SYNTHETIC GUARDED DOMAIN THEORY AND CLASSIFYING TOPOI

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1. Introduction

This is an internal analysis on synthetic guarded domain theory [1].

2. The theory of sequential propositions

Let $\overline{\omega}$ be the poset of extended natural numbers $\mathbf{N} \cup \{\infty\}$. As a poset it is a meet-semi-lattice, thus corresponds to an essentially algebraic theory, which we call $\mathbb{P}$. Since it is a lattice, it is also localic, and can be presented as a theory with infinitely many propositional letters $p_0, p_1, \cdots$, with infinitely many sequents as axioms

$$p_0 \vdash p_1 \vdash p_2 \vdash \cdots$$

We also consider a geometric quotient $\mathbb{P}_\omega$ of $\mathbb{P}$, which is obtained by adding an additional geometric sequent

$$\top \vdash \bigvee_{n:\mathbf{N}} p_n.$$

The classifying topos $\mathbf{Set}[\mathbb{P}]$ is given by the presheaf category

$$\mathbf{Set}[\mathbb{P}] \simeq \mathbf{Psh}(\overline{\omega}).$$

Inside, the universal model $\mathbb{w}$ is given by the sequence of representables,

$$\mathbb{w} := \text{よ}_0 \hookrightarrow \text{よ}_1 \hookrightarrow \cdots$$

The classifying topos for $\mathbb{P}_\omega$ is again of presheaf type, given by

$$\mathbf{Set}[\mathbb{P}_\omega] \simeq \mathbf{Psh}(\omega).$$

As a subtopos $\mathbf{Psh}(\omega) \hookrightarrow \mathbf{Psh}(\overline{\omega})$, since $\bigvee_{n:\mathbf{N}} n = \infty$ is a universal effective colimit in $\overline{\omega}$, all representables are still sheaves in $\mathbf{Psh}(\omega)$. Thus, the universal model $\mathbb{w}$ is again the generic model in $\mathbf{Psh}(\omega)$. Below we will mainly work with the sheaf subtopos $\mathbf{Psh}(\omega)$, and we will denote it as $\mathcal{S}$.

Proposition 2.1. *Let $\overline{\omega}_\perp$ be the poset $\overline{\omega}$ with a bottom element added, and $\mathbf{Inf}(\omega, \overline{\omega}_\perp)$ denote the poset of unbounded monotone functions from ω to $\overline{\omega}_\perp$. Then we have*

$$\mathbf{Topoi}(\mathcal{S}, \mathcal{S}) \simeq \mathbb{P}_\omega\text{-}\mathbf{Mod}(\mathcal{S}) \simeq \mathbf{Inf}(\omega, \omega_\perp).$$

Proof. The first equivalence is due to the universal property of the classifying topos. We note that a model of $\mathbb{P}_\omega$ in $\mathcal{S}$ is a sequence of increasing propositions whose union is 1. Furthermore, we know that in $\mathcal{S}$ we have $\mathrm{Sub}(1) \cong \overline{\omega}_\perp$, since a proposition can either be よ_k for some k, or $0, 1$. This explains the second equivalence. □

We can describe the correspondence more concretely: For any $\mathfrak{F} \in \mathbb{P}_\omega\text{-}\mathbf{Mod}(\mathcal{S})$, it is a sequence of propositions in $\mathcal{S}$

$$\mathfrak{F}_0 \hookrightarrow \mathfrak{F}_1 \hookrightarrow \cdots$$

such that $\bigvee_{n:\mathbf{N}} \mathfrak{F}_n = 1$. It induces a geometric morphism

$$\begin{tikzcd} \mathbf{Psh}(\omega) \arrow[r, bend left, "\lhd"] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(\omega) \arrow[l, bend left, "\rhd"] \end{tikzcd}$$

