Asymptotic Growth Rate

Slides and figures have been collected from various publicly available Internet sources for preparing the lecture slides of IT2001 course. I acknowledge and thank all the original authors for their contribution to prepare the content.

Growth Functions

- The running time of an algorithm as input size approaches infinity is called the *asymptotic running time*
- We shall study different notations for asymptotic efficiency.
- In particular, we shall study tight bounds, upper bounds and lower bounds.

The functions

- Let $f(n)$ and $g(n)$ be *asymptotically nonnegative* functions whose domains are the set of natural numbers N={0,1,2,…}.

- A function $g(n)$ is *asymptotically nonnegative*,

  if $g(n) \geq 0$ for all $n \geq n_0$ where $n_0 \in N$

Asymptotic Upper Bound: O

Definition:

Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. We say $f(n)$ is in $O(g(n))$ if there is a real positive constant c and a positive integer n_0 such that for every $n \geq n_0$

$$0 \leq f(n) \leq c * g(n).$$

Or using more mathematical notation

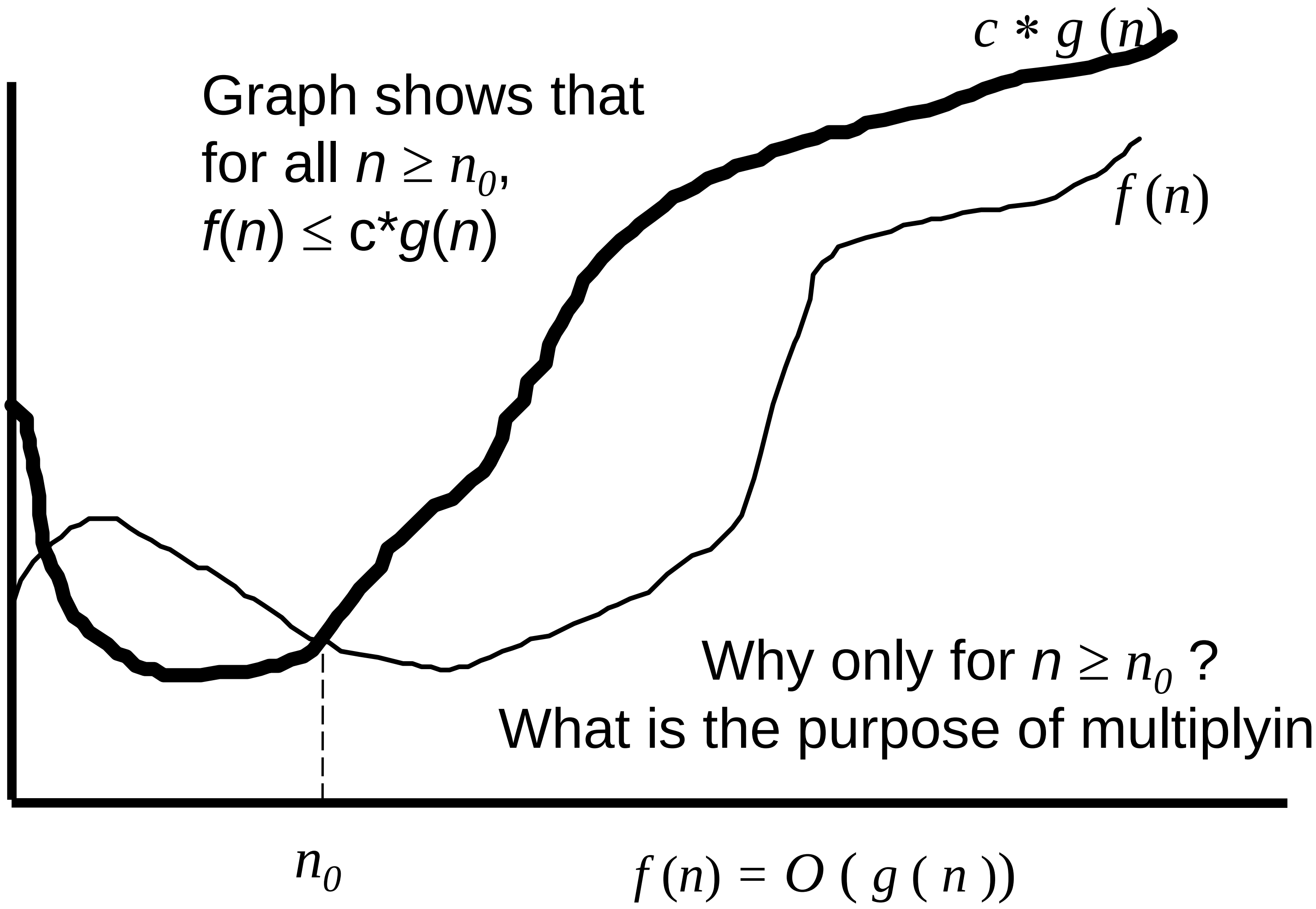
$O(g(n)) =$

$\{ f(n) \mid$ there exist positive constant c and a positive integer n_0 such that

$$0 \leq f(n) \leq c * g(n) \text{ for all } n \geq n_0 \}$$

Asymptotic Upper Bound: big *O*

Graph shows that
for all $n \geq n_0$,
$f(n) \leq c*g(n)$



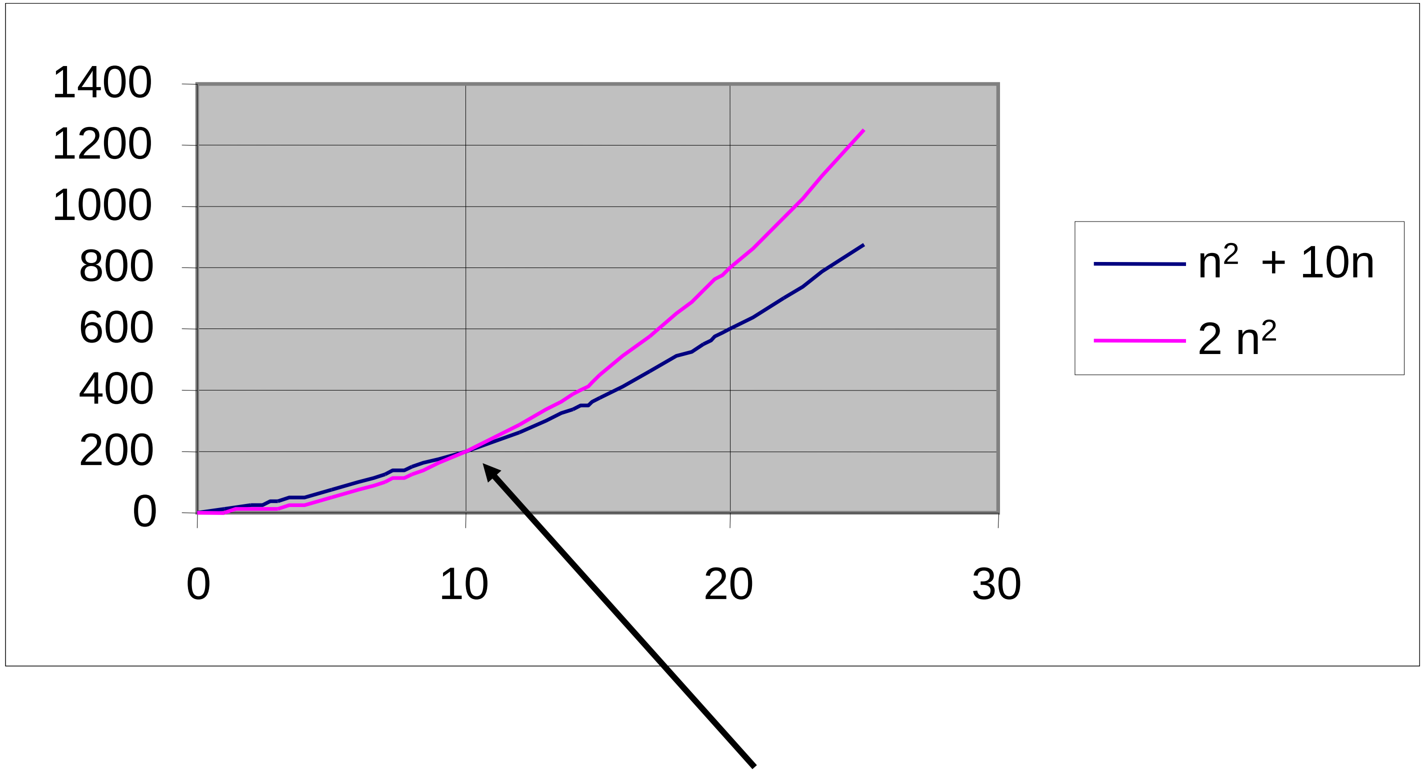
Why only for $n \geq n_0$?
What is the purpose of multiplying by $c > 0$?

