Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\, \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\, \mu C$.

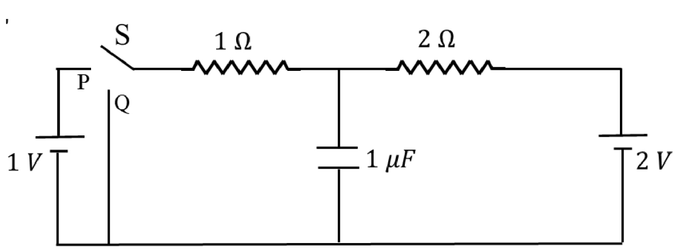


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:

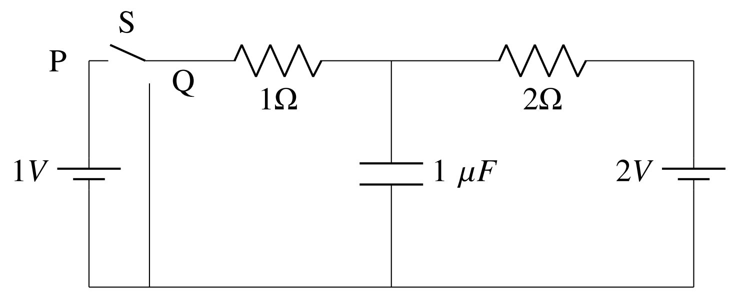
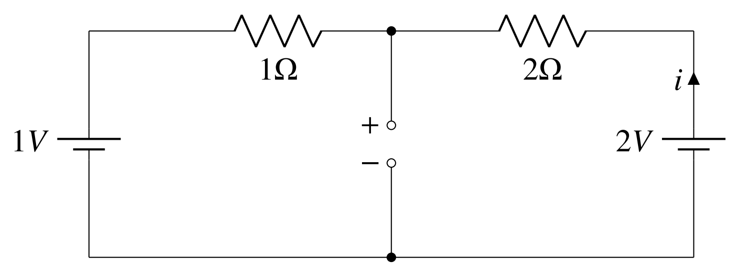


Fig. 2.2: Circuit diagram of the question

3. Find q_1.
Solution: Since the switch S is closed for a long time at P, the circuit at steady state looks like, Now if we apply KVL,



$$2 - 2i - i - 1 = 0 \tag{2.1}$$

$$\implies i = \frac{1}{3} \tag{2.2}$$

The potential difference across capacitor is,

$$V_C = 2 - 2i \tag{2.3}$$

$$= \frac{4}{3} \tag{2.4}$$

Therefore the charge on capacitor will be,

$$q_1 = \frac{4}{3}\mu F \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: From 1.2 we can write laplace trans-

form of unit step function $u(t)$ as,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.6}$$
$$= \int_{0}^{\infty} e^{-} stdt \tag{2.7}$$
$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} \tag{2.8}$$

For the laplace transform to exist,$Re\,(s) > 0$ using that

$$\mathcal{L}\{u(t)\} = \frac{1}{s}\text{with ROC } Re\,(s) > 0 \tag{2.9}$$

The ROC plot looks like 2.3,

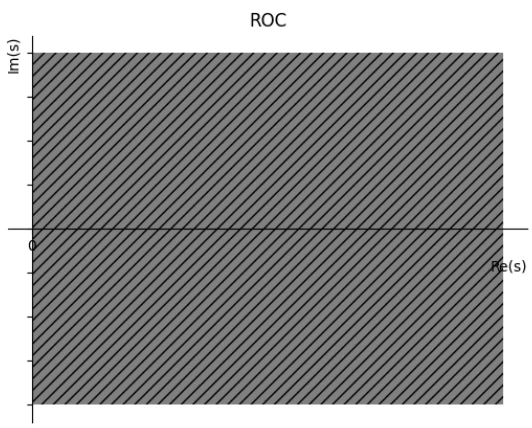


Fig. 2.3: ROC of laplace transform of $u(t)$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.10}$$

and find the ROC.
Solution: The laplace transform will be,

$$\mathcal{L}\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-(a+s)t}u(t)dt \tag{2.11}$$
$$= \int_{0}^{\infty} e^{-(a+s)t}dt \tag{2.12}$$
$$= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty} \tag{2.13}$$

