A GEOMETRIC APPROACH TO DOMAINS

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Contents

1. Introduction

Let $\mathbb{T}$ be a Horn theory. The type system we work with in this document is based on the quasi-coherence principle available in the classifying topoi of $\mathbb{T}$. This principle originated from the observation made in [1] The [3, 2]. (The envelopping ∞-topoi of) such topoi provide models for the type theory we work with in this paper.

In this paper we will be working in a dependent type theory with a univalent universe Type. Recall the notion of h-levels [15]. For us the most important h-levels are -1 and 0, which are *propositions* and *sets*. In particular, we can define subuniverses Prop, Set.

We will also assume the existence of propositional truncation. This allows us to define existential quantifier of a family $P : X \to \mathsf{Prop}$,

$$\exists x : X.\, P(x) := \| \sum_{x : X} P(x) \|$$

and disjunction of propositions,

$$P \vee Q := \|P + Q\|.$$

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Note that by univalence, Prop is already closed under dependent product. But to emphasis that the result is again a proposition, we may also write dependent products of propositions as $\forall x\!:\!X.\, P(x)$.

The universe Prop is used to define subtypes, i.e. we identify a subtype of X as a family of propositions $A : X \to \mathsf{Prop}$ indexed over X, where the projection $\sum_{x:X} A(x) \rightarrowtail X$ is the inclusion of this subtype. We often write the sum as a collection,

$$\{\, x : X \mid A(x) \,\} := \sum_{x:X} A(x).$$

When X in fact is a set, then any of its subtype is also a set, and we will emphasis by calling them *subsets*.

Sets are used to define *algebras* for a Horn theory. Given any Horn theory $\mathbb{T}$, by a $\mathbb{T}$-*model* we will always mean a *set* A equipped with operations according to the signature of $\mathbb{T}$, such that it satisfies the axioms of $\mathbb{T}$. Since A is a set, being a $\mathbb{T}$-model is again a proposition for a given family of operations.

At the end, we remark that $\mathbf{N}$ is the usual inductive type of natural numbers. For any $n : \mathbf{N}$, we will also use n to denote the finite type with n elements, which is viewed as an initial segment of $\mathbf{N}$. We stress that this is only syntactic sugar, and can be made more precise if need be.

2. Quasi-coherence and affine spaces

In this section we discuss the notion of quasi-coherent algebra and affine spaces. The technical results presented in this section is essentially contained in [6], based on the work in [3].

As a starting point, we assume there is a generic $\mathbb{T}$-model in our type system:

Axiom. There is a set $\mathbb{I}$ equipped with a $\mathbb{T}$-model structure.

Quasi-coherence is a condition on $\mathbb{I}$-*algebras*: An $\mathbb{I}$-algebra is a $\mathbb{T}$-model A equipped with a homomorphism $\mathbb{I} \to A$. The type of $\mathbb{I}$-algebras will be denoted as $\mathbb{I}$-$\mathbf{Alg}$. More generally, for any $\mathbb{T}$-algebras A, B, we will use $\mathbb{I}$-$\mathbf{Alg}(A, B)$ to denote the set of homomorphisms of $\mathbb{I}$-algebras from A to B. For instance, this can be used to define the *spectrum* A,

$$\operatorname{Spec} A := \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}).$$

A lot of sets can be naturally viewed as the spectrum of some $\mathbb{I}$-algebra. For the simplest example, since $\mathbb{I}$ is the initial $\mathbb{I}$-algebra,

$$\operatorname{Spec} \mathbb{I} = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}, \mathbb{I}) = 1.$$

Now if we use $\mathbb{I}[x]$ to denote the free $\mathbb{I}$ algebra on one generator, we can identify $\mathbb{I}$ itself as a spectrum,

$$\mathrm{Spec}\,\mathbb{I}[x] = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}[x], \mathbb{I}) = \mathbb{I}.$$

However, notationally we will distinguish $\mathbb{I}$ as an algebraic gagdet and as a spectrum, which we view as a geometrical object. The latter will be denoted as $\mathbb{I}$, and the equivalence $\mathbb{I} \to \mathbb{I}$ takes $i : \mathbb{I}$ to the evaluation map $\mathrm{ev}_i : \mathbb{I}[x] \to \mathbb{I}$. More generally, we have:

Example 2.1. For any $n : \mathbf{N}$, the n-th cube $\mathbb{I}^n$ is the following spectrum,

$$\mathrm{Spec}\,\mathbb{I}[n] = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}[n], \mathbb{I}) = \mathbb{I}^n.$$

Definition 2.2. A is *quasi-coherent*, if the canonical evaluation

$$A \to \mathbb{I}^{\mathrm{Spec}\,A}$$

is an equivalence of $\mathbb{I}$-algebras, where $\mathbb{I}^{\mathrm{Spec}\,A}$ has the point-wise $\mathbb{I}$-algebra structure. We write $\mathsf{isQC}(A)$ for the proposition of being quasi-coherent.

Remark 2.3. If A is quasi-coherent, then it is in particular *replete*, in the sense that for any map $f : X \to Y$ where $\mathbb{I}^f$ is an equivalence, A^f is also an equivalence.

Example 2.4. $\mathbb{I}$ itself by definition is quasi-coherent. We have seen that $\mathrm{Spec}\,\mathbb{I} = 1$. Under this equivalence, the canonical map

$$\mathbb{I} \to \mathbb{I}^{\mathrm{Spec}\,\mathbb{I}} = \mathbb{I}$$

is exactly the identity on $\mathbb{I}$, hence is an equivalence.

Note by construction, Spec is a contravariant functor on $\mathbb{I}$-algebras. For quasi-coherent algebras, we have a more general duality result:

Proposition 2.5. *If B is quasi-coherent, then for any $\mathbb{I}$-algebras A, we have*

$$\mathbb{I}\text{-}\mathbf{Alg}(B, A) = \mathrm{Spec}\,B^{\mathrm{Spec}\,A}.$$

Proof. By quasi-coherence of B, we have the following equivalences,

$$\mathbb{I}\text{-}\mathbf{Alg}(A, B) = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}^{\mathrm{Spec}\,B}) = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I})^{\mathrm{Spec}\,B} = \mathrm{Spec}\,A^{\mathrm{Spec}\,B}.$$

The second identity holds since by quasi-coherence the $\mathbb{I}$-algebra structure on $\mathbb{I}^{\mathrm{Spec}\,B}$ is point-wise. □

The above duality result tells us that we can infer equivalence of algebras via equivalence of their spectrum:

Corollary 2.6. *For any $f : \mathbb{I}\text{-}\mathbf{Alg}(A, B)$, if A, B are quasi-coherent, then f is an equivalence iff it induces an equivalence* $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$.

To illustrate the usefulness of the above duality result, let us also introduce the following notion:

Definition 2.7. We say an $\mathbb{I}$-algebra A is *stably quasi-coherent* if all finitely generated principle congruences of A are also quasi-coherent, i.e. for any finite $n : \mathbf{N}$ and for any $a, b : n \to A$, $A/a = b$ is again quasi-coherent.

By definition, any finitely generated principle congruences of a stably quasi-coherent algebra will again be stably quasi-coherent. For a stably quasi-coherent $\mathbb{I}$-algebra A, its spectrum always "has enough points" to distinguish elements:

Lemma 2.8. *For any stably quasi-coherent $\mathbb{I}$-algebra A and for any $a, b : A$,*

$$a = b \leftrightarrow \forall x \colon \operatorname{Spec} A.\ xa = xb.$$

Proof. The $\Rightarrow$ direction is trivial. For $\Leftarrow$, by the universal property of the quotient $A/a = b$, we have

$$\operatorname{Spec}(A/a = b) = \{\, x : \operatorname{Spec} A \mid xa = xb \,\} \subseteq \operatorname{Spec} A.$$

If $\forall x \colon \operatorname{Spec} A.\ xa = xb$ holds, then the inclusion $\operatorname{Spec}(A/a = b) \rightarrowtail \operatorname{Spec} A$ is an equivalence. Since by assumption $A/a = b$ is also quasi-coherent, by Corollary 2.6 the quotient $A \twoheadrightarrow A/a = b$ is an equivalence. This exactly says $a = b$. ☐

Dually, we can use quasi-coherence to define the notion of *affine spaces*:

Definition 2.9. We say a set X is *affine*, if $\mathbb{I}^X$ is quasi-coherent, and the canonical evaluation map

$$X \to \operatorname{Spec} \mathbb{I}^X$$

is an equivalence. Similarly, we say X is *stably affine* if $\mathbb{I}^X$ is furthermore stably quasi-coherent.

