

QUASI-COHERENCE IS (ALMOST) ALL YOU NEED FOR SYNTHETIC DOMAIN THEORY

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1. INTRODUCTION

Let \mathbb{T} be a Horn theory. The type system we work with is based on the quasi-coherence principle available in the classifying topoi of subcanonical quotients of \mathbb{T} [2, 1]. (The envelopping ∞ -topoi of) such topoi provide models for the type theory we work with in this paper.

In this paper we will be working in a dependent type theory with a univalent universe \mathbf{Type} . Recall the notion of h -levels [13]. For us the most important h -levels are -1 and 0 , which are *propositions* and *sets*. In particular, we can define subuniverses \mathbf{Prop} , \mathbf{Set} .

We will also assume the existence of propositional truncation. This allows us to define existential quantifier of a family $P : X \rightarrow \mathbf{Prop}$,

$$\exists x : X. P(x) := \|\sum_{x : X} P(x)\|$$

and disjunction of propositions,

$$P \vee Q := \|P + Q\|.$$

Note that by univalence, \mathbf{Prop} is already closed under dependent product. But to emphasis that the result is again a proposition, we may also write dependent products of propositions as $\forall x : X. P(x)$.

The universe \mathbf{Prop} is used to define subtypes, i.e. we identify a subtype of X as a family of propositions $A : X \rightarrow \mathbf{Prop}$ indexed over X , where the projection $\sum_{x : X} A(x) \rightarrowtail X$ is the inclusion of this subtype. We often write the sum as a collection,

$$\{x : X \mid A(x)\} := \sum_{x : X} A(x).$$

When X in fact is a set, then any of its subtype is also a set, and we will emphasis by calling them *subsets*.

Sets are used to define *algebras* for a Horn theory. Given any Horn theory \mathbb{T} , by a \mathbb{T} -*model* we will always mean a *set* A equipped with operations according to the signature of \mathbb{T} , such that it satisfies the axioms of \mathbb{T} . Since

A is a set, being a \mathbb{T} -model is again a proposition for a given family of operations.

At the end, we remark that \mathbb{N} is the usual inductive type of natural numbers. For any $n : \mathbb{N}$, we will also use n to denote the finite type with n elements, which is viewed as an initial segment of \mathbb{N} . We stress that this is only syntactic sugar, and can be made more precise if need be.

2. QUASI-COHERENCE AND AFFINE SPACES

In this section we discuss the notion of quasi-coherent algebra and affine spaces. The technical results presented in this section is essentially contained in [5], based on the work in [2].

As a starting point, we assume there is a generic \mathbb{T} -model in our type system:

Axiom 1. There is a set \mathbb{I} equipped with a \mathbb{T} -model structure.

Quasi-coherence is a condition on \mathbb{I} -algebras: An \mathbb{I} -algebra is a \mathbb{T} -model A equipped with a homomorphism $\mathbb{I} \rightarrow A$. The type of \mathbb{I} -algebras will be denoted as $\mathbb{I}\text{-Alg}$. More generally, for any \mathbb{T} -algebras A, B , we will use $\mathbb{I}\text{-Alg}(A, B)$ to denote the set of homomorphisms of \mathbb{I} -algebras from A to B . For instance, this can be used to define the *spectrum* A ,

$$\text{Spec } A := \mathbb{I}\text{-Alg}(A, \mathbb{I}).$$

A lot of sets can be naturally viewed as the spectrum of some \mathbb{I} -algebra. For the simplest example, since \mathbb{I} is the initial \mathbb{I} -algebra,

$$\text{Spec } \mathbb{I} = \mathbb{I}\text{-Alg}(\mathbb{I}, \mathbb{I}) = 1.$$

Now if we use $\mathbb{I}[x]$ to denote the free \mathbb{I} algebra on one generator, we can identify \mathbb{I} itself as a spectrum,

$$\text{Spec } \mathbb{I}[x] = \mathbb{I}\text{-Alg}(\mathbb{I}[x], \mathbb{I}) = \mathbb{I}.$$

However, notationally we will distinguish \mathbb{I} as an algebraic gadget and as a spectrum, which we view as a geometrical object. The latter will be denoted as \mathbb{I} , and the equivalence $\mathbb{I} \rightarrow \mathbb{I}$ takes $i : \mathbb{I}$ to the evaluation map $\text{ev}_i : \mathbb{I}[x] \rightarrow \mathbb{I}$. More generally, we have:

Example 2.1. For any $n : \mathbb{N}$, the n -th cube \mathbb{I}^n is the following spectrum,

$$\text{Spec } \mathbb{I}[n] = \mathbb{I}\text{-Alg}(\mathbb{I}[n], \mathbb{I}) = \mathbb{I}^n.$$

Definition 2.2. A is *quasi-coherent*, if the canonical evaluation

$$A \rightarrow \mathbb{I}^{\text{Spec } A}$$

is an equivalence of \mathbb{I} -algebras, where $\mathbb{I}^{\text{Spec } A}$ has the point-wise \mathbb{I} -algebra structure. We write $\text{isQC}(A)$ for the proposition of being quasi-coherent.

Example 2.3. \mathbb{I} itself by definition is quasi-coherent. We have seen that $\text{Spec } \mathbb{I} = 1$. Under this equivalence, the canonical map

$$\mathbb{I} \rightarrow \mathbb{I}^{\text{Spec } \mathbb{I}} = \mathbb{I}$$

is exactly the identity on \mathbb{I} , hence is an equivalence.

Note by construction, Spec is a contravariant functor on \mathbb{I} -algebras. For quasi-coherent algebras, we have a more general duality result:

Proposition 2.4. *If B is quasi-coherent, then for any \mathbb{I} -algebras A , we have*

$$\mathbb{I}\text{-Alg}(B, A) = \text{Spec } B^{\text{Spec } A}.$$

Proof. By quasi-coherence of B , we have the following equivalences,

$$\mathbb{I}\text{-Alg}(A, B) = \mathbb{I}\text{-Alg}(A, \mathbb{I}^{\text{Spec } B}) = \mathbb{I}\text{-Alg}(A, \mathbb{I})^{\text{Spec } B} = \text{Spec } A^{\text{Spec } B}.$$

The second identity holds since by quasi-coherence the \mathbb{I} -algebra structure on $\mathbb{I}^{\text{Spec } B}$ is point-wise. \square

The above duality result tells us that we can infer equivalence of algebras via equivalence of their spectrum:

Corollary 2.5. *For any $f : \mathbb{I}\text{-Alg}(A, B)$, if A, B are quasi-coherent, then f is an equivalence iff it induces an equivalence $\text{Spec } B \rightarrow \text{Spec } A$.*

To illustrate the usefulness of the above duality result, let us also introduce the following notion:

Definition 2.6. We say an \mathbb{I} -algebra A is *stably quasi-coherent* if all finitely generated principle congruences of A are also quasi-coherent, i.e. for any finite $n : \mathbb{N}$ and for any $a, b : n \rightarrow A$, $A/a = b$ is again quasi-coherent.

By definition, any finitely generated principle congruences of a stably quasi-coherent algebra will again be stably quasi-coherent. For a stably quasi-coherent \mathbb{I} -algebra A , its spectrum always “has enough points” to distinguish elements:

Lemma 2.7. *For any stably quasi-coherent \mathbb{I} -algebra A and for any $a, b : A$,*

$$a = b \leftrightarrow \forall x : \text{Spec } A. xa = xb.$$

Proof. The \Rightarrow direction is trivial. For \Leftarrow , by the universal property of the quotient $A/a = b$, we have

$$\text{Spec}(A/a = b) = \{x : \text{Spec } A \mid xa = xb\} \subseteq \text{Spec } A.$$

If $\forall x : \text{Spec } A. xa = xb$ holds, then the inclusion $\text{Spec}(A/a = b) \hookrightarrow \text{Spec } A$ is an equivalence. Since by assumption $A/a = b$ is also quasi-coherent, by Corollary 2.5 the quotient $A \twoheadrightarrow A/a = b$ is an equivalence. This exactly says $a = b$. \square

Dually, we can use quasi-coherence to define the notion of *affine spaces*:

Definition 2.8. We say a set X is *affine*, if \mathbb{I}^X is quasi-coherent, and the canonical evaluation map

$$X \rightarrow \operatorname{Spec} \mathbb{I}^X$$

is an equivalence. Similarly, we say X is *stably affine* if \mathbb{I}^X is furthermore stably quasi-coherent.

