

Random Numbers

Mannem Charan

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Abstract—This manual provides solutions for random numbers assignment.

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files.

```
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/uniform.c
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

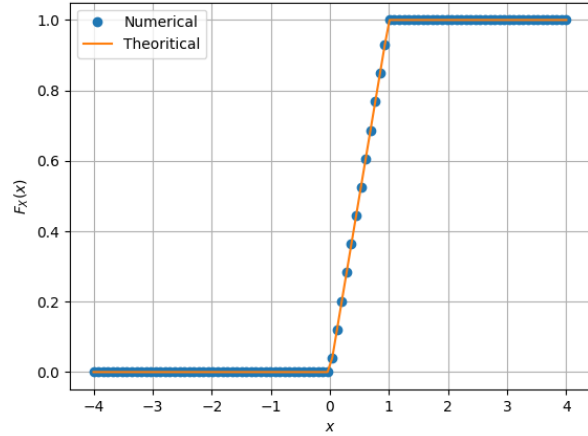


Fig. 1.1: The CDF of U

Solution: Use the following code to plot Fig. 1.1

```
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/
uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given that, random variable U is uniformly distributed in interval $(0, 1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \quad (1.2)$$

$$= 1 \quad (1.3)$$

So for $x \in (0, 1)$, the probability distribution

function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x) dx \quad (1.4)$$

$$= \int_0^x 1 dx \quad (1.5)$$

$$= x \quad (1.6)$$

For $x < 0$,

$$F_U(x) = \Pr(U \leq x) = 0 (\because f_U(x) = 0) \quad (1.7)$$

And for $x > 1$

$$F_U(x) = \Pr(U \leq x) = 1 (\because f_U(x) = 0) \quad (1.8)$$

Overall,

$$F_U(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.9)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.10)$$

Write a C program to find the mean and variance of U .

Solution: Download the following code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/mean_var_uniform.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c -lm
./a.out
```

We will get output as,

$$\text{mean} = 0.500007 \quad (1.11)$$

$$\text{variance} = 0.083301 \quad (1.12)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.13)$$

Solution: Already we know that,

$$F_U(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

So, the given integral solves down to,

$$E[U^k] = \int_0^1 x^k dx \quad (1.14)$$

Since $F_U(x)$ is constant w.r.t x for $x \geq 1$ and $x \leq 0$. For mean,

$$E[U] = \int_0^1 x dx \quad (1.15)$$

$$= \left\{ \frac{x^2}{2} \right\}_0^1 \quad (1.16)$$

$$= 0.5 \quad (1.17)$$

Now for variance, we know that

$$\text{Var}[U] = E[U - E[U]]^2 \quad (1.18)$$

$$= E[U^2 + E[U]^2 - 2E[U]U] \quad (1.19)$$

$$(1.20)$$

Since expected value is a linear operator, we can write

$$= E[U^2] + E[U]^2 - 2E[U]^2 \quad (1.21)$$

$$= E[U^2] - E[U]^2 \quad (1.22)$$

To get variance we will find,

$$E[U^2] = \int_0^1 x^2 dx \quad (1.23)$$

$$= \left\{ x^3/3 \right\}_0^1 \quad (1.24)$$

$$= \frac{1}{3} \quad (1.25)$$

Therefore,

$$\text{Var}[U] = \frac{1}{3} - \left(\frac{1}{2} \right)^2 \quad (1.26)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.27)$$

$$= \frac{1}{12} \quad (1.28)$$

$$= 0.0833 \quad (1.29)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the code below

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the below code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.1.

Properties Of CDF:

- CDF is monotonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as,
 $Q(x) = \Pr(X > x)$
- The CDF, $F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian_pdf_plot.py
```

