Classifying topoi and domain theory

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Contents

Preliminaries

1. Internal category theory

CHAPTER 1

General quasi-coherence

1. The enriched endo profunctors

Let $\mathcal{C}$ be a category. We write $\mathbf{Prof}(\mathcal{C},\mathcal{C})$ as the category of endo profunctors on $\mathcal{C}$. We adopt the convention that subscripts will be *covariant*, and superscripts will be *contravariant*. This way, for the profunctor よ, $よ_c$ will be $\mathcal{C}(-,c)$, and $よ^c$ will be $\mathcal{C}(c,-)$.

One way to view $\mathbf{Prof}(\mathcal{C},\mathcal{C})$ is the functor category $[\mathcal{C},\mathbf{Psh}(\mathcal{C})]$, which we also denote as $\mathbf{Diag}_{\mathcal{C}}$. This notation mainly signifies that we are integrating along the covariant variable,

$$\mathbf{Diag}_{\mathcal{C}}(F,G) \cong \int_{c\in\mathcal{C}} \mathbf{Psh}(\mathcal{C})(F_c,G_c).$$

Now a diagram equivalently can be viewed as an internal diagram for the constant internal category $\Delta\mathcal{C}$ in $\mathbf{Psh}(\mathcal{C})$. We organise this internal perspective into a $\mathbf{Psh}(\mathcal{C})$-*enriched* category $\mathbb{D}\mathbf{iag}_{\mathcal{C}}$ of internal diagrams over $\Delta\mathcal{C}$. Given two diagrams $F,G : \mathcal{C} \to \mathbf{Psh}(\mathcal{C})$, we define

$$\mathbb{D}\mathbf{iag}_{\mathcal{C}}(F,G) := \int_{c\in\mathcal{C}} G_c^{F_c}.$$

In other words, for any $d \in \mathcal{C}$ we have

$$\begin{aligned}\mathbb{D}\mathbf{iag}_{\mathcal{C}}(F,G)^d &\cong \mathbf{Psh}(\mathcal{C})\Big(よ_d, \int_{c\in\mathcal{C}} G_c^{F_c}\Big)\\ &\cong \int_{c\in\mathcal{C}} \mathbf{Psh}(\mathcal{C})(F_c, G_c^{よ_d})\\ &\cong \mathbf{Diag}_{\mathcal{C}}(F, G^{よ_d})\end{aligned}$$

Here $G^{よ_d}$ is the constant power of the diagram G, i.e. $(G^{よ_d})_c \cong G_c^{よ_d}$. In this case, the underlying category of $\mathbb{D}\mathbf{iag}_{\mathcal{C}}$ is $\mathbf{Diag}_{\mathcal{C}}$ itself,

$$\left|\mathbb{D}\mathbf{iag}_{\mathcal{C}}\right| \simeq \mathbf{Diag}_{\mathcal{C}} \simeq \mathbf{Prof}(\mathcal{C},\mathcal{C}).$$

Proposition 1.1. *$\mathbb{D}\mathbf{iag}_{\mathcal{C}}$ is both tensored and powered over $\mathbf{Psh}(\mathcal{C})$.*

Proof. For any $X \in \mathbf{Psh}(\mathcal{C})$, we define $X \times F$ to be the constant tensor,

$$(X\times F)_c \cong X \times F_c.$$

This way, we have

$$\begin{aligned}\mathbb{D}\mathbf{iag}_{\mathcal{C}}(X\times F,G) &\cong \int_{c\in\mathcal{C}} G_c^{X\times F_c}\\ &\cong \int_{c\in\mathcal{C}} (G_c^{F_c})^X\end{aligned}$$

$$\cong \left(\int_{c \in C} G_c^{F_c} \right)^X$$
$$\cong \mathbb{D}\mathbf{iag}_C(F, G)^X$$

