Mat Geo Presentation

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January 16, 2025

Problem Statement

Find the area of the region in the first quadrant enclosed by x-axis, line $y = x$ and the circle $x^2 + y^2 = 32$.

Matrix Equation

$$g(x) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.1}$$

$$\begin{aligned} \mathbf{V} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{u} &= \mathbf{0} \\ f &= -32 \end{aligned} \tag{3.2}$$

$$Line : \mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{3.3}$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.4}$$

$$\mathbf{m} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.5}$$

$$\mathbf{x_i} = \mathbf{h} + k_i\mathbf{m} \tag{3.6}$$

Point of Intersection

By Substituting (3.6) in (3.1) we get two values of k_1 and k_2

$$k_1 = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-m^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - g(\mathbf{h}) \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)} \right) \tag{3.7}$$

$$k_2 = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-m^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) - \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - g(\mathbf{h}) \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)} \right) \tag{3.8}$$

so we get

$$k_i = \pm 4\sqrt{2} \tag{3.9}$$

$$x_i = \pm \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{3.10}$$

As we are only considering the first quadrant, we take the point of intersection as $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Area

$$Area : A = A_1 + A_2 \tag{3.11}$$

$$A_1 = \int_0^4 x \, dx \tag{3.12}$$

$$= \left[\frac{x^2}{2} \right]_0^4 \tag{3.13}$$

$$= 8 \tag{3.14}$$

$$A_2 = \int_4^{4\sqrt{2}} \sqrt{32 - x^2} \, dx \tag{3.15}$$

$$= \left[\frac{x\sqrt{32 - x^2}}{2} + 16 \sin^{-1} \frac{x}{4\sqrt{2}} \right]_4^{4\sqrt{2}} \tag{3.16}$$

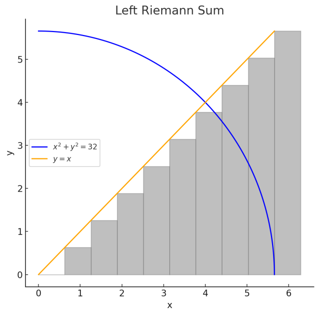
$$= 4\pi - 8 \tag{3.17}$$

$$\therefore A = A_1 + A_2 = 4\pi \approx 12.56637 \tag{3.18}$$

computational solutions

Area Approximation can be done by various methods:
1) Left Riemann sum
2) Right Riemann sum
3) Mid Point Rule
4) Trapezoidal Rule
5) Simpson's Rule
6) Method of Exhaustion

Left Riemann

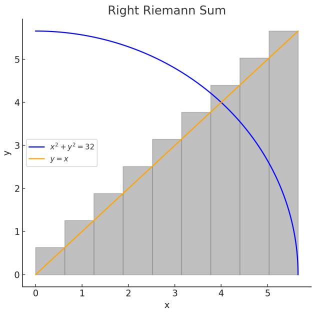


The Left Riemann Sum approximates the area by using rectangles whose heights are determined by the left endpoint of each subinterval. Let h be the step size, and A_n be the area till x_n, then:

$$A_n = h \sum_{i=0}^{n-1} y(x_i), \tag{3.19}$$

$$A_{n+1} = A_n + hy(x_n). \tag{3.20}$$

Right Riemann

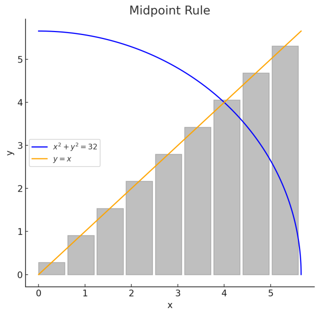


The Right Riemann Sum approximates the area by using rectangles whose heights are determined by the right endpoint of each subinterval. Let h be the step size, and A_n be the area till x_n, then:

$$A_n = h \sum_{i=1}^{n} y(x_i), \tag{3.21}$$

$$A_{n+1} = A_n + h y(x_{n+1}). \tag{3.22}$$

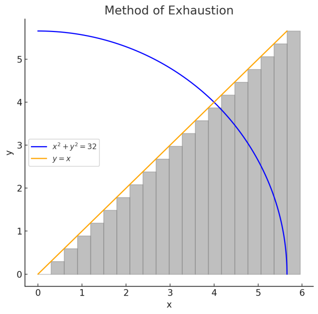
Mid Point Rule



The Midpoint Rule approximates the area by using rectangles whose heights are determined by the midpoint of each subinterval. Let h be the step size, and A_n be the area till x_n, then:

