Robust Kalman Bucy Filter

***Abstract*—Development of a robust estimator for uncertain stochastic systems under persistent excitation is presented. The given continuous-time stochastic formulation assumes norm bounded parametric uncertainties and excitations. When there are no system uncertainties, the performance of the proposed robust estimator is similar to that of the Kalman-Bucy filter and the proposed approach asymptotically recovers the desired optimal performance in the presence of uncertainties and or persistent excitation.**

I. Introduction

Although the Kalman-Bucy filter is inherently robust, its performance may suffer in the presence of system uncertainties. This robust estimation problem centers on recovering unmeasured state variables when the plant model and noise statistics are uncertain. Robust estimators based on H_∞ filtering aim to minimize the worst-case H_∞ norm of the transfer function from noise inputs to estimation error outputs. However, this worst-case design sacrifices average filter performance. The H_∞ formulation involves deregularization.

This paper introduces a robust estimator for uncertain linear stochastic systems under persistent excitation. The proposed estimator ensures asymptotic convergence of the estimation error, even in the presence of persistent excitation. In contrast to existing approaches, the main contributions of this manuscript are outlined below:

- Most of the existing robust estimator schemes are in discrete-time or assumes deterministic noise scenario. Therefore, a complete stochastic formulation of a continuous time robust estimator is presented here.
- Present formulation considers uncertainties in both system and output matrices as well as an unknown persistently exciting signal.
- Proposed approach asymptotically recovers the desired optimal performance.

The structure of this technical note is as follows. A detailed problem formulation and the development of the robust estimator are first given in Sections II and III, respectively. Afterwards, numerical simulations are presented in Section IV to further illustrate the performance of the proposed robust estimator. Finally concluding remarks are given in Section V.

II. Problem Formulation

Consider the following linear time-invariant stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$:

$$d\mathbf{X}_t = \{A\mathbf{X}_t + \mathbf{W}_t\}\, dt + d\mathcal{B}_t, \qquad t \in [t_0, \infty)\} \tag{1}$$

with initial condition $\mathbf{X}_{t_0} = \mathbf{X}_0$. Here, $\mathbf{X}_t[t_0,\infty) \times \Omega \mapsto \Re^n$ is the state vector, $\mathbf{W}_t : [t_0,\infty) \times \Omega \mapsto \Re^n$ is an unknown persistent excitation, and $\mathbf{B}_t : [t_0,\infty) \times \Omega \mapsto \Re^n$ is an n-dimensional $\mathcal{F}_t$-Wiener process with zero mean and the correlation of increments

$$E\left[\{\mathcal{B}(\tau) - \mathcal{B}(\zeta)\}\{\mathcal{B}(\tau) - \mathcal{B}(\zeta)\}^T\right] = \mathcal{Q}|\tau - \zeta|$$

Elements of the system matrix $\mathbf{A} \in \Re^{n\times n}$, and the noise intensity matrix $\mathbf{Q} \in \Re^{n\times n}$, are assumed to be unknown. The measurement process is given as

$$d\mathbf{Y}_t = C\mathbf{X}_t dt + d\mathcal{V}_t \tag{2}$$

where $\mathbf{Y}_t : [t_0,\infty) \times \Omega \mapsto \Re^m$ is the output vector, and $\mathbf{V}_t : [t_0,\infty) \times \Omega \mapsto \Re^m$, is assumed to be an $\mathcal{F}_t$-Wiener process with zero mean and the correlation of increments

$$E\left[\{\mathcal{V}(\tau) - \mathcal{V}(\zeta)\}\{\mathcal{V}(\tau) - \mathcal{V}(\zeta)\}^T\right] = R|\tau - \zeta|$$

Definition 1: The stochastic process W_t is persistently exciting with level α if

$$\left(\int_t^{t+T} \mathbf{W}_\tau \mathbf{W}_\tau^T d\tau \geq \alpha T I_{n\times n}\right) = 1 \tag{3}$$

for some $\alpha > 0$, $T > 0$, and $\forall t > t_0$. Since the true system state matrix, A, and the output matrix, C, are unknown, the assumed (known) system matrix and output matrix are given as A_m and C_m, respectively. Define

$$M_t = \{A - A_m\}\mathbf{X}_t + \mathbf{W}_t, \tag{4}$$

$$N_t = \{C - C_m\}\mathbf{X}_t. \tag{5}$$

Now the system equation and the output equation may be rewritten in terms of the assumed system matrix, A_m, and the assumed output matrix, C_m, as

$$d\mathbf{X}_t = \{A_m\mathbf{X}_t + M_t\}\, dt + d\mathcal{B}_t, \tag{6}$$

$$d\mathbf{Y}_t = \{C_m\mathbf{X}_t + N_t\}\, dt + d\mathcal{V}_t. \tag{7}$$

Assumption 1: The persistent disturbance, $\mathbf{W}_t$, is an $\mathcal{F}_t$-adapted process and is almost surely (a.s.) upper bounded as follows

$$\left(\sup_{t\geq t_0} |\mathbf{W}_t| < \infty\right) = 1 \tag{8}$$

where $|.|$ is the one norm i.e. $|\mathbf{x}| = \mathbf{\Sigma_i}|\mathbf{x_i}|$ *Assumption 2:* The matrices, A and A_m are Hurwitz. The assumption of the stability of the uncertain system is a weak one, as it is rare that an estimator be applied 'open-loop' to an unstable system. Furthermore, if A is only marginally stable, and the the persistent excitation is such that the system states are bounded, then an equivalent representation of system (1) similar to the one in (6) can be selected such that the state matrix is Hurwitz.

