



COMP9020

Foundations of Computer Science

Lecture 5: Relations

Administrivia

- No lecture next Monday; Quiz and Formatif due on Wednesday October 4
- First assignment released at the end of this week, due October 20
- Question(s) of the week
 - Haozhou Ni
 - Farzaneh Aghajari
 - Bowen Lu
- Feedback: Recording ending too soon

Relations and Functions

Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

Functions capture the idea of transforming *inputs* into *outputs*.

In general, functions and relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

Applications in Computer Science

- Relations are the building blocks of nearly all Computer Science structures
- Databases are collections of relations
- Any ordering is a relation
- Common data structures (e.g. graphs) are relations
- Functions/procedures/programs compute relations between their input and output

Applications in Computer Science

Many binary relations (i.e. relationships between two entities) that appear in CS fall into three broad categories:

Functions (relating inputs to outputs)

- Most programming languages use function calls
- Programs are functions

Equivalence relations (generalizing “equality”):

- Programs that exhibit the same behaviour
- Logically equivalent statements
- The `.equals()` method in Java

Partial orders (generalizing “less than or equal to”):

- Object inheritance
- Simulation
- The `.compareTo()` method in Java

Outline

Definition and Examples

Binary Relations

Properties of Binary Relations

Functions

Relations

Definition

An **n-ary relation** is a subset of the Cartesian product of n sets.

$$R \subseteq S_1 \times S_2 \times \dots \times S_n$$

To show tuples related by R we write:

$$(x_1, x_2, \dots, x_n) \in R \quad \text{or} \quad R(x_1, x_2, \dots, x_n)$$

If $n = 2$ we have a **binary** relation $R \subseteq S \times T$ and to show pairs related by R we write:

$$(x, y) \in R \quad \text{or} \quad R(x, y) \quad \text{or} \quad xRy$$

$\mathcal{U} = S_1 \times S_2 \times \dots \times S_n$ is the **domain** of R , and we say R is a **relation on \mathcal{U}** (or **on S** if $S_1 = \dots = S_n = S$ and n is clear).

Examples

Examples

- Equality: $=$
- Inequality: $\leq, \geq, <, >, \neq$
- Divides relation: $|$
- Element of: \in
- Subset, superset: $\subseteq, \subset, \supseteq, \supset$
- Congruence modulo n : $m =_{(n)} p$

Database Examples

Example (Course enrolments)

S = set of CSE students

(S can be a subset of the set of all students)

C = set of CSE courses

(likewise)

E = enrolments = $\{ (s, c) : s \text{ takes } c \}$

$$E \subseteq S \times C$$

In practice, almost always there are various 'onto' (nonemptiness) and 1-1 (uniqueness) constraints on database relations.

Example (Class schedule)

C = CSE courses

T = starting time (hour & day)

R = lecture rooms

S = schedule =

$$\{ (c, t, r) : c \text{ is at } t \text{ in } r \} \subseteq C \times T \times R$$

Example (sport stats)

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

Defining Relations

Just as with sets R can be defined by

- explicit enumeration of interrelated k -tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $S_1 \times S_2 \times \dots \times S_k$;
- construction from other relations (e.g. union, intersection, complement etc).

Outline

Definition and Examples

Binary Relations

Properties of Binary Relations

Functions

Binary relations

A **binary relation between S and T** is a subset of $S \times T$: i.e. a set of ordered pairs.

Also: over S and T ; from S to T ; on S (if $S = T$).

Example (Special (Trivial) Relations)

- **Identity:** (diagonal, equality) $I = \{ (x, x) : x \in S \}$
- **Empty:** \emptyset
- **Universal:** $U = S \times S$

Defining binary relations: Set-based definitions

Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1, 1), (2, 3), (3, 2)\}$
- Set comprehension: $\{(x, y) \in [1, 3] \times [1, 3] : 5 \mid xy - 1\}$
- Construction from other relations:
 $\{(1, 1)\} \cup \{(2, 3)\} \cup \{(2, 3)\}^{\leftarrow}$

Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:

Rows enumerated by elements of S , columns by elements of T :

Examples

- The relation $\{(1, 1), (2, 3), (3, 2)\} \subseteq [1, 3] \times [1, 3]$:

