SYNTHETIC GUARDED DOMAIN THEORY AND CLASSIFYING TOPOI

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1. Introduction

This is an internal analysis on synthetic guarded domain theory [1]. Let $\mathbb{W}$ be the theory of filters on the poset $\omega = (\mathbf{N}, \leq) = \{0 < 1 < \cdots\}$. For convenience, we will in fact use the poset $\omega_\perp = \{\perp < 0 < 1 < \cdots\}$. Explicitly, $\mathbb{W}$ has propositional constants $\mathsf{p}_\perp, \mathsf{p}_0, \mathsf{p}_1, \cdots$, with axioms:

(1) $\mathsf{p}_\perp \vdash \perp$;
(2) For all $n \leq m \in \omega_\perp$, $\mathsf{p}_n \vdash \mathsf{p}_m$;
(3) $\top \vdash \bigvee_{n \in \omega} \mathsf{p}_n$.

We work internally in the classifying topos $\mathbf{Set}[\mathbb{W}] \simeq \mathbf{Psh}(\omega)$. We will use Prop to denote the type of propositions. We will also assume a univalent type universe Type. We will use $\omega_\perp$ to denote the poset internalised in $\mathbf{Set}[\mathbb{W}]$ as well. For any $n : \omega_\perp$, we use $[\![n]\!]$ to denote the representable proposition.

Axiom (QC)**.** For any $n, m : \omega_\perp$,

$$[\![n]\!] \to [\![m]\!] = \begin{cases} [\![m]\!] & m < n \\ 1 & \text{otherwise} \end{cases}$$

Axiom (GR)**.** Every proposition is definable: $\forall p : \mathsf{Prop}.\ \exists n : \omega_\perp.\ p = [\![n]\!]$.

Remark 1.1. Notice that (GR) plus (QC) implies the sequence of propositions is a model of $\mathbb{W}$. By (GR), $\emptyset = [\![n]\!]$ for some $n : \overline{\omega}$, and by (QC) $[\![\perp]\!] \to [\![n]\!] = 1$, which implies $[\![\perp]\!] \to \emptyset$, thus $[\![\perp]\!] = \emptyset$. By (GR) again, $\exists n : \omega_\perp.\ 1 = [\![n]\!]$, while $1 = [\![n]\!]$ is equivalent to $[\![n]\!]$, thus this implies $\bigvee_{n : \omega_\perp} [\![n]\!]$.

As an easy consequence, we can show the generic model is indeed generic in the following sense:

Lemma 1.1. *For any $n : \omega$, $\neg\neg[\![n]\!]$.*

Proof. By (QC), given $n : \omega$, $[\![n]\!] \to [\![\perp]\!] = [\![\perp]\!] = \emptyset$. Equivalently, $\neg\neg[\![n]\!]$. $\square$

2. Internal later modality on propositions

We can construct an internal later modality from the data of a certain endo-function on $\omega_\perp$:

Construction 2.1 (Later modality on propositions)**.** Let $\theta : \omega_\perp \to \omega_\perp$ be a montone, non-decreasing function. Then the *θ-later modality* $\square_\theta : \mathsf{Prop} \to \mathsf{Prop}$ takes $p : \mathsf{Prop}$ to $[\![\theta n]\!]$, where $p = [\![n]\!]$ by (GR).

We first show this construction is well-defined.

Proposition 2.1. $\square_\theta : \mathsf{Prop} \to \mathsf{Prop}$ *is well-defined, i.e. we have a diagram*

$$\begin{tikzcd} \omega_\perp \arrow[r, "\theta"] \arrow[d, two heads] & \omega_\perp \arrow[d, two heads] \\ \mathsf{Prop} \arrow[r, dashed, "\square_\theta"'] & \mathsf{Prop} \end{tikzcd}$$

Proof. Suppose $p = [\![n]\!] = [\![m]\!]$ for some $n, m : \omega_\perp$, and we need to show $[\![\theta n]\!] = [\![\theta m]\!]$ as well. We may assume $m < n$, since the case $m = n$ is trivial. By (QC), $[\![n]\!] \to [\![m]\!]$ is equivalent to $[\![m]\!]$, thus $[\![m]\!]$ holds. Since θ is non-decreasing, by (QC) $[\![\theta m]\!]$ holds as well, which by monotonicity of θ, $[\![\theta n]\!]$ holds as well. Thus shis shows $[\![\theta n]\!] = [\![\theta m]\!]$. □

From now on, fix a monotone and non-decreasing function $\theta : \omega_\perp \to \omega_\perp$.