Now for the integral to exist,

$$Re\,(s+a) > 0 \tag{2.14}$$
$$Re\,(s) > -a \tag{2.15}$$

So the ROC will be $Re\,(s) > -a$ and with that

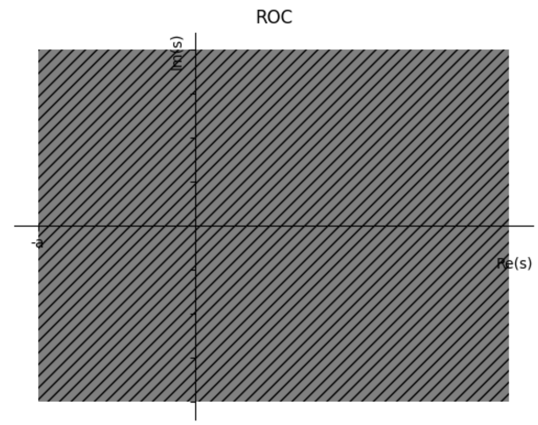


Fig. 2.4: ROC of laplace transform of $e^{-at}u\,(t)$

ROC,

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{a+s} \tag{2.16}$$

And the ROC plot looks like 2.4,

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

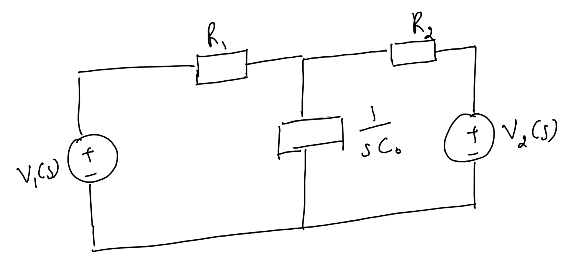


Fig. 2.5

$$u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_1(s) \tag{2.17}$$
$$2u(t) \overset{\mathcal{H}}{\longleftrightarrow} LV_2(s) \tag{2.18}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: From the earlier proved results,

$$V_1(s) = \mathcal{L}\,(u(t)) \tag{2.19}$$
$$= \frac{1}{s} \tag{2.20}$$
$$V_2(s) = \mathcal{L}\,(2u(t)) \tag{2.21}$$
$$= \frac{2}{s} \tag{2.22}$$

Note that ROC here is $Re\,\{s\} > 0$.

And the let the ends of resistive capacitor has voltages of V_{C_0} and 0. The same can be seen in figure 2.6, If we apply Kirchoff's Junction

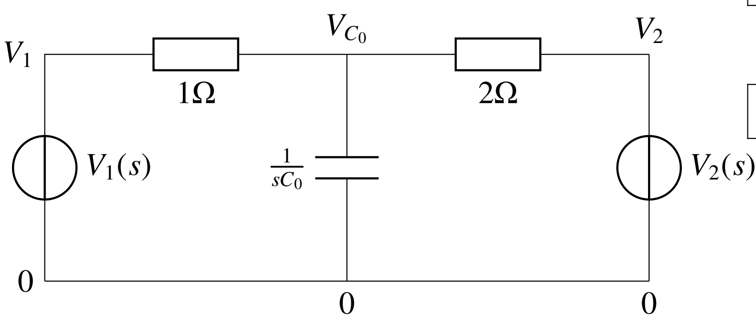


Fig. 2.6

law,

$$\frac{V_{C_0} - V_1(s)}{1} + \frac{V_{C_0} - 0}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{2} = 0 \tag{2.23}$$

$$V_{C_0}(s)\left(\frac{3}{2} + sC_0\right) = V_1(s) + \frac{V_2(s)}{2} \tag{2.24}$$

Substituting $V_1(s), V_2(s)$ and C_0, you will get

$$V_{C_0}(s) = \frac{4}{s\left(3 + 2x10^{-6}s\right)} \tag{2.25}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Now we can find the voltage of capacitor in time domain using inverse laplace transform,

$$v_{C_0}(t) = \mathcal{L}^{-1}\left[\frac{4}{s\left(3 + 2x10^{-6}s\right)}\right] \tag{2.26}$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3}\left[\frac{1}{s} - \frac{1}{10^{-6}s + \frac{3}{2}}\right]\right] \tag{2.27}$$

$$= \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s + \frac{3x10^6}{2}}\right] \tag{2.28}$$

$$= \frac{4}{3}\left(u(t) - u(t)e^{-\frac{3x10^6 t}{2}}\right) \tag{2.29}$$

$$\implies v_{C_0}(t) = \frac{4}{3}\left(1 - e^{-\frac{3t}{2}x10^6}\right)u(t) \tag{2.30}$$

Note that the ROC here is $Re\{s\} > 0$.
The plot 2.7 of the same can be viewed using the python code in the following link,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.7.py
```

Then run the following command,

