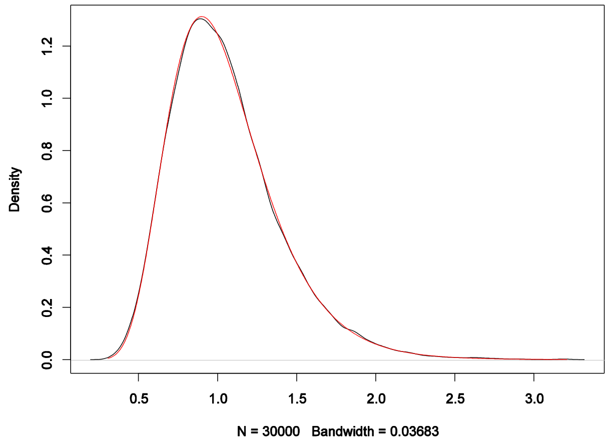
I observed the similarity between the lognormal distribution and the distribution of the sample median that comes from a lognormal population. The chart below shows the density (in black) of the sample median (with sample size 15) coming from $LogN(0,1)$ and the fitted lognormal density (in red) using method of moments, with $\mu = -0.0022, \sigma = 0.3206$.



Since this is not a journal article, I don't write it in an academic manner. Still, for the sake of readiness, I use the following notations for certain special functions.

The CDF of normal distribution	$\Phi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} \exp\left(-t^2\right) dt$
Hypergeometric function	${}_2F_1(a,b,c;w) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{w^n}{n!}; \quad (a)_n = \begin{cases} 1 & n=0 \\ a(a+1)\cdots(a+n-1) & n>0 \end{cases}$
Error function	$erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w \exp(-t^2) dt$
Beta function	$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$

Also, given the CDF of the parent population $G(x)$, the sample median distribution is given by,

$$\int_{-\infty}^{x} \frac{1}{B(k+1,k+1)} G^k(y)\left[1-G(y)\right]^k dG(y)$$

where sample size $n = 2k+1$. In this paper, $G(x) = \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)$. Also, having this substituted then solving this integral gives the exact sample median distribution,

$$\frac{\Phi^{k+1}\left(\frac{\ln(x)-\mu}{\sigma}\right)}{(k+1)B(k+1,k+1)} \, {}_2F_1\left(k+1,-k,k+2;\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)\right)$$

Formally I found the lognormal approximation to the distribution of the sample median that comes from a lognormal parent,

$$\lim_{k\to\infty}\frac{\Phi^{k+1}\left(\frac{\ln(x)-\mu}{\sigma}\right)}{(k+1)B(k+1,k+1)}\,{}_2F_1\left(k+1,-k,k+2;\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)\right)=\Phi\left(\frac{\ln(x)-\mu}{\sigma B(k+1,k+1)4^k\sqrt{2}}\right)$$

That is, the sample median distribution can be approximated by a lognormal distribution with the two parameters as μ and $\sigma B(k+1,k+1)4^k$.

Alternatively, in terms of PDF,

$$\frac{\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(k+1,k+1)}\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)^k\left[1-\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)\right]^k\approx\frac{\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma B(k+1,k+1)4^k}\right)^2\right\}}{x\sigma B(k+1,k+1)4^k\sqrt{2\pi}}$$

The proof requires the following approximation.

Proposition (Approximating the error function):

$$\operatorname{erf}(w)\approx\operatorname{sgn}(w)\sqrt{1-\exp\left\{-w^2\frac{4}{\pi}\right\}}$$

with the maximum absolute error of 6.29×10^{-3} at $w\approx1.169$ and $\operatorname{sgn}(\cdot)$ is the sign function.

This approximation is based on the uniform approximation in Winitzki (2013) with the use of Padé approximant which is given by,

$$\operatorname{erf}(w)\approx\operatorname{sgn}(w)\sqrt{1-\exp\left\{-w^2\frac{P(w)}{Q(w)}\right\}}$$

where $P(w)$ and $Q(w)$ are polynomials.