Proposition 2.9. *For any set X , X is affine iff there exists a quasi-coherent A such that $X = \operatorname{Spec} A$.*

Proof. The only if direction holds by definition, since A can be taken as \mathbb{I}^X . For the if direction, it suffices to show $\operatorname{Spec} A$ is affine whenever A is quasi-coherent. Now by quasi-coherence, $A = \mathbb{I}^{\operatorname{Spec} A}$ thus $\mathbb{I}^{\operatorname{Spec} A}$ is quasi-coherent. Furthermore, the canonical map $\operatorname{Spec} A \rightarrow \operatorname{Spec} \mathbb{I}^{\operatorname{Spec} A}$ is an equivalence again by quasi-coherence of A . \square

At the end of this section, we describe finite limits of affine spaces. By Example 2.3, the terminal type 1 is affine. More generally, pullbacks of affine spaces can be computed as a spectrum:

Proposition 2.10. *Let A, B, C be \mathbb{I} -algebras. Given $f : \mathbb{I}\text{-Alg}(B, A)$ and $g : \mathbb{I}\text{-Alg}(B, C)$, the pullback of $\operatorname{Spec} f$ and $\operatorname{Spec} g$ is given by the spectrum*

$$\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} C = \operatorname{Spec}(A \otimes_B C),$$

where $A \otimes_B C$ is the following pushout of \mathbb{I} -algebras,

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ g \downarrow & & \downarrow \\ C & \longrightarrow & A \otimes_B C \end{array}$$

Proof. This essentially holds by the universal property of the pushout. \square

Remark 2.11. Notice that in the above result, if $\operatorname{Spec} B$ is affine, then by Proposition 2.4 any map from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ is induced by an \mathbb{I} -algebra morphism, and similarly for $\operatorname{Spec} C$. Thus, any such pullback will be computed as a spectrum.

Following [5], the above result provides an extremely easy formula to compute the coproducts of quasi-coherent algebras:

Corollary 2.12. *For any \mathbb{I} -algebra A, B , if B is quasi-coherent, then there is an equivalence*

$$\mathbb{I}^{\operatorname{Spec} A \otimes B} = B^{\operatorname{Spec} A}.$$

Proof. Since 1 is affine, thus $\text{Spec } A \times \text{Spec } B$ can be computed as $\text{Spec } A \otimes B$. Then by quasi-coherence of B , we have

$$\mathbb{I}^{\text{Spec}(A \otimes B)} = \mathbb{I}^{\text{Spec } A \times \text{Spec } B} = B^{\text{Spec } A}.$$

Following this equivalence, the evaluation map $A \otimes B \rightarrow \mathbb{I}^{\text{Spec } A \otimes B}$ is exactly the one given above. \square

Under the above identification, if B is quasi-coherent, then the canonical map $A \otimes B \rightarrow \mathbb{I}^{\text{Spec } A \otimes B}$ can be equivalently viewed as the following map

$$c \mapsto \lambda x : \text{Spec } A. (x \otimes B)(c) : A \otimes B \rightarrow B^{\text{Spec } A}.$$

Of course when $A \otimes B$ is quasi-coherent, then the above map will again be an equivalence. We can similarly calculate the canonical inclusions. For the left inclusion $A \rightarrow A \otimes B$, given any $a : A$ and $x : \text{Spec } A$,

$$(x \otimes B)(a \otimes 1) = xa,$$

where we view $xa : \mathbb{I}$ as an element in B using its \mathbb{I} -algebra. For the right inclusion $B \rightarrow A \otimes B = B^{\text{Spec } A}$, for any $b : B$ and $x : \text{Spec } A$,

$$(x \otimes B)(1 \otimes b) = b,$$

which implies the image of b is simply the constant function on b .

3. OPEN PROPOSITIONS AND DOMINANCE

Notice that up till this point we have not used any special property of \mathbb{T} rather than the fact that it is a Horn theory. However, to move closer to the intended application in synthetic domain theory, we start by assuming our theory \mathbb{T} is *propositional*:

Definition 3.1. We say a Horn theory \mathbb{T} is *propositional*, if it extends the theory of meet-semi-lattices, and truth of an element is computed by slicing: For any \mathbb{T} -model A and $a : A$, the quotient $A/a = 1$ is given by

$$a \wedge - : A \twoheadrightarrow A/a,$$

where A/a by definition is $\downarrow a := \{ b : A \mid b \leq a \}$.

Remark 3.2. The theory \mathbb{M} of meet-semi-lattices, \mathbb{D} of distributive lattices, \mathbb{H} of Heyting algebras, and \mathbb{B} of Boolean algebras are all examples of propositional theories in the above sense. In fact, all finitary quotients of \mathbb{H} or \mathbb{B} will again be propositional. More generally, for any propositional theory \mathbb{T} and any \mathbb{T} -model D , the theory of D -algebras will again be propositional. We call such a theory a theory of \mathbb{T} -algebra.

For a propositional theory \mathbb{T} , we think of the generic model \mathbb{I} as certain interval object, since it is equipped with a partial order. For propositional theories, more sets can be realised as spectrums. The important examples are *simplices*:

Example 3.3. For any $n : \mathbb{N}$, let $\Delta^n \subseteq \mathbb{I}^n$ be the following subset,

$$\Delta^n := \{i : n \rightarrow \mathbb{I} \mid i_1 \geq \dots \geq i_n\}.$$

This type is indeed a spectrum, since by definition we have

$$\Delta^n = \text{Spec } \mathbb{I}[i_1, \dots, i_n] / i_1 \geq \dots \geq i_n.$$

To simplify later discussion, we might also introduce the following types isomorphic to simplices above,

$$\Delta_n := \{i : n \rightarrow \mathbb{I} \mid i_1 \leq \dots \leq i_n\}.$$

The constant $1 : \mathbb{I}$, which is the top element in \mathbb{I} , induces a predicate

$$t : \mathbb{I} \rightarrow \text{Prop},$$

which takes $i : \mathbb{I}$ to the proposition $i = 1$. The first observation is that t takes $i : \mathbb{I}$ to a spectrum:

Lemma 3.4. *For any $i : \mathbb{I}$, $ti = \text{Spec } \mathbb{I}/i$.*

Proof. By definition, $\text{Spec } \mathbb{I}/i = \mathbb{I}\text{-Alg}(\mathbb{I}/i, \mathbb{I})$. Since $\mathbb{I} \twoheadrightarrow \mathbb{I}/i$ is a quotient, there is a homomorphism from \mathbb{I}/i to \mathbb{I} iff $i = 1$, and in this case the map is unique. \square

To say something about the propositions that lies in t , we assume a minimal amount of axioms:

Axiom 2 (SQCI). \mathbb{I} is stably quasi-coherent.

We already know that \mathbb{I} is quasi-coherent. If it is stably so, then in particular for any $i : \mathbb{I}$, the principle congruence \mathbb{I}/i will also be (stably) quasi-coherent. This implies the following result:

Lemma 3.5 (SQCI). *The interval \mathbb{I} is conservative,*

$$\forall i, j : \mathbb{I}. (ti \leftrightarrow tj) \leftrightarrow i = j.$$

Proof. It suffices to show

$$\forall i, j : \mathbb{I}. (ti \rightarrow tj) \rightarrow i \leq j.$$

Take i, j with $ti \rightarrow tj$. By Lemma 3.4 and (SQCI), each ti and tj will be affine. Then we get a restriction map between \mathbb{I} -algebras,

$$\mathbb{I}/j = \mathbb{I}^{tj} \rightarrow \mathbb{I}^{ti} = \mathbb{I}/i.$$

By the universal property of the quotient and the fact that \mathbb{T} is a propositional theory, such a map exists iff $i \leq j$, which is the desired result. \square

Thus, under (SQCI), we may view the generic algebra \mathbb{I} as a subuniverse of *open* propositions via the embedding t :

Definition 3.6 (SQCI). Given a proposition p , we say it is *open* if p is of the form ti for some i in \mathbb{I} ,

$$\text{isopen}(p) := \sum_{i:\mathbb{I}} p \leftrightarrow ti.$$

By Lemma 3.5, the i that $p \leftrightarrow ti$ will be unique, thus being open is a proposition. Also, open propositions are evidently closed under finite conjunctions, since t preserves them. More generally, we may define the notion of open subtypes:

Definition 3.7 (SQCI). A subtype U of X is *open*, if for any $x : X$ the proposition $U(x)$ is open,

$$\text{isOpen}(U) := \forall x : X. \text{isopen}(U(x)).$$

In other words, an open proposition is equivalently a map $U : X \rightarrow \mathbb{I}$, where the subtype is classified as the following pullback,

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \mathbb{I} \end{array}$$

This also implies that open subtypes are closed under pullbacks. Note that this crucial depends on conservativity of \mathbb{I} .