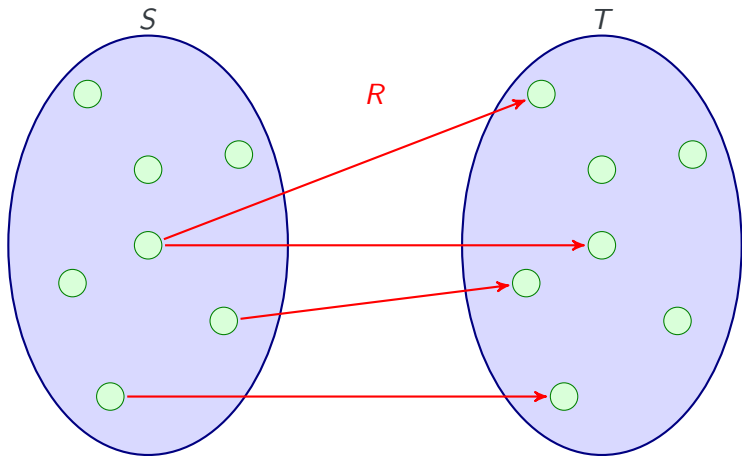
$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 2)\} \subseteq [1, 3] \times [1, 4]$:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{bmatrix}$$

Defining binary relations: Graphical representation

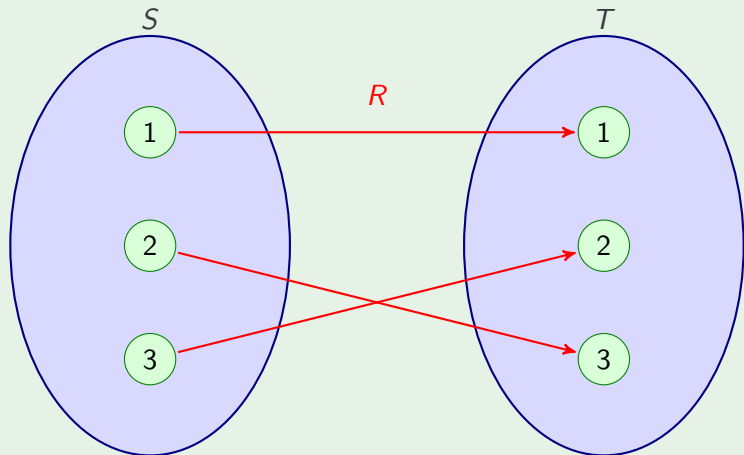
Defining a relation $R \subseteq S \times T$:



Defining binary relations: Graphical representation

Example

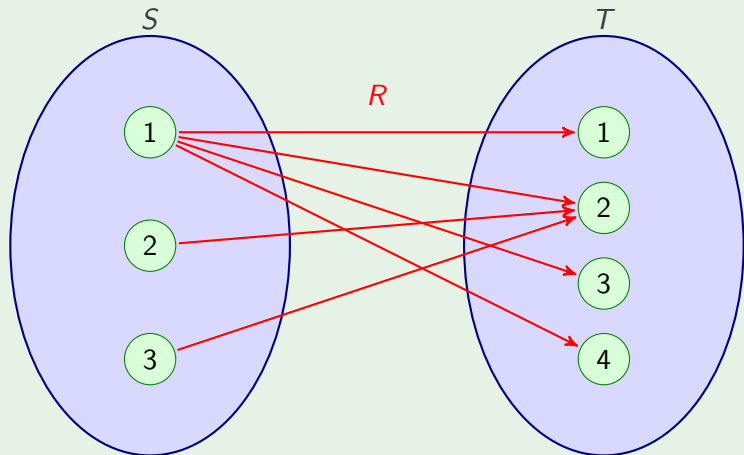
$$R = \{(1, 1), (2, 3), (3, 2)\} \subseteq [1, 3] \times [1, 3]:$$



Defining binary relations: Graphical representation

Example

$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 2)\} \subseteq [1, 3] \times [1, 4]$:



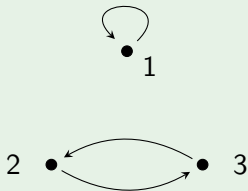
Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

- Nodes: Elements of S
- Edges: Elements of R

Example

$$R = \{(1, 1), (2, 3), (3, 2)\} \subseteq [1, 3] \times [1, 3]:$$



Operations for binary relations

Relations are sets, so the standard set operations (\cap , \cup , \setminus , \oplus , etc) can be used to build new relations.

