Formatif_P_week3

Haofan Xu

September 2023

Show that for any set S, and any three binary relations $R_1, R_2, R_3 \subseteq S \times S$: $(R_1; R_2); R_3 = R_1; (R_2; R_3)$

Proof:
$" \Rightarrow "$:
Since we have $(R_1; R_2); R_3$, by the definition of composition:
$(R_1; R_2); R_3 \overset{\text{def}}{=} \{s_1, s_4 \in S \times S \ \textit{there exists} \ s_3 \in S, \ \textit{such that} \ (s_1, s_3) \in (R_1; R_2) \ \textit{and} \ (s_3, s_4) \in R_3\}$
$(R_1; R_2) \overset{\text{def}}{=} \{s_1, s_3 \in S \times S \ \textit{there exists} \ s_2 \in S, \ \textit{such that} \ (s_1, s_2) \in R_1 \ \textit{and} \ (s_2, s_3) \in R_2\}$
$\implies (s_1, s_2) \in R_1, (s_2, s_3) \in R_2, (s_3, s_4) \in R_3$ where $s_1, s_2, s_3, s_4 \in S$
$\implies (s_1, s_2) \in R_1, (s_2, s_4) \in (R_2; R_3)$
$\implies (s_1, s_4) \in R_1; (R_2; R_3)$
$\implies (R_1; R_2); R_3 \subseteq R_1; (R_2; R_3)$ (1)
$" \Leftarrow "$:
Since we have $R_1; (R_2; R_3)$, by the definition of composition:
$R_1; (R_2; R_3) \overset{\text{def}}{=} \{s_1, s_4 \in S \times S \ \textit{there exists} \ s_2 \in S, \ \textit{such that} \ (s_1, s_2) \in R_1 \ \textit{and} \ (s_2, s_4) \in (R_2; R_3)\}$
$(R_2; R_3) \overset{\text{def}}{=} \{s_2, s_4 \in S \times S \ \textit{there exists} \ s_3 \in S, \ \textit{such that} \ (s_2, s_3) \in R_2 \ \textit{and} \ (s_3, s_4) \in R_3\}$
$\implies (s_1, s_2) \in R_1, (s_2, s_3) \in R_2, (s_3, s_4) \in R_3$ where $s_1, s_2, s_3, s_4 \in S$
$\implies (s_1, s_3) \in (R_1; R_3), (s_3, s_4) \in R_3$
$\implies (s_1, s_4) \in (R_1; R_2); R_3$
$\implies R_1; (R_2; R_3) \subseteq (R_1; R_2); R_3$ (2)
by (1) and (2), we get $(R_1; R_2); R_3 = R_1; (R_2; R_3)$