PORTFOLIO OPTIMIZATION PROJECT
Implement and Compare Financial Optimization Models

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PARAMETR ESTIMATION

Investment universe consists of 20 stocks (n=20) including F, CAT, DIS, MCD, KO, PEP, WMT, C, WFC, JPM, AAPL, IBM, PFE, JNJ, XOM, MRO, ED, T, VZ and NEM.

For the given period, 54 historical monthly return (r_{it}) for each asset could be derived by using the following formular.

$r_{it} = \frac{P_{k+1} - P_k}{P_k}$, where P_k is the adjusted closing price at the last trading day of the month

Using monthly return (r_{it}) to compute the **mean**, **variance** and **covariances** between all assets.

Arithmetic mean $\overline{r_i} = \frac{1}{T}\sum_{t=1}^{T} r_{it}$

Geometric mean $\mu_i = (\prod_{t=1}^{T}(1 + r_{it}))^{\frac{1}{T}} - 1$

Covariance $\sigma_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \overline{r_i})(r_{jt} - \overline{r_j})$

where T is sample size. Screenshots of computed results are shown in the Table 1 (only part of covariance due to limited space)

	Arithmetic Mean	Geometric Mean	Sample variance
F	-0.014276	-0.021158	0.014721
CAT	0.008088	0.006003	0.004283
DIS	0.004131	0.003179	0.001953
MCD	0.017955	0.016680	0.002626
KO	0.008147	0.007545	0.001234
PEP	0.009273	0.008628	0.001304
WMT	0.005057	0.004109	0.001960
C	-0.011948	-0.015202	0.006352
WFC	0.009518	0.007098	0.005304
JPM	0.009691	0.006873	0.005800
AAPL	0.037063	0.028424	0.016627
IBM	0.006492	0.004888	0.003253
PFE	-0.003225	-0.004837	0.003286
JNJ	0.004407	0.003931	0.000979
XOM	0.012579	0.010835	0.003717
MRO	0.023319	0.018531	0.009963
ED	0.004553	0.003832	0.001452
T	0.007281	0.005584	0.003385
VZ	0.000829	-0.000519	0.002751
NEM	0.001306	-0.002289	0.007417

	F	CAT	DIS	MCD	KO	PEP
F	0.014721	0.001548	0.001057	0.002166	7.086874e-04	-0.000391
CAT	0.001548	0.004283	0.000862	0.001250	4.969156e-04	-0.000104
DIS	0.001057	0.000862	0.001953	0.000913	5.041601e-04	0.000338
MCD	0.002166	0.001250	0.000913	0.002626	1.013901e-03	0.000560
KO	0.000709	0.000497	0.000504	0.001014	1.234342e-03	0.000708
PEP	-0.000391	-0.000104	0.000338	0.000560	7.084121e-04	0.001304
WMT	0.001724	-0.000161	0.000173	0.000155	6.927632e-07	-0.000157
C	0.004618	0.000337	0.000499	0.001143	5.748997e-04	0.000575
WFC	0.001966	-0.000827	-0.000515	0.000282	2.643581e-04	0.000483
JPM	0.003574	-0.000236	-0.000423	0.000909	7.300467e-04	0.000724
AAPL	0.001562	0.003675	0.001410	0.004075	1.176363e-03	0.000849

Table 1 (Asset mean, Variance, Covariance)

Risk-Free Rate

Treasury bills are hardly to default since the probability of country go bankrupt is very low. The 13 Week Treasury Bill (^IRX) is considered as risk-free rate. Collect daily adjusted closing price at the last trading day of the months from 30-Dec-2004 to 30-Sep-2008. Take the mean and convert to monthly risk-free rate, derived $rf = 0.002953550731358321$.

Risk Aversion

Risk aversion labeled λ refers to preference for certainty over uncertainty, which is positive for risk-averse investor and negative for risk-seeking investors. Taking the suggestion from Black-Litterman paper and setting λ as

$$\lambda = \frac{E(r_{mkt}) - r_f}{\sigma_{mkt}^2}$$

Assume selected 20 stocks represent the market, construct market portfolio using market capitalizations of assets on 30-Sep-2008, estimate mean and variance to compute risk aversion, all computed quantities are showed in Table 2.

Market portfolio x_{mkt} $\qquad$ $x_i = \frac{market\ cap\ of\ the\ asset\ i}{market\ cap\ of\ the\ market}$

Market return $\qquad$ $E(r_{mkt}) = \mu\, x_{mkt}$

Market variance $\sigma^2_{mkt} = x^T_{mkt} Q x_{mkt}$

	Output
monthly risk-free rate	0.002954
Expected Market return	0.005974
Market Variance	0.000841
risk_aversion	3.593889

Table 2 (Estimated Parameters)

PORTFOLIO OPTIMIZATION

Portfolio selection based on the desired investment goals. Use estimated parameters defined below to optimize assets weight x of MVO, robust MVO and risk parity. And market portfolio where the weights are based on market capitalization.

