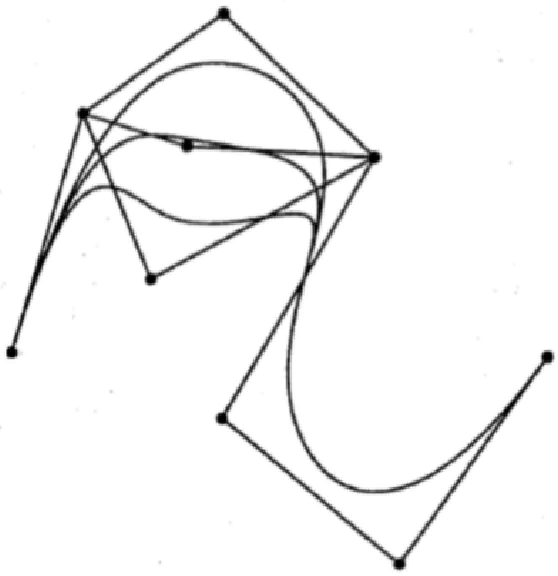
结向量$T = (t_0, t_1, \cdots, t_{2k-1}) = (\underbrace{0, \cdots, 0}_{k}, \underbrace{1, \cdots, 1}_{k})$下的$k$阶B样条函数$N_{i,k}(i = 0, \cdots, k-1)$为$k-1$次Bernstein多项式$B_i^{k-1}$

2.2.2. 基本性质

- 给定
  - $T = (\underbrace{t_0, \cdots, t_0}_{k \text{ times}}, t_k, \cdots, t_n, \underbrace{t_{n+1}, \cdots, t_{n+1}}_{k \text{ times}})$
  - de Boor多边形$\boldsymbol{d}_0, \cdots, \boldsymbol{d}_n$
- 相应的B样条曲线$\boldsymbol{x}(t)$有以下性质：
  - $\boldsymbol{x}(t_0) = \boldsymbol{d}_0$，$\boldsymbol{x}(t_{n+1}) = \boldsymbol{d}_n$（边界点插值）
  - $\dot{\boldsymbol{x}}(t_0) = \frac{k-1}{t_k - t_0}(\boldsymbol{d}_1 - \boldsymbol{d}_0)$（$\boldsymbol{d}_0$处的切线方向与$\boldsymbol{d}_n$处相似）
  - $\boldsymbol{x}(t)$由$n-k+2$个$k-1$次多项式曲线段构成
  - 多重内部结$\Rightarrow$减小了$\boldsymbol{x}(t)$的连续阶数

    l重内部结$(1 \le l < k)$意味着C^{k-l-1}阶连续
  - de Boor点的局部影响：移动$\boldsymbol{d}_i$只会改变曲线的$[t_i, t_{i+k}]$区间部分
  - 插入新的de Boor点不会改变曲线段的多项式阶数

2.2.3. B样条曲线的局部性



2.2.4. B样条曲线的升阶

- 使用B样条函数
- 使用de Boor算法

  与Bezier曲线的de Casteljau算法类似，在Boor多边形上进行一系列的线性插值

2.3. de Boor算法

- 给定：

  de Boor点：$\boldsymbol{d}_0, \cdots, \boldsymbol{d}_n$

  结向量：$(t_0, \cdots, t_{k-1} = t_0, t_k, t_{k+1}, \cdots, t_n, t_{n+1}, \cdots, t_{n+k} = t_{n+1})$
- 目标：k结B样条曲线的曲线点$\boldsymbol{x}(t)$
- 算法流程：

1. Search index r with $t_r \leq t < t_{r+1}$
2. for $i = r - k + 1, \dots, r$

   $d_i^0 = d_i$

- for $j = 1, \dots, k-1$

  for $i = r - k + 1 + j, \dots, r$

  $$d_i^j = \left(1 - \alpha_i^j\right) \cdot d_{i-1}^{j-1} + \alpha_i^j \cdot d_i^{j-1}$$