Lemma 1 Given assumptions 1 and 2, the process, X_t, is a.s. upper bounded as follows:

$$\left(\sup_{t\geq t_0} |\mathbf{X}_t| < c\right) = 1$$

where c< ∞ is a constant.

Proof : Matrix A generates an exponentially stable evolution operator $\Phi_A(t-t_0)$ and the solution of (1), $X(t)$, Integrating (1) from time t to t_0 we get

$$\mathbf{X}(t) = \Phi_A(t-t_0)\mathbf{X}(t_0) + \int_{t_0}^{t} \Phi_A(t-\tau)\mathbf{W}(\tau)d\tau + \int_{t_0}^{t} \Phi_A(t-\tau)d\mathcal{B}(\tau).$$

Since $\Phi_A(t-t_0)$ is exponentially stable.

$$\mathcal{M}_t = \int_{t_0}^{t} \Phi_A(t-\tau)d\mathcal{B}(\tau).$$

Note that $\mathcal{M}_t$ is a supermartingale and based on Doob's martingale inequality it is a.s. bounded. Thus

$$\begin{aligned}
|\mathbf{X}(t)| \leq & \parallel \Phi_A(t-t_0) \parallel |\mathbf{X}(t_0)| \\
& + \int_{t_0}^{t} \parallel \Phi_A(t-\tau) \parallel |\mathbf{W}(\tau)| d\tau + |\mathcal{M}_t|, \\
& \leq \lambda_0 e^{-a(t-t_0)} |\mathbf{X}(t_0)| \\
& + \int_{t_0}^{t} \lambda_0 e^{-a(t-\tau)} |\mathbf{W}(\tau)| d\tau + |\mathcal{M}_t|, \\
& \leq \lambda_0 e^{-a(t-t_0)} |\mathbf{X}(t_0)| \\
& + \int_{t_0}^{t} \lambda_0 e^{-(a-a_0/2)(t-\tau)} e^{-a_0/2(t-\tau)} |\mathbf{W}(\tau)| d\tau \\
& + |\mathcal{M}_t|.
\end{aligned}$$

The last inequality is obtained by expressing $e^{-a(t-\tau)}$ as $e^{-(a-a_0/2)(t-\tau)}$, where $a_o < 2a$ is a positive constant. Applying the Schwartz inequality yields

$$\begin{aligned}
|\mathbf{X}(t)| \leq & \lambda_0 e^{-a(t-t_0)} |\mathbf{X}(t_0)| \\
& + \lambda_0 \left(\int_{t_0}^{t} e^{-(2a-a_0)(t-\tau)} d\tau\right)^{1/2} \\
& \times \left(\int_{t_0}^{t} e^{-a_0(t-\tau)} |\mathbf{W}(\tau)|^2 d\tau\right)^{1/2} + |\mathcal{M}_t|.
\end{aligned}$$

Simplifying the integral term

$$\begin{aligned}
|\mathbf{X}(t)| \leq & \lambda_0 e^{-a(t-t_0)} |\mathbf{X}(t_0)| \\
& + \frac{\lambda_0}{\sqrt{(2a-a_0)}} \left(\int_{t_0}^{t} e^{-a_0(t-\tau)} |\mathbf{W}(\tau)|^2 d\tau\right)^{1/2} + |\mathcal{M}_t|.
\end{aligned}$$

Since $|\mathbf{X}_0|$ is almost surely bounded, from assumption 1 stated above $|\mathbf{W}_t|$ and since $|\mathcal{M}_t|$ is bounded as it is an integral of a exponentially stable(finite) function $\Phi_A(t-t_0)$, we can say that there exist a positive constant c< ∞

$$\left(\sup_{t\geq t_0} |\mathbf{X}_t| < c\right) = 1.$$

Given the system parameter uncertainties are bounded and the system states are almost surely bounded, an upper bound on $\mathbf{M}_t$ can be obtained as

$$(|M_t| \leq \bar{\mu}(t)) = 1, \quad \forall t \geq t_0. \tag{9}$$

Where $\bar{\mu(t)}$ is defined as the upper bound of $\mathbf{M}_t$. Now consider (5) where $\mathbf{N}_t$ is defined. Differentiate that equation and substitute the value of $d\mathbf{X}_t$ from (1)

$$dN_t = \{C - C_m\}\{A_m\mathbf{X}_t + M_t\} dt + \{C - C_m\} d\mathcal{B}_t. \tag{10}$$