μ Asset expected returns (use geometric mean)
Q Asset covariance matrix
$\lambda > 0$ Risk aversion

1. Mean-Variance Optimization (short selling is allowed)

$$\min_x \quad \boldsymbol{\lambda x^T Q x - \mu^T x}$$
$$\boldsymbol{s.t.} \quad \boldsymbol{1^T x = 1}$$
$$(x \geq 0)$$

Harry Markowitz introduced the idea rational investors seek after maximum return for a given level of risk, or minimum risk for a given amount of return. Mean-variance Optimization (MVO) is developed based on this idea, assets will be selected in an efficient way. And min variance portfolio, max return portfolio and efficient frontier can be produced by MVO. There are also some major limitations. MVO is very sensitive to the expected asset mean return, which is difficult to estimate accurately. Inaccurately expect retune can lead to inefficient portfolios. Moreover, MVO sometimes create concentrated portfolios, which can be tell from the Figure 1 (portfolio weights of MVO with no short selling).

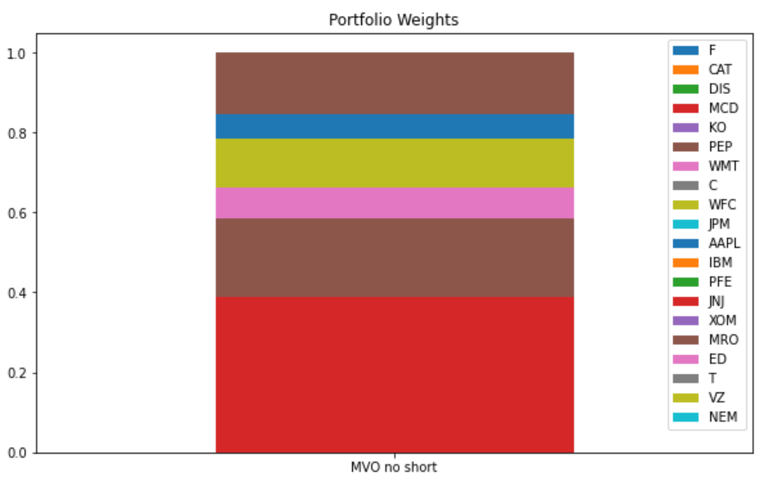

Figure 1

2. Robust Mean-Variance Optimization (short selling is allowed)

$$\min_{x} \quad \lambda x^T Q x - \mu^T x + \epsilon_2 \left|\Theta^{\frac{1}{2}} x\right|_2$$
$$s.t. \quad \mathbf{1}^T x = 1$$
$$(x \geq 0)$$

Incorporate uncertainty into MVO help to reduce the impact caused by inaccurate estimate expect return. Consider use an ellipsoidal uncertainty set

$$\mu^{true} = \{\mu^{true} \in R^n : (\mu^{true} - \mu)^T \Theta (\mu^{true} - \mu)^T \leq \epsilon_2^2\}$$

$\epsilon_2 = x_n^2(\alpha)$ is the distance between estimated and true values, $\Theta = \frac{1}{T} diag(diag(Q))$ measure the uncertainty where $\left(\Theta^{\frac{1}{2}}\right)_{ii} = \frac{\sigma_i}{\sqrt{T}}$ *, and* $\left(\Theta^{\frac{1}{2}}\right)_{ij} = 0 \; for \; i \neq j$.

Figure 2 shows portfolios generated by robust MVO with and without short selling are well diversified.

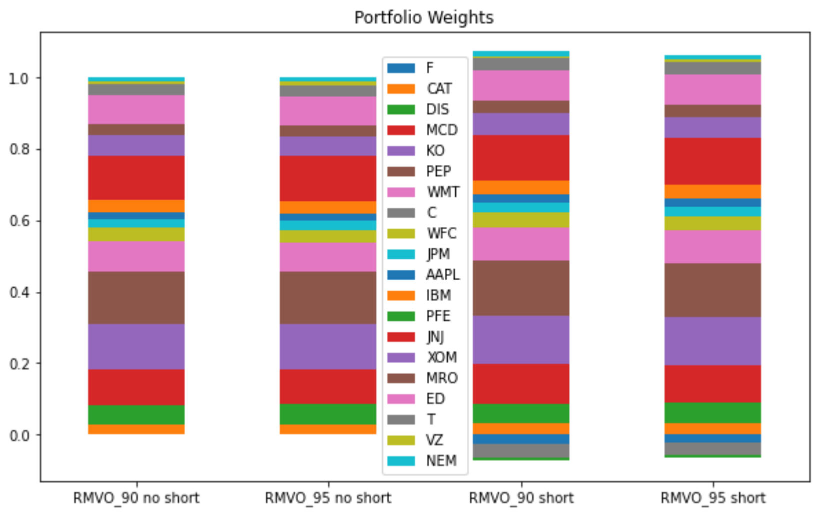

Figure 2

3. Risk Parity Optimization (no short selling)

$$\min_{x,\theta} \quad \sum_{i=1}^{n} (x_i (Qx)_i - \theta)^2$$
$$s.t. \quad \mathbf{1}^T x = 1$$
$$x \geq 0$$

Risk parity optimization is also called equal risk contribution optimization (ERC), another approach to eliminate the impact of inaccurate estimate expect return.