Two operations that apply to binary relations uniquely:

- **Converse:** If $R \subseteq S \times T$ is a relation, then $R^{\leftarrow} \subseteq T \times S$:

$$R^{\leftarrow} \stackrel{\text{def}}{=} \{(t, s) \in T \times S : (s, t) \in R\}$$

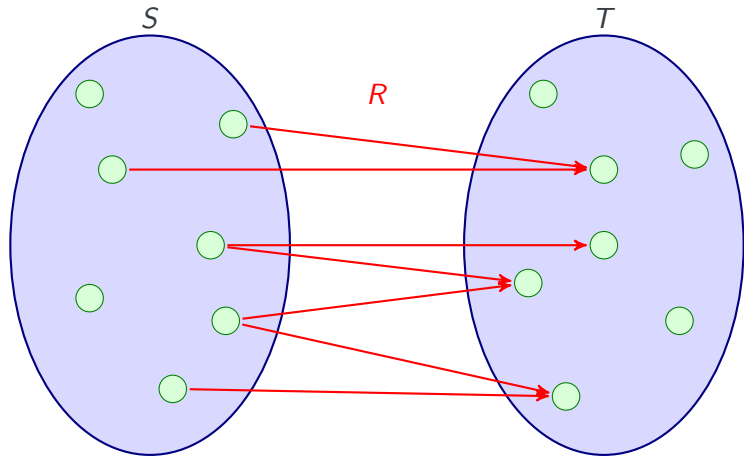
- **Composition:** If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ then $R_1; R_2 \subseteq S \times U$:

$$R_1; R_2 \stackrel{\text{def}}{=} \{(s, u) \in S \times U : \text{there exists } t \in T \text{ such that } (s, t) \in R_1 \text{ and } (t, u) \in R_2\}.$$

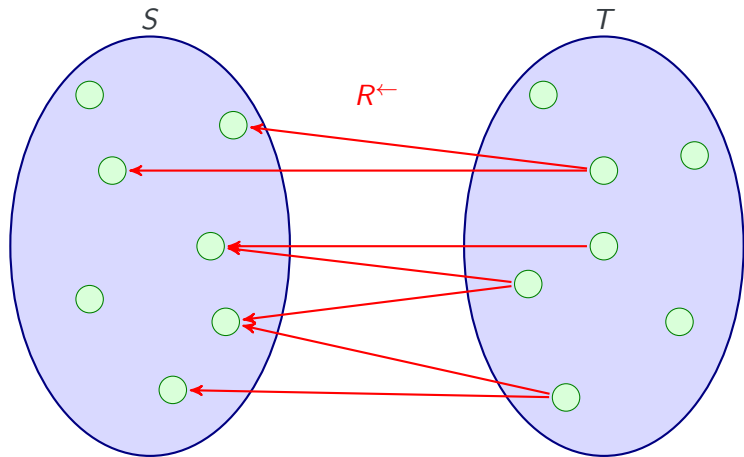
Fact

$$(R^{\leftarrow})^{\leftarrow} = R$$

Binary relation: Graphical representation



Binary relation: Graphical representation



Relational images

Given $R \subseteq S \times T$, $A \subseteq S$, and $B \subseteq T$.

Definition

- Relational image of A , $R(A)$:

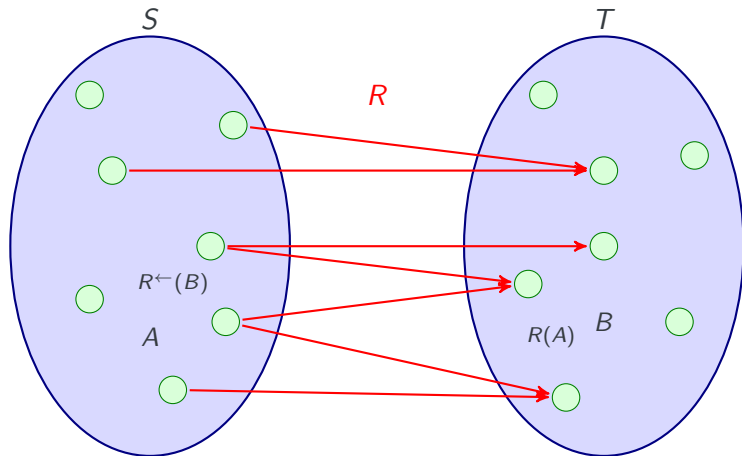
$$R(A) \stackrel{\text{def}}{=} \{t \in T : (s, t) \in R \text{ for some } s \in A\}$$

