Formatif_HD_week3

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September 2023

Prove or disprove: For any set S and any three binary relations $R_1, R_2, R_3 \subseteq S \times S$:
(a) $(R_1 \cap R_2); R_3 = (R_1; R_3) \cap (R_2; R_3)$
Suppose $R_1 \cap R_2 = \emptyset$
we have $(x, y) \in R_1, (x, y') \in R_2, (y, z), (y', z) \in R_3$ for all $x, y, y', z \in S$
So $(R_1 \cap R_2); R_3 = \emptyset; R_3 = \emptyset$
$\Rightarrow (x, z) \notin (R_1 \cap R_2); R_3$ (1)
$R_1; R_3 \overset{def}{=} \{x, z \in S \times S\ there\ exists\ y \in S,\ such\ that\ (x, y) \in R_1\ and\ (y, z) \in R_3\}$
$\Rightarrow (x, z) \in R_1; R_3$
$R_2; R_3 \overset{def}{=} \{x, z \in S \times S\ there\ exists\ y' \in S,\ such\ that\ (x, y') \in R_2\ and\ (y', z) \in R_3\}$
$\Rightarrow (x, z) \in R_2; R_3$
$\Rightarrow (x, z) \in (R_1; R_3) \cap (R_2; R_3)$ (2)
These is a contradiction for (1) and (2), then disproved.

(b) $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$
" $\Rightarrow$ " :
$(R_1 \cup R_2); R_3 \overset{def}{=} \{x, z \in S \times S\ there\ exists\ y \in S,\ such\ that\ (x, y) \in (R_1 \cup R_2)\ and\ (y, z) \in R_3\}$
Since $(x, y) \in (R_1 \cup R_2) \Rightarrow (x, y) \in R_1$ or R_2
If $(x, y) \in R_1$ then $(R_1 \cup R_2); R_3 \equiv R_1; R_3$
$\Rightarrow (R_1 \cup R_2); R_3 \subseteq (R_1; R_3)$ (3)
If $(x, y) \in R_2$ then $(R_1 \cup R_2); R_3 \equiv R_2; R_3$
$\Rightarrow (R_1 \cup R_2); R_3 \subseteq (R_1; R_3)$ (4)
$\Rightarrow (R_1 \cup R_2); R_3 \subseteq (R_1; R_3)$ by using (3) and (4)
" $\Leftarrow$ " :
$(R_1; R_3) \cup (R_2; R_3) \overset{def}{=} \{x, z \in S \times S\ there\ exists\ y \in S,\ such\ that\ case\ 1 : (x, y) \in R_1\ and\ (y, z) \in R_3\ or\ case\ 2 : (x, y) \in R_2\ and\ (y, z) \in R_3\}$
For case 1: we get $(x, z) \in (R_1; R_3)$
$\Rightarrow (x, z) \in (R_1; R_3) \subseteq (R_1 \cup R_2); R_3$
$\Rightarrow (x, z) \in (R_1 \cup R_2); R_3$ (5)
For case 2: we get $(x, z) \in (R_2; R_3)$
$\Rightarrow (x, z) \in (R_2; R_3) \subseteq (R_1 \cup R_2); R_3$

$\Rightarrow (x, z) \in (R_1 \cup R_2); R_3$
$\Rightarrow (x, z) \in (R_1 \cup R_2); R_3$ (6)
$\Rightarrow (R_1; R_3) \cup (R_2; R_3) \subseteq (R_1 \cup R_2); R_3$ by using (5) and (6).

Therefore $(R_1 \cup R_2); R_3 \subseteq (R_1; R_3), (R_1; R_3) \cup (R_2; R_3) \subseteq R_1 \cup R_2); R_3 \Rightarrow$ $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$