Proposition 2.10. *For any set X, X is (stably) affine iff there exists a (stably) quasi-coherent A such that $X = \operatorname{Spec} A$.*

Proof. The only if direction holds by definition, since A can be taken as $\mathbb{I}^X$. For the if direction, it suffices to show $\operatorname{Spec} A$ is affine whenever A is quasi-coherent. Now by quasi-coherence, $A = \mathbb{I}^{\operatorname{Spec} A}$ thus $\mathbb{I}^{\operatorname{Spec} A}$ is quasi-coherent. Furthermore, the canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{I}^{\operatorname{Spec} A}$ is an equivalence again by quasi-coherence of A. ☐

At the end of this section, we describe finite limits of affine spaces. By Example 2.4, the terminal type 1 is affine. More generally, pullbacks of affine spaces can be computed as a spectrum:

Proposition 2.11. *Let A, B, C be $\mathbb{I}$-algebras. Given $f : \mathbb{I}\text{-}\mathbf{Alg}(B, A)$ and $g : \mathbb{I}\text{-}\mathbf{Alg}(B, C)$, the pullback of* $\operatorname{Spec} f$ *and* $\operatorname{Spec} g$ *is given by the spectrum*

$$\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} C = \operatorname{Spec}(A \otimes_B C),$$

where $A \otimes_B C$ is the following pushout of $\mathbb{I}$-algebras,

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d] \\ C \arrow[r] & A \otimes_B C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Proof. This essentially holds by the universal property of the pushout. □

Remark 2.12. Notice that in the above result, if $\operatorname{Spec} B$ is affine, then by Proposition 2.5 any map from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ is induced by an $\mathbb{I}$-algebra morphism, and similarly for $\operatorname{Spec} C$. Thus, any such pullback will be computed as a spectrum.

Following [6], the above result provides an extremely easy formula to compute the coproducts of quasi-coherent algebras:

Corollary 2.13. *For any $\mathbb{I}$-algebra A, B, if B is quasi-coherent, then there is an equivalence*

$$\mathbb{I}^{\operatorname{Spec} A \otimes B} = B^{\operatorname{Spec} A}.$$

Proof. Since 1 is affine, $\operatorname{Spec} A \times \operatorname{Spec} B$ can be computed as $\operatorname{Spec} A \otimes B$. Then by quasi-coherence of B, we have

$$\mathbb{I}^{\operatorname{Spec}(A \otimes B)} = \mathbb{I}^{\operatorname{Spec} A \times \operatorname{Spec} B} = B^{\operatorname{Spec} A}.$$

Following this equivalence, the evaluation map $A \otimes B \to \mathbb{I}^{\operatorname{Spec} A \otimes B}$ is exactly the one given above. □

Under the above identification, if B is quasi-coherent, then the canonical map $A \otimes B \to \mathbb{I}^{\operatorname{Spec} A \otimes B}$ can be equivalently viewed as the following map

$$c \mapsto \lambda x \colon \operatorname{Spec} A.\, (x \otimes B)(c) \,:\, A \otimes B \to B^{\operatorname{Spec} A}.$$

Of course when $A \otimes B$ is quasi-coherent, then the above map will again be an equivalence. We can similarly calculate the canonical inclusions. For the left inclusion $A \to A \otimes B$, given any $a : A$ and $x : \operatorname{Spec} A$,

$$(x \otimes B)(a \otimes 1) = xa,$$

where we view $xa : \mathbb{I}$ as an element in B using its an $\mathbb{I}$-algebra. For the right inclusion $B \to A \otimes B = B^{\operatorname{Spec} A}$, for any $b : B$ and $x : \operatorname{Spec} A$,

$$(x \otimes B)(1 \otimes b) = b,$$

which implies the image of b is simply the constant function on b.

3. Open propositions and dominance

Notice that up till this piont we have not used any spectial property of $\mathbb{T}$ rather than the fact that it is a Horn theory. However, to move closer to the intended application in domain theory, we start by assuming our theory $\mathbb{T}$ is *propositional*:

Definition 3.1. We say a Horn theory $\mathbb{T}$ is *propositional*, if it extends the theory of meet-semi-lattices, and truth of an element is computed by slicing: For any $\mathbb{T}$-model A and $a : A$, the quotient $A/a = 1$ is given by

$$a \wedge - : A \twoheadrightarrow A/a,$$

where A/a by definition is $\downarrow a := \{ b : A \mid b \leq a \}$.

Remark 3.2. The theory $\mathbb{M}$ of meet-semi-lattices, $\mathbb{D}$ of distributive lattices, $\mathbb{H}$ of Heyting algebras, and $\mathbb{B}$ of Boolean algebras are all examples of propositional theories in the above sense. In fact, all finitary quotients of $\mathbb{H}$ or $\mathbb{B}$ will again be propositional. More generally, for any propositional theory $\mathbb{T}$ and any $\mathbb{T}$-model D, the theory of D-algebras will again be propositional. We call such a theory a theory of $\mathbb{T}$-algebra.

For a propositional theory $\mathbb{T}$, we think of the generic model $\mathbb{I}$ as certain interval object, since it is equipped with a partial order. For propositional theories, more sets can be realised as spectrums. The important examples are *simplices*:

Example 3.3. For any $n : \mathbf{N}$, let $\Delta^n \subseteq \mathbb{I}^n$ be the following subset,

$$\Delta^n := \{ i : n \to \mathbb{I} \mid i_1 \geq \cdots \geq i_n \}.$$

This type is indeed a spectrum, since by definition we have

$$\Delta^n = \operatorname{Spec} \mathbb{I}[i_1, \cdots, i_n]/i_1 \geq \cdots \geq i_n.$$

To simplify later discussion, we might also introduce the following types isomorphic to simplices above,

$$\Delta_n := \{ i : n \to \mathbb{I} \mid i_1 \leq \cdots \leq i_n \}.$$

Remark 3.4. All the cubes $\mathbb{I}^n$ and simplices Δ^n, Δ_n are replete. The former holds since replete objects are closed under limits, and the latter holds since they are retracts of $\mathbb{I}^n$ using the meet $\wedge$ operation.

The constant $1 : \mathbb{I}$, which is the top element in $\mathbb{I}$, induces a predicate

$$\mathsf{t} : \mathbb{I} \to \mathsf{Prop},$$

which takes $i : \mathbb{I}$ to the proposition $i = 1$. The first observation is that t takes $i : \mathbb{I}$ to a spectrum:

Lemma 3.5. *For any $i : \mathbb{I}$, $\mathsf{t}i = \operatorname{Spec} \mathbb{I}/i$.*

Proof. By definition, $\operatorname{Spec}\mathbb{I}/i = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}/i, \mathbb{I})$. Since $\mathbb{I} \twoheadrightarrow \mathbb{I}/i$ is a quotient and $\mathbb{I}$ is the initial $\mathbb{I}$-algebra, there is a homomorphism from $\mathbb{I}/i$ to $\mathbb{I}$ iff $i = 1$, and in this case the map is unique. □

To say something about the propositions that lies in t, we assume a minimal amount of axioms:

Axiom (SQCI)**.** $\mathbb{I}$ is stably quasi-coherent.

