Digital Signal Processing

Mannem Charan AI21BTECH11019

Contents

***Abstract*—This document provides the solution of Sound 1 Assignment.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/Sound%20With%20
    ReducedNoise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
from scipy.fft import fftfreq
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs= sf.read("Assignment 1/
    Codes/filter_codes_Sound_Noise.wav")

#sampling frequency of Input signal
sampl_freq = fs

# order of the filter
order = 4

#cutoff frequency 4kHz
cutoff_freq = 4000

#digital frequency
Wn = 2*cutoff_freq/sampl_freq

#b and a are numerator and denominator
    polynomials respectively.
b,a = signal.butter(order,Wn,'low')

#filter the input signal with butterworth filter.
output_signal = signal.filtfilt(b,a,input_signal
    )
#output_signal = signal.lfilter(b,a,
    input_signal)

#write the output signal into .wav file.
sf.write('Assignment 1/Codes/Sound With 
    ReducedNoise.wav',output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav.Play the file in the spectrogram in Problem 2.2. What

do you observe?
Solution: The key strokes as well as background noise is subdued in the audio.Also,the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The plot of $x(n)$ is given in 3.2

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/xnyn.py
```

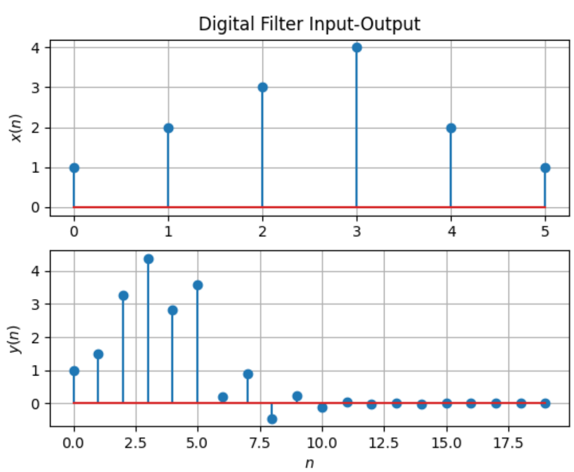

Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution: Download the C code from the below link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/xnyn.c
```

Then run the follwing command in terminal

```
cc xnyn.c
./a.out
```

Then for the plot 3.3 download the python file from the below link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/xnyn2.py
```

Then run the command

```
python3 xnyn2.py
```

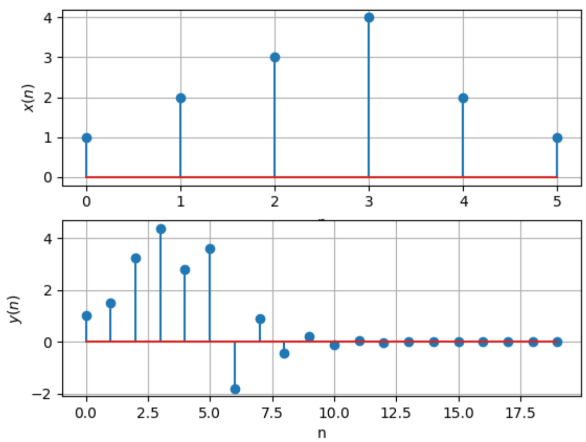

Fig. 3.3: Plot using C code

4 $\mathcal{Z}$-transform

4.1 The $\mathcal{Z}$-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

Take $k = n - 1$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \tag{4.7}$$

$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{4.8}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$

$$= z^{-1} X(z) \tag{4.10}$$

resulting in (4.2) and similarly following the above steps you will get,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(n) \tag{4.11}$$

Hence proved.

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:** Now we will find Z transform of the signal $x(n)$,from (3.1),

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n) z^{-n} \tag{4.12}$$

$$= 1z^{0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \tag{4.13}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.14}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.15}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Now we will rewrite (3.2),

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \tag{4.16}$$

Now since Z-transform is a linear operator we can write that,

$$Y(n) + \frac{1}{2} Y(n-1) = X(n) + X(n-2) \tag{4.17}$$

From (4.11),

$$Y(n) + \frac{z^{-1}}{2} Y(n) = X(n) + z^{-2} X(n) \tag{4.18}$$

$$\implies \frac{Y(n)}{X(n)} = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{4.19}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.21}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.22}$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.23}$$

$$= \delta(0) z^{0} + 0 \ \text{(Using (4.20))} \tag{4.24}$$

$$= 1 \tag{4.25}$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.26}$$

$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \tag{4.27}$$

$$= 1 + z^{-1} + z^{-2} + \ldots \tag{4.28}$$

Above is a infinite geometric series with z^{-1} as common ratio , so we can write it as

$$U(n) = \frac{1}{1 - z^{-1}} \because |z| > 1 \tag{4.29}$$

4.5 Show that

$$a^{n} u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - a z^{-1}} \quad |z| > |a| \tag{4.30}$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^{n} u(n)\} = \sum_{n=0}^{\infty} a^{n} z^{-n} \tag{4.31}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^{2} + \ldots. \tag{4.32}$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - \frac{a}{z}} \because |a| < |z| \tag{4.33}$$

Therefore,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.34}$$

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.35}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: Download the code for the plot 4.6 from the link below

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/dtft.py
```

