Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\, \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\, \mu C$.

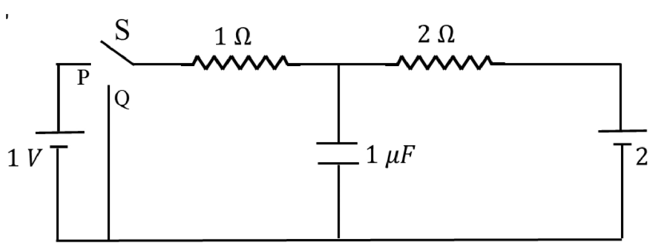


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:

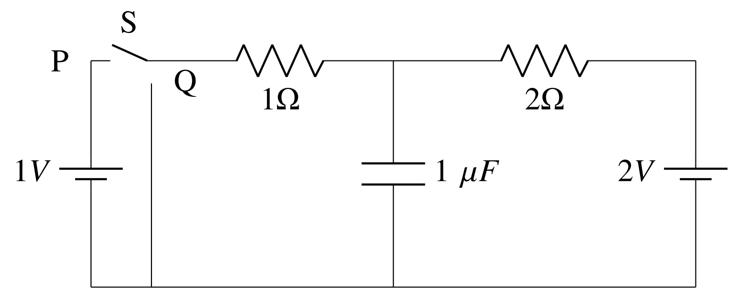
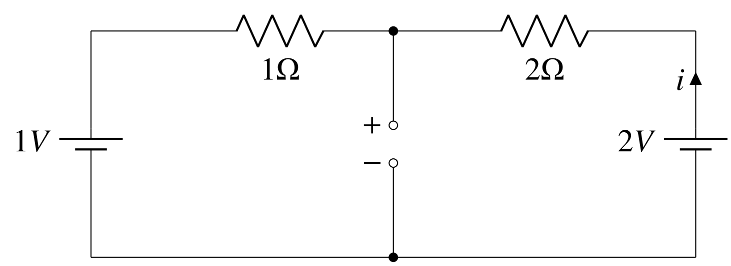


Fig. 2.2: Circuit diagram of the question

3. Find q_1.
Solution: Since the switch S is closed for a long time at P, the circuit at steady state looks like, Now if we apply KVL,



$$2 - 2i - i - 1 = 0 \tag{2.1}$$

$$\implies i = \frac{1}{3} \tag{2.2}$$

The potential difference across capacitor is,

$$V_C = 2 - 2i \tag{2.3}$$

$$= \frac{4}{3} \tag{2.4}$$

Therefore the charge on capacitor will be,

$$q_1 = \frac{4}{3}\mu F \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: From 1.2 we can write laplace trans-

form of unit step function $u(t)$ as,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.6}$$

$$= \int_{0}^{\infty} e^{-}stdt \tag{2.7}$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} \tag{2.8}$$

For the laplace transform to exist,$Re\,(s) > 0$ using that

$$\mathcal{L}\{u(t)\} = \frac{1}{s}\text{with ROC } Re\,(s) > 0 \tag{2.9}$$

The ROC plot looks like 2.3,

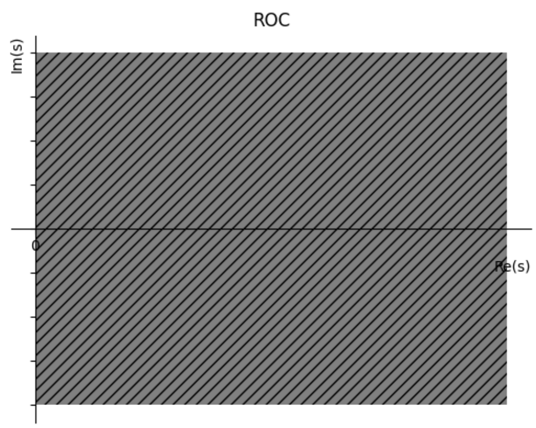


Fig. 2.3: ROC of laplace transform of $u(t)$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.10}$$

and find the ROC.
Solution: The laplace transform will be,

$$\mathcal{L}\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-(a+s)t}u(t)dt \tag{2.11}$$

$$= \int_{0}^{\infty} e^{-(a+s)t}dt \tag{2.12}$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty} \tag{2.13}$$

