Random Numbers

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***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/uniform.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Use the following code to plot Fig. 1.2

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

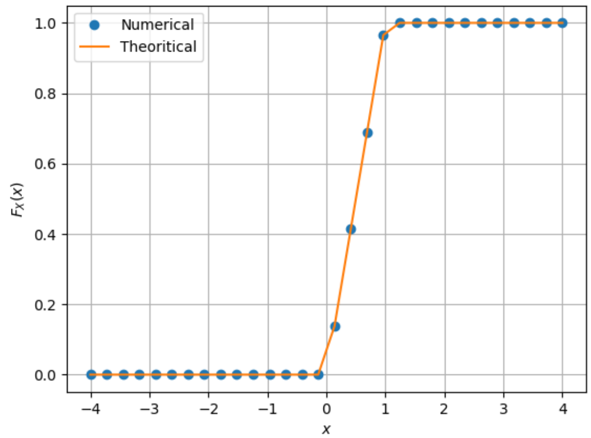

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given that, random variable U is uniformly distributed in interval $(0, 1)$. So we can write that, the probability density function

$$f_x(x) = \frac{1}{1-0} \tag{1.2}$$
$$= 1 \tag{1.3}$$

So for $U \in (0, 1)$, the probability distribution function $F_x(x)$ can be calculated as,

$$F_x(x) = \int_0^x f_x(x)\,dx \tag{1.4}$$
$$= \int_0^x 1dx \qquad = x \tag{1.5}$$

Since U is distributed from 0 to 1, we can write

$$F_x(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.7}$$

Write a C program to find the mean and variance of U.
Solution: Download the following code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/mean_var_uniform.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c -lm
./a.out
```

We will get output as,

$$mean = 0.500007 \tag{1.8}$$
$$variance = 0.083301 \tag{1.9}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.10}$$

Solution: Already we know that,

$$F_x(x) = \begin{cases} 0 & ,x \le 0 \\ x & ,0 < x < 1 \\ 1 & ,x \ge 1 \end{cases}$$

So, the given integral solves down to,

$$E\left[U^k\right] = \int_0^1 x^k dx \tag{1.11}$$

Since $F_x(x)$ is constant w.r.t x for $x \ge 1$ amd $x \le 0$. For mean,

$$E[U] = \int_0^1 xdx \tag{1.12}$$
$$= \left\{\frac{x^2}{2}\right\}_0^1 \qquad = 0.5 \tag{1.13}$$

Now for variance,we know that

$$Var[U] = E[U - E[U]]^2 \tag{1.14}$$
$$= E\left[U^2 + E[U]^2 - 2E[U]U\right] \tag{1.15}$$
$$= E\left[U^2\right] + E[U]^2 - 2E[U]^2 \tag{1.16}$$
$$= E\left[U^2\right] - E[U]^2 \tag{1.17}$$

And

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.18}$$
$$= \left\{x^3/3\right\}_0^1 \tag{1.19}$$
$$= \frac{1}{3} \tag{1.20}$$

Therefore,

$$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad = 0.0833 \tag{1.21}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the code below

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/gaussian.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/coeffs.h
```

Run the following command

```
cc gaussian.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the below code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.2.
Properties Of CDF:

a) The Q function is defined as

$$Q(x) = \Pr(X > x) \tag{2.2}$$

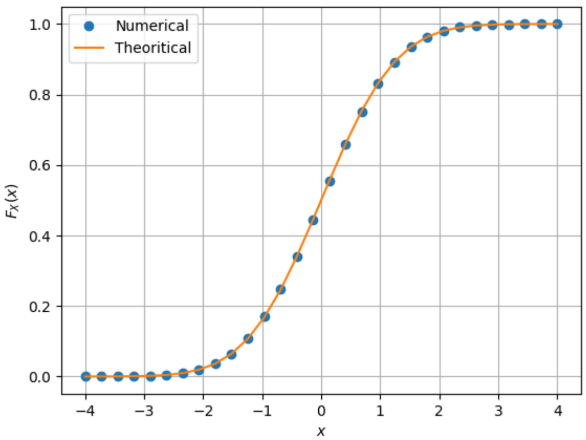

Fig. 2.2: The CDF of X

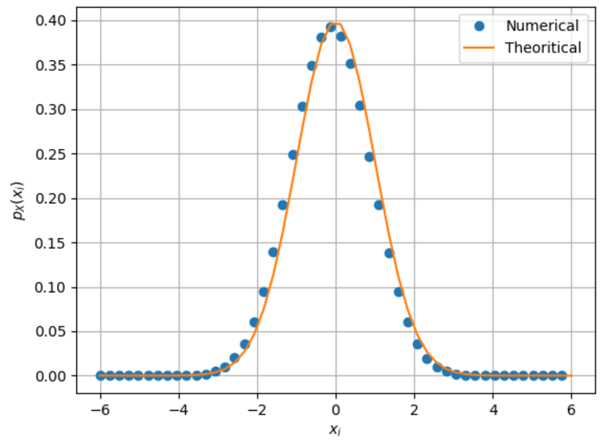

Fig. 2.3: The PDF of X

b) So we can write,

$$F_X(x) = 1 - Q(x) \tag{2.3}$$

c) $F_X(x) = P(X \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx.$
d) $\lim_{x\to\infty} F_x(x) = 1, \quad \lim_{x\to-\infty} F_x(x) = 0$
e) $F_X(x) = \frac{1}{2}$
f) $F_X(-x) = 1 - F_X(x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.4}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    gaussian_pdf_plot.py
```