We already know that $\mathbb{I}$ is quasi-coherent. If it is stably so, then in particular for any $i : \mathbb{I}$, the principle congruence $\mathbb{I}/i$ will also be (stably) quasi-coherent. This implies the following result:

Lemma 3.6 (SQCI)**.** *The interval* $\mathbb{I}$ *is conservative,*

$$\forall i, j : \mathbb{I}.\ (\mathsf{t}i \leftrightarrow \mathsf{t}j) \leftrightarrow i = j.$$

Proof. It suffices to show

$$\forall i, j : \mathbb{I}.\ (\mathsf{t}i \to \mathsf{t}j) \to i \leq j.$$

Take i, j with $\mathsf{t}i \to \mathsf{t}j$. By Lemma 3.5 and (SQCI), each $\mathsf{t}i$ and $\mathsf{t}j$ will be affine. Then we get a restriction map between $\mathbb{I}$-algebras,

$$\mathbb{I}/j = \mathbb{I}^{\mathsf{t}j} \to \mathbb{I}^{\mathsf{t}i} = \mathbb{I}/i.$$

By the universal property of the quotient and the fact that $\mathbb{T}$ is a propositional theory, such a map exists iff $i \leq j$. □

Thus, under (SQCI), we may view the generic algebra $\mathbb{I}$ as a subuniverse of *open* propositions via the embedding t:

Definition 3.7 (SQCI)**.** Given a proposition p, we say it is *open* if p is of the form $\mathsf{t}i$ for some i in $\mathbb{I}$,

$$\mathsf{isopen}(p) := \sum_{i:\mathbb{I}} p \leftrightarrow \mathsf{t}i.$$

By Lemma 3.6, the i that $p \leftrightarrow \mathsf{t}i$ will be unique, thus being open is a proposition. Also, open propositions are evidently closed under finite conjunctions, since t preserves them. More generally, we may define the notion of open subtypes:

Definition 3.8 (SQCI)**.** A subtype U of X is *open*, if for any $x : X$ the proposition $U(x)$ is open,

$$\mathsf{isOpen}(U) := \forall x : X.\ \mathsf{isopen}(U(x)).$$

In other words, an open proposition is equivalently a map $U : X \to \mathbb{I}$, where the subtype is classified as the following pullback,

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d] \\ X \arrow[r] & \mathbb{I} \end{tikzcd}$$

This also implies that open subtypes are closed under pullbacks. Note that this crucial depends on conservativity of $\mathbb{I}$.

If X is affine, open subtypes are easy to classify:

Lemma 3.9 (SQCI)**.** *Let $X = \operatorname{Spec} A$ be affine. Then any open subset of X is of the form D_a for some $a : A$, where*

$$D_a := \{ x : X \mid \mathsf{t}(xa) \} = \operatorname{Spec} A/a.$$

If X is stably affine, then an open subset of it is again stably affine.

Proof. This follows from the fact that the type of opens of X is $\mathbb{I}^X = A$ since X is affine. If X is stably affine, then so is D_a. □

One observation from [6] is that the type of open propositions forms a *dominance* in the sense of [13]. In other words, open propositions are closed under truth and dependent product. The *loc. cit.* uses certain choice principle on affine spaces to show this. However, in our situation the conservativity of the interval $\mathbb{I}$ allows direct computation of the type of open subsets of affine spaces as shown above. Hence, our proof does not require any additional choice principle:

Proposition 3.10 (SQCI)**.** *The type of opens $\mathbb{I}$ forms a dominance.*

Proof. Suppose p is an open proposition. By definition $p = \mathsf{t}i$ for some $i : \mathbb{I}$. Now if q is an open subset of p, since $p = \mathsf{t}i$ is affine, Lemma 3.9 implies that $q = D_j$ for some $j : \mathbb{I}/i$. Since $\mathbb{T}$ is propositional, equivalently j can be viewed as an element $j : \mathbb{I}$ with $j \leq i$. This way, $q = \mathsf{t}j$. □

Corollary 3.11. *Open subtypes are transitive.*

4. Lifting

Given then dominance structure $\mathbb{I}$, we can construct internally the lifting functor. For any type X, its lift is given by

$$X_\perp := \sum_{i:\mathbb{I}} X^{\mathsf{t}i}.$$

The functoriality is easy to express: For any $f : X \to Y$, we have

$$f_\perp(i, x) := (i, \lambda w{:}\mathsf{t}i.\ f x w).$$

There is an evident unit $\eta_\perp : X \to X_\perp$, where

$$\eta := \lambda x{:}X.\,(1, \lambda\text{-}{:}1.\, x).$$

The dominance structure on $\mathbb{I}$ also gives a multiplication $\mu : (X_\perp)_\perp \to X_\perp$, where μ takes any (i, u) with $i : \mathbb{I}$ and $u : \mathsf{t}i \to X_\perp$ first to (j, x), where j is the dependent sum

$$j := \sum_{w : \mathsf{t}i} (uw)_0,$$

and $x : \mathsf{t}j \to X$ is the partial element such that for $w : \mathsf{t}i$ and $v : (uw)_0$

$$x(w, v) := (uw)_1(v).$$

Example 4.1. By definition, it is easy to see that

$$1_\perp = \sum_{i : \mathbb{I}} \mathsf{t}i = \mathbb{I}.$$

For synthetic domain theory, the object of particular importance is the lift of the type of open propositions $\mathbb{I}$ itself. In fact, the lifting of any stably quasi-coherent $\mathbb{I}$-algebra can be computed fairely explicitly:

Lemma 4.2. *If A is stably quasi-coherent, then we have*

$$A_\perp = \{\, i : \mathbb{I}, a : A \mid a \leq i \,\}.$$

Proof. Note for any $i : \mathbb{I}$, the quotient A/i is again quasi-coherent. Now notice that we do have

$$A/i = A \otimes \mathbb{I}/i.$$

By Corollary 2.13 and quasi-coherence of A/i,

$$A/i = A \otimes \mathbb{I}/i = A^{\operatorname{Spec} \mathbb{I}/i} = A^{\mathsf{t}i}.$$

This way, it follows that

$$A_\perp = \sum_{i : \mathbb{I}} A^{\mathsf{t}i} = \sum_{i : \mathbb{I}} A/i = \{\, i : \mathbb{I}, a : A \mid a \leq i \,\}. \qquad \square$$

Corollary 4.3 (SQCI)**.** *For the interval* $\mathbb{I}$, $\mathbb{I}_\perp = \Delta^2$.

Proof. By Lemma 4.2,

$$\mathbb{I}_\perp = \{\, i, j : \mathbb{I} \mid i \geq j \,\} = \Delta^2.$$

The second equivalence again uses the fact that $\mathbb{T}$ is propositional. $\square$

Remark 4.4. One interesting thing to notice here is that, though $\mathbb{I}_\perp$ by computation is a dependent sum of algebras, it is naturally equivalent to a *spectrum*, which is a geometric object. In some sense the source is the assumption that $\mathbb{T}$ is propositional, which allows us to identify the algebraic object $\mathbb{I}/i$ as a subset $\{\, j : \mathbb{I} \mid j \leq i \,\}$.

More generally, for domain theoretic applications we would want to compute the liftings of the simplices introduced as spectrums in Example 3.3. It is first of all easy to see that the lifting of any affine space is also straight forward to compute by duality:

Lemma 4.5. *If $X = \operatorname{Spec} A$ is affine, then the lifting of X is given by*

$$X_\perp = \sum_{i:\mathbb{I}} \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}/i).$$

Proof. By Proposition 2.5, since $X = \operatorname{Spec} A$ is affine,

$$X_\perp = \sum_{i:\mathbb{I}} X^{ti} = \sum_{i:\mathbb{I}} \operatorname{Spec} A^{\operatorname{Spec}\mathbb{I}/i} = \sum_{i:\mathbb{I}} \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}/i). \qquad \square$$

Thus, motivated by domain theory, we find ourselves naturally move towards the following axiomatisation:

Axiom (SQCF)**.** All finitely generated free $\mathbb{I}$-algebras, i.e. $\mathbb{I}[n]$ for $n : \mathbf{N}$, are stably quasi-coherent.

Equivalently, (SQCF) says that any finitely presented $\mathbb{I}$-algebra is (stably) quasi-coherent, where an $\mathbb{I}$-algebra is finitely presented if it is merely of the form $\mathbb{I}[n]/s = t$ with $s, t : m \to \mathbb{I}[n]$ for some finite n, m.

Of course, (SQCF) implies (SQCI) when taking n to be 0. Furthermore, (SQCF) implies that the simplices Δ^n are now (stably) affine as well. This way, we can indeed compute their lifts:

Lemma 4.6 (SQCF)**.** *For any $n : \mathbf{N}$, we have*

$$\Delta^n_\perp = \Delta^{n+1}.$$

Proof. By Lemma 4.5, for the affine space $\Delta^n = \operatorname{Spec} \mathbb{I}[n]/i_1 \geq \cdots \geq i_n$,

$$\begin{aligned} \Delta^n_\perp &= \sum_{i:\mathbb{I}} \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}[n]/i_1 \geq \cdots \geq i_n, \mathbb{I}/i) \\ &= \{\, i, i_1, \cdots, i_n : \mathbb{I} \mid i \geq i_1 \geq \cdots \geq i_n \,\} \\ &= \Delta^{n+1} \end{aligned}$$

The second equality again uses the fact that $\mathbb{T}$ is propositional. $\square$

In particular, from a geometric perspective, the unit

$$\eta : \Delta^n \rightarrowtail \Delta^n_\perp = \Delta^{n+1}$$

takes $i_1 \geq \cdots \geq i_n$ in Δ^n to $1 \geq i_1 \geq \cdots \geq i_n$ in Δ^{n+1}.