The right adjoint is easy to define, which is given by

$$\rhd X(k) \cong \mathcal{S}(\mathfrak{F}_k, X) \cong X(f(k)),$$

where here we view $f : \omega \to \overline{\omega}_\perp$ as $\mathfrak{F}$ under the equivalence in Proposition 2.1, and we define the value of X on $\perp, \infty$ to be

$$X(\perp) := 1, \quad X(\infty) := \varprojlim_{n:\mathbf{N}} X(n).$$

The left adjoint $\lhd$ is a left Kan extension,

$$\lhd X \cong \varinjlim_{x \in X(n)} \mathfrak{F}_n.$$

However, we also have a more concrete description of $\lhd$: For any $k \in \mathbf{N}$, we define

$$k^p := \min\{\, n : \mathbf{N} \mid k \leq f(n) \,\}.$$

This is well-defined exactly because $\bigvee_{n:\mathbf{N}} \mathfrak{F}_n = 1$.

Lemma 2.2. *For any presheaf $X \in \mathcal{S}$,*

$$\lhd X(k) \cong X(k^p).$$

Proof. By construction, since colimits in $\mathcal{S}$ are computed pointwise,

$$\lhd X(k) \cong \varinjlim_{x \in X(n)} \mathcal{S}(\text{よ}_k, \mathfrak{F}_n).$$

Notice that $\mathcal{S}(\text{よ}_k, \mathfrak{F}_n) \cong 1$ when $k \leq f(n)$, and is empty otherwise. This makes $X(k^p)$ terminal in the above colimit. □

Example 2.3. The universal model $\mathbb{w}$ induces the identity on $\mathcal{S}$.

Example 2.4. Consider the model $\mathbb{w}[-1]$ defined by

$$\mathbb{w}[-1] := 0 \hookrightarrow \text{よ}_0 \hookrightarrow \text{よ}_1 \hookrightarrow \cdots$$

The adjoint pair $\lhd \dashv \rhd$ induced by $\mathbb{w}[-1]$ is the usual modalities used in synthetic guarded domain theory [1].

3. Guarded recursion externally

Definition 3.1. We say a model $\mathfrak{F}$ is *subgeneric* if $\mathfrak{F} \leq \mathbb{w}$.

Under the equivalence in Proposition 2.1, a subgeneric model $\mathfrak{F}$ induces the following natural transformations,

$$\sigma : \lhd \Rightarrow \mathrm{id}, \quad \eta : \mathrm{id} \Rightarrow \rhd.$$

Using η, we can in fact turn $\rhd$ into an $\mathcal{S}$-indexed functor:

Proposition 3.2. *Let $\rhd_X : \mathcal{S}/X \to \mathcal{S}/X$ sends $f : Y \to X$ to the following pullback,*

$$\begin{tikzcd}
\rhd_X Y \arrow[r] \arrow[d, "\rhd_X f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \rhd Y \arrow[d, "\rhd f"] \\
X \arrow[r, "\eta"'] & \rhd X
\end{tikzcd}$$

This gives a well-defined $\mathcal{S}$-indexed functor.

Proof. See [1]. □

Definition 3.3. We say a model $\mathfrak{F}$ is *inductive*, if $f(n) < n$ for all $n \in \omega$.

Theorem 3.4. *For any inductive model $\mathfrak{F}$, Löb's induction holds in $\mathcal{S}$,*

$$\forall \varphi \in \Omega.\, (\rhd\varphi \to \varphi) \to \varphi.$$

Proof. Using the Kripke-Joyal semantics, suppose we have $\varphi \in \Omega(n)$ such that

$$n \vDash \rhd\varphi \to \varphi.$$

Now we have

$$n \vDash \varphi \Leftrightarrow n \vDash \rhd\varphi \Leftrightarrow f(n) \vDash \varphi \Leftrightarrow f(f(n)) \vDash \varphi \cdots \Leftrightarrow \bot \vDash \varphi.$$

which then implies that $n \vDash \varphi$ since $\bot \vDash \varphi$ always holds. □

4. Guarded recursion internally

Equivalently, the geometric theory $\mathbb{P}_\omega$ is the theory of *filters* on ω, and for any filter $\mathfrak{F}$, the proposition $\mathfrak{F}_n$ can be identified as $n \in \mathfrak{F}$. In this section we will solely use the latter notation.

