SYNTHETIC CATEGORIES AND SYNTHETIC DOMAINS WITH CLASSIFYING TOPOI

1. Axioms for the presheaf geometry

We work in a type theory where we can develop a synthetic theory for duality of finitely presented lattice-like algebras. We work with the theory of distributive lattices. Thus, we will assume we have a generic model in our type theory

$$(\mathbb{I} \in \mathsf{Set}, 1, \wedge, \cdots)$$

which satisfies finite version of quasi-coherence:

Axiom 1 (QC)**.** For any f.p. $\mathbb{I}$-algebra A, the canonical map is an equivalence of $\mathbb{I}$-algebras,

$$A \to \mathbb{I}^{\mathrm{Spec}\, A},$$

where $\mathrm{Spec}\, A$ is internally defined again as follows,

$$\mathrm{Spec}\, A :\equiv \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}).$$

Notice that Spec is a contravariant functor. (QC) in fact implies the following more general duality result:

Proposition 1.1 (QC)**.** *For any f.p.* $\mathbb{I}$*-algebras* A, B*, we have*

$$\mathbb{I}\text{-}\mathbf{Alg}(B, A) = \mathrm{Spec}\, B^{\mathrm{Spec}\, A}.$$

Proof. We have the following equivalences,

$$\mathbb{I}\text{-}\mathbf{Alg}(A, B) = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}^{\mathrm{Spec}\, B}) = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I})^{\mathrm{Spec}\, B} = \mathrm{Spec}\, A^{\mathrm{Spec}\, B}. \qquad \square$$

Definition 1.2. For any type X, we define

$$\mathsf{isAff}(X) :\equiv \sum_{A : \mathbb{I}\text{-}\mathbf{Alg}_{\mathrm{f.p.}}} X = \mathrm{Spec}\, A.$$

If $\mathsf{isAff}(X)$ holds, we say X is an *affine space.*

By (QC), if X is affine, then the f.p. $\mathbb{I}$-algebra A such that $X = \mathrm{Spec}\, A$ is unique, with $A = \mathbb{I}^X$. Hence, being affine is a proposition.

Lemma 1.3. *Any affine space* X *is a set.*

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Proof. Let $X = \operatorname{Spec} A$. By function extensionality, given $x, y : X$,

$$(x = y) = \prod_{a : A} xa = ya.$$

Since $\mathbb{I}$ is a set, this is a proposition, thus X is again a set. □

We give some examples of affine spaces. First, all the cubes are affine:

Example 1.4. For any $n : \mathbb{N}$, $\mathbb{I}^n$ is affine,

$$\operatorname{Spec} \mathbb{I}[n] = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}[n], \mathbb{I}) = \mathbb{I}^n.$$

In particular, this means that $1 = \mathbb{I}^0$ and $\mathbb{I} = \mathbb{I}^1$ are affine.

As another example, we can also construct the simplices as affine spaces:

Example 1.5. Let $\Delta^n \subseteq \mathbb{I}^n$ be the following subtype,

$$\Delta^n := \{ x_1, \cdots, x_n : \mathbb{I} \mid x_1 \geq \cdots \geq x_n \}.$$

This type is indeed affine, since it by definition we have

$$\Delta^n = \operatorname{Spec} \mathbb{I}[x_1, \cdots, x_n]/x_1 \geq \cdots \geq x_n.$$

One important consequence of (QC) is that the spectrum of a f.p. $\mathbb{I}$-algebra always "have enough points" to detect equalities between elements in the algebra:

Proposition 1.6 (QC)**.** *For $A : \mathbb{I}\text{-}\mathbf{Alg}_{\text{f.p.}}$ and $a, b : n \to A$,*

$$a = b \Leftrightarrow \forall x \colon \operatorname{Spec} A.\ xa = xb.$$

Proof. The $\Rightarrow$ direction is trivial. For $\Leftarrow$, by the universal property of the quotient $A/a = b$, we have

$$\operatorname{Spec}(A/a = b) = \{ x : \operatorname{Spec} A \mid xa = xb \} \subseteq \operatorname{Spec} A.$$

If $\forall x \colon \operatorname{Spec} A.\ xa = xb$ holds, then $\operatorname{Spec}(A/a = b) = \operatorname{Spec} A$, which by (QC) we have $(A/a = b) = A$, which means $a = b$ holds in A. □

To account for the fact that in the presheaf topos $\mathbf{Set}[\mathbb{T}]$ the generic model is representable, we also assume the following choice principle, which we denote as *presheaf-choice*:

Axiom 2 (PC)**.** Let X be affine and let $B : X \to \mathsf{Type}$ be a family of types. Then we have an element of the following type,

$$\mathsf{pc} : \prod_{x : X} \|B(x)\| \to \|\prod_{x : X} B(x)\|.$$

Remark 1.7. In fact, the above axiom is quite strong, which highlights the fact that representables preserves *all* colimits, thus the spectrums for f.p. $\mathbb{I}$-algebras will be supercompact. Later for other applications in mind, we might weaken the choice principle to account for working in a subtopos of $\mathbf{Set}[\mathbb{T}]$ (cf. Section 6).

The aim of this chapter is to show that, starting from the above two simple axioms, we can derive a lot of structures for the classifying topos $\mathbf{Set}[\mathbb{T}]$ synthetically.