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

a) PDF is symmetric about the mean, in this case at $x = 0$
b) It has bell shaped graph.
c) The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program. **Solution:** Download the C code from the links below,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    mean_var_gauss.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/coeffs.h
```

Then run the following command in linux terminal

```
cc mean_var_gauss.c −lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically. **Solution:** Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.6}$$

We know that,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x]\, dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

Since $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an odd function.We can write,

$$E[x] = 0 \tag{2.9}$$

Consider the following expression,

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p_X[x]\,dx \tag{2.10}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \tag{2.11}$$

To solve the above integral, we will use integration by parts, i.e,

$$\int uv dx = u \int v dx - \int \left(u' \left(\int v\right)\right) dx \tag{2.12}$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} x\left(xe^{-\frac{x^2}{2}}\right) dx\right) \tag{2.13}$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int xe^{-\frac{x^2}{2}} dx - \int \left(\int e^{-\frac{x^2}{2}} dx\right)\right) \tag{2.14}$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \tag{2.15}$$

$$dt = x dx \tag{2.16}$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t)\, dt \tag{2.17}$$

$$= -\exp(-t) + c \tag{2.18}$$

$$\implies = -\exp\left(-\left(\frac{x^2}{2}\right)\right) + c \tag{2.19}$$

Using (2.19),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(-xe^{\left(-\frac{x^2}{2}\right)} + \int e^{\left(\frac{-x^2}{2}\right)} dx\right) \tag{2.20}$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(\frac{-x^2}{2}\right)} = 1 \tag{2.21}$$

Now putting limits and using (2.9),(2.21),

$$E\left[x^2\right] = 1 \tag{2.22}$$

Using (1.17) we can write,

$$Var[x] = 1 - 0 \tag{2.23}$$

$$= 1 \tag{2.24}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

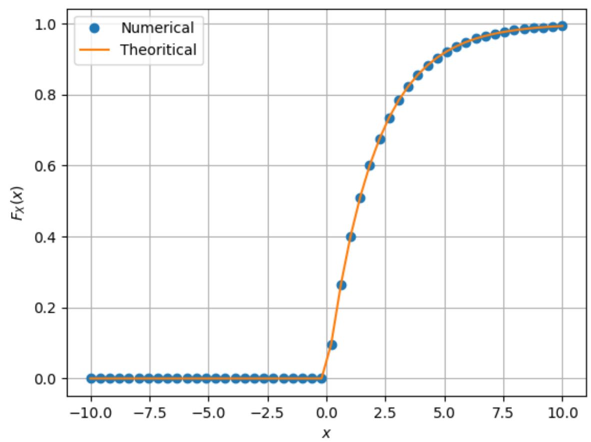

Fig. 3.1: The CDF of V

and plot its CDF.
Solution: Download the C code from the link below to generate samples of V from uni.dat file

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/V.c
```

Run the following command,

```
cc rayleigh.c −lm
./a.out
```

Then download the below python file get CDF

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/V_cdf_plot.
    py
```

Then run the following command

```
python3 rayleigh_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Given

$$V = -2\ln(1 - U) \tag{3.2}$$

For,

$$F_V(x) = \Pr(V \le x) \tag{3.3}$$

we will use (3.2)

$$
\begin{aligned}
F_V(x) &= \Pr(-2\ln(1-U) \leq x) && (3.4)\\
&= \Pr\left(\ln(1-U) \geq \frac{-x}{2}\right) && (3.5)\\
&= \Pr\left(1-U \geq \exp\left(\frac{-x}{2}\right)\right) && (3.6)\\
&= \Pr\left(U \leq 1-\exp\left(\frac{-x}{2}\right)\right) && (3.7)\\
&= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) && (3.8)\\
& && (3.9)
\end{aligned}
$$

Using CDF of U,

$$
F_v(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{\frac{-x}{2}} & , x \geq 0 \end{cases}
$$