Exercise 10.39

XarisGeorgiadis

We know that $Y = e^X$, where $X \sim U(0,1)$. Thus, we can easily conclude that:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Since e^X increases exponentially while X is increasing and $R_X = [0,1]$, then $R_Y = [1,e]$ since for $X = 0$ we get $Y = 1$ and for $X = 1$ we get $Y = e$.
So we have:

$$F_Y(y) = P(Y \leq y) = 0, \ y < 1,$$
$$F_Y(y) = P(Y \leq y) = 1, \ y \geq e.$$

Thus $\forall y \in [1,e]$ we get:

$$F_Y(y) = P(Y \leq y) = P(e^x \leq y) = P(X \leq \ln y) = F_x(\ln y) = \ln y$$

Finally:

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \ln y, & 1 \leq y < e \\ 1, & y \geq e \end{cases}$$

Since $F_Y(y)$ is a continuous function, we can find its derivative in order to find the PDF of Y:

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{y}, & 1 \leq y \leq e \\ 0, & \text{otherwise} \end{cases}$$