If X is affine, open subtypes are easy to classify:

Lemma 3.8 (SQCI). *Let $X = \text{Spec } A$ be affine. Then any open subset of X is of the form D_a for some $a : A$, where*

$$D_a := \{x : X \mid t(xa)\} = \text{Spec } A/a.$$

If X is stably affine, then an open subset of it is again stably affine.

Proof. This follows from the fact that the type of opens of X is $\mathbb{I}^X = A$ since X is affine. If X is stably affine, then so is D_a . \square

One observation from [5] is that the type of open propositions forms a *dominance* in the sense of [11]. In other words, open propositions are closed under truth and dependent product. The *loc. cit.* uses certain choice principle on affine spaces to show this. However, in our situation the conservativity of the interval \mathbb{I} allows direct computation of the type of open subsets of affine spaces as shown above. Hence, our proof does not require any additional choice principle:

Proposition 3.9 (SQCI). *The type of opens \mathbb{I} forms a dominance.*

Proof. Suppose p is an open proposition. By definition $p = ti$ for some $i : \mathbb{I}$. Now if q is an open subset of p , since $p = ti$ is affine, Lemma 3.8 implies that $q = D_j$ for some $j : \mathbb{I}/i$. Since \mathbb{T} is propositional, equivalently j can be viewed as an element $j : \mathbb{I}$ with $j \leq i$. This way, $q = tj$. \square

Corollary 3.10. *Open subtypes are transitive.*

4. LIFTING

Given then dominance structure \mathbb{I} , we can construct internally the lifting functor. For any type X , its lift is given by

$$X_{\perp} := \sum_{i:\mathbb{I}} X^{ti}.$$

The functoriality is easy to express: For any $f : X \rightarrow Y$, we have

$$f_{\perp}(i, x) := (i, \lambda w:ti. f x w).$$

There is an evident unit $\eta_{\perp} : X \rightarrow X_{\perp}$, where

$$\eta := \lambda x:X. (1, \lambda -:1. x).$$

The dominance structure on \mathbb{I} also gives a multiplication $\mu : (X_{\perp})_{\perp} \rightarrow X_{\perp}$, where μ takes any (i, u) with $i : \mathbb{I}$ and $u : ti \rightarrow X_{\perp}$ first to (j, x) , where j is the dependent sum

$$j := \sum_{w:ti} (uw)_0,$$

and $x : tj \rightarrow X$ is the partial element such that for $w : ti$ and $v : (uw)_0$

$$x(w, v) := (uw)_1(v).$$

Example 4.1. By definition, it is easy to see that

$$1_{\perp} = \sum_{i:\mathbb{I}} ti = \mathbb{I}.$$

For synthetic domain theory, the object of particular importance is the lift of the type of open propositions \mathbb{I} itself. In fact, the lifting of any stably quasi-coherent \mathbb{I} -algebra can be computed fairedly explicitly:

Lemma 4.2. *If A is stably quasi-coherent, then we have*

$$A_{\perp} = \{i : \mathbb{I}, a : A \mid a \leq i\}.$$

Proof. Note for any $i : \mathbb{I}$, the quotient A/i is again quasi-coherent. Now notice that we do have

$$A/i = A \otimes \mathbb{I}/i.$$

By Corollary 2.12 and quasi-coherence of A/i ,

$$A/i = A \otimes \mathbb{I}/i = A^{\text{Spec } \mathbb{I}/i} = A^{ti}.$$

This way, it follows that

$$A_{\perp} = \sum_{i:\mathbb{I}} A^i = \sum_{i:\mathbb{I}} A/i = \{i : \mathbb{I}, a : A \mid a \leq i\}. \quad \square$$

Corollary 4.3 (SQCI). *For the interval \mathbb{I} , $\mathbb{I}_{\perp} = \Delta^2$.*

Proof. By Lemma 4.2,

$$\mathbb{I}_{\perp} = \{i, j : \mathbb{I} \mid i \geq j\} = \Delta^2.$$

The second equivalence again uses the fact that \mathbb{T} is propositional. \square

Remark 4.4. One interesting thing to notice here is that, though \mathbb{I}_{\perp} by computation is a dependent sum of algebras, it is naturally equivalent to a *spectrum*, which is a geometric object. In some sense the source is the assumption that \mathbb{T} is propositional, which allows us to identify the algebraic object \mathbb{I}/i as a subset $\{j : \mathbb{I} \mid j \leq i\}$.

More generally, for domain theoretic applications we would want to compute the liftings of the simplices introduced as spectrums in Example 3.3. It is first of all easy to see that the lifting of any affine space is also straight forward to compute by duality:

Lemma 4.5. *If $X = \text{Spec } A$ is affine, then the lifting of X is given by*

$$X_{\perp} = \sum_{i:\mathbb{I}} \mathbb{I}\text{-Alg}(A, \mathbb{I}/i).$$

Proof. By Proposition 2.4, since $X = \text{Spec } A$ is affine,

$$X_{\perp} = \sum_{i:\mathbb{I}} X^i = \sum_{i:\mathbb{I}} \text{Spec } A^{\text{Spec } \mathbb{I}/i} = \sum_{i:\mathbb{I}} \mathbb{I}\text{-Alg}(A, \mathbb{I}/i). \quad \square$$

Thus, motivated by domain theory, we find ourselves naturally move towards the following axiomatisation:

Axiom 3 (SQCF). All finitely generated free \mathbb{I} -algebras, i.e. $\mathbb{I}[n]$ for $n : \mathbb{N}$, are stably quasi-coherent.

Equivalently, (SQCF) says that any finitely presented \mathbb{I} -algebra is (stably) quasi-coherent, where an \mathbb{I} -algebra is finitely presented if it is merely of the form $\mathbb{I}[n]/s = t$ with $s, t : m \rightarrow \mathbb{I}[n]$ for some finite n, m .

Of course, (SQCF) implies (SQCI) when taking n to be 0. Furthermore, (SQCF) implies that the simplices Δ^n are now (stably) affine as well. This way, we can indeed compute their lifts:

Lemma 4.6 (SQCF). *For any $n : \mathbb{N}$, we have*

$$\Delta_{\perp}^n = \Delta^{n+1}.$$

Proof. By Lemma 4.5, for the affine space $\Delta^n = \text{Spec } \mathbb{I}[n]/i_1 \geq \dots \geq i_n$,

$$\begin{aligned} \Delta_\perp^n &= \sum_{i:\mathbb{I}} \mathbb{I}\text{-Alg}(\mathbb{I}[n]/i_1 \geq \dots \geq i_n, \mathbb{I}/i) \\ &= \{i, i_1, \dots, i_n : \mathbb{I} \mid i \geq i_1 \geq \dots \geq i_n\} \\ &= \Delta^{n+1} \end{aligned}$$

The second equality again uses the fact that \mathbb{T} is propositional. \square

In particular, from a geometric perspective, the unit

$$\eta : \Delta^n \rightarrow \Delta_\perp^n = \Delta^{n+1}$$

takes $i_1 \geq \dots \geq i_n$ in Δ^n to $1 \geq i_1 \geq \dots \geq i_n$ in Δ^{n+1} .

5. INTRINSIC ORDER AND A LOCALITY AXIOM

Following [7], given the dominance \mathbb{I} there is indeed a canonical way to define an order on any type X :

Definition 5.1. The *intrinsic order* on a type X is defined as follows:

$$x \preceq y := \forall U : X \rightarrow \mathbb{I}. U(x) \leq U(y).$$

By definition, the intrinsic order is reflexive and transitive. As already observed in *loc. cit.*, one important property of the intrinsic order is that every map is monotone w.r.t. this order:

Lemma 5.2. For $f : X \rightarrow Y$, $x \preceq y$ in X implies $fx \preceq fy$ in Y .

Proof. This simply follows from compositionality of functions. \square

Just as how we have computed the lifting of affine spaces in Section 4, the quasi-coherence principle also determines the intrinsic order on affine spaces. For instance, if $X = \text{Spec } A$ is affine, then $\mathbb{I}^X = A$, which means for any $x, y : X$,

$$x \preceq y \leftrightarrow \forall a : A. xa \leq ya.$$

In other words, the order \preceq on an affine space is the point-wise order in $X = \mathbb{I}\text{-Alg}(A, \mathbb{I})$. In practice, we usually have an explicit presentation of the algebra A , and in this case we have a more explicit description:

Lemma 5.3. Let $A = \mathbb{I}[X]/R$ be a quasi-coherent \mathbb{I} -algebra for some generating set X and relation R . Then the intrinsic order on $\text{Spec } A$ is induced as a subspace of the point-wise order on $\text{Spec } \mathbb{I}[X] = \mathbb{I}^X$.

