Random Numbers

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Contents

***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/uniform.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

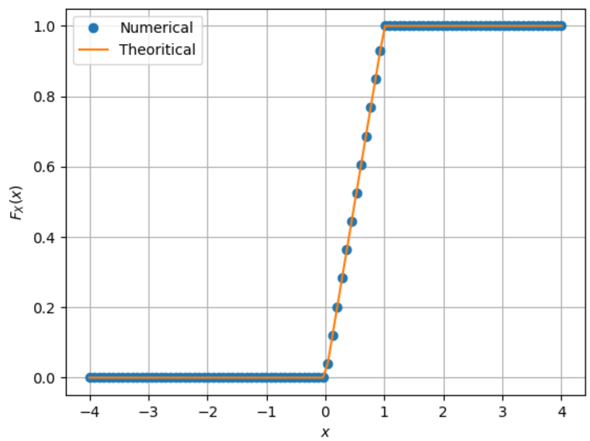

Fig. 1.1: The CDF of U

Solution: Use the following code to plot Fig. 1.1

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given that, random variable U is uniformly distributed in interval $(0, 1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \tag{1.2}$$
$$= 1 \tag{1.3}$$

So for $x \in (0, 1)$,the probability distribution function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x)\,dx \tag{1.4}$$
$$= \int_0^x 1dx \tag{1.5}$$
$$= x \tag{1.6}$$

For $x < 0$,

$$F_U(x) = \Pr(U \le x) \quad = 0\,(\because f_U(x) = 0) \tag{1.7}$$

And for $x > 1$

$$F_U(x) = \Pr(U \le x) \quad = 1\,(\because f_U(x) = 0) \tag{1.8}$$

Overall,

$$F_U(x) = \begin{cases} 0 & , x \le 0 \\ x & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.
Solution: Download the following code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q1/
    mean_var_uniform.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q1/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c −lm
./a.out
```

We will get output as,

$$mean = 0.500007 \tag{1.11}$$
$$variance = 0.083301 \tag{1.12}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.13}$$

Solution: Already we know that,

$$F_U(x) = \begin{cases} 0 & , x \le 0 \\ x & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

So, the given integral solves down to,

$$E\left[U^k\right] = \int_0^1 x^k dx \tag{1.14}$$

Since $F_U(x)$ is constant w.r.t x for $x \ge 1$ amd $x \le 0$. For mean,

$$E[U] = \int_0^1 xdx \tag{1.15}$$
$$= \left\{\frac{x^2}{2}\right\}_0^1 \tag{1.16}$$
$$= 0.5 \tag{1.17}$$

Now for variance,we know that

$$Var[U] = E[U - E[U]]^2 \tag{1.18}$$
$$= E\left[U^2 + E[U]^2 - 2E[U]U\right] \tag{1.19}$$
$$\tag{1.20}$$

Since expected value is a linear operator, we can write

$$= E\left[U^2\right] + E[U]^2 - 2E[U]^2 \tag{1.21}$$
$$= E\left[U^2\right] - E[U]^2 \tag{1.22}$$

To get variance we will find,

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.23}$$
$$= \left\{x^3/3\right\}_0^1 \tag{1.24}$$
$$= \frac{1}{3} \tag{1.25}$$

Therefore,

$$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.26}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.27}$$
$$= \frac{1}{12} \tag{1.28}$$
$$= 0.0833 \tag{1.29}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: Download the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/gaussian.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the below code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.1.
Properties Of CDF:

- CDF is montonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as, $Q(x) = \Pr(X > x)$
- The CDF,$F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    gaussian_pdf_plot.py
```