5. Distributive lattice and locality

One important finitary axiom for synthetic domain theory is the *Phoa's principle*, which as an observation of [7] is a consequence of the quasi-coherence principle for distributive lattices. Thus, we now work in the theory $\mathbb{T}$ of distributive lattices, or more generally the theory of D-algebra for some distributive lattice D.

In this case, $\mathbb{I}$ also has a minimal element 0. Notice that the constant 0 also induces a predicate on $\mathbb{I}$,

$$\mathsf{f} : \mathbb{I} \to \mathsf{Prop},$$

which takes any $i : \mathbb{I}$ to $i = 0$. Notice that the theory of distributive lattices also satisfies the *dual* property of being propositional. Hence, previous developments would also hold when we exchange 1 for 0 and $\wedge$ for $\vee$.

For instance, the same proof as in Lemma 3.6 also implies the following:

Corollary 5.1 (SQCI)**.** $\mathsf{f} : \mathbb{I} \to \mathsf{Prop}$ *is an embedding.*

Remark 5.2. Notice that this does *not* hold for the theory of Heyting algebras, since the map $a \vee - : A \to A/a = 0$ is *not* a quotient morphism for A a Heyting algebra. Hence, the proof of Lemma 3.6 does not apply. For instance, under (SQCI), the affine space $\mathbb{I}/i = 0$ in this case is equivalent to $\mathbb{I}/\neg i$. It then follows by Lemma 3.6 that $\mathsf{f}i = \mathsf{f}j$ iff $\neg i = \neg j$, which not necessarily implies $i = j$.

We will now call propositions in the image of f *closed*, and a subtype classified by f a *closed subtype*. Completely similar to the previous Lemma 3.9, closed subtypes of affine spaces are again of the following form:

Corollary 5.3 (SQCI)**.** *If* $X = \operatorname{Spec} A$ *is affine, then closed subtypes of* X *are all of the form*

$$C_a := \{ x : X \mid \mathsf{f}(xa) \} = \operatorname{Spec} a/A,$$

for some $a : A$*. If* X *is stably affine, then so is* C_a*.*

Corollary 5.4 (SQCI)**.** f *forms a dominance.*

There is an accompanying *colifting* monad $(-)_\top$, defined by

$$X_\top := \sum_{i:\mathbb{I}} X^{\mathsf{f}i}.$$

And similarly, under (SQCF), we can explicitly compute the colifting of the simplices Δ^n,

$$\Delta^n_\top = \Delta^{n+1},$$

where now the unit $\eta : \Delta^n \rightarrowtail \Delta^n_\top = \Delta^{n+1}$ takes a sequence $i_1 \geq \cdots \geq i_n$ to $i_1 \geq \cdots \geq i_n \geq 0$ in Δ^{n+1}. To distinguish the two inclusions $\Delta^n \rightarrowtail \Delta^n_\perp = \Delta^{n+1}$ and $\Delta^n \rightarrowtail \Delta^n_\top = \Delta^{n+1}$, we will denote the former as $\eta_\perp$ and the latter as $\eta_\top$.

To model divergent computation, we would like the dominance $\mathbb{I}$ to be closed under falsum. Hence, we may further introduce the following minimal amount of non-triviality:

Axiom (NT)**.** For $\mathbb{I}$, $0 \neq 1$.

Remark 5.5. The above axiom semantically corresponds to working with certain subtopoi of the classifying topos $\mathbf{Set}[\mathbb{T}]$. For instance, the minimal topology on the underlying site $\mathbb{T}\text{-}\mathbf{Mod}_{\mathrm{f.p.}}$ we might choose for (NT) to hold is to assert that the trivial $\mathbb{T}$-model is covered by the empty sieve. Since the trivial $\mathbb{T}$-model is a strict terminal object, this topology will be subcanonical, thus (SQCF) still holds.

Let us first observe some elementary consequence of (NT). For instance, the lifting monad $(-)_\perp$ defined in Section 4 is now also pointed, where for any X we can define

$$\perp := (0, ?) : X_\perp,$$

where $? : \mathsf{t}0 = \varnothing \to X$ is the unique map from $\varnothing$.

As another consequence, now $\varnothing = \mathsf{t}0$ is affine, in fact both an open and closed proposition. For any $\mathbb{I}$-algebra A, we say A is *trivial* if it is equivalent to the trivial $\mathbb{I}$-algebra, which we denote as 0. Equivalently, this is saying that $0 = 1$ in A. In the case of (NT), we have the following weak form of nullstellensatz:

Lemma 5.6 (NT)**.** *For any affine $X = \operatorname{Spec} A$, $X = \varnothing$ iff A is trivial.*

Proof. By assumption, $A = \mathbb{I}^{\operatorname{Spec} A} = \mathbb{I}^{\varnothing} = 0$. □

As a simple application of the above weak nullstellensatz, $\mathbb{I}$ is almost a field in the following sense:

Corollary 5.7 (NT, SQCI)**.** *For any $i : \mathbb{I}$*

$$\neg \mathsf{t}i \leftrightarrow \mathsf{f}i, \quad \neg \mathsf{f}i \leftrightarrow \mathsf{t}i.$$

In particular, the embedding $2 \rightarrowtail \mathbb{I}$ induced by $0, 1$ is $\neg\neg$-dense,

$$\forall i : \mathbb{I}.\ \neg\neg(\mathsf{t}i \vee \mathsf{f}i).$$

Proof. If $\mathsf{f}i$ then $\neg \mathsf{t}i$ by (NT). On the other hand, by the conservativity result in Lemma 3.6 and (NT), if $i \neq 1$ then $i = 0$ since $0 \neq 1$. The dual case is completely similar. □

This allows us to observe that the open and closed propositions are exactly complementary to each other:

Corollary 5.8 (NT, SQCI)**.** *For any proposition p, p is open iff $\neg p$ is closed and vice versa. Furthermore, open and closed propositions are $\neg\neg$-stable.*

As another consequence, we will see that $\mathbb{I}$ is *internally connected*. To show this, we consider the following notion of *specialisation order*:

Definition 5.9. The *specialisation order* on a type X is defined as follows:

$$x \preceq y \;:=\; \forall U\colon X \to \mathbb{I}.\ U(x) \leq U(y).$$

By definition, the specialisation order is reflexive and transitive. As already observed in *loc. cit.*, one important property of the specialisation order is that *every* map is monotone w.r.t. this order:

Lemma 5.10. *For $f : X \to Y$, $x \preceq y$ in X implies $fx \preceq fy$ in Y.*

Proof. This simply follows from compositionality of functions. ☐

For $X = \operatorname{Spec} A$ an affine space, $\mathbb{I}^X = A$, which means for any $x, y : X$,

$$x \preceq y \leftrightarrow \forall a\colon A.\ xa \leq ya.$$

In other words, the order $\preceq$ on an affine space is the point-wise order in $X = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I})$. In particular, we have:

Lemma 5.11 (SQCF)**.** *The specialisation order on $\mathbb{I}^n$ is point-wise for $n : \mathbf{N}$.*

Proof. Note that $\mathbb{I}^n = \operatorname{Spec} \mathbb{I}[n]$. Since all variables are in particular polynomials, the specialisation order on $\mathbb{I}^n$ will be point-wise. ☐

Furthermore, subspaces of affine spaces induced by quotients inherits the specialisation order:

Lemma 5.12. *Let A, B be quasi-coherent and $A \twoheadrightarrow B$ a surjection of $\mathbb{I}$-algebras. Then for the inclusion $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$, for any $x, y : \operatorname{Spec} B$ we have $x \preceq y$ in $\operatorname{Spec} B$ iff $x \preceq y$ in $\operatorname{Spec} A$.*

Proof. By monotonicity in Lemma 5.10, it suffices to show for $x, y : \operatorname{Spec} B$, if $x \preceq y$ in $\operatorname{Spec} A$, then $x \preceq y$ in $\operatorname{Spec} B$. However, this follows from the fact that $\mathbb{I}$ is weakly orthogonal to the inclusion $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$,

