Digital Signal Processing

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Contents

***Abstract*—This document provides the solution of Assignment 1.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/main/
    Assignment−1/Codes/
    filter_codes_Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
from scipy.fft import fftfreq
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs= sf.read("Assignment 1/
    Codes/filter_codes_Sound_Noise.wav")

#sampling frequency of Input signal
sampl_freq = fs

# order of the filter
order = 4

#cutoff frequency 4kHz
cutoff_freq = 4000

#digital frequency
Wn = 2*cutoff_freq/sampl_freq

#b and a are numerator and denominator
    polynomials respectively.
b,a = signal.butter(order,Wn,'low')

#filter the input signal with butterworth filter.
output_signal = signal.filtfilt(b,a,input_signal
    )
#output_signal = signal.lfilter(b,a,
    input_signal)

#write the output signal into .wav file.
sf.write('Assignment 1/Codes/Sound With 
    ReducedNoise.wav',output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav.Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as

background noise is subdued in the audio.Also,the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

Solution: The plot of $x(n)$ is given in 3.2

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/main/
    Assignment−1/Codes/xnyn.py
```

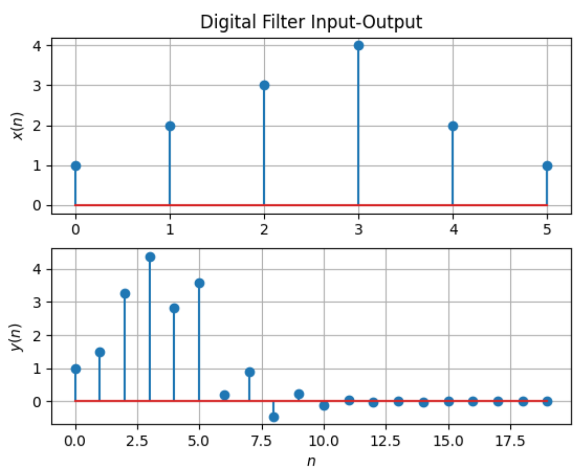

Fig. 3.2

3.3 Repeat the above exercise using a C code.

Solution: Download the C code from the below link,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/master/
    Assignment%201/Codes/xnyn.c
```

Then run the follwing command in terminal

```
cc xnyn.c
./a.out
```

Then for the plot 3.3 download the python file from the below link,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/master/
    Assignment%201/Codes/xnyn2.py
```

Then run the command

```
python3 xnyn2.py
```

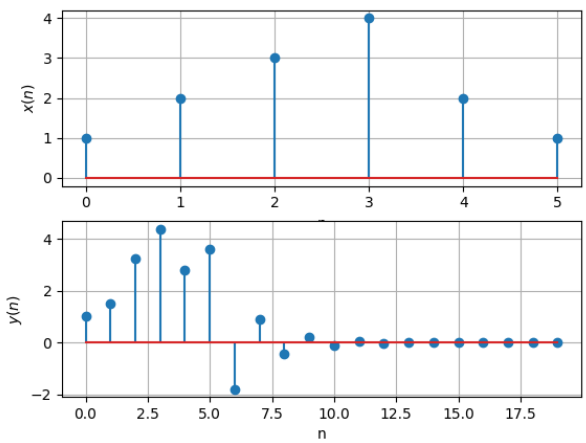

Fig. 3.3: Plot using C code

4 $\mathcal{Z}$-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

Take $k = n - 1$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \quad (4.7)$$

$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \quad (4.8)$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (4.9)$$

$$= z^{-1} X(z) \quad (4.10)$$

resulting in (4.2) and similarly following the above steps you will get,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(n) \quad (4.11)$$

Hence proved. Now we will find Z transform of the signal $x(n)$,from (3.1),

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n) z^{-n} \quad (4.12)$$

$$= 1z^0 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \quad (4.13)$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.14)$$

4.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.15)$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.11) in (3.2),

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \quad (4.16)$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \quad (4.17)$$

Solution: Now we will rewrite (3.2),

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \quad (4.18)$$

Now since Z-transform is a linear operator we can write that,

$$Y(n) + \frac{1}{2} Y(n-1) = X(n) + X(n-2) \quad (4.19)$$

From (4.11),

$$Y(n) + \frac{z^{-1}}{2} Y(n) = X(n) + z^{-2} X(n) \quad (4.20)$$

$$\implies \frac{Y(n)}{X(n)} = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \quad (4.21)$$

4.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.22)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.23)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.24)$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \quad (4.25)$$

$$= \delta(0) z^0 + 0 \ \text{(Using (4.22))} \quad (4.26)$$

$$= 1 \quad (4.27)$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \quad (4.28)$$

$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \quad (4.29)$$

$$= 1 + z^{-1} + z^{-2} + \dots \quad (4.30)$$

Above is a infinite geometric series with z^{-1} as common ratio , so we can write it as

$$U(n) = \frac{1}{1 - z^{-1}} \because |z| > 1 \quad (4.31)$$

4.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.32)$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.33)$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots. \quad (4.34)$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\left\{a^n u(n)\right\} = \frac{1}{1 - \frac{a}{z}} \because |a| < |z| \tag{4.35}$$

Therefore,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.36}$$

4.5 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.37}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: Download the code for the plot 4.5 from the link below

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/main/
    Assignment-1/Codes/dtft.py
```

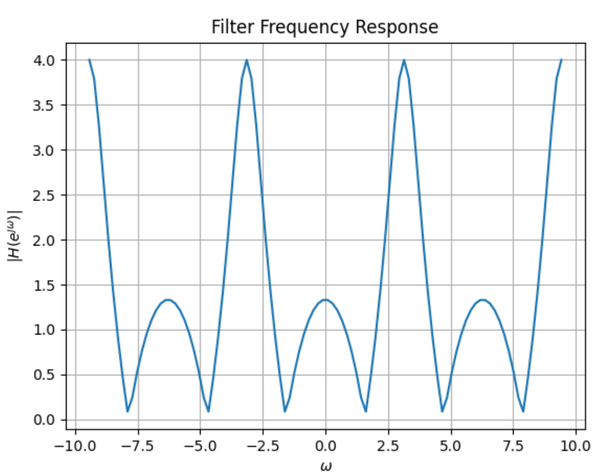

Fig. 4.5: $\left|H\left(e^{j\omega}\right)\right|$

Now using (4.17), we will find $\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.38}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.39}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.40}$$

$$= \frac{|1 + \cos 2\omega + j\sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.41}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \tag{4.42}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \tag{4.43}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.44}$$

Using the above expression we can say that graph is symmetric about origin and has a period of 2π.