$n^2 + 10\,n = O(n^2)$ Why?

take c = 2

$n_0 = 10$

$2n^2 > n^2 + 10\,n$ for all $n>=10$



Does $5n+2 = O(n)$?

Proof: From the definition of Big Oh, there must exist c>0 and integer n_0 >0 such that $0 \leq 5n+2 \leq cn$ for all $n \geq n_0$.

Dividing both sides of the inequality by n>0 we get:

$0 \leq 5+2/n \leq c$.

$2/n \leq 2$, $2/n>0$ becomes smaller when n increases

There are many choices here for c and n_0.

If we choose $n_0=1$ then $c \geq 5+2/1 = 7$.

If we choose $c=6$, then $0 \leq 5+2/n \leq 6$. So $n_0 \geq 2$.

In either case (we only need one!) we have a c>o and n_0>0 such that $0 \leq 5n+2 \leq cn$ for all $n \geq n_0$. So the definition is satisfied and $5n+2 = O(n)$

Does $n^2 = O(n)$? No.

We will prove by contradiction that the definition cannot be satisfied.
Assume that $n^2=O(n)$.
From the definition of Big Oh, there must exist c>0 and integer n_0 >0 such that
$0 \le n^2 \le cn$ for all $n \ge n_0$.

Dividing the inequality by n>0, we get $0 \le n \le c$ for all $n \ge n_0$.

$n \le c$ cannot be true for any $n > \max\{c, n_0\}$, contradicting our assumption
So, there is no constant c>0 such that $n \le c$ is satisfied for all $n \ge n_0$, and
$n^2 = O(n)$

$O(g(n)) =$

$\{ f(n) \mid$ there exist positive constant c and positive integer n_0 such that $0 \le f(n) \le c * g(n)$ for all $n \ge n_0 \}$

1. $1{,}000{,}000\ n^2 = O(n^2)$ why/why not?

2. $(n - 1)n / 2 = O(n^2)$ why /why not?

3. $n / 2 = O(n^2)$ why /why not?

4. $\lg(n^2) = O(\lg n)$ why /why not?

5. $n^2 = O(n)$ why /why not?

Asymptotic Lower Bound: Ω

Definition:

