Pingala Assignment

Mannem Charan AI21BTECH11019

Contents

Abstract—**This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution: Download the following python code from the below link,

```
wget https://github.com/Charanyash/EE3900–
    Digital_Signal_Processing/tree/master/
    pingala/Codes/1.py
```

Then run the following command,

```
python3 1.py
```

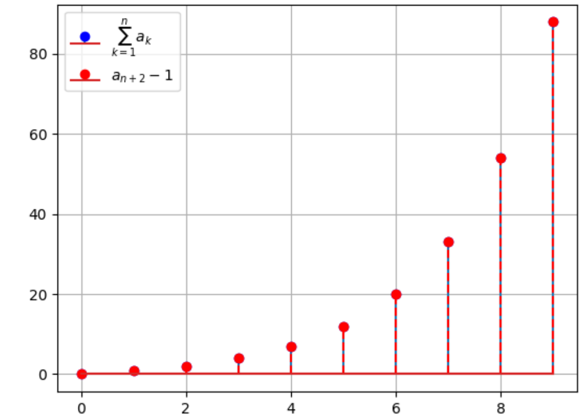

Fig. 1.1

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution:

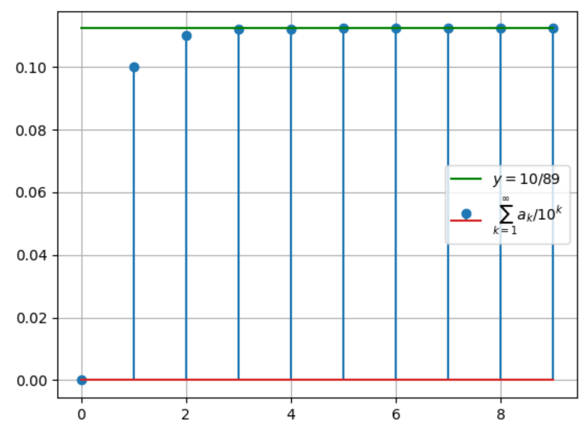

Fig. 1.2

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution:

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: As you can see in the 1.4, the summation is converging to $\frac{12}{89}$ but not $\frac{8}{89}$.

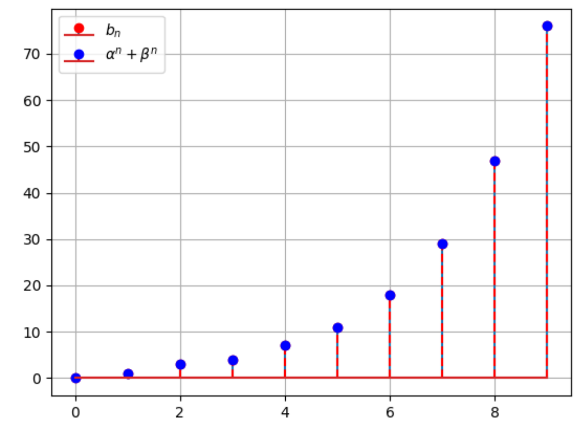

Fig. 1.3

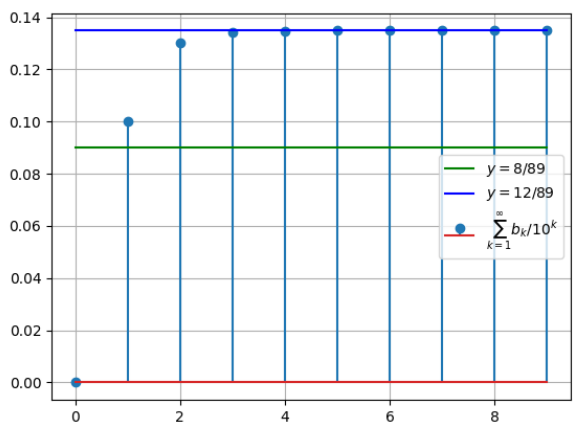

Fig. 1.4

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: Download the below python code,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    pingala/Codes/2.py
```

Then run the following command,