5 Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1 + \frac{z^{-1}}{2}} + \frac{z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{5.2}$$

From (4.32) we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.3}$$

5.2 Sketch $h(n)$. Is it bounded? Convergent?

Solution: Download the code for the plot 5.2 from the below link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/blob/main/
    Assignment-1/Codes/hn.py
```

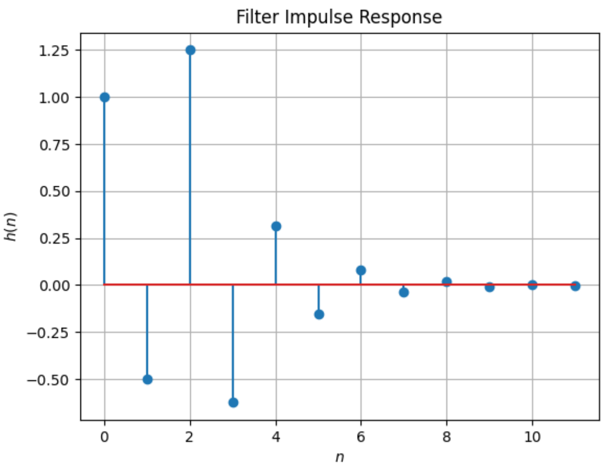

Fig. 5.2: $h(n)$ as inverse of $H(n)$

From the plot 5.2, we can say that $h(n)$ is bounded and converges to 0 as n increases.And theoritically we can the same using (5.3),

$$h(n) = \begin{cases} 0 & , n \leq 0 \\ \left(\frac{-1}{2}\right)^n & , 0 < n < 2 \\ 5\left(\frac{1}{2}\right)^n & , n \geq 2 \end{cases} \tag{5.4}$$

As $5\left(\frac{1}{2}\right)^n \to 0$ as $n \to \infty$ we can say $h(n)$ is bounded and Convergent.

5.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.5}$$

Is the system defined by (3.2) stable for the impulse response in (5.1)?

Solution: From (5.3),

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \right) \tag{5.6}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.7}$$

$$= \frac{4}{3} \tag{5.8}$$

$\therefore$ the system is stable.

5.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.9}$$

This is the definition of $h(n)$.

Solution: Download the code for the plot 5.4 from the below link,

```
wget https://github.com/Charanyash/EE3900-Digital_Signal_Processing/blob/main/Assignment-1/Codes/hndef.py
```

Note that this is same as 5.2.

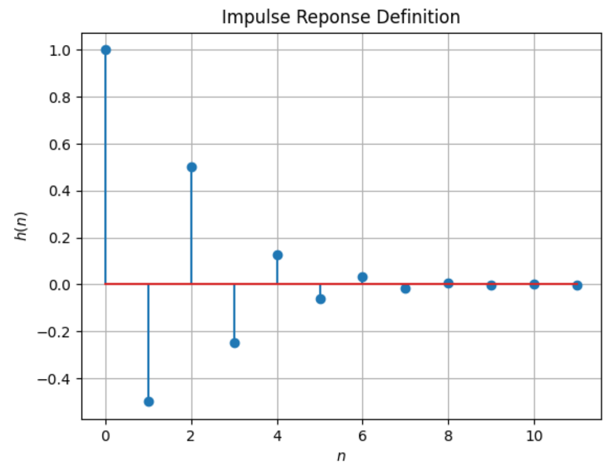

Fig. 5.4: From the definition of $h(n)$

5.5 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.10}$$

Comment. The operation in (5.10) is known as *convolution*.

Solution: Download the code for plot 5.5 from the below link

```
wget https://github.com/Charanyash/EE3900-Digital_Signal_Processing/blob/main/Assignment-1/Codes/ynconv.py
```

Note that the plot is same that as in 3.2.

5.6 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.11}$$

Solution: Substitute $k := n-k$ in (5.10), we

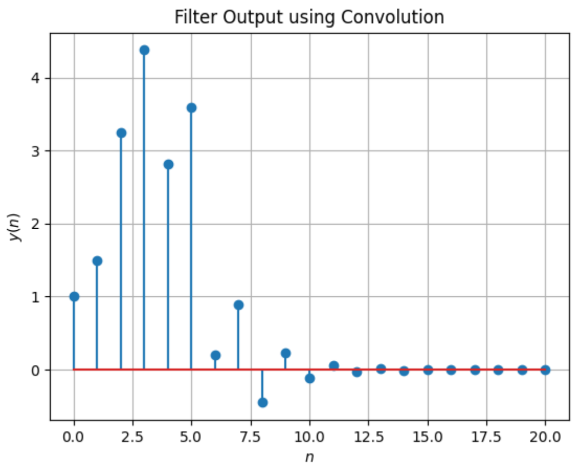

Fig. 5.5: $y(n)$ using the convolution definition

will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.12}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.13}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.14}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

6.5 Wherever possible, express all the above equations as matrix equations.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{7.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.