Define

$$U_t = \{C - C_m\}\{[A_m + I_{n\times n}]\mathbf{X}_t + M_t\}. \tag{11}$$

Since the system parametric uncertainties, system states, and $\mathbf{M}_t$ are assumed to be upper bounded, a conservative upper bound on the stochastic process $\mathbf{U}_t$ can easily be obtained, i.e.

$$(|U_t| \leq \bar{v}(t)) = 1, \quad \forall t \geq t_0. \tag{12}$$

After appending d$\mathbf{N}_t$ to system (1) can be rewritten as

$$d\begin{bmatrix} X_t \\ N_t \end{bmatrix} = \left[\begin{bmatrix} A_m & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} X_t \\ N_t \end{bmatrix} + \begin{bmatrix} M_t \\ U_t \end{bmatrix}\right] dt + \begin{bmatrix} I \\ \Delta C \end{bmatrix} d\mathcal{B}_t \tag{13}$$

where $\Delta C = CC_m$. Define $Z_t = [X_t^T \ N_t^T]^T$, $F = \left[\begin{smallmatrix} A_m & 0 \\ 0 & -I_m \end{smallmatrix}\right]$,G=$\left[\begin{smallmatrix} I_n \\ \Delta C \end{smallmatrix}\right]$ and $\Gamma^t = [M_t^T \ U_t^T]^T$. Now (13) may be written as

$$d\mathbf{Z}_t = \{F\mathbf{Z}_t + \Gamma_t\} dt + Gd\mathcal{B}_t \tag{14}$$

The output equation (2) can be written as

$$d\mathbf{Y}_t = H\mathbf{Z}_t dt + d\mathcal{V}_t \tag{15}$$

where $H = [C_m \quad I_{m\times m}]$. Note that the process $\Gamma_t = [M_t^T \quad U_t^T]^T$ may be upper bounded by $\gamma(t)$, where

$$\gamma(t) = \max\{\bar{\mu}(t), \quad \bar{v}(t)\}. \tag{16}$$

Since Γ_t is an augmented matrix derived from M_t^T and U_t^T the upper bound of Γ_t has to the max value of the upper bounds of the M_t^T and U_t^T.

Assume that there are no uncertainties, i.e., $A_m = A, C_m = C, W_t = 0$, and $\mathcal{B}_t = \bar{\mathcal{B}}_t$, where $\bar{\mathcal{B}}_t$ is an $\mathcal{F}_t$-Wiener process with zero mean and known correlation of increments Q. From the definition of a Wiener process:

$$\left[\left\{\bar{\mathcal{B}}(\tau) - \bar{\mathcal{B}}(\zeta)\right\}\left\{\bar{\mathcal{B}}(\tau) - \bar{\mathcal{B}}(\zeta)\right\}^T\right] = Q|\tau - \zeta|.$$

Now Γ_t is identically zero. So (14) can be written as

$$d\mathbf{Z}_{m_t} = F\mathbf{Z}_{m_t}dt + G_m d\bar{\mathcal{B}}_t \tag{17}$$

where $G_m = [I_{n\times n} \quad 0_{n\times m}]^T$. The corresponding measurement process is given as

$$d\mathbf{Y}_{m_t} = H\mathbf{Z}_{m_t}dt + d\mathcal{V}_t. \tag{18}$$

For the system in (17), an optimal estimator, such as a Kalman-Bucy filter, of the following form can be designed:

$$d\hat{\mathbf{Z}}_{m_t} = F\hat{\mathbf{Z}}_{m_t}dt + K\left[d\mathbf{Y}_{m_t} - H\hat{\mathbf{Z}}_{m_t}dt\right] \tag{19}$$

where $K \in \Re^{(n+m)\in m}$ is the steady-state Kalman gain and is calculated as

$$K = PH^T R^{-1} \tag{20}$$

Innovative Process refers to a time series analysis which involves finding the difference between the observed value of a variable at time t and the optimal forecast of that value based on information available prior to time t. In the equation above R is called the innovative covariance (covariance of a innovative process) which defined as:

$$R_t = E\left[(\hat{y} - y)(\hat{y} - y)^T\right] \in \Re^{mm}$$

where $P \in \Re^{(n+m)\times(n+m)}$ can be obtained by solving the algebraic Riccati equation

$$FP + PF^T - PH^T R^{-1}HP + G_m Q G_m^T = 0. \tag{21}$$

Let $\tilde{Z}_{mt} = Z_{mt} - \hat{Z}_{mt}$. From (17) and (19) we infer that

$$d\tilde{\mathbf{Z}}_{m_t} = [F - KH]\tilde{\mathbf{Z}}_{m_t}dt - Kd\mathcal{V}_t + G_m d\bar{\mathcal{B}}_t \tag{22}$$