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "b"] \arrow[d, hook] & \mathbb{I} \\
\operatorname{Spec} A \arrow[ur, dashed, "\exists a"'] &
\end{tikzcd}$$

because by quasi-coherence $\mathbb{I}^{\operatorname{Spec} A} = A \twoheadrightarrow B = \mathbb{I}^{\operatorname{Spec} B}$ is a surjection. ☐

Corollary 5.13 (SQCF)**.** *The specialisation order on simplices Δ^n and Δ_n are also point-wise.*

We can also classify the specialisation order on discrete sets, i.e. sets with decidable equality:

Lemma 5.14 (NT, SQCF)**.** *If M is a discrete set, then the specialisation order on M is discrete, in the sense that for $n, m : M$,*

$$n \preceq m \to n = m.$$

Proof. Suppose $n \preceq m$. If $n \neq m$, we can construct a map $f : M \to \mathbb{I}$ with $f(n) = 1$ and $f(m) = 0$, contradictory to $n \preceq m$. Hence, $n = m$. $\square$

Corollary 5.15 (NT, SQCF)**.** *For any type X whose specialisation order is connected, the map $M \to M^X$ is an equivalence for a discrete set M.*

Proof. When X is merely inhabited, the map $M \to M^X$ is always an embedding, thus it suffices to show it is surjective. Suppose we have $f : X \to M$. Since X is connected, there merely exists $x : X$, and for any $y : X$, by monotonicity in Lemma 5.10 and by Lemma 5.14 $f(x) = f(y)$. Hence, f is the constant function on x, which implies $M \to M^X$ is an equivalence. $\square$

This is in particularly true for cubes $\mathbb{I}^n$ and simplices Δ^n, Δ_n, since their specialisation order has a top and bottom element. To further determine the space 2^X for an arbitrary affine space X, we introduce a further locality axiom:

Axiom (L)**.** $\mathbb{I}$ is local, i.e. $0 \neq 1$ and $\mathsf{t}(i \vee j) \leftrightarrow \mathsf{t}i \vee \mathsf{t}j$ for all $i, j : \mathbb{I}$.

(L) implies that the dominance $\mathbb{I}$ is closed under finite disjunction, and that open subtypes are closed under finite unions. For open subsets of affine spaces this can be described more clearly:

Lemma 5.16 (L, SQCI)**.** *If $X = \operatorname{Spec} A$ is affine, then $D_0 = \emptyset$, and for any $a, b : A$, $D_a \cup D_b = D_{a \vee b}$.*

Proof. For any $D_0 = \emptyset$ follows from (NT). For any $x : X$,

$$(D_a \cup D_b)(x) = \mathsf{t}(xa) \vee \mathsf{t}(xb) = \mathsf{t}(x(a \vee b)) = D_{a \vee b}(x). \qquad \square$$

Using this characterisation, we can classify the space 2^X for any affine space X:

Lemma 5.17 (L)**.** *If $X = \operatorname{Spec} A$ is stably affine, then 2^X is equivalent to the set of complemented elements in A.*

Proof. Notice that a map $X \to 2 \to \mathbb{I}$ corresponds to an open subset of X, which is of the form D_a for some $a : A$. By swapping $\neg : 2 \to 2$, we get another open subset D_b with $D_a \cap D_b = \emptyset$ and $D_a \cup D_b = X$. Now this imparticular means that

$$D_a \cap D_b = D_{a \wedge b} = \emptyset, \quad D_a \cup D_b = D_{a \vee b} = X,$$

which by Lemma 2.8 $a \wedge b = 0$ and $a \vee b = 1$. $\square$

As a consequence, we may conclude the following:

Corollary 5.18 (L, SQCI)**.** *For any $i : \mathbb{I}$, if i has a complement, then $\mathsf{t}i \vee \mathsf{f}i$. In particular, $\mathbb{I}$ is not Boolean.*

Proof. Since $2 = 2^1$ where $1 = \operatorname{Spec}\mathbb{I}$, Lemma 5.17 then implies that the only complemented element in $\mathbb{I}$ are $0, 1$. Now if $\mathbb{I}$ is Boolean, this implies that $\mathbb{I} = 2$, which contradicts Corollary 5.15. ☐

However, not all affine spaces will be internally connected:

Example 5.19. Consider the algebra classifying complemented elements,

$$A := \mathbb{I}[x, y]/x \vee y = 1, x \wedge y = 0.$$

The two variables x, y are complemented to each other in the above algebra, hence under (L) and (SQCF) by Lemma 5.17 we know $\operatorname{Spec} A$ can be decomposed into a non-trivial coproduct,

$$\operatorname{Spec} A = D_x + D_y,$$

Furthermore, we observe that

$$D_x = \operatorname{Spec}\mathbb{I}[x, y]/x \vee y = 1, x \wedge y = 0, x = 1 = \operatorname{Spec}\mathbb{I} = 1,$$

and similarly for D_y. Hence, now we have $\operatorname{Spec} A = 2$, and now 2 is *affine.*

As another example of an affine space (L) allows us to define:

Example 5.20. Consider the right outer horn Λ^2_2 as a pushout,

$$\begin{tikzcd} 1 \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{I} \arrow[d] \\ \mathbb{I} \arrow[r] & \Lambda^2_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By viewing Λ^2_2 as a subspace of $\mathbb{I}^2$, we may identify it as follows,

$$\Lambda^2_2 = \{\, i, j : \mathbb{I} \mid \mathsf{t}i \vee \mathsf{t}j \,\}.$$

Assuming (L), $\mathsf{t}i \vee \mathsf{t}j = \mathsf{t}(i \vee j)$, it follows that

$$\Lambda^2_2 = \operatorname{Spec}\mathbb{I}[i, j]/i \vee j.$$

Dually, one can assume the colocality:

Axiom (cL)**.** $\mathbb{I}$ is colocal, i.e. $0 \neq 1$ and $\mathsf{f}(i \wedge j) \leftrightarrow \mathsf{f}i \vee \mathsf{f}j$ for all $i, j : \mathbb{I}$.

Similalry to Example 5.20, this allows to define the left ourter horn Λ^2_0 as an affine space. Of course we can combine (L) and (cL), which gives us both outer horns.

A stronger localicy axiom than (L) and (cL) is the simplicial axiom:

Axiom (SL)**.** $\mathbb{I}$ is *linear*, i.e. $0 \neq 1$ and for all $i, j : \mathbb{I}$, $i \vee j \vee j \leq i$.

The linearity axiom is saying that Δ^2, Δ_2 covers the square,

$$\Delta^2 \cup \Delta_2 = \mathbb{I}^2.$$

An even stronger locality axiom states that the simplicial structure is truncated to level 1:

Axiom (1T). $\mathbb{I}$ is *1-truncated*, i.e. $0 \neq 1$ and for $i, j : \mathbb{I}$, $i \leq j$ iff $\mathsf{f}i \vee \mathsf{t}j \vee i = j$.

The above exactly says that the 2-simplice Δ_2 is covered by its three edges, which means that $\partial\Delta_2 = \Delta_2$.

6. Phoa's principle

Given the specialisation order on X, one natural question is its relation with the function space $X^{\mathbb{I}}$:

Definition 6.1. We say a type X is *Phoa*, if

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : X^{\mathbb{I}} \to \preceq$$

is an equivalence over $X \times X$, and $\preceq$ is anti-symmetric on X, in the sense that for $x, y : X$,

$$x = y \to x \preceq y \wedge y \preceq x$$

is an equivalence. In particular, X will be a set.

Remark 6.2. In [11], a type X is called *linked* if $X^{\mathbb{I}} \to \preceq$ is a surjection.

Notice that assuming (SQCF), the specialisation order on $\mathbb{I}$ coincides with its usual order, thus by monotonicity for any $f : \mathbb{I} \to X$ we indeed have $f(0) \preceq f(1)$, which means the above notion is well-defined.

Example 6.3. For instance, assuming (NT) and (SQCF), Lemma 5.14 and Corollary 5.15 implies all discrete sets will be Phoa.

