probability integrals as

$$\mu_x(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)p\,(x(t))\;dx(t),$$

$$\mu_x(k) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(k)p\,(x(k))\;dx(k),$$

which can be defined element-wise as

$$\underset{x(t)\in X(t)}{\mathrm{E}} \langle x_i(t)\rangle = \int_{-\infty}^{\infty} x_i(t)\;p(x_i(t))\;dx_i(t) \quad i = 1,\;2,\;\ldots\;,n$$

$$\underset{x(k)\in X(k)}{\mathrm{E}} \langle x_i(k)\rangle = \int_{-\infty}^{\infty} x_i(k)\;p(x_i(k))\;dx_i(k) \quad i = 1,\;2,\;\ldots\;,n.$$

d

4.3.2.1 ***Zero-Mean Processes in Kalman Filtering*** Dynamic models for K filtering separate zero-mean inputs from other inputs—usually labeled as ("control inputs" or (unknown) "slow variables." Any nonzero-mean RP would be separated into its mean component and its (unknown) zero-mean compon the mean of an RP were to change unpredictably and slowly over time, then its would be modeled as a separate "slow variable," not as part of a short-term RP. for such slow variables are an integral part of Kalman filtering.

Therefore, the fundamental RP models used in Kalman filtering will be *zero-process models*. Models for all other slowly varying parameters or variables built up from zero-mean RP models—which are usually i.i.d. processes, as we

4.3.3 Time Correlation and Covariance

The *time correlation* of the n-vector-valued process $X(t)$ between any two tim and t_2 is defined as the $n \times n$ matrix

$$\underset{\substack{x(t_1)\in X(t_1)\\ x(t_2)\in X(t_2)}}{\mathrm{E}} \langle x(t_1)x^T(t_2)\rangle = \underset{\substack{x(t_1)\in X(t_1)\\ x(t_2)\in X(t_2)}}{\mathrm{E}} \left\langle \begin{bmatrix} x_1(t_1)x_1(t_2) & \cdots & x_1(t_1)x_n(t_2) \\ \vdots & \ddots & \vdots \\ x_n(t_1)x_1(t_2) & \cdots & x_n(t_1)x_n(t_2) \end{bmatrix} \right\rangle$$

$$= \begin{bmatrix} \mathrm{E}\,\langle x_1(t_1)x_1(t_2)\rangle & \cdots & \mathrm{E}\,\langle x_1(t_1)x_n(t_2)\rangle \\ \vdots & \ddots & \vdots \\ \mathrm{E}\,\langle x_n(t_1)x_1(t_2)\rangle & \cdots & \mathrm{E}\,\langle x_n(t_1)x_n(t_2)\rangle \end{bmatrix},$$

where, if the distributions can be defined in terms of probability density functi

$$\mathrm{E}\,\langle x_i(t_1)x_j(t_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i(t_1)x_j(t_2)p[x_i(t_1),x_j(t_2)]dx_i(t_1)dx_j(t_2).$$

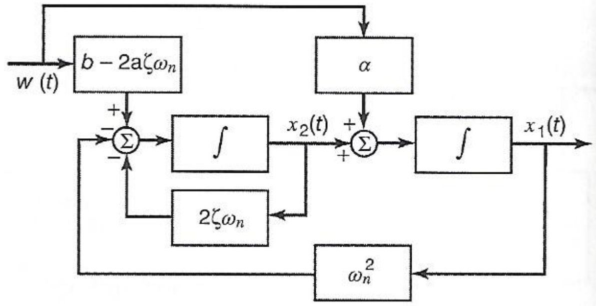


Figure 4.3 Diagram of a second-order Markov process

In practice, σ^2, θ, ζ, and w_n are chosen to fit empirical data (see Problem 4.4). The PSD corresponding to the $\psi_x(\tau)$ will have the form

$$\Psi_x(\omega) = \frac{a^2\,\omega^2 + b^2}{\omega^4 + 2\,\omega_n^2(2\zeta^2 - 1)\,\omega^2 + \omega_n^4}$$

(The peak of this PSD will not be at the "natural" (undamped) frequency ω_n, but at the "resonant" frequency defined in Example 2.6.)

The block diagram corresponding to the state-space model is shown in Figure 4.3.

The *mean power* of a scalar RP is given by the equations

$$\begin{aligned}
\mathop{\mathrm{E}}_t\langle x^2(t)\rangle &= \lim_{T\to\infty} \int_{-T}^{T} x^2(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_x(\omega)\,d\omega \\
&= \sigma^2.
\end{aligned}$$

The *cross power spectral density* between an RP $X(t)$ and an RP $Y(t)$ is given by the formula

$$\Psi_{xy}(\omega) = \int_{-\infty}^{\infty} \psi_{xy}(\tau)e^{-j\omega\tau}\,d\tau.$$

4.4 LINEAR RANDOM PROCESS MODELS

Linear system models of the type illustrated in Figure 4.4 are defined by the equation

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\,d\tau,$$

where $X(t)$ is input and $h(t,\tau)$ is the linear system *weighting function* (see Figure 4.4). If the system is time invariant (i.e., h does not depend on t), then Equation 4.5

process has the same formulation, except that $\acute{u}_k$ is a white sequence. Note that this formula for an autoregressive process can be transition matrix (STM) form as

$$
= \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \acute{x}_k \\ \acute{x}_{k-1} \\ \acute{x}_{k-2} \\ \vdots \\ \acute{x}_{k-n+1} \end{bmatrix} + \begin{bmatrix} \cancel{\acute{x}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \acute{u}_k, \tag{4.75}
$$

$$
= \Phi x_k + u_k, \tag{4.76}
$$

is the n-vector of the last n samples of the signal and the covariance associated process noise $\acute{u}_k$ will be filled with zeros, except for the

FILTERS (SF) AND STATE AUGMENTATION

a long and interesting history. In the early years of "wireless transmission, a spark provided a white-noise source which could circuit" (a shaping filter), the output of which was the radio transmission frequency band of interest.

section is on the use of shaping filters to model nonwhite-noise RPs, using white-noise processes as inputs. For many physical in practice, it may not be justified to assume that all noises noise processes. It can be useful to generate an autocorrelation from real data and then develop an appropriate noise model using difference equations. These models are called *shaping filters,* a in works by Hendrik Wade Bode (1905–1982) and Claude (1916–2001) [12] and by Lotfi Asker Zadeh and John Ralph [13]. They are filters driven by noise with a flat spectrum), which they shape to represent the spectrum of the actual in the previous section that a linear time-invariant system driven by WSS white Gaussian noise provides such a model. The "augmented" by appending to it the state vector components with the resulting model having the form of a linear dynamic white noise.

