MIE 1621 Computational Project Part 2

Convex quadratic constrained optimization $\min c^T x + \frac{1}{2} x^T Q x$
Subject to $Ax = b, x \geq 0$
(where Q is positive semi-definite, A has full row rank)

Barrier Reformulation (modified primal problem)

$$\min c^T x + \frac{1}{2} x^T Q x - \mu \sum_{i=1}^{n} In\,(x_i) \qquad (1)$$
$$\text{Subject to } Ax = b$$
$$(\text{where } \mu > 0)$$

The Lagrangian of (1)

$$L(x,\pi) = c^T x + \frac{1}{2} x^T Q x - \mu \sum_{i=1}^{n} In\,(x_i) + \pi^T (b - Ax)$$
$$\frac{dL(x,\pi)}{dx} = c + Qx - \mu X^{-1} - A^T \pi = 0 \qquad (2)$$
$$\frac{dL(x,\pi)}{d\pi} = b - Ax = 0 \qquad (3)$$

Where $X = diag(x)$

Let $z = \mu X^{-1} e$, then $XZe = \mu e$, where $Z = diag(z)$

$$-Qx + A^T \pi + z = c \qquad (4)$$
$$Ax = b \qquad (5)$$
$$XZe = \mu e \qquad (6)$$

Central path $\Gamma = \{(x,y,z) \mid -Qx + A^T\pi + z = c, Ax = b, XZe = \mu e, \mu > 0\}$

Newton-Raphson Method

The conditions (4) – (6) for given penalty parameter $\mu > 0$ can be represented as:

$$F(x,\pi,z) = \begin{bmatrix} -Qx + A^T\pi + z \\ Ax - b \\ XZe \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu e \end{bmatrix}$$

$\mu = \tau y$

where centering parameter $\tau \in (0,1)$, $y = \frac{1}{n} \sum_{i=1}^{n} x_i z_i$

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} d_x \\ d_\pi \\ d_z \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -XZe + \mu e \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -XZe + \tau y e \end{bmatrix} \qquad (8)$$

Start with infeasible interior point

Define the residual: Primal residual $r_p = Ax - b$
Dual residual $r_d = -Q^T x + A^T \pi + z - c$

Predictor Step

Set $\tau = 0$

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} d_x^{aff} \\ d_\pi^{aff} \\ d_z^{aff} \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -XZe \end{bmatrix}$$

Solve affine scaling direction $d^{aff} = \begin{bmatrix} d_x^{aff} & d_\pi^{aff} & d_z^{aff} \end{bmatrix}$

step length $\alpha^{aff} = \min\left\{1, \min_{i:\left(d_x^{aff}\right)_i<0} -\frac{x_i}{\left(d_z^{aff}\right)_i}, \min_{i:\left(d_z^{aff}\right)_i<0} -\frac{z_i}{\left(d_z^{aff}\right)_i}\right\}$

center parameter: $\tau = \left(\frac{y^{aff}}{y}\right)^3$

where $\quad y_{aff} = \frac{\left(x+\alpha^{aff}d_x^{aff}\right)^T\left(z^{(k)}+\alpha^{aff}d_z^{aff}\right)}{3}$

$y = \frac{(x)^T z}{3}$

Correct and centering step

Add back terms τy and $-D_x D_z$ where $D_x = diag(d_x^{aff}), D_z = diag(d_z^{aff})$

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} d_x \\ d_\pi \\ d_z \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -XZe - D_x D_z e + \tau y e \end{bmatrix}$$

Solve direction $d = [d_x \quad d\_\pi \quad d_z]$

step length $\alpha = \min\{1, \eta\alpha_x^{max}, \eta\alpha_z^{max}\}$ where dampening parameter $\eta \in (0.9,1)$

The implementation of primal-dual interior point method (predictor-corrector version)

(explicit formula can be found above)

1. Start with an initial interior point $(x^{(0)}, \pi^{(0)}, z^{(0)})$
   Let tolerance ϵ be small positive number

2. Solve for affine scaling direction= $[d_{x^{(k)}}^{aff} \quad d_{\pi^{(k)}}^{aff} \quad d_{z^{(k)}}^{aff}]$

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ Z^{(k)} & 0 & X^{(k)} \end{bmatrix} \begin{bmatrix} d_{x^{(k)}}^{aff} \\ d_{\pi^{(k)}}^{aff} \\ d_{z^{(k)}}^{aff} \end{bmatrix} = \begin{bmatrix} -r_d^{(k)} \\ -r_p^{(k)} \\ -X^{(k)} Z^{(k)} e \end{bmatrix}$$