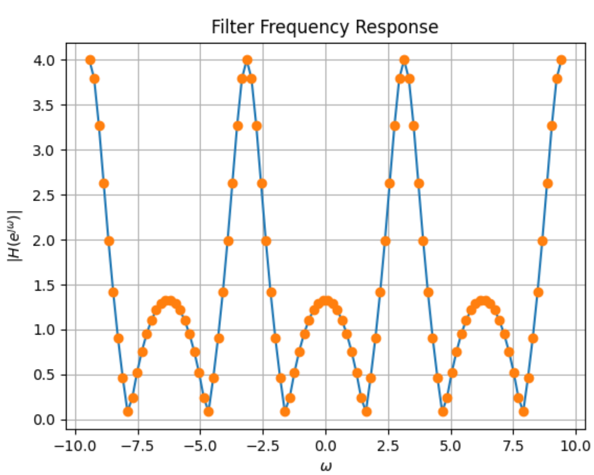

Fig. 4.6: $|H(e^{j\omega})|$

Now using (4.19), we will find $\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.36}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.37}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.38}$$

$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.39}$$

$$= \frac{\left|4\cos^2(\omega) + 4j \sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \tag{4.40}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \tag{4.41}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.42}$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{j\omega}) = H(e^{j(-\omega)}) \text{ (cos is an even function)} \tag{4.43}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5 + 4\cos(\omega)}$ is 2π. So the period of division of both will be,

$$lcm(\pi, 2\pi) = 2\pi \tag{4.44}$$

This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.

Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.45}$$

and

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega \tag{4.46}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.47}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.48}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.49}$$

$$= \frac{1}{2\pi}\left\{\sum_{k\neq n} h(k)\frac{e^{j\omega(n-k)}}{j(n-k)}\Bigg]_{-\pi}^{\pi} + h(n)\int_{-\pi}^{\pi} d\omega\right\} \tag{4.50}$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \tag{4.51}$$

$$= h(n) \tag{4.52}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.19).
Solution: From (4.19), we can write

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{5.2}$$

$$\begin{array}{r|llll}
 & 2z^{-1} & -4 & & \\
\hline
1 + z^{-1}/2 & 1 & +z^{-2} & & \\
 & 2z^{-1} & +z^{-2} & & \\
\hline
 & & 1 & -2z^{-1} & \\
 & & -4 & -2z^{-1} & \\
\hline
 & & & 5 &
\end{array}$$

So we can replace (4.19) as,

$$\frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} = 2z^{-1} - 4 + \frac{5}{1 + z^{-1}/2} \tag{5.3}$$