Proof. By the previous discussion, for any $x, y : \text{Spec } A$ we have

$$x \preceq y \leftrightarrow \forall p : \mathbb{I}[X]/R. xp \leq yp.$$

Now if we view x, y as objects in $\text{Spec } \mathbb{I}[X] = \mathbb{I}^X$, then for any $p : \mathbb{I}[X]/R$, xp is simply the evaluation of p on x . Of course then $xp \leq yp$ for all p iff $x \leq y$ point-wise, since p can be chosen as the generating variables. \square

Corollary 5.4 (SQCF). *The intrinsic order on cubes \mathbb{I}^n and on the simplices Δ^n are exactly the one we expect. In particular, the maps between them are all monotone.*

Given the intrinsic order on X , one natural question is to ask when it is classified by the interval object \mathbb{I} itself. However, to even formulate this property, we should at least require that \mathbb{I} has an additional boundary point 0 , which is the minimal element.

Besides this, motivated by applications in domain theory, the existence of a minimal element in the dominance \mathbb{I} is also accounted for divergent computation. For this reason, we will now make an additional assumption on our theory \mathbb{T} :

Definition 5.5. We say a propositional theory \mathbb{T} admits *falsum*, if it also has a constant 0 denoting the minimal element in the order induced by the meet-semi-lattice structure.

Remark 5.6. A minimal example of such \mathbb{T} is the theory of meet-semi-lattices equipped with an additional constant 0 . Of course, the theory of distributive lattices, Heyting algebras and Boolean algebras are all of such type, and so is any theory of a \mathbb{T} -algebra for a propositional theory \mathbb{T} with falsum.

From now on, we work with a propositional theory \mathbb{T} that admits falsum. To make the dominance \mathbb{I} closed under falsum, we may further assume the following non-triviality axiom:

Axiom 4 (NT). For \mathbb{I} , $0 \neq 1$.

Remark 5.7. The above axiom semantically corresponds to working with certain subtopoi of the classifying topos $\text{Set}[\mathbb{T}]$. For instance, the minimal topology on the underlying site $\mathbb{T}\text{-Mod}_{\text{f.p.}}$ we might choose for (NT) to hold is to assert that the trivial \mathbb{T} -model is covered by the empty sieve. Since the trivial \mathbb{T} -model is a strict terminal object, this topology will be subcanonical, thus (SQCF) still holds.

Before we return to our previous question, let us first observe some elementary consequence of (NT). For instance, the lifting monad $(-)_\perp$ defined in Section 4 is now also pointed, where for any X we can define

$$\perp := (0, ?) : X_\perp,$$

where $? : t0 = \emptyset \rightarrow X$ is the unique map from \emptyset .

As another consequence, now $\emptyset = t0$ is affine, in fact an open proposition. For any \mathbb{I} -algebra A , we say A is *trivial* if it is equivalent to the trivial \mathbb{I} -algebra, which we denote as 0 . Equivalently, this is saying that $0 = 1$ in A . In the case of (NT), we have the following weak form of nullstellensatz:

Lemma 5.8 (NT). *For any affine $X = \text{Spec } A$, $X = \emptyset$ iff A is trivial.*

Proof. By assumption, $A = \mathbb{I}^{\text{Spec } A} = \mathbb{I}^\emptyset = 0$. \square

Let us look at a simple example of the application of the above weak nullstellensatz. Notice that the constant 0 also induces a predicate on \mathbb{I} ,

$$f : \mathbb{I} \rightarrow \text{Prop},$$

which takes any $i : \mathbb{I}$ to $i = 0$.

Lemma 5.9 (NT, SQCI). *For any $i : \mathbb{I}$, $ti \leftrightarrow \neg fi$.*

Proof. If fi then $\neg ti$ by (NT). On the other hand, by the conservativity result in Lemma 3.5 and (NT), if $i \neq 1$ then $i = 0$ since $0 \neq 1$. \square

Remark 5.10. At this point, we do not have the converse $fi \leftrightarrow \neg ti$, since as shown in the above proof it requires f also being an embedding. But the proof of Lemma 3.5 heavily relies on the fact that \mathbb{T} is propositional. For the dual case to hold for f , it will require \mathbb{T} to also have join \vee , and furthermore the quotient $A/a = 0$ is computed by the coslice a/A for any \mathbb{T} -model A . This will be true for distributive lattices and Boolean algebras.

For the theory of Heyting algebras this is *not* true. For instance, under (SQCI), the affine space $\mathbb{I}/i = 0$ is equivalent to $\mathbb{I}/\neg i$. It then follows by Lemma 3.5 that $fi = fj$ iff $\neg i = \neg j$, which not necessarily implies $i = j$.

Now let us get back to the discussion of the intrinsic order. Under the current set up, the interval \mathbb{I} has a minimal element 0 . Hence, it now makes sense to ask when the intrinsic order to be classified by maps from \mathbb{I} , which is usually called the *Phoa's principle* in the literature:

Definition 5.11. We say a type X satisfies the *Phoa's principle*, or simply is *Phoa*, if its intrinsic order is classified by $\langle \text{ev}_0, \text{ev}_1 \rangle : X^\mathbb{I} \rightarrow X \times X$.

The Phoa's principle has many consequences. For instance, if X is Phoa, since the intrinsic order on a type is in particular a relation, the evaluation pair $X^\mathbb{I} \rightarrow X \times X$ will be an *embedding*. Intuitively, it means that any map from the interval $f : \mathbb{I} \rightarrow X$ is completely determined by its boundary. Furthermore, since \leq is also *transitive*, there will then be a well-defined *composition* on functions from \mathbb{I} to X . These are indeed certain internal *orthogonality* classes that can be formulated in our type system. We will consider them in more detail in Section 8.

Another consequence of the Phoa's principle is that, it also implies an extensionality principle on intrinsic orders on function spaces:

Lemma 5.12 (SQCF). *For any X, Y , if X is Phoa, then the intrinsic order on X^Y is induced point-wise by the intrinsic order on X , and is also Phoa.*

Proof. Since $\mathbb{I} \rightarrow X^Y$ is equivalent to $Y \rightarrow X^{\mathbb{I}}$, it suffices to show that the intrinsic order on X^Y is point-wise. For any $f, g : X^Y$, suppose $f \leq g$. For any $y : Y$, since the evaluation map $\text{ev}_y : X^Y \rightarrow X$ is monotone by Lemma 5.2, we have $fy \leq gy$. On the other hand, suppose for any $y : Y$ we have $fy \leq gy$. This way, by the Phoa's principle on X , we get a map $Y \rightarrow X^{\mathbb{I}}$, whose transpose

$$[f, g] : \mathbb{I} \rightarrow X^Y,$$

satisfies $[f, g](0) = f$ and $[f, g](1) = g$. Then consider any $U : X^Y \rightarrow \mathbb{I}$. Notice that by Corollary 5.4 the composite $U[f, g]$ is monotone, thus

$$Uf = U[f, g](0) \leq U[f, g](1) = Ug.$$

Hence, indeed we have $f \leq g$. \square

As a more or less trivial example, let us first consider discrete sets:

Lemma 5.13 (NT, SQCF). *If M is a set with decidable equality, then the intrinsic order on M is discrete, in the sense that for $n, m : M$,*

$$n \leq m \rightarrow n = m.$$

In particular, M is Phoa, in the sense that $M \rightarrow M^{\mathbb{I}}$ is an equivalence.

Proof. Suppose $n \leq m$. If $n \neq m$, we can construct a map $f : M \rightarrow \mathbb{I}$ with $f(n) = 1$ and $f(m) = 0$, contradictory to $n \leq m$. Hence, $n = m$.

Now suppose $f : \mathbb{I} \rightarrow M$. For any $i : \mathbb{I}$, by monotonicity in Lemma 5.2, $f(i) \leq f(1)$, which implies $f(i) = f(1)$. Hence, f is the constant function on 1, which implies $M \rightarrow M^{\mathbb{I}}$ is an equivalence. \square

Corollary 5.14 (NT, SQCF). *For any type X whose intrinsic order is connected, the map $M \rightarrow M^X$ is an equivalence for a discrete set M .*

For instance, the above result applies to the affine spaces \mathbb{I}^n and Δ^n , since by Corollary 5.4 they have both a top and bottom element in the intrinsic order, thus in particular are connected.