The third equivalence has used the fact that $(-)^X$ is a right adjoint, hence preserves end. Completely similarly, we may define the power G^X to be the constant power,

$$(G^X)_c \cong G_c^X.$$

This way, again we have

$$\mathbb{D}\mathbf{iag}_C(F, G^X) \cong \int_{c \in C} (G_c^X)^{F_c} \cong \left(\int_{c \in C} G_c^{F_c} \right)^X \cong \mathbb{D}\mathbf{iag}_C(F, G)^X.$$

This shows $\mathbb{D}\mathbf{iag}_C$ is both tensored and powered by $\mathbf{Psh}(C)$. □

2. The internal theory of flat functors

Let $\mathbb{C}$ denote the theory of flat functors on C. We know that

$$\mathbb{C}\text{-}\mathbf{Mod} \simeq \mathbf{Flat}(C, \mathbf{Set}) \simeq \mathbf{Ind}(C^{\mathrm{op}}).$$

We know $\mathbf{Psh}(C)$ is the classifying topos of $\mathbb{C}$. There are interesting structures in the category of $\mathbb{C}$-models in $\mathbf{Psh}(C)$,

$$\mathbb{C}\text{-}\mathbf{Mod}(\mathbf{Psh}(C)) \simeq \mathbf{Flat}(C, \mathbf{Psh}(C)) \simeq [C^{\mathrm{op}}, \mathbb{C}\text{-}\mathbf{Mod}] \subseteq \mathbf{Prof}(C, C),$$

which is a full subcategory of the endo profunctors on C. We will write the full subcategory $\mathbf{Flat}(C, \mathbf{Psh}(C))$ of $\mathbf{Diag}_C$ as $\mathbf{Flat}_C$. The flatness condition is a condition on the covariant entry for an endo profunctor.

Example 1.2. There is a generic $\mathbb{C}$-model in $\mathbf{Flat}_C$, since $\mathbf{Psh}(C)$ is the classifying topos of $\mathbb{C}$. This is nothing but the Yoneda embedding,

$$よ : C \to \mathbf{Psh}(C).$$

We have constructed a $\mathbf{Psh}(C)$-enriched category $\mathbb{D}\mathbf{iag}_C$ with the underlying category $\mathbf{Diag}_C$. With $\mathbf{Flat}_C$ as a full subcategory, it will also be the underlying category of a full enriched subcategory of $\mathbb{D}\mathbf{iag}_C$, which we denote as $\mathbb{F}\mathbf{lat}_C$.

Proposition 1.3. $\mathbb{F}\mathbf{lat}_C$ *inherits the power of* $\mathbf{Psh}(C)$ *over* $\mathbb{D}\mathbf{iag}_C$.

Proof. Since the power F^X is pointwise on the covariant entry, i.e. $(F^X)_c \cong F_c^X$, and it will be flat if F is, since $(-)^X$ preserves limits. □

The more interesting enriched category is the coslice $よ/\mathbb{F}\mathbf{lat}_C$, whose objects are flat diagrams F with a map $よ \to F$. The enriched hom in $よ/\mathbb{F}\mathbf{lat}_C$ for two flat diagrams F, G with $f : よ \to F$ and $g : よ \to G$ is given by the pullback

$$\begin{tikzcd}
よ/\mathbb{F}\mathbf{lat}_C(F, G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{F}\mathbf{lat}_C(F, G) \arrow[d, "\mathbb{F}\mathbf{lat}_C(f, G)"] \\
1 \arrow[r, "g"'] & \mathbb{F}\mathbf{lat}_C(よ, G)
\end{tikzcd}$$

In fact, the coslice $よ/\mathbb{F}\mathbf{lat}_C$ is the enriched category of models of an internal theory $\mathbb{U}_{\mathbb{C}}$ of $よ$-*algebra*, and henceforth we will write it as $\mathbb{U}_{\mathbb{C}}$-$\mathbf{Mod}$. By construction we have

$$|\mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}| \simeq よ/\mathbf{Flat}_C.$$

Proposition 1.4. *The enriched forgetful functor* $\mathbb{U}_{\mathbb{C}}$-**Mod** $\to \mathbb{F}\mathbf{lat}_{\mathcal{C}}$ *creates power by* $\mathbf{Psh}(\mathcal{C})$.