Now we can expand the second term of above expression as an infinite geometric series,

$$\frac{5}{1 + z^{-1}/2} = 5\left(1 + \left(\frac{-1}{2z}\right) + \left(\frac{-1}{2z}\right)^2 + \dots\right) \tag{5.4}$$

where we assume $\left|\frac{1}{2z}\right| < 1$. So (5.3) will become,

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.5}$$

$$= 1z^{0} + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.6}$$

Now to get $h(n)$ for $n < 5$ we will compare (5.6) with the below equation,

$$H(z) = \sum_{n=-\infty}^{n=\infty} h(n)z^{-n} \tag{5.7}$$

$h(n)$ will be the coefficient of z^{-n}.
Using this, from (5.6) we can write,

$$h(0) = 1 \tag{5.8}$$

$$h(1) = \frac{-1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = \frac{-5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

And for $n < 0$ $h(n) = 0$.
For $n > 5$, we can get h(n) from the geometric series,

$$h(n) = 5\left(\frac{-1}{2}\right)^n \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1 + \frac{z^{-1}}{2}} + \frac{z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{5.15}$$

From (4.30) we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

5.3 Sketch $h(n)$. Is it bounded? Justify Theoritically.

Solution: Download the code for the plot 5.3 from the below link,

```
wget https://github.com/Charanyash/EE3900-Digital_Signal_Processing/blob/master/Sound%201/Codes/hn.py
```

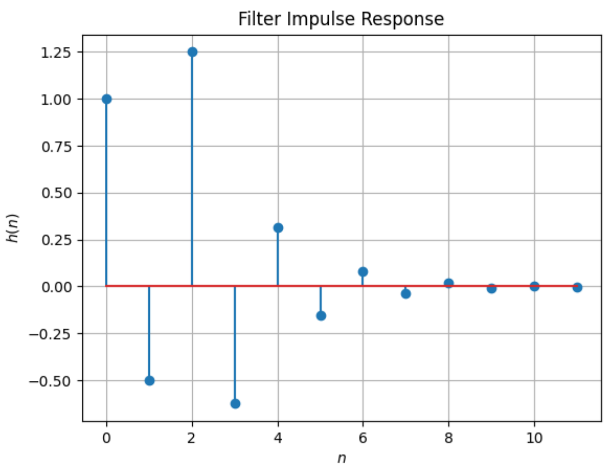

Fig. 5.3: $h(n)$ as inverse of $H(n)$

From the plot it seems like $h(n)$ is bounded and becomes smaller in magnitude as n increases. Using (5.16), we can get theoritical expression as,

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \leq n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \geq 2 \end{cases} \tag{5.17}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number M such that,

$$|x_n| \leq M, \forall n \in \mathcal{N} \tag{5.18}$$

So to say $h(n)$ is bounded we should able to find the M which satisfies (5.18).
For $n < 0$,

$$|h(n)| \leq 0 \tag{5.19}$$

For $0 \leq n < 2$,

$$|h(n)| = \left|\frac{-1}{2}\right|^n \tag{5.20}$$

$$= \left(\frac{1}{2}\right)^n \leq 1 \tag{5.21}$$

And for $n \geq 2$,

$$|h(n)| = \left|5\left(\frac{-1}{2}\right)\right|^n \tag{5.22}$$

$$= \left(\frac{5}{2}\right)^n \leq \frac{5}{4} \tag{5.23}$$

From above three cases, we can get M as,

$$M = max\left\{0, 1, \frac{5}{4}\right\} \tag{5.24}$$

$$= \frac{5}{4} \tag{5.25}$$

Therefore, $h(n)$ is bounded using (5.18) with $M = \frac{5}{4}$ i.e.,

$$|h(n)| \leq \frac{5}{4} \forall n \in \mathcal{N} \tag{5.26}$$

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.27}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty} \left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^n}\right| \tag{5.28}$$

$$= \lim_{n\to\infty} \left|\frac{-1}{2}\right| \tag{5.29}$$

$$= \frac{1}{2} \tag{5.30}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.31}$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?

Solution: From (5.16),

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right) \tag{5.32}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.33}$$

$$= \frac{4}{3} \tag{5.34}$$

$\therefore$ the system is stable.