2. Elementary Properties for the Interval

The constant $1 : \mathbb{I}$ induces a predicate

$$\mathsf{t} : \mathbb{I} \to \mathsf{Prop},$$

which takes $i : \mathbb{I}$ to $i = 1$. The first observation is that this map is always an embedding since we have assumed that $\mathbb{T}$ is meet-distributive:

Proposition 2.1 (QC)**.** *$\mathbb{I}$ is conservative,*

$$\forall x, y{:}\mathbb{I}.\ (\mathsf{t}x \leftrightarrow \mathsf{t}y) \leftrightarrow x = y.$$

Proof. It suffices to show

$$\forall x, y{:}\mathbb{I}.\ (\mathsf{t}x \to \mathsf{t}y) \to x \leq y.$$

Take x, y with $\mathsf{t}x \to \mathsf{t}y$. Consider the f.p. $\mathbb{I}$-algebra $\mathbb{I}/x = 1$. By construction, we have

$$\mathrm{Spec}(\mathbb{I}/x = 1) = \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}/x = 1, \mathbb{I}) = \mathsf{t}x,$$

Hence, by (QC), if $\mathsf{t}x \to \mathsf{t}y$, we get a restriction map

$$(\mathbb{I}/y = 1) = \mathbb{I}^{\mathsf{t}y} \to \mathbb{I}^{\mathsf{t}x} = (\mathbb{I}/x = 1).$$

Now by the universal property of the quotient, this map exists iff $x \leq y$, which is the desired result. $\square$

Equipped with (PC), we may show that $\mathbb{I}$ is internally *supercompact.* For any type X and a family of subtypes $U_i : X \to \mathsf{Prop}$ indexed by another type I, we say $\{ U_i \}_{i \in I}$ is a *cover* of X if

$$\forall x{:}X.\ \exists i{:}I.\ U_i(x).$$

Proposition 2.2 (PC)**.** *For any affine X, and a cover $\{ U_i \}_i$ of a type Y. Then the canonical map*

$$\left(\sum_{i:I} U_i \right)^X \to Y^X$$

is surjective.

Proof. Suppose we have a map $f : X \to Y$. The cover $\{ U_i \}_{i:I}$ induces a cover on X, where $\prod_{x:X} \| \sum_{i:I} U_i(fx) \|$. Hence by (PC), $\| \prod_{x:X} \sum_{i:I} U_i(fx) \|$ holds. □

As a consequence, each spectrum is internally connected:

Corollary 2.3 (PC)**.** *For any affine X and any $\varphi : X \to \mathsf{Prop}$,*

$$\forall x{:}X.\ \varphi(x) \vee \neg\varphi(x) \to (\forall x{:}X.\ \varphi(x)) \vee (\forall x{:}X.\ \neg\varphi(x)).$$

Proof. This follows from the fact that by assumption, φ and $\neg\varphi$ is a cover of X, which by Proposition 2.2 it merely holds that one of them already covers X. □

3. Exactness Properties of the Category of Affine Spaces

The first easy consequence is that, since f.p. $\mathbb{I}$-algebras are closed under finite colimits, affine spaces would be closed under finite limits. Indeed, 1 is affine, and we can show the following result:

Proposition 3.1 (QC)**.** *Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$ be affine. Then the pullback of $f : X \to Y$ and $g : Z \to Y$ is also affine,*

$$X \times_Y Z = \operatorname{Spec}(A \otimes_B C),$$

where $A \otimes_B C$ is the following pushout,

$$\begin{tikzcd} B \arrow[r, "f^*"] \arrow[d, "g^*"'] & A \arrow[d] \\ C \arrow[r] & A \otimes_B C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Proof. Recall the pullback is defined as follows,

$$X \times_Y Z = \{ x : X, z : Z \mid fx = gz \}.$$

This is indeed a subtype since X, Y, Z are affine, hence in particular sets. In particular, an element in $X \times_Y Z$ are exactly two maps $x : A \to \mathbb{I}$, $z : C \to \mathbb{I}$, where $xf^* = zg^*$. By universal property of the pushout $A \otimes_B C$, this exactly corresponds to a map $(x, z) : A \otimes_B C \to \mathbb{I}$. □

The above result provides an extremely easy formula to compute the coproducts of f.p. $\mathbb{I}$-algebras internally:

Corollary 3.2 (QC)**.** *For any f.p. $\mathbb{I}$-algebra A, B, $A \otimes B = B^{\operatorname{Spec} A}$.*

Proof. By (QC) and Proposition 3.1,

$$A \otimes B = \mathbb{I}^{\operatorname{Spec}(A \otimes B)} = \mathbb{I}^{\operatorname{Spec} A \times \operatorname{Spec} B} = B^{\operatorname{Spec} A}.$$ □

Concretely, the equivalence takes any element c in $A \otimes B$ to the map

$$\lambda x\colon \operatorname{Spec} A.\, (x \otimes B)(c) \,:\, A \otimes B \to B^{\operatorname{Spec} A}.$$

Under this identification, for the left inclusion $A \to A \otimes B$, the induced map $A \to B^{\operatorname{Spec} A}$ is give by

$$a \mapsto \lambda x\colon \operatorname{Spec} A.\, (x \otimes B)(a \otimes 1) = i_B x a \,:\, B^{\operatorname{Spec} A},$$

where $i_B : \mathbb{I} \to B$ is its structure map. For the right inclusion $B \to A \otimes B = B^{\operatorname{Spec} A}$, it is simply given by

$$b \mapsto \lambda x\colon \operatorname{Spec} A.\, (x \otimes B)(1 \otimes b) = b \,:\, B^{\operatorname{Spec} A},$$

which is the constant function on B.

With the above calculation, we can prove an exactness property of the category $\mathbb{I}$-$\mathbf{Alg}_{\text{f.p.}}$. For any f.p. $\mathbb{I}$-algebra, we define the following notion:

Definition 3.3. For $A : \mathbb{I}\text{-}\mathbf{Alg}_{\text{f.p.}}$, we say A is *faithful*, if $i_A : \mathbb{I} \to A$ is an embedding.