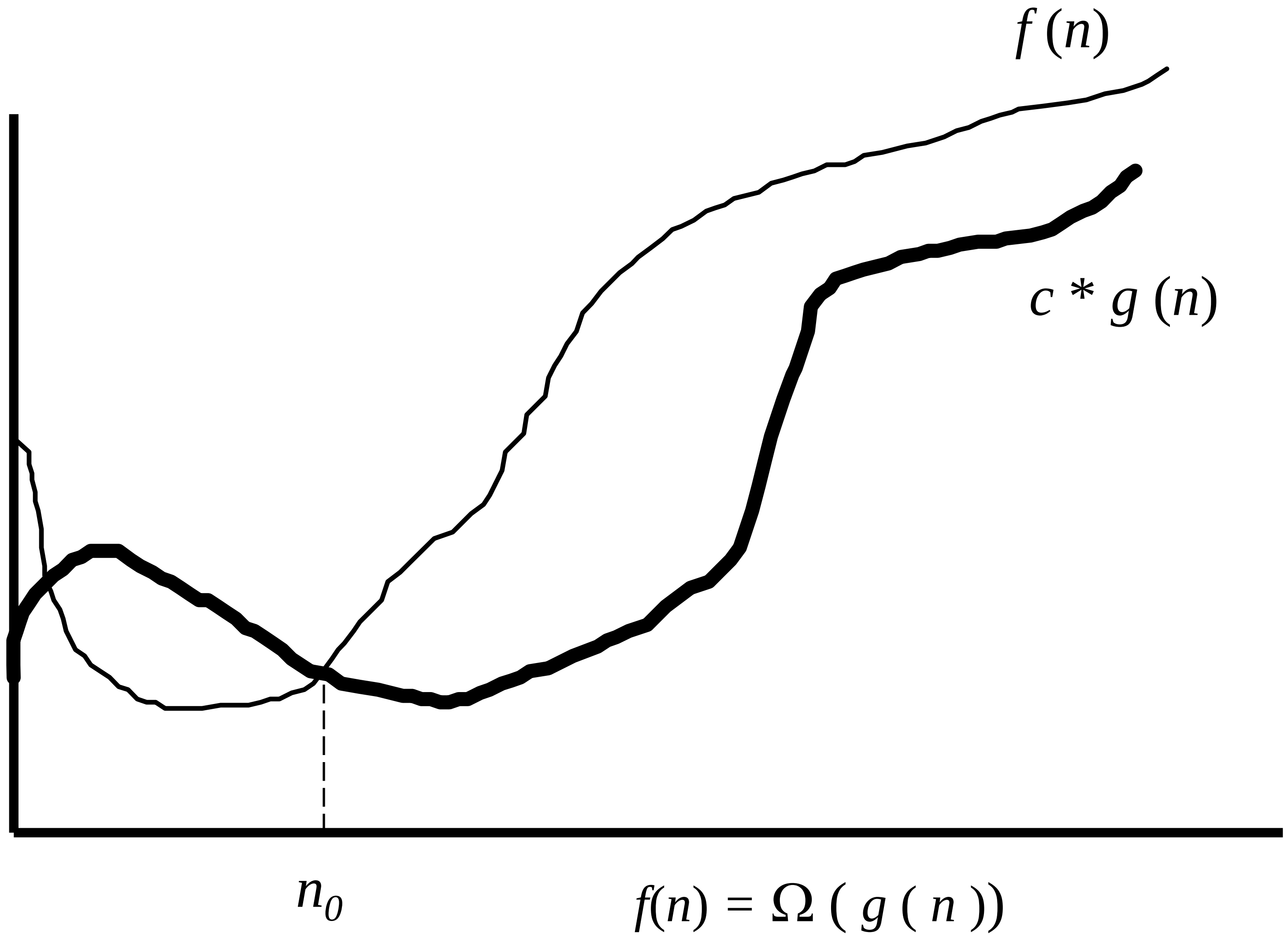
Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. We say $f(n)$ is $\Omega(g(n))$ if there is a positive constant c and a positive integer n_0 such that for every $n \geq n_0$

$$0 \leq c * g(n) \leq f(n).$$

Or using more mathematical notation

$$\Omega(g(n)) =$$
$$\{ f(n) \mid \text{there exist positive constant } c \text{ and a positive integer } n_0 \text{ such that } 0 \leq c * g(n) \leq f(n) \text{ for all } n \geq n_0 \}$$

Asymptotic Lower Bound, Omega: Ω



Is $5n$-20= Ω (n)?

Proof: From the definition of Omega, there must exist c>0 and integer n_0 >0 such that $0 \leq cn \leq 5n-20$ for all $n \geq n_0$

Dividing the inequality by n>0 we get: $0 \leq c \leq 5-20/n$ for all $n \geq n_0$.

20/n $\leq$ 20, and 20/n becomes smaller as n grows.

There are many choices here for c and n_0.

Since $c > 0$, 5 – 20/n >0 and n_0 >4

For example, if we choose c=4, then 5 – 20/n $\geq$ 4 and $n_0 \geq 20$

In this case we have a c>o and n_0 >0 such that $0 \leq cn \leq 5n-20$ for all $n \geq n_0$. So, the definition is satisfied and $5n-20 = \Omega(n)$

$\Omega(g(n)) =$

$\{ f(n) \mid$ there exist positive constant c and a positive integer n_0 such that $0 \le c * g(n) \le f(n)$ for all $n \ge n_0 \}$

1. $1{,}000{,}000\ n^2 = \Omega(n^2)$ why /why not?

2. $(n - 1)n / 2 = \Omega(n^2)$ why /why not?

3. $n / 2 = \Omega(n^2)$ why /why not?

4. $\lg(n^2) = \Omega(\lg n)$ why /why not?

5. $n^2 = \Omega(n)$ why /why not?