5.6 Verify the above result using a python code.
Solution: Download the python code from the below link

```
wget https://github.com/Charanyash/EE3900
    -Digital_Signal_Processing/blob/
    master/Sound%201/Codes/hnstable.py
```

Then run the following command,

```
python3 hn_stable.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.35}$$

This is the definition of $h(n)$.
Solution: Download the code for the plot 5.7 from the below link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/hndef.py
```

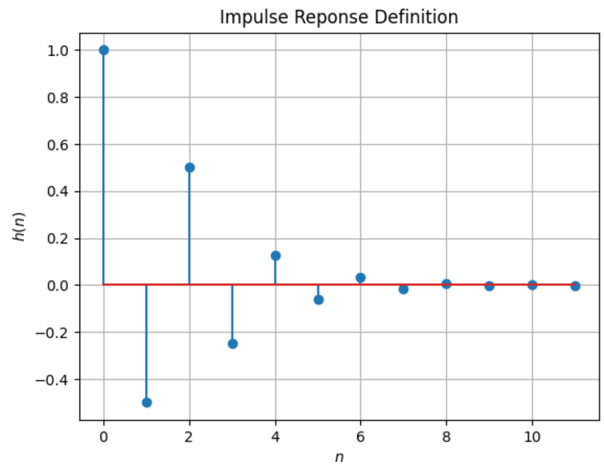

Fig. 5.7: From the definition of $h(n)$

Note that this is same as 5.3.
For $n < 0$, $h(n) = 0$ and,

$$h(0) = \delta(0) \tag{5.36}$$

$$= 1 \tag{5.37}$$

For $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) \tag{5.38}$$

$$\implies h(1) = -\frac{1}{2}h(0) \tag{5.39}$$

$$= -\frac{1}{2} \tag{5.40}$$

$n = 2$,

$$h(2) + \frac{1}{2}h(1) = 0 + \delta(0) \tag{5.41}$$

$$h(2) = 1 + \frac{1}{4} \tag{5.42}$$

$$= \frac{5}{4} \tag{5.43}$$

And for $n > 2$ RHS will be 0 so,

$$h(n) = -\frac{1}{2}h(n-1) \tag{5.44}$$

Overall

$$h(n) = \begin{cases} 0 & , n < 0 \\ 1 & , n = 0 \\ -\frac{1}{2} & , n = 1 \\ \frac{5}{4} & , n = 2 \\ -\frac{1}{2}h(n-1) & , n > 2 \end{cases} \tag{5.45}$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.46}$$

Comment. The operation in (5.46) is known as *convolution*.
Solution: Download the code for plot 5.8 from the below link

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/ynconv.py
```

Note that the plot is same that as in 3.2.

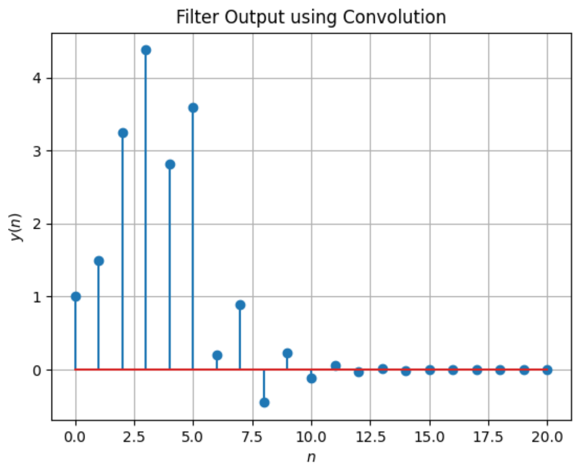

Fig. 5.8: $y(n)$ using the convolution definition

5.9 Express the above convolution using a Toeplitz matrix.
Solution: Download the python code from the below link for the plot 5.9,

```
wget https://github.com/Charanyash/EE3900–
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/ynconv_toeplitz.py
```

Then run the following command,