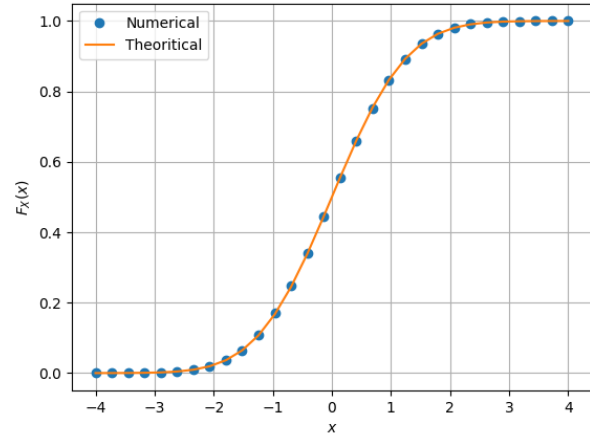


Fig. 2.1: The CDF of X

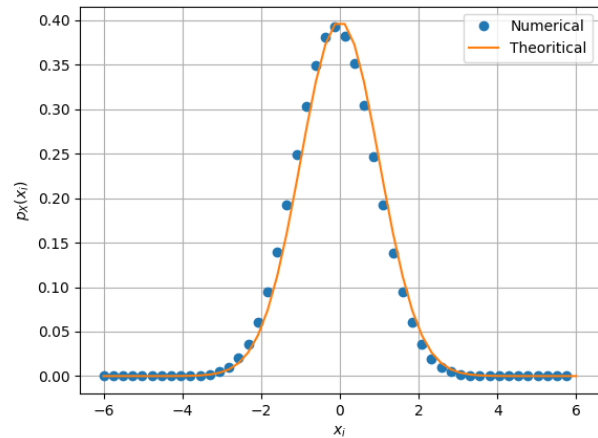


Fig. 2.2: The PDF of X

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

- $\forall x \in \mathbb{R}, p_X(x) \geq 0$
- PDF is symmetric about the mean, in this case at $x = 0$
- The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/mean_var_gauss.c
```

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/coeffs.h

Then run the following command in linux terminal

```
cc mean_var_gauss.c -lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x] dx \quad (2.5)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function. We can write,

$$E[x] = 0 \quad (2.7)$$

Consider the following expression,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X[x] dx \quad (2.8)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

To solve the above integral, we will use integration by parts, i.e.,

$$\int u v dx = u \int v dx - \int u' \left(\int v dx \right) dx \quad (2.10)$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}} \right) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx \right) \right) \quad (2.12)$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \quad (2.13)$$

$$dt = x dx \quad (2.14)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \quad (2.15)$$

$$= -\exp(-t) + c \quad (2.16)$$

$$\Rightarrow = -\exp\left(-\frac{x^2}{2}\right) + c \quad (2.17)$$

Using (2.17), we can write

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \right) \quad (2.18)$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = 1 \quad (2.19)$$

Now putting limits and using (2.7), (2.19),

$$E[x^2] = 1 \quad (2.20)$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \quad (2.21)$$

$$= 1 \quad (2.22)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the C code from the link below to generate samples of V from uni.dat file

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q3/V.c

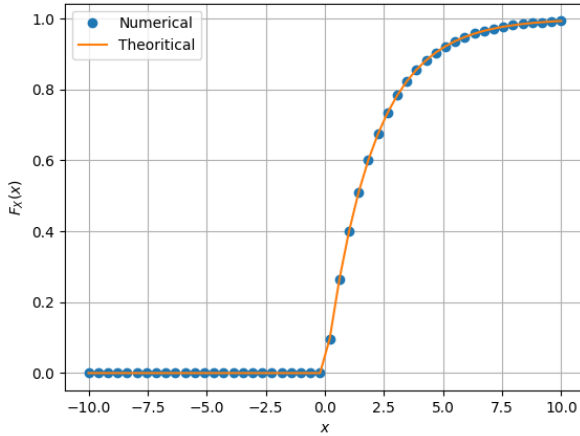
Run the following command,

```
cc V.c -lm
./a.out
```

Then download the below python file to get CDF

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q3/V_cdf_plot.py

Then run the following command

Fig. 3.1: The CDF of V

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: Given

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$F_V(x) = \Pr(V \leq x) \quad (3.3)$$

we will use (3.3)

$$F_V(x) = \Pr(-2 \ln(1 - U) \leq x) \quad (3.4)$$

$$= \Pr\left(\ln(1 - U) \geq \frac{-x}{2}\right) \quad (3.5)$$

$$= \Pr\left(1 - U \geq \exp\left(\frac{-x}{2}\right)\right) \quad (3.6)$$

$$= \Pr\left(U \leq 1 - \exp\left(\frac{-x}{2}\right)\right) \quad (3.7)$$

$$= F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (3.8)$$

For $x > 0$, $1 - e^{\frac{-x}{2}} < 1$ and $x < 0$, $1 - e^{\frac{-x}{2}} < 0$

$$F_V(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \quad (3.9)$$

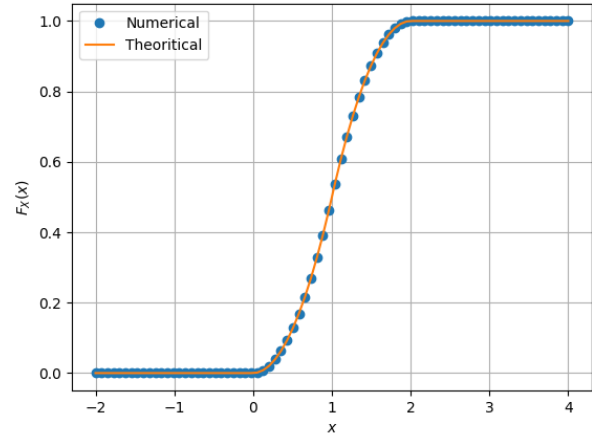
4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the below code,