Definition 4.1. We say a filter $\mathfrak{F}$ is *sup-generic*, if $\mathbb{w} \subseteq \mathfrak{F}$. The *spectrum* of a sup-generic filter $\mathfrak{F}$ is simply

$$\operatorname{Spec}\mathfrak{F} \ :\equiv \mathfrak{F} \subseteq \mathbb{w} \leftrightarrow \mathfrak{F} = \mathbb{w}.$$

Proposition 4.2. *For any sup-generic filter $\mathfrak{F}$, we have*

$$\mathfrak{F} = \mathbb{w}^{\operatorname{Spec}\mathfrak{F}}.$$

In other words for any $n : \omega$,

$$n \in \mathfrak{F} \Leftrightarrow \mathfrak{F} = \mathbb{w} \to n \in \mathbb{w}.$$

Proof. This is exactly quasi-coherence for the theory $\mathbb{P}$. □

Theorem 4.3. *There is an adjunction between Ω and $\mathrm{SFil}(\omega)$ of sup-generic filters on ω,*

$$\begin{tikzcd}
\mathrm{SFil}(\omega)^{\mathrm{op}} \arrow[rr, bend right, "\mathrm{Spec}"'] \arrow[rr, phantom, "\perp"] & & \Omega \arrow[ll, bend right, "\mathbb{w}^{-}"']
\end{tikzcd}$$

which identifies $\mathrm{SFil}(\omega)^{\mathrm{op}}$ as a reflective sub-poset of Ω.

Proof. For any proposition $\varphi : \Omega$ and sup-generic filter $\mathfrak{F}$, if $\varphi \to \operatorname{Spec}\mathfrak{F}$, then

$$\begin{aligned}
n \in \mathfrak{F} &\Leftrightarrow \operatorname{Spec}\mathfrak{F} \to n \in \mathbb{w} \\
&\Rightarrow \varphi \to n \in \mathbb{w} \\
&\Leftrightarrow n \in \mathbb{w}^{\varphi}
\end{aligned}$$

which implies $\mathfrak{F} \subseteq \mathbb{w}^{\varphi}$. On the other hand, suppose $\mathfrak{F} \subseteq \mathbb{w}^{\varphi}$. If φ holds, then $\mathbb{w}^{\varphi} = \mathbb{w}$, which implies $\mathfrak{F} = \mathbb{w}$ since $\mathfrak{F}$ is sup-generic. Hence, $\operatorname{Spec}\mathfrak{F}$ also holds. □

Definition 4.4. We say a proposition is *affine* if it belongs to the image of Spec. By Theorem 4.3, affine propositions are

Example 4.5.

Proposition 4.6. *Affine propositions are closed under arbitrary meets.*

Proof. □

Remark 4.7. The above explains perfectly what it means for $\mathbb{w}$ to be the *generic filter.*

Proposition 4.8. *For any $n : \omega$, $\neg\neg n \in \mathbb{w}$.*

Proof. Suppose $n \notin \mathbb{w}$. Then consider the filter $\mathfrak{P}_n$ defined by

$$\mathfrak{P}_n := \{ k : \omega \mid n \leq k \}.$$

For any $k \in \mathbb{w}$ we know that $n \leq k$, since if $k < n$ and $k \in \mathbb{w}$ then $n \in \mathbb{w}$, contradictory. Hence, $n \leq k$, since this proposition is decidable. It follows that $\mathfrak{P}_n$ is sup-generic, which by Proposition 4.2 implies for any $k : \omega$

$$k \in \mathfrak{P}_n \Leftrightarrow n \leq k \Leftrightarrow \mathfrak{P}_n = \mathbb{w} \to k \in \mathbb{w}.$$

□

Furthermore, the topos $\mathcal{S}$ can be viewed as an internal way to doing *forcing* on ω:

Definition 4.9. For any $n : \omega$ and $\varphi : \Omega$, we define the forcing relation

$$n \Vdash \varphi := \mathbb{w}_n \to \varphi.$$

Lemma 4.10. *For any $n : \omega$, the forcing relation $n \Vdash (-) : \Omega \to \Omega$ has a right adjoint.*