Now for the integral to exist,

$$Re\,(s+a) > 0 \tag{2.14}$$

$$Re\,(s) > -a \tag{2.15}$$

So the ROC will be $Re\,(s) > -a$ and with that

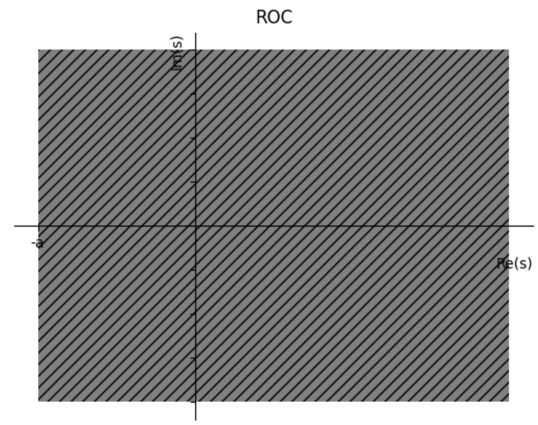


Fig. 2.4: ROC of laplace transform of $e^{-at}u\,(t)$

ROC,

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{a+s} \tag{2.16}$$

And the ROC plot looks like 2.4,

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

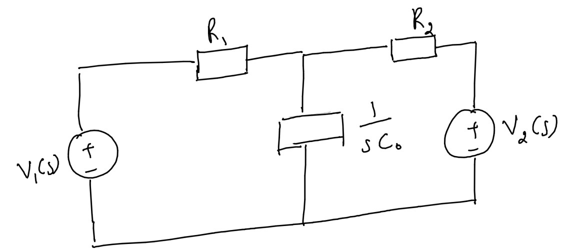


Fig. 2.5

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.17}$$

$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.18}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: From the earlier proved results,

$$V_1(s) = \mathcal{L}\,(u(t)) \tag{2.19}$$

$$= \frac{1}{s} \tag{2.20}$$

$$V_2(s) = \mathcal{L}\,(2u(t)) \tag{2.21}$$

$$= \frac{2}{s} \tag{2.22}$$

Note that ROC here is $Re\,\{s\} > 0$.

And the let the ends of resistive capacitor has voltages of V_{C_0} and 0. The same can be seen in figure 2.6, If we apply Kirchoff's Junction

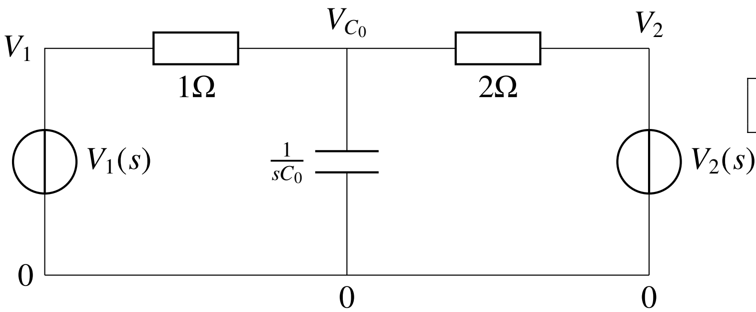


Fig. 2.6

law,

$$\frac{V_{C_0} - V_1(s)}{1} + \frac{V_{C_0} - 0}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{2} = 0 \quad (2.23)$$

$$V_{C_0}(s)\left(\frac{3}{2} + sC_0\right) = V_1(s) + \frac{V_2(s)}{2} \quad (2.24)$$