4. Market Portfolio

Assume selected 20 stocks represent the market, construct market portfolio using market capitalizations of assets on 30-Sep-2008, weight of $asset_i$ follows formula $x_i = \frac{market\ cap\ of\ the\ asset\ i}{market\ cap\ of\ the\ market}$.

Figure 3 shows risk parity and market portfolios, which are well diversified.

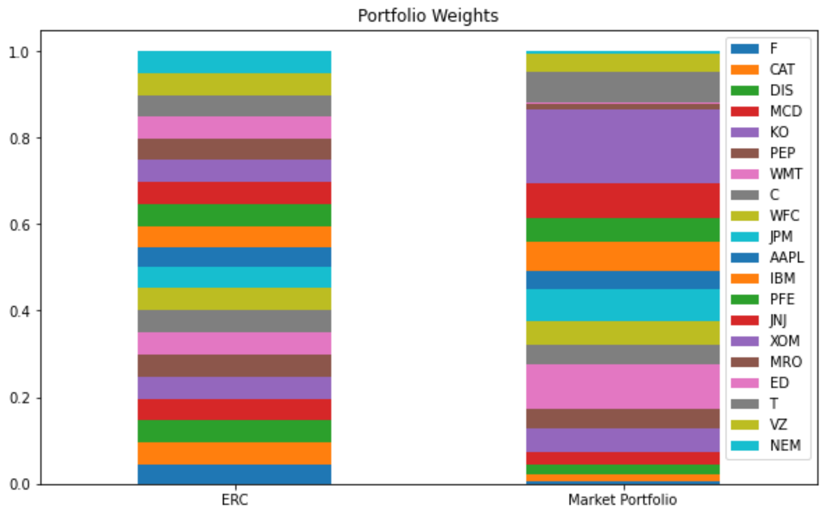

Figure 3

Use Matlab to implement the optimization models, results are presented in the Table 3.

	MVO no short	MVO short	RMVO_90 no short	RMVO_95 no short	RMVO_90 short	RMVO_95 short	ERC	Market Portfolio
F	0.0000	-0.2009	0.0000	0.0000	-0.0247	-0.0226	0.0450	0.005276
CAT	0.0001	-0.0264	0.0302	0.0304	0.0320	0.0320	0.0500	0.015359
DIS	0.0001	0.6748	0.0531	0.0542	0.0554	0.0564	0.0512	0.023900
MCD	0.3881	1.2014	0.0997	0.0974	0.1094	0.1056	0.0500	0.029378
KO	0.0001	-0.9811	0.1280	0.1275	0.1351	0.1338	0.0513	0.052219
PEP	0.1978	0.6750	0.1469	0.1451	0.1553	0.1522	0.0515	0.047833
WMT	0.0769	0.0692	0.0843	0.0837	0.0917	0.0901	0.0513	0.100654
C	0.0000	-1.4137	0.0000	0.0000	-0.0377	-0.0340	0.0495	0.047749
WFC	0.1203	1.5304	0.0356	0.0351	0.0416	0.0402	0.0518	0.053250
JPM	0.0001	0.5058	0.0243	0.0243	0.0295	0.0288	0.0509	0.074356
AAPL	0.0630	0.0411	0.0213	0.0207	0.0230	0.0221	0.0425	0.043015
IBM	0.0000	-0.7729	0.0341	0.0347	0.0366	0.0370	0.0507	0.067128
PFE	0.0000	-1.2104	0.0002	0.0003	-0.0089	-0.0057	0.0510	0.053119
JNJ	0.0001	-0.0935	0.1238	0.1252	0.1291	0.1301	0.0514	0.082709
XOM	0.0001	0.9144	0.0550	0.0543	0.0604	0.0588	0.0499	0.168761
MRO	0.1531	0.3581	0.0324	0.0315	0.0350	0.0336	0.0491	0.012025
ED	0.0001	-0.5307	0.0790	0.0802	0.0843	0.0850	0.0513	0.005020
T	0.0001	0.6501	0.0323	0.0328	0.0342	0.0345	0.0503	0.070290
VZ	0.0000	-0.3298	0.0067	0.0090	0.0057	0.0085	0.0504	0.040684
NEM	0.0000	-0.0609	0.0131	0.0135	0.0129	0.0134	0.0511	0.007273

Table 3 (Portfolio Weights)

PART A & B

Compute the major portfolio quantities for each portfolio, using the realized return for each stock for the month of Oct. 2008 and Nov.2008. Results are presented in the Table 4 & 5.