Process Noise Models

Filters for Dynamic Disturbance Noise Let a system model be

$$
= F(t)x(t) + G(t)w_1(t), \quad z(t) = H(t)x(t) + v(t), \tag{4.77}
$$

From Table 4.3,

$$\begin{aligned}
\mathrm{E}\dot{x}(t) &= -\mathrm{E}x(t) + 1 \\
\mathrm{E}x(0) &= 0 \\
\mathrm{E}x(t) &= 1 - e^{-t}, \ t \geq 0.
\end{aligned}$$

The covariance equation is then given by

$$\begin{aligned}
\dot{P}(t) &= -2P(t) + 1 \\
P(0) &= 0 \\
P(t) &= e^{-2t}P(0) + \int_0^t e^{-2(t-\tau)}1 \quad d\tau \\
&= \frac{1}{2}(1 - e^{-2t}).
\end{aligned}$$

The steady-state covariance is then

$$\begin{aligned}
P(\infty) &= \lim_{t\to+\infty} \frac{1}{2}(1 - e^{-2t}) \\
&= \frac{1}{2}.
\end{aligned}$$

Example 4.8 (Scalar System in Discrete Time) A discrete-time model is given as

$$\left.\begin{array}{rcll}
x_k^1 & = & -x_{k-1}^1 + w_{k-1}^1 & \\
\mathrm{E}w_{k-1}^1 & = & 0 & \text{(non-white)} \\
\Psi_w^1 & = & e^{-(k_1-k_2)} & \text{(autocorrelation)}
\end{array}\right\} (a)$$

The model of non-white noise evolves into scalar difference equation (given in Table 4.1), called a shaping filter:

$$x_k^2 = \mathrm{e}^{-1}x_{k-1}^2 + \sqrt{1-e^{-2}} \qquad w_{k-1} \quad (b),$$

where w_{k-1} is white noise with zero mean and covariance equal to 1.

Combining equation (a) and (b), as given by Equation ~~4.98~~, 4.81

$$x_k = \begin{bmatrix} -1 & 1 \\ 0 & e^{-1} \end{bmatrix} x_{k-1} + \begin{bmatrix} 0 \\ \sqrt{1-e^{-2}} \end{bmatrix} w_{k-1}$$

$$P_k = \begin{bmatrix} -1 & 1 \\ 0 & e^{-1} \end{bmatrix} \quad P_{k-1} \begin{bmatrix} -1 & 0 \\ 1 & e^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{1-e^{-2}} \end{bmatrix} 1 \begin{bmatrix} 0 & \sqrt{1-e^{-2}} \end{bmatrix}$$

$$P_k^{11} = +P_{k-1}^{11} - 2P_{k-1}^{12} + P_{k-1}^{22}$$

$$= -e^{-1}P_{k-1}^{12} + e^{-1}P_{k-1}^{22}$$

$$= e^{-2}P_{k-1}^{22} + (1 - e^{-2})$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

...state solution will be

$$P_{\infty}^{22} = \frac{(1 - e^{-2})}{1 - e^{-2}} = 1$$

$$P_{\infty}^{12} = \frac{\left[e^{-1}\frac{(1-e^{-2})}{1-e^{-2}}\right]}{1 + e^{-1}} = \frac{e^{-1}}{1+e^{-1}}$$

$$P_{\infty}^{11} = \text{underdetermined}.$$

...4.9 **(Steady-State Covariance of a Harmonic Resonator)** The stochas-... model

$$\dot{x}(t) = Fx(t) + w(t),$$

$$\mathrm{E}\,\langle w(t_1)w^{\mathrm{T}}(t_2)\rangle = \delta(t_1 - t_2)Q,$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$$

...underdamped harmonic resonator driven by zero-mean white *acceleration* ...

...of interest to find whether the covariance of the process $x(t)$ reaches a finite ...state value $P(\infty)$. Not every RP has a finite steady-state value, but it will turn ...finite in this example.

...that the state-space model for the mass-spring harmonic resonator from ...2.2, 2.3, and 2.7 has as its dynamic coefficient matrix the 2×2 constant ...

$$F = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{k_d}{m} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\omega_r^2 - \omega_d^2 & -2\omega_d \end{bmatrix},$$

...the supported mass, k_s is the spring elastic constant, and k_d is the dashpot ...coefficient. The alternate model parameters are

$$\omega_r = \sqrt{\frac{k_s}{m} - \frac{k_d^2}{4m^2}} \quad \text{(undamped resonant frequency)},$$

at time $t = s$. Consequently, the indefinite integral matrix

$$
\begin{aligned}
\Psi(t) &= \int_0^t \Phi^{-1}(s) \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \Phi^{\mathrm{T}-1}(s)\, ds \\
&= \frac{q}{\omega^2\tau^2} \int_0^t \begin{bmatrix} \tau^2 S(s)^2 & -\tau S(s)[\omega\tau C(s) + S(s)] \\ -\tau S(s)[\omega\tau C(s) + S(s)] & [\omega\tau C(s) + S(s)]^2 \end{bmatrix} e^{2s/\tau}\, ds \\
&= \begin{bmatrix} \dfrac{q\tau\{-\omega^2\tau^2 + [2S(t)^2 - 2C(t)\omega S(t)\tau + \omega^2\tau^2]\zeta^2\}}{4\omega^2(1+\omega^2\tau^2)} & \dfrac{-qS(t)^2\zeta^2}{2\omega^2} \\ \dfrac{-qS(t)^2\zeta^2}{2\omega^2} & \dfrac{q\{-\omega^2\tau^2 + [2S(t)^2 + 2C(t)\omega S(t)\tau + \omega^2\tau^2]\zeta^2\}}{4\omega^2\tau} \end{bmatrix}, \\
\zeta &= e^{t/\tau}.
\end{aligned}
$$

The discrete-time covariance matrix Q_{k-1} can then be evaluated as (see Section 4.7)

$$
\begin{aligned}
Q_{k-1} &= \Phi(\Delta t)\Psi(\Delta t)\Phi^{\mathrm{T}}(\Delta t) \\
&= \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \\
q_{11} &= \frac{q\tau\{\omega^2\tau^2(1 - e^{-2\Delta t/\tau}) - 2S(\Delta t)e^{-2\Delta t/\tau}[S(\Delta t) + \omega\tau C(\Delta t)]\}}{4\omega^2(1+\omega^2\tau^2)}, \\
q_{12} &= \frac{qe^{-2\Delta t/\tau}S(\Delta t)^2}{2\omega^2}, \\
q_{21} &= q_{12}, \\
q_{22} &= \frac{q\{\omega^2\tau^2(1 - e^{-2\Delta t/\tau}) - 2S(\Delta t)e^{-2\Delta t/\tau}[S(\Delta t) - \omega\tau C(\Delta t)]\}}{4\omega^2\tau}.
\end{aligned}
$$

Note that the *structure* of the discrete-time process noise covariance Q_{k-1} for this example is quite different from the continuous-time process noise Q. In particular, Q_{k-1} is a full matrix, although Q is a sparse matrix.