Intuitively, i_A being an embedding says that A is non-trivial. Using the above fact, we can prove the following exactness property:

Theorem 3.4 (QC)**.** *For any $A : \mathbb{I}\text{-}\mathbf{Alg}_{\text{f.p.}}$, A is faithful iff for all $B : \mathbb{I}\text{-}\mathbf{Alg}_{\text{f.p.}}$, $B \to A \otimes B$ is an embedding.*

Proof. For the if direction, take B to be $\mathbb{I}$. Then $i_A : \mathbb{I} \to A \otimes \mathbb{I} = A$ is an embedding. For the other direction, take any $B : \mathbb{I}\text{-}\mathbf{Alg}_{\text{f.p.}}$. By the previous calculation, the map $B \to A \otimes B = A^{\operatorname{Spec} B}$ is given by

$$b \mapsto \lambda x\colon \operatorname{Spec} B.\, i_A x b \,:\, B \to A^{\operatorname{Spec} B}.$$

For this map to be an embedding, it suffices to show for all $a, b : B$,

$$\forall x\colon \operatorname{Spec} B.\, i_A x a = i_A x b \to a = b,$$

But this follows from i_A being an embedding, and Proposition 1.6. □

Remark 3.5. Now if we look at $A \otimes B$ as $B^{\operatorname{Spec} A}$, Theorem 3.4 says that for any f.p. $\mathbb{I}$-algebra B, if A is faithful, then the constant map

$$c : B \to B^{\operatorname{Spec} A}$$

is an embedding. This is saying that $\operatorname{Spec} A$ is in some sense weakly inhabited, though it is *not* in general true that in this case we have $\|\operatorname{Spec} A\|$.

Remark 3.6. Externalising the above result, it in particular gives a constructive proof that for such a theory $\mathbb{T}$, the pushouts in $\mathbb{T}$-$\mathbf{Mod}_{\text{f.p.}}$ preserves monomorphisms.

4. Open Subtypes and Dominance

Definition 4.1. Given a proposition p, we say it is *open* if p is merely of the form $\mathsf{t}x$ for some x in $\mathbb{I}$.

Since by Proposition 2.1 $\mathbb{I}$ is conservative, which implies if a proposition p is open, then the x in $\mathbb{I}$ that $p = \mathsf{t}x$ is unique. Hence, we can identify $\mathbb{I}$ itself as the subset of open propositions via the embedding

$$\mathsf{t} : \mathbb{I} \rightarrowtail \mathsf{Prop}.$$

More generally, we may define an open subtype:

Definition 4.2. A subtype U on X is open, if for all $x : X$ the proposition $U(x)$ is open.

For any f.p. $\mathbb{I}$-algebra A and $a : A$, we may introduce *standard opens* D_a of $\operatorname{Spec} A$ as the following subset,

$$D_a := \lambda x \colon \operatorname{Spec} A.\, \mathsf{t}(xa) : \operatorname{Spec} A \to \mathsf{Prop}.$$

More generally, we say a subset U of $\operatorname{Spec} A$ is a standard open, if there merely exists an element a that $U = D_a$ as subsets,

$$\mathsf{isStOpen}(U) := \exists a \colon A.\, \forall x \colon \operatorname{Spec} A.\, U(x) = D_a(x) = \mathsf{t}(xa).$$

Standard opens are themselves affine open subspaces:

Proposition 4.3. *Any standard open U of an affine space* $\operatorname{Spec} A$ *is itself affine, and an open subspace of* $\operatorname{Spec} A$.

Proof. Since being affine is a proposition, we can assume an $a : A$ that $U = D_a$. Now D_a is affine,

$$D_a = \sum_{x : \operatorname{Spec} A} \mathsf{t}(xa) = \mathbb{I}\text{-}\mathbf{Alg}(A/a = 1, \mathbb{I}) = \operatorname{Spec}(A/a = 1).$$

The second equivalence holds by the universal property of $A/a = 1$. □

Using (PC), we can first show that for affine spaces, an open subset coincide with a standard open:

Lemma 4.4 (PC). *Let $X = \operatorname{Spec} A$ be an affine space. Let U be an open subset of X, then U is a standard open of X.*

Proof. Let $L : \operatorname{Spec} A \to \mathsf{Type}$ be the following type family,

$$L(x) = \sum_{a : A} U(x) = \mathsf{t}a.$$

By assumption that U is open, we have an element of $\prod_{x:\mathrm{Spec}\,A} \|L(x)\|$, which by (PC), it means $\prod_{x:\mathrm{Spec}\,A} L(x)$ is inhabited. Since U being a standard open is a proposition, we can assume a section $s : \prod_{x:\mathrm{Spec}\,A} L(x)$. Equivalently, we can view s as a function

$$s : \mathrm{Spec}\,A \to A,$$

with $U(x) = \mathrm{t}s(x)$ for all $x : X$. This way, it follows that $D_s = U$, hence U is standard open. □

Proposition 4.5 (PC)**.** *Open subsets are transitive on affine spaces, i.e. if X is affine with $U \subseteq X$ open and $V \subseteq U$ open, then $V \subseteq X$ is also open.*

Proof. According to Lemma 4.4 it suffices to prove it for affine opens of X. Given Proposition 4.3, if $U = D_a \subseteq X = \mathrm{Spec}\,A$ and furthermore $V = D_b \subseteq U = \mathrm{Spec}(A/a = 1)$, then we can view b as an element in A with $b \leq a$ by meet-distributivity. Thus, $V = D_b \subseteq X$ is open. □

Corollary 4.6 (QC, PC)**.** *The type of opens $\mathbb{I}$ forms a* dominance.