```
python3 2.7.py
```

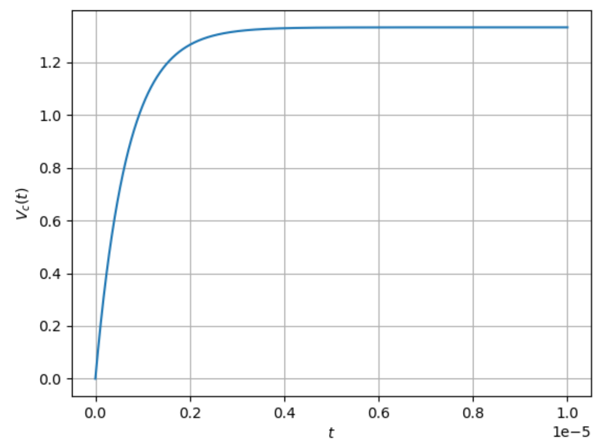


Fig. 2.7: The plot of $V_c(t)$ vs t

8. Verify your result using ngspice.
Solution: Download the codes from the below links,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.8.cir
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /2.8.py
```

Then run the following command,

```
ngspice 2.8.cir
python3 2.8.py
```

9. Obtain Fig. 2.5 using the equivalent differential equation.
Solution: Using Kirchoff's Junction Law in 2.1,

$$\frac{v_c(t) - v_1(t)}{1} + \frac{v_c(t) - v_2(t)}{2} + \frac{dq}{dt} = 0 \tag{2.31}$$

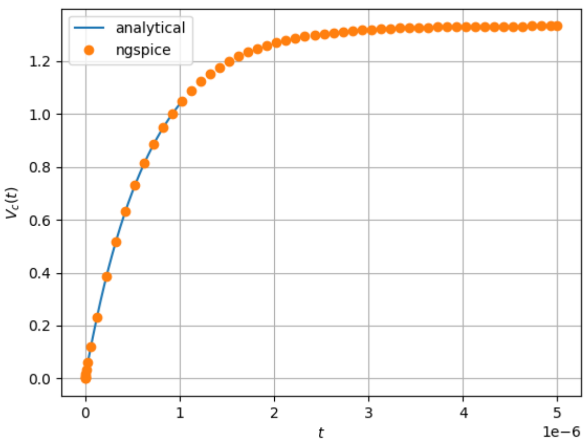


Fig. 2.8: The plot of $V_c(t)$ vs t using ngspice

where $q(t)$ is the charge on capacitor. And can be written as ,

$$q_c(t) = Cv_c(t) \tag{2.32}$$

Now applying laplace transform of (2.31) using (2.32),

$$(V_c(s) - V_1(s)) + \frac{1}{2}(V_c(s) - V_2(s)) + C(sV_c(s) - v_c(0^-)) = 0 \tag{2.33}$$

We know,

$$v_c(0^-) = v_c(0) = 0 \tag{2.34}$$

Using that,

$$\frac{V_{C_0} - V_1(s)}{1} + \frac{V_{C_0} - 0}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{2} = 0 \tag{2.35}$$

With this equation, we can write the equivalent resistive circuit as shown in Fig 2.5.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: After closing switch at Q for long time the circuit looks like 3.1, Now we will use Kirchoff's Junction Law in this circuit,

$$\frac{V_c - 0}{1} + \frac{V_c - 2}{2} = 0 \tag{3.1}$$

$$\implies V_c = \frac{2}{3}V \tag{3.2}$$

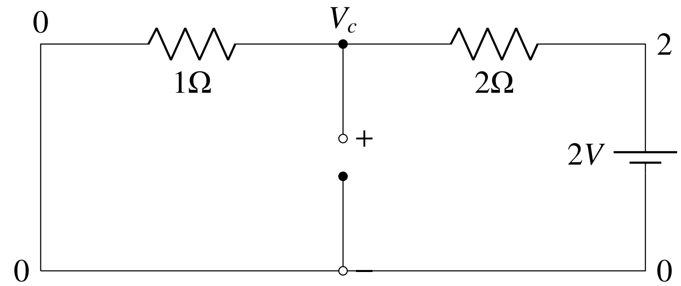


Fig. 3.1

And the charge on the capacitor will be,

$$q_2 = \frac{2}{3}\mu C \tag{3.3}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution: The equivalent s-domain resistive circuit looks like 3.2. The battery $\frac{4}{3s}$ is added