Proof. Suppose we have two $\mathbb{U}_{\mathbb{C}}$-models F, G and $X \in \mathbf{Psh}(\mathcal{C})$. From Proposition 1.3 we know G^X is again flat, and admits a canonical よ-algebra structure via the constant map $G \to G^X$. This way, since $(-)^X$ preserves limits, we have

$$\begin{aligned} \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(F,G)^X &\cong (1 \times_{\mathbb{F}\mathbf{lat}_{\mathcal{C}}(\text{よ},G)} \mathbb{F}\mathbf{lat}_{\mathcal{C}}(F,G))^X \\ &\cong 1 \times_{\mathbb{F}\mathbf{lat}_{\mathcal{C}}(\text{よ},G)^X} \mathbb{F}\mathbf{lat}_{\mathcal{C}}(F,G)^X \\ &\cong 1 \times_{\mathbb{F}\mathbf{lat}_{\mathcal{C}}(\text{よ},G^X)} \mathbb{F}\mathbf{lat}_{\mathcal{C}}(F,G^X) \\ &\cong \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(F,G^X) \end{aligned}$$

Hence, G^X is again the power in $\mathbb{U}_{\mathbb{C}}$-**Mod**. □

3. Quasi-coherence for left exact categories

Let us now assume $\mathcal{C}$ is a left exact category. Then for any topos $\mathcal{E}$, we know

$$\mathbf{Flat}(\mathcal{C},\mathcal{E}) \simeq \mathbf{Lex}(\mathcal{C},\mathcal{E}).$$

In particular, $\mathbb{C}$-**Mod** $\simeq \mathbf{Lex}(\mathcal{C},\mathbf{Set})$. Furthermore, it is well-known that in this case, $\mathbb{C}$-**Mod** is locally finitely presentable, thus in particular complete and cocomplete.

Lemma 1.5. *The generic model* よ *as a functor*

$$\text{よ}^{-} : \mathcal{C}^{\mathrm{op}} \to \mathbb{C}\text{-}\mathbf{Mod}$$

takes finite limits in $\mathcal{C}$ *to finite colimits in* $\mathbb{C}$-**Mod**.

Proof. Suppose we have a finite limit $c \cong \varprojlim_{i\in I} c_i$ in $\mathcal{C}$. For any flat functor $F \in \mathbb{C}$-**Mod**, by Yoneda

$$\begin{aligned} \mathbb{C}\text{-}\mathbf{Mod}(\text{よ}^c, F) &\cong F(c) \\ &\cong \varprojlim_{i\in I} F(c_i) \\ &\cong \varprojlim_{i\in I} \mathbb{C}\text{-}\mathbf{Mod}(\text{よ}^{c_i}, F) \\ &\cong \mathbb{C}\text{-}\mathbf{Mod}(\varinjlim_{i\in I} \text{よ}^{c_i}, F) \end{aligned}$$

Hence, $\text{よ}^c \cong \varinjlim_{i\in I} \text{よ}^{c_i}$ in $\mathbb{C}$-**Mod**. □

Furthermore, we construct a functor

$$\widetilde{-} : \mathbb{C}\text{-}\mathbf{Mod} \to \text{よ}/\mathbf{Flat}_{\mathcal{C}},$$

sending each flat functor F to the よ-algebra

$$\widetilde{F} := \text{よ} \sqcup \Delta F.$$

Here $\Delta F : \mathcal{C}^{\mathrm{op}} \to \mathbb{C}$-**Mod** is the constant functor on F, and $\sqcup$ is the coproduct in $[\mathcal{C}^{\mathrm{op}}, \mathbb{C}\text{-}\mathbf{Mod}]$, which is induced by the pointwise coproduct $\sqcup$ in $\mathbb{C}$-**Mod**. In other words, for any $c \in \mathcal{C}$,

$$\widetilde{F}^c \cong \text{よ}^c \sqcup F.$$

Proposition 1.6. *There is a reflective adjunction*

$$\widetilde{-} \dashv (-)^1 : \mathbb{C}\text{-}\mathbf{Mod} \to \text{よ}/\mathbf{Flat}_{\mathcal{C}},$$