Proof. Apply Proposition 4.5 to the unit type $1 = \mathrm{Spec}\,\mathbb{I}$. □

Notice that with the dominance structure on $\mathbb{I}$ established, the conclusion of Proposition 4.5 now will in fact apply to *any* type X whatsoever, not just affine ones, since we can show that the transitivity on 1 is a *generic* composition of open subtypes. In fact, open partial maps will be classified, as shown in the following section.

5. Lifting

Given then dominance $\mathbb{I}$, we can construct internally the lifting functor. For any type X, its lift is given by

$$X_\perp := \sum_{i:\mathbb{I}} X^{\mathrm{t}i}.$$

The functoriality is easy to express: For any $f : X \to Y$, we have

$$f_\perp(i, x) := (i, \lambda w{:}\mathrm{t}i.\ f x w).$$

There is an evident unit $\eta : X \to X_\perp$, where

$$\eta := \lambda x{:}X.\ (\top, \lambda\text{-}{:}1.\ x).$$

Notice that even without the dominance structure on $\mathbb{I}$ we can still perform the above construction. However, $\mathbb{I}$ being a dominance implies that the lifting also has a multiplication structure $\mu : (X_\perp)_\perp \to X_\perp$, where μ takes any (i, u) with $i : \mathbb{I}$ and $u : \mathrm{t}i \to X_\perp$ first to the dependent sum

$$j := \sum_{w:\mathrm{t}i} (uw)_0 : \mathbb{I},$$

and a partial element $x : \mathsf{t}j \to X$, such that for $w : \mathsf{t}i$ and $v : (uw)_0$

$$x(w, v) := (uw)_1(v).$$

Example 5.1. By definition, it is easy to see that

$$1_\perp = \sum_{i:\mathbb{I}} \mathsf{t}i = \mathbb{I}.$$

For the synthetic theory of domains, the object of particular importance is the lift $\mathbb{I}_\perp$. In fact, we can compute the lift for any affine space $X = \operatorname{Spec} A$ using (QC):

$$X_\perp = \sum_{i:\mathbb{I}} X^{\mathsf{t}i} = \sum_{i:\mathbb{I}} \operatorname{Spec} A^{\operatorname{Spec} I/i=\top} = \sum_{i:\mathbb{I}} \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}/i = \top).$$

For instance, for the affine space $\mathbb{I} = \operatorname{Spec} \mathbb{I}[x]$ we have

$$\mathbb{I}_\perp = \sum_{i:\mathbb{I}} \mathbb{I}/i = \top = \sum_{i:\mathbb{I}} \sum_{j:\mathbb{I}} i \geq j = \Delta^2,$$

which implies that $\mathbb{I}_\perp$ also classifies the order on $\mathbb{I}$, i.e. $\mathbb{I}_\perp = \mathbb{I}^{\mathbb{I}} = \mathbb{I}[x]$. More generally, we have:

Lemma 5.2 (QC)**.** *For any $n : \mathbb{N}$, we have*

$$\Delta^n_\perp = \Delta^{n+1}.$$

Proof. Recall from Example 1.5 that Δ^n is affine,

$$\Delta^n = \operatorname{Spec} \mathbb{I}[x_1, \cdots, x_n]/x_1 \geq \cdots \geq x_n.$$

This way, we have

$$\begin{aligned}
\Delta^n_\perp &= \sum_{i:\mathbb{I}} \mathbb{I}\text{-}\mathbf{Alg}(\mathbb{I}[x_1, \cdots, x_n]/x_1 \geq \cdots \geq x_n, \mathbb{I}/i = \top) \\
&= \{\, i, x_1, \cdots, x_n : \mathbb{I} \mid i \geq x_1 \geq \cdots \geq x_n \,\} \\
&= \Delta^{n+1}
\end{aligned}$$

□

In particular, from a geometric perspective, the inclusion

$$\Delta^n \rightarrowtail \Delta^n_\perp = \Delta^{n+1}$$

takes $i_1 \geq \cdots \geq i_n$ in Δ^n to $1 \geq i_1 \geq \cdots \geq i_n$ in Δ^{n+1}.

6. Nullstellensatz and Local Geometry

At the end of this chapter, let us briefly discuss the situation when the theory $\mathbb{T}$ has slightly more structure. For this, we assume $\mathbb{T}$ extends $\mathbb{D}$, the theory of distributive lattices. In particular, it has a further constant 0 and a binary join $\vee$, which distributes over $\wedge$. For instance, the theory of distributive lattices $\mathbb{D}$, of de Morgan algebra $\mathbb{dM}$, of Heyting algebra $\mathbb{H}$, and of Boolean algebra $\mathbb{B}$ are all of this type.

In this case, we may further assume that the generic interval is non-trivial:

Axiom 3 (NT)**.** For $\mathbb{I}$, $0 \neq 1$.

In this case, we are working in a non-trivial subtopos of $\mathbf{Set}[\mathbb{T}]$. Note that the choice principle (PC) as stated Axiom 2 will no longer be adequate due to the non-trivial topology. However, it is not far from being true. We will replace it with the following choice principle for *merely inhabited* affine spaces:

Axiom 4 (NPC)**.** Let X be a merely inhabited affine space. Let $B : X \to \mathsf{Type}$ be a family of types. Then we have an element of the following type,

$$\mathsf{npc} : \prod_{x:X} \|B(x)\| \to \|\prod_{x:X} B(x)\|.$$

Note that if we restrict previous results obtained by applying (PC) to merely inhabited affine spaces, the same would still holds by (NPC). In particular, recall the proof of Corollary 4.6 that $\mathbb{I}$ is a dominance. We only need to apply the choice principle to the affine space $\mathbb{I}$, which is inhabited. Hence, the same result still holds. Similar comments apply to other places where we have applied (PC), e.g. Proposition 2.2 or Corollary 2.3.