The estimation error for any system is the same $E[\tilde{Z}_{mt}] = 0$, and the steady-state value of the error covariance is given as

$$\lim_{t\infty} E\left[\tilde{\mathbf{Z}}_{m_t}\tilde{\mathbf{Z}}_{m_t}^T\right] = P. \tag{23}$$

Given next is the formulation of a robust estimator which guarantees the asymptotic convergence of the true state estimation error to the desired optimal error $\tilde{Z}_{mt}$

III. Robust Estimator

The proposed robust estimator is

$$d\hat{\mathbf{Z}}_t = \left\{F\hat{\mathbf{Z}}_t + \eta(t)\right\}dt + K\left[d\mathbf{Y}_t - H\hat{\mathbf{Z}}_t dt\right] \tag{24}$$

where $\eta(t)$ is a signal whose details would be explained shortly. The robust estimator formulation presented here follows the typical robust controller formulation in that the estimator "input," $\eta(t)$, is selected such that the true estimator error dynamics given in (25) asymptotically tracks the desired error dynamics given in (22). A systematic approach for the selection of $\eta(t)$ constitutes the main result of this section and is presented later in Theorem 1.

Let $\tilde{Z}_t$ = $Z_t - \hat{Z}_t$. Subtracting (24) and (14) we get estimator error dynamics i.e. d$\tilde{Z}_t$

$$d\tilde{\mathbf{Z}}_t = \left\{[F - KH]\tilde{\mathbf{Z}}_t + \Gamma_t - \eta(t)\right\}dt - Kd\mathcal{V}_t + Gd\mathcal{B}_t. \tag{25}$$

Let $\mathcal{Z}_t$ be the difference between the desired estimation error in (22) and the true estimation error in (25), i.e., $Z_t = \tilde{Z}_t - \tilde{Z}_{mt}$. After subtracting (22) from (25), $\mathcal{Z}_t$ can be written as

$$d\mathcal{Z}_t = \{[F - KH]\mathcal{Z}_t + \Gamma_t - \eta(t)\}dt + Gd\mathcal{B}_t - G_m d\bar{\mathcal{B}}_t \tag{26}$$

The solution of $\mathcal{Z}$can be written as

$$\mathcal{Z}_t = \mathcal{Z}_0 + \int_{t_0}^t \{[F - KH]\mathcal{Z}_s + \Gamma_s - \eta(s)\}ds + \int_{t_0}^t Gd\mathcal{B}_s - \int_{t_0}^t G_m d\bar{\mathcal{B}}_s.$$

Let $\tilde{Y}_t$ denote a innovations process

$$d\tilde{\mathbf{Y}}_t = \left[d\mathbf{Y}_t - H\hat{\mathbf{Z}}_t dt\right].$$

$\mathcal{Z}_t$ can be defined as $\mathcal{Z}_t$ = $E[\mathcal{Z}_t|\mathcal{F}_t^{\tilde{Y}}]$, where $\mathcal{F}_t^{\tilde{Y}}$ is the filtration generated by $\tilde{Y}_t$. Note $E[d\mathcal{B}_t|\mathcal{F}_t^{\tilde{Y}}] = 0$ and $E[d\bar{\mathcal{B}}_t|\mathcal{F}_t^{\tilde{Y}}] = 0$ because $\mathcal{B}_t$ and $\bar{\mathcal{B}}_t$ are wiener processes and the expectation of a wiener process is zero, thus $\mathcal{Z}_t$ can be written as

$$\hat{\mathcal{Z}}_t = \hat{\mathcal{Z}}_0 + E\left[\int_{t_0}^t \left\{[F - KH]\hat{\mathcal{Z}}_s + \Gamma_s - \eta(s)\right\}ds\middle|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right] \tag{27}$$

Sine the $\mathcal{Z}_t$ is bounded it is riemann mean square integrable and from the properties of riemann integration the order of integral and expectation can be swapped and the expectation obeys linearity property. Therefore (27) can be rewritten as

$$\hat{\mathcal{Z}}_t = \hat{\mathcal{Z}}_0 + \int_{t_0}^t [F - KH]\hat{\mathcal{Z}}_s ds + \int_{t_0}^t E\left[\Gamma_s\middle|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]ds - \int_{t_0}^t E\left[\eta(s)\middle|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]ds.$$

Differentiating the above equation we get

$$d\hat{\mathcal{Z}}_t = [F - KH]\hat{\mathcal{Z}}_t dt + E\left[\Gamma_t\middle|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt - E\left[\eta(t)\middle|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt. \tag{28}$$

Theorem 1: Assume there exist a positive definite symmetric matrix $\mathcal{X} \in \Re^{(n+m)\times(n+m)}$, such that

$$[F - KH]^T\mathcal{X} + \mathcal{X}[F - KH] + S \leq \mathbf{0} \tag{29}$$

and

$$\mathcal{X} = H^T\Lambda^T \tag{30}$$

where $S \in \Re^{(n+m)\times(n+m)}$ is a positive definite matrix and $\Lambda \in \Re^{(n+m)m}$. Then the robust estimator error dynamics in (25) is globally asymptotically stable in the first moment, i.e.

$$\lim_{t\infty} E\left[\tilde{\mathbf{Z}}_t\right] = \mathbf{0}$$

and we have mean square convergence of the true conditional error to the desired conditional error, i.e.

$$\lim_{t\infty} E\left[|\hat{\mathcal{Z}}_t|^2\right] = 0$$

if the estimator inputs, $\eta(t)$, are selected as

$$\eta(t) = \text{sgn}\left\{\Lambda d\tilde{\mathbf{Y}}_t\right\}\gamma(t) \tag{31}$$

where sgn denotes the signum function or the sign function and $\gamma(t)$ is an upper bound on the process Γ_t.