Proof. For any $F \in \mathbb{C}\text{-}\mathbf{Mod}$ and $G \in \text{よ}/\mathbf{Flat}_C$, we have

$$\begin{aligned}
\text{よ}/\mathbf{Flat}_C(\widetilde{F}, G) &\cong \text{よ}/\mathbf{Flat}_C(\text{よ} \sqcup \Delta F, G) \\
&\cong [C^{\mathrm{op}}, \mathbb{C}\text{-}\mathbf{Mod}](\Delta F, G) \\
&\cong \int_{c \in C} \mathbb{C}\text{-}\mathbf{Mod}(F, G^c) \\
&\cong \mathbb{C}\text{-}\mathbf{Mod}(F, G^1)
\end{aligned}$$

The adjunction is reflective, since for any $F \in \mathbb{C}\text{-}\mathbf{Mod}$, by construction $\widetilde{F}^1 \cong \text{よ}^1 \sqcup F \cong F$, since by Lemma 1.5 よ^1 is initial in $\mathbb{C}\text{-}\mathbf{Mod}$. ☐

Definition 1.7. For any $\mathbb{U}_{\mathbb{C}}$-model G in $\mathbf{Psh}(C)$, we define its *spectrum* as the enriched hom

$$\operatorname{Spec} G := \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(G, \text{よ}).$$

This gives us an enriched functor

$$\operatorname{Spec} : \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod} \to \mathbf{Psh}(C).$$

Lemma 1.8. *For any $c \in C$, we have*

$$\operatorname{Spec} \widetilde{\text{よ}^c} \cong \text{よ}_c.$$

Proof. For any $d \in C$, we have the following natural isomorphisms,

$$\begin{aligned}
(\operatorname{Spec} \widetilde{\text{よ}^c})^d &\cong \mathbf{Psh}(C)(\text{よ}_d, \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(\widetilde{\text{よ}^c}, \text{よ})) \\
&\cong \text{よ}/\mathbf{Flat}_C(\widetilde{\text{よ}^c}, \text{よ}^{\text{よ}_d}) \\
&\cong \mathbb{C}\text{-}\mathbf{Mod}(\text{よ}^c, (\text{よ}^{\text{よ}_d})^1) \\
&\cong (\text{よ}_c^{\text{よ}_d})^1 \\
&\cong \text{よ}_c^d
\end{aligned}$$

It follows that $\operatorname{Spec} \widetilde{\text{よ}^c} \cong \text{よ}_c$. ☐

Proposition 1.9. *There is an enriched adjunction*

$$\begin{tikzcd}
\mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}^{\mathrm{op}} \arrow[rr, bend right, "\operatorname{Spec}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(C) \arrow[ll, bend right, "\mathcal{O}"']
\end{tikzcd}$$

where $\mathcal{O}$ sends $X \in \mathbf{Psh}(C)$ to the algebra $\mathcal{O}\, X \cong \text{よ}^X$.

Proof. For any $F \in \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}$ and $X \in \mathbf{Psh}(C)$, we have

$$\operatorname{Spec} F^X \cong \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(F, \text{よ})^X \cong \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}(F, \text{よ}^X),$$

where the last step holds by Proposition 1.4. ☐

Definition 1.10. We call $F \in \mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}$ *quasi-coherent*, if the counit $F \to \mathcal{O} \operatorname{Spec} F$ is an isomorphism in $\mathbb{U}_{\mathbb{C}}\text{-}\mathbf{Mod}$. Similarly, we say $X \in \mathbf{Psh}(C)$ is *affine*, if the unit $X \to \operatorname{Spec} \mathcal{O}\, X$ is an isomorphism in $\mathbf{Psh}(C)$.