Proposition 2.2. *$\Box_\theta$ is pointed.*

Proof. For any p : Prop, by (GR) let $p = [\![n]\!]$. By construction, $\Box_\theta p = [\![\theta n]\!]$, which by θ being non-decreasing we have $p \to \Box_\theta p$. □

Proposition 2.3. *$\Box_\theta$ is left exact.*

Proof. For 1, by (GR) let $1 = [\![n]\!]$. Then since θ is non-decreasing, $\Box_\theta 1 = [\![\theta n]\!]$ holds as well. Preserving binary meets is easy. □

Theorem 2.1. *If θ is strictly increasing, then $\Box_\theta$ satisfies Löb induction,*

$$\forall p : \mathsf{Prop}.\, (\Box_\theta p \to p) \to p.$$

Proof. Take p : Prop, and by (GR) let $p = [\![n]\!]$. Assume $\Box_\theta p \to p$, viz. $[\![\theta n]\!] \to [\![n]\!]$. Since θ is strictly increasing, by (QC) this implication is equivalent to $[\![n]\!]$, viz. p, holds. □

3. Internal later modality on types

For any type X : Type, we need a way to relate the function spaces $X^{[\![n]\!]}$ for $n : \omega_\perp$ to X itself. It turns out we can write any type as a *coend*:

Theorem 3.1. *For any X : Type, it is isomorphic to the coend*

$$X \cong \int^{n : \omega_\perp} [\![n]\!] \times X^{[\![n]\!]}.$$

Proof. This is a consequence of (GR), where we can assume $1 = [\![n]\!]$ for some $n : \omega_\perp$. This way, the coend becomes constant, thus is isomorphic to X. □

Construction 3.1 (Later modality on types)**.** Let $\theta : \omega_\perp \to \omega_\perp$ be a monotone, non-decreasing map on $\omega_\perp$. The *θ-later modality* $\Box_\theta$: Type → Type is constructed as follows: For any X : Type,

$$\Box_\theta X := \int^{n : \omega_\perp} [\![\theta n]\!] \times X^{[\![n]\!]}.$$

Again, let us fix a monotone and non-decreasing map $\theta : \omega_\perp \to \omega_\perp$.

Proposition 3.1. *The modality $\Box_\theta$: Type → Type given in Construction 3.1 restricts to the modality $\Box_\theta$: Prop → Prop on propositions given in Construction 2.1.*

Proof. By (GR), it suffices to show $\Box_\theta[\![n]\!]$, according to Construction 3.1, will be equal to $[\![\theta n]\!]$ for all $n : \omega_\perp$. By construction and (QC), we have

$$\begin{aligned}\Box_\theta[\![n]\!] &\equiv \int^{m : \omega_\perp} [\![\theta m]\!] \times [\![n]\!]^{[\![m]\!]} \\ &= \bigvee_{m : \omega_\perp} [\![\theta m]\!] \wedge ([\![m]\!] \to [\![n]\!]) \\ &= \bigvee_{m \le n} [\![\theta m]\!] \vee \bigvee_{m > n} [\![\theta m]\!] \wedge [\![n]\!] \\ &= [\![\theta n]\!]\end{aligned}$$

The last step holds by the fact that θ is monotone and non-decreasing, thus $[\![\theta m]\!] \wedge [\![n]\!] = [\![n]\!]$ for all $m > n$. □

