Formatif_D_week3

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Prove that for any set S and any binary relation $R \subseteq S \times S$:

- (a) R is reflexive if and only if $I \subseteq R$ *where* $I = \{(x, x) : x \in S\}$ is the identity relation
  $" \Rightarrow "$:
  If R is reflexive, for all $x \in S : (x, x) \in R$
  Since $I = \{(x, x) : x \in S\} \Rightarrow (x, x) \in R$
  $\Rightarrow I \subseteq R$
  $" \Leftarrow "$:
  If $I \subseteq R \Rightarrow \{(x, x) : x \in S\} \subseteq R$
  $\Rightarrow (x, x) \in R$ for all $x \in S$
  $\Rightarrow R$ is reflexive

  $\Rightarrow R$ is reflexive if and only if $I \subseteq R$

- (b) R is symmetric if and only if $R = R^{\leftarrow}$
  $" \Rightarrow "$:
  If R is symmetric, for all $x, y \in S : if\ (x, y) \in R\ then\ (y, x) \in R$
  Since $(y, x) \in R$ for all $x, y \in S \Rightarrow (y, x) \in S \times S$ where $(x, y) \in R \Rightarrow (y, x) \in R^{\leftarrow} \Rightarrow R \subseteq R^{\leftarrow}, R^{\leftarrow} \subseteq R \Rightarrow R = R^{\leftarrow}$
  $" \Leftarrow "$:
  If $R = R^{\leftarrow}$, we have $(x, y) \in R$ and $(x, y) \in R^{\leftarrow}$
  If $(x, y) \in R^{\leftarrow} \Rightarrow (x, y) \in S \times S$ where $(y, x) \in R$
  $\Rightarrow$ if $(x, y) \in R\ then\ (y, x) \in R$ for all $x, y \in S$
  $\Rightarrow$ R is symmetric.

  $\Rightarrow$ R is symmetric if and only if $R = R^{\leftarrow}$

- (c) R is transitive if and only if $R; R \subseteq R$
  $" \Rightarrow "$:
  If R is transitive then: for all $x, y, z \in S$ if $(x, y) \in R$ and $(y, x) \in R$ then $(x, z) \in R$
  $\Rightarrow (x, z) \in R$
  Since $R; R \overset{def}{=} \{x, z \in S \times S\ there\ exists\ y \in S,\ such\ that\ (x, y) \in R\ and\ (y, z) \in R\}$
  $\Rightarrow (x, z) \in R; R$

$\Rightarrow R; R \subseteq R$

$\Leftarrow$ " :
If $R; R \subseteq R$, by the definition of composition:
$R; R \stackrel{def}{=} \{x, z \in S \times S \; there \; exists \; y \in S, \; such \; that \; (x, y) \in R \; and \; (y, z) \in R\}$
$\Rightarrow (x, z) \in R; R$
$\Rightarrow (x, z) \in R$ Since $R; R \subseteq R$
$\Rightarrow$ for all $x, y, z \in S$ if $(x, y) \in R$ and $(y, x) \in R$ then $(x, z) \in R$
$\Rightarrow$ R is transitive.

$\Rightarrow$ R is transitive if and only if $R; R \subseteq R$