Proof: Proof of this theorem is based on the stochastic Lyapunov stability analysis. Consider a stochastic Lyapunov function

$$V(\mathcal{Z}_t) = \hat{\mathcal{Z}}_t^T \mathcal{X} \hat{\mathcal{Z}}_t.$$

Differentiating the above equation using matrix differentiating we get

$$dV(\hat{\mathcal{Z}}_t) = d\hat{\mathcal{Z}}_t^T \mathcal{X}\hat{\mathcal{Z}}_t + \hat{\mathcal{Z}}_t^T \mathcal{X} d\hat{\mathcal{Z}}_t$$

From (28) substitute the value of $d\hat{\mathcal{Z}}_t$ (Note $(AB)^T = B^T A^T$ and $\mathcal{X}$ is a symmetric matrix)

$$\begin{aligned} dV(\hat{\mathcal{Z}}_t) &= \hat{\mathcal{Z}}_t^T\left\{[F-KH]^T\mathcal{X} + \mathcal{X}[F-KH]\right\}\hat{\mathcal{Z}}_t dt \\ &\quad +2E\left[\hat{\mathcal{Z}}_t^T\mathcal{X}\Gamma_t|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt - 2E\left[\hat{\mathcal{Z}}_t^T\mathcal{X}\eta(t)|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt. \end{aligned}$$

Substituting value of $\mathcal{X}$ from (30) and inequailty from (31)

$$\begin{aligned} dV(\hat{\mathcal{Z}}_t) &= \hat{\mathcal{Z}}_t^T\left\{[F-KH]^T\mathcal{X} + \mathcal{X}[F-KH]\right\}\hat{\mathcal{Z}}_t dt \\ &\quad +2E\left[\hat{\mathcal{Z}}_t^T\mathcal{X}\Gamma_t|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt - 2E\left[\hat{\mathcal{Z}}_t^T\mathcal{X}\eta(t)|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]dt. \end{aligned}$$

From the definition give above (28) we know that

$$\begin{aligned} H\hat{\mathcal{Z}}_t &= E\left[H\mathcal{Z}_t|\mathcal{F}_t^{\tilde{Y}}\right] \\ &= E\left[(H\tilde{Z}_t - H\tilde{Z}_{mt})|\mathcal{F}_t^{\tilde{Y}}\right] \\ &= E\left[(d\tilde{Y}_t - d\tilde{Y}_{mt})|\mathcal{F}_t^{\tilde{Y}}\right] \\ &= d\tilde{Y}_m t - E\left[d\tilde{Y}_{mt}|\mathcal{F}_t^{\tilde{Y}}\right] \end{aligned}$$

where $d\tilde{Y}_m t$=$[dY_{mt} - H\hat{Z}_{mt}dt]$. Given $\hat{Z}_m(t_0) = Z_m(t_0)$, one could conclude that $\tilde{Y}_{mt}$ is a zero mean processes and

$$E\left[d\tilde{\mathbf{Y}}_{m_t}|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right] = E\left[d\tilde{\mathbf{Y}}_{m_t}\right] = 0.$$

Therefore $dV(\hat{\mathcal{Z}}_t)$ can be bounded as

$$\begin{aligned} dV(\hat{\mathcal{Z}}_t) &\le -\hat{\mathcal{Z}}_t^T S\hat{\mathcal{Z}}_t dt \\ &\quad +2E\left[\left\{d\tilde{\mathbf{Y}}_t^T\Lambda^T\Gamma_t - d\tilde{\mathbf{Y}}_t^T\Lambda^T\eta(t)\right\}|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]. \end{aligned}$$

After substituting (31), an upper bound on $\hat{\mathcal{Z}}_t$ can be obtained as

$$\begin{aligned} dV(\hat{\mathcal{Z}}_t) &\le -\hat{\mathcal{Z}}_t^T S\hat{\mathcal{Z}}_t dt \\ &\quad +2E\left[\left\{d\tilde{\mathbf{Y}}_t^T\Lambda^T\Gamma_t - d\tilde{\mathbf{Y}}_t^T\Lambda^T\text{sgn}\left\{\Lambda d\tilde{\mathbf{Y}}_t\right\}\gamma(t)\right\}|\mathcal{F}_t^{\tilde{\mathbf{Y}}}\right]. \end{aligned}$$