Asymptotic Bound Theta: Θ

Definition:

- Let $f(n)$ and $g(n)$ be asymptotically non-negative functions.
- We say $f(n)$ is $\Theta(g(n))$ if there are positive constants c, d and a positive integer n_0 such that for every $n \geq n_0$

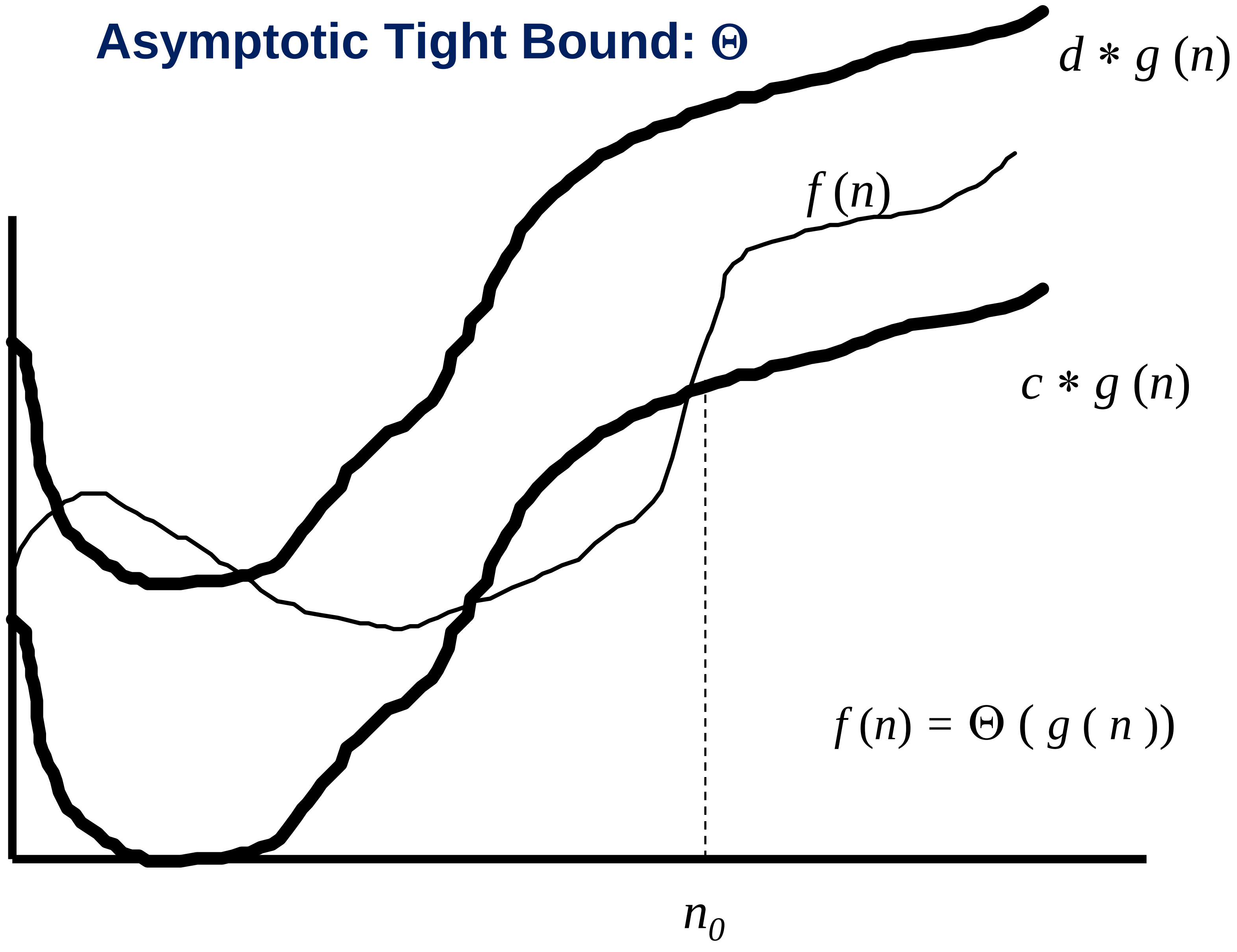
$$0 \leq c * g(n) \leq f(n) \leq d * g(n).$$

Or using more mathematical notation

$\Theta(g(n)) =$

$$\{ f(n) \mid \text{there exist positive constants } c, d \text{ and a positive integer } n_0 \text{ such that } 0 \leq c * g(n) \leq f(n) \leq d * g(n). \text{ for all } n \geq n_0 \}$$

Asymptotic Tight Bound: Θ



More on Θ

- We will use this definition:

  $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$

$$\text{Does } \frac{1}{2}n^2 - 3n = \Theta(n^2)?$$

- We show:

1. $$\frac{1}{2}n^2 - 3n = O(n^2)$$

2. $$\frac{1}{2}n^2 - 3n = \Omega(n^2)$$

Does $\dfrac{1}{2}n^2 - 3n = O(n^2)$?