Substituting $V_1(s), V_2(s)$ and C_0, you will get

$$V_{C_0}(s) = \frac{4}{s\,(3 + 2x10^{-6}s)} \quad (2.25)$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Now we can find the voltage of capacitor in time domain using inverse laplace transform,

$$v_{C_0}(t) = \mathcal{L}^{-1}\left[\frac{4}{s\,(3 + 2x10^{-6}s)}\right] \quad (2.26)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3}\left[\frac{1}{s} - \frac{1}{10^{-6}s + \frac{3}{2}}\right]\right] \quad (2.27)$$

$$= \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s + \frac{3x10^6}{2}}\right] \quad (2.28)$$

$$= \frac{4}{3}\left(u(t) - u(t)e^{-\frac{3x10^6 t}{2}}\right) \quad (2.29)$$

$$\implies v_{C_0}(t) = \frac{4}{3}\left(1 - e^{-\frac{3t}{2}x10^6}\right)u(t) \quad (2.30)$$

Note that the ROC here is $Re\{s\} > 0$.
The plot 2.7 of the same can be viewed using the python code in the following link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.7.py
```

Then run the following command,

```
python3 2.7.py
```

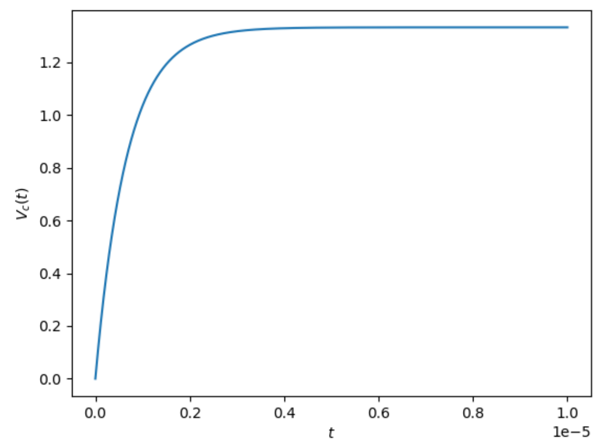


Fig. 2.7: The plot of $V_c(t)$ vs t

8. Verify your result using ngspice.
Solution: Download the codes from the below links,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.8.cir
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.8.py
```

Then run the following command,

```
ngspice 2.8.cir
python3 2.8.py
```

9. Obtain Fig. 2.5 using the equivalent differential equation.
Solution: Using Kirchoff's Junction Law in 2.1,

$$\frac{v_c(t) - v_1(t)}{1} + \frac{v_c(t) - v_2(t)}{2} + \frac{dq}{dt} = 0 \quad (2.31)$$

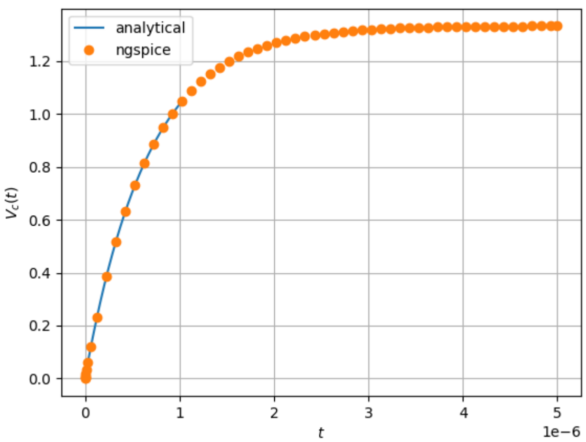


Fig. 2.8: The plot of $V_c(t)$ vs t using ngspice

where $q(t)$ is the charge on capacitor. And can be written as ,

$$q_c(t) = Cv_c(t) \tag{2.32}$$

Now applying laplace transform of (2.31) using (2.32),

$$\begin{aligned}(V_c(s) - V_1(s)) + \frac{1}{2}(V_c(s) - V_2(s)) \\ +C\left(sV_c(s) - v_c(0^-)\right) = 0\end{aligned} \tag{2.33}$$

We know,

$$v_c(0^-) = v_c(0) = 0 \tag{2.34}$$

Using that,

$$\frac{V_{C_0} - V_1(s)}{1} + \frac{V_{C_0} - 0}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{2} = 0 \tag{2.35}$$

