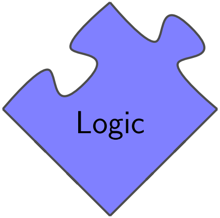
COMP9020

Foundations of Computer Science

Lecture 12: Boolean Logic

Topic 3: Logic



		[LLM]	[RW]	[Rosen]
Week 8	Boolean Logic	Ch. 3	Ch. 2, 10	Ch. 12
Week 8	Propositional Logic	Ch. 3	Ch. 2	Ch. 1

What is logic?

Logic is about **formalizing reasoning** and **defining truth**

- Adding rigour
- Removing ambiguity
- Mechanizing the process of reasoning

Loose history of logic

- (Ancient times): Logic exlusive to philosophy
- Mid-19th Century: Logical foundations of Mathematics (Boole, Jevons, Schröder, etc)
- 1910: Russell and Whitehead's Principia Mathematica
- 1928: Hilbert proposes *Entscheidungsproblem*
- 1931: Gödel's Incompleteness Theorem
- 1935: Church's Lambda calculus
- 1936: Turing's Machine-based approach
- 1930s: Shannon develops Circuit logic
- 1960s: Formal verification; Relational databases

Logic in Computer Science

Computation $=$ Calculation $+$ Symbolic manipulation

Logic in Computer Science

$$\text{Computation} \quad = \quad \text{Calculation} + \text{Symbolic manipulation}$$

Logic as 2-valued computation (Boolean logic):

- Circuit design
- Code optimization
- Boolean algebra
- Nand game

Logic in Computer Science

$$\text{Computation} \quad = \quad \text{Calculation} + \text{Symbolic manipulation}$$

Logic as symbolic reasoning (Propositional logic, and beyond):

- Formal verification
- Proof assistance
- Knowledge Representation and Reasoning
- Automated reasoning
- Databases

Outline

Outline

Boolean logic

Boolean logic is about performing calculations in a "simple" mathematical structure.

- complex calculations can be built entirely from these simple ones
- can help identify simplifications that improve performance at the circuit level
- can help identify simplifications that improve presentation at the programming level

The Boolean Algebra $\mathbb{B}$

Definition

The (two-element) **Boolean algebra** is defined to be the set $\mathbb{B} = \{0, 1\}$, together with the functions $! : \mathbb{B} \to \mathbb{B}$, $\&\& : \mathbb{B}^2 \to \mathbb{B}$, and $\| : \mathbb{B}^2 \to \mathbb{B}$, defined as follows:

$$!x = (1 - x) \qquad x \;\&\&\; y = \min\{x, y\} \qquad x \;\|\; y = \max\{x, y\}$$

The Boolean Algebra $\mathbb{B}$ – Alternative definition

Definition

The (two-element) **Boolean algebra** is defined to be the set $\mathbb{B} = \{\text{false}, \text{true}\}$, together with the functions $! : \mathbb{B} \to \mathbb{B}$, $\&\&\colon \mathbb{B}^2 \to \mathbb{B}$, and $\|\colon \mathbb{B}^2 \to \mathbb{B}$, defined as follows:

x	$!x$
false	true
true	false

x	y	$x \mathbin{\&\&} y$
false	false	false
false	true	false
true	false	false
true	true	true

x	y	$x \parallel y$
false	false	false
false	true	true
true	false	true
true	true	true

Alternative notation

Commonly, the following alternative notation is used:

For $\mathbb{B}$:	$\{F, T\}$
For $!x$:	$\overline{x}$, x', $\sim x$, $\neg x$
For x && y:	xy, $x \wedge y$
For $x \parallel y$:	$x + y$, $x \vee y$

Properties

We observe that $!$, $\&\&$, and $\|$ satisfy the following:

For all $x, y, z \in \mathbb{B}$:

Commutativity	$x \| y = y \| x$
	$x \;\&\&\; y = y \;\&\&\; x$
Associativity	$(x \| y) \| z = x \| (y \| z)$
	$(x \;\&\&\; y) \;\&\&\; z = x \;\&\&\; (y \;\&\&\; z)$
Distribution	$x \| (y \;\&\&\; z) = (x \| y) \;\&\&\; (x \| z)$
	$x \;\&\&\; (y \| z) = (x \;\&\&\; y) \| (x \;\&\&\; z)$
Identity	$x \| 0 = x$
	$x \;\&\&\; 1 = x$
Complementation	$x \| (!x) = 1$
	$x \;\&\&\; (!x) = 0$

Examples

Examples

- Calculate $x \mathbin{\&\&} x$ for all $x \in \mathbb{B}$
- Calculate $((1 \mathbin{\&\&} 0) \parallel ((!1) \mathbin{\&\&} (!0))$

Outline

Boolean Functions

Definition

An n-**ary Boolean function** is a map $f : \mathbb{B}^n \to \mathbb{B}$.

Question

How many unary Boolean functions are there?
How many binary functions?
n-ary?