For types that are not necessarily discrete, one might expect that quasi-coherence will imply all affine spaces would be Phoa. As a minimal requirement, one crucial axiom of synthetic domain theory is that the interval \mathbb{I} itself should be Phoa; cf. [7]. However, under the current assumption, this is not necessarily the case. Given an affine space $X = \text{Spec } A$, it is easy to compute $X^{\mathbb{I}}$ by duality,

$$X^{\mathbb{I}} = \mathbb{I}\text{-Alg}(A, \mathbb{I}[x]).$$

By the characterisation previously, this coincides with the intrinsic order on X iff the following pair of evaluation on polynomials

$$\langle \text{ev}_0, \text{ev}_1 \rangle : \mathbb{I}[x] \rightarrow \mathbb{I} \times \mathbb{I}$$

classifies the order on \mathbb{I} . What's interesting is that this is a purely *algebraic* statement. Hence, we should look for a Horn theory where this property holds.

A canonical family of examples consists of the theory of *distributive lattices*, or more generally the theory of D -algebras for any distributive lattice D . The observation that this special property of distributive lattices can be used to show Phoa's principle is already made in [6]. Hence, from now on, we will specifically assume we work in a theory of D -algebras for some distributive lattice D .

6. DISTRIBUTIVE LATTICE AND PHOA'S PRINCIPLE

Working with distributive lattices not only gives us the Phoa's principle. As already mentioned in Remark 5.10, the theory of distributive lattices also satisfies the dual axiom of being propositional. Hence, previous developments would also hold when we exchange 1 for 0 and \wedge for \vee .

For instance, the same proof as in Lemma 3.5 also implies the following:

Corollary 6.1 (SQCI). $f : \mathbb{I} \rightarrow \text{Prop}$ is an embedding.

We will now call propositions in the image of f as *closed propositions*, and a subtype classified by f a *closed subtype*. Completely similar to Lemma 3.8, closed subtypes of affine spaces are again of the following form:

Corollary 6.2 (SQCI). *If $X = \text{Spec } A$ is affine, then closed subtypes of X are all of the form*

$$C_a := \{x : X \mid f(xa)\} = \text{Spec } a/A,$$

for some $a : A$. If X is stably affine, then so is C_a .

Corollary 6.3 (SQCI). f forms a dominance.

There is an accompanying *colifting* monad $(-)_\top$, defined by

$$X_\top := \sum_{i:\mathbb{I}} X^{fi}.$$

And similarly, under (SQCF), we can explicitly compute the colifting of the simplices Δ^n ,

$$\Delta_\top^n = \Delta^{n+1},$$

where now the unit $\eta : \Delta^n \rightarrow \Delta_\top^n = \Delta^{n+1}$ takes a sequence $i_1 \geq \dots \geq i_n$ to $i_1 \geq \dots \geq i_n \geq 0$ in Δ^{n+1} . To distinguish the two inclusions $\Delta^n \rightarrow \Delta_\perp^n = \Delta^{n+1}$ and $\Delta^n \rightarrow \Delta_\top^n = \Delta^{n+1}$, we will denote the former as η_\perp and the latter as η_\top .

Furthermore, the dual statement of Lemma 5.9 now also holds:

Corollary 6.4 (NT, SQCI). \mathbb{I} is a field in the sense that

$$\forall i: \mathbb{I}. \neg ti \leftrightarrow fi, \quad \forall i: \mathbb{I}. \neg fi \leftrightarrow ti.$$

In particular, the embedding $2 \hookrightarrow \mathbb{I}$ induced by $0, 1$ is $\neg\neg$ -dense,

$$\forall i: \mathbb{I}. \neg\neg(ti \vee fi).$$

This allows us to observe that the open and closed propositions are exactly complementary to each other:

Corollary 6.5 (NT, SQCI). For any proposition p , p is open iff $\neg p$ is closed and vice versa. Furthermore, open and closed propositions are $\neg\neg$ -stable.

Remark 6.6. Notice that the above result only uses the fact that \mathbb{T} also satisfies the dual axiom of propositional, thus in particular applies to the case of Boolean algebras as well. However, the content will in some sense be trivialised in that case. By the presence of a classical negation, a proposition is open iff it is closed.

More importantly, as indicated at the end of the previous section, working with distributive lattices allows us to show that all affine spaces will satisfy the Phoa's principle:

Theorem 6.7. Any affine space X is Phoa.

Proof. As mentioned, for any affine space X we have

$$X^{\mathbb{I}} = \mathbb{I}\text{-Alg}(A, \mathbb{I}[i]).$$

Hence it suffices to show that the pair of evaluation maps

$$\langle \text{ev}_0, \text{ev}_1 \rangle : \mathbb{I}[i] \rightarrow \mathbb{I} \times \mathbb{I}$$

classifies the order on \mathbb{I} . This follows from a normal form result of polynomials for distributive lattices, i.e. any polynomial p is of the form

$$p = p(0) \vee x \wedge p(1).$$

See e.g. [9, Thm. 10.11]. This way, a map $\mathbb{I} \rightarrow X$ is exactly described as two points $x, y : X$ such that $\forall a: A. xa \leq ya$, which coincides with $x \leq y$. \square

Remark 6.8. Crucially, we emphasize that the above proof does *not* in fact rely on the affineness of the interval \mathbb{I} itself. Rather, it is purely the consequence of the fact that for any distributive lattice A , the free algebra $A[i]$ is equivalent to the order on A . This is a perfect example of how an algebraic property of a theory has a non-trivial effect on the internal logic of its classifying topos.

As a consequence, we can also show that the algebraic objects also satisfy the Phoa's principle:

Theorem 6.9 (SQCF). *For any quasi-coherent \mathbb{I} -algebra A , the intrinsic order on A coincides with its usual order, and it is Phoa.*

Proof. Since A is quasi-coherent, we have $A = \mathbb{I}^{\text{Spec } A}$. By Theorem 6.7 the affine space \mathbb{I} is Phoa, hence Lemma 5.12 implies that the intrinsic order on A coincides with the point-wise order on $\mathbb{I}^{\text{Spec } A}$, which is indeed the order on A . Lemma 5.12 then also implies A satisfies the Phoa's principle. \square

At the end of this section, we describe another interesting perspective arising from the proof of Theorem 6.7. We have seen that the dualising object \mathbb{I} has a double role: It is both an algebra and a spectrum. The proof of Phoa's principle gives us many more such examples. For instance, $\mathbb{I}[i]$ classifies the order on \mathbb{I} , which by definition is also a spectrum. Now since for us the simplex Δ^2 is the dual order, we will then identify it with $\mathbb{I}[i]$ along the following map,

$$\langle \text{ev}_1, \text{ev}_0 \rangle : \mathbb{I}[i] \rightarrow \Delta^2.$$

In fact, *all* the simplices in this case have an algebraic description:

Proposition 6.10. *For any $n : \mathbb{N}$ and \mathbb{I} -algebra A , for any $a_0 \geq \dots \geq a_n$ in A , define a polynomial in $A[i_1, \dots, i_n]$ as follows,*

$$a_0 \wedge i_1 \vee a_1 \wedge \dots \wedge i_n \vee a_n.$$

This is well-defined in the quotient $A[n]/i_1 \leq \dots \leq i_n$, in the sense that no matter how you arrange the parenthesis there is a unique value in the quotient $A[n]/i_1 \leq \dots \leq i_n$. In fact this gives us an equivalence

$$\Delta[A]^{n+1} = A[n]/i_1 \leq \dots \leq i_n,$$

where we write $\Delta[A]^{n+1}$ as $\{a_0, \dots, a_n : A \mid a_0 \geq \dots \geq a_n\}$.

Proof. This again follows from a normal form result on algebras of the form $A[n]/i_1 \leq \dots \leq i_n$, which is a consequence of the general normal form for multivariable polynomials on distributive lattices; see [9, Thm. 10.21]. \square

Now we can identify the simplices Δ^n as a finitely presented \mathbb{I} -algebra as shown above. Under such equivalences, the inclusion η_\perp simply corresponds to the canonical inclusion as shown below,

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\eta_\perp} & \Delta^{n+2} \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{I}[n]/i_1 \leq \dots \leq i_n & \xrightarrow{\quad} & \mathbb{I}[n+1]/i_0 \leq \dots \leq i_n \end{array}$$

7. INFINITARY DOMAIN THEORY

Untill this point, we have seen that all the finitary axioms for synthetic domain theory is a consequence of (SQCF) for a theory of distributive algebra, with potentially the assumption of (NT).