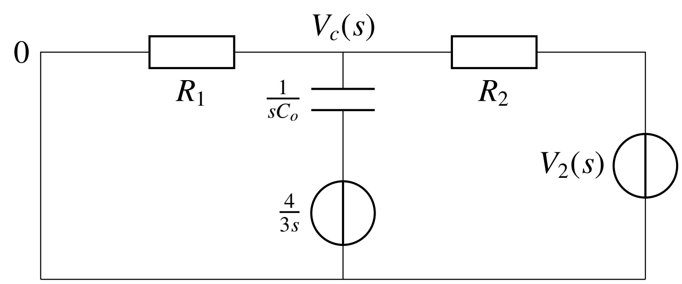


Fig. 3.2: Circuit after closing switch to Q in s-domain

series to C_0 in s- domain by taking consideration of initial charge on capacitor $q_1 = \frac{4}{3}\mu C$ before closing switch to Q.

3. $V_{C_0}(s) = ?$
Solution: Apply kirchoff's junction law in the s-domain circuit 3.2,taking $R_1 = 1\Omega$ and $R_2 =$

2Ω

$$\frac{V_{C_0}(s)-0}{1}+\frac{V_{C_0}(s)-\frac{4}{3s}}{\frac{1}{sC_0}}+\frac{V_{C_0}(s)-V_2(s)}{2}=0 \tag{3.4}$$

$$\implies V_{C_0}(s)\left(\frac{3}{2}+sC_0\right)=\frac{V_2(s)}{2}+\frac{4C_0}{3} \tag{3.5}$$

$$\implies V_{C_0}(s)=\frac{\frac{1}{s}+\frac{4C_0}{3}}{\frac{3}{2}+sC_0} \tag{3.6}$$

$$\therefore V_{C_0}(s)=\frac{2\,(3+4sC_0)}{3s\,(3+2sC_0)} \tag{3.7}$$

4. $v_{C_0}(t)$ = ? Plot using python.
Solution: We can obtain the potential difference across the capacitor in time domain by applying inverse laplace transform under right ROC conditions,

$$v_{C_0}(t)=\mathcal{L}^{-1}\left[\frac{2\,(3+4sC_0)}{3s\,(3+2sC_0)}\right] \tag{3.8}$$

$$=\mathcal{L}^{-1}\left[\frac{2}{3s}+\frac{4C_0}{3\,(3+2sC_0)}\right] \tag{3.9}$$

$$=\mathcal{L}^{-1}\left[\frac{2}{3s}\right]+\mathcal{L}^{-1}\left[\frac{4C_0}{3\,(3+2sC_0)}\right] \tag{3.10}$$

With ROC condition $Re\{s\}>0$,

$$v_{C_0}(t)=\frac{2u(t)}{3}+\frac{2u(t)}{3}e^{-\frac{3}{2C_0}} \tag{3.11}$$

Substituting $C_0 = 1\mu F$,

$$v_{C_0}(t)=\frac{2}{3}\left(1+e^{-\frac{3x10^6}{2}}\right)u(t) \tag{3.12}$$

We can verify the same using the python code from the below link,

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.4.py
```

Then run the following command,

```
wget python3 3.4.py
```

5. Verify your result using ngspice.
Solution: Download the codes from the below links

```
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.5.py
wget https://github.com/Charanyash/EE3900-
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /3.5.cir
```

Then run the following commands,