With this equation, we can write the equivalent resistive circuit as shown in Fig 2.5.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: After closing switch at Q for long time the circuit looks like 3.1,
Now we will use Kirchoff's Junction Law in this circuit,

$$\frac{V_c - 0}{1} + \frac{V_c - 2}{2} = 0 \tag{3.1}$$

$$\implies V_c = \frac{2}{3}V \tag{3.2}$$

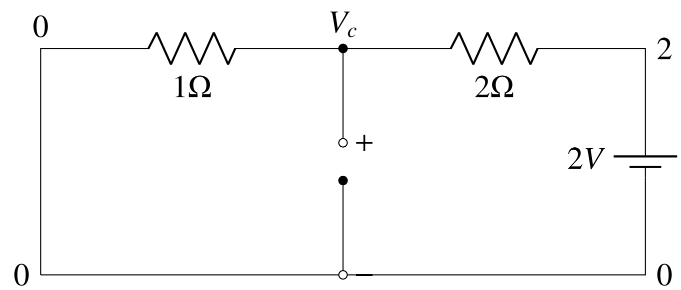


Fig. 3.1

And the charge on the capacitor will be,

$$q_2 = \frac{2}{3}\mu C \tag{3.3}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution: The equivalent s-domain resistive circuit looks like 3.2. The battery $\frac{4}{3s}$ is added

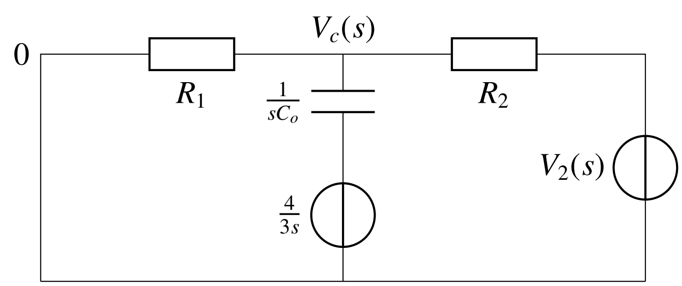


Fig. 3.2: Circuit after closing switch to Q in s-domain

series to C_0 in s- domain by taking consideration of initial charge on capacitor $q_1 = \frac{4}{3}\mu C$ before closing switch to Q.
3. $V_{C_0}(s) = ?$
Solution: Apply kirchoff's junction law in the s-domain circuit 3.2,taking $R_1 = 1\Omega$ and $R_2 =$

2Ω

$$\frac{V_{C_0}(s) - 0}{1} + \frac{V_{C_0}(s) - \frac{4}{3s}}{\frac{1}{sC_0}} + \frac{V_{C_0}(s) - V_2(s)}{2} = 0 \tag{3.4}$$

$$\implies V_{C_0}(s)\left(\frac{3}{2} + sC_0\right) = \frac{V_2(s)}{2} + \frac{4C_0}{3} \tag{3.5}$$

$$\implies V_{C_0}(s) = \frac{\frac{1}{s} + \frac{4C_0}{3}}{\frac{3}{2} + sC_0} \tag{3.6}$$

$$\therefore V_{C_0}(s) = \frac{2\,(3 + 4sC_0)}{3s\,(3 + 2sC_0)} \tag{3.7}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: We can obtain the potential difference across the capacitor in time domain by applying inverse laplace transform under right ROC conditions,

$$v_{C_0}(t) = \mathcal{L}^{-1}\left[\frac{2\,(3 + 4sC_0)}{3s\,(3 + 2sC_0)}\right] \tag{3.8}$$

$$= \mathcal{L}^{-1}\left[\frac{2}{3s} + \frac{4C_0}{3\,(3 + 2sC_0)}\right] \tag{3.9}$$

$$= \mathcal{L}^{-1}\left[\frac{2}{3s}\right] + \mathcal{L}^{-1}\left[\frac{4C_0}{3\,(3 + 2sC_0)}\right] \tag{3.10}$$

With ROC condition $Re\{s\} > 0$,

$$v_{C_0}(t) = \frac{2u(t)}{3} + \frac{2u(t)}{3}e^{-\frac{3}{2C_0}} \tag{3.11}$$

Substituting $C_0 = 1\mu F$,

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\frac{3x10^6}{2}}\right)u(t) \tag{3.12}$$