There is one final crucial infinitary axiom of synthetic domain theory. From the observation in [7, 8], we can define internally in type theory the initial algebra and final coalgebra for the lifting functor $(-)_\perp$, which we denote as ω and $\bar{\omega}$, respectively. There is a canonical inclusion $\omega \hookrightarrow \bar{\omega}$, and the final axiom for synthetic domain theory states that the interval \mathbb{I} is *complete*, in the sense that the induced map $\mathbb{I}^{\bar{\omega}} \rightarrow \mathbb{I}^\omega$ is an *equivalence*. The goal of this section is to explain that, this completeness axiom can also be realised as a consequence of quasi-coherence.

Since $(-)_\perp$ by construction preserves all connected limits, the final coalgebra $\bar{\omega}$ can be easily characterised as a limit. As shown in [7], it can be identified as the object of infinite descending sequences in \mathbb{I} ,

$$\bar{\omega} := \{ i : \mathbf{N} \rightarrow \mathbb{I} \mid \forall n : \mathbf{N}. i_n \geq i_{n+1} \},$$

which can be viewed as the following sequential limit,

$$\dots \longrightarrow \Delta^2 \xrightarrow{\perp_\perp} \mathbb{I} \xrightarrow{!} 1$$

where the transition map $\Delta^{n+1} \rightarrow \Delta^n$ takes the sequence $i_0 \geq \dots \geq i_n$ to the final segment $i_1 \geq \dots \geq i_n$.

According to the above description, $\bar{\omega}$ is in fact a *spectrum*,

$$\bar{\omega} = \text{Spec}(\mathbb{I}[\mathbf{N}]/\forall n. i_n \geq i_{n+1}).$$

However, the corresponding algebra is no longer finitely presented, but countably presented. By a countably presented, or c.p. in short, \mathbb{I} -algebra, we mean an \mathbb{I} -algebra of the form $\mathbb{I}[I]/s = t$ for some *decidable* subsets I, J of \mathbf{N} with $s, t : J \rightarrow \mathbb{I}[I]$. In particular, all finitely presented \mathbb{I} -algebra will also be countably presented. Motivated by the above characterisation of $\bar{\omega}$, we naturally consider the following stronger quasi-coherence principle:

Axiom 5 (SQCC). All c.p. \mathbb{I} -algebras are (stably) quasi-coherent.

Remark 7.1. The quasi-coherence principle for countably presented algebras is investigated in [4]. Just like the finitary version (SQCF), which is true in the classifying topos $\mathbf{Set}[\mathbf{T}] = [\mathbf{T}\text{-}\mathbf{Mod}_{\text{f.p.}}, \mathbf{Set}]$, the countable version (SQCC) will be valid in a larger presheaf topos $[\mathbf{T}\text{-}\mathbf{Mod}_{\text{c.p.}}, \mathbf{Set}]$, or any subtopos induced by a subcanonical topology. For instance, the semantics of [4] is based on the topos of light condensed sets introduced by Clausen and Scholze, which is a subtopos of $[\mathbf{B}\text{-}\mathbf{Mod}_{\text{c.p.}}, \mathbf{Set}]$.

There are many immediate consequences of (SQCC). For instance, the description of the intrinsic order for affine spaces in Lemma 8.3 now applies to $\bar{\omega}$ as well, which implies it has both a top and bottom element. Then Corollary 5.14 now also applies to $\bar{\omega}$, and $M \rightarrow M^{\bar{\omega}}$ is an equivalence for any discrete set M .

There are more logical consequences of (SQCC). Let us observe that it implies the following form of Markov principle:

Lemma 7.2 (NT, SQCC). *For any $i : \bar{\omega}$, we have*

$$\neg \forall n : \mathbb{N}. ti_n \rightarrow \exists n : \mathbb{N}. fi_n.$$

Proof. Let $i : \bar{\omega}$. Notice that similar to Lemma 3.4, the proposition $\forall n : \mathbb{N}. ti_n$ by construction is the following affine space,

$$\text{Spec } \mathbb{I}/i = \mathbb{I}\text{-Alg}(\mathbb{I}/i, \mathbb{I}) = \forall n : \mathbb{N}. ti_n,$$

where we have abbreviated the c.p. \mathbb{I} -algebra $\mathbb{I}/\forall k. i_k = 1$ as \mathbb{I}/i . Now if we have $\neg \forall n : \mathbb{N}. ti_n$, then $\text{Spec } \mathbb{I}/i = \emptyset$ which implies \mathbb{I}/i is trivial by Lemma 5.8. But this algebra is trivial iff there merely exists $n : \mathbb{N}$ that fi_n holds, thus $\exists n : \mathbb{N}. fi_n$. \square

Now let us consider the initial algebra ω for the lifting functor. It is shown in [8] that we can identify $\omega \rightarrow \bar{\omega}$ as the following subset,

$$\omega := \{ i : \bar{\omega} \mid \forall \phi : \text{Prop}. (\forall n : \mathbb{N}. (ti_n \rightarrow \phi) \rightarrow \phi) \rightarrow \phi \}.$$

For another proof, see e.g. [14]. In the presence of the Markov principle above, this description can be drastically simplified:

Lemma 7.3 (NT, SQCC). *ω is equivalent to the following subset of $\bar{\omega}$,*

$$\omega = \{ i : \bar{\omega} \mid \exists n : \mathbb{N}. fi_n \}.$$

Proof. Let $i : \bar{\omega}$. It suffices to show that

$$(\forall \phi : \text{Prop}. (\forall n : \mathbb{N}. (ti_n \rightarrow \phi) \rightarrow \phi) \rightarrow \phi) \rightarrow \exists n : \mathbb{N}. fi_n,$$

We can take ϕ to be \emptyset . By assumption we have $\neg \forall n : \mathbb{N}. \neg ti_n$, which by Corollary 6.5 is equivalent to $\neg \forall n : \mathbb{N}. ti_n$. Then Lemma 7.2 implies this is $\exists n : \mathbb{N}. fi_n$. \square

By [14, Cor. 1.10], the above result exactly means that ω is indeed the internal colimit of the following sequence,

$$\emptyset \xrightarrow{?} 1 \xrightarrow{?_{\perp} = \eta_{\top}} \mathbb{I} \xrightarrow{\eta_{\top}} \Delta^2 \longrightarrow \dots$$

Using this, we can now show the most important infinitary axiom for synthetic domain theory:

Theorem 7.4 (NT, SQCC). *The canonical map $\mathbb{I}^{\bar{\omega}} \rightarrow \mathbb{I}^{\omega}$ is an equivalence.*

Proof. Since $\bar{\omega}$ is now affine, we have

$$\mathbb{I}^{\bar{\omega}} = \mathbb{I}[\mathbf{N}] / \forall n. i_n \geq i_{n+1}.$$

On the other hand, since ω is internally the colimit of Δ^n , we have

$$\mathbb{I}^\omega = \varprojlim_{n:\mathbf{N}} \mathbb{I}^{\Delta^n} = \varprojlim_{n:\mathbf{N}} \mathbb{I}[n] / i_1 \geq \dots \geq i_n.$$

Note that the transition maps induced by $\eta_\top : \Delta^n \rightarrow \Delta^{n+1}$ under quasi-coherence gives us the following maps on algebras:

$$\begin{array}{ccc} \mathbb{I}^{\Delta^{n+1}} & \xrightarrow{\mathbb{I}\eta_\top} & \mathbb{I}^{\Delta^n} \\ \simeq \uparrow & & \uparrow \simeq \\ \mathbb{I}[n+1] / i_1 \geq \dots \geq i_{n+1} & \xrightarrow{i_{n+1} \mapsto 0} & \mathbb{I}[n] / i_1 \geq \dots \geq i_n \end{array}$$

Now taking the limit of the above sequence of algebras gives us

$$\varprojlim_{n:\mathbf{N}} \mathbb{I}[n] / i_1 \geq \dots \geq i_n = \mathbb{I}[\mathbf{N}] / \forall n. i_n \geq i_{n+1} = \mathbb{I}^{\bar{\omega}}.$$

This shows that $\mathbb{I}^\omega = \mathbb{I}^{\bar{\omega}}$. \square

Finally, at the end of this section, we observe that the algebraic description of the simplices given at the end of Section 6 can be extended to the above infinitary case:

Corollary 7.5. $\mathbb{I}^{\bar{\omega}} = \mathbb{I}[\mathbf{N}] / \forall n. i_n \geq i_{n+1}$ is equivalent to Δ_∞ ,

$$\Delta_\infty := \{i : \mathbf{N} \rightarrow \mathbb{I} \mid \forall n:\mathbf{N}. i_n \leq i_{n+1}\}.$$

Proof. Following the discussion at the end of Section 6, we have the following equivalences,

$$\begin{array}{ccc} \Delta_{n+1} & \xrightarrow{\eta_\top} & \Delta_{n+2} \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbb{I}[n] / i_1 \geq \dots \geq i_n & \xrightarrow{\quad} & \mathbb{I}[n+1] / i_1 \geq \dots \geq i_{n+1} \end{array}$$

Then taking the sequential colimit on both sides, we get an equivalence

$$\Delta_\infty = \mathbb{I}[\mathbf{N}] / \forall n. i_n \geq i_{n+1}. \quad \square$$

8. INTRINSIC POSETS

Definition 8.1. We say X is an *intrinsic poset*, if the intrinsic order on X is anti-symmetric, and it satisfies the Phoa's principle.