From the definition there must exist $c > 0$, and $n_0 > 0$ such that

$$0 \le \frac{1}{2}n^2 - 3n \le cn^2 \text{ for all } n \ge n_0.$$

Dividing the inequality by $n^2 > 0$ we get :

$$0 \le \frac{1}{2} - \frac{3}{n} \le c \text{ for all } n \ge n_0.$$

Since $3/n > 0$ for finite n, $c < 1/2$.

Choose $c = 1/4$.

So $\dfrac{1}{2} - \dfrac{3}{n} \le \dfrac{1}{4}$, and $n_0 \ge 12$

Does $\frac{1}{2}n^2 - 3n = \Omega(n^2)$?

There must exist $c > 0$ and $n_0 > 0$ such that

$0 \le cn^2 \le \frac{1}{2}n^2 - 3n$ for all $n \ge n_0$

Dividing by $n^2 > 0$ we get

$$0 \le c \le \frac{1}{2} - \frac{3}{n}.$$

Since $c > 0$, $0 < \frac{1}{2} - \frac{3}{n}$ and $n_0 > 6$.

Since $3/n > 0$ for finite n, $c < 1/2$. Choose $c = 1/4$.

$\frac{1}{4} \le \frac{1}{2} - \frac{3}{n}$ for all $n_0 \ge 12$.

So $c = 1/4$ and $n_0 = 12$.

More Θ

1. $1{,}000{,}000\ n^2 = \Theta(n^2)$ why /why not?
2. $(n - 1)n / 2 = \Theta(n^2)$ why /why not?
3. $n / 2 = \Theta(n^2)$ why /why not?
4. $\lg(n^2) = \Theta(\lg n)$ why /why not?
5. $n^2 = \Theta(n)$ why /why not?

Limits can be used to determine Order

$$\text{if } \lim_{n\to\infty} f(n)/g(n) = \begin{cases} c>0 \text{ then } f(n) = \Theta(g(n)) \\ 0 \text{ or } c>0 \text{ then } f(n) = O(g(n)) \\ \infty \text{ or } c>0 \text{ then } f(n) = \Omega(g(n)) \end{cases}$$

- The limit must exist

Example using limits

$$5n^3 + 3n = \Omega(n^2)\text{since,}$$

$$\lim_{n\to\infty} \frac{5n^3 + 3n}{n^2} = \lim_{n\to\infty} \frac{5n^3}{n^2} + \lim_{n\to\infty} \frac{3n}{n^2} = \infty$$

L'Hopital's Rule

If $f(x)$ and $g(x)$ are both differentiable with derivatives $f'(x)$ and $g'(x)$, respectively, and if

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} f(x) = \infty \text{ then}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

whenever the limit on the right exists

Example using limits

$$10n^3 - 3n = \Theta(n^3)\text{since,}$$

$$\lim_{n\to\infty} \frac{10n^3 - 3n}{n^3} = \lim_{n\to\infty} \frac{10n^3}{n^3} - \lim_{n\to\infty} \frac{3n}{n^3} = 10$$

$$n\log_e n \in O(n^2)\text{since,}$$

$$\lim_{n\to\infty} \frac{n\log_e n}{n^2} = \lim_{n\to\infty} \frac{\log_e n}{n} = ?\quad \text{Use L'Hopital's Rule :}$$

$$\lim_{n\to\infty} \frac{(\log_e n)'}{(n)'} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

Example using limits

$$\lg n = O(n)$$

$$\lg n = \frac{\ln n}{\ln 2} \qquad (\lg n)' = \left(\frac{\ln n}{\ln 2}\right)' = \frac{1}{n \ln 2}$$

$$\lim_{n\to\infty} \frac{\lg n}{n} = \lim_{n\to\infty} \frac{(\lg n)'}{n'} = \lim_{n\to\infty} \frac{1}{n \ln 2} = 0$$

Example using limits

$$n^k = O(2^n)$$

where k is a positive integer.