Since

$$d\tilde{\mathbf{Y}}_t^T\Lambda^T\text{sgn}\left\{\Lambda d\tilde{\mathbf{Y}}_t\right\} = |\Lambda d\tilde{\mathbf{Y}}_t|$$

And $d\tilde{Y}_t^T\Lambda^T\Gamma_t \le |d\tilde{Y}_t^T\Lambda^T||\Gamma_t| \le |d\tilde{Y}_t^T\Lambda^T|\gamma(t)$ we get

$$dV(\hat{\mathcal{Z}}_t) \le -\hat{\mathcal{Z}}_t^T S\hat{\mathcal{Z}}_t dt.$$

Now it can be concluded that for every initial value $\hat{\mathcal{Z}}_t$, the solution, $\hat{\mathcal{Z}}_t$, of (28) has the property that $\hat{\mathcal{Z}}_t \mapsto 0$ almost surely as $t \mapsto \infty$, i.e.

$$\left(\lim_{t\infty}|\hat{\mathcal{Z}}_t| = 0\right) = 1.$$

From equations derived previously $\forall t \ge t_0$, we have

$$V(\hat{\mathcal{Z}}_t) \le V(\hat{\mathcal{Z}}_0) \Rightarrow E\left[V(\hat{\mathcal{Z}}_t)\right] \le E\left[V(\hat{\mathcal{Z}}_0)\right] < \infty.$$

Thus $\forall t \ge t_0$, $\hat{\mathcal{Z}}_t \in \mathcal{L}^2$, i.e., $E[||\hat{\mathcal{Z}}_t||^2] < \infty$. Therefore, we could conclude that

$$E\left[\sup_{t\ge t_0}\left|\hat{\mathcal{Z}}_t\right|^2\right] < \infty$$

Lebesgue's dominated convergence theorem states that

$$\lim_{n\to\infty}\int_S g_n\,d\mu = \int_S g\,d\mu$$

Now based on the Lebesgue's dominated convergence theorem, a.s. convergence of $\hat{\mathcal{Z}}_t$ implies

$$\lim_{t\infty} E\left[|\hat{\mathcal{Z}}_t|^2\right] = 0.$$

Moreover, $\hat{\mathcal{Z}}_t \in \mathcal{L}^2$ implies $\hat{\mathcal{Z}}_t \in \mathcal{L}^1$ and

$$\lim_{t\infty} E\left[|\hat{\mathcal{Z}}_t|\right] = 0.$$

Also

$$E\left[|\hat{\mathcal{Z}}_t|\right] \ge \left|E\left[\hat{\mathcal{Z}}_t\right]\right| = \left|E\left[\tilde{\mathbf{Z}}_t\right] - E\left[\tilde{\mathbf{Z}}_{m_t}\right]\right|.$$

Thus, the asymptotic convergence in the mean follows from the fact that $\tilde{Z}_{mt}$ is a zero mean process.

The design of proposed robust estimator requires the calculation of matrices $\mathcal{X}$ *and* Λ that satisfies the conditions given in (29) and(30). Matrices $\mathcal{X}$ *and* Λ that satisfies the conditions (29) and(30) can be obtained by solving the following linear matrix inequality:

$$\begin{bmatrix} \{F-KH\}^T\mathcal{X} + \mathcal{X}\{F-KH\} + S & \mathcal{X} - H^T\Lambda^T \\ \mathcal{X}^T - \Lambda H & 0_{(n+m)\times(n+m)} \end{bmatrix} \tag{32}$$

IV. Numerical Simulations

Performance of the proposed estimator is evaluated here though numerical simulations. For simulation purposes consider a stochastic system of following form:

$$d\begin{bmatrix}X_{1t}\\X_{2t}\\X_{3t}\\X_{4t}\end{bmatrix}=\begin{bmatrix}0&0&-1&0\\0&0&0&-2\\2.3&1.33&-8.5&-1.62\\1.78&2.030&-1.020&-9.5\end{bmatrix}\begin{bmatrix}X_{1t}\\X_{2t}\\X_{3t}\\X_{4t}\end{bmatrix}dt$$

$$+[0_{1\times 2}W_{1t}W_{2t}]^T dt$$

$$+[0_{1\times 2}dcalB_{1t}dcalB_{2t}]^T$$

The measurement equation is of the following form:

$$d\begin{bmatrix}Y_{1t}\\Y_{2t}\\Y_{3t}\\Y_{4t}\end{bmatrix}=\begin{bmatrix}0.43&0&-2.31&0\\0&-2.4&0&1.54\\3.64&2.74&0&0\\0&2.42&1.12&4.34\end{bmatrix}\begin{bmatrix}Y_{1t}\\Y_{2t}\\Y_{3t}\\Y_{4t}\end{bmatrix}dt$$
$$+\,[d\mathcal{V}_{1t}\ d\mathcal{V}_{2t}\ d\mathcal{V}_{3t}\ d\mathcal{V}_{4t}]$$