```
python3 2.py
```

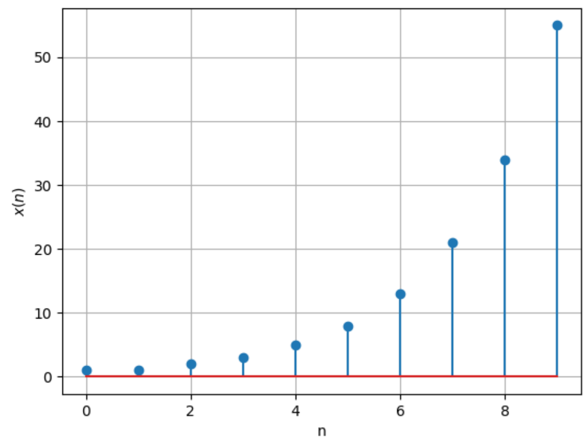

Fig. 2.2

2.3 Find $X^+(z)$.
Solution: Consider the eq :(2.2),

$$x(n+2) = x(n+1) + x(n) \tag{2.3}$$

Now apply one-sided Z-transform on both sides,(using linearity)

$$\sum_{n=0}^{\infty} x(n+2)z^{-n} = \sum_{n=0}^{\infty} x(n+1)z^{-n} + \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.4}$$

$$z^2\left(X^+(z) - x(1)z^{-1} - x(0)\right) = z\left(X^+(z) - x(0)\right) + X^+(z) \tag{2.5}$$

$$X^+(z) = \frac{x(1)z + x(0)\left(z^2 - z\right)}{z^2 - z - 1} \tag{2.6}$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (|z| \neq 0) \tag{2.7}$$

2.4 Find $x(n)$.
Solution: We know that,

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.8}$$

Using partial fractions,

$$= \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} \tag{2.9}$$

$$= \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha z^{-1}} - \frac{\beta}{1-\beta z^{-1}}\right) \tag{2.10}$$

where α,β are the roots of the equation,

$$z^2 - z - 1 = 0 \tag{2.11}$$

Using the result of Z-transform,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{2.12}$$

$$\tag{2.13}$$

We can write,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \tag{2.14}$$

with ROC as,

$$|z| > \max\{\alpha, \beta\} \tag{2.15}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.16}$$

Solution: Download the below python code,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/tree/master/
    pingala/Codes/2.py
```

Then run the following command,

```
python3 2.py
```

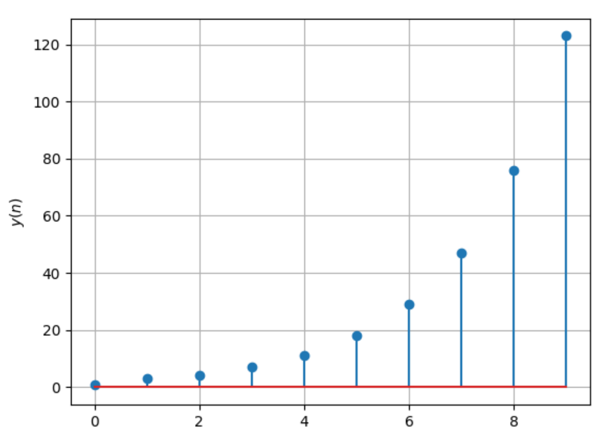

Fig. 2.5

2.6 Find $Y^+(z)$.
Solution: Apply the one-sided Z-transform on (2.16),

$$\sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{n=0}^{\infty} x(n-1)z^{-n} + \sum_{n=0}^{\infty} x(n+1)z^{-n} \tag{2.17}$$

$$= z^{-1}\left(X^+(z) - x(-1)\right) + z\left(X^+(z) - x(0)\right) \tag{2.18}$$

Using (2.2) and (2.7),

$$Y^+(z) = \frac{z^{-1}}{1-z^{-1}-z^{-2}} + \frac{1+z^{-1}}{1-z^{-1}-z^{-2}} \tag{2.19}$$

$$\implies Y^+z = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} \tag{2.20}$$

2.7 Find $y(n)$.
Solution: Consider (2.20),

$$Y^+z = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} \tag{2.21}$$

$$= X^+(z) + 2z^{-1}X^+(z) \tag{2.22}$$

Using (2.14) and the property of z-transform that,

$$x(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} z^{-1}X^+(z) \tag{2.23}$$

we can write the inverse z-transform of $Y^+(z)$ (using ROC as $|z| > \max\{\alpha,\beta\}$) as,

$$y(n) = x(n) + 2x(n-1) \tag{2.24}$$

$$= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} + 2\frac{\alpha^n - \beta^n}{\alpha-\beta} \tag{2.25}$$

$$= \frac{\alpha^{n+2} + \alpha^n - \beta^{n+2} - \beta^n}{\alpha-\beta} \tag{2.26}$$