The true strength of working with distributive lattices, besides the symmetry between $1, \wedge$ and $0, \vee$, is the following fact:

Theorem 6.4. *Any affine space X will be Phoa.*

Proof. For any affine space X we have

$$X^{\mathbb{I}} = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}[i]).$$

Hence it suffices to show that the pair of evaluation maps

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : \mathbb{I}[i] \to \mathbb{I} \times \mathbb{I}$$

classifies the order on $\mathbb{I}$. This follows from a normal form result of polynomials for distributive lattices, i.e. any polynomial p is of the form

$$p = p(0) \vee x \wedge p(1).$$

See e.g. [10, Thm. 10.11]. This way, a map $\mathbb{I} \to X$ is exactly described as two points $x, y : X$ such that $\forall a : A.\ xa \leq ya$, which coincides with $x \preceq y$. Furthermore, this order is anti-symmetric by definition. □

In the literature, the above statement restricted to $\mathbb{I}$ is usually denoted as the *Phoa's principle*. From Theorem 6.4, this property can be generalised to all affine spaces.

Remark 6.5. Crucially, we emphasis that the above proof does *not* in fact rely on the affineness of the interval $\mathbb{I}$ itself. Rather, it is purely the consequence of the fact that for any distributive lattice A, the free algebra $A[i]$ is equivalent to the order on A. This is a perfect example of how an algebraic property of a theory has a non-trivial effect on the internal logic of its classifying topos.

The class of Phoa types is indeed *reflective*:

Proposition 6.6 (SQCF). *The specialisation order on a limit of Phoa types is point-wise. Furthermore, Phoa types are reflective, with the reflector of X being the largest sub Phoa type of $\mathbb{I}^{\mathbb{I}^X}$ containing X.*

Proof. See [11, Sec. 5.4]. □

Corollary 6.7 (SQCF). *For a quasi-coherent $\mathbb{I}$-algebra A, the specialisation order is its usual order, and it is Phoa.*

At the end of this section, we describe another interesting perspective arising from the proof of Theorem 6.4. We have seen that the dualising object $\mathbb{I}$ has a double role: It is both an algebra and a spectrum. The proof of Phoa's principle gives us many more such examples. For instance, $\mathbb{I}[i]$ classifies the order on $\mathbb{I}$, which by definition is the spectrum Δ_2. In fact, *all* the simplices in this case have an algebraic description:

Proposition 6.8. *For any $n : \mathbf{N}$ and $\mathbb{I}$-algebra A, for any $a_0 \geq \cdots \geq a_n$ in A, define a polynomial in $A[i_1, \cdots, i_n]$ as follows,*

$$a_0 \wedge i_1 \vee a_1 \wedge \cdots \wedge i_n \vee a_n.$$

This is well-defined in the quotient $A[n]/i_1 \leq \cdots \leq i_n$, in the sense that the value does not depend on how one arranges parenthesis. In fact this gives us an equivalence

$$\Delta[A]^{n+1} = A[n]/i_1 \leq \cdots \leq i_n,$$

where we write $\Delta[A]^{n+1}$ as $\{\, a_0, \cdots, a_n \ :\ A \mid a_0 \geq \cdots \geq a_n \,\}$. Completely similarly, we have an equivalence

$$\Delta[A]_{n+1} = A[n]/i_1 \geq \cdots \geq i_n.$$

Proof. This again follows from a normal form result on algebras of the form $A[n]/i_1 \leq \cdots \leq i_n$, which is a consequence of the general normal form for polynomials on distributive lattices; see [10, Thm. 10.21]. □

Now we can identify the simplices Δ^n as a finitely presented $\mathbb{I}$-algebra as shown above. Under such equivalences, the inclusion $\eta_\perp$ simply corresponds to the canonical inclusion as shown below,

$$\begin{tikzcd}
\Delta^{n+1} \arrow[r, tail, "\eta_\perp"] \arrow[d, "\simeq"'] & \Delta^{n+2} \arrow[d, "\simeq"] \\
\mathbb{I}[n]/i_1 \leq \cdots \leq i_n \arrow[r, tail] & \mathbb{I}[n+1]/i_0 \leq \cdots \leq i_n
\end{tikzcd}$$

7. Infinitay domain theory

Untill this point, we have seen that all the finitary axioms for synthetic domain theory is a consequence of (SQCF) for a theory of distributive algebra, with potentially the assumption of (NT).

There is one final crucial infinitary axiom of synthetic domain theory. From the observation in [8, 9], we can define internally in type theory the initial algebra and final coalgebra for the lifting functor $(-)_\perp$, which we denote as ω and $\overline{\omega}$, respectively. There is a canonical inclusion $\omega \rightarrowtail \overline{\omega}$, and the final axiom for synthetic domain theory states that the interval $\mathbb{I}$ is *complete*, in the sense that the induced map $\mathbb{I}^{\overline{\omega}} \to \mathbb{I}^{\omega}$ is an *equivalence*. The goal of this section is to explain that, this completeness axiom can also be realised as a consequence of quasi-coherence.

Since $(-)_\perp$ by construction preserves all connected limits, the final coalgebra $\overline{\omega}$ can be easily characterised as a limit. As shown in [8], it can be identified as the object of infinite descending sequences in $\mathbb{I}$,

$$\overline{\omega} := \{\, i : \mathbf{N} \to \mathbb{I} \mid \forall n{:}\mathbf{N}.\, i_n \geq i_{n+1} \,\},$$

which can be viewed as the following sequential limit,

$$\cdots \longrightarrow \Delta^2 \xrightarrow{!_\perp} \mathbb{I} \xrightarrow{!} 1$$

where the transition map $\Delta^{n+1} \to \Delta^n$ takes the sequence $i_0 \geq \cdots \geq i_n$ to the final segment $i_1 \geq \cdots \geq i_n$.

According to the above description, $\overline{\omega}$ is in fact a *spectrum*,

$$\overline{\omega} = \mathrm{Spec}(\mathbb{I}[\mathbf{N}]/\forall n.\, i_n \geq i_{n+1}).$$

However, the corresponding algebra is no longer finitely presented, but countably presented. By a countably presented, or c.p. in short, $\mathbb{I}$-algebra, we mean an $\mathbb{I}$-algebra of the form $\mathbb{I}[I]/s = t$ for some *decidable* subsets I, J of $\mathbf{N}$ with $s, t : J \to \mathbb{I}[I]$. In particular, all finitely presented $\mathbb{I}$-algebra will also be countably presented. Motivated by the above characterisation of $\overline{\omega}$, we naturally consider the following stronger quasi-coherence principle:

Axiom (SQCC)**.** All c.p. $\mathbb{I}$-algebras are (stably) quasi-coherent.

Remark 7.1. The quasi-coherence principle for countably presented algebras is investigated in [5]. Just like the finitary version (SQCF), which is true in the classifying topos $\mathbf{Set}[\mathbb{T}] = [\mathbb{T}\text{-}\mathbf{Mod}_{\text{f.p.}}, \mathbf{Set}]$, the countable version (SQCC) will be valid in a larger presheaf topos $[\mathbb{T}\text{-}\mathbf{Mod}_{\text{c.p.}}, \mathbf{Set}]$, or any subtopos induced by a subcanonical topology. For instance, the semantics of [5] is based on the topos of light condensed sets introduced by Clausen and Scholze, which is a subtopos of $[\mathbb{B}\text{-}\mathbf{Mod}_{\text{c.p.}}, \mathbf{Set}]$.

There are many immediate consequences of (SQCC). For instance, the description of the specialisation order for affine spaces in Lemma 5.12 now applies to $\overline{\omega}$ as well, which implies it has both a top and bottom element. Then Corollary 5.15 now also applies to $\overline{\omega}$, and $M \to M^{\overline{\omega}}$ is an equivalence for any discrete set M.