  with $\alpha_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$

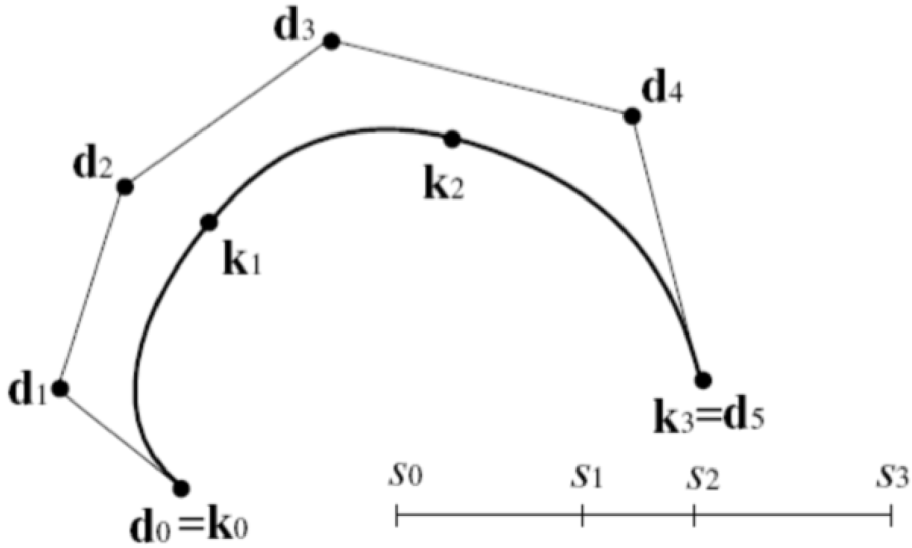
  Then: $d_r^{k-1} = x(t)$

- 中间系数$\boldsymbol{d}_i^j(t)$可以表示为一个下三角矩阵——de Boor图

$$\begin{array}{lllll}
\boldsymbol{d}_{r-k+1} = \boldsymbol{d}_{r-k+1}^0 & & & & \\
\boldsymbol{d}_{r-k+2} = \boldsymbol{d}_{r-k+2}^0 & \boldsymbol{d}_{r-k+2}^1 & & & \\
\vdots & & & & \\
\boldsymbol{d}_{r-1} = \boldsymbol{d}_{r-1}^0 & \boldsymbol{d}_{r-1}^1 & \cdots & \boldsymbol{d}_{r-1}^{k-2} & \\
\boldsymbol{d}_r = \boldsymbol{d}_r^0 & \boldsymbol{d}_r^1 & \cdots & \boldsymbol{d}_r^{k-2} & \boldsymbol{d}_r^{k-1} = \boldsymbol{x}(t)
\end{array}$$

2.4. B样条曲线插值

- 给定：$n+1$个控制点$\boldsymbol{k}_0,\cdots,\boldsymbol{k}_n$，结序列$s_0,\cdots,s_n$
- 目标：分段三次插值B样条曲线$\boldsymbol{x}$
- 方法：分段三次$\Rightarrow k=4$
    - $\boldsymbol{x}(t)$由n段组成$\Rightarrow n+3$个de Boor点
- 实例：$n=3$



- 若选择结向量

$$
\begin{aligned}
T &= (t_0,t_1,t_2,t_3,t_4,\cdots,t_{n+2},t_{n+3},t_{n+4},t_{n+5},t_{n+6}) \\
&= (s_0,s_0,s_0,s_0,s_1,\cdots,s_{n-1},s_n,s_n,s_n,s_n)
\end{aligned}
$$

- 插值条件：

$$
\begin{aligned}
\boldsymbol{x}(s_0) &= \boldsymbol{k}_0 = \boldsymbol{d}_0 \\
\boldsymbol{x}(s_i) &= \boldsymbol{k}_i = N_{i,4}(s_i)\boldsymbol{d}_i + N_{i+1,4}(s_i)\boldsymbol{d}_{i+1} + N_{i+2,4}(s_i)\boldsymbol{d}_{i+2} \\
& \quad \text{for } i=1,\cdots,n-1 \\
\boldsymbol{x}(s_n) &= \boldsymbol{k}_n = \boldsymbol{d}_{n+2}
\end{aligned}
$$

- 共计：$n+1$个条件解$n+3$个未知的de Boor点

  $\Rightarrow$ 2个终值条件

- natural end condition

$$
\begin{aligned}
\ddot{\boldsymbol{x}}(s_0) = 0 &\Leftrightarrow \frac{\boldsymbol{d}_2-\boldsymbol{d}_1}{s_2-s_0} = \frac{\boldsymbol{d}_1-\boldsymbol{d}_0}{s_1-s_0} \\
\ddot{\boldsymbol{x}}(s_n) = 0 &\Leftrightarrow \frac{\boldsymbol{d}_{n+2}-\boldsymbol{d}_{n+1}}{s_n-s_{n-1}} = \frac{\boldsymbol{d}_{n+1}-\boldsymbol{d}_n}{s_n-s_{n-2}}
\end{aligned}
$$

- 结果可以表示为求解对角系统方程：

$$
\begin{pmatrix}
1 & & & & & & & \\
\alpha_0 & \beta_0 & \gamma_0 & & & & & \\
 & \alpha_1 & \beta_1 & \gamma_1 & & & & \\
 & & & \ddots & & & & \\
 & & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} & \\
 & & & & & \alpha_n & \beta_n & \gamma_n \\
 & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
\boldsymbol{d}_0 \\ \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_n \\ \boldsymbol{d}_{n+1} \\ \boldsymbol{d}_{n+2}
\end{pmatrix}
=
\begin{pmatrix}
\boldsymbol{k}_0 \\ \boldsymbol{0} \\ \boldsymbol{k}_1 \\ \vdots \\ \boldsymbol{k}_{n-1} \\ \boldsymbol{0} \\ \boldsymbol{k}_n
\end{pmatrix}
$$