We can verify the same using the python code from the below link,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.4.py
```

Then run the following command,

```
wget python3 3.4.py
```

5. Verify your result using ngspice.
Solution: Download the codes from the below links

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.5.py
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.5.cir
```

Then run the following commands,

```
ngspice 3.5.cir
python3 3.5.py
```

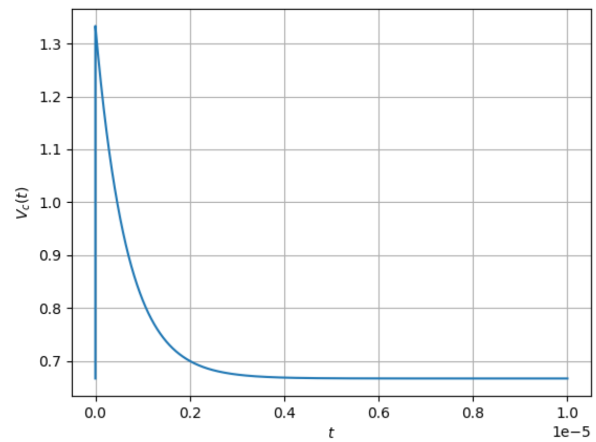


Fig. 3.3: The plot of $V_c(t)$ vs t

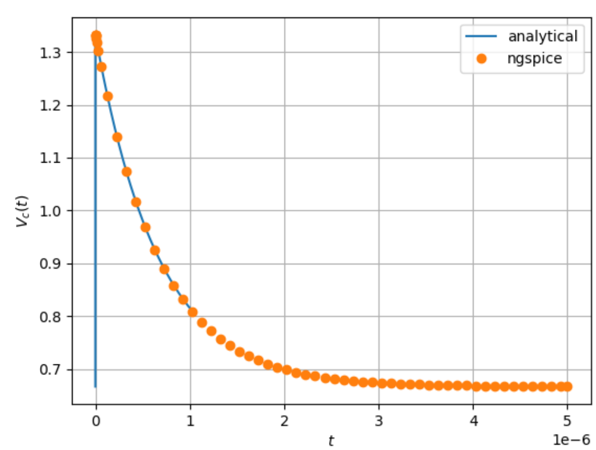


Fig. 3.4: The plot of $V_c(t)$ vs t using ngspice

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: In that case when $t < 0$ switch is not closed to Q so the circuit will be in steady

state (switch at P),

$$v_{C_0}(0-) = \left[\frac{4}{3}\left(1 - e^{-\frac{3x10^6}{2}}\right)u(t)\right]_{t=\infty} \tag{3.13}$$

$$= \frac{4}{3}V \tag{3.14}$$

And for $t = 0+$ and $t = \infty$, we can use (3.12),

$$v_{C_0}(0+) = \frac{4}{3}V \tag{3.15}$$

$$v_{C_0}(\infty) = \frac{2}{3}V \tag{3.16}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution: Using Kirchoff's Junction Law ,

$$\frac{v_c(t) - v_1(t)}{1} + \frac{v_c(t) - v_2(t)}{2} + \frac{dq}{dt} = 0 \tag{3.17}$$

where $q(t)$ is the charge on capacitor. And can be written as ,

$$q_c(t) = Cv_c(t) \tag{3.18}$$

Now applying laplace transform ,

$$(V_c(s) - V_1(s)) + \frac{1}{2}(V_c(s) - V_2(s)) + C\left(sV_c(s) - v_c(0^-)\right) = 0 \tag{3.19}$$

We know,

$$v_c(0^-) = \frac{4}{3}V \tag{3.20}$$

Using that,

$$\frac{V_{C_0} - V_1(s)}{R_1} + \frac{V_{C_0} - \frac{4}{3s}}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{R_2} = 0 \tag{3.21}$$

With this equation, we can write the equivalent resistive circuit as shown in Fig 3.2.

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
2. Find $H(s)$ considering the ouput voltage at the capacitor.
3. Plot $H(s)$. What kind of filter is it?
4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.1}$$

5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$?