$$2^n = e^{n \ln 2}$$

$$(2^n)' = (e^{n \ln 2})' = \ln 2 \; e^{n \ln 2} = 2^n \ln 2$$

$$\lim_{n\to\infty} \frac{n^k}{2^n} = \lim_{n\to\infty} \frac{kn^{k-1}}{2^n \ln 2} =$$

$$= \lim_{n\to\infty} \frac{k(k-1)n^{k-2}}{2^n \ln^2 2} = \ldots = \lim_{n\to\infty} \frac{k!}{2^n \ln^k 2} = 0$$

Another upper bound "little oh": *o*

Definition:

Let $f(n)$ and $g(n)$ be asymptotically non-negative functions.. We say $f(n)$ is $o(g(n))$ if for every positive real constant c there exists a positive integer n_0 such that for all $n \geq n_0$

$$0 \leq f(n) < c * g(n).$$

$o(g(n))$ =

$\{f(n)$: for any positive constant $c > 0$, there exists a positive integer $n_0 > 0$ such that $0 \leq f(n) < c * g(n)$ for all $n \geq n_0\}$

- "little omega" can also be defined

Main difference between *O* and *o*

$O\,(g(n)) =$

$\{\ f(n) \mid$ there exist positive constant c and a positive integer n_0 such that $0 \leq f(n) \leq c * g(n)$ for all $n \geq n_0\ \}$

$o\,(g(n)) =$

$\{\ f(n) \mid$ for any positive constant $c > 0$, there exists a positive integer n_0 such that $0 \leq f(n) < c * g(n)$ for all $n \geq n_0\ \}$

For 'o' the inequality holds for all positive constants.

Whereas for 'O' the inequality holds for some positive constants.

Lower-order terms and constants

- Lower order terms of a function do not matter since lower-order terms are dominated by the higher order term.
- Constants (multiplied by highest order term) do not matter, since they do not affect the asymptotic growth rate
- All logarithms with base b >1 belong to $\Theta(\lg n)$ since

$$\log_b n = \frac{\lg n}{\lg b} = c \lg n$$

Asymptotic notation in equations

What does

$$n^2 + 2n + 99 = n^2 + \Theta(n)$$

mean?

$$n^2 + 2n + 99 = n^2 + f(n)$$

Where the function $f(n)$ is from the set $\Theta(n)$.
In fact $f(n) = 2n + 99$.

Using notation in this manner can help to eliminate non-affecting details and clutter in an equation.

$$2n^2 + 5n + 21 = 2n^2 + \Theta(n) = \Theta(n^2)$$

Transitivity:

If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ then $f(n) = \Theta(h(n))$.

If $f(n) = O(g(n))$ and $g(n) = O(h(n))$ then $f(n) = O(h(n))$.

If $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ then $f(n) = \Omega(h(n))$.

If $f(n) = o(g(n))$ and $g(n) = o(h(n))$ then $f(n) = o(h(n))$.

If $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ then $f(n) = \omega(h(n))$

Reflexivity:

$$f(n) = \Theta(f(n)).$$

$$f(n) = O(f(n)).$$

$$f(n) = \Omega(f(n)).$$

“o” is not reflexive

Symmetry and Transpose symmetry:

- Symmetry:
  $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

- Transpose symmetry:
  $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
  $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Analogy between asymptotic comparison of functions and comparison of real numbers.