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/codes/
Q4/coeffs.h
```

Fig. 4.1: The CDF of T

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/codes/
Q4/triangular.c
```

and run the following command,

```
cc triangular.c -lm
./a.out
```

You will get required generated random numbers in tri.dat file.

4.2 Find the CDF of T .

Solution: Download the below files,

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri.dat
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri_cdf_plot.py
```

Run the following command,

```
python3 tri_cdf_plot.py
```

4.3 Find the PDF of T .

Solution: Download the below files,

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri.dat
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri_pdf_plot.py
```

Run the following command,

```
python3 tri_pdf_plot.py
```

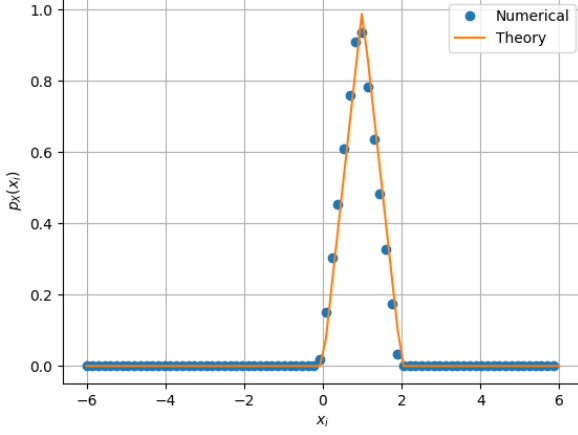


Fig. 4.2: The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: Given that,

$$T = U1 + U2 \quad (4.2)$$

where $U1, U2$ are uniform random variables $\in (0, 1)$.

Calculation of CDF The CDF of T is defined as,

$$F_T(t) = \Pr(T \leq t) \quad (4.3)$$

Now from (4.2) we can write,

$$F_T(t) = \Pr(U1 + U2 \leq t) \quad (4.4)$$

Case -1 : For $t > 2$.

The $\Pr(U1 + U2 \leq t) = 1$, because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 < 2$,

$$\Rightarrow F_T(t) = 1 \quad (4.5)$$

Case -2 : For $t < 0$.

The $\Pr(U1 + U2 \leq t) = 0$ because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 > 0$,

$$\Rightarrow F_T(t) = 0 \quad (4.6)$$

Case - 3: For, $t \in (0, 2)$.

We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U1 = x$ where $x \in (0, t)$.

So in this case CDF will be,

$$F_T(t) = \Pr(U1 + U2 \leq t) \quad (4.7)$$

$$= \Pr(U1 = x, U2 \leq t - x) \quad (4.8)$$

$$= \Pr(U1 = x) \Pr(U2 \leq t - x) \quad (4.9)$$

Since $U1, U2$ are i.i.d.

Now note that x is a variable and varies in $(0, t)$, so we have to take integral over x to evaluate the $\Pr(U1 = x)$,

$$F_T(t) = \int_0^t f_U(x) \Pr(U2 \leq t - x) dx \quad (4.10)$$

$$= \int_0^t f_U(x) F_U(t - x) dx \quad (4.11)$$

Case - 1 For $t \in (0, 1)$, we know $f_U(x) = 1$ so,

$$F_T(t) = \int_0^t 1 \cdot F_U(t - x) dx \quad (4.12)$$

As, $x < t$, $0 < t - x < t < 1$ using (1), we can write

$$F_T(t) = \int_0^t (t - x) dx \quad (4.13)$$

$$= \left\{ tx - \frac{x^2}{2} \right\}_0^t \quad (4.14)$$

$$= \frac{t^2}{2} \quad (4.15)$$

Case -2 For $t \in (1, 2)$, we know $f_U(x) = 0$ at $x > 1$, so the integral solves down to,