In particular, now $\varnothing$ is affine,

$$\varnothing = \operatorname{Spec} \mathbb{I}/0 = 1.$$

For any $\mathbb{I}$-algebra A, we say A is *trivial* if $0 = 1$ in A. In the case of (NT), we have the following Nullstellensatz result:

Proposition 6.1 (QC, NT)**.** *For any affine* $\operatorname{Spec} A$*,* $\operatorname{Spec} A = \varnothing$ *iff* A *is trivial.*

Proof. This follows from $\varnothing$ is affine, thus if $\operatorname{Spec} A = \varnothing$, then

$$A = \mathbb{I}^{\operatorname{Spec} A} = \mathbb{I}^{\varnothing} = 0.$$ □

In fact, much more is true when (NT) and (QC) are combined together. Since the meet and join structure on $\mathbb{I}$ are dual to each other, Proposition 2.1 still applies to

For instance, $\mathbb{I}$ satisfies the following field axiom as a consequence of Nullstellensatz:

Corollary 6.2. $\mathbb{I}$ *is a field in the sense that*

$$\forall x : \mathbb{I}.\ \neg \mathsf{t}x \leftrightarrow \mathsf{f}x,$$

and vice versa by exchanging t *and* f*. In particular,* $\mathbb{I}$ *is not not* 2*,*

$$\forall x : \mathbb{I}.\ \neg\neg(\mathsf{t}x \vee \mathsf{f}x).$$

Proof. By conservativity in Proposition 2.1 and (NT), if $x \neq \top$ then $x = \bot$ since $\bot \neq \top$. Now suppose $\neg(\mathsf{t}x \vee \mathsf{f}x)$, then $\neg\mathsf{t}x \wedge \neg\mathsf{f}x$, which equals $\mathsf{f}x \wedge \mathsf{t}x$, contradictory. Thus, $\neg\neg(\mathsf{t}x \vee \mathsf{f}x)$ always holds. □

This also implies that open propositions are $\neg\neg$-closed:

Lemma 6.3. *For any $p : \mathsf{Prop}$, p is open iff $\neg p$ is closed and vice versa. Furthermore, open and closed propositions are $\neg\neg$-closed.*

Proof. This follows from Corollary 6.2. □

7. Intrinsic Order and Phoa's Principle

For synthetic domain theory, we work with the meet-distributive theory $\mathbb{D}$ of distributive lattices. In particular, now we will assume the generic interval $\mathbb{I}$ has furthermore 0 and $\vee$, which distributes over 1 and $\wedge$.

We will see in the next section that the algebraic properties of distributive lattices has important consequences for the internal logic.

Definition 7.1. The *intrinsic order* on a type X is defined as follows:

$$x \preceq y \ := \forall U : X \to \mathbb{I}.\ U(x) \leq U(y).$$

One important property of intrinsic order is that every map is monotone w.r.t. this order:

Proposition 7.2. *For $f : X \to Y$, $x \preceq y$ in X implies $fx \preceq fy$ in Y.*

Proof. This simply follows from compositionality of functions. □

As a first example, we will show that the intrinsic order on $\mathbb{I}$ coincide with its canonical order. In fact this holds for all f.p. $\mathbb{I}$-algebras due to the following Phoa's principle:

Proposition 7.3 (Phoa's Principle)**.** *For any f.p. $\mathbb{I}$-algebra A, the boundary $\partial : A^{\mathbb{I}} \to A \times A$ is equivalent to $\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : A[x] \to A \times A$, which classifies the order on A.*

Proof. By Corollary 3.2 we have

$$A^{\mathbb{I}} = A \otimes \mathbb{I}[x] = A[x].$$

Hence, it suffices to show that the two projection

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : A[x] \to A \times A$$

is equivalent to the order $\leq$. It factors through it, since $\mathrm{ev}_0(p) \leq \mathrm{ev}_1(p)$ for any polynomial $p \in A[x]$. On the other hand, recall for distributive lattices we have a normal form for its polynomials given by Proposition A.1,

$$p = \mathrm{ev}_1(p) \wedge x \vee \mathrm{ev}_0(p).$$

This proves the factorisation will be an equivalence. □

Remark 7.4. In some sense, the above result really highlights the special property of distributive lattices. This is a perfect example of how algebraic facts affects the internal logic of its classifying topos.

For instance, we have $\mathbb{I}^{\mathbb{I}} = \mathbb{I}[x] = \Delta^2$. Also, following the above proof, the evaluation $\mathbb{I} \times A^{\mathbb{I}} \to A$, under the equivalence $A^{\mathbb{I}} = \{\, a, b \,:\, A \mid a \leq b \,\}$, is given as follows,

$$(i, a \geq b) \mapsto a \wedge i \vee b.$$

Notice we have seen in Example 1.5 that Δ^2 is *affine*, but here it also has an algebraic structure $\mathbb{I}[x]$. This dual identity generalises to all simplices:

Proposition 7.5. *For any $n : \mathbb{N}$, the evaluation map*

$$\mathbb{I}[x_1, \cdots, x_n]/x_1 \leq \cdots \leq x_n \to \Delta^{n+1}$$

which takes $p : \mathbb{I}[x_1, \cdots, x_n]/x_1 \leq \cdots \leq x_n$ to

$$(p(1, \cdots, 1), p(0, 1, \cdots, 1), \cdots, p(0, \cdots, 0)) : \Delta^{n+1}$$

is an equivalence.

Proof. This again follows from the normal form for $A[x, y]/x \leq y$ given in Proposition A.3. □

From this algebraic perspective, the inclusion $\Delta^n \rightarrowtail \Delta^n_\perp = \Delta^{n+1}$ corresponds to the map

$$\mathbb{I}[x_1, \cdots, x_n]/x_1 \leq \cdots \leq x_n \to \mathbb{I}[x_0, \cdots, x_n]/x_0 \leq \cdots \leq x_n$$

which takes x_i to $x_0 \vee x_i$.

