MIE 1622H Assignment 4 - Asset Pricing

1. (60 %) Implement pricing functions in Python

$S_0 = 100, K = 105, \mu = 0.05, \sigma = 0.2, r = 0.05, T = 1, Sb = 110$

European option (non-dividend)

1. Black-Scholes formula (non-dividend)

Pricing call & put option:

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$
$$P(S,t) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}[In\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)]$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

2. Monte Carlo simulations

Apply Geometric Brownian Motion (GBM) with constant drift μ and volatility σ, which is described by Geometric Random Walk with small time increment

$$S_{t+1} = S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma \cdot \epsilon_t}$$

Determine the payoff for simulate paths
Average call and put payoffs and discount back to current time with the riskless short rate r

$$C(S,t) = mean(max(K - S_T, 0)) \cdot e^{-rT}$$
$$P(S,t) = mean(max(S_T - K, 0)) \cdot e^{-rT}$$

Barrier knock-in option

A knock-in option becomes a standard European option if the predetermined barrier is crossed during the life of the option, otherwise expires worthless. The remaining process is same as above step.

2. (30 %) Analyze your results:

Produce Black-Scholes call and put price for the given European option.

Compute one-step MC call and put price for the given European option.
Compute multi-step MC call and put price for the given European option.
Compute one-step MC call and put price for the given Barrier option.
Compute multi-step MC call and put price for the given Barrier option.

For the Monte Carlo pricing procedure choose a number of time steps and a number of scenarios.

1. For all case: set number of paths to 1000000 to cover more scenarios.
2. For one-step MC: set number of steps to 1, compute the price of an underlying only once at the end of the life of an option
3. For multiple-step MC: set number of steps to 252, compute the price of underlying each business day. In this way, the time increment is small and the assumption to apply GBM could be satisfied.

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Black-Scholes price of an European call option is 8.021352235143176
Black-Scholes price of an European put option is 7.9004418077181455
One-step MC price of an European call option is 8.015212933670396
One-step MC price of an European put option is 7.889813108168575
Multi-step MC price of an European call option is 8.00196059121057
Multi-step MC price of an European put option is 7.915066262365608
One-step MC price of an Barrier call option is 7.81502543181406
One-step MC price of an Barrier put option is 0.0
Multi-step MC price of an Barrier call option is 8.000693483498955
Multi-step MC price of an Barrier put option is 2.0157995036943803
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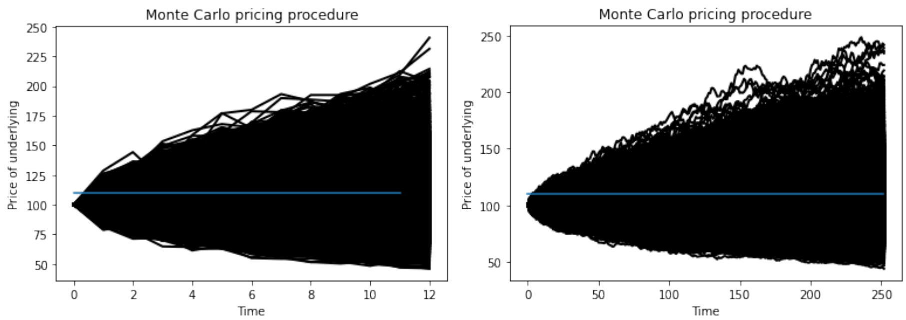
Plot one chart in Python that illustrates your Monte Carlo pricing procedure in the best way.

Plot the Monte Carlo pricing procedure when number of steps equal to 12(left) and 252 (right). The horizonal blue line represents the barrier. The more larger the number of steps, the more fluctuate the asset price during the life of options.



Compare three pricing strategies for European option and discuss their performance relative to each other.

European	Call	Put
Black-Scholes	8.0214	7.9004
One-step MC	8.0152	7.8898
Multi-step MC	8.0020	7.9151

The option prices for European option derived by three strategies are very close. When calculate European option price for single asset, Black Scholes method is quite accurate, and the computational time is fast. Rank the option price from highest to lowest:

Call: Black-Scholes > One-step MC > Multi-step MC
Put: Multi-step MC > One-step MC > Black-Scholes

Compare to the option prices derived by one-step MC, the price of multi-step MC call option is lower and put option is higher, imply that the mean asset price at the end of the life for multi-step MC is lower than one-step MC.

Explain the difference between call and put prices obtained for European and Barrier options.

Strategy	Option	Call	Put
Black-Scholes	European	8.0214	7.9004
One-step MC	European	8.0152	7.8898
	Barrier	7.8150	0
Multi-step MC	European	8.0020	7.9151
	Barrier	8.0007	2.0158

The Barrier options will only be exercised if the asset price hits the barrier during the option life. Otherwise, the option is worthless, and the payoff is zero for that path. The holders of the Barrier options will take more risk. As a result, the call and put price of Barrier options will lower than European options.

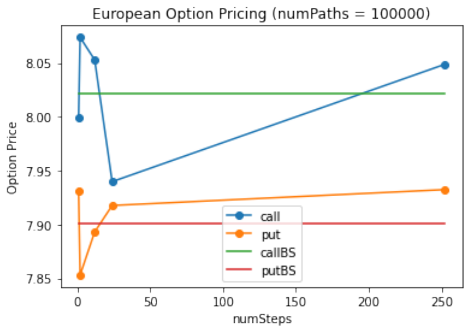
Compute prices of Barrier options with volatility increased and decreased by 10% from the original inputs.

σ	Strategy	Call	Put
One-step MC	0.18	6.9916	0
	0.2	7.8150	0
	0.22	8.6047	0
Multi-step MC	0.18	7.1975	1.5604
	0.2	8.0007	2.0158
	0.22	8.7836	2.5056

The larger the volatility, the higher the chance that the price of underlying will hit the predetermined barrier during the lifetime of option. The holders will take less risk, the call and put options price for Barrier options will increase.

3. (10 %) Discuss possible strategies to obtain the same prices from two procedures:

Design your own procedure for choosing a number of time steps and a number of scenarios in Monte Carlo pricing for European option to get the same price (up to the cent) as given by the Black-Scholes formula.



Set number of paths to 100000, compute the MC pricing for European option for number of paths equal to [1, 2, 12, 24, 252]. Visualize the results, above plot illustrates that two procedures obtain the same price when number of paths within the range between 12 and 24. Set the number of paths equal to np.arange (12,24), recompute the MC pricing for European option, break the procedure when the option prices obtained by two procedures equal.

When numSterps = 12
MC pricing for European call = 8.02
Black-Scholes price of a European call option = 8.02

When numSterps = 19
MC pricing for European put = 7.9
Black-Scholes price of a European put option = 7.9