The assumed matrices are

$$A_m=\begin{bmatrix}0&0&-1&0\\0&0&0&-2\\2.3&1.33&-8.5&-1.62\\1.78&2.030&-1.020&-9.5\end{bmatrix}$$

$$C_m=\begin{bmatrix}0.43&0&-2.31&0\\0&-2.4&0&1.54\\3.64&2.74&0&0\\0&2.42&1.12&4.34\end{bmatrix}$$

The processes $W_{1t} and W_{2t}$ are given in Fig. 1. Note that the system uncertainty is only associated with the dynamics of the last two states and the output matrix error is only associated with the last two outputs. Since the two measurement uncertainties are only associated with the last two outputs, $N_t \in \Re^2$ and thus the matrices F and H are defined as $F=\begin{bmatrix}A_m & 0_(4\times 2)\\ 0_(4\times 2) & -I_(2\times 2)\end{bmatrix}$ and $H=[C_m\ [0_(22)\ I_(22)]^T]$, respectively. After defining $G_m=[I_(22)\ 0_(24)]^T$, the steady-state Kalman gain can be calculated.

Since there is no uncertainties or external disturbances acting on the first two states Γ_t is defined as $\Gamma_t = \Xi[M_{1t}^T\ \ M_{2t}^T\ \ U_{1t}^T\ \ U_{2t}^T]^T, where\ \ \Xi = [0_{(4\times 2)}\ \ I_{(4\times 4)}]^T$. Now the robust estimator may be designed based on the premises of Theorem 1 after replacing the condition $\mathcal{X} = H^T\Lambda^T with \mathcal{X}\Xi = H^T\Lambda^T$. System process noise covariance and the measurement noise covariance matrices are selected as $R = 10^2 \times I_{(44)} and Q = \begin{bmatrix}0.25 & 0\\ 0 & 0.35\end{bmatrix}$. Two different simulations are considered here. The first simulation scenario considers the case where there is no system uncertainties. For the second scenario, aforementioned uncertainties as assumed. True initial states are selected to be $X_0 = [1\ 2\ 5\ 0]^T$ and the assumed initial states are $X_{m0} = 0_{(4\times 1)}$.