There are more logical consequences of (SQCC). Let us observe that it implies the following form of Markov principle:

Lemma 7.2 (NT, SQCC)**.** *For any $i : \overline{\omega}$, we have*

$$\neg\forall n{:}\mathbf{N}.\ \mathsf{t}i_n \to \exists n{:}\mathbf{N}.\ \mathsf{f}i_n.$$

Proof. Let $i : \overline{\omega}$. Notice that similar to Lemma 3.5, the proposition $\forall n{:}\mathbf{N}.\ \mathsf{t}i_n$ by construction is the following affine space,

$$\operatorname{Spec}\mathbb{I}/i = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}/i, \mathbb{I}) = \forall n{:}\mathbf{N}.\ \mathsf{t}i_n,$$

where we have abbreviated the c.p. $\mathbb{I}$-algebra $\mathbb{I}/\forall k.\, i_k = 1$ as $\mathbb{I}/i$. Now if we have $\neg\forall n{:}\mathbf{N}.\ \mathsf{t}i_n$, then $\operatorname{Spec}\mathbb{I}/i = \varnothing$ which implies $\mathbb{I}/i$ is trivial by Lemma 5.6. But this algebra is trivial iff there merely exists $n : \mathbf{N}$ that $\mathsf{f}i_n$ holds, thus $\exists n{:}\mathbf{N}.\ \mathsf{f}i_n$. □

Now let us consider the initial algebra ω for the lifting functor. It is shown in [9] that we can identify $\omega \rightarrowtail \overline{\omega}$ as the following subset,

$$\omega := \{\, i : \overline{\omega} \mid \forall \phi{:}\mathsf{Prop}.\ (\forall n{:}\mathbf{N}.\ (\mathsf{t}i_n \to \phi) \to \phi) \to \phi \,\}.$$

For another proof, see e.g. [16]. In the presence of the Markov principle above, this description can be drastically simplified:

Lemma 7.3 (NT, SQCC)**.** *ω is equivalent to the following subset of $\overline{\omega}$,*

$$\omega = \{\, i : \overline{\omega} \mid \exists n{:}\mathbf{N}.\ \mathsf{f}i_n \,\}.$$

Proof. Let $i : \overline{\omega}$. It suffices to show that

$$(\forall \phi{:}\mathsf{Prop}.\ (\forall n{:}\mathbf{N}.\ (\mathsf{t}i_n \to \phi) \to \phi) \to \phi) \to \exists n{:}\mathbf{N}.\ \mathsf{f}i_n,$$

We can take ϕ to be $\varnothing$. By assumption we have $\neg\forall n{:}\mathbf{N}.\ \neg\neg\mathsf{t}i_n$, which by Corollary 5.8 is equivalent to $\neg\forall n{:}\mathbf{N}.\ \mathsf{t}i_n$. Then Lemma 7.2 implies this is $\exists n{:}\mathbf{N}.\ \mathsf{f}i_n$. ☐

By [16, Cor. 1.10], the above result exactly means that ω is indeed the internal colimit of the following sequence,

$$\varnothing \xrightarrow{?} 1 \xrightarrow{?_\perp = \eta_\top} \mathbb{I} \xrightarrow{\eta_\top} \Delta^2 \longrightarrow \cdots$$

Using this, we can now show the most important infinitary axiom for synthetic domain theory:

Theorem 7.4 (NT, SQCC)**.** *The canonical map* $\mathbb{I}^{\overline{\omega}} \to \mathbb{I}^{\omega}$ *is an equivalence.*

Proof. Since $\overline{\omega}$ is now affine, we have

$$\mathbb{I}^{\overline{\omega}} = \mathbb{I}[\mathbf{N}]/\forall n.\, i_n \geq i_{n+1}.$$

On the other hand, since ω is internally the colimit of Δ^n, we have

$$\mathbb{I}^{\omega} = \varprojlim_{n:\mathbf{N}} \mathbb{I}^{\Delta^n} = \varprojlim_{n:\mathbf{N}} \mathbb{I}[n]/i_1 \geq \cdots \geq i_n.$$

Note that the transition maps induced by $\eta_\top : \Delta^n \to \Delta^{n+1}$ under quasi-coherence gives us the following maps on algebras:

$$\begin{tikzcd}
\mathbb{I}^{\Delta^{n+1}} \arrow[r, "\mathbb{I}^{\eta_\top}"] & \mathbb{I}^{\Delta^n} \\
\mathbb{I}[n+1]/i_1 \geq \cdots \geq i_{n+1} \arrow[u, "\simeq"] \arrow[r, "i_{n+1}\mapsto 0"'] & \mathbb{I}[n]/i_1 \geq \cdots \geq i_n \arrow[u, "\simeq"']
\end{tikzcd}$$

Now taking the limit of the above sequence of algebras gives us

$$\varprojlim_{n:\mathbf{N}} \mathbb{I}[n]/i_1 \geq \cdots \geq i_n = \mathbb{I}[\mathbf{N}]/\forall n.\, i_n \geq i_{n+1} = \mathbb{I}^{\overline{\omega}}.$$

This shows that $\mathbb{I}^{\omega} = \mathbb{I}^{\overline{\omega}}$. ☐

Finally, at the end of this section, we observe that the algebraic description of the simplices given at the end of Section 6 can be extended to the above infinitary case:

Corollary 7.5. $\mathbb{I}^{\overline{\omega}} = \mathbb{I}[\mathbf{N}]/\forall n.\, i_n \geq i_{n+1}$ *is equivalent to* Δ_∞*,*

$$\Delta_\infty := \{\, i : \mathbf{N} \to \mathbb{I} \mid \forall n{:}\mathbf{N}.\ i_n \leq i_{n+1} \,\}.$$

Proof. Following the discussion at the end of Section 6, we have the following equivalences,

$$\begin{tikzcd}
\Delta_{n+1} \arrow[r, tail, "\eta_\top"] \arrow[d, "\simeq"'] & \Delta_{n+2} \arrow[d, "\simeq"] \\
\mathbb{I}[n]/i_1 \geq \cdots \geq i_n \arrow[r, tail] & \mathbb{I}[n+1]/i_1 \geq \cdots \geq i_{n+1}
\end{tikzcd}$$

Then taking the sequential colimit on both sides, we get an equivalence

$$\Delta_\infty = \mathbb{I}[\mathbf{N}]/\forall n.\, i_n \geq i_{n+1}. \qquad \square$$

8. Local properties for affine spaces

In this section we review some of the locality axioms we have introduced in Section 5. The observation is that, even if we do not assume them to be true globally for the internval $\mathbb{I}$, we can still show the maps representing the locality axiom to be *orthogonal* to $\mathbb{I}$, or more generally for all affine spaces. Again, these orthogonality conditions are completely due to the algebraic properties of distributive lattices.

As a starting example, let us observe that the inner horn Λ^2_1 can be constructed as the following pushout,

$$\begin{tikzcd} 1 \arrow[r, "1"] \arrow[d, "0"'] & \mathbb{I} \arrow[d] \\ \mathbb{I} \arrow[r] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

It is essentially observed in [12] that the inclusion $\Lambda^2_1 \hookrightarrow \Delta_2$ can be defined in type theory as follows,

$$\Lambda^2_1 = \{\, (i, j) \,:\, \Delta_2 \mid \mathsf{f} i \vee \mathsf{t} j \,\}.$$

Definition 8.1. We say X is *Segal* if it is orthogonal to $\Lambda^2_1 \hookrightarrow \Delta^2$.

As a first example, all affine spaces will be Segal:

Lemma 8.2. *Any affine space X is Segal.*

Proof. Let $X = \operatorname{Spec} A$ be affine, and let $f, g : \mathbb{I} \to X$ be two maps with $f(1) = g(0)$. By Proposition 2.5, these are equivalently two maps between $\mathbb{I}$-algebras

$$f^*, g^* \,:\, A \to \mathbb{I}[x],$$

with $f^*(a)(1) = g^*(a)(0)$ for all $a : A$. Now to define $h : \Delta_2 \to X$, equivalently we want to define $h^* : A \to \mathbb{I}[x, y]/x \leq y$,

$$h^*(a) := f^*(a)(0) \vee y \wedge f^*(a)(1) \vee x \wedge g^*(a)(1).$$

This is well-defined by Proposition 6.8. $\square$

Furthermore, we verify that affine spaces are 1-truncated in the following sense:

Definition 8.3. X is *1-truncated*, if it is orthogonal to $\partial\Delta_2 \hookrightarrow \Delta_2$.

Lemma 8.4. *Affine spaces are 1-truncated.*

Proof. Let $X = \operatorname{Spec} A$ be affine. By quasi-coherence, $[f, h, g] : \partial\Delta_2 \to X$ consists of three maps

$$f^*, h^*, g^* : A \to \mathbb{I}[x],$$

where for any $a : A$,

$$f^*(a)(0) = h^*(a)(0), \quad f^*(a)(1) = g^*(a)(0), \quad h^*(a)(1) = g^*(a)(1).$$

Now to give $\Delta_2 \to X$, equivalently we need $u : A \to \mathbb{I}[x, y]/x \leq y$, which by Proposition 6.8 we may define it to be

$$u(a) : f^*(a) \vee y \wedge h^*(a) \vee x \wedge g^*(a).$$

This is well-defined, since by definition $f^*(a) \leq h^*(a) \leq g^*(a)$. ☐

Definition 8.5. We say X is *simplicial* if it is orthogonal to $\Delta^2 \cup \Delta_2 \hookrightarrow \mathbb{I}^2$.