As a corollary of Phoa's principle, the intrinsic order coincide with the point-wise order for a wide range of types:

Corollary 7.6. *For any type X, the intrinsic order on $\mathbb{I}^X$ coincides with the point-wise induced order on $\mathbb{I}$.*

Proof. Given $f, g : \mathbb{I}^X$, suppose $f \preceq g$. For $x : X$ there is an evaluation function

$$\mathrm{ev}_x : \mathbb{I}^X \to \mathbb{I},$$

and by assumption $f(x) = \mathrm{ev}_x(f) \leq \mathrm{ev}_x(g) = g(x)$, which implies $f \leq g$ for the point-wise order.

Now suppose $\forall x{:}X.\ fx \leq gx$. Consider any $U : \mathbb{I}^X \to \mathbb{I}$. By Phoa's principle, we get a map

$$[f, g] : \mathbb{I} \to \mathbb{I}^X,$$

where its transpose $X \to \mathbb{I}^{\mathbb{I}}$ takes any $x : X$ to (fx, gx) in $\mathbb{I}^{\mathbb{I}} = \Delta^2$. In particular, by definition

$$f = [f, g](0), \quad g = [f, g](1).$$

Then it follows that the composite $U[f,g] : \mathbb{I} \to \mathbb{I}$ satisfies

$$Uf = U[f,g](0) \leq U[f,g](1) = Ug,$$

which implies $f \preceq g$. $\square$

This can be particularly applied to any f.p. $\mathbb{I}$-algebra A, since $A = \mathbb{I}^{\operatorname{Spec} A}$ as an $\mathbb{I}$-algebra, it follows that the intrinsic order on A also coincides with its canonical order. On the other hand, we can also generalise the Phoa's principle to affine space:

Corollary 7.7. *For affine $X = \operatorname{Spec} A$, the intrinsic order on X is represented by $X^{\mathbb{I}}$, which coincide with the point-wise order on $X = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I})$.*

Proof. By (QC), recall $\mathbb{I}^X = A$, thus for any $x, y : X$,

$$x \preceq y \Leftrightarrow \forall a{:}A.\ x(a) \leq y(a),$$

which is exactly the point-wise order on X. $\square$

8. Markov's Principle

Equipped with (CQC), we also have the Markov principle:

Lemma 8.1 (Markov Principle)**.**

$$\forall x{:}\overline{\omega}.\ \left(\neg\forall n{:}\mathbb{N}.\ \mathsf{t}x_n \to \sum_{n:\mathbb{N}} \mathsf{f}x_n\right).$$

Proof. Let $x : \overline{\omega}$. If $\neg\forall n{:}\mathbb{N}.\ \mathsf{t}x_n$, then

$$\operatorname{Spec}_\infty \mathbb{I}/\bigwedge_{n:\mathbb{N}} x_n = \top = \varnothing,$$

which implies $\mathbb{I}/\bigwedge_{n:\mathbb{N}} x_n = \top$ is trivial by Proposition 6.1. This way, there exists n that $\mathsf{f}x_n$, and we can take n to be the least one. $\square$

Now let ω be the initial algebra for the lifting functor L. As a consequence, we always have ω as the internal colimit of Δ^n:

Corollary 8.2. *ω can be viewed internally as a subtype of $\overline{\omega}$ as follows,*

$$\omega = \{\, x : \overline{\omega} \mid \exists n{:}\mathbb{N}.\ \mathsf{f}x_n \,\}.$$

Proof. Following [3], we need to show that for any $x : \overline{\omega}$,

$$(\forall \phi{:}\mathsf{Prop}.\ (\forall n{:}\mathbb{N}.\ (\mathsf{t}x_n \to \phi) \to \phi) \to \phi) \to \exists n{:}\mathbb{N}.\ \mathsf{f}x_n.$$

But this follows from Lemma 6.3 and 8.1 by taking ϕ to be $\varnothing$. $\square$

Theorem 8.3. *$\mathbb{I}$ is complete, i.e. the canonical map $\mathbb{I}^{\overline{\omega}} \to \mathbb{I}^{\omega}$ is an equivalence.*

Proof. Since $\overline{\omega}$ is affine, we now have

$$\mathbb{I}^{\overline{\omega}} = \mathbb{I}[\mathbb{N}]/\bigwedge_{n:\mathbb{N}} x_n \geq x_{n+1}.$$

On the other hand, since ω is internally the colimit of Δ^n, we have

$$\mathbb{I}^{\omega} = \varprojlim_{n:\mathbb{N}} \mathbb{I}^{\Delta^n} = \varprojlim_{n:\mathbb{N}} \mathbb{I}[x_1, \cdots, x_n]/x_1 \geq \cdots \geq x_n$$

Note that the transition map is given by the canonical inclusion

$$\mathbb{I}[x_1, \cdots, x_n]/x_1 \geq \cdots \geq x_n \to \mathbb{I}[x_1, \cdots, x_{n+1}]/x_1 \geq \cdots \geq x_{n+1},$$

which implies that we indeed have

$$\mathbb{I}^{\omega} = \mathbb{I}[\mathbb{N}]/\bigwedge_{n:\mathbb{N}} x_n \geq x_{n+1} = \mathbb{I}^{\overline{\omega}}.$$ □

9. Synthetic Posets

In type theory we can also define general shapes of boundaries $\partial\Delta^n$ and horns Λ^n_k are definable. For instance, the horn Λ^2_1 will be constructed as the following pushout,

$$\begin{tikzcd} 1 \arrow[r, "\top"] \arrow[d, "\bot"'] & \mathbb{I} \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Lambda^2_1 \end{tikzcd}$$

This gives us an embedding $\Lambda^2_1 \to \Delta^2$, which we can identify with

$$\Lambda^2_1 = \{\, j \geq i \,:\, \mathbb{I} \mid \mathsf{t}j \vee \mathsf{f}i \,\}.$$

A more complex construction is the walking equivalence, which we will denote as E.