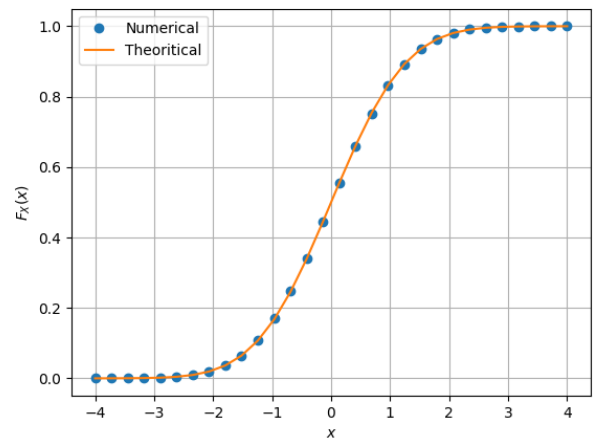

Fig. 2.1: The CDF of X

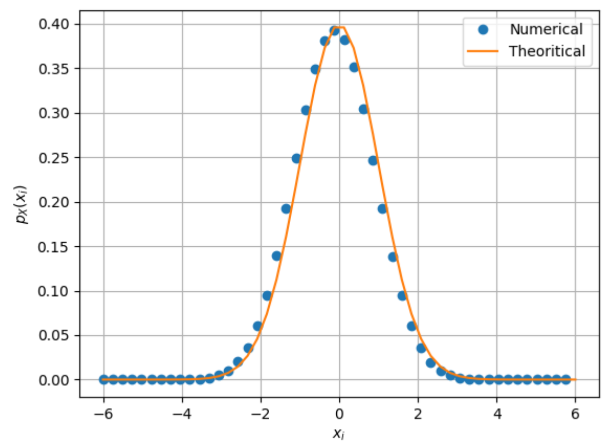

Fig. 2.2: The PDF of X

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) PDF is symmetric about the mean, in this case at $x = 0$
c) The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    mean_var_gauss.c
```

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/coeffs.h
```

Then run the following command in linux terminal

```
cc mean_var_gauss.c −lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x]\,dx \tag{2.5}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.6}$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.We can write,

$$E[x] = 0 \tag{2.7}$$

Consider the following expression,

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p_X[x]\,dx \tag{2.8}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \tag{2.9}$$

To solve the above integral, we will use integration by parts, i.e,

$$\int uv dx = u \int v dx - \int u' \left(\int v dx\right) dx \tag{2.10}$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}}\right) dx \tag{2.11}$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx\right)\right) \tag{2.12}$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \tag{2.13}$$

$$dt = x dx \tag{2.14}$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t)\,dt \tag{2.15}$$

$$= -\exp(-t) + c \tag{2.16}$$

$$\implies = -\exp\left(-\frac{x^2}{2}\right) + c \tag{2.17}$$

Using (2.17),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{\frac{-x^2}{2}} dx\right) \tag{2.18}$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} = 1 \tag{2.19}$$

Now putting limits and using (2.7),(2.19),

$$E\left[x^2\right] = 1 \tag{2.20}$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \tag{2.21}$$

$$= 1 \tag{2.22}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \tag{3.1}$$

and plot its CDF.

Solution: Download the C code from the link below to generate samples of V from uni.dat file

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q3/V.c
```

Run the following command,

```
cc V.c −lm
./a.out
```

Then download the below python file to get CDF

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q3/
    V_cdf_plot.py
```

Then run the following command

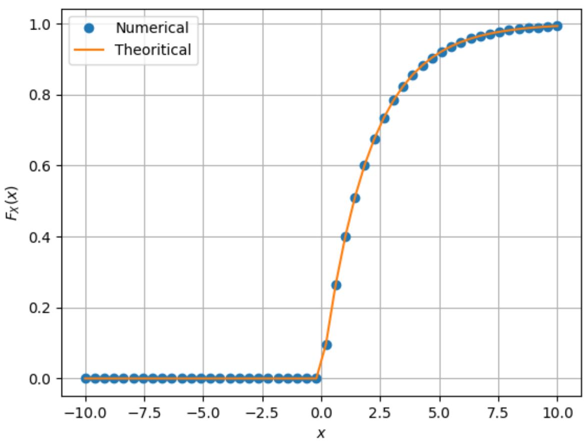

Fig. 3.1: The CDF of V

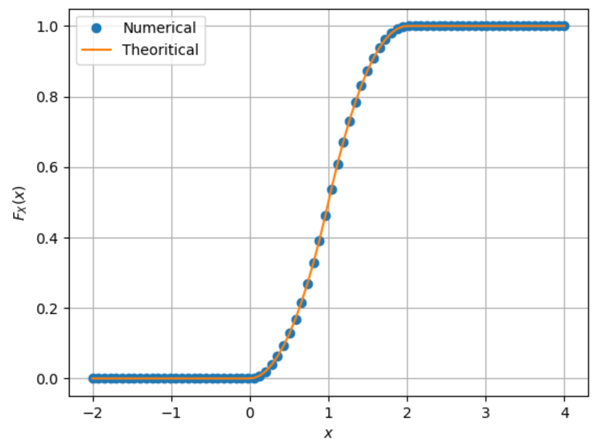

Fig. 4.1: The CDF of T

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Given

$$V = -2\ln(1-U) \tag{3.2}$$
$$F_V(x) = \Pr(V \le x) \tag{3.3}$$

we will use (3.3)

$$F_V(x) = \Pr(-2\ln(1-U) \le x) \tag{3.4}$$
$$= \Pr\left(\ln(1-U) \ge \frac{-x}{2}\right) \tag{3.5}$$
$$= \Pr\left(1-U \ge \exp\left(\frac{-x}{2}\right)\right) \tag{3.6}$$
$$= \Pr\left(U \le 1-\exp\left(\frac{-x}{2}\right)\right) \tag{3.7}$$
$$= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) \tag{3.8}$$

For$x > 0, 1 - e^{\frac{-x}{2}} < 1$ and $x < 0, 1 - e^{\frac{-x}{2}} < 0$

$$F_V(x) = \begin{cases} 0 & x \le 0 \\ 1-\exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \tag{3.9}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the below code,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/codes/
    Q4/coeffs.h
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/codes/
    Q4/triangular.c
```