1. Portfolio Return $r_p = \sum_{i=1}^{n} x_i r_i$
2. Portfolio Variance $\sigma_p^2 = \sum_{i=1}^{n} x_i x_j \sigma_{ij}$

   Portfolio standard deviation σ_p

3. Sharpe ratio $SR_p = \frac{r_p - r_f}{\sigma_p}$

	portfolio return	portfolio var	portfolio sd	Sharpe ratio
MVO_without_short	-0.124502	0.001281	0.035796	-3.560637
MVO_with_short	0.328795	0.010965	0.104712	3.111786
RMVO_90_without_short	-0.125830	0.000586	0.024207	-5.320151
RMVO_95_without_short	-0.125649	0.000582	0.024131	-5.329426
RMVO_90_with_short	-0.107001	0.000570	0.023870	-4.606427
RMVO_95_with_short	-0.108346	0.000564	0.023755	-4.685286
RP	-0.159970	0.000894	0.029898	-5.449223
mkt_portfolio	-0.112932	0.000841	0.028992	-3.997221

Table 4 (Oct. 2008)

In this part, we aim to compare the performance of portfolios generated by different strategies. The easiest way is to compare the portfolio return or variance. However, it only gives partial picture, portfolio with higher return may also have higher risk. Sharp ratio, a risk-adjusted measure of return that can be used to evaluate performance of portfolio. It can tell how much additional return the investor is receiving for the additional volatility of holding the risky portfolio over a risk-free asset. The higher the Sharpe ratio, the better the investment strategy. More specifically, a Sharpe ratio less than 1 is considered bad, between 1 to 2 is considered good, and greater than 3 is considered excellent.

For Oct. 2008 Results

The Market portfolio has negative portfolio return and Sharpe ratio. Under the assumption market are presented by those selected 20 stocks, the market portfolio can well explain the market, and the expected return of the market portfolio is same as the expected return of the market. The large negative Shape ratio means the portfolio return is smaller than the risk-free interest rate, indicates the economy situation is bad.

The MVO with short selling has the highest return but also the highest variance. It is the only portfolio with positive return, which make it has the highest sharp ratio. Short selling allows investor to sell borrowed assets, which have to be bought back and returned at later time. When the market drop, this trading strategy can be profitable. The MVO without short selling will be far less profitable, the variance will be smaller, but not small enough since the portfolio is not well diversified.

The robust MVO portfolios have small variance and moderate return among all portfolios. With short selling is better than without short selling, the return and Sharpe ratio is higher. Cases of 90% confidence level and 95% confidence level, returns similar results. Among all robust MVO portfolios, 90% with short selling has the best performance.

The Risk parity portfolio has the lowest return and Sharp ratio. This strategy aims to minimize the portfolio variance and distribute risk equally among assets. The portfolio return hasn't been considered while constructing the portfolio, it very depends on the situation of the economy. In this case, the portfolio is less risky, but the return is low due to a weak economy.

For Nov. 2008 Results

	portfolio return	portfolio var	portfolio sd	Sharpe ratio
MVO_without_short	-0.020516	0.001281	0.035796	-0.655660
MVO_with_short	-0.055900	0.010965	0.104712	-0.562048
RMVO_90_without_short	-0.011427	0.000586	0.024207	-0.594079
RMVO_95_without_short	-0.011298	0.000582	0.024131	-0.590585
RMVO_90_with_short	-0.008602	0.000570	0.023870	-0.484116
RMVO_95_with_short	-0.008726	0.000564	0.023755	-0.491681
RP	-0.013050	0.000894	0.029898	-0.535257
mkt_portfolio	-0.019298	0.000841	0.028992	-0.767534

Table 5 (Nov.2008)

Compare to Oct results, the Market portfolio has higher portfolio return and Sharpe ratio in Nov, which might mean the economy is start to recovery. This point can be validated from the Figure 4, assets price start to rise around the month of Nov.2008. Among all portfolios, the market portfolio has lower return and the lowest Sharpe ratio, this is because this strategy does not contain smart investment strategy like max return, min variance or max Sharpe ratio. When the economy starts to grow, other strategy will make more profit and beat the market portfolio from the Shape ratio perspective.

Additional Information:

Figure 4 visualize the trend of daily adjusted closing price of all assets from 2005 to 2013.