$$F_T(t) = \int_0^1 f_U(x) F_U(t - x) dx \quad (4.16)$$

$$= \int_0^1 1 \cdot F_U(t - x) dx \quad (4.17)$$

$$(4.18)$$

To solve the above integral we will use integration by substitution,

$$k = t - x \quad (4.19)$$

$$dk = -dx \quad (4.20)$$

$$F_T(t) = \int_t^{t-1} F_U(k) (-dk) \quad (4.21)$$

$$= \int_{t-1}^t F_U(k) dk \quad (4.22)$$

As $1 \leq t \leq 2$, $0 \leq t - 1 \leq 1$ we will

break integral at 1 because $F_U(k)$ changes at 1. Using (1),

$$F_T(t) = \int_{t-1}^1 F_U(k) dk + \int_1^t F_U(k) dk \quad (4.23)$$

$$= \int_{t-1}^1 k dk + \int_1^t 1 dk \quad (4.24)$$

$$= \left\{ \frac{k^2}{2} \right\}_{t-1}^1 + t - 1 \quad (4.25)$$

$$= \frac{1}{2} - \left(\frac{(t-1)^2}{2} \right) + t - 1 \quad (4.26)$$

$$= 2t - \frac{t^2}{2} - 1 \quad (4.27)$$

Overall we can write the CDF of $F_T(x)$ as,

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \leq x \leq 1 \\ 2t - \frac{t^2}{2} - 1 & , 1 \leq x \leq 2 \\ 1 & , x > 2 \end{cases} \quad (4.28)$$

Calculation of PDF Now we will find PDF of T ,

As,

$$T = U1 + U2 \quad (4.29)$$

We will use method of convolution to get PDF of T as $U1$ and $U2$ are i.i.d.

$$f_T(t) = \int_{-\infty}^{\infty} f_{U1}(x) f_{U2}(t-x) dx \quad (4.30)$$

Since $U1, U2$ are of same distribution we can write,

$$f_{U1}(x) = f_{U2}(x) = f_U(x) \quad (4.31)$$

$$\Rightarrow f_T(t) = \int_{-\infty}^{\infty} f_U(x) f_U(t-x) dx \quad (4.32)$$

From the PDF of U , we can write

$$f_T(t) = \int_0^1 f_U(x) f_U(t-x) dx \quad (4.33)$$

$$= \int_0^1 1 \cdot f_U(t-x) dx \quad (4.34)$$

$$(4.35)$$

we will solve the above integral using sub-

stitution.

$$z = t - x \quad (4.36)$$

$$dz = -dx \quad (4.37)$$

$$\Rightarrow f_T(t) = \int_t^{t-1} f_U(z) (-dz) \quad (4.38)$$

$$= \int_{t-1}^t f_U(z) dz \quad (4.39)$$

Case -1 For $t < 0$ as $z < t$, the PDF $f_U(z) = 0$. So,

$$f_T(t) = 0 \quad (4.40)$$

Case -2 For $0 \leq t \leq 1$, we will break the integral at $z = 0$, since $f_U(z)$ changes at 0.

$$f_T(t) = \int_{t-1}^0 f_U(z) dz + \int_0^t f_U(z) dz \quad (4.41)$$

$$= 0 + \int_0^t 1 dz \quad (4.42)$$

$$= t \quad (4.43)$$

Case-3 Similarly for $1 \leq t \leq 2$, we will break the integral at $z = 1$,

$$f_T(t) = \int_{t-1}^1 f_U(z) dz + \int_1^t f_U(z) dz \quad (4.44)$$

$$= \int_{t-1}^1 1 \cdot dz + 0 \quad (4.45)$$

$$= 2 - t \quad (4.46)$$

Case-4 For $t > 2$, as $z > t - 1 > 1$, the PDF $f_U(z) = 0$. So,

$$f_T(z) = 0 \quad (4.47)$$

Overall, the PDF of T will be,

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leq x \leq 1 \\ 2 - x & , 1 \leq x \leq 2 \\ 0 & , x > 2 \end{cases} \quad (4.48)$$

4.5 Verify your results through a plot.

Solution: This is already done in 4.1, 4.2.

5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable $X \in \{1, -1\}$.

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

- 5.3 Plot Y using a scatter plot.
 5.4 Guess how to estimate X from Y .
 5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.3)$$

- 5.6 Find P_e assuming that X has equiprobable symbols.
 5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
 5.8 Now, consider a threshold δ while estimating X from Y . Find the value of δ that maximizes the theoretical P_e .
 5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.4)$$

- 5.10 Repeat the above exercise using the MAP criterion.

6 GAUSSIAN TO OTHER

- 6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

- 6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

- 6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.3)$$

7 CONDITIONAL PROBABILITY

- 7.1 Plot

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

- 7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$.
 7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

- 7.4 Plot P_e in problems 7.7.2 and 7.7.2 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

- 8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

- 8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

- 8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

- 8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.