Lemma 8.2. Any *intrinsic poset* is a set.

Proof. By definition, for any intrinsic poset X and $x, y : X$,

$$x = y \leftrightarrow x \preceq y \wedge y \preceq x,$$

thus is a proposition. \square

Lemma 8.3. *All affine spaces are intrinsic posets.*

Proof. As shown in Theorem 6.7, any affine space $X = \text{Spec } A$ is Phoa, and the intrinsic order on X is induced by the point-wise order on $\mathbb{I}\text{-Alg}(A, \mathbb{I})$, which is anti-symmetric. \square

Lemma 8.4 (SQCF). *All quasi-coherent \mathbb{I} -algebras are intrinsic posets.*

Proof. Again, by Theorem 6.9, a quasi-coherent algebra A is Phoa, and the intrinsic order coincides with the order on A , hence is anti-symmetric. \square

Intrinsic posets satisfy the closure properties of an orthogonality class:

Lemma 8.5 (SQCF). *The intrinsic order on limits of intrinsic posets is point-wise, and are again intrinsic posets. They in fact form an exponential ideal.*

Proof. Suppose we have a limit $\varprojlim_{r:I} X_r$ of intrinsic posets. Given $(x_r)_{r:I}$ and $(y_r)_{r:I}$ in the limit, by monotonicity in Lemma 5.2, if $(x_r)_{r:I} \preceq (y_r)_{r:I}$, then the projection maps imply that $x_r \preceq y_r$ for all $r : I$. On the other hand, if $x_r \preceq y_r$ for all $r : I$, then since each X_r is Phoa, we get a unique map $f_r : \mathbb{I} \rightarrow X_r$. We then need to show that this gives us $(f_r)_{r:I} : \mathbb{I} \rightarrow X_r$. This indeed follows from the Phoa's principle on each X_r . Hence, the order on the limit is again point-wise, and the limit is Phoa. Being an exponential ideal follows from 5.12. \square

Perhaps more surprisingly, intrinsic posets will also be closed under various colimit constructions. To show this, we observe the following result:

Lemma 8.6 (NT, SQCF). *Let X be a type with connected intrinsic order. $(-)^X$ preserves dependent sums over discrete sets.*

Proof. Suppose we have a discrete set M and a family P of types over M . Consider the canonical map

$$\sum_{m:M} P_m^X \rightarrow \left(\sum_{m:M} P_m \right)^X.$$

Since X is connected, there merely is $x : X$. For any $f : X \rightarrow \sum_{m:M} P_m$, by Corollary 5.14, $\pi_M f(y) = \pi_M f(x)$ for all $y : X$, thus f factors through P_m^X , which then implies the fibres of the above map is contractible since M is a discrete set. \square

Corollary 8.7 (NT, SQCF). *Intrinsic posets are closed under dependent sums over discrete sets.*

Definition 8.8. A type X is *projective*, if for any family P of sets over X ,

$$\prod_{x:X} \|P_x\| \rightarrow \left\| \prod_{x:X} P_x \right\|.$$

Lemma 8.9 (NT, SQCF). *Let X be a projective type with connected intrinsic order. $(-)^X$ preserves sequential colimits.*

Proof. For a family of sets Y_n indexed by \mathbf{N} with maps $f_n : Y_n \rightarrow Y_{n+1}$, recall from [12] the colimit $\varinjlim_{n:\mathbf{N}} Y_n$ is again a set, which is constructed as in usual set theory. Consider the canonical map

$$\varinjlim_{n:\mathbf{N}} Y_n^X \rightarrow \left(\varinjlim_{n:\mathbf{N}} Y_n \right)^X.$$

To show this is injective, it suffices to assume we have two elements $[(n, y)]$ and $[(m, z)]$. If they agree in the target, which means

$$\forall x:X. \exists l:\mathbf{N}. f_{ln}(yx) = f_{lm}(zx).$$

By projectivity of X , we merely have a section $l : X \rightarrow \mathbf{N}$. By Corollary 5.14, this section is indeed constant at $l = l(x)$ for the mere element $x : X$. It then follows that $[(n, y)] = [(m, z)]$.

On the other hand, by projectivity the above map is also surjective, since if we have $f : X \rightarrow \varinjlim_{n:\mathbf{N}} Y_n$, then we have

$$\forall x:X. \exists n:\mathbf{N}. \exists y:Y_n. fx = [(i, y)].$$

By projectivity again, we merely have a section $X \rightarrow \mathbf{N}$, which is again constant by Corollary 5.14. Thus we merely get a preimage. \square

Corollary 8.10 (NT, SQCF). *Assuming \mathbb{I} is projective, intrinsic posets are closed under sequential colimits.*

Remark 8.11. This is the first place in this work where we require a form of projectivity of \mathbb{I} . \mathbb{I} will indeed be projective, in fact ∞ -projective in the sense that Definition 8.8 not only applies to sets but arbitrary types, in all the models we care about.

Example 8.12. Notice that as we have shown in Section 7 that, under (NT) and (SQCC), ω is the colimit of simplices Δ_n . By Corollary 8.10, assuming \mathbb{I} is projective, ω will also be an intrinsic poset.

9. LIFTING CONTINUED

We would like to furthermore show that intrinsic posets are closed under lifting. To this end, we introduce a stronger locality axiom:

Axiom 6 (L). \mathbb{I} is local, i.e. $0 \neq 1$ and $t(i \vee j) \leftrightarrow ti \vee tj$ for all $i, j : \mathbb{I}$.

We observe some elementary consequences of (L):

Lemma 9.1 (L, SQCF). \mathbb{I} is not Boolean.

Proof. If \mathbb{I} is Boolean, then $\mathbb{I} = 2$ by Corollary 6.4, since in that case we have for any $i : \mathbb{I}$,

$$ti \vee fi = ti \vee t\neg i = t(i \vee \neg i) = 1.$$

However, this contradicts Lemma 5.13, because if $\mathbb{I} = 2$ then $2^{\mathbb{I}} = 2^2$, which cannot be equivalent to 2. \square

Furthermore, (L) also allows us to define more affine spaces:

Example 9.2 (L). Consider the right outer horn Λ_2^2 as a pushout,

$$\begin{array}{ccc} 1 & \xrightarrow{1} & \mathbb{I} \\ 1 \downarrow & \lrcorner & \downarrow \\ \mathbb{I} & \longrightarrow & \Lambda_2^2 \end{array}$$

By viewing Λ_2^2 as a subspace of \mathbb{I}^2 , we may identify it as follows,

$$\Lambda_2^2 = \{i, j : \mathbb{I} \mid ti \vee tj\}.$$

Since $ti \vee tj = t(i \vee j)$, it follows that

$$\Lambda_2^2 = \text{Spec } \mathbb{I}[i, j]/i \vee j.$$

The most important consequence of (L) is that it allows us to show intrinsic posets are closed under dependent sums over certain affine spaces:

Definition 9.3 (SQCF). We say an affine space X is *finitary*, if it is of the form $\text{Spec } \mathbb{I}[n]/s = t$ for some finitely presented \mathbb{I} -algebra. X is thus a subspace of \mathbb{I}^n , and we say X is *topped*, if $1 : \mathbb{I}^n$ belongs to X .

Example 9.4. All the finitary examples of affine spaces considered so far, including the cubes \mathbb{I}^n , the simplices Δ^n and Δ_n , and the outer horn Λ_2^2 , are all topped.