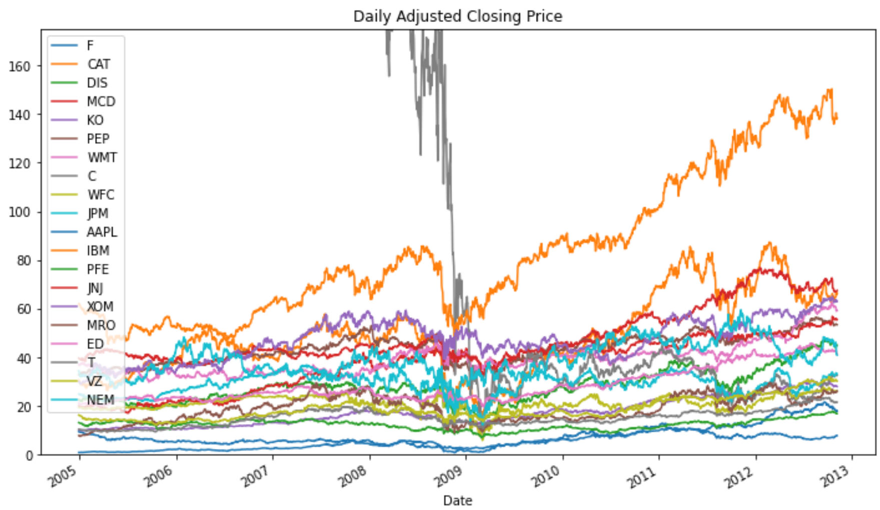

Figure 4

The MVO with short selling has the lowest return and the largest variance. Short selling is a speculation strategy, investors can make profit if the asset price go down, otherwise investors will loss money. In the

Nov.2008, price started to rise, and portfolio return became negative. Large portfolio variance didn't make the Shape ratio look too bad. The MVO without short selling has higher return than with short selling case. However, the Shape ration is smaller since the portfolio variance is far less than with short selling case.

The robust MVOs have higher portfolio return and smaller variance, lead to better performance with higher Sharp ratio. With short selling is better than without short selling, the return and Sharpe ratio is higher. Among all robust MVO portfolios, 90% with short selling still has the best performance.

Compare to Oct results, the Risk parity portfolio has higher return and Sharp ratio. The last session has disused that this strategy is quiet depend on the market since the model doesn't take return in the consideration. In the prosperous economy, this portfolio has low return, low risk and moderate Shape ratio.

PART C

Efficient frontier is a set of efficient portfolios which has the minimum portfolio volatilities at different levels of portfolio returns. Efficient frontier also illustrates a non-linear relationship between portfolio return and portfolio volatility. A portfolio with higher return tends to have higher volatility, enduring higher risk.

In this project, efficient frontiers are built based on varying risk aversion(lambda). Mean-variance optimization model (MVO) and Robust MVO model use these different risk aversions to generate different portfolios with different asset weights, which can be used to compute different portfolio returns and volatilities for building efficient frontier. These portfolios are efficient portfolios since the optimization models try to minimize portfolio volatilities while maximizing portfolio returns with corresponding risk aversion level. Our team decides to use a sample of 20 different risk aversions to construct efficient frontiers.

There are three types of efficient frontier our team implemented.

- Estimated Efficient Frontier

  Optimization models use estimated parameters and different risk aversions to generate different portfolios. Portfolio returns and volatilities are computed based on estimated parameters.

- Actual Efficient Frontier

  Optimization models use estimated parameters and different risk aversions to generate different portfolios. Portfolio returns and volatilities are computed based on realized assets returns.

- True Efficient Frontier

  Optimization models use realized assets returns and different risk aversions to generate different portfolios. Portfolio returns and volatilities are computed based on realized assets returns.

In this project, estimated parameters are the 20 assets monthly returns from 30-Dec-2004 to 30-Sep-2008 described at the previous session. The realized assets returns are 20 assets monthly returns at Oct-2008. Since there is only one monthly return for every asset at Oct-2008, and covariance among assets wouldn't change much for one month regardless of how the market moves, our team decides to use the same covariance obtained from 30-Dec-2004 to 30-Sep-2008 in every calculation of portfolio volatility.

Our team constructs two sets of efficient frontiers. One set allows short sale, and another set only permits positive asset weights. Both sets include (1) estimated MVO efficient frontier, (2) actual MVO efficient frontier, (3) true MVO efficient frontier, (4) estimated robust MVO with 90% confidence interval efficient frontier, (5) actual robust MVO with 90% confidence interval efficient frontier, (6) estimated robust MVO with 95% confidence interval efficient frontier, (7) actual robust MVO with 95% confidence interval efficient frontier.

The following figures show these two sets of efficient frontiers.