4.7.2.2 *First-order approximation of Q_k for constant F and G* The justification of a truncated power series expansion for Q_k when F and G are constant is as follows:

$$
Q_k = \sum_{i=1}^{\infty} \frac{Q^i \Delta t^i}{i!}. \qquad (4.1[illegible])
$$

Consider the Taylor series expansion of Q_k about t_{k-1}, where

$$
Q^i = \left. \frac{d^i Q}{dt^i} \right|_{t=t_{k-1}},
$$

where the ratio

$$r = \frac{\cancel{\mathrm{E}\langle xz\rangle}}{\sigma_x\sigma_z} \quad \frac{E[(x-E(x))(z-E(z))]}{\sigma_x \sigma_z} \tag{4.157}$$

is called the *correlation coefficient* of x and z, and σ_x, σ_z are standard deviations of x and z, respectively.

Suppose α_1 is specified. Then

$$\frac{d}{d\alpha_0}\mathrm{E}\left\langle [x - \alpha_0 - \alpha_1 z]^2 \right\rangle = 0 \tag{4.158}$$

and

$$\alpha_0 = \mathrm{E}\langle x\rangle - \alpha_1 \mathrm{E}\langle z\rangle. \tag{4.159}$$

Substituting the value of α_0 in $\mathrm{E}\langle [x - \alpha_0 - \alpha_1 z]^2\rangle$ yields

$$\begin{aligned}
\mathrm{E}\langle [x - \alpha_0 - \alpha_1 z]^2\rangle &= \mathrm{E}\langle [x - \mathrm{E}\langle x\rangle - \alpha_1(z - \mathrm{E}\langle z\rangle)]^2\rangle \\
&= \mathrm{E}\langle [(x - \mathrm{E}\langle x\rangle) - \alpha_1(z - \mathrm{E}\langle z\rangle)]^2\rangle \\
&= \mathrm{E}\langle [x - \mathrm{E}\langle x\rangle]^2\rangle + \alpha_1^2 \mathrm{E}\langle [z - \mathrm{E}\langle z\rangle]^2\rangle \\
&\quad - 2\alpha_1 \mathrm{E}\langle (x - \mathrm{E}\langle x\rangle)(z - \mathrm{E}\langle z\rangle)\rangle,
\end{aligned}$$

and differentiating with respect to α_1 as

$$\begin{aligned}
0 &= \frac{d}{d\alpha_1}\mathrm{E}\langle [x - \alpha_0 - \alpha_1 z]^2\rangle \\
&= 2\alpha_1 \mathrm{E}\langle (z - \mathrm{E}\langle z\rangle)^2\rangle - 2\mathrm{E}\langle (x - \mathrm{E}\langle x\rangle)(z - \mathrm{E}\langle z\rangle)\rangle, && (4.160)\\
\alpha_1 &= \frac{\mathrm{E}\langle (x - \mathrm{E}\langle x\rangle)(z - \mathrm{E}\langle z\rangle)\rangle}{\mathrm{E}\langle (z - \mathrm{E}\langle z\rangle)^2\rangle} \\
&= \frac{r\sigma_x\sigma_z}{\sigma_z^2} \\
&= \frac{r\sigma_x}{\sigma_z}, && (4.161)\\
e_{\min} &= \sigma_x^2 - 2r^2\sigma_x^2 + r^2\sigma_x^2 \\
&= \sigma_x^2(1 - r^2).
\end{aligned}$$

Note that if one assumes that x and z have zero means,

$$\mathrm{E}\langle x\rangle = \mathrm{E}\langle z\rangle = 0, \tag{4.162}$$

then we have the solution

$$\alpha_0 = 0. \tag{4.163}$$

Orthogonality Principle The co

e

is such that $x - \alpha_1 z$ is *orthogonal* t

E

and the value of the minimum MS

e_m

4.8.3 A Geometric Interpretati

Consider all RVs as vectors in abs
taken as the second moment $\mathrm{E}\langle xz$

is the square of the length of x. T
Figure 4.7.

The MS error $\mathrm{E}\langle (x - \alpha_1 z)^2\rangle$ is
minimum if $x - \alpha_1 z$ is orthogonal

E

We will apply the orthogonality p

4.9 SUMMARY

4.9.1 Important Points to Rem

Events Form a Sigma Algebra of O
is an undertaking with an uncertai

Figure 4.

... of the distribution and P is the covariance matrix of deviations ...

... *process in continuous time* with state x and state covariance P ...

$$
\begin{aligned}
\dot{x}(t) &= F(t)x(t) + G(t)w(t), \\
z(t) &= H(t)x(t) + v(t), \\
\dot{P}(t) &= F(t)P(t) + P(t)F^{\mathrm{T}}(t) + G(t)Q(t)G^{\mathrm{T}}(t),
\end{aligned}
$$

... the covariance of zero-mean *plant noise* $w(t)$. A *discrete-time linear* ... has the model equations

$$
\begin{aligned}
x_k &= \Phi_{k-1}x_{k-1} + G_{k-1}\,w_{k-1}, \\
z_k &= H_k x_k + v_k, \\
P_k &= \Phi_{k-1}P_{k-1}\Phi_{k-1}^{\mathrm{T}} + G_{k-1}Q_{k-1}G_{k-1}^{\mathrm{T}},
\end{aligned}
$$

... state, z is the system output, w is the zero-mean uncorrelated ... is its covariance of w_{k-1}, and v is the zero-mean uncorrelated *mea-* ... noise is also called *process noise*. These models may also have ... *filters* are models of these types that are used to represent RPs ... of spectral properties or temporal correlations.