- Relational pre-image of B , $R^{\leftarrow}(B)$:

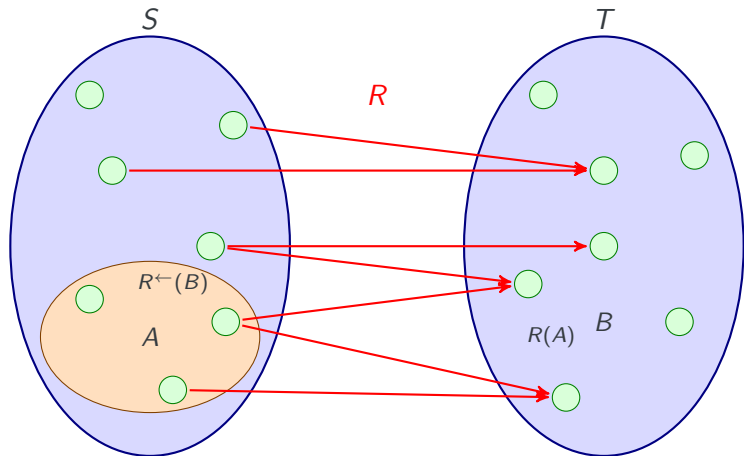
$$R^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S : (s, t) \in R \text{ for some } t \in B\}$$

Observe that the relational pre-image is the relational image of the converse relation.

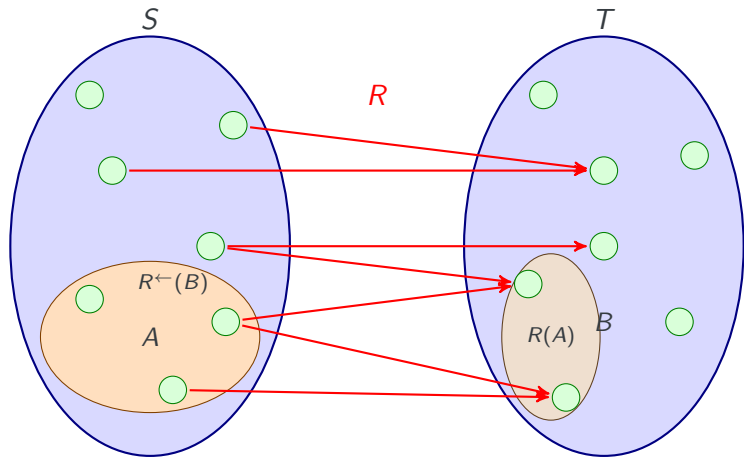
Binary relation: Graphical representation



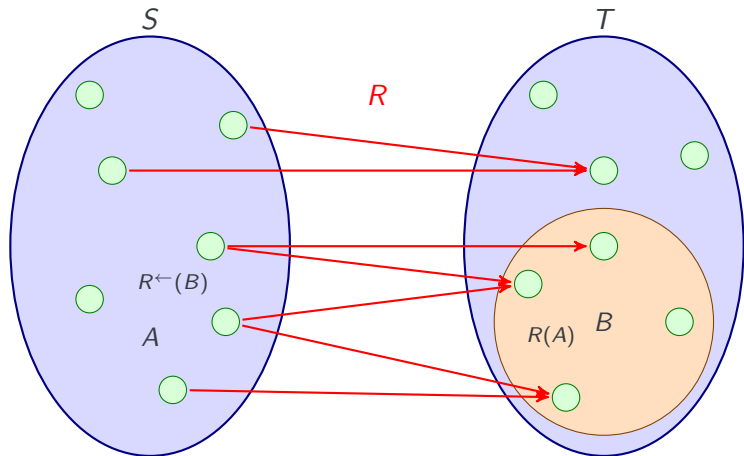
Binary relation: Graphical representation