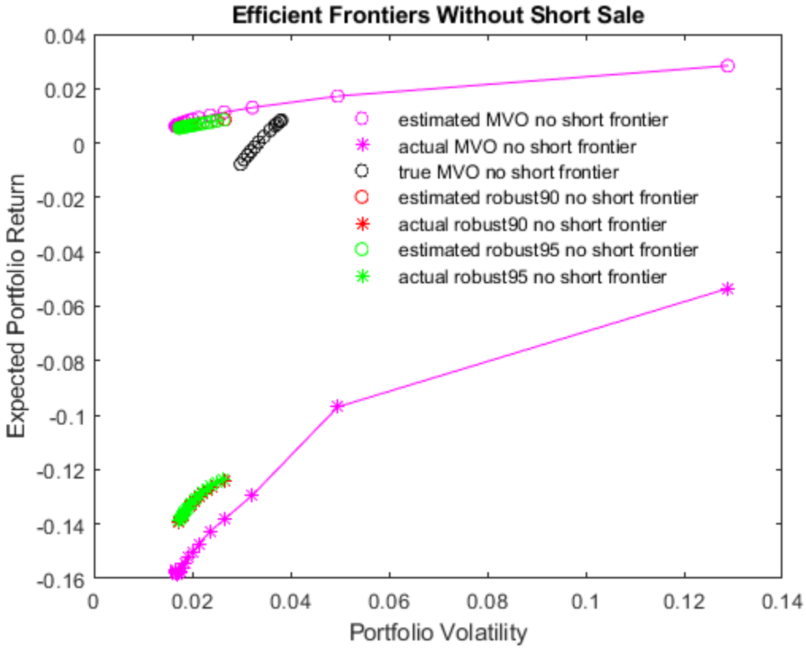

Figure 5 All efficient frontiers without short sale

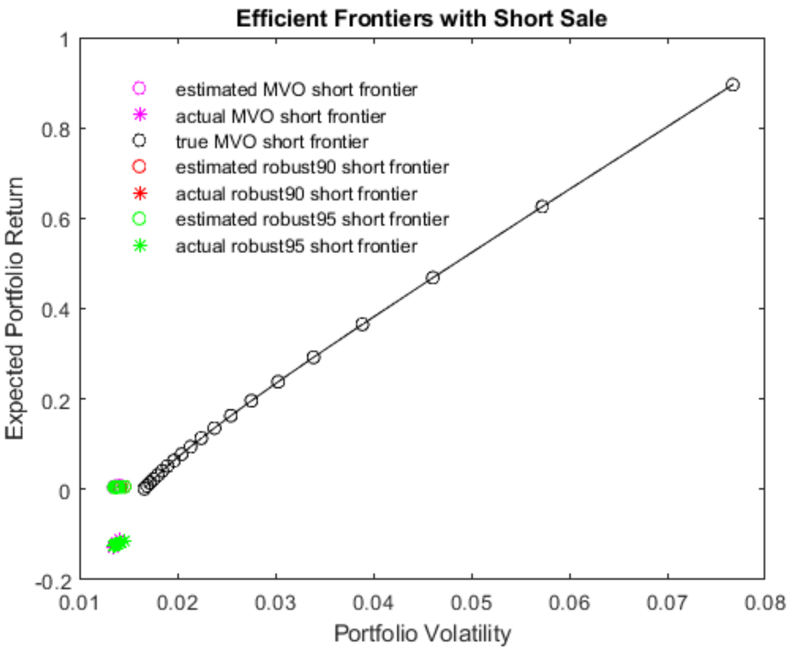

Figure 6 All efficient frontiers with short sale

Figure 5 illustrates all the efficient frontiers without short sales. In this set, lambda ranges from 0.04 to 40. Lambda of 0.04 would let the estimated MVO efficient frontier reach the highest expected monthly return among 20 assets obtained from 30-Dec-2004 to 30-Sep-2008.

Our team sets the upper bound of lambda to 40, as when lambda is small, the true MVO efficient frontier is one point, which is the only positive asset return at Oct 2008. Other assets returns are all negative at Oct 2008. Therefore, our team increases lambda so that optimization models will emphasize more on the risk part, leading to lower portfolio volatility and lower portfolio return, which forces the true MVO efficient frontier to extend downwards to become a line.

In no allowance for short sale case, as most asset expected monthly returns from 30-Dec-2004 to 30-Sep-2008 are positive, all estimated efficient frontiers have positive portfolio returns, and as most asset monthly returns at Oct 2008 are negative, all actual efficient frontiers have negative portfolio returns. Thus, estimated efficient frontiers lie a lot higher than actual efficient frontiers. The true MVO efficient frontier lies at the middle of estimated and actual efficient frontiers.