... discrete-time model

$$... + w_{k-1},\ Ew_{k-1} = 0,\ Q_{k-1} = \Delta(k_1 - k_2),\ P_0 = 1, \text{ and } \mathop{\mathrm{E}}_{x}\langle x_0\rangle = 1,$$

... P_k, and P_∞.

... amplitude-modulated signal is specified by

$$y(t) = [1 + mx(t)]\cos(\Omega t + \lambda).$$

... is a WSS RP independent of λ, which is an RV uniformly distributed ... $[0, 2\pi]$. We are given that

$$\psi_x(\tau) = \frac{1}{\tau^2 + 1}.$$

(a) ... that $\psi_x(\tau)$ is an autocorrelation.

(b) ... have the autocorrelation given above. Using the direct method for ... the spectral density, calculate Ψ_y.

... an autocorrelation function of a WSS process?

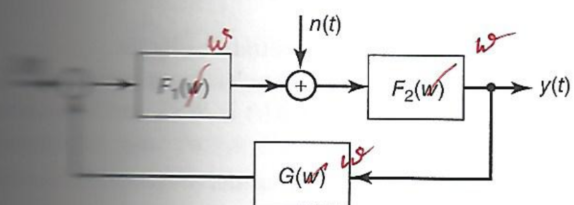
$$\psi_x(\tau) = 1.5e^{-|\tau|} + (11/3)e^{-3|\tau|}.$$

... following:

... between stationarity and wide-sense stationarity.

... character of the cross-correlation function of two processes ... periodic with periods mT and nT, respectively.

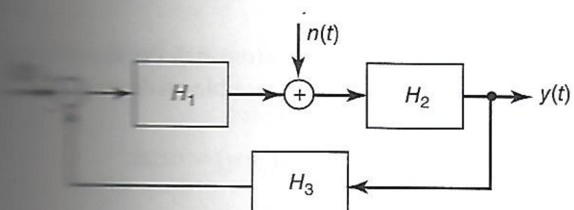
... function can sometimes be experimentally determined by ... noise $w(t)$ and measuring the cross-correlation between the ... and the white noise. Here we consider the following system:



n(t)

... known, $S(t)$ and $\cancel{w(t)}$ independent, and $\Psi_n(\omega) = 1$. Find ... $y(t) = y_S(t) + y_n(t)$, where y_S and y_n are the parts of the ... respectively.

... real stationary uncorrelated RPs, each with mean zero.



..., $H_2(j2\pi\omega)$, and $H_3(j2\pi\omega)$ are transfer functions of ... linear systems and $S_0(t)$ is the output when $w(t)$ is zero ... output when $S(t)$ is zero. Find the output signal-to-noise ratio, ... $/\mathrm{E}\langle n_0^2(t)\rangle$.

4.21 Let $x(t)$ be the solution of

$$\dot{x} + x = w(t)$$

with initial condition $x(0) = x_0$. It is assumed that $w(t)$ is white noise with spectral density 2π and is turned on at $t = 0$. The initial condition x_0 is an RV independent of $w(t)$ and with zero mean.

(a) If x_0 has variance σ^2, what is $\psi_x(t_1, t_2)$? Derive the result.

(b) Find that value of σ (call it σ_0) for which $\psi_x(t_1, t_2)$ is the same for all $t \geq 0$. Determine whether, with $\sigma = \sigma_0$, $\psi_x(t_1, t_2)$ is a function only of $t_1 - t_2$.

(c) If the white noise had been turned on at $t = -\infty$ and the initial condition has zero mean and variance σ_0^2 as above, is $x(t)$ WSS? Justify your answer by appropriate reasoning and/or computation.

4.22 Let

$$\dot{x}(t) = F(t)x(t) + w(t),$$
$$x(a) = x_a, t \geq a,$$

where x_a is a zero-mean RV with covariance matrix P_a and

$$\mathrm{E}\langle w(t)\rangle = 0 \quad \forall t,$$
$$\mathrm{E}\langle w(t)w^{\mathrm{T}}(s)\rangle = Q(t)\delta(t-s) \quad \forall t, s$$
$$\mathrm{E}\langle x(a)w^{\mathrm{T}}(t)\rangle = 0 \quad \forall t.$$

(a) Determine the mean m(t) and covariance P(t, t) for the process x(t).

(b) Derive a differential equation for P(t, t).

4.23 Find the covariance matrix P(t) and its steady-state value $P(\infty)$ for the following continuous systems:

(a) $\dot{x} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1, \end{bmatrix} w(t), \quad P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1, \end{bmatrix},$

(b) $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \end{bmatrix} w(t), \quad P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $w \in \mathcal{N}(0, 1)$ and white.

4.24 For the continuous-time system

$$\dot{x}(t) = -x(t) + w(t)$$
$$E\, x(0) = 1$$
$$P(0) = 1$$
$$E\, w(t) = 2$$
$$Q(t) = e^{-2}\delta(t - \tau).$$

Find $\mathrm{E}\langle x(t)\rangle$, $P_x(t)$, P_∞.

4.25 Find the covariance matrix P_k and its steady-state value P_∞ for the
discrete system:

$$x_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $w_k \in N(0, 1)$ and white.

4.26 Find the steady-state covariance for the state-space model
Example ~~3.4.~~ 4.2

4.27 Show that the continuous-time steady-state algebraic equation

$$0 = FP(\infty) + P(\infty)F^{\mathrm{T}} + GQG^{\mathrm{T}}$$

has no nonnegative solution for the scalar case with $F = Q = G = 1$

4.28 Show that the discrete-time steady-state algebraic equation

$$P_\infty = \Phi P_\infty \Phi^{\mathrm{T}} + Q$$

has no solution for the scalar case with $\Phi = Q = 1$.

4.29 Find the covariance of x_k as a function of k and its steady-state
system

$$x_k = -2x_{k-1} + w_{k-1},$$

where $\mathrm{E}w_{k-1} = 0$ and $\mathrm{E}(w_k w_j) = e^{-|k-j|}$. Assume the initial
covariance (P_0) is 1.

4.30 Find the covariance of $x(t)$ as a function of t and its steady-state
system

$$\dot{x}(t) = -2x(t) + w(t),$$

where $\mathrm{E}w(t) = 0$ and $\mathrm{E}(w(t_1) \quad w(t_2)) = e^{-|t_1 - t_2|}$. Assume the initial
the covariance (P_0) is 1.

