10.3.1.1

EE24BTECH11052 - Rongali Charan

Question:
Solve the following pairs of equations by reducing them to a pair of linear equations:

$$\frac{5}{x-1} + \frac{1}{y-2} = 2$$
$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

Solution:
Let's solve this using LU decomposition. First, let's substitute:

$$\frac{1}{x-1} = u \tag{0.1}$$
$$\frac{1}{y-2} = v \tag{0.2}$$

Then our equations become:

$$5u + v = 2 \tag{0.3}$$
$$6u - 3v = 1 \tag{0.4}$$

This can be written in matrix form as:

$$\begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.5}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{0.6}$$

The upper triangular matrix U is found by row reducing A:

$$\begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{6}{5}R_1} \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{0.7}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{0.8}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = \frac{6}{5}$.

$$L = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \tag{0.9}$$

Now,

$$A = \begin{pmatrix} 5 & 1 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \tag{0.10}$$

We can solve this using two steps:

$$L\mathbf{y} = \mathbf{b} \tag{0.11}$$

$$U\mathbf{x} = \mathbf{y} \tag{0.12}$$

Using forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.13}$$

This gives:

$$y_1 = 2 \tag{0.14}$$

$$\frac{6}{5}(2) + y_2 = 1 \tag{0.15}$$

$$y_2 = -\frac{7}{5} \tag{0.16}$$

Now using back substitution:

$$\begin{pmatrix} 5 & 1 \\ 0 & -\frac{21}{5} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{7}{5} \end{pmatrix} \tag{0.17}$$

This gives:

$$v = \frac{1}{3} \tag{0.18}$$

$$5u + \frac{1}{3} = 2 \tag{0.19}$$

$$u = \frac{1}{3} \tag{0.20}$$

Therefore:

$$\frac{1}{x-1} = \frac{1}{3} \implies x = 4 \tag{0.21}$$

$$\frac{1}{y-2} = \frac{1}{3} \implies y = 5 \tag{0.22}$$

The solution is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{0.23}$$

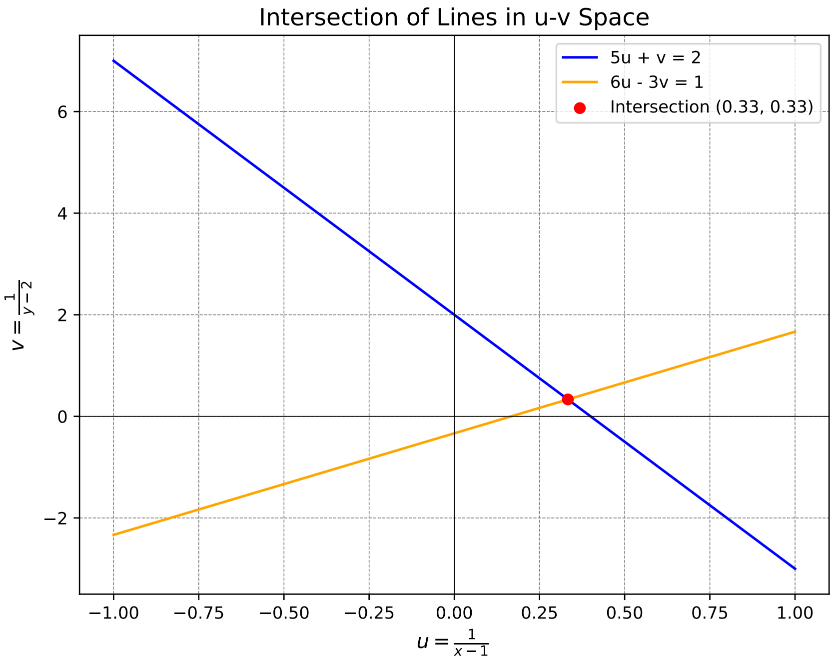

Fig. 0.1: Graph of the solution