The notion of synthetic posets is formulated internally as certain orthogonality conditions:

Definition 9.1. For any type X, we say it is

- $\mathbb{I}$-*separated*, if $X^{\mathbb{I}} \to X^2$ is an embedding;
- *Segal*, if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.
- *Rezk*, if $X \to X^E$ is an equivalence.

X is a (synthetic) category, if X Segal and Rezk. X is a (synthetic) poset, if it is furthermore $\mathbb{I}$-separated. These are propositions.

For $\mathbb{I}$-separatedness, it is in fact equivalently to being separated for the double negation topology, at least for sets:

Proposition 9.2. *A set X is $\mathbb{I}$-separated iff it is separated, i.e. for any x, y, $x = y$ is $\neg\neg$-closed.*

Proof. By Corollary 6.2 the inclusion $\mathbf{2} \rightarrowtail \mathbb{I}$ is $\neg\neg$-dense, thus being separated implies being $\mathbb{I}$-separated. On the other hand, $\square$

Furthermore, from the above definition, it immediately follows that posets are closed under limits and retracts, and in fact forms an exponential ideal. For a non-trivial example, let us first show that $\mathbb{I}$ is a poset:

Lemma 9.3. $\mathbb{I}$ *is a poset.*

Proof. By Phoa's principle, $\mathbb{I}$ is $\mathbb{I}$-separated. To show it is segal, consider a map $[f, g] : \Lambda^2_1 \to \mathbb{I}$, which is equivalently two maps $f : \mathbb{I} \to \mathbb{I}$ and $g : \mathbb{I} \to \mathbb{I}$ with $f(\bot) = g(\top)$. By Phoa's principle again, this is equivalently a sequence $g(\bot) \leq g(\top) = f(\bot) \leq f(\top)$. Now the pair $g(\bot) \leq f(\top)$ defines a map $\Delta^2 \to \mathbb{I}$, which is easily seen to be unique. $\square$

Corollary 9.4. *All the simplices* Δ^n *are posets.*

Proof. They are retracts of cubes $\mathbb{I}^n$, and since posets form an exponential ideal, $\mathbb{I}^n$ are posets due to Lemma 9.3. $\square$

For another type of objects, all the algebraic objects we care about will be posets, and in fact they also satisfies the Phoa's principle:

Corollary 9.5. *Any f.p.* $\mathbb{I}$*-algebra* A *is a poset, and in fact the canonical order on* A *coincide with* $A^{\mathbb{I}}$*.*

Proof. By (QC), for any f.p. $\mathbb{I}$-algebra A we have $A = \mathbb{I}^{\operatorname{Spec} A}$, which is a poset since they are closed under exponentials. $\square$

However, perhaps the more interesting examples of posets are affine spaces. For instance, $\mathbb{I}$ is the canonical example of an affine space being a poset. In fact, again by (QC), we can show *all* affine spaces are posets:

Proposition 9.6. *Any affine space* $\operatorname{Spec} A$ *is a poset.*

Proof. By the duality given in Proposition 1.1, we have

$$\operatorname{Spec} A^{\mathbb{I}} = \operatorname{Spec} A^{\operatorname{Spec} \mathbb{I}[x]} = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}[x]),$$

and under this equivalence, it is easy to see that the boundary map is now given by

$$\langle \mathrm{ev}_\bot, \mathrm{ev}_\top \rangle : \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}[x]) = \operatorname{Spec} A^{\mathbb{I}} \to \operatorname{Spec} A^2 = \mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I})^2.$$

But then $\operatorname{Spec} A$ being a poset trivially follows from the Phoa's principle of $\mathbb{I}$, since $\mathbb{I}\text{-}\mathbf{Alg}(A, \mathbb{I}[x])$ now exactly classifies maps $f, g : A \to \mathbb{I}$ such that $f \leq g$, because as the order $\mathbb{I}[x] \rightarrowtail \mathbb{I} \times \mathbb{I}$, $\mathbb{I}[x]$ is in fact a *subalgebra.* $\square$

For all the above examples, they are indeed special cases of *schemes*, which arise as glueing of affine spaces along open subsets. Intuitively, since $\mathbb{I}$ is tiny enough to make being a poset a local property, schemes would themselves again be posets.

10. Zariski Geometry and Disjunction

References

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Appendix A. Distributive Lattice

Let $\mathbb{D}$ be the theory of distributive lattices. We use **DL** to denote the category of distributive lattices.