Proposition 1.11. *For any $c \in C$, $\widetilde{\text{よ}^c}$ is quasi-coherent, i.e. よ_c is affine.*

Proof. For any $d \in C$, by Lemma 1.5 we have

$$(\widetilde{\text{よ}^c})^d \cong \text{よ}^c \sqcup \text{よ}^d \cong \text{よ}^{c \times d}.$$

On the other hand, for the power $\text{よ}^{\text{よ}_c}$ we have

$$(\text{よ}^{\text{よ}_c})^d \cong \mathbf{Psh}(C)(\text{よ}_d, \text{よ}^{\text{よ}_c}) \cong \mathbf{Psh}(C)(\text{よ}_{c \times d}, \text{よ}) \cong \text{よ}^{c \times d}.$$

Hence, we have the above equivalence. □

4. Quasi-coherent diagrams

To characterise quasi-coherent algebras in $\mathbb{U}_{\mathbb{C}}$-**Mod**, let us look at the concrete description of the counit $F \to \mathcal{O}\,\mathrm{Spec}\,F$ for some よ-algebra F. We start by giving a more concrete description of the よ-algebra structure on a diagram:

Lemma 1.12. *For any $F \in \mathbf{Diag}_C$, morphisms from よ is given by the end*

$$\mathbf{Diag}_C(\text{よ}, F) \cong \int_{c \in C} F^c_c.$$

Proof. By construction,

$$\mathbf{Diag}_C(\text{よ}, F) \cong \int_{c \in C} \mathbf{Psh}(C)(\text{よ}_c, F_c) \cong \int_{c \in C} F^c_c. \qquad \square$$

Hence, an よ-algebra structure $t : \text{よ} \to F$ is equivalently an element t in the end $\int_{c \in C} F^c_c$, i.e. a family of elements $t(c) \in F^c_c$ such that for any $f : c \to d$ in C,

$$F^d_f(t(c)) = F^f_c(t(d)).$$

Example 1.13. Interestingly, we have the computation that

$$\mathbf{Diag}_C(\text{よ}, \text{よ}) \cong \int_{c \in C} \text{よ}^c_c \cong \int_{c \in C} C(c, c).$$

When C is a one-object category, i.e. a monoid, this is exactly the *centre*, i.e. an element in it is one that commutes with all other elements. More generally for $t \in \int_{c \in C} C(c, c)$, $t(c)$ must lie in the centre of the endomorphism monoid $C(c, c)$.

Lemma 1.14. *For any $(F, t) \in \text{よ}/\mathbf{Flat}_C$ and $c \in C$, we have*

$$(\mathrm{Spec}\,F)^c \cong$$

Proof. By construction, we know that

$$(\mathrm{Spec}\,F)^c \cong \text{よ}/\mathbf{Flat}_C(F, \text{よ}^{\text{よ}_c}).$$

On one hand, we know that

$$\begin{aligned} \mathbf{Flat}_C(F, \text{よ}^{\text{よ}_c}) &\cong \int_{d \in C} \mathbf{Psh}(C)(F_d, \text{よ}^{\text{よ}_c}_d) \\ &\cong \end{aligned}$$