Taking $P(w)$ and $Q(w)$ with the degree of two, Nealy, et al (2010) then approximates that,

$$\operatorname{erf}(w)\approx\operatorname{sgn}(w)\sqrt{1-\exp\left\{-w^2\frac{\frac{4}{\pi}+aw^2}{1+aw^2}\right\}}$$

where $a=\frac{8(\pi-3)}{3\pi(4-\pi)}$. This approximation used here further reduces the degree of the polynomials to one.

Following the above error function approximation, the main proof is given below,

$$\frac{\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(k+1,k+1)}\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)^k\left[1-\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right)\right]^k$$
$$=\frac{\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(k+1,k+1)}\left[1-erf^2\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)\right]^k$$

Then,

$$\frac{\exp\left\{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2\right\}}{x\sigma\sqrt{2\pi}B(k+1,k+1)}\left[1-erf^2\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)\right]^k\approx\frac{\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2B^2(k+1,k+1)4^{2k}}\right\}}{xB(k+1,k+1)4^k\sigma\sqrt{2\pi}}$$
$$\Rightarrow\left[1-erf^2\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)\right]^k\approx\frac{1}{xB(k+1,k+1)4^k\sigma\sqrt{2\pi}}\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2B^2(k+1,k+1)4^{2k}}\right\}\frac{B(k+1,k+1)4^k}{f(x;\mu,\sigma)}$$
$$\Rightarrow\left[1-erf^2\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)\right]^k\approx\frac{\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2B^2(k+1,k+1)4^{2k}}\right\}}{\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}}$$
$$\Rightarrow\left[1-erf^2\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)\right]^k\approx\left[\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}\right]^{\left(\frac{1}{H}-1\right)}$$

where $H=4^{2k}B^2(k+1,k+1)$.

For x cannot be negative, apply the error function approximation on the LHS and rearrange to get,

$$\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\frac{4k}{\pi}\right\}\approx\left[\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}\right]^{\left(\frac{1}{H}-1\right)}$$
$$\Rightarrow\left[\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}\right]^{\frac{4k}{\pi}}\approx\left[\exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}\right]^{\left(\frac{1}{H}-1\right)}$$

As noted earlier, the maximum absolute error of the error function approximation is 6.29×10^{-3}. The resulting error is not as large as it seems; $|\mathrm{erf}\,(w)|$ is strictly smaller than 1 hence the error resulted from $[1-\mathrm{erf}\,(w)]^k$ is much smaller as k increases.

We now have to show that $\frac{4k}{\pi}\sim\frac{1}{H}-1$. Given that $H=2^{4k}B^2(k+1,k+1)$, we first simplify the beta function.

$B(k+1,k+1)=B(k,k)\dfrac{k^2}{4k^2+2k}$. Then by Stirling's formula,

$$B(k+1,k+1)=B(k,k)\frac{k^2}{4k^2+2k}\approx\sqrt{2\pi}\,\frac{k^{2k-1}}{(2k)^{2k-0.5}}\frac{k^2}{4k^2+2k}=\frac{\sqrt{\pi k}}{2^{2k}(2k+1)}$$

Therefore, $H=2^{4k}B^2(k+1,k+1)\approx\dfrac{\pi k}{(2k+1)^2}$ and it is straight forward that $\lim_{k\to\infty}\dfrac{\frac{1}{H}-1}{4k/\pi}=1$

A final note regarding the approximation error and the rate of convergence:

- The maximum absolute error depends only on the sample size.
- It has faster rate of convergence than the asymptotically normal sample median distribution introduced in Cramer (1946, p.369).
- (Hence, I still haven't shown how the lognormal approximation proposed here eventually converges to the one in Cramer (1946, p.369), recalling the normal approximation to lognormal distribution)

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Nealy JE, Chang CK, Norman RB, Blattnig SR, Badavi FF, Adamczyk AM. (2010) A deterministic transport code for space environment electrons, NASA Technical Documents, NASA/TP-2010-216168, Hampton: NASA Langley Research Center.

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