```
ngspice 3.5.cir
python3 3.5.py
```

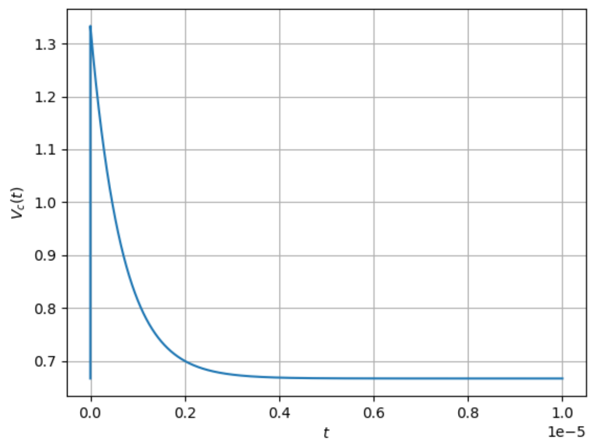


Fig. 3.3: The plot of $V_c(t)$ vs t

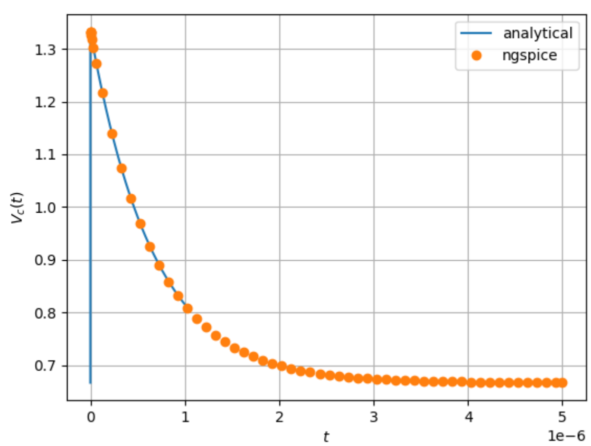


Fig. 3.4: The plot of $V_c(t)$ vs t using ngspice

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: In that case when $t < 0$ switch is not closed to Q so the circuit will be in steady

state (switch at P),

$$v_{C_0}(0-) = \left[\frac{4}{3}\left(1 - e^{-\frac{3x10^6}{2}}\right)u(t)\right]_{t=\infty} \tag{3.13}$$
$$= \frac{4}{3}V \tag{3.14}$$

And for $t = 0+$ and $t = \infty$, we can use (3.12),

$$v_{C_0}(0+) = \frac{4}{3}V \tag{3.15}$$
$$v_{C_0}(\infty) = \frac{2}{3}V \tag{3.16}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution: Using Kirchoff's Junction Law ,

$$\frac{v_c(t) - v_1(t)}{1} + \frac{v_c(t) - v_2(t)}{2} + \frac{dq}{dt} = 0 \tag{3.17}$$

where $q(t)$ is the charge on capacitor. And can be written as ,

$$q_c(t) = Cv_c(t) \tag{3.18}$$

Now applying laplace transform ,

$$(V_c(s) - V_1(s)) + \frac{1}{2}(V_c(s) - V_2(s)) + C\left(sV_c(s) - v_c(0^-)\right) = 0 \tag{3.19}$$

We know,

$$v_c(0^-) = \frac{4}{3}V \tag{3.20}$$

Using that,

$$\frac{V_{C_0} - V_1(s)}{R_1} + \frac{V_{C_0} - \frac{4}{3s}}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2(s)}{R_2} = 0 \tag{3.21}$$

With this equation, we can write the equivalent resistive circuit as shown in Fig 3.2.

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
Solution: When the switch S is switched to Q right from the beginning, the circuit looks like 4.1 So similar what we did earlier we use Kirchoff's Junction Law,

$$\frac{v_c(t) - 0}{1} + \frac{dq}{dt} + \frac{v_c(t) - v_2(t)}{2} = 0 \tag{4.1}$$

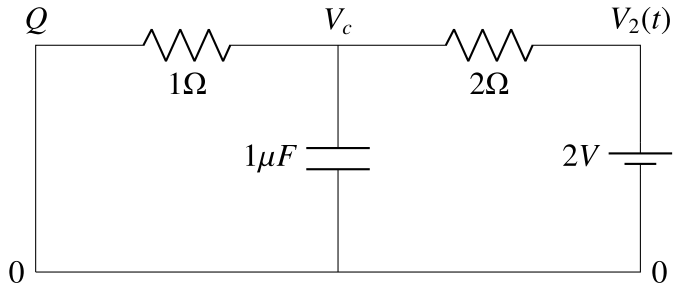


Fig. 4.1

where $q(t)$ is the charge on capacitor at time t with initial conditions as,

$$q(0^-) = q(0) = 0 \tag{4.2}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
Solution: Here $H(s)$ means transfer function, it relates the output (response) of the system to the given input. Here we are asked to consider voltage at the capacitor as the output.
For that first we will do laplace transform of the above differential equation (4.1),

$$\frac{V_c(s) - 0}{1} + C\left(sV(s) - v_c(0^-)\right) + \frac{V_c(s) - V_2(s)}{2} = 0 \tag{4.3}$$

We know that $v_c(0^-) = 0$ using that,

$$V_c(s)\left(sC + \frac{3}{2}\right) = \frac{V_2(s)}{2} \tag{4.4}$$
$$\frac{V_c(s)}{V_2(s)} = \frac{1}{(2sC + 3)} \tag{4.5}$$

This will be the transfer function $H(s)$ and substituting $C = 1\mu F$, you will get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.6}$$

3. Plot $H(s)$. What kind of filter is it?
Solution: Download the python code from the below link for the plot,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /4.3.py
```