A.1. **Finitely Presented Distributive Lattices.** The theory $\mathbb{D}$ is constructively well-behaved. The first indicator is that free distributive lattices have a normal form theorem:

Proposition A.1. *For a distributive lattice A, every polynomial $p : A[x]$ has a normal form,*

$$p = \mathrm{ev}_\bot(p) \vee x \wedge \mathrm{ev}_\top(p).$$

In particular, for any $p, q : A[x]$,

$$p \le q \Leftrightarrow \mathrm{ev}_\bot(p) \le \mathrm{ev}_\bot(q) \;\&\; \mathrm{ev}_\top(p) \le \mathrm{ev}_\top(q).$$

Proof. See [1, Thm. 10.11]. □

Proposition A.2. *Finitely presentable $\mathbb{D}$-models are finite.*

Proof. We already know all finitely generated free distributive lattices are finite. Since a general finitely presentable model is a finite quotient of a finite algebra, it will still be finite. □

In fact, to show the above result, we can provide more explicit computation of how to do quotient:

Proposition A.3. *For a distributive lattice A, $p : A[x, y]/x \le y$ also has a normal form*

$$p = \mathrm{ev}_{\bot,\bot}(p) \vee x \wedge \mathrm{ev}_{\bot,\top}(p) \vee y \wedge \mathrm{ev}_{\top,\top}(p).$$

Proof. This can be directly proved from Proposition A.1. □

Theorem A.4. *The category $\mathbf{DL}_{\text{f.p.}}$ is dually equivalent to $\mathbf{Pos}_{\text{f.p.}}$.*

Proof. Since all finitely presentable distributive lattice is finite, proving the duality theorem is constructive. □

A.2. **Quotients.** Another fundamental operation is to form quotients of a distributive lattice A. One general way of generating a quotient is by considering two family of terms $s, t : X \to A$, such that the quotient algebra, which we denote as $A/_{\langle s,t\rangle}$, satisfies the following universal property

$$\mathbf{DL}(A/_{\langle s,t\rangle}, B) = \{\, f : \mathbf{DL}(A, B) \mid f s = f t \,\}.$$

Notice that by univalence, the equality between functions is characterised by the Π-type,

$$(f s = f t) = \forall x : X.\ f s(x) = f t(x),$$

which is a proposition. Thus, maps out of the quotient $A/_{\langle s,t\rangle}$ can be viewed as a subset of maps out of A.

For distributive lattices, we have some special types of quotients. By a *filter* F on A, we mean a subset $F \subseteq A$ which is closed under finite meets. We say a filter F is *principle*, if there exists $a : A$ with $F = {\uparrow} a$, where ${\uparrow} a$ is the subset $\{\, b : A \mid a \leq b \,\}$. It is easy to see that such a is necessarily unique, thus being principle is a proposition.

Any filter F generates a quotient by taking the two family of maps to be $\iota, \top : F \to A$, where ι is the canonical inclusion and $\top$ is the constant map on $\top$. In other words, we are collapsing elements in the filter F to the top. We will denote this quotient by $A/_F$. The importance about quotients over a filter is that there is a canonical way of representing the generated congruence of the quotient:

Lemma A.5. *The relation R_F on A defined by*

$$R_F(a, b) :\equiv \exists x : F.\ a \wedge x = b \wedge x$$

is a congruence on A which respects the two maps $\iota, \top : F \to A$.

Proof. By construction R_F is an equivalence relation. It is also easy to see that the distributive lattice axioms makes it into a congruence. Furthermore, given any $x : F$, evidently $R_F(x, \top)$, since $x \wedge x = x = \top \wedge x$. $\square$

Proposition A.6. *he quotient $A/_F$ is the quotient of A w.r.t. the congruence R_F. In other words, for the quotient map $q : A \twoheadrightarrow A/_F$ and $a, b : A$,*

$$q(a) = q(b) \Leftrightarrow \exists x : F.\ a \wedge x = b \wedge x.$$

Proof. Right to left is evident. For left to right, since by Lemma A.5 the congruence R_F respects the two maps $\iota, \top : F \to A$, by universal property of the quotient, it must validates this congruence. $\square$

Corollary A.7. *For a principle filter $F = {\uparrow} x$ on A, the quotient $A/_{\uparrow x}$ can be identified as the map*

$$x \wedge - : A \twoheadrightarrow {\downarrow} x.$$

Proof. In the special case of the principle filter $\uparrow x$, it is easy to see that that congruence $R_{\uparrow x}$ specialises to an equality,

$$R_{\uparrow x}(a, b) \Leftrightarrow x \wedge a = x \wedge b.$$

Thus, the congruence $R_{\uparrow x}$ is represented by the idempotent operator $x \wedge -$ on A, which allows us to conclude [2, Lem. 6.10.8]. □

Hence, the nice thing about the quotient $A/_{\uparrow x}$ is that, it can be naturally identified as a *subset* of A, making it into a retract.

Appendix B. Heyting Algebra

Note that a Heyting algebra A is a distributive lattice such that $a \wedge -$ has a right adjoint for any $a : A$. In this section we will use $A[X]$ to denote the Heyting algebra freely generated by X over A, while use $A\langle X\rangle$ to denote the free distributive lattice generated by A.

Lemma B.1. *For any Heyting algebra A, $A\langle n\rangle$ is also a Heyting algebra for any $n : \mathbb{N}$.*

Proof. By induction, it suffices to show that $A\langle x\rangle$ is a Heyting algebra. By the normal form given in Proposition A.1, it suffices to compute the evaluation of $p \to q$ at $0, 1$ for $p, q : A\langle x\rangle$. We claim that

$$\begin{aligned}(p \to q)(0) &= (p(0) \to q(0)) \wedge (p(1) \to q(1))\\ (p \to q)(1) &= p(1) \to q(1)\end{aligned}$$

It is easy to verify that for any $r : A\langle x\rangle$,

$$r \wedge p \leq q \Leftrightarrow r(0) \wedge p(0) \leq q(0) \ \&\ r(1) \wedge p(1) \leq q(1),$$

which by the above definition translates to

$$r(0) \leq (p \to q)(0) \ \&\ r(1) \leq (p \to q)(1).$$

Hence $A\langle x\rangle$ is also a Heyting algebra. □

Lemma B.2. *For any Heyting algebra A and $n : \mathbb{N}$, $A\langle n\rangle$ is a finitely presented Heyting algebra over A.*

Proof. Again by induction, it suffices for $A\langle x\rangle$ to be finitely presented as a Heyting algebra over A. □