Digital Signal Processing

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Contents

Abstract—**This document provides the solution of Assignment 1.**

Question 3.6.e : Determine

1) the inverse z-transform using both the methods-partial fraction expansion and power series expansion.In addition, indicate whether the Fourier transform exists or not.
2) And then consider the result from inverse z-transform as the impulse response of the a system $(h(n))$,check whether that sytem system is causal or not and stable or not.

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \ |z| > \left|\frac{1}{a}\right| \tag{1}$$

Solution: Given z-transform of a signal $x(n)$ as,

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \ |z| > \left|\frac{1}{a}\right| \tag{2}$$

1) Now to find inverse z-transform of eq (2),we will use following two methods,
   a) Using partial fractions
   b) Using power series expansion

   Using partial fractions:

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \ |z| > \left|\frac{1}{a}\right| \tag{3}$$

$$= \frac{z^1 (z - a)}{z^1 (1 - az^1)} \tag{4}$$

From the region of convergence i.e., ROC $x(n)$ is a right-sided sequence and since $M = N = 1$ and pole is first order,we can write $X(z)$ as,

$$X(z) = b_0 + \frac{A_1}{1 - rz^{-1}} \tag{5}$$

Now

$$X(z) = \frac{1 - az^{-1}}{-a\left(1 - \frac{z^{-1}}{a}\right)} \tag{6}$$

Comparing eqs (5) and (6), we will get $r = \frac{1}{a}$, and

$$b_0 r = -1 \tag{7}$$

$$\implies b_0 = -a \text{ and} \tag{8}$$

$$b_0 + A_1 = \frac{-1}{a} \tag{9}$$

$$\implies A_1 = \frac{-1}{a} + a \tag{10}$$

So $X(z)$ will be,

$$X(z) = -a - \frac{\frac{1}{a} - a}{1 - \frac{z^{-1}}{a}} \tag{11}$$

Since ROC $|z| > \left|\frac{1}{a}\right|$,

$$-a \overset{\mathcal{Z}}{\rightleftharpoons} -a\delta(n) \tag{12}$$

$$\frac{\frac{1}{a} - a}{1 - \frac{z^{-1}}{a}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(\frac{1}{a} - a\right)\left(\frac{1}{a}\right)^n u(n) \tag{13}$$

$$\tag{14}$$

From the linearity of z-transform,

$$x(n) = -a\delta(n) - \left(1 - a^2\right) a^{-(n+1)} u(n) \tag{15}$$

Using power series expansion Here we will try to find the power series itself which results in z-transform.And compare it with the below expression to get $x(n)$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{16}$$

We will do that using long division method,

$$\begin{array}{r|lll}
 & -\frac{1}{a} \quad -\frac{1}{a}(\frac{1}{a} - a)z^{-1} & -\frac{1}{a^2}\left(\frac{1}{a} - a\right)z^{-2} & \ldots \\
\hline
z^{-1} - a & 1 \quad -az^{-1} & & \\
 & 1 \quad -\frac{z^{-1}}{a} & & \\
\hline
 & \quad \frac{z^{-1}}{a} - az^{-1} & & \\
 & \quad \left(\frac{1}{a} - a\right)z^{-1} & -\frac{1}{a}\left(\frac{1}{a} - a\right)z^{-2} & \\
\hline
 & & \frac{1}{a}\left(\frac{1}{a} - a\right)z^{-2} & \\
 & & \ldots &
\end{array}$$

The power series will be

$$X(z) = -\frac{1}{a} + -\left(\frac{1}{a} - a\right)\sum_{n=1}^{\infty}\left(\frac{1}{a}\right)^{n} z^{-n} \tag{17}$$

Comparing with (16),

$$x(n) = -a\delta(n) - \left(\frac{1}{a} - a\right)\left(\frac{1}{a}\right)^{n} u(n) \tag{18}$$

$$= -a\delta(n) - \left(1 - a^2\right)a^{-(n+1)}u(n) \tag{19}$$

And for $x(n)$, fourier transform exists when ROC contains the unit circle ($|z| = 1$),this will happen when

$$\left|\frac{1}{a}\right| < 1 \tag{20}$$

$$\implies |a| > 1 \tag{21}$$

$\therefore$ For $|a| > 1$, given signal $x(n)$ has Fourier transform.

2) Consider a system whose impulse response is given by,

$$h(n) = -a\delta(n) - \left(1 - a^2\right)a^{-(n+1)}u(n) \tag{22}$$

We can say that the system is stable when,

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{23}$$

It is possible when ROC contains the unit circle,

$$\left|\frac{1}{a}\right| < 1 \tag{24}$$

$$|a| > 1 \tag{25}$$

$\therefore$ The system is stable if $|a| > 1$.
To check whether the system is causal or not we need to check whether the ROC for $H(z)$ is outside the outermost pole.Since the ROC here is,

$$|z| > \left|\frac{1}{a}\right| \tag{26}$$

It is clearly outside the outer most pole $\left(|z| = \frac{1}{a}\right)$. Hence the given system is causal.