Then run the following command

```
wget python3 4.3.py
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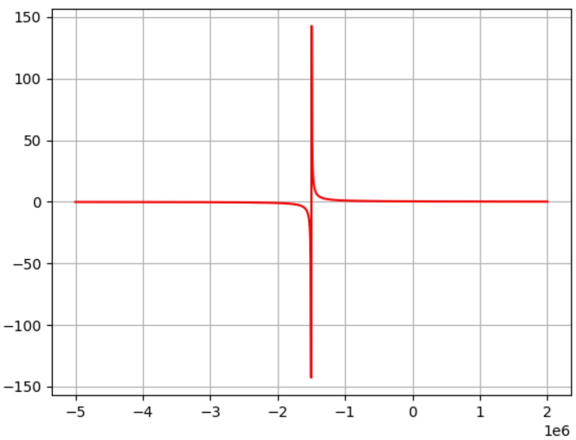

Fig. 4.2: The plot of transfer function $H(s)$

Now if we consider frequency-domain transfer function by putting $s = j\omega$,

$$H(j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \tag{4.7}$$

$$\implies |H(j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + (1.5 \times 10^6)^2}} \tag{4.8}$$

As you can see , when ω increases the magnitude of transfer function $|H(j\omega)|$ decreases.
In other words, when the high frequency signals are passed as input their amplitudes will get diminished and filtered out.
So this filter acts like a low-pass filter.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.9}$$

Solution: To derive the difference equation, we will consider the differntial equation (4.1),

$$\frac{v_c(t) - 0}{1} + \frac{dq}{dt} + \frac{v_c(t) - v_2(t)}{2} = 0 \tag{4.10}$$

$$\frac{v_c(t) - 0}{1} + C\frac{dv_c(t)}{dt} + \frac{v_c(t) - v_2(t)}{2} = 0 \tag{4.11}$$

$$C\frac{dv_c(t)}{dt} = \frac{v_2(t) - 3v_c(t)}{2} \tag{4.12}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \frac{1}{C}\int_n^{n+1} \frac{v_2(t) - 3v_c(t)}{2} dt \tag{4.13}$$

Here we can use trapezoidal rule of integration for $n = 1$(no.of sub intervals),

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right) \tag{4.14}$$

Along with this rule,considering that $y(t) = v_c(t)$ and $y(n) = y(t)|_{t=n}$ we can write that

$$\begin{aligned} y(n+1) - y(n) = \frac{1}{2C}&\left(\frac{v_2(n) + v_2(n+1)}{2}\right) \\ &- \frac{1}{2C}\frac{3\left(y(n) + y(n+1)\right)}{2} \end{aligned} \tag{4.15}$$

Now using the fact that $v_2(t) = 2u(t)$,

$$\begin{aligned} y(n+1) - y(n) = \frac{1}{2C}&\left(u(n) + u(n+1)\right) \\ &- \frac{1}{2C}\frac{3\left(y(n) + y(n+1)\right)}{2} \end{aligned} \tag{4.16}$$

$$\begin{aligned} \implies y(n+1)\left(1 + \frac{3}{4C}\right) = y(n)\left(1 - \frac{3}{4C}\right) \\ + \frac{u(n) + u(n+1)}{2C} \end{aligned} \tag{4.17}$$

This will be the difference equation.

5. Find $H(z)$.
For that we will find the z transform of $y(n)$ by applying z-transform on the difference equation,

$$zY(z)\left(1 + \frac{3}{4C}\right) = Y(z)\left(1 - \frac{3}{4C}\right) + \frac{1}{2C}\left(\frac{1+z}{1-z^{-1}}\right) \tag{4.18}$$

$$\implies Y(z) = \frac{\frac{1}{2C}\left(\frac{1+z}{1-z^{-1}}\right)}{\frac{3}{4C}(z+1) + z - 1} \tag{4.19}$$

Now assume that

$$x(n) = x(t)|_{t=n} \tag{4.20}$$

And here,

$$x(t) = v_2(t) = 2u(t)\,(()\because v_2(t) = 2V\forall t \geq 0) \tag{4.21}$$

$$\implies x(n) = 2u(n) \tag{4.22}$$

So the z-transform of $x(n)$ will be,

$$X(z) = \frac{2}{1 - z^{-1}} \tag{4.23}$$

with ROC being

$$|z| > 1 \tag{4.24}$$

Now our function of interest,transfer function in z-domain $H(z)$ will be

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.25}$$

$$= \frac{\frac{1}{2C}(1+z)}{\frac{3}{2C}(z+1) + 2(z-1)} \tag{4.26}$$

Using that $C = 1\mu F$,

$$H(z) = \frac{\left(1+z^{-1}\right)2.5 \times 10^5}{z^{-1}\left(7.5 \times 10^5 - 1\right) + 7.5 \times 10^5 + 1} \tag{4.27}$$

with ROC being

$$|z| > max\left\{1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right\} \tag{4.28}$$

$$\implies |z| > 1 \tag{4.29}$$