Examples

Examples

- ! is a unary Boolean function
- &&, $\|$ are binary Boolean functions
- $f(x, y) =!(x \mathbin{\&\&} y)$ is a binary boolean function (NAND)
- $\textsc{And}(x_0, x_1, \ldots) = (\cdots((x_0 \mathbin{\&\&} x_1) \mathbin{\&\&} x_2) \cdots)$ is a (family) of Boolean functions
- $\textsc{Or}(x_0, x_1, \ldots) = (\cdots((x_0 \parallel x_1) \parallel x_2) \cdots)$ is a (family) of Boolean functions

Application: Adding two one-bit numbers

How can we implement:

$$\mathsf{add} : \mathbb{B}^2 \to \mathbb{B}^2$$

defined as

x	y	$\mathsf{add}(x, y)$
0	0	00
0	1	01
1	0	01
1	1	10

Use two Boolean functions!

NB

Digital circuits are just sequences of Boolean functions.

Outline

Conjunctive and Disjunctive normal form

Definition

- A **literal** is a unary Boolean function
- A **minterm** is a Boolean function of the form $\textsc{And}(l_1(x_1), l_2(x_2), \ldots, l_n(x_n))$ where the l_i are literals
- A **maxterm** is a Boolean function of the form $\textsc{Or}(l_1(x_1), l_2(x_2), \ldots, l_n(x_n))$ where the l_i are literals
- A **CNF Boolean function** is a function of the form $\textsc{And}(m_1, m_2, \ldots)$, where the m_i are maxterms.
- A **DNF Boolean function** is a function of the form $\textsc{Or}(m_1, m_2, \ldots)$, where the m_i are minterms.

Examples

Examples

- $f(x, y, z) = (x \mathbin{\&\&} (!y) \mathbin{\&\&} z) \parallel (x \mathbin{\&\&} (!y) \mathbin{\&\&} (!z)) = x\,\overline{y}\,z + x\,\overline{y}\,\overline{z}$: DNF , but not CNF
- $g(x, y, z) = (x \parallel (!y) \parallel z) \mathbin{\&\&} (x \parallel (!y) \parallel (!z)) = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$: CNF function, but not DNF
- $h(x, y, z) = (x \mathbin{\&\&} (!y) \mathbin{\&\&} z) = x\,\overline{y}\,z$: both CNF and DNF
- $j(x, y, z) = x + y(z + x)$: Neither CNF nor DNF

NB

CNF: product of sums; DNF: sum of products

Theorem

Every Boolean function can be written as a function in DNF/CNF

Proof...

Canonical DNF

Given an n-ary boolean function $f : \mathbb{B}^n \to \mathbb{B}$ we construct an equivalent DNF boolean function as follows:
For each $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{B}^n$ we define the minterm

$$m_{\mathbf{b}} = \textsc{And}(l_1(x_1), l_2(x_2), \dots, l_n(x_n))$$

where

$$l_i(x_i) = \begin{cases} x_i & \text{if } b_i = 1 \\ !x_i & \text{if } b_i = 0 \end{cases}$$

We then define the DNF formula:

$$f_{\mathrm{DNF}} = \sum_{f(\mathbf{b})=1} m_{\mathbf{b}},$$

that is, f_{DNF} is the disjunction (or) over all minterms corresponding to elements $\mathbf{b} \in \mathbb{B}$ where $f(\mathbf{b}) = 1$.

Canonical DNF

Theorem

f and f_{DNF} are the same function.

Exercise

Exercises

RW: 10.2.3 Find the canonical DNF form of each of the following expressions in variables x, y, z

- xy
- $\overline{z}$
- $xy + \overline{z}$
- $f(x, y, z) = 1$

Outline

Karnaugh Maps

For up to four variables (propositional symbols) a diagrammatic method of simplification called **Karnaugh maps** works quite well.
For every propositional function of $k = 2, 3, 4$ variables we construct a rectangular array of 2^k cells. We mark the squares corresponding to the value `true` with eg "$+$" and try to cover these squares with as few rectangles with sides 1 or 2 or 4 as possible.

Example

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	$+$	$+$		$+$
$\bar{x}$	$+$		$+$	$+$

$$E = (xy) \vee (\bar{x}\bar{y}) \vee z$$

Canonical form would consist of writing all cells separately (6 clauses).

Karnaugh Maps

For optimisation, the idea is to cover the $+$ squares with the minimum number of rectangles. One *cannot* cover any empty cells.

- The rectangles can go ‘around the corner’/the actual map should be seen as a *torus*.
- Rectangles must have sides of 1, 2 or 4 squares (three adjacent cells are useless).

Example

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	$+$	$+$		$+$
$\bar{x}$	$+$		$+$	$+$

$$E = ({\color{red}xy}) \vee ({\color{orange}\bar{x}\bar{y}}) \vee {\color{blue}z}$$

Canonical form would consist of writing all cells separately (6 clauses).