$$= \frac{\alpha^{n+2} - \beta^{n+2} - \beta\alpha^{n+1} + \beta^{n+1}\alpha}{\alpha-\beta} \quad (\because \alpha\beta = -1) \tag{2.27}$$

$$\implies y(n) = \alpha^{n+1} + \beta^{n+1} \tag{2.28}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: The first part of the equation is trivial,

$$\because a_k = x(k-1) \tag{3.2}$$

For the second part,

$$x(n) * u(n-1) = \sum_{k=-\infty}^{\infty} x(k)u(n-k-1) \tag{3.3}$$

$$= \sum_{k=0}^{\infty} x(k)u(n-k-1) \tag{3.4}$$

$$(\because x(k) = 0, k < 0)$$

$$= \sum_{k=0}^{n-1} x(k) \tag{3.5}$$

$$(\because u(n-k-1) = 0 \;\; k >= n)$$

Hence proved.

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.7}$$

Solution: Here we have,

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.8}$$

We know that,

$$a_k = x(k-1) \tag{3.9}$$

Using that,

$$x(n+1) - 1 \quad n+1 \geq 1 \tag{3.10}$$

We can generalise the above signal for all n, using $u(n)$

$$(x(n+1) - 1)\, u(n) \tag{3.11}$$

Hence proved.

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.12}$$

Solution: Using (3.9), we can get the first part.

Now for the second part,

$$\frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} x(k)(10)^{-k} \tag{3.13}$$

$$= \frac{1}{10} X^+(10) \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.

Solution: Put $n := k+1$,

$$\alpha^{k+1} + \beta^{k+1}, \quad k \geq 0 \tag{3.17}$$

$$\tag{3.18}$$

We will represent the above expression $\forall k$ as,

$$w(k) = \left(\alpha^{k+1} + \beta^{k+1}\right) u(n) \tag{3.19}$$

Change of variable,

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.20}$$

Hence proved.

Now we will try to find the z-transform of $w(n)$,

$$\sum_{n=-\infty}^{\infty} w(n) z^{-n} = \sum_{n=0}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) z^{-n} \tag{3.21}$$

$$= \alpha \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n + \beta \sum_{n=0}^{\infty} \left(\frac{\beta}{z}\right)^n \tag{3.22}$$

$$= \frac{\alpha}{1 - \frac{\alpha}{z}} + \frac{\beta}{1 - \frac{\beta}{z}} \tag{3.23}$$

With ROC $|z| > \max\alpha, \beta$, and on solving

$$W(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.24}$$

$\because \alpha, \beta$ are roots of $z^2 - z - 1 = 0$.

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.25}$$

Solution: We get the first part using the following relation,

$$b_k = y(k-1) \tag{3.26}$$

Now for the second part,

$$\frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} y(k)(10)^{-k} \tag{3.27}$$

$$= \frac{1}{10}Y^+(10) \tag{3.28}$$

Hence proved.

3.6 Solve the JEE 2019 problem.
Solution: As we proved earlier,

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.29}$$

$$a_{n+2} - 1 = (x(n+1) - 1)\,u(n) \tag{3.30}$$

with $n \geq 1$.

$$x(n) * u(n-1) = \sum_{k=0}^{\infty} x(k)u(n-k-1) \tag{3.31}$$

$$= \sum_{k=0}^{n-1} x(k) \tag{3.32}$$

Now since $x(n)$ is a pingala series,

$$\sum_{k=0}^{n-1} x(k) = x(n+1) - 1 \tag{3.33}$$

$$\tag{3.34}$$

So we can write that,

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1\,(n >= 1) \tag{3.35}$$

And using (2.7),

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k} = \frac{1}{10}X^+(10) \tag{3.36}$$

$$= \frac{1}{10}\left(\frac{1}{1-10^{-1}-10^{-2}}\right) \tag{3.37}$$

$$= \frac{10}{89} \tag{3.38}$$

And similarly,

$$\sum_{k=1}^{\infty}\frac{b_k}{10^k} = \frac{1}{10}Y^+(10) \tag{3.39}$$

$$= \frac{1}{10}\left(\frac{1+2(10)^{-1}}{1-10^{-1}-10^{-2}}\right) \tag{3.40}$$

$$= \frac{12}{89} \tag{3.41}$$

And we know,

$$b_n = y(n-1) \tag{3.42}$$

$$y(n) = \alpha^{n+1} + \beta^{n+1} \tag{3.43}$$

$$\implies b_n = \alpha^n + \beta^n \forall n \geq 1 \tag{3.44}$$

Therefore all the options are correct except the last one.