□

Proof. By definition, we directly compute for $c, d \in C$ that

$$\begin{aligned} (\text{よ}^{\mathrm{Spec}\,F})^d_c &\cong (\text{よ}^{\mathrm{Spec}\,F}_c)^d \\ &\cong \mathbf{Psh}(C)(\text{よ}_d \times \mathrm{Spec}\,F, \text{よ}_c) \\ &\cong \int_{e \in C} \mathbf{Set}(\text{よ}^e_d \times (\mathrm{Spec}\,F)^e, \text{よ}^e_c) \end{aligned}$$

$$
\begin{aligned}
\mathbb{D}\mathbf{iag}_{C}(F, \text{よ})^c &\cong \mathbf{Psh}(C)\Big(\text{よ}_c, \int_{d \in C} \text{よ}_d^{F_d}\Big) \\
&\cong \int_{d \in C} \mathbf{Psh}(C)(F_d \times \text{よ}_c, \text{よ}_d) \\
&\cong \int_{d,e \in C} \mathbf{Set}(F_d^e \times \text{よ}_c^e, \text{よ}_d^e) \\
&\cong \int_{d \in C} \mathbf{Set}(F_d^c, \text{よ}_d^c)
\end{aligned}
$$

□

Lemma 1.15. *For* $(F, t) \in \text{よ}/\mathbf{Flat}_C$*, we have*

$$\operatorname{Spec} F$$

Proof. We first compute $\mathbb{F}\mathbf{lat}_C(F, \text{よ})$,

$$\mathbb{F}\mathbf{lat}_C(F, \text{よ})$$

□

Proposition 1.16. *For any* $F \in \mathbb{U}_{\mathbb{C}}$-$\mathbf{Mod}$*,* F *is quasi-coherent iff*

Proof. By construction, for any $c, d \in C$ we have

$$
\begin{aligned}
(\text{よ}^{\operatorname{Spec} F})_c^d &\cong \text{よ}_c^{\operatorname{Spec} F}(d) \\
&\cong \mathbf{Psh}(C)(\text{よ}_d \times \operatorname{Spec} F, \text{よ}_c)
\end{aligned}
$$

On the other hand, □

CHAPTER 2

Modality in classifying topoi

1. Modality from models

For any diagram $F \in \mathbf{Diag}_C$, it induces an adjoint pair

$$\begin{tikzcd} \mathbf{Psh}(C) \arrow[r, bend right, "{[F]}"'] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(C) \arrow[l, bend right, "\langle F\rangle"'] \end{tikzcd}$$

where $\langle F\rangle$ is given by left Kan extension,

$$\langle F\rangle X \cong \int^{c\in C} X^c \times F_c,$$

and the right adjoint $[F]$ is given by

$$[F]X \cong \mathbf{Psh}(C)(F, X) \cong \int_{d\in C} \mathbf{Set}(F^d, X^d).$$

In fact, we get an equivalence. Let us use $\mathbf{RAdj}_C$ to denote the category of right adjoints from $\mathbf{Psh}(C)$ to itself, and similarly $\mathbf{LAdj}_C$ for the dual category.

Proposition 2.1. *There is an equivalence of categories*

$$\langle - \rangle : \mathbf{Diag}_C \simeq \mathbf{LAdj}_C,$$

Equivalently, we have

$$[-] : \mathbf{Diag}_C \simeq \mathbf{RAdj}_C^{\mathrm{op}}.$$

Proof. We construct the inverse: For a left adjoint $A : \mathbf{Psh}(C) \to \mathbf{Psh}(C)$ we send it to the diagram

$$A\text{よ} : C \to \mathbf{Psh}(C).$$

This way, for any diagram F and any left adjoint A, we have

$$\mathbf{LAdj}_C(\langle F\rangle, A) \cong \mathbf{Diag}_C(F, A\text{よ}),$$

because $\langle F\rangle$ is the left Kan extension. Hence, we have an adjunction. Furthermore, since よ is fully faithful, for any diagram F, $\langle F\rangle \circ \text{よ} \cong F$. Also, since $\mathbf{Psh}(C)$ is the free cocompletion of C, $\langle A\text{よ}\rangle \cong A$. This shows the above is an equivalence. □

Example 2.2. For the diagram $\text{よ} \in \mathbf{Diag}_C$, we have $\langle \text{よ} \rangle \cong \mathrm{id}$ and $[\text{よ}] \cong \mathrm{id}$.