6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Bilinear Transform is a first-order approximation of natural logarithm function that is an exact mapping between z-plane to the s-plane.
Here to get z-transform of a discrete-time sequence from its corresponding laplace transform, where each element is attached with corresponding unit impulse, we will approximate the below fact,

$$s = \frac{\ln(z)}{T} \tag{4.30}$$

where T is the step size.
If we expand the natural logarithm,

$$s = \frac{2}{T}\left[\left(\frac{z-1}{z+1}\right) + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \cdots\right] \tag{4.31}$$

$$s \approx \frac{2}{T}\frac{z-1}{z+1} \tag{4.32}$$

$$s \approx \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{4.33}$$

Here $T = 1$ .So,

$$H(s)\Big|_{s=2\frac{1-z^{-1}}{1+z^{-1}}} = \frac{5 \times 10^5}{2\frac{1-z^{-1}}{1+z^{-1}} + 1.5 \times 10^6} \tag{4.34}$$

$$= \frac{2.5 \times 10^5\left(1+z^{-1}\right)}{\left(1-z^{-1}\right) + 7.5 \times 10^5\left(1+z^{-1}\right)} \tag{4.35}$$

$$= \frac{\left(1+z^{-1}\right)2.5 \times 10^5}{z^{-1}\left(7.5 \times 10^5 - 1\right) + 7.5 \times 10^5 + 1} \tag{4.36}$$

$$= H(z) \tag{4.37}$$

7. Find $y(n)$ from $H(z)$ and verify whether $y(n) = y(t)|_{t=n}$.
Solution:

$$Y(z) = H(z)X(z) \tag{4.38}$$

$$= \frac{2.5 \times 10^5\left(1+z^{-1}\right)}{z^{-1}\left(7.5 \times 10^5 - 1\right) + 7.5 \times 10^5 + 1}\frac{2}{1-z^{-1}} \tag{4.39}$$

$$= \frac{2}{3}\left[\frac{1}{1-z^{-1}} - \frac{2}{3}\frac{1}{\left(7.5 \times 10^5 - 1\right)z^{-1} + \left(7.5 \times 10^5 + 1\right)}\right] \tag{4.40}$$

Now if we apply inverse z-transform by taking ROC $|z| > 1$,

$$y(n) = \frac{2}{3}u(n) - \frac{2}{3}\left(-\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right)^n u(n) \tag{4.41}$$

Here we used the following properties of z-transform,

$$u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-z^{-1}}, |z| > 1 \tag{4.42}$$

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}}, |z| > |a| \tag{4.43}$$

Here we are sampling the signal at high frequency means at small intervals of time let say for $n = 0.5 \times 10^{-5}, 1 \times 10^{-5} \cdots$,for that we can approximate $y(n)$ as,

$$y(n) \approx \frac{2}{3}\left(1 - \frac{1 - 7.5 \times 10^5 n}{1 + 7.5 \times 10^5 n}\right)u(n) \tag{4.44}$$

Now we will verify it from $y(t)$, for that consider

$$Y(s) = H(s)X(s) \tag{4.45}$$
$$= \frac{5\times 10^5}{s+1.5\times 10^6}\frac{2}{s} \tag{4.46}$$
$$= \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s+1.5\times 10^6}\right) \tag{4.47}$$

Now we will apply inverse laplace transform with ROC being $\mathcal{R}\{s\} > 0$ on b.s,

$$y(t) = \frac{2}{3}\left(1 - e^{-1.5\times 10^6 t}\right)u(t) \tag{4.48}$$

Now

$$e^{-1.5\times 10^6 t} = \frac{e^{-7.5\times 10^5 t}}{e^{7.5\times 10^5 t}} \tag{4.49}$$
$$\approx \frac{1-7.5\times 10^5 t}{1+7.5\times 10^5 t}, \text{when } t << 10^{-6} \tag{4.50}$$

Using that,

$$y(t) = \frac{2}{3}\left(1 - \frac{1-7.5\times 10^5 t}{1+7.5\times 10^5 t}\right)u(t) \tag{4.51}$$

And

$$y(t)|_{t=n} = \frac{2}{3}\left(1 - \frac{1-7.5\times 10^5 n}{1+7.5\times 10^5 n}\right)u(n) \tag{4.52}$$

As you can see both are turn to be the same. Download the following codes for the simulation and plot **??**

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /4.7.cir
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    Circuits%20and%20Transforms/Codes
    /4.7.py
```