```
python3 ynconv_toeplitz.py
```

From (5.46),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.47}$$

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.46)

$$y(0) = x(0)\, h(0) \tag{5.48}$$

$$y(1) = x(0)\, h(1) + x(1)\, h(0) \tag{5.49}$$

$$y(2) = x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \tag{5.50}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.51}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.52}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.53}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.54}$$

Now from (3.1) we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.55}$$

And from (5.17)

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.56}$$

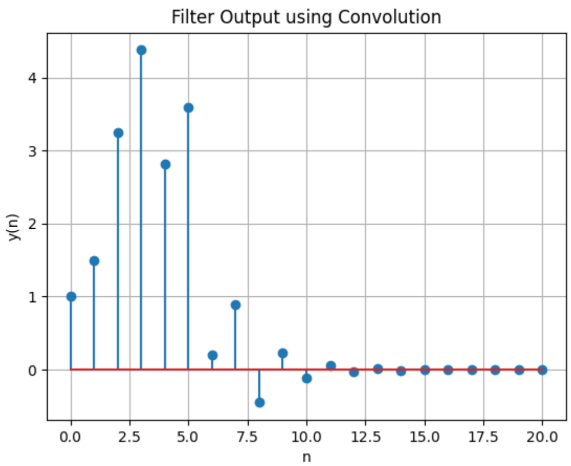

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

Now using (5.54),

$$y(n) = x(n) * h(n) \tag{5.57}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.58}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.59}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.60}$$

Solution: Substitute $k := n - k$ in (5.46), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.61}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.62}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.63}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: Download the below python code for the plot 6.1,

```
wget https://github.com/Charanyash/EE3900-
     Digital_Signal_Processing/blob/master/
     Sound%201/Codes/dft.py
```

And run the following command in the terminal,

```
python3 dft.py
```

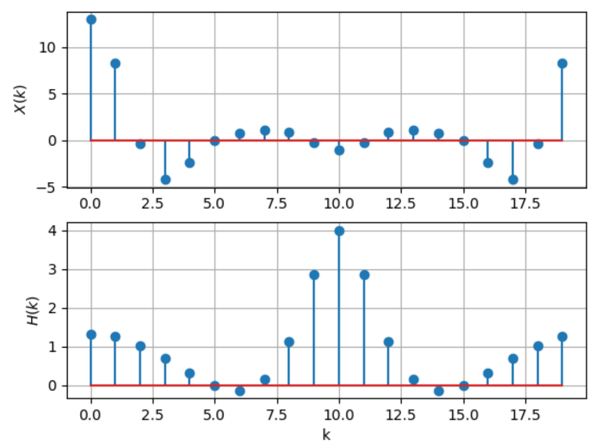

Fig. 6.1: Plot of real part of Discrete Fourier Transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download the below python code for the plot 6.2,

```
wget https://github.com/Charanyash/EE3900-
     Digital_Signal_Processing/blob/master/
     Sound%201/Codes/Y_K.py
```

Then run the following command in the terminal,

```
python3 Y_K.py
```

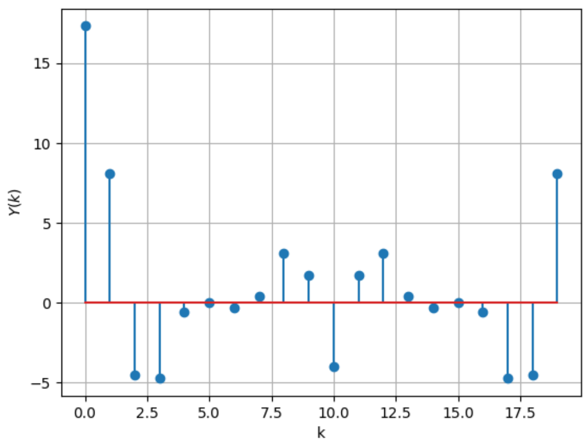

Fig. 6.2: $Y(k)$ as the product of $X(k)$ and $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: Download the below python code for the plot 6.3,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/yndft_dif.py
```