Definition 2.3. A *pointer* for a diagram F is a map $\sigma : F \to \text{よ}$ in $\mathbf{Diag}_C$. Equivalently, this is a natural transformation $\eta : \mathrm{id} \to [F]$.

Using the pointer, we can in fact turn $[F]$ into a $\mathbf{Psh}(C)$-indexed functor: Let F be a diagram equipped with a pointer, with the corresponding natural transformation $\eta : \mathrm{id} \to [F]$. For any $X \in \mathbf{Psh}(X)$, define

$$[F]_X : \mathbf{Psh}(C)/X \to \mathbf{Psh}(C)/X$$

sending $f : Y \to X$ to the following pullback,

$$\begin{tikzcd}
{[F]_X Y} \arrow[r] \arrow[d, "{[F]_X f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[F]Y} \arrow[d, "{[F]f}"] \\
X \arrow[r, "\eta"'] & {[F]X}
\end{tikzcd}$$

Proposition 2.4. *The family $[F]_-$ provides a well-defined $\mathbf{Psh}(\mathcal{C})$-indexed functor.*

Proof. We need to verify the following diagram commutes up to natural isomorphism for any $f : Y \to X$,

$$\begin{tikzcd}
\mathbf{Psh}(\mathcal{C})/X \arrow[r, "f^*"] \arrow[d, "{[F]_X}"'] & \mathbf{Psh}(\mathcal{C})/Y \arrow[d, "{[F]_Y}"] \\
\mathbf{Psh}(\mathcal{C})/X \arrow[r, "f^*"'] & \mathbf{Psh}(\mathcal{C})/Y
\end{tikzcd}$$

Given any $g : Z \to X$, consider the following diagram,

$$\begin{tikzcd}
{[F]_Y f^*Z} \arrow[rr] \arrow[dr] \arrow[dd, "{[F]_Y f^*g}"'] & & {[F]f^*Z} \arrow[dr] \arrow[dd, "{[F]f^*g}" near start] & \\
& {[F]_X Z} \arrow[rr] \arrow[dd, "{[F]_X g}" near start] & & {[F]Z} \arrow[dd, "{[F]g}"] \\
Y \arrow[rr, "\eta" near start] \arrow[dr, "f"'] & & {[F]Y} \arrow[dr, "{[F]f}"'] & \\
& X \arrow[rr, "\eta"'] & & {[F]X}
\end{tikzcd}$$

By construction, the front and back squares are pullbacks. The right square is a pullback since $[F]$ preserves limits. This implies the left square must also be a pullback by 2-out-of-3. Hence,

$$f^*[F]_X g \cong [F]_Y f^* g,$$

which implies $[F]_-$ is a well-defined $\mathbf{Psh}(\mathcal{C})$-indexed functor. ☐

The above shows that we can view $[F]$ as a modality in the internal logic of $\mathbf{Psh}(\mathcal{C})$. This way, we can connect the property of the diagram F with the property of the modality $[F]$.

Remark 2.5. In general, the dual functor $\langle F \rangle$ does *not* internalise, unless the corresponding map $\langle F \rangle \to \mathrm{id}$ is *Cartesian.*

Proposition 2.6. *The internal modality $[F]$ is left exact.*

Proof. As an indexed functor, being left exact is equivalent to each $[F]_X$ being left exact for all $X \in \mathbf{Psh}(\mathcal{C})$. This follows directly from construction. ☐

Remark 2.7. In particular, this shows the modality $[F]$ is *normal*, i.e. it satisfies necessitation, i.e. $[F]1 \cong 1$, and the (K) axiom,

$$\forall \varphi, \psi : \Omega.\ [F](\varphi \to \psi) \to ([F]\varphi \to [F]\psi).$$

Furthermore, since we have $\mathrm{id} \Rightarrow [F]$, it also satisfies the (C) axiom,

$$\forall \varphi : \Omega.\ \varphi \to [F]\varphi.$$

Remark 2.8. However, $[F]$ cannot preserve all internal limit, since this is equivalent to the existence of a $\mathbf{Psh}(C)$-indexed left adjoint. This holds iff $\langle F\rangle$ can be made into an indexed functor.