One interesting fact is that, since \mathbb{I} is a dominance, it will always be orthogonal to the following family of maps:

Lemma 9.5 (SQCF). Let X be a finitary topped affine space, $Y : X \rightarrow \text{Type}$ a type family. Then \mathbb{I} is weakly orthogonal to the embedding $Y_1 \hookrightarrow \sum_{i:X} Y_i$,

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & \mathbb{I} \\ \downarrow & \nearrow \tilde{f} & \\ \sum_{i:X} Y_i & & \end{array}$$

Proof. Using the dominance structure, we define \tilde{f} as follows,

$$\tilde{f}(i, y) := \sum_{* : ti} f(y).$$

This is well-defined, and evidently extends f . \square

For any finitary topped X , by viewing it as a subspace of \mathbb{I}^n , we may define the generalised lifting functor over X as follows:

$$L_X Y := \sum_{i : X} Y^{ti}.$$

Of course, $L_{\mathbb{I}}$ is the usual lifting functor $(-)_\perp$. The most crucial observation of this section is that intrinsic posets are closed under generalised liftings:

Theorem 9.6 (L, SQCF). *Let X be a finitary topped affine space and Y an intrinsic poset. Then the intrinsic order on $L_X Y$ is given by*

$$(i, x) \preceq (j, y) \leftrightarrow ti \rightarrow (tj \wedge x \preceq_Y y).$$

In particular, $L_X Y$ is again an intrinsic order.

Proof. Recall from Lemma 5.3 the intrinsic order on X is induced from the point-wise order on \mathbb{I}^n . For the left to right direction, if $(i, x) \preceq (j, y)$, then by monotonicity $i \leq j$. If ti holds, then tj holds, and $x \preceq_Y y$ since by Lemma 9.5 any map $Y \rightarrow \mathbb{I}$ extends to one on $L_X Y$.

On the other hand, suppose we have $(i, x) \preceq (j, y)$ in $L_X Y$. We construct a map $f : \mathbb{I} \rightarrow L_X Y$. On the base, it must take k to $i \vee k \wedge j$. Hence, assuming $t(i \vee k \wedge j)$, we need an element in Y . By (L), equivalently $ti \vee tk \wedge tj$. Thus, we can define this by case distinction:

- If ti holds, then since $x \preceq_Y y$, by the fact that Y is an intrinsic poset, we get a unique map $g : \mathbb{I} \rightarrow X$, and we define $f(k) := g(k)$.
- If $tk \wedge tj$ holds, we simply put $f(k) := y$.

This is evidently well-defined, thus we get a map $f : \mathbb{I} \rightarrow L_X Y$. \square

Of course, we can also consider the dual notion of *bottomed* finitary affine spaces, which are affine subspaces of \mathbb{I}^n containing 0. However, to develop the same arguments, we would have to assume furthermore the *colocality* axiom as follows,

Axiom 7 (cL). \mathbb{I} is colocal, i.e. $0 \neq 1$ and $f(i \wedge j) \leftrightarrow fi \vee fj$.

And in this case, for any bottomed finitary affine space X we can construct the generalised colifting of a type Y as

$$T_X Y := \sum_{i : X} Y^{fi}.$$

We then similarly have the following result:

Theorem 9.7 (cL, SQCF). *Let X be a finitary bottomed affine space and Y an intrinsic poset. Then the intrinsic order on $T_X Y$ is given by*

$$(i, x) \leq (j, y) \leftrightarrow f j \rightarrow (f i \wedge x \leq_Y y).$$

In particular, $T_X Y$ is again an intrinsic order.

Of course, it is perfectly fine to assume both (L) and (cL), and we may refer to the combination of them as (bL). Working in the context of (bL), we again recover the full duality between the meet and join structure as the presheaf geometry.

10. SYNTHETIC CATEGORY THEORY

Following [6], we develop some amount of synthetic category theory in our framework. In this section, we will more generally work in the following axiomatic framework:

Axiom 8 (PF). The affine spaces \mathbb{I}^n and Δ^n are ∞ -projective for all $n : \mathbb{N}$.

To formulate the Segal condition, we observe that the horn Λ_1^2 can be constructed as the following pushout,

$$\begin{array}{ccc} 1 & \xrightarrow{1} & \mathbb{I} \\ 0 \downarrow & \lrcorner & \downarrow \\ \mathbb{I} & \longrightarrow & \Lambda_1^2 \end{array}$$

It is essentially observed in [10] that the inclusion $\Lambda_1^2 \hookrightarrow \Delta_2$ can be defined in type theory as follows,

$$\Lambda_1^2 = \{ (i, j) : \Delta_2 \mid f i \vee t j \}.$$

Definition 10.1. We say X is *Segal* if it is orthogonal to $\Lambda_1^2 \hookrightarrow \Delta^2$.

Following [3], we can also define the type of free equivalence \mathbb{E} as the following colimit,

$$\begin{array}{ccccc} & \mathbb{I} & & \mathbb{I} & \\ & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\ 1 & & \Delta_2 & & \Delta_2 & & 1 \\ & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\ & \mathbb{I} & & \mathbb{I} & \end{array}$$

(Note: The diagram above is a simplified representation of the colimit defining E. The actual diagram shows three copies of the horn Λ_1^2 being pushed out to Δ_2, with maps i ↦ (i, i), i ↦ (0, i), and i ↦ (i, 1).)

Definition 10.2. We say X is *Rezk*, if it is orthogonal to $\mathbb{E} \rightarrow 1$.

Finally, we can also construct \mathbb{I}_{\Rightarrow} as the following pushout,

$$\begin{array}{ccc} 2 & \longrightarrow & \mathbb{I} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{I} & \longrightarrow & \mathbb{I}_{\Rightarrow} \end{array}$$

Definition 10.3. We say X is \mathbb{I} -separated, if $X^{\mathbb{I}} \rightarrow X^{\mathbb{I} \Rightarrow}$ is an equivalence.

Equivalently, X is \mathbb{I} -separated iff $X^{\mathbb{I}} \rightarrow X \times X$ is an embedding. The notion of synthetic categories and synthetic posets are formulated internally as these orthogonality classes:

Definition 10.4. For any type X , we say it is a *synthetic category*, if it is Segal and Rezk. We say it is a *synthetic poset*, if it is also \mathbb{I} -separated.

Lemma 10.5 (SQCF). *Any intrinsic poset is a synthetic poset.*

Proof. This holds simply because for any intrinsic poset X , by Phoa's principle $X^{\mathbb{I}}$ classifies the intrinsic order, which is a reflexive, transitive, and anti-symmetric relation. \square

Theorem 10.6 (NT, SQCF, PF). *Synthetic categories and synthetic posets form exponential ideals, which are closed under dependent sums over discrete sets and sequential colimit.*

Proof. This follows from the fact that they are described by orthogonality classes, and the codomains of the generating maps are in particular strongly connected and ∞ -projective. \square

Remark 10.7. It is shown in [] that, by additionally assuming (L), synthetic categories and synthetic posets are closed under lifting.

Furthermore, there is a strengthening of the Segal condition, which is obtained by displaying the inner horn inclusion $\Lambda_1^2 \hookrightarrow \Delta_2$ over \mathbb{I} ,

$$\begin{array}{ccc} 1 \star \mathfrak{t}j & \longrightarrow & \Lambda_1^2 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{I}/j & \longrightarrow & \Delta_2 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{j} & \mathbb{I} \end{array}$$

where $1 \star \mathfrak{t}j$ type-theoretically can be realised as the subtype

$$1 \star \mathfrak{t}j = \{i : \mathbb{I} \mid \mathfrak{f}i \vee \mathfrak{t}j\}.$$

Definition 10.8. A type X is *based Segal*, if it is orthogonal to $1 \star \mathfrak{t}j \rightarrow \mathbb{I}/j$ for all $j : \mathbb{I}$.

Proposition 10.9 (L, SQCF). *Let X be an affine subspace of a cube \mathbb{I}^n which is closed under bottom and dependent sums. Then X is based Segal.*

Proof. We follow the proof strategy in []. To extend a map $f : 1 \star \mathbf{t}j \rightarrow X$ with X a closed subspace of \mathbb{I}^n ,

$$\begin{array}{ccc} 1 \star \mathbf{t}j & \xrightarrow{f} & X \\ \downarrow & \nearrow \tilde{f} & \\ \mathbb{I}/j & & \end{array}$$

we define \tilde{f} as follows:

$$\tilde{f}(i) := f(0) \vee \sum_{* : \mathbf{t}i} f(j).$$

By assumption this is well-defined, and the proof that \tilde{f} is a unique extension follows the same as in []. \square

Example 10.10. By the above result, any closed subspace of \mathbb{I}^n will be based Segal. For instance, this include closed propositions fi and i/\mathbb{I} for all i . More generally, the simplices Δ^n will also be based Segal. In fact, if we assume (SQCC), the affine space $\overline{\omega}$ will also be based Segal.

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