Proposition 3.2. *$\Box_\theta$ is left exact.*

Proof. $\Box_\theta$ preserves 1 by Proposition 3.1 and Proposition 2.3. For pullback, suppose we have $f : Y \to X$ and $g : Z \to X$. We then have

$$\begin{aligned}\Box_\theta(Y \times_X Y) &\equiv \int^{n:\omega_\perp} [\![\theta n]\!] \times (Y \times_X Z)^{[\![n]\!]} \\ &\cong \int^{n:\omega_\perp} [\![\theta n]\!] \times Y^{[\![n]\!]} \times_{X^{[\![n]\!]}} Z^{[\![n]\!]} \\ &\cong \Box_\theta Y \times_{\Box_\theta X} \Box_\theta Z\end{aligned}$$

The final step uses the fact that finite limits commutes with the filtered colimit. □

Proposition 3.3. *$\Box_\theta$ is well-pointed.*

Proof. To construct the unit, by (GR) we may assume $1 = [\![n]\!]$, thus by the universal property there is a comparison map

$$X \xrightarrow{\cong} [\![n]\!] \times X^{[\![n]\!]} \longrightarrow [\![\theta n]\!] \times X^{[\![n]\!]} \longrightarrow \Box_\theta X.$$

This is well-defined due to the fact that $\Box_\theta X$ is a coend, and naturality is evident. Checking well-pointedness is routine. □

Corollary 3.1. *If θ is strictly increasing, then $\Box_\theta : \mathsf{Type} \to \mathsf{Type}$ supports guarded recursion fixed-point construction.*

Proof. This follows from Proposition 3.3, Proposition 3.1, and [2]. □

4. A non-localic model of guarded recursion

Consider **Inj**, the category of finite sets with injections between them. Notice that this category has pullback but not a terminal object. There is an evident functor

$$1 + - : \mathbf{Inj} \to \mathbf{Inj},$$

which induces an adjunction

$$\begin{tikzcd} \mathbf{Psh(Inj)} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathbf{Psh(Inj)} \arrow[l, bend right, "L"'] \end{tikzcd}$$

where L is given by precomposition with $1 + -$ and R is obtained via right Kan extension. Since L is also continuous, this gives us a geometric morphism

$$R : \mathbf{Psh(Inj)} \to \mathbf{Psh(Inj)}.$$

Proposition 4.1. *We have a self-indexed path family*

$$\begin{tikzcd} \mathbf{Psh(Inj)} \arrow[r, bend left, "R"] \arrow[r, bend right, equal] \arrow[r, phantom, "\Downarrow\theta"] & \mathbf{Psh(Inj)} \end{tikzcd}$$

Proof. It suffices to provide a natural transformation $\theta : L \Rightarrow \mathrm{id}$. But such a natural transformation is induced by the natural inclusion $\mathrm{id} \Rightarrow 1 + -$. □

Theorem 4.1. *The self-indexing family in Proposition 4.1 is contractive, which means the internal modality associated to this self-indexing family satisfies Löb induction.*

Proof. To show contractivity, it suffices to observe that for any representable よ_n, there exists some k that $L^k \text{よ}_n \cong \emptyset$. Now we first show that

$$L \text{よ}_n \cong n \cdot \text{よ}_{n-1}.$$

For any $i \in n$, let $\hat{n}^i$ denote $n - \{ i \}$. By construction we have

$$L \text{よ}_n(m) \cong \mathbf{Inj}(m+1, n) \cong \sum_{i:n} \mathbf{Inj}(m, \hat{n}^i) \cong n \cdot \text{よ}_{n-1}(m).$$

It is worth explicitly specifying the last isomorphism: given a pair $(i, f : m \rightarrowtail n-1)$, the induced injection $(i, f) : m+1 \rightarrowtail n$ is given by

$$(i,f)(j) = \begin{cases} f(j) & j < m \wedge f(j) < i \\ f(j)+1 & j < m \wedge f(j) \geq i \\ i & j = m \end{cases}$$

This isomorphism is natural: Given any $f : l \rightarrowtail m$, we verify

$$\begin{tikzcd} n \times \mathbf{Inj}(m, n-1) \arrow[r, "\cong"] \arrow[d, "n\times(-\circ f)"'] & \mathbf{Inj}(m+1, n) \arrow[d, "-\circ(f+1)"] \\ n \times \mathbf{Inj}(l, n-1) \arrow[r, "\cong"'] & \mathbf{Inj}(l+1, n) \end{tikzcd}$$