$$f(n) = O(g(n)) \approx a \leq b$$

$$f(n) = \Omega(g(n)) \approx a \geq b$$

$$f(n) = \Theta(g(n)) \approx a = b$$

$$f(n) = o(g(n)) \approx a < b$$

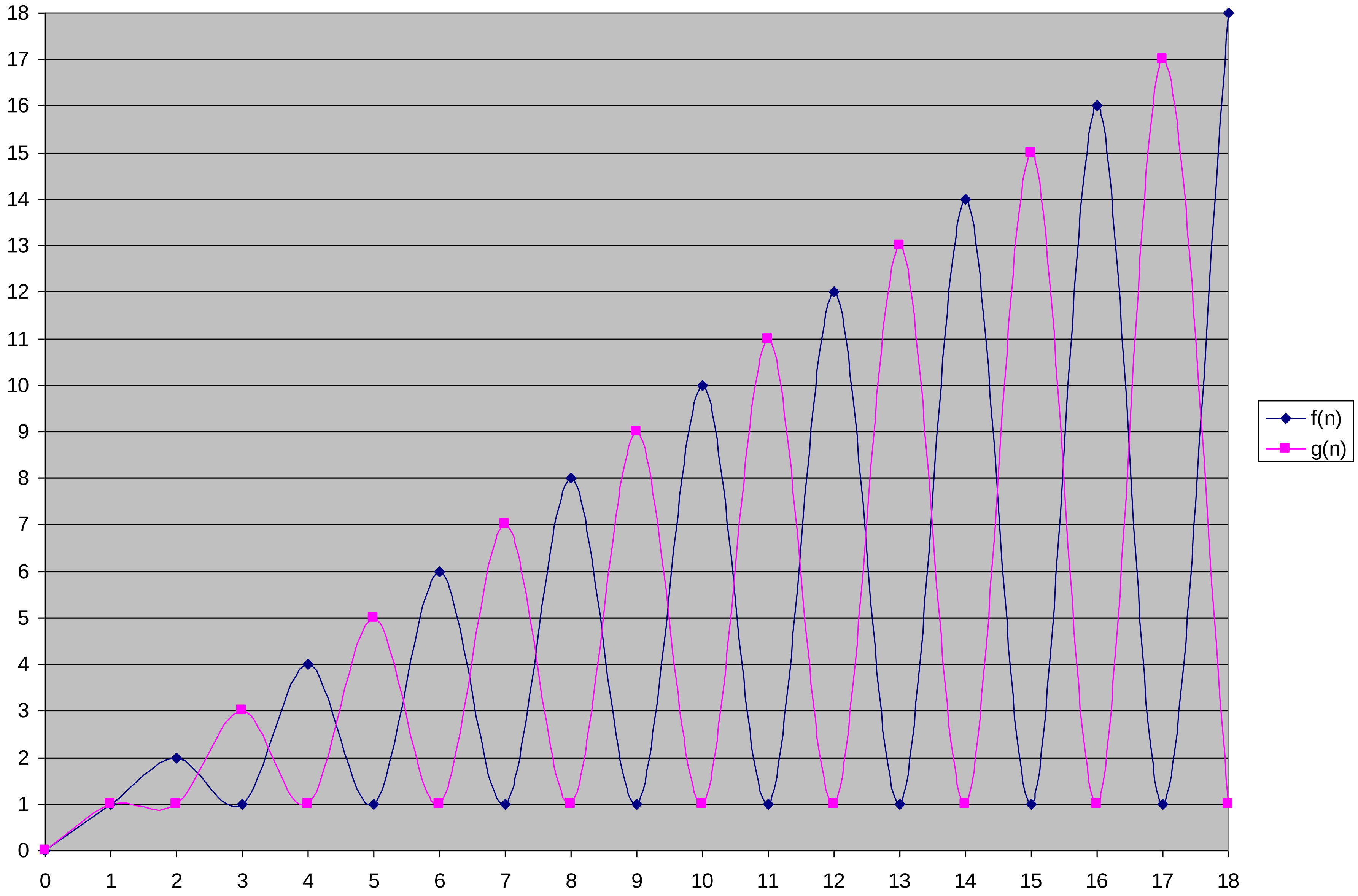
$$f(n) = \omega(g(n)) \approx a > b$$

Not All Functions are Comparable

The following functions are not asymptotically comparable:

$$f(n) = \begin{cases} n \text{ for even } n \\ 1 \text{ for odd } n \end{cases} \qquad g(n) = \begin{cases} 1 \text{ for even } n \\ n \text{ for odd } n \end{cases}$$

$$f(n) \neq O(g(n)), \text{ and } f(n) \neq \Omega(g(n)),$$



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f(n)
g(n)

Is O(g(n)) = Θ(g(n)) ∪ o(g(n))?

We show a counter example:

The functions are:

$$g(n) = n$$

and

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

$f(n) = O(n)$ but $f(n) \neq \Theta(n)$ and $f(n) \neq o(n)$

Conclusion:

$\mathbf{O}(g(n)) \neq \Theta(g(n)) \cup \mathbf{o}(g(n))$

General Rules

- We say a function $f(n)$ is polynomially bounded if
  $f(n) = O(n^k)$ for some positive constant k

- We say a function $f(n)$ is polylogarithmic bounded if
  $f(n) = O(\lg^k n)$ for some positive constant k

- Exponential functions
  - grow faster than positive polynomial functions
  - grow faster than polylogarithmic functions