Then run the following command,

```
python3 yndft_dif.py
```

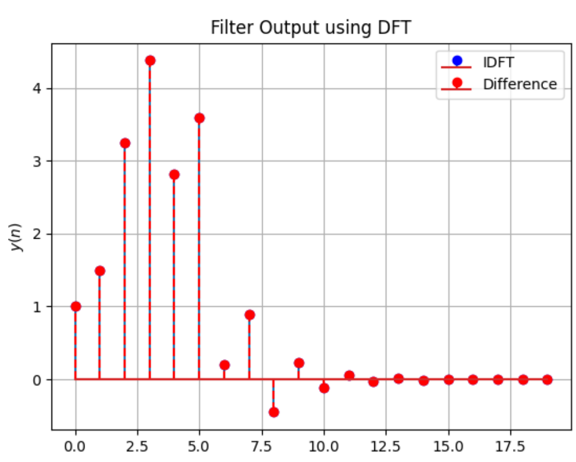

Fig. 6.3: $y(n)$ using IDFT and difference equation

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

Solution: Download the below python code for the plot 6.4,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/master/
    Sound%201/Codes/yn_ifft.py
```

Then run the following command,

```
python3 yn_ifft.py
```

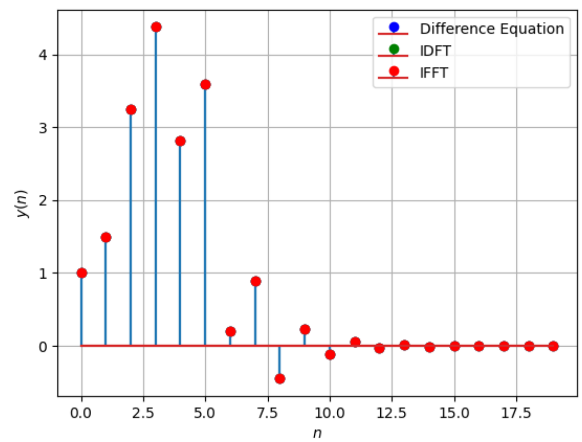

Fig. 6.4: The plot of $y(n)$ using IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFTdiagonalmatrix* is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From (7.2),
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
Consider,
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Hence proved.

6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$
Solution: From the eq (7.5),
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.13}$$
Clearly $\mathbf{P}_4$ is an elementary matrix of $\mathbf{I}_4$, so on multiplication with a matrix it will interchange the rows/columns of matrix depending on positions of unit vectors.
From that it follows ,
$$\mathbf{P}_4^2 = \mathbf{I}_4 \tag{7.14}$$
So it is similar to prove that,
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.15}$$
Now from (7.3),
$$\mathbf{F}_2 = \begin{bmatrix} W_2^{0.0} & W_2^{0.1} \\ W_2^{1.0} & W_2^{1.1} \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.17}$$
Using the result (7.11), we can write
$$\mathbf{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.18}$$
And $\mathbf{D}_2$ is a diagonal matrix,
$$\mathbf{D}_2 = diag\left(W_4^0, W_4^1\right) \tag{7.19}$$
$$= diag\,(1, W_4) \tag{7.20}$$
Then,
$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.21}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.22}$$
And for $k \in \mathcal{N}$ and N be a even integer we know that,
$$W_N^{Nk} = 1 \tag{7.23}$$
$$W_N^{Nk+N/2} = -1 \tag{7.24}$$
Using that we can write,
$$-\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.25}$$
And from (7.3),
$$\mathbf{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.26}$$
And
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix} \tag{7.27}$$
This is same as,
$$\begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.28}$$
$$\implies \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.29}$$
Hence proved.
7. Show that
$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.30}$$
Solution: As we saw earlier, it is similar to prove that
$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \tag{7.31}$$

Assuming that N is even, consider LHS

$$\mathbf{F}_N\mathbf{P}_N = \begin{bmatrix} W_N^{0\times0} & W_N^{0\times2} & .. & W_N^{0\times1} & W_N^{0\times3}.. \\ W_N^{1\times0} & W_N^{1\times2} & .. & W_N^{1\times1} & W_N^{1\times3}.. \\ & & .. & & \\ & & .. & & \\ W_N^{N/2\times0} & W_N^{N/2\times2} & .. & W_N^{N/2\times1} & W_N^{N/2\times3}.. \\ & & .. & & \\ & & .. & & \\ W_N^{N-1\times0} & W_N^{N-1\times2} & .. & W_N^{N-1\times1} & W_N^{N-1\times3}.. \end{bmatrix} \tag{7.32}$$

On multiplying with $\mathbf{P}_N$ (permutation matrix), the odd-numbered columns of $\mathbf{F}_N$ shifted towards left.
Now we can divide the above matrix (7.32), into four sub-matrices as,