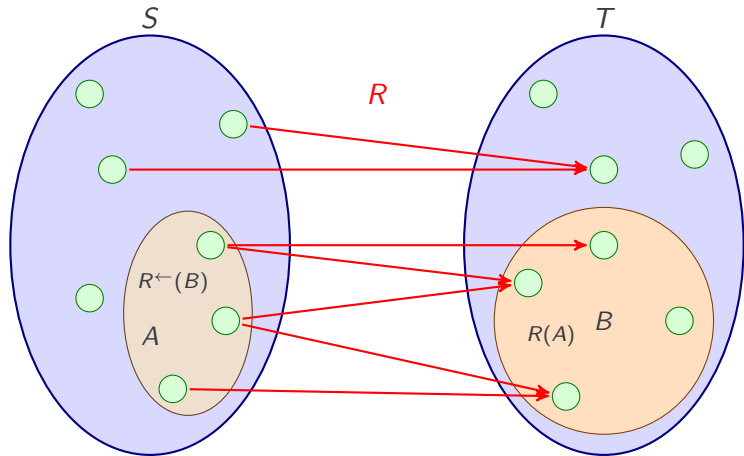
Binary relation: Graphical representation



Binary relation: Graphical representation



Binary relation: Graphical representation



Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$,
 $M = \{A, B, C\}$, $N = \{A, B, C, X\}$.

- $|$ on X :
- \in on $X \times M$:
- \subseteq^{\leftarrow} on N :
- $|\vdash$:
- $<(\{2\})$ (on X):

Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$,
 $M = \{A, B, C\}$, $N = \{A, B, C, X\}$.

- $|$ on X : $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- \in on $X \times M$: $\{(1, A), (2, A), (2, B), (3, B), (3, C), (4, C)\}$
- \subseteq^{\leftarrow} on N : $\{(A, A), (X, A), (B, B), (X, B), (C, C), (X, C), (X, X)\}$
- $|\in$:
 $\{(1, A), (1, B), (1, C), (2, A), (2, B), (2, C), (3, B), (3, C), (4, C)\}$
- $<(\{2\})$ (on X): $\{3, 4\}$

Outline

Definition and Examples

Binary Relations

Properties of Binary Relations

Functions

Properties of Binary Relations $R \subseteq S \times T$

A binary relation $R \subseteq S \times T$ is:

Definition

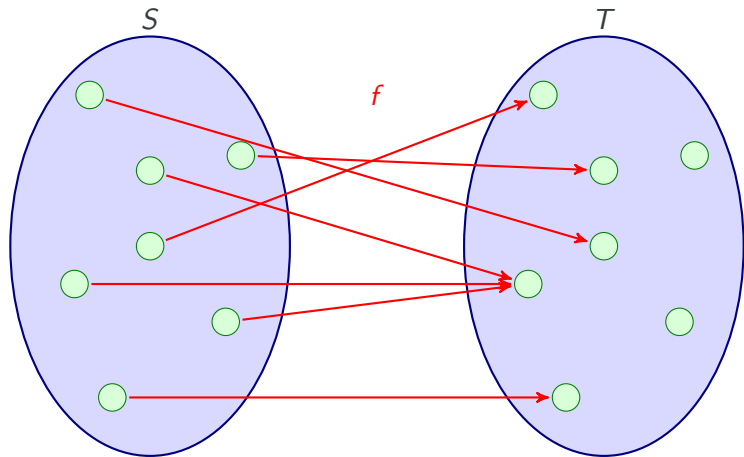
(Fun)	functional	For all $s \in S$ there is at most one $t \in T$ such that $(s, t) \in R$
(Tot)	total	For all $s \in S$ there is at least one $t \in T$ such that $(s, t) \in R$
(Inj)	injective	For all $t \in T$ there is at most one $s \in S$ such that $(s, t) \in R$
(Sur)	surjective	For all $t \in T$ there is at least one $s \in S$ such that $(s, t) \in R$
(Bij)	bijective	Injective and surjective