Estimated Robust MVO with 90% confidence interval efficient frontier lies below the estimated MVO efficient frontier, and lies above the estimated Robust MVO with 95% confidence interval efficient frontier. The actual efficient frontiers follow a reverse order. Actual Robust MVO with 90% confidence interval efficient frontier lies above the actual MVO efficient frontier and lies below the actual Robust MVO with 95% confidence interval efficient frontier. This situation is expected as Robust MVO with the higher confidence interval efficient frontier should be closer to the true MVO efficient frontier, and MVO efficient frontier should be the farthest to the true MVO efficient frontier. The distance between estimated or actual efficient frontier and the true MVO efficient frontier represents how much consideration the efficient frontier has on uncertainty of expected return. The longest distance means the efficient frontier has no consideration, which generally are the MVO efficient frontiers, so that they would generate unstable portfolios with respect to disturbance on expected return. While Robust MVO with higher confidence interval efficient frontiers generally have shorter distances so that they would generate more stable portfolios in terms of disturbance on expected return as they put more consideration on uncertainty of expected asset returns.

Figure 6 illustrates all the efficient frontiers with short sale. In this set, lambda ranges from 90 to 700.

Our team decides to set lambda so high since the true MVO efficient frontier can be brought down so that it can lie along with the other estimated and actual efficient frontiers more properly. Because in the case of allowance for short sale, as most assets' monthly returns at Oct 2008 are negative, the true MVO efficient frontier can be enormously high. While as most asset expected monthly returns from 30-Dec-2004 to 30-Sep-2008 are positive, estimated efficient frontiers have positive returns which is however extremely small compared to the true MVO efficient frontier. As for the actual efficient frontiers, the resulting portfolios generated from optimizations models using estimated parameters have some negative asset weights so that the actual MVO efficient frontier can have higher portfolio returns than the estimated MVO efficient frontier with low limit on risk (lambda is low). That’s another reason to have high lambda so that actual MVO efficient frontier can stay below the estimated MVO efficient frontier under high emphasis on risk. Actual Robust MVO efficient frontiers don’t show this kind of behavior. They stay below the estimated Robust MVO efficient frontier regardless of lambda’s value, as Robust MVO are more stable in terms of changes in expected assets returns.

The order of efficient frontiers is similar to the no short sale case, except the actual efficient frontiers. The actual Robust MVO with 90% confidence interval efficient frontier lies below the actual MVO efficient frontier and lies above the actual Robust MVO with 95% confidence interval efficient frontier, which is opposite to no short sale case. This is also expected as the true MVO efficient frontier lies towards the upper right of the graph. The actual efficient frontiers lie this way so that actual Robust MVO with 95% confidence interval efficient frontier is the closest to the true MVO efficient frontier and the actual MVO efficient frontier is the farest to the true MVO efficient frontier.

The zoom in views of unclear parts of Figure 5 and Figure 6 are shown in the Appendix.

Appendix:

