① # Poisson Distribution ($Pois(\lambda)$)

A discrete random variable X is said to have a Poisson distribution with parameter $\lambda > 0$ if pmf given by

$$p(i) = P(X=i) = \frac{\lambda^i e^{-\lambda}}{i!} \quad — ①$$

, $\lambda > 0$ - rate parameter

i = no. of occurences, $i = 0, 1, \ldots\ldots$

We write $X \sim Pois(\lambda)$

X - models no. of occurences of some event in fixed interval of time, space or length or area etc.

Assumptions

① Event occurs independently

② " at a constant rate λ

③ Probability of more than one event in a tiny sub-interval is negligible.

$$E(X) = \lambda$$

$$Var(X) = \lambda$$

Verify that ① is a pmf!

Need to show $\sum_{i=0}^{\infty} p(i) = 1$!

$$\sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}.$$

Note that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ (Taylor series)

We have $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Now by taking $x = \lambda$, $k = i$, we can get

$$e^{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

, Hence $\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \cdot e^{\lambda} = e^0 = 1$

$\Rightarrow$ $p(i)$ is a valid pmf!

(2) Poisson distribution from Binomial distribution.

Let $X \sim B(n, p)$;

Assume $n \to \infty$, $p \to 0$, and take $\lambda = np$.

Then for each $i = 0, \dots n$,

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} \quad ; \quad p = \frac{\lambda}{n}$$

$$= \frac{n!}{(n-i)!\, i!} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!\, i!} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\dots(n-i+1)}{n^i\, i!} \lambda^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\dots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

Note that as $n \to \infty$, λ moderate

$$\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}, \quad \frac{n(n-1)\dots(n-i+1)}{n^i} \approx 1$$

$$\& \left(1-\frac{\lambda}{n}\right)^i \approx 1$$

Hence

$$P(X=i) \approx \frac{\lambda^i}{i!} e^{-\lambda}$$ for large values of n

($n \to \infty$, $p \to 0$)

p is small to make np constant

Example: (1) Suppose the no. of typographical errors in a single page of a book follows poisson distribution with parameter $\lambda = 1/2$. Calculate the probability that there is at least one error in a randomly selected page of the book.

let X - denote the no. of errors on a single page.

$X \sim Pois(\lambda = 1/2)$

Then we need to compute

$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X=0)$

$$P(X=0) = \frac{e^{-1/2}(1/2)^0}{0!} = e^{-1/2}$$

Hence $P(X \geq 1) = 1 - e^{-0.5} \approx 0.393$

(Convention follows for chosing n s.t that Binomial can be approximated by Poisson is $n \geq 20$ & $p \leq 0.05$ or $n \geq 100$, & $np \leq 10$ ie $p \leq 0.1$)

(Ideally, $np \leq 10$)

2) Exercise: Determine $P(X \leq 90)$, where X follows Poisson with mean 100 ?