Gradient Ascent

CHARAN RONGALI
Electrical Engineering,
IIT Hyderabad

February 5, 2025

Problem Statement

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius R is $\frac{4R}{3}$.

Geometric Analysis

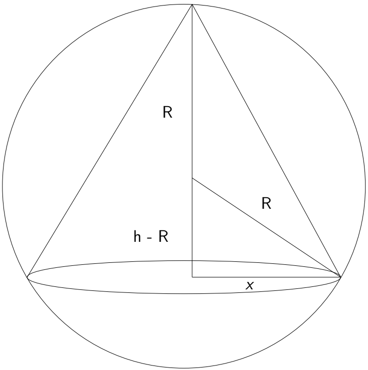
Let R be the radius of the sphere, and let h be the height (altitude) of the inscribed cone. Let x be the radius of the base of the cone.

By considering a cross-section of the sphere and the inscribed cone, we relate x, h, and R using the Pythagorean theorem. The center of the sphere lies on the axis of the cone.

$$x^2 + (h - R)^2 = R^2 \tag{3.1}$$

$$x^2 + h^2 - 2hR + R^2 = R^2 \tag{3.2}$$

$$x^2 = 2hR - h^2 \tag{3.3}$$



Volume Maximization

The volume V of the cone is given by:

$$V = \frac{1}{3}\pi x^2 h \tag{3.4}$$

$$V = \frac{1}{3}\pi(2hR - h^2)h \tag{3.5}$$

$$V = \frac{1}{3}\pi(2Rh^2 - h^3) \implies [CostFunction] \tag{3.6}$$

To maximize the volume, we take $\frac{dV}{dh} = 0$:

$$\frac{dV}{dh} = \frac{1}{3}\pi(4Rh - 3h^2) \tag{3.7}$$

$$0 = h(4R - 3h) \implies h \neq 0 \tag{3.8}$$

$$4R - 3h = 0 \tag{3.9}$$

$$h = \frac{4R}{3} \tag{3.10}$$

Second Derivative Test

To confirm a maximum, we take the second derivative:

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4R - 6h) \tag{3.11}$$

Substituting $h = \frac{4R}{3}$:

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4R - 8R) \tag{3.12}$$

$$= -\frac{4\pi R}{3} \tag{3.13}$$

Since this is negative, $h = \frac{4R}{3}$ gives a maximum volume.

Computational Approach

Using Gradient Ascent, the update rule is:

$$h_{n+1} = h_n + \alpha \frac{dV}{dh}\bigg|_{h=h_n} \tag{3.14}$$

For our case:

$$h_{n+1} = h_n + \alpha \frac{\pi}{3}(4Rh_n - 3h_n^2) \tag{3.15}$$

Choosing:

$$R = 1, \quad \alpha = 0.0001, \quad h_0 = 0.5 \tag{3.16}$$

This converges to $h = \frac{4}{3}$.

Graphical Representation

After a few iterations, the value of h will converge to approximately $\frac{4}{3} \approx 1.333$.