  其中，

$$
\begin{aligned}
\alpha_0 &= s_2 - s_0 \\
\beta_0 &= -(s_2 - s_0) - (s_1 - s_0) \\
\gamma_0 &= s_1 - s_0
\end{aligned}
$$

$$
\begin{aligned}
\alpha_n &= s_n - s_{n-1} \\
\beta_n &= -(s_n - s_{n-1}) - (s_n - s_{n-2}) \\
\gamma_n &= s_n - s_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i &= N_{i,4}(s_i) \\
\beta_i &= N_{i+1,4}(s_i) \\
\gamma_i &= N_{i+2,4}(s_i) \\
\text{for } i &= 1, \cdots, n-1
\end{aligned}
$$

- 解法
  - 托马斯算法——解决对角系统方程
  - 复杂度$O(n)$
  - 仅适用于对角占优矩阵

  对于对角系统方程

$$
\begin{pmatrix}
b_1 & c_1 & & & 0 \\
a_2 & b_2 & c_2 & & \\
 & a_3 & b_3 & \cdot & \\
 & & \cdot & \cdot & c_{n-1} \\
0 & & & a_n & b_n
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
=
\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}
$$

  求解流程：

  1. 前向消除阶段

$$
\begin{aligned}
&\text{for } k = 2:n \\
&\qquad m = \frac{a_k}{b_{k-1}} \\
&\qquad b_k = b_k - m c_{k-1} \\
&\qquad d_k = d_k - m d_{k-1} \\
&\text{end}
\end{aligned}
$$

  2. 后向替代阶段

$$
\begin{aligned}
&x_n = \frac{d_n}{b_n} \\
&\text{for } k = 2:n \\
&\qquad x_k = \frac{d_k - c_k x_{k+1}}{b_k} \\
&\text{end}
\end{aligned}
$$

2.5. Bezier曲线转B样条曲线

- 给定

  控制点：$\boldsymbol{k}_0, \cdots, \boldsymbol{k}_n$

  结序列：$t_0, \cdots, t_n$

  2个终值条件

  $\boldsymbol{b}_0, \cdots, \boldsymbol{b}_{3n}$：用于$C^2$连续插值三次Bezier样条曲线的Bezier点
- 目标：一些B样条形式的曲线

- 结向量

$$T=(t_0,t_0,t_0,t_0,t_1,\cdots,t_{n-1},t_n,t_n,t_n,t_n)$$

- $\boldsymbol{d}_0,\cdots,\boldsymbol{d}_{n+2}$由下列式子决定：

$$\begin{aligned}\boldsymbol{d}_0&=\boldsymbol{b}_0\\ \boldsymbol{d}_1&=\boldsymbol{b}_1\\ \boldsymbol{d}_i&=\boldsymbol{b}_{3i-4}+\frac{\Delta_{i-1}}{\Delta_{i-2}}(\boldsymbol{b}_{3i-4}-\boldsymbol{b}_{3i-5})\text{ for }i=2,\cdots,n\\ \boldsymbol{d}_{n+1}&=\boldsymbol{b}_{3n-1}\\ \boldsymbol{d}_{n+2}&=\boldsymbol{b}_{3n}\end{aligned}$$

  其中，$\Delta_i=t_{i+1}-t_i\quad\text{for}\quad i=0,\cdots,n-1$

- 逆问题同样可解

3. Bezier和B样条曲线小结

1. 由$n+1$个控制点$\boldsymbol{b}_0,\cdots,b_n$决定的Bezier曲线
   - n次多项式曲线
   - 由控制点唯一确定
   - 边界点作插值，其他点作逼近
   - 控制点的伪局部影响
2. 由控制点$\boldsymbol{k}_0,\cdots,\boldsymbol{k}_n$插值的三次Bezier样条曲线
   - 由n个分段三次曲线段组成
   - 控制点处有C^2连续性
   - 由参数化（如结序列）和两个终值条件唯一确定
   - 插值所有控制点
   - 控制点的伪局部影响
3. 由控制点$\boldsymbol{d}_0,\cdots,\boldsymbol{d}_n$和结向量$T=(t_0,t_0,t_0,t_0,t_1,\cdots,t_{n-1},t_n,t_n,t_n,t_n)$得到的分段三次B样条曲线
   - 由在结点处具有C^2连续状态的$n-2$条分段三次曲线段组成
   - 由$\boldsymbol{d}_i$和T唯一确定
   - 边界点插值，其余点逼近
   - de Boor点局部影响
4. 通过控制点$\boldsymbol{k}_0,\cdots,k_n$插值三次B样条
   - 对每个x,y,z分量，可以使用2个终值条件和求解一个对角线矩阵系统方程来实现类似3.的效果
   - 与2. 的曲线相同