and run the following command,

```
cc triangular.c −lm
./a.out
```

You will get required generated random numbers in tri.dat file.

4.2 Find the CDF of T.
Solution: Download the below files,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri.dat
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri_cdf_plot.py
```

Run the following command,

```
python3 tri_cdf_plot.py
```

4.3 Find the PDF of T.
Solution: Download the below files,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri.dat
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri_pdf_plot.py
```

Run the following command,

```
python3 tri_pdf_plot.py
```

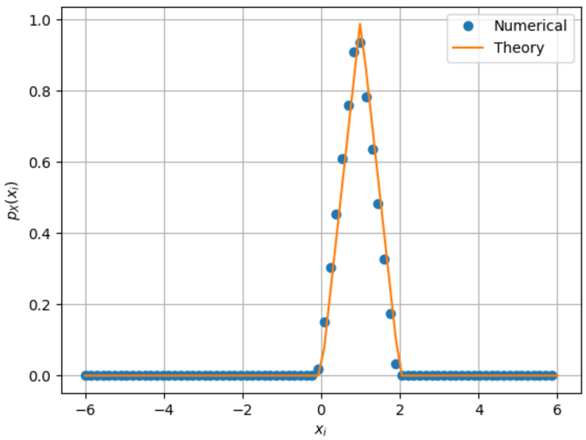

Fig. 4.2: The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Given that,

$$T = U1 + U2 \tag{4.2}$$

where$U1, U2$ are uniform random variables $\in (0,1)$.
The CDF of T is defined as,

$$F_T(t) = \Pr(T \le t) \tag{4.3}$$

Now from (4.2) we can write,

$$F_T(t) = \Pr(U1 + U2 \le t) \tag{4.4}$$

Case -1 : For $t > 2$.
The $\Pr(U1 + U2 \le t) = 1$, because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 < 2$,

$$\implies F_T(t) = 1 \tag{4.5}$$

Case -2 : For $t < 0$.
The $\Pr(U1 + U2 \le t) = 0$ because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 > 0$,

$$\implies F_T(t) = 0 \tag{4.6}$$

Case - 3: For, $t \in (0,2)$.
We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U1 = x$ where $x \in (0,t)$.
So in this case CDF will be,

$$F_T(t) = \Pr(U1 + U2 \le t) \tag{4.7}$$
$$= \Pr(U1 = x, U2 \le t - x) \tag{4.8}$$
$$= \Pr(U1 = x)\Pr(U2 \le t - x) \tag{4.9}$$

Since U1,U2 are i.i.d.
Now note that x is a variable and varies in $(0,t)$, so we have to take integral over x to evaluate the $\Pr(U1 = x)$,

$$F_T(t) = \int_0^t f_U(x)\Pr(U2 \le t - x)\,dx \tag{4.10}$$
$$= \int_0^t f_U(x)F_U(t - x)\,dx \tag{4.11}$$

Case - 1 For $t \in (0,1)$,we know $f_U(x) = 1$ so,

$$F_T(t) = \int_0^t 1.F_U(t - x)\,dx \tag{4.12}$$

As, $x < t$, $0 < t - x < t < 1$ using (1),we can write

$$F_T(t) = \int_0^t (t - x)\,dx \tag{4.13}$$
$$= \left\{tx - \frac{x^2}{2}\right\}_0^t \tag{4.14}$$
$$= \frac{t^2}{2} \tag{4.15}$$