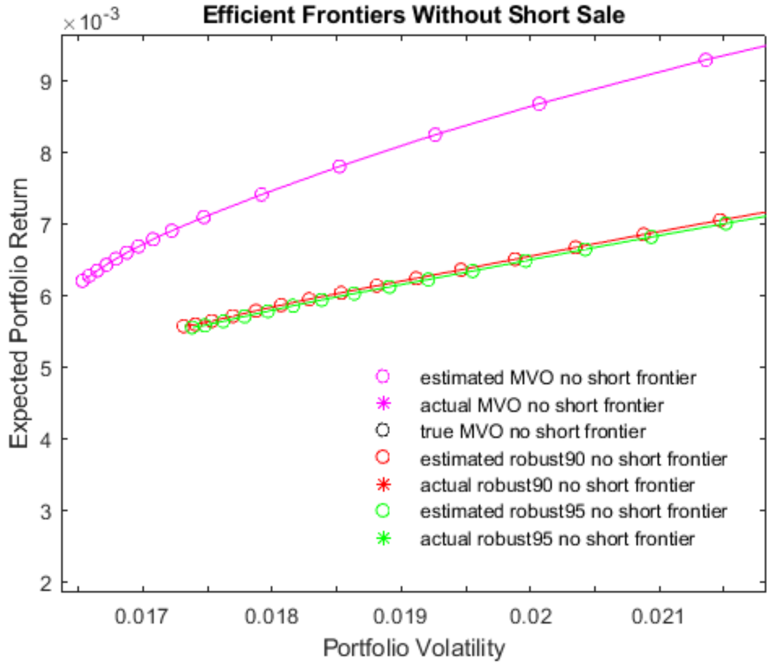

Figure 6 Zoom in estimated efficient frontiers in no short case

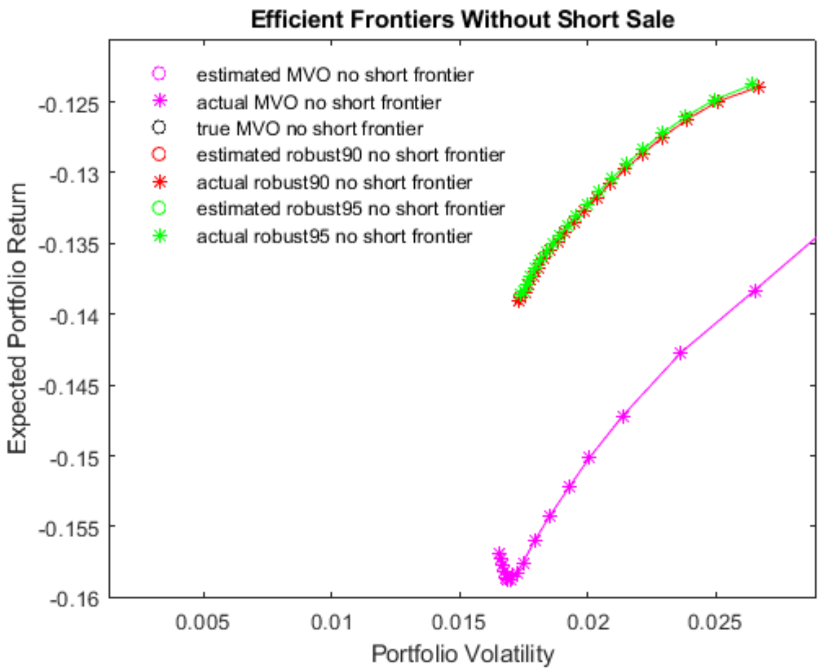

Figure 7 Zoom in actual efficient frontiers in no short case

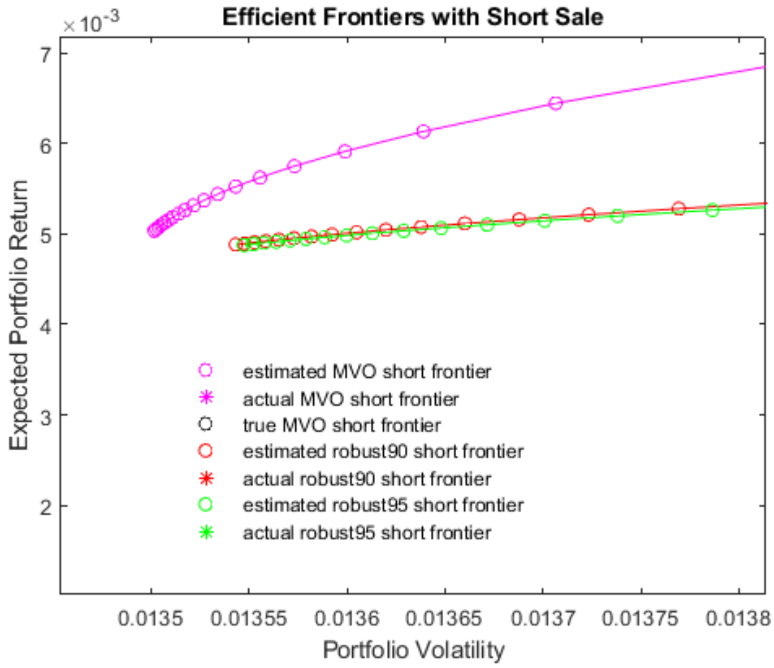

Figure 8 Zoom in estimated efficient frontiers with short sale case

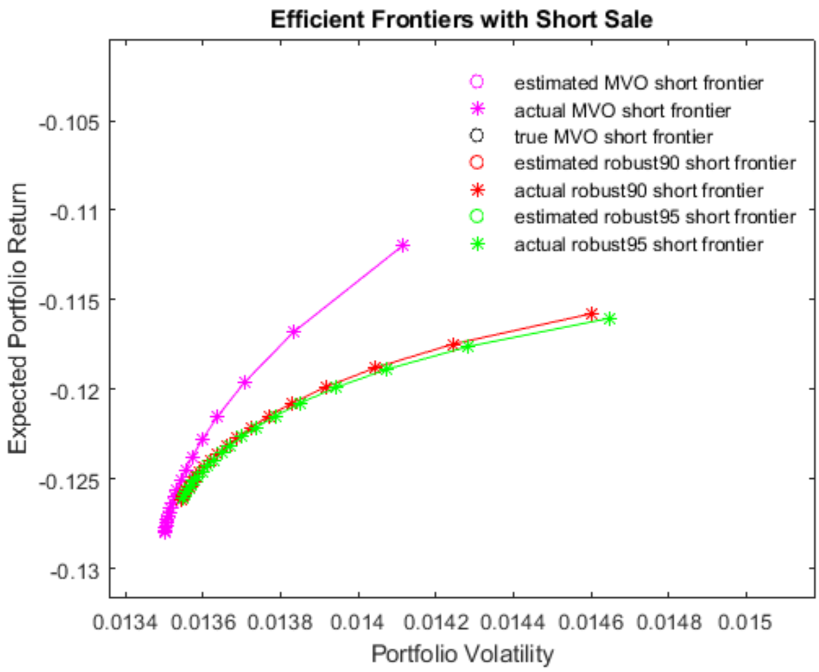

Figure 9 Zoom in actual efficient frontiers with short sale case