Corollary 8.6. *Any affin space are simplicial.*

Proof. This is a consequence of being Segal and 1-truncated. ☐

Following [4], we can define the type of free equivalence $\mathbb{E}$ as the following colimit,

$$\begin{tikzcd}
 & \mathbb{I} \arrow[dl] \arrow[dr, "d_1"] & & \mathbb{I} \arrow[dl, "d_2"'] \arrow[dr, "d_0"] & & \mathbb{I} \arrow[dl, "d_1"'] \arrow[dr] & \\
1 & & \Delta_2 & & \Delta_2 & & 1
\end{tikzcd}$$

Definition 8.7. We say X is *Rezk*, if it is orthogonal to $\mathbb{E} \to 1$.

Lemma 8.8. *Affine spaces are Rezk.*

Proof. Since an affine space X is 1-truncated, a map $\mathbb{E} \to X$ is completely determined by three edges $f, g, h : \mathbb{I} \to X$, where g, h are inverses to f. Now X is Rezk follows from the fact that its order is a poset. ☐

Finally, we can also construct $\mathbb{I}_{\rightrightarrows}$ as the following pushout,

$$\begin{tikzcd}
2 \arrow[r] \arrow[d] & \mathbb{I} \arrow[d] \\
\mathbb{I} \arrow[r] & \mathbb{I}_{\rightrightarrows} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Definition 8.9. We say X is $\mathbb{I}$-*separated*, if $X^{\mathbb{I}} \to X^{\mathbb{I}_{\rightrightarrows}}$ is an equivalence.

Equivalently, X is $\mathbb{I}$-separated iff $X^{\mathbb{I}} \to X \times X$ is an embedding. The following is evident:

Lemma 8.10. *Any affine space, or any specialisation preorder, is* $\mathbb{I}$*-separated.*

Proof. Evident from definition. ☐

Remark 8.11. Unless we assume the very strong locality principle (1T), Phoa types will in general *not* be Segal, since there is in general no control of maps $\Delta_2 \to X$ for a specialisation poset X.

The notion of categories and posets are formulated internally as these orthogonality classes:

Definition 8.12. A type X is a *category*, if it is simplitial, Segal, and Rezk. We say it is a *poset*, if it is also $\mathbb{I}$-separated.

Theorem 8.13. *Any affine space X is a poset.*

Corollary 8.14 (SQCF)**.** *Quasi-coherent $\mathbb{I}$-algebras will be posets.*

Proof. This follows from the fact that $\mathbb{I}$ is affine, hence a poset, and that if A is quasi-coherent then $A = \mathbb{I}^{\operatorname{Spec} A}$. □

At the end of this section, we would like to show that the previous orthogonality classes have various closure properties. In particular, we would like to show they are closed under *dependent sums over discrete sets*, and *sequential colimits.*

Lemma 8.15 (NT, SQCF)**.** $\prod_{i:\mathbf{N}}$ *commutes with* $\sum_{m:M}$ *with M discrete, i.e. for any type family P over $\mathbb{I}, M$, the canonical map*

$$\sum_{m:M}\prod_{i:\mathbf{N}} P_{i,m} \to \prod_{i:\mathbb{I}}\sum_{m:M} P_{i,m}$$

is an equivalences.

Proof. By definition, this map is an embedding, hence it suffices to show surjectivity. But this follows from Corollary 5.15. □

Proposition 8.16 (NT, SQCF)**.** *For an orthogonality condition $f : X \to Y$, if both X, Y are connected colimits of cubes or simplices, then f-local objects will be closed under dependent sums indexed by discrete sets.*

Proof. Let $Y = \varprojlim_r Y_r$ be a connected limit of cubes or simplices,

$$\left(\sum_{m:M} P_m\right)^Y \simeq \varprojlim_r \left(\sum_{m:M} \varprojlim_r P_m\right)^{Y_r} \simeq \sum_{m:M} \varprojlim_r P_m^{Y_r} \simeq \sum_{m:M} P_m^Y.$$

The second equivalence uses the fact that finite connected limits commutes with dependent sums, and by Lemma 8.15 Y_r also commutes with dependent sums indexed by M. Similarly for X. Hence, if each P_m is f-local, then so is $\sum_{m:M} P_m$. □

Corollary 8.17. *All the orthogonality classes introduced in this section are closed under dependent sums indexed by discrete sets.*

Completely similarly, we consider the case for sequential colimits. To show this closure property, we need to assume a choice principle:

Axiom (PF)**.** $\mathbb{I}$ satisfies choice: For any type family P over $\mathbb{I}$,

$$\prod_{i:\mathbb{I}} \|P(i)\| \to \|\prod_{i:\mathbb{I}} P(i)\|.$$

Similarly for cubes $\mathbb{I}^n$ and simplices Δ^n.

Lemma 8.18 (NT, PF, SQCF)**.** *$\prod_{i:\mathbf{N}}$ commutes with sequential colimit $\varinjlim_{n:\mathbf{N}}$ for all k-truncated type families: If $P_{i,n}$ is a k-truncated type family with transition maps $\tau_{i,n} : P_{i,n} \to P_{i,n+1}$ for all $i : \mathbb{I}, n : \mathbf{N}$, then the canonical map*

$$\varinjlim_{n:\mathbf{N}} \prod_{i:\mathbb{I}} P_{i,n} \to \prod_{i:\mathbb{I}} \varinjlim_{n:\mathbf{N}} P_{i,n}$$

is an equivalence, and similarly for $\mathbb{I}^n, \Delta^n$.

Proof. We prove this by induction on the truncation level. For $P_{i,n}$ contractible, by [14, Lem. 7.2] both $\varinjlim_{n:\mathbf{N}} \prod_{i:\mathbb{I}} P_{i,n}$ and $\prod_{i:\mathbb{I}} \varinjlim_{n:\mathbf{N}} P_{i,n}$ are contractible, hence an equivalence.

Now suppose P is a family of $k+1$-types. Given two canonical elements $[(0,f)], [(0,g)]$ in the colimit $\varinjlim_{n:\mathbf{N}} \prod_{i:\mathbb{I}} P_{i,n}$, recall from [14], we have

$$[(0,f)] = [(0,g)] \simeq \varinjlim_{n:\mathbf{N}} \tau_n f = \tau_n g \simeq \varinjlim_{n:\mathbf{N}} \prod_{i:\mathbb{I}} \tau_n f i = \tau_n g i.$$

On the other hand, we have

$$\prod_{i:\mathbb{I}} [(0, fi)] = [(0, gi)] = \prod_{i:\mathbb{I}} \varinjlim_{n:\mathbf{N}} \tau_n f i = \tau_n g i.$$

The two types are equivalent by induction hypothesis, since the identity types are k-truncated. This means the map is an embedding.

To show surjectivity, suppose we have $f : \prod_{i:\mathbb{I}} \varinjlim_{n:\mathbf{N}} P_{i,n}$. Now by construction,

$$\forall i:\mathbb{I}.\ \exists n:\mathbf{N}.\ \exists y:P_{i,n}.\ f(i) = [(n,y)].$$

By (PF), and Corollary 5.15, we merely get a section $f : \prod_{i:\mathbf{N}} P_{i,n}$ for some n, which shows surjectivity. $\square$

Remark 8.19. Notice that the first half of the proof of Lemma 8.18 does not rely on any properties of $\mathbb{I}$, hence this shows that for *any* type X,

$$\varinjlim_{n:\mathbf{N}} \prod_{x:X} P_{x,n} \to \prod_{x:X} \varinjlim_{n:\mathbf{N}} P_{x,n}$$

is an embedding, for P a k-truncated family.

Proposition 8.20 (NT, PF, SQCF)**.** *For $f : X \to Y$, if both X, Y are finite colimits of cubes or simplices, then k-truncated f-local objects are closed under sequential colimits for all k.*

Proof. The proof is similar to Proposition 8.16, by using Lemma 8.18 and realising that finite limits commutes with sequential colimits. □

Example 8.21. As mentioned in Section 7, $\omega = \varinjlim_{n:\mathbb{N}} \Delta^n$ is the sequential colimit of the simplices. Hence, assuming (NT), (PF), and (SQCF), ω is also a poset. In fact, by Theorem 7.4, ω will also be Phoa.

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