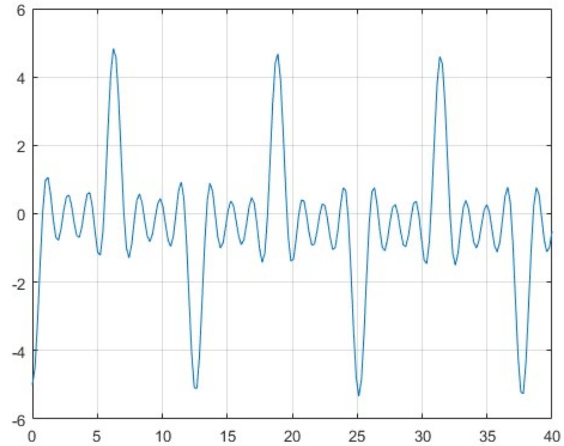

Fig. 1. Persistently exciting disturbances W_{1t}

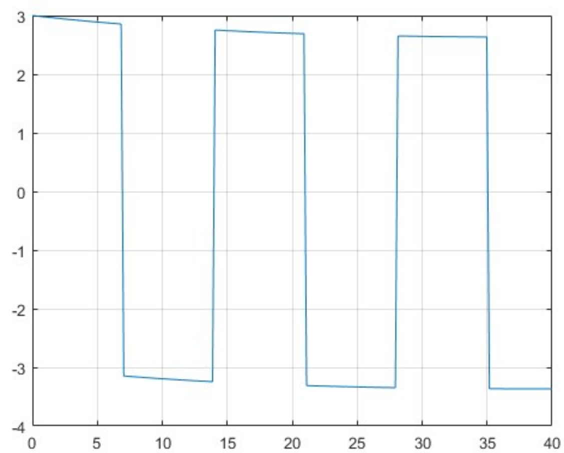

Fig. 2. Persistently exciting disturbances W_{2t}

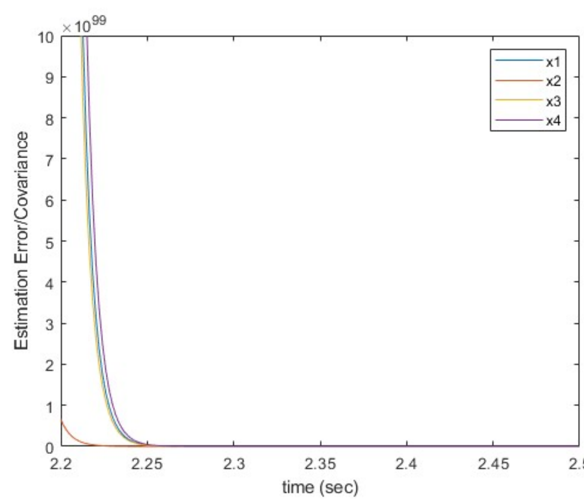

Fig. 3. State estimation error for a Kalman-Bucy Filter

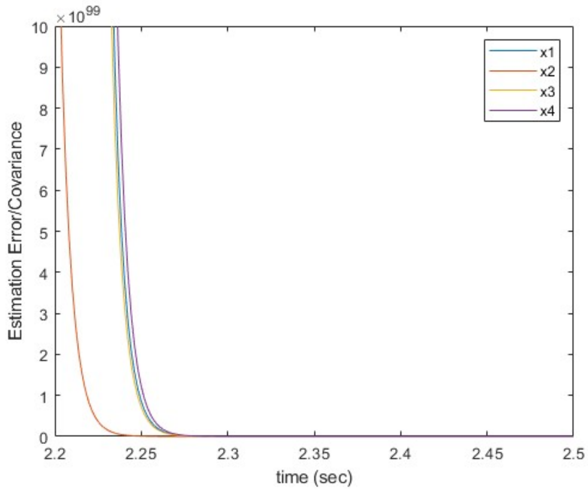

Fig. 4. State estimation error for a robust Kalman-Bucy Filter

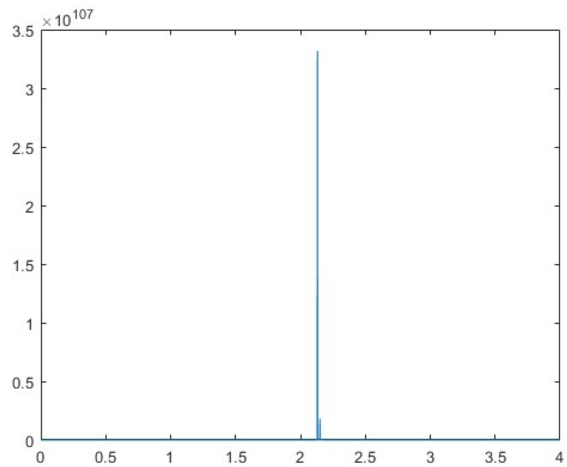

Fig. 5. State Estimation Error Norm for a Kalman-Bucy Filter

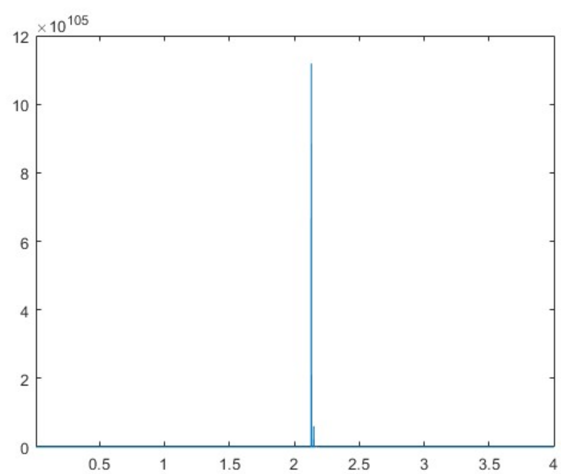

Fig. 6. State Estimation Error Norm for a robust Kalman-Bucy Filter

A. Simulation I

For the first simulation, consider a scenario where there is no model uncertainties and external disturbances, i.e., $A = A_m, C = C_m and W_t = 0$. For the first simulation, the state estimation error obtained using the Kalman-Bucy filter is given in Fig. 3 and the estimation error obtained from the proposed robust estimator is presented in Fig. 4. Fig. 3 shows the estimation error norm obtained for both estimators. Results shown in Figs. 3, 4, 5 and 6 indicate that the performance of the proposed robust estimator is identical to that of the Kalman-Bucy filter when there is no system uncertainties.

B. Simulation II

The uncertain system is considered for the second simulation. Fig. contains the model-error vector, Γ_t, corresponding to the aforementioned uncertainties. Note that at any time, $|\Gamma_t|$ is upper bounded by $\gamma = 8$. The state estimation error obtained using the Kalman-Bucy filter is given in Fig. and the estimation error obtained from the robust estimator is presented in Fig. . Fig. shows the estimation error norm obtained for both estimators. Results shown in Figs. indicate that the performance of the proposed robust estimator is superior to that of the Kalman-Bucy filter when there is system uncertainties. Finally note that the performance of the robust estimator given in Figs. are similar to the ones given in Figs. .

V. Conclusions

The construction of a resilient estimator for uncertain stochastic systems under continuous excitation is presented in this technical note. The stochastic continuous-time formulation presented here is based on the assumption of norm-bounded parametric uncertainty and excitations. The state estimation error's asymptotic convergence is guaranteed by the robust estimator that has been proposed. The simulation results presented here show that in the absence of system uncertainties and excitations, the performance of the suggested robust estimator is comparable to that of the Kalman-Bucy filter. The numerical simulation results demonstrate that, even in the presence of persistent excitation and system uncertainties, the technique asymptotically recovers the intended optimal performance. Future research might expand on this idea by applying constraints to the nonlinear stochastic differential equation's drift and diffusion factors in order to address the nonlinear estimation challenge.