4.31 Suppose that $x(t)$ has autocorrelation function $\psi_x(\tau) = e^{-c|\tau|}$. It is
predict $x(t + \alpha)$ on the basis of the past and present of $x(t)$, that is,
may use $x(s)$ for all $s \leq t$.

(a) Show that the minimum mean-square error linear prediction is

$$\hat{x}(t + \alpha) = e^{-c\alpha} x(t).$$

(b) Find the MS error corresponding to the above. *Hint:* Use the
principle.

5.2 KALMAN FILTER

5.2.1 Observational Update Problem for System State Estimator

Suppose that a measurement has been made at time t_k and that the information it provides is to be applied in updating the estimate of the state x of a stochastic system at time t_k. It is assumed that the measurement is linearly related to the state by an equation of the form $z_k = Hx_k + v_k$, where H is the *measurement sensitivity matrix* and v_k is the *measurement noise*.

5.2.2 Estimator in Linear Form

4.8.1

The optimal linear estimate is equivalent to the general (nonlinear) optimal estimator if the variates x and z are jointly Gaussian (see Section ~~5.8.1~~). Therefore, it suffices to seek an updated estimate $\hat{x}_{k(+)}$—based on the observation z_k—that is a *linear function* of the a priori estimate and the measurement z:

$$\hat{x}_{k(+)} = K_k^1 \hat{x}_{k(-)} + \overline{K}_k z_k, \tag{5.7}$$

where $\hat{x}_{k(-)}$ is the a priori estimate of x_k and $\hat{x}_{k(+)}$ is the a posteriori value of the estimate.

5.2.3 Solving for the Kalman Gain

The matrices K_k^1 and $\overline{K}_k$ are as yet unknown. We seek those values of K_k^1 and $\overline{K}_k$ such that the new estimate $\hat{x}_{k(+)}$ will satisfy the orthogonality principle of Section 4.8.2. This orthogonality condition can be written in the form

$$\mathrm{E}\langle [x_k - \hat{x}_{k(+)}] z_i^{\mathrm{T}} \rangle = 0, \quad i = 1, 2, \dots, k-1, \tag{5.8}$$

$$\mathrm{E}\langle [x_k - \hat{x}_{k(+)}] z_k^{\mathrm{T}} \rangle = 0. \tag{5.9}$$

If one substitutes the formula for x_k from Equation 5.1 (in Table 5.1) and for $\hat{x}_{k(+)}$ from Equation 5.7 into Equation 5.8, then one will observe from Equations 5.1 and 5.2 that the data $z_1, \dots, z_k$ do not involve the noise term w_k. Therefore, because the random sequences w_k and v_k are uncorrelated, it follows that $\mathrm{E}\langle w_k z_i^{\mathrm{T}} \rangle = 0$ for $1 \le i \le k$. (See Problem 5.6.)

Using this result, one can obtain the following relation:

$$\mathrm{E}\langle [\Phi_{k-1} x_{k-1} + w_{k-1} - K_k^1 \hat{x}_{k(-)} - \overline{K}_k z_k] z_i^{\mathrm{T}} \rangle = 0, \quad i = 1, \dots, k-1. \tag{5.10}$$

But because $z_k = H_k x_k + v_k$, Equation 5.10 can be rewritten as

$$\mathrm{E}\langle [\Phi_{k-1} x_{k-1} - K_k^1 \hat{x}_{k(-)} - \overline{K}_k H_k x_k - \overline{K}_k v_k] z_i^{\mathrm{T}} \rangle = 0, \quad i = 1, \dots, k-1. \tag{5.11}$$

for the propagation of the estimation error, $\tilde{x}$. Postmultiply it by $\tilde{x}_k^{\mathrm{T}}(-)$ (on both sides of the equation) and take the expected values. Use the fact that $\mathrm{E}\tilde{x}_{k-1}\, w_{k-1}^{\mathrm{T}} =$ … obtain the results

$$\begin{aligned} P_{k(-)} &\stackrel{\text{def}}{=} \mathrm{E}\langle \tilde{x}_{k(-)}\tilde{x}_{k(-)}^{\mathrm{T}}\rangle \\ &= \Phi_{k-1}\mathrm{E}\langle \tilde{x}_{k-1(+)}\tilde{x}_{k-1(+)}^{\mathrm{T}}\rangle\Phi_{k-1}^{\mathrm{T}} + \mathrm{E}\langle w_{k-1}\, w_{k-1}^{\mathrm{T}}\rangle \\ &= \Phi_{k-1}P_{k-1}^{(+)}\Phi_{k-1}^{\mathrm{T}} + Q_{k-1}, \end{aligned} \qquad (5.2$$

which gives the a priori value of the covariance matrix of estimation uncertainty as a function of the previous a posteriori value.

5.2.4 Kalman Gain from Gaussian Maximum Likelihood

The original derivation of the Kalman gain makes the fewest possible assumptions but requires the most mathematical rigor to obtain the most general result. The essential elements of that proof have been covered in the previous sections and Chapter 4.

An alternative derivation using the *linear Gaussian maximum-likelihood estimator* (*LGMLE* or *GMLE*) makes much more restrictive assumptions about the distributions of the state vector and measurements, but the resulting formula for the Kalman gain is the same. This derivation was introduced after some instructors observed symptoms resembling post traumatic stress disorder among students struggling with the more rigorous derivation.

In essence, this approach uses the mean μ_x and the information matrix $Y_{xx} =$ … as parameters for Gaussian distributions, then dismisses the Gaussian normalizing factors to allow Y_{xx} to represent measurements that would render it singular.

The resulting functions are no longer probability functions, because their integrals are no longer 1 (one) and are not necessarily finite. They are called *Gaussian likelihood functions*, and they have properties similar to those of Gaussian probability distributions for joint and independent likelihoods. However, because integrals of likelihood functions are not necessarily defined, *expectation* can no longer be used to characterize likelihoods the way it characterizes probability measures. Least-mean-squared estimation error is not defined for likelihood functions, so it is replaced by *maximum likelihood* as a criterion for optimal estimation.

The means and information matrices of Gaussian likelihood functions are the parameters used in deriving the Kalman gain.