Exercise

Exercise

RW: 10.6.6(c)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	$+$	$+$		$+$
$w\bar{x}$	$+$	$+$	$+$	$+$
$\bar{w}\bar{x}$			$+$	$+$
$\bar{w}x$	$+$			$+$

Outline

Definition: Boolean Algebra

Definition

A **Boolean algebra** is a structure $(T, \vee, \wedge, ', \mathbb{0}, \mathbb{1})$ where

- $\mathbb{0}, \mathbb{1} \in T$
- $\vee, \wedge : T \times T \to T$ (called **join** and **meet** respectively)
- $' : T \to T$ (called **complementation**)

and the following laws hold for all $x, y, z \in T$:

Commutativity: $x \vee y = y \vee x, \quad x \wedge y = y \wedge x$

Associativity:
$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$

Distributivity:
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Identity: $x \vee \mathbb{0} = x, \quad x \wedge \mathbb{1} = x$

Complementation: $x \vee x' = \mathbb{1}, \quad x \wedge x' = \mathbb{0}$

Examples of Boolean Algebras

Example

The set of subsets of a set X:

$$\begin{aligned} T &: \quad \mathrm{Pow}(X) \\ \vee \text{ (join)} &: \quad \cup \\ \wedge \text{ (meet)} &: \quad \cap \\ {}' \text{ (complement)} &: \quad \cdot^{c} \\ \mathbb{0} &: \quad \emptyset \\ \mathbb{1} &: \quad X \quad (\mathcal{U}) \end{aligned}$$

The Laws of Boolean algebra follow from the Laws of Set Operations.

Examples of Boolean Algebras

Example

The two element Boolean Algebra :

$$\mathbb{B} = (\{\text{true}, \text{false}\}, \|, \&\&, !, \text{false}, \text{true})$$

where $!, \&\&, \|$ are defined as:

- $!\text{true} = \text{false}$; $!\text{false} = \text{true}$,
- $\text{true} \;\&\&\; \text{true} = \text{true}$; ...
- $\text{true} \;\|\; \text{true} = \text{true}$; ...

Examples of Boolean Algebras

Example

Cartesian products of $\mathbb{B}$, that is n-tuples of 0's and 1's with Boolean operations, e.g. $\mathbb{B}^4$:

$$\begin{aligned} \textit{join:} \quad & (1,0,0,1) \vee (1,1,0,0) = (1,1,0,1) \\ \textit{meet:} \quad & (1,0,0,1) \wedge (1,1,0,0) = (1,0,0,0) \\ \textit{complement:} \quad & (1,0,0,1)' = (0,1,1,0) \\ \mathbb{0}: \quad & (0,0,0,0) \\ \mathbb{1}: \quad & (1,1,1,1). \end{aligned}$$

Examples of Boolean Algebras

Example

Functions from any set S to $\mathbb{B}$; that is, $\mathbb{B}^S$

If $f, g : S \longrightarrow \mathbb{B}$ then

$$\begin{array}{rcl} (f \vee g) : S \to \mathbb{B} & \text{defined by} & s \mapsto f(s) \parallel g(s) \\ (f \wedge g) : S \to \mathbb{B} & \text{defined by} & s \mapsto f(s)\ \&\&\ g(s) \\ f' : S \to \mathbb{B} & \text{defined by} & s \mapsto !f(s) \\ \mathbb{0} : S \to \mathbb{B} & \text{is the function} & s \mapsto 0 \\ \mathbb{1} : S \to \mathbb{B} & \text{is the function} & s \mapsto 1 \end{array}$$

Proofs in Boolean Algebras

Show an identity holds using the laws of Boolean Algebra, then that identity holds **in all Boolean Algebras**.

Example

In all Boolean Algebras

$$x \wedge x = x$$

for all $x \in T$.

Proof:

$$\begin{aligned} x &= x \wedge \mathbb{1} && \text{[Identity]} \\ &= x \wedge (x \vee x') && \text{[Complement]} \\ &= (x \wedge x) \vee (x \wedge x') && \text{[Distributivity]} \\ &= (x \wedge x) \vee \mathbb{0} && \text{[Complement]} \\ &= (x \wedge x) && \text{[Identity]} \end{aligned}$$

Duality

Definition

If E is an expression defined using variables (x, y, z, etc), constants (0 and 1), and the operations of Boolean Algebra ($\wedge$, $\vee$, and $'$) then $\mathrm{dual}(E)$ is the expression obtained by replacing $\wedge$ with $\vee$ (and vice-versa) and 0 with 1 (and vice-versa).

Definition

If $(T, \vee, \wedge, ', \mathbb{0}, \mathbb{1})$ is a Boolean Algebra, then $(T, \wedge, \vee, ', \mathbb{1}, \mathbb{0})$ is also a Boolean algebra, known as the **dual** Boolean algebra.

Theorem (Principle of duality)

If you can show $E_1 = E_2$ using the laws of Boolean Algebra, then $dual(E_1) = dual(E_2)$.

Duality

Example

We have shown $x \wedge x = x$.

By duality: $x \vee x = x$.