   Compute $(\alpha^{aff})^{(k)}$, $y^{(k)}, y_{aff}^{(k)}, \tau^{(k)}$

   Solve for affine scaling direction= $[d_{x^{(k)}} \quad d_{\pi^{(k)}} \quad d_{z^{(k)}}]$

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ Z^{(k)} & 0 & X^{(k)} \end{bmatrix} \begin{bmatrix} d_{x^{(k)}} \\ d_{\pi^{(k)}} \\ d_{z^{(k)}} \end{bmatrix} = \begin{bmatrix} -r_d^{(k)} \\ -r_p^{(k)} \\ -X^{(k)} Z^{(k)} e - D_x^{(k)} D_z^{(k)} e + \tau^{(k)} y^{(k)} e \end{bmatrix}$$

   Compute $\alpha^{(k)}$

   Update solution: $x^{(k+1)} = x^{(k)} + \alpha^{(k)} d_{x^{(k)}}$
   $\pi^{(k+1)} = \pi^{(k)} + \alpha^{(k)} d_{\pi^{(k)}}$
   $z^{(k+1)} = z^{(k)} + \alpha^{(k)} d_{z^{(k)}}$

3. Repeat step 2 until $\left\|Ax^{(k+1)} - b\right\| < \epsilon$
   $\left\|-Qx^{(k+1)} + A^T\pi^{(k+1)} + z^{(k+1)} - c\right\| < \epsilon$
   $(x^{(k+1)})^T z^{(k+1)} < \epsilon$

Markowitz mean variance optimization

$$\min c^T x + \frac{1}{2} x^T Q x$$
$$\text{Subject to } Ax = b, x \geq 0$$

$$\text{Where } c = [0 \quad 0 \quad 0], Q = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.0002 \\ 0.00021 & -0.0002 & 0.00115 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 0.1073 & 0.0737 & 0.0627 \\ 1 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 0.0650 \\ 1 \end{bmatrix}$$

Q is PD since all leading principal minors are positive
A is full row rank since all rows are linear independent

start with an infeasible solution: $x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \pi^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, z^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Set tolerance $\epsilon = 10^{-8}, k = 0, \eta = 0{,}95$

Primal solution of QP

	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$c^T x$
0	1.000000000	1.000000000	1.000000000	0.000000000
1	0.050000000	0.448614116	0.683928296	0.000000000
2	0.017312985	0.169889624	0.825590825	0.000000000
3	0.015198689	0.149016882	0.836424100	0.000000000
4	0.015894912	0.144761318	0.839392084	0.000000000
5	0.022028593	0.119789584	0.858187852	0.000000000
6	0.025939248	0.103919780	0.870141274	0.000000000
7	0.026285546	0.102515005	0.871199464	0.000000000
8	0.026303047	0.102444010	0.871252944	0.000000000

Dual solution of QP

	$\pi_1^{(k)}$	$\pi_2^{(k)}$	$z_1^{(k)}$	$z_2^{(k)}$	$z_3^{(k)}$	$b^T \pi$
0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.065000000
1	-20.206745134	1.258530294	1.102350132	0.423671324	0.197299163	-0.054908140
2	-16.968481268	1.068164992	0.767140851	0.197373397	0.009864958	-0.034786290
3	-1.104876954	0.070352951	0.050038710	0.013283595	0.000517464	-0.001464051
4	-0.049884829	0.004078903	0.002501935	0.001150727	0.000038364	0.000836389
5	0.003776684	0.000731681	0.000125097	0.000241857	0.000005320	0.000977166
6	0.006962107	0.000548763	0.000010007	0.000020385	0.000000349	0.001001300
7	0.007231352	0.000533489	0.000000500	0.000001028	0.000000017	0.001003527
8	0.007244848	0.000532724	0.000000025	0.000000051	0.000000001	0.001003639

The residuals and τ

	τ	$\|\|Ax-b\|\|$	**residual2**	$(x^{(k)})^T z$
0		2.007968e+00	3.577221e+00	3.000000e+00
1	0.00589407	1.832696e-01	3.264973e-01	3.801209e-01
2	0.00662218	1.284440e-02	2.288247e-02	5.495761e-02
3	0.00114608	6.422200e-04	1.144123e-03	3.172822e-03
4	1.96069e-05	4.850730e-05	8.641640e-05	2.385510e-04
5	0.0104547	6.053190e-06	1.078384e-05	3.629293e-05
6	0.00690251	3.026595e-07	5.391919e-07	2.681929e-06
7	1.1781e-05	1.513756e-08	2.696776e-08	1.337910e-07
8	1.7116e-09	7.568802e-10	1.348392e-09	6.686523e-09

Observe as $k \to 8, c^T x - b^T \pi \to 0, \tau, residuals \to 0$