LU decomposition

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Problem Statement

Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{5}{x-1} + \frac{1}{y-2} = 2$$
$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

Matrix representation

Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{x-1} = u \tag{3.1}$$

$$\frac{1}{y-2} = v \tag{3.2}$$

Then our equations become:

$$5u + v = 2 \tag{3.3}$$

$$6u - 3v = 1 \tag{3.4}$$

This can be written in matrix form as:

$$\begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.5}$$

$$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b} \tag{3.6}$$

Doolittle's Algorithm

This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:
Elements of the U Matrix:
For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{3.7}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{3.8}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{3.9}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{3.10}$$

LU factorization

By doing the following steps and solving we get :

$$\mathbf{U} = \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{3.11}$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \tag{3.12}$$

Now,

$$A = \begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{3.13}$$

We can solve this using two steps:

$$L\mathbf{y} = \mathbf{b} \tag{3.14}$$

$$U\mathbf{x} = \mathbf{y} \tag{3.15}$$

Substitution

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{3.16}$$

$$\mathbf{y} = \begin{pmatrix} 2 \\ -\frac{7}{5} \end{pmatrix} \tag{3.17}$$

Now using back substitution:

$$\begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{7}{5} \end{pmatrix} \tag{3.18}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \tag{3.19}$$

By equating this $\frac{1}{x-1} = u$ and $\frac{1}{y-2} = v$ we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{3.20}$$

Graphical Representation