AI1110 : Probability and Random Variables

Assignment 8

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Outline

Question

Question Exercise 6.52: Show that, if the correlation coefficient $r_{xy} = 1$, then $y = ax + b$.

Solution

Solution : The correlation coefiicient for any random variables x, y is defined as,

$$r_{xy} = \frac{E\left\{(x-\eta_x)\left(y-\eta_y\right)\right\}}{\sigma_x\sigma_y} \tag{1}$$

where,

$$E\{x\} = \eta_x \tag{2}$$

$$E\{y\} = \eta_y \tag{3}$$

$$\sigma_x^2 = E\left\{(x-\eta_x)^2\right\} \tag{4}$$

$$\sigma_y^2 = E\left\{\left(y-\eta_y\right)^2\right\} \tag{5}$$

Given that,

$$r_{xy} = 1 \tag{6}$$

Then we can write that,

$$E\left\{(x-\eta_x)\left(y-\eta_y\right)\right\}^2 = E\left\{(x-\eta_x)^2\right\} E\left\{\left(y-\eta_y\right)^2\right\} \tag{7}$$

Now consider the below expression,

$$E\left\{\left[a\left(x-\eta_x\right)-\left(y-\eta_y\right)\right]^2\right\} \tag{8}$$

$$E\left\{\left[a^2\left(x-\eta_x\right)^2+\left(y-\eta_y\right)^2-2a\left(x-\eta_x\right)\left(y-\eta_y\right)\right]\right\} \tag{9}$$

$$\begin{aligned} \implies E\left\{\left(a\left(x-\eta_x\right)\right)^2\right\} + E\left\{\left(y-\eta_y\right)^2\right\} \\ -2aE\left\{\left(x-\eta_x\right)\left(y-\eta_y\right)\right\} \end{aligned} \tag{10}$$

Using (4), (5), we will get

$$\implies a^2\sigma_x^2 - 2aE\left\{\left(x-\eta_x\right)\left(y-\eta_y\right)\right\} + \sigma_y^2 \tag{11}$$

If we consider the discriminant of above quadratic equation,

$$\Delta = 4E\left\{(x-\eta_x)\left(y-\eta_y\right)\right\}^2 - 4\sigma_x^2\sigma_y^2 \tag{12}$$

From (7) we can write,

$$\Delta = 0 \tag{13}$$

This is possible only if the quadratic is zero for some $a = a_o$.
This shows that,

$$a\left(x-\eta_x\right) - \left(y-\eta_y\right) = 0 \tag{14}$$

from some $a = a_o$, which can be represented as

$$y = ax + b \tag{15}$$

for some arbitary constants a, b.