AI1110 : Probability and Random Variables

Assignment 7

Mannem Charan(AI21BTECH11019)

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Outline

1. Question
2. Solution

Question

The random variable x takes the values $0, 1, ...$ with $\Pr(x = k) = p_k$. Show that if $y = (x - 1)\, U(x - 1)$ then

$$\Gamma_y(z) = p_0 + z^{-1}\left[\Gamma_x(z) - p_0\right]$$
$$\eta_y = \eta_x - 1 + p_0$$
$$E\left\{y^2\right\} = E\left\{x^2\right\} - 2\eta_x + 1 - p_0$$

Solution

Given that,

$$y = (x - 1)\,U(x - 1) \tag{1}$$

where $U(x-1)$ is step unit function.
So random variable y , can be wriitten as,

$$y = \begin{cases} x - 1 & , x > 1 \\ 0 & , 0 \le x \le 1 \end{cases}$$

Now

$$\Pr(y = 0) = \Pr(x = 0) + \Pr(x = 1) \tag{2}$$

$$= p_0 + p_1 \tag{3}$$

And

$$\Pr\left(y=k\right)=\Pr\left(x=k+1\right)=p_{k+1}\left(k\geq 1\right) \tag{4}$$

The moment generating function $\Gamma_x\left(z\right)$ for any lattice type random variable x is defined as,

$$\Gamma_x\left(z\right)=\sum_{k=-\infty}^{\infty}\Pr\left(x=k\right)z^k \tag{5}$$

In this case,

$$\Gamma_x\left(z\right)=\sum_{k=0}^{\infty}p_k z^k \tag{6}$$

$$=p_0+p_1 z+p_2 z^2+.... \tag{7}$$

Now writing the moment generating function for random variable y,

$$\Gamma_y(z) = \sum_{k=0}^{\infty} \Pr(y = k) z^k \tag{8}$$

$$= \Pr(y = 0) + \sum_{k=1}^{\infty} \Pr(y = k) z^k \tag{9}$$

Using (3),(4),(7),

$$\implies = p_0 + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k \tag{10}$$

$$= p_0 + z^{-1} \{\Gamma_x(z) - p_0\} \tag{11}$$

And for any discrete integer type random variable x,

$$\eta_x = \sum_{k=-\infty}^{\infty} k \Pr(x = k) \tag{12}$$

In this case,

$$\eta_x = \sum_{k=0}^{\infty} k p_k \tag{13}$$

For random variable y we can write,

$$\eta_y = \sum_{k=1}^{\infty} k p_{k+1} \tag{14}$$

$$= \sum_{k=1}^{\infty} k p_k - \sum_{k=1}^{\infty} p_k \tag{15}$$

$$= \eta_x - (1 - p_0) \tag{16}$$

$$= \eta_x - 1 + p_0 \tag{17}$$

For any random variable x,

$$E\left\{x^2\right\} = \sum_{k=-\infty}^{k=\infty} k^2 \Pr(x = k) \tag{18}$$

In this case,

Now for random variable y,

$$E\left\{y^2\right\} = \sum_{k=1}^{k=\infty} k^2 p_{k+1} \tag{20}$$

$$= \sum_{r=1}^{r=\infty} (r-1)^2 p_r \tag{21}$$

$$= \sum_{r=1}^{r=\infty} \left(r^2 - 2r + 1\right) p_r \tag{22}$$

$$= \sum_{r=1}^{r=\infty} r^2 p_r - 2\sum_{r=1}^{r=\infty} r p_r + \sum_{r=1}^{r=\infty} p_r \tag{23}$$

$$= E\left\{x^2\right\} - 2\eta_x + (1 - p_0) \tag{24}$$

$$= E\left\{x^2\right\} - 2\eta_x + 1 - p_0 \tag{25}$$