Example 5.1 (Combining Independent Gaussian Likelihoods) Consider the two-dimensional Gaussian likelihood function with a singular information matrix

$$Y_a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

in which case the 2D Gaussian likelihood function will have a shape like that shown in Figure 5.1(a), with the direction of "no information" shown by the double-ended arrow indicating the direction of the zero eigenvector of Y_{xx}.

$$\mathcal{X}_k = P_{k-1}H_k^{\mathrm{T}}\mathcal{Y}_k \tag{5.101}$$

$$= P_{k-1}H_k^{\mathrm{T}}[R_k + H_kP_{k-1}H_k^{\mathrm{T}}]^{-1} \tag{5.102}$$

$$\stackrel{\text{def}}{=} \overline{K}_k, \tag{5.103}$$

so that Equation 5.94 becomes

$$\hat{x}_k = \hat{x}_{k-1} - \overline{K}_kH_k\hat{x}_{k-1} + \overline{K}_kz_k \tag{5.104}$$

$$= \hat{x}_{k-1} + \overline{K}_k(z_k - H_k\hat{x}_{k-1}). \tag{5.105}$$

That is, the formula for the recursive LMS estimate in Equation 5.105 is exactly the same as the Kalman measurement update formula, and with the same Kalman gain, as given by Equation ~~5.88.~~ 5.49

This completes the derivation of the Kalman gain from the recursive LMS estimator.

5.2.6 Summary of Equations for the Discrete-Time Kalman Estimator

The equations derived in the previous section are summarized in Table 5.3. In this formulation of the filter equations, G has been combined with the plant covariance by multiplying G_{k-1} and G_{k-1}^{T}, for example,

$$Q_{k-1} = G_{k-1}\mathrm{E}\langle w_{k-1}w_{K-1}^{\mathrm{T}}\rangle G_{k-1}^{\mathrm{T}}$$
$$= G_{k-1}\overline{Q}_{k-1}G_{k-1}^{\mathrm{T}}.$$

The relation of the filter to the system is illustrated in the block diagram of Figure 5.3.

The basic steps of the computational procedure for the discrete-time Kalman estimator are as follows:

1. Compute $P_{k(-)}$ using $P_{(k-1)(+)}$, Φ_{k-1}, and Q_{k-1}.

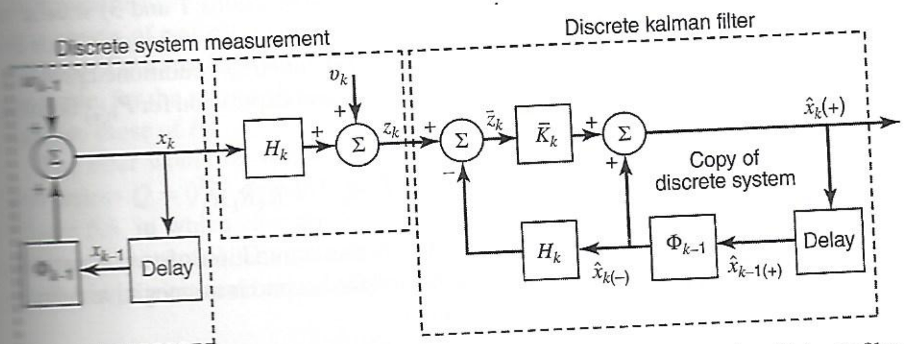


Figure 5.3 Block diagram of system, measurement model, and discrete-time Kalman filter.

Resonator Tracking) Consider a linear, underdamped, second-order displacement $x_1(t)$, rate $x_2(t)$, damping ratio ζ and (undamped) natof 5 rad/s, and constant driving term of 12.0 with additive white normally distributed. The second-order continuous-time dynamic

$$\ddot{x}_1(t) + 2\zeta w\dot{x}_1(t) + \omega^2 x_1(t) = 12 + w(t)$$

in state-space form via state-space techniques of Chapter 2:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 12 \end{bmatrix}.$$

equation is

$$z(t) = x_1(t) + v(t).$$

a trajectory of 100 simulated data points with plant noise and meaequal to zero using the following initial condition and parameter

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & \text{ft} \\ 0 & \text{ft/s} \end{bmatrix},$$

$$P(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$Q = 4.47\cancel{(\text{ft/s})^2}\ \text{ft}^2/\text{s}^3, \quad R = 0.01(\text{ft})^2$$

$$\zeta = 0.2, \quad \omega = 5 \quad \text{rad/s}.$$

5.24, 5.25, and 5.26 were programmed in MATLAB® software on a A) to estimate $\hat{x}_1(t)$ and $\hat{x}_2(t)$. Figure 5.8 shows the resulting estiand velocity using the noise-free data generated from simulating order equation. The estimated and actual values are identical in this shows the corresponding RMS uncertainties in position and velocity correlation coefficient between position and velocity (Figure 5.9(b)), (Figure 5.9(c)). These results were generated from the accompaprogram exam57.m described in Appendix A with sample interval

(Radar Tracking) This example is that of a pulsed *radar tracking* system, radar pulses are sent out and return signals are processed by in order to determine the position of maneuvering airborne objects ple's equations are drawn from IEEE papers [48, 49].

(damping coefficient) is an unknown constant. Therefore, the damping coefficient can be modeled as a state variable and its value estimated using an EKF.

Let

$$x_3(t) = \zeta$$

and

$$\dot{x}_3(t) = 0.$$

Then the otherwise linear dynamic equation in continuous time becomes nonlinear:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\omega^2 x_1 - 2x_2x_3 \quad \omega \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}.$$

The observation equation is still linear, however:

$$z(t) = x_1(t) + v(t)$$

One hundred data points were simulated with random plant noise and measurement noise, $\zeta = 0.1$, $\omega = 10$ rad/s, and initial conditions

$$\begin{bmatrix} x^1(0) \\ x^2(0) \\ x^3(0) \end{bmatrix} = \begin{bmatrix} 0 \text{ ft} \\ 0 \text{ ft/s} \\ 0 \end{bmatrix}, \qquad P(0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

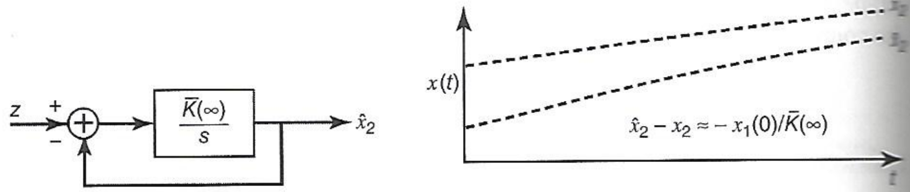
$$Q = 4.47\cancel{(\text{ft/s})^2}; \ \text{ft}^2/\text{s}^3 \qquad R = 0.001(\text{ft}^2).$$

The discrete nonlinear plant and linear observation equations for this model are

$$\begin{aligned}
x_k^1 &= x_{(k-1)}^1 + T x_{(k-1)}^2 \\
x_k^2 &= -25 T x_{(k-1)}^1 + (1 - 10 T x_{(k-1)}^3) x_{(k-1)}^2 + 12T + T w_{k-1} \\
x_k^3 &= x_{(k-1)}^3 \\
z_k &= x_k^1 + v_k.
\end{aligned}$$

Figure 8.3 shows the estimated position, velocity, and damping factor states using an EKF. This is implemented in MATLAB®script `DampParamEst.m`, which runs successive calls with independent random samples. (Because this is a Monte Carlo simulation, successive calls will not produce the same results.)