$$= \begin{bmatrix} \left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right] \\ \left[W_N^{(n+\frac{N}{2})\times(2m)}\right] & \left[W_N^{(n+\frac{N}{2})\times(2m+1)}\right] \end{bmatrix} \tag{7.33}$$

where, $0 \le n, m \le \dfrac{N}{2} - 1$

$$= \begin{bmatrix} \left[\left(W_N^{n\times m}\right)^2\right] & \left[W_N^{n}\left(W_N^{n\times m}\right)^2\right] \\ \left[W_N^{Nm}\left(W_N^{n\times m}\right)^2\right] & \left[W_N^{Nm+N/2}W_N^{n}\left(W_N^{n\times m}\right)^2\right] \end{bmatrix} \tag{7.34}$$

Using (7.23), (7.24) and (7.11)

$$= \begin{bmatrix} \left[W_{\frac{N}{2}}^{n\times m}\right] & \left[W_N^{n}W_{\frac{N}{2}}^{n\times m}\right] \\ \left[W_{\frac{N}{2}}^{n\times m}\right] & \left[-W_N^{n}W_{\frac{N}{2}}^{n\times m}\right] \end{bmatrix} \tag{7.35}$$

Now from def (7.3) and (7.6), we can write,

$$= \begin{bmatrix} \mathbf{F}_{\frac{N}{2}} & \mathbf{D}_{\frac{N}{2}}\mathbf{F}_{\frac{N}{2}} \\ \mathbf{F}_{\frac{N}{2}} & -\mathbf{D}_{\frac{N}{2}}\mathbf{F}_{\frac{N}{2}} \end{bmatrix} \tag{7.36}$$

$$\implies \mathbf{F}_N\mathbf{P}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \tag{7.37}$$

Hence proved.
Note : If we want to do the above matrix decomposition recursively the value of N should in the form of 2^k.

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.38}$$

Solution: Let $\mathbf{x}$,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.39}$$

and $\mathbf{P}_4$ is 4 - point permutation matrix. So,

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.40}$$

$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \tag{7.41}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.42}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: From (7.1),

$$X(k) = \sum_{n=0}^{N-1} x(n)W^{kn} \tag{7.43}$$

Now we will try to convert the above expression into matrix equations,

$$X(0) = \sum_{n=0}^{N-1} x(n)W^{0.n} \tag{7.44}$$

$$= \begin{pmatrix} W^{0.0} \\ W^{0.1} \\ W^{0.2} \\ W^{0.(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.45}$$

$$X(1) = \begin{pmatrix} W^{1.0} \\ W^{1.1} \\ W^{1.2} \\ W^{1.(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.46}$$

.

.

$$X(N-1) = \begin{pmatrix} W^{(N-1)\times 0} \\ W^{(N-1)\times 1} \\ W^{(N-1)\times 2} \\ W^{(N-1)\times(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.47}$$

$\mathbf{X} =$

$$\begin{bmatrix} W_N^{0\times 0} & W_N^{0\times 1} & .. & W_N^{0\times N-1} \\ .. & .. & .. & .. \\ W_N^{N-1\times 0} & W_N^{N-1\times 1} & .. & W_N^{N-1\times N-1} \end{bmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.48}$$

From def (7.3),

$$\mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{7.49}$$

Hence proved.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.51}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.54}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.55}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.56}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.62}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.63}$$

compute the DFT using (7.42)

12. Repeat the above exercise using the FFT after zero padding **x**.
13. Write a C program to compute the 8-point FFT.

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.