Then run the following command,

```
ngspice 4.7.cir
python3 4.7.py
```

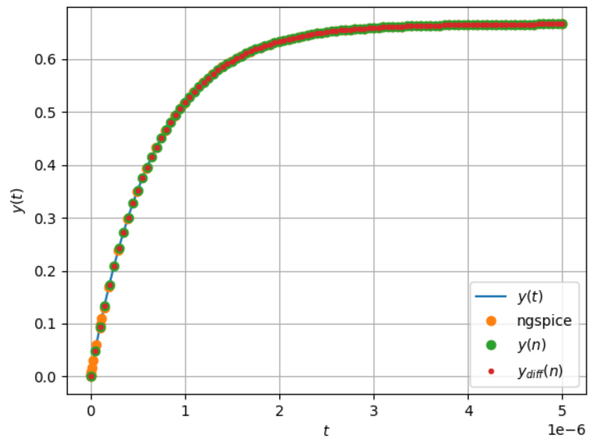

Fig. 4.3: Output of the filter vs time

8. Derive the expression for $y(n)$ using difference equation.
Solution: Consider the difference equation (4.18),

$$y(n+1)\left(1+\frac{3}{4C}\right) = y(n)\left(1-\frac{3}{4C}\right) + \frac{u(n)+u(n+1)}{2C} \tag{4.53}$$

$$\implies y(n+1) = y(n)\left(\frac{1-\frac{3}{4C}}{1+\frac{3}{4C}}\right) + \frac{u(n)+u(n+1)}{2C+\frac{3}{2}} \tag{4.54}$$

Since our discussion is in $n > 0$, we can write

$$u(n) + u(n+1) = 2 \tag{4.55}$$

Now let say a,b are coefficients in difference equation,

$$a = \frac{1-\frac{3}{4C}}{1+\frac{3}{4C}} \tag{4.56}$$
$$= \frac{1-7.5\times 10^5}{1+7.5\times 10^5} \tag{4.57}$$
$$b = \frac{2}{2C+\frac{3}{2}} \tag{4.58}$$
$$= \frac{10^6}{1+7.5\times 10^5} \tag{4.59}$$

So our difference equation looks like,

$$y(n+1) = ay(n) + b \tag{4.60}$$

$$y(n) = ay(n-1) + b \tag{4.61}$$

$$= a\left(ay(n-2) + b\right) + b \tag{4.62}$$

$$= a^2 y(n-2) + ab + b \tag{4.63}$$

$$..$$

$$= a^n y(0) + b\left(1 + a + \cdots + a^{n-1}\right) \tag{4.64}$$

We know that $y(0) = 0$,

$$y(n) = b\frac{1-a^n}{1-a} \tag{4.65}$$

And substituting the values of a and b,

$$y(n) = \frac{10^6}{1+7.5\times10^5}\frac{1-\left(\frac{1-7.5\times10^5}{1+7.5\times10^5}\right)^n}{1-\left(\frac{1-7.5\times10^5}{1+7.5\times10^5}\right)} \tag{4.66}$$

$$= \frac{10^6}{1+7.5\times10^5}\frac{1-\left(\frac{1-7.5\times10^5}{1+7.5\times10^5}\right)^n}{\frac{1.5\times10^6}{1+7.5\times10^5}} \tag{4.67}$$

$$= \frac{2}{3}\left(1-\left(\frac{1-7.5\times10^5}{1+7.5\times10^5}\right)^n\right) \tag{4.68}$$

And for $n << 10^{-6}$,

$$y(n) \approx \frac{2}{3}\left(1-\frac{1-7.5\times10^5 n}{1+7.5\times10^5 n}\right) \tag{4.69}$$

In general,we can write

$$y(n) = \frac{2}{3}\left(1-\frac{1-7.5\times10^5 n}{1+7.5\times10^5 n}\right) \tag{4.70}$$

As you can see it is similar to what we got earlier.