$$A_n = h \sum_{i=1}^{n} y\left(x_{i-\frac{1}{2}}\right), \quad x_{i-\frac{1}{2}} = x_{i-1} + \frac{h}{2}, \tag{3.23}$$

$$A_{n+1} = A_n + h \cdot y\left(x_n + \frac{h}{2}\right). \tag{3.24}$$



Method of Exhaustion

This method divides the area into smaller shapes (rectangles, trapeziums, etc.) and calculates the total area as the limit of the sum as $n \to \infty$. The area A_n is given by:

$$A_n = \sum_{i=1}^{n} A_i, \quad \text{where } A_i \text{ is the area of each small shape.} \tag{3.25}$$

For rectangles:

$$A_i = h \cdot y(x_i), \quad \text{where } h \to 0, \tag{3.26}$$

$$A_{n+1} = A_n + hy(x_n). \tag{3.27}$$

For trapeziums:

$$A_i = \frac{h}{2}\left(y(x_{i-1}) + y(x_i)\right), \tag{3.28}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_n) + y(x_n + h)\right). \tag{3.29}$$

Trapezoidal Rule

In the Trapezoidal method, We split the area into multiple small trapeziums (like small strips), and we sum up all the trapezium areas to find the total area.
We discretize the range of x-coordinates with uniform step-size $h \to 0$, such that the discretized points are x_0, x_1, ..., x_n and $x_{n+1} = x_n + h$.
Let the sum of trapizoidal areas till x_n be A_n and $y = y(x)$, then we write the **difference equation**,

$$A_n = \frac{h}{2}\left(y\left(x_0\right) + y\left(x_1\right)\right) + \frac{h}{2}\left(y\left(x_1\right) + y\left(x_2\right)\right) + \cdots + \frac{h}{2}\left(y\left(x_{n-1}\right) + y\left(x_n\right)\right) \tag{3.30}$$

$$A_n = h\left(\frac{y\left(x_0\right)}{2} + y\left(x_1\right) + y\left(x_2\right) \ldots \frac{y\left(x_n\right)}{2}\right) \tag{3.31}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_{n+1}\right) + y\left(x_n\right)\right),\ x_{n+1} = x_n + h \tag{3.32}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{3.33}$$

By the first principle of derivative,

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \tag{3.34}$$

$$y(x+h) = y(x) + h\left(y'(x)\right),\ h \to 0 \tag{3.35}$$

Rewriting the difference equation, we get,

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_n) + hy'(x_n) + y(x_n)\right) \tag{3.36}$$

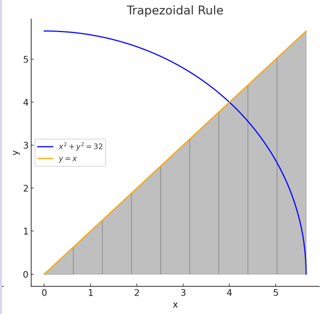
$$A_{n+1} = A_n + h\left(y(x_n) + \frac{h}{2}y'(x_n)\right) \tag{3.37}$$

$$A_{n+1} = A_n + hy(x_n) + \frac{h^2}{2}y'(x_n) \tag{3.38}$$

For the given area enclosed, we take

$$y(x) = \begin{cases} x & 0 < x < 4 \\ \sqrt{32 - x^2} & 4 < x < 4\sqrt{2} \end{cases} \tag{3.39}$$

Trapezoidal Rule



Substituting $y(x)$, the equation becomes,

$$A_{n+1} = \begin{cases} A_n + hx_n + \frac{h^2}{2} & 0 < x_n < 4 \\ A_n + h\sqrt{32 - x_n^2} + \frac{h^2}{2}\left(\frac{-x_n}{32 - x_n^2}\right) & 4 < x_n < 4\sqrt{2} \end{cases} \tag{3.40}$$

$$x_{n+1} = x_n + 1 \tag{3.41}$$

Computational Area: 12.56576
Theoritical Area: 12.56637
Plotting the given equations, we get the following plot.

Plot