For any pair (i, g) with $i \in m$ and $g : m \rightarrowtail n-1$, and for $j \in l+1$ we have

$$\begin{aligned} (i,g)((f+1)(j)) &= \begin{cases} gf(j) & j < l \wedge gf(j) < i \\ gf(j)+1 & j < l \wedge gf(j) \geq i \\ i & j = l \end{cases} \\ &= (i, gf)(j) \end{aligned}$$

This shows that the isomorphism is natural, thus $L\,よ_n \cong n \cdot よ_{n-1}$. This way, since L preserves colimits, it follows that

$$L^{n+1}\,よ_n \cong \emptyset,$$

which completes the proof. ☐

5. Another interesting example

Consider $\mathbf{Psh}(\mathbf{Fin})$, where $\mathbf{Fin}$ is the category of finite sets, and consider the functor

$$1 + - : \mathbf{Fin} \to \mathbf{Fin},$$

and consider the functor $L : \mathbf{Psh}(\mathbf{Fin}) \to \mathbf{Psh}(\mathbf{Fin})$ given by

$$LX(n) \cong X(n+1).$$

L is cocontinuous, thus in particular has a right adjoint $R : \mathbf{Psh}(\mathbf{Fin}) \to \mathbf{Psh}(\mathbf{Fin})$. To compute the right adjoint, it suffices to compute what L does on representables.

Lemma 5.1. *For any $n \in \mathbf{Fin}$, we have*

$$L\,よ_n \cong n \cdot よ_n,$$

where $n \cdot -$ denotes the tensor of the finite set n with the presheaf $よ_n$.

Proof. By construction, for any $m \in \mathbf{Fin}$,

$$(L\,よ_n)(m) \cong n^{m+1} \cong n \times n^m \cong n \times よ_n(m).$$

This shows that $L\,よ_n \cong n \cdot よ_n$. ☐

Proposition 5.1. *For any $X \in \mathbf{Psh}(\mathbf{Fin})$, we have*

$$(RX)(n) \cong X(n)^n,$$

where for any $f : n \to m$, the action on f takes $a \in X(m)^m$ to the following composite,

$$\begin{tikzcd} n \arrow[r, dashed, "a\cdot f"] \arrow[d, "f"'] & X(n) \\ m \arrow[r, "a"'] & X(m) \arrow[u, "-\cdot f"'] \end{tikzcd}$$

Proof. This follows from Lemma 5.1, where by construction

$$(RX)(n) \cong \mathbf{Psh}(\mathbf{Fin})(L\,\text{よ}n, X) \cong \mathbf{Psh}(\mathbf{Fin})(n \cdot \text{よ}_n, X) \cong X(n)^n.$$

The action on morphisms in **Fin** follows from routine computation. □

Proposition 5.2. *R is well-pointed.*

Proof. The point is easy to provide, where for any $X \in \mathbf{Psh}(\mathbf{Fin})$, the unit $\eta_X : X \to RX$ is given by the constant function for any $n \in \mathbf{Fin}$,

$$\eta_X := c_{-} : X(n) \to X(n)^n.$$

Naturality is evident. For well-pointedness, this is routine to check. □

Theorem 5.1. *$R : \mathbf{Psh}(\mathbf{Fin}) \to \mathbf{Psh}(\mathbf{Fin})$ restricts to identity $\mathrm{id} : \Omega \to \Omega$ on propositions.*

Proof. For any $n \in \mathbf{Fin}$, let $p \in \Omega(n)$ be a sieve on n in **Fin**. Over よ_n, the fibred modality R acting on p is given by the pullback

$$\begin{tikzcd} \Box_{\text{よ}_n} p \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Rp \arrow[d, hook] \\ \text{よ}_n \arrow[r, "\eta"'] & R\,\text{よ}_n \end{tikzcd}$$

By construction, for any $f : m \to n$, $f \in \Box_{\text{よ}_n} p$ iff □

References

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