Functions and function properties

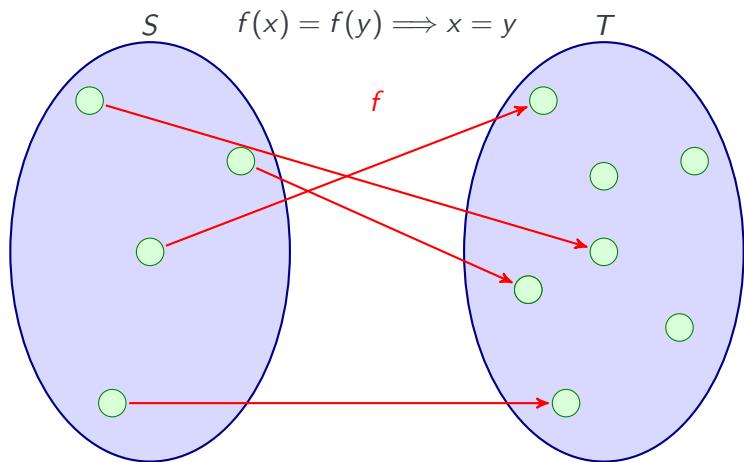
Definition

- **partial function** is a binary relation that is (Fun).
- A **function** is a binary relation that is (Fun) and (Tot).
- An **injection** is a function that is (Inj).
- A **surjection** is a function that is (Sur).
- A **bijection** is a function that is (Bij).

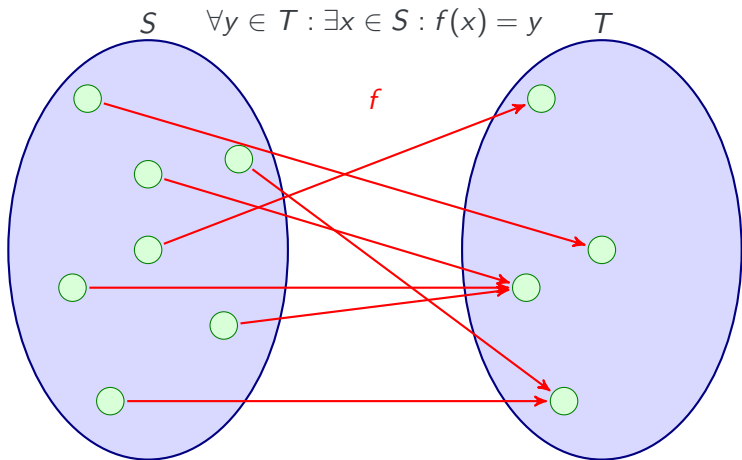
Graphical representation: Function



Graphical representation: Injection



Graphical representation: Surjection



Properties of Binary Relations $R \subseteq S \times S$

Definition

(R)	reflexive	For all $x \in S$: $(x, x) \in R$
(AR)	antireflexive	For all $x \in S$: $(x, x) \notin R$
(S)	symmetric	For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$
(AS)	antisymmetric	For all $x, y \in S$: If (x, y) and $(y, x) \in R$ then $x = y$
(T)	transitive	For all $x, y, z \in S$: If (x, y) and $(y, z) \in R$ then $(x, z) \in R$

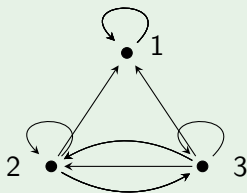
NB

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

Relation properties: Examples

Examples

- (R)** Reflexivity: $(x, x) \in R$ for all x
- (AR)** Antireflexivity: $(x, x) \notin R$ for all x
- (S)** Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all x, y
- (AS)** Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all x, y
- (T)** Transitivity: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all x, y, z .



●○	○	○
○●	●○	○●
●○	○●	●○

Interaction of Properties

A relation *can* be both symmetric and antisymmetric. Namely, when R consists only of some pairs $(x, x), x \in S$.

A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$\left. \begin{array}{l} \text{nonreflexive} \\ \text{nonsymmetric} \end{array} \right\}$ is not the same as $\left\{ \begin{array}{l} \text{antireflexive/irreflexive} \\ \text{antisymmetric} \end{array} \right.$

Exercises

Exercises

RW: 3.1.1 The following relations are on $S = \{1, 2, 3\}$. Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

- (a) $(m, n) \in R$ if $m + n = 3$?
- (e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

Exercises

Exercises

RW: 3.1.1 The following relations are on $S = \{1, 2, 3\}$. Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?

(a) $(m, n) \in R$ if $m + n = 3$? (AR) and (S)

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$? (S)

Exercises

Exercises

RW: 3.1.10 Give examples of relations with specified properties.

(a) (AS), (T), not (R)

(b) (S), not (R), not (T)

Exercises

Exercises

RW: 3.1.10 Give examples of relations with specified properties.

