Random Numbers

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Contents

***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/Q1/uniform.c
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/Q1/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

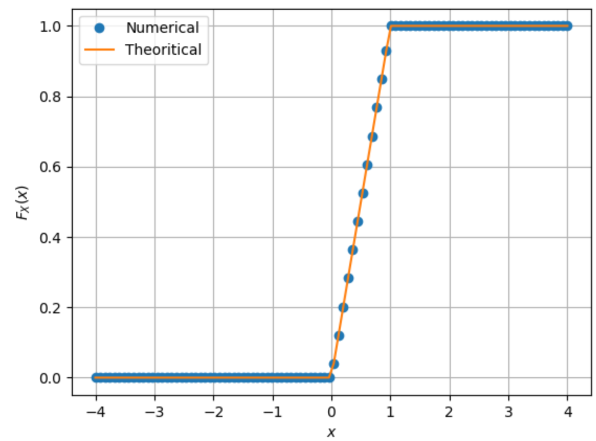

Fig. 1.2: The CDF of U

Solution: Use the following code to plot