Proof. Given a family $\varphi : I \to \Omega$ of propositions, consider the sup-generic filter Φ,

$$n \in \Phi := n \in \mathbb{w} \vee \bigvee_{i:I} \varphi_i.$$

This way, we have

$$\Phi = \mathbb{w} \Leftrightarrow \left(\bigvee_{i:I} \varphi_i \right) \to \mathbb{w}_n = \bigwedge_{i:I} \varphi_i \to \mathbb{w}_n.$$

By Proposition 4.2, we have that

$$n \in \Phi \Leftrightarrow \mathbb{w}_n \vee \bigvee_{i:I} \varphi_i \Leftrightarrow \left(\bigwedge_{i:I} \varphi_i \to \mathbb{w}_n \right) \to \mathbb{w}_n.$$

$$n \Vdash \bigvee_{i:I} \varphi_i \Leftrightarrow \mathbb{w}_n \to \bigvee_{i:I} \varphi_i.$$

□

This inspires the following construction:

Construction 1. Any $\mathbb{P}_\omega$-model $\mathfrak{F}$ induces two modalities on Ω, where for $\varphi : \Omega$

$$\lhd\varphi := \bigvee_{\mathbb{w}_n \to \varphi} \mathfrak{F}_n, \quad \rhd\varphi := .$$

Lemma 4.11. *For all $n : \omega$, $\lhd\mathbb{w}_n = \mathfrak{F}_n$.*

Proof. Notice that $\mathbb{w}_n \to \mathbb{w}_n$, thus $\mathfrak{F}_n \to \bigvee_{\mathbb{w}_m \to \mathbb{w}_m} \mathfrak{F}_m = \lhd\mathbb{w}_n$. On the other hand, for any m that $\mathbb{w}_m \to \mathbb{w}_n$, we have $m \leq n$. Thus, $\mathfrak{F}_m \to \mathfrak{F}_n$, and $\lhd\mathbb{w}_n = \bigvee_{\mathbb{w}_m \to \mathbb{w}_n} \mathfrak{F}_m \to \mathfrak{F}_n$. □

Proposition 4.12. *The two modalities are adjoint to each other $\lhd \dashv \rhd$.*

Proof. For any $\varphi, \psi : \Omega$, we have

$$\lhd\varphi \leq \psi \Leftrightarrow \forall k \in \mathbf{N}.\, (\mathbb{w}_n \to \varphi) \to (\mathfrak{F}_n \to \psi).$$
$$\Leftrightarrow \forall k \in \mathbf{N}.\, (\mathbb{w}_n \to \varphi) \wedge \mathfrak{F}_n \to \psi$$

On the other hand,

$$\varphi \leq \rhd\psi \Leftrightarrow \forall k \in \mathbf{N}.\, p_k \subseteq \psi \to k \in \varphi.$$

□

Lemma 4.13. *For any*

$$p \leftrightarrow \mathbb{w}^{\operatorname{Spec} p}.$$

Proof. This holds by quasi-coherence. □

Theorem 4.14. *For any $\varphi : \Omega$, we have*

$$\varphi \vee \neg\varphi \vee \exists n \in \mathbf{N}.\ \varphi = \mathbb{w}_n.$$

Proof. □

Definition 4.15. By an $\mathbb{w}$-algebra we mean an increasing proposition $p : \mathbf{N} \to \Omega$ such that $\mathbb{w} \leq p$. The *spectrum* of p is defined to be

$$\mathrm{Spec}\, p := \forall n \in \mathbf{N}.\ p_n \leq \mathbb{w}_n \leftrightarrow p = \mathbb{w}.$$

Example 4.16. For $\mathbb{w}_n$, we have

$$\lhd \mathbb{w}_n = \bigvee_{\mathbb{w}_n \leq p_k} \mathbb{w}_k$$

References

[1] Birkedal, L., Møgelberg, R. E., Schwinghammer, J., and Støvring, K. (2012). First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8.

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