(a) (AS), (T), not (R)

- Strict order of numbers $x < y$
- \leq but with some pairs (x, x) removed
- Being a prime divisor: $(p, n) \in R$ iff p is prime and $p|n$
 - Not reflexive: $(1, 1) \notin R$
 - Transitivity is meaningful only for the pairs $(p, p), (p, n) p|n$ for p prime

(b) (S), not (R), not (T)

Simplest example - inequality

Exercises

Exercises

RW: 3.6.10 (supp)

R is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$
 $(m, n) R (p, q)$ if $m =_{(3)} p$ or $n =_{(5)} q$.

- (a) Is R reflexive?
- (b) Is R symmetric?
- (c) Is R transitive?

Exercises

Exercises

RW: 3.6.10 (supp)

R is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$

$(m, n) R (p, q)$ if $m =_{(3)} p$ or $n =_{(5)} q$.

- (a) Is R reflexive? Yes: $m =_{(3)} m$ so $(m, n) R (m, n)$.
- (b) Is R symmetric? Yes: by symmetry of $\cdot =_{(n)} \cdot$.
- (c) Is R transitive? No: Consider $(1, 1)$, $(1, 4)$ and $(2, 4)$.

Exercises

Exercises

Complete the following table of common relations (over \mathbb{Z}) and their properties:

	(R)	(AR)	(S)	(AS)	(T)
$=$					
\leq					
$<$					
\emptyset					
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$					
$ $					
$\equiv_{(3)}$					

Exercises

Exercises

Complete the following table of common relations (over \mathbb{Z}) and their properties:

	(R)	(AR)	(S)	(AS)	(T)
$=$	✓		✓	✓	✓
\leq	✓			✓	✓
$<$		✓		✓	✓
\emptyset		✓	✓	✓	✓
$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$	✓		✓		✓
$ $	✓			*	✓
$=_{(3)}$	✓		✓		✓

* True for $| \subseteq \mathbb{N} \times \mathbb{N}$

Outline

Definition and Examples

Binary Relations

Properties of Binary Relations

Functions

Functions

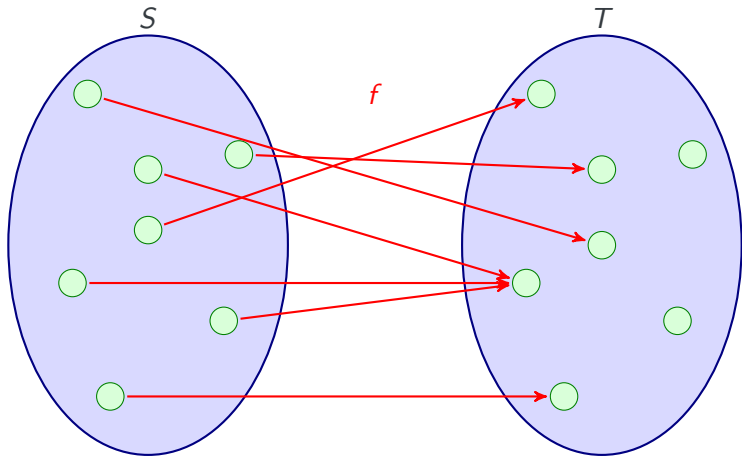
Definition

A **function**, $f : S \rightarrow T$, is a binary relation $f \subseteq S \times T$ that satisfies (Fun) and (Tot). That is, for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

We write $f(s)$ for the unique element related to s .

We write T^S for the set of all functions from S to T .

Graphical representation



Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$

		Symbol	
S	domain of f	$\text{Dom}(f)$	(inputs)
T	co-domain of f	$\text{Codom}(f)$	(<i>possible</i> outputs)
$f(S)$	image of f	$\text{Im}(f)$	(<i>actual</i> outputs)
$= \{ f(x) : x \in \text{Dom}(f) \}$			

Important!

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \rightarrow \mathbb{Z} \quad \text{given by} \quad f(x) \mapsto x^2$$

and

$$g : \mathbb{N} \rightarrow \mathbb{N} \quad \text{given by} \quad g(x) \mapsto x^2$$

are different functions even though they have the same behaviour!

Converse of a function

Question

f^{\leftarrow} is a relation; when is it a function?