Case -2 For $t \in (1,2)$, we know $f_U(x) = 0$ at $x > 1$,so the integral solves down to,

$$F_T(t) = \int_0^1 f_U(x)F_U(t - x)\,dx \tag{4.16}$$
$$= \int_0^1 1.F_U(t - x)\,dx \tag{4.17}$$
$$\tag{4.18}$$

To solve the above integral we will use integration by substitution,

$$k = t - x \tag{4.19}$$
$$dk = -dx \tag{4.20}$$
$$F_T(t) = \int_t^{t-1} F_U(k)(-dk) \tag{4.21}$$
$$= \int_{t-1}^t F_U(k)\,dk \tag{4.22}$$

As $1 \le t \le 2, 0 \le t - 1 \le 1$ we will break integral at 1 because $F_U(k)$ changes at 1.Using (1),

$$F_T(t) = \int_{t-1}^{1} F_U(k)\,dk + \int_{1}^{t} F_U(k)\,dk \tag{4.23}$$

$$= \int_{t-1}^{1} kdk + \int_{1}^{t} 1dk \tag{4.24}$$

$$= \left\{\frac{k^2}{2}\right\}_{t-1}^{1} + t - 1 \tag{4.25}$$

$$= \frac{1}{2} - \left(\frac{(t-1)^2}{2}\right) + t - 1 \tag{4.26}$$

$$= 2t - \frac{t^2}{2} - 1 \tag{4.27}$$

Overall we can write the CDF of $F_T(x)$ as,

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \le x \le 1 \\ 2t - \frac{t^2}{2} - 1 & , 1 \le x \le 2 \\ 1 & , x > 2 \end{cases} \tag{4.28}$$

Now we will find PDF of T,
As,

$$T = U1 + U2 \tag{4.29}$$

We will use method of convolution to get PDF of T as $U1$ and $U2$ are i.i.d.

$$f_T(t) = \int_{-\infty}^{\infty} f_{U1}(x)\, f_{U2}(t-x)\,dx \tag{4.30}$$

Since $U1, U2$ are of same distribution we can write,

$$f_{U1}(x) = f_{U2}(x) = f_U(x) \tag{4.31}$$

$$\implies f_T(t) = \int_{-\infty}^{\infty} f_U(x)\, f_U(t-x)\,dx \tag{4.32}$$

From the PDF of U,we can write

$$f_T(t) = \int_{0}^{1} f_U(x)\, f_U(t-x)\,dx \tag{4.33}$$

$$= \int_{0}^{1} 1.f_U(t-x)\,dx \tag{4.34}$$

$$\tag{4.35}$$

we will solve the above integral using substitution.

$$z = t - x \tag{4.36}$$

$$dz = -dx \tag{4.37}$$

$$\implies f_T(t) = \int_{t}^{t-1} f_U(z)(-dz) \tag{4.38}$$

$$= \int_{t-1}^{t} f_U(z)\,dz \tag{4.39}$$

Case -1 For $t < 0$ as $z < t$,the PDF $f_U(z) = 0$. So,

$$f_T(t) = 0 \tag{4.40}$$

Case -2 For $0 \le t \le 1$, we will break the integral at $z = 0$, since $f_U(z)$ changes at 0.

$$f_T(t) = \int_{t-1}^{0} f_U(z)\,dz + \int_{0}^{t} f_U(z)\,dz \tag{4.41}$$

$$= 0 + \int_{0}^{t} 1dz \tag{4.42}$$

$$= t \tag{4.43}$$

Case-3 Similarly for $1 \le t \le 2$, we will break the integral at $z = 1$,

$$f_T(t) = \int_{t-1}^{1} f_U(z)\,dz + \int_{1}^{t} f_U(z) \tag{4.44}$$

$$= \int_{t-1}^{1} 1.dz + 0 \tag{4.45}$$

$$= 2 - t \tag{4.46}$$

Case-4 For $t > 2$,as $z > t - 1 > 1$,the PDF $f_U(z) = 0$. So,

$$f_T(z) = 0 \tag{4.47}$$

Overall, the PDF of T will be,

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 2 - x & , 1 \le x \le 2 \\ 0 & , x > 2 \end{cases} \tag{4.48}$$

4.5 Verify your results through a plot.
Solution: This is already done in 4.1 ,4.2.

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

5.3 Plot Y using a scatter plot.
5.4 Guess how to estimate X from Y.
5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.6 Find P_e assuming that X has equiprobable symbols.
5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.4}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.
6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.
7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.
7.4 Plot P_e in problems 7.7.2 and 7.7.2 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.
8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.
8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.