2. The Löb's axiom externally

Let us now fix a diagram F on C equipped with a pointer $\sigma : F \to \text{よ}$. We also use σ to denote the induced natural transformation $\sigma : \langle F\rangle \to \mathrm{id}$.

Lemma 2.9. *For any $X \in \mathbf{Psh}(C)$ and $\varphi \in \mathrm{Sub}(X)$,*

$$X \vDash [F]\varphi \Leftrightarrow \langle F\rangle X \vDash \sigma_X^* \varphi,$$

i.e. $\sigma_X : \langle F\rangle X \to X$ factors through φ.

Proof. By the Kripke-Joyal semantics, $X \vDash [F]\varphi$ iff $[F]_X\varphi \cong X$, i.e. we have a factorisation

$$\begin{tikzcd} & {[F]\varphi} \arrow[d, hook] \\ X \arrow[ur, dashed] \arrow[r, "\eta"'] & {[F]X} \end{tikzcd}$$

By the adjunction $\langle F\rangle \dashv [F]$, this is equivalent to σ_X factors through φ. □

Definition 2.10. We say a diagram F is *inductive*, if for any $c \in C$, there exists a finite number n such that $\langle F\rangle^n \text{よ}_c \cong \varnothing$.

Proposition 2.11. *For any inductive diagram F with a pointer $\sigma : F \to \text{よ}$, the internal modality $[F]$ satisfies (Löb),*

$$\forall \varphi : \Omega.\ ([F]\varphi \to \varphi) \to \varphi,$$

Proof. Suppose we have $c \in C$ and $\varphi \in \Omega(c)$ that

$$c \vDash [F]\varphi \to \varphi.$$

This way, since $[F]$ satisfies (C), by Lemma 2.9 we have

$$c \vDash \varphi \Leftrightarrow c \vDash [F]\varphi \Leftrightarrow \langle F\rangle \text{よ}_c \vDash \sigma_c^* \varphi.$$

Furthermore, since again $\langle F\rangle \text{よ}_c \vDash [F]\sigma_c^*\varphi \to \sigma_c^*\varphi$ by stability, it follows that

$$\langle F\rangle \text{よ}_c \vDash \sigma_c^*\varphi \Leftrightarrow \langle F\rangle \text{よ}_c \vDash [F]\sigma_c^*\varphi \Leftrightarrow \langle F\rangle^2 \text{よ}_c \vDash \sigma^*_{\langle \text{よ}\rangle_c} \sigma_c^* \varphi.$$

It follows that for any $n \in \mathbb{N}$,

$$c \vDash \varphi \Leftrightarrow \langle F\rangle^n \text{よ}_c \vDash \sigma^*_{\langle F\rangle^{n-1}\text{よ}_c} \cdots \sigma_c^* \varphi.$$

This way, if F is inductive, then for some n we have $\langle F\rangle^n \text{よ}_c \cong \varnothing$, which implies $\langle F\rangle^n \text{よ}_c \vDash \sigma^*_{\langle F\rangle^{n-1}\text{よ}_c} \cdots \sigma_c^* \varphi$. Hence $c \vDash \varphi$. This shows (Löb) holds in $\mathbf{Psh}(C)$. □

3. The Löb's axiom internally

For any category C, we can in fact look at the *algebraic theory* of C-diagrams. Quasi-coherence for C-diagrams holds in $\mathbf{Psh}(C)$, since the theory of flat diagrams of C is a subcanonical quotient of the theory of C-diagrams:

Proposition 2.12. *For a C-diagram*