Digital Signal Processing

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Abstract—**This document provides the solution of Assignment 2.**

Question 2.12 : Consider a system with input $x[n]$ and output $y[n]$ that satisfy the differential equation

$$y[n] = ny[n-1] + x[n] \tag{1}$$

The system is causal and satisfies initial-rest conditions i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

(a) If $x[n] = \delta[n]$, determine $y[n]$ for all n.
(b) Is the system linear? Justify your answer.
(c) Is the system time invariant? Justify your answer.

Solution: Given that system is causal which means the output $y[n]$ only depends on present and past inputs ($x[n]$) but not the future inputs and also it satisfies initial-rest conditions.

(a) Given that $x[n] = \delta[n]$,

$$\implies x[n] = \begin{cases} 1 & , n = 0 \\ 0 & , \text{otherwise} \end{cases} \tag{2}$$

Now since system satisfies initial-rest conditions,

$$y[n] = 0 \text{for} n < 0 \tag{3}$$
$$\because x[n] = 0 \text{for} n < 0$$

Now put $n = 0$ for (1),

$$y[0] = 0 + x[0] \tag{4}$$
$$= 1 \tag{5}$$

Now for $n > 0$,

$$y[n] = ny[n-1] + 0 \tag{6}$$
$$\implies = n(n-1)\,y[n-2] \tag{7}$$
$$= n(n-1)(n-2)\cdots y[0] \tag{8}$$
$$\implies = n!\,(\because y[0] = 1) \tag{9}$$

So, overall the output $y[n]$ will be,

$$y[n] = \begin{cases} 0 & , n < 0 \\ n! & , n \geq 0 \end{cases} \tag{10}$$
$$\implies y[n] = n!u[n] \tag{11}$$

(b) To determine whether a system is linear, it should satisfy the following condition,

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_1[n]\} \tag{12}$$
$$= ay_1[n] + by_2[n] \tag{13}$$

where T is underlying transformation from input to output in the system.
So here, take the input signal as superposition of two Dirac-delta functions,

$$x[n] = a\delta[n] + b\delta[n] \tag{14}$$

Similarly we will calculate the output signal as above,

$$y[n] = 0, n < 0 \tag{15}$$
$$y[0] = x[0] \tag{16}$$
$$= a + b \tag{17}$$

And for $n > 0$,

$$y[n] = ny[n-1] \tag{18}$$
$$= n!\,(a+b) \tag{19}$$

So for all n,

$$y[n] = (a+b)\,n!u\{n\} \tag{20}$$
$$\implies T\{a\delta[n] + b\delta[n]\} = a(n!u[n]) + b(n!u[n]) \tag{21}$$
$$= aT\{\delta[n]\} + bT\{\delta[n]\} \tag{22}$$

$\therefore$ The given system is **linear**.

(c) To determine whether a system is Time-Invariant, the delay of n_0 in the input signal should result in same delay of n_0 in the output signal.

Here take the input signal as,

$$x[n] = \delta[n - n_0] \tag{23}$$

So using initial-rest conditions,

$$y[n] = 0 \text{ for } n < n_0 \tag{24}$$

And using the (1),

$$\begin{aligned} y[n_0] &= 0 + x[n_0] \quad &(25)\\ &= 1 \quad &(26) \end{aligned}$$

And for $n > n_0$,

$$\begin{aligned} y[n] &= ny[n-1] + 0 \quad &(27)\\ &= n(n-1)(n-2)\cdots(n_0+1) \quad &(28)\\ &= \frac{n!}{(n_0)!} \quad &(29) \end{aligned}$$

So overall,

$$y[n] = \left(\frac{n!}{n_0!}\right) u[n - n_0] \tag{30}$$

Whereas using (11), you will get,

$$\begin{aligned} y'[n] &= (n - n_0)!u[n - n_0] \quad &(31)\\ &\neq y[n] \quad &(32) \end{aligned}$$

$\therefore$ The system is not a **Time-Invariant** system.