Example 8.5 (Nonlinear Freeway Traffic Modeling with EKF) This is an application of extended Kalman filtering to estimating parameters of an already nonlinear dynamic model. The objective of this particular task was to estimate certain key parameters of a macroscopic freeway traffic model designed to simplify a driver-level "microscopic" model [4].



(a) (b)

Figure 9.10 Type-1 servo (a) and estimates (b).

The error in $x_2(t)$ is

$$\tilde{x}_2(t) = \hat{x}_2(t) - x_2(t).$$

Taking the Laplace of the equation and substituting the value of $\hat{x}_2(s)$ from Equation 9.50 give

$$\begin{aligned}\tilde{x}_2(s) &= \frac{\overline{K}(\infty)}{s + \overline{K}(\infty)} x_2(s) - x_2(s) \\ &= \left[-\frac{s}{s + \overline{K}(\infty)} \right] x_2(s).\end{aligned}$$

Applying the final-value theorem, one gets

$$\begin{aligned}\tilde{x}_2(\infty) = [\hat{x}_2(\infty) - x_2(\infty)] &= \lim_{s\to 0} s[\hat{x}_2(s) - x_2(s)] \\ &= \lim_{s\to 0} s \left[-\frac{s}{s + \overline{K}(\infty)} \right] [x_2(s)] \\ &= \lim_{s\to 0} s \left[-\frac{s}{s + \overline{K}(\infty)} \right] \left[\frac{x_2(0)}{s} + \frac{x_1(0)}{s^2} \right] \\ &= -\frac{x_1(0)}{\overline{K}(\infty)} \text{(a bias)}.\end{aligned}$$

This type of behavior is shown in Figure 9.10(b).

If the steady-state bias in the estimation error is unsatisfactory with the approach
9.13 in Example ~~7.11~~, one can go one step further by adding another state variable and fictitious process noise to the system model assumed by the Kalman filter.

Example 9.14 (Effects of Adding States and Process Noise to the Kalman Filter Model) Suppose that the model of Example 7.11 was modified to the following form:

Real World	Kalman Filter Model
$\dot{x}_1 = 0$	$\dot{x}_1 = w$
$\dot{x}_2 = x_1$	$\dot{x}_2 = x_1$
$z = x_2 + v$	$z = x_2 + v$

(9.5

where p is the period of the sinusoid. For simplicity, let us choose the sampling time as $T = Np$, an integer multiple of the period, so that

$$\hat{x} = \begin{bmatrix} \dfrac{Np}{2} & 0 \\ 0 & \dfrac{Np}{2} \end{bmatrix}^{-1} \begin{bmatrix} \int_0^{Np} \sin(\omega t) z(t) \quad dt \\ \int_0^{Np} \cos(\omega t) z(t) \quad dt \end{bmatrix} = \begin{bmatrix} \frac{2}{Np}\int_0^{Np} z(t)\sin\omega t \quad dt \\ \frac{2}{Np}\int_0^{Np} z(t)\cos\omega t \quad dt \end{bmatrix}.$$

Concentrating on the first component $\hat{x}_1$, one can obtain its solution as

$$\begin{aligned} \hat{x}_1 = {} & \left\{ \frac{2}{Np} \left[\frac{\sin(\omega-\Omega)t}{2(\omega-\Omega)} - \frac{\sin(\omega+\Omega)t}{2(\omega+\Omega)} \right] \Bigg|_{t=0}^{t=Np} \right\} x_1 \\ & + \left\{ \frac{2}{Np} \left[\frac{-\cos(\omega-\Omega)t}{2(\omega-\Omega)} - \frac{\cos(\omega+\Omega)t}{2(\omega+\Omega)} \right] \Bigg|_{t=0}^{t=Np} \right\} x_2 \\ & + \frac{2}{Np} \int_0^{Np} v(t)\sin\omega t \quad dt. \end{aligned}$$

By setting $v = 0$ (ignoring the estimation error due to measurement noise), one obtains the result

$$\begin{aligned} \hat{x}_1 = {} & \frac{2}{Np} \left[\frac{\sin(\omega-\Omega)Np}{2(\omega-\Omega)} - \frac{\sin(\omega+\Omega)Np}{2(\omega+\Omega)} \right] x_1 \\ & + \frac{2}{Np} \left[\frac{1-\cos(\omega-\Omega)Np}{2(\omega-\Omega)} + \frac{1-\cos(\omega+\Omega)\overset{Np}{\cancel{t}}}{2(\omega+\Omega)} \right] x_2. \end{aligned}$$

For the case that $\omega \to \Omega$,

$$\left. \begin{aligned} \frac{\sin(\omega-\Omega)Np}{2(\omega-\Omega)} &= \frac{Np}{2}\frac{\sin x}{x}\Big|_{x\to 0} &&= \frac{Np}{2}, \\ \frac{\sin(\omega+\Omega)Np}{2(\omega+\Omega)} &= \frac{\sin[(4\pi/p)Np]}{2\,\Omega} &&= 0, \\ \frac{1-\cos(\omega-\Omega)Np}{2(\omega-\Omega)} &= \frac{1-\cos x}{x}\Big|_{x\to 0} &&= 0, \\ \frac{1-\cos(\omega+\Omega)Np}{2(\omega+\Omega)} &= \frac{1-\cos[(4\pi/p)Np]}{2\,\Omega} &&= 0, \end{aligned} \right\}$$

and $\hat{x}_1 = x_1$. In any other case, $\hat{x}_1$ would be a biased estimate of the form

$$\hat{x}_1 = \Upsilon_1 x_1 + \Upsilon_2 x_2, \qquad \text{where} \quad \Upsilon_1 \neq 1, \quad \Upsilon_2 \neq 0.$$

Similar behavior occurs with $\hat{x}_2$.

The number of scalar arithmetic operations required for computation of matrix exponentials grows as the cube of the matrix dimension, so partitioning the matrix in this way saves considerable time and effort.

If the matrix M in this case is the dynamic coefficient matrix F for a linear dynamic system, the associated state-transition matrix is defined by Equation 10.14. Consequently, if any diagonal block F_{ii} of F is time invariant, its exponential

$$\Phi_{(k-1)\ ii} = \exp\left(\int_{t_{k-1}}^{t_k} F_{ii}\, dt\right) \tag{10.17}$$

$$= \exp\,(F_{ii}\,\Delta t) \tag{10.18}$$

is also time invariant.

This will be the case for all diagonal submatrices in the dynamic coefficient matrices used for solving the GNSS navigation problem. However, it will not be the case when we get to integrating GNSS with INS. In that case, only the time-varying blocks associated with inertial navigation error propagation need to be recomputed as a function of time. The blocks with time invariant F_{ii} can be left un~~molested~~. changed

Dimensions of these diagonal subblocks will be determined by which of several possible models is chosen, as will the dimensions of the H and R matrices.

10.3.4.5 *Host Vehicle Dynamic Models* Timing-based GNSS navigation is defined in terms of having a certain receiver antenna location at the time a satellite signal timing mark is received, and the timing-based navigation correction component for that measurement is always in the direction between the satellite and the receiver location. If the antenna were to remain in that same location long enough for similar corrections to be make for all available satellites, then (assuming good GDOP) that location would be observable from the ensemble of measurements. This may not be a problem in surveying applications, for which the receiver antenna is held at a fixed position on Earth. It is a problem for other applications, however.

The GNSS geometry shown in Figure 10.5 takes into account the motion of the satellite antennas, which is determined by the short-term ephemerides transmitted by the satellites. There is no equivalent predetermined trajectory for the receiver antenna.

Tracking Filters The problem gets a bit more complicated for navigation aboard maneuvering vehicles, in that the location of the receiver antenna now becomes an unknown function of time. This is the same sort of problem faced in the early 1950s, when radar systems were being integrated with real-time computers for detecting and tracking aircraft as part of an air defense system [11], and the GNSS navigation solution can use the same types of "tracking filters" developed around that time. The Kalman filter was not available in the 1950s, but today's tracking filters use both the position and velocity of the receiver antenna[6] as state variables in a Kalman

[6] Air defense jargon uses the term *target* for the object being tracked.

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ion, so partitioning the matrix in th

efficient matrix F for a linear dyna
s defined by Equation 10.14. Conse-
ariant, its exponential

$$F_{ii}\, dt\Big) \qquad (10.17)$$

$$\Delta t) \qquad (10.18)$$

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Timing-based GNSS navigation
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tenna[6] as state variables in a Kalma

g tracked.

filter. In the case of GNSS navigation, performance can be improved if the parameters of the filter are well matched to the dynamics of the host vehicle carrying the receiver antenna. These parameters include the dynamic coefficient matrix F and the mean-squared dynamic disturbance covariance Q.

Mathematical Formulas Table 10.2 lists some of the parametric models used for modeling a single component of the motion of a host vehicle. These are specified in terms of the dynamic coefficient matrix F and disturbance noise covariance Q for a single axis of motion. Depending on the application, the host vehicle dynamic model may have different models for different axes. Surface ships, for example, may assume constant altitude and only estimate the north and east axes of position and velocity.

Model Descriptions The model parameters listed in Table 10.2 include standard deviations σ and correlation time constants τ of position, velocity, acceleration, and jerk (derivative of acceleration). Those parameters labeled as "independent" can be specified by the designer. Those labeled as "dependent" will depend on the values specified for the independent variables. (See Reference 2 for more details.)

TABLE 10.2 Host Vehicle Dynamic Models

Model No.	model parameters (each axis): F	Q	independent parameters	dependent parameters
1	0	0	none	none
2	$\begin{bmatrix} 0 & 1 \\ -\dfrac{\sigma_{\mathrm{vel}}^2}{\sigma_{\mathrm{pos}}^2} & \dfrac{-2\,\sigma_{\mathrm{vel}}}{\sigma_{\mathrm{pos}}} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & \dfrac{4\,\sigma_{\mathrm{vel}}^3}{\sigma_{\mathrm{pos}}} \end{bmatrix}$	σ_{pos}^2 σ_{vel}^2	δ (damping) σ_{acc}^2
3	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\mathrm{acc}}^2\,\Delta t^2 \end{bmatrix}$	σ_{acc}^2	$\sigma_{\mathrm{pos}}^2 \to \infty$ $\sigma_{\mathrm{vel}}^2 \to \infty$
4	$\begin{bmatrix} 0 & 1 \\ 0 & -1/\tau_{\mathrm{vel}} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\mathrm{acc}}^2\,\Delta t^2 \end{bmatrix}$	σ_{vel}^2 τ_{vel}	$\sigma_{\mathrm{pos}}^2 \to \infty$ σ_{acc}^2
5	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & \dfrac{-1}{\tau_{\mathrm{vel}}} & 1 \\ 0 & 0 & \dfrac{-1}{\tau_{\mathrm{acc}}} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{\mathrm{jerk}}^2\,\Delta t^2 \end{bmatrix}$	σ_{vel}^2 σ_{acc}^2 τ_{acc}	$\sigma_{\mathrm{pos}}^2 \to \infty$ τ_{vel} $\rho_{\mathrm{vel,\,acc}}$ σ_{jerk}^2
6	$\begin{bmatrix} \dfrac{-1}{\tau_{\mathrm{pos}}} & 1 & 0 \\ 0 & \dfrac{-1}{\tau_{\mathrm{vel}}} & 1 \\ 0 & 0 & \dfrac{-1}{\tau_{\mathrm{acc}}} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{\mathrm{jerk}}^2\,\Delta t^2 \end{bmatrix}$	σ_{pos}^2 σ_{vel}^2 σ_{acc}^2 τ_{acc}	τ_{pos} τ_{vel} $\rho_{\mathrm{pos,\,vel}}$ $\rho_{\mathrm{pos,\,acc}}$ $\rho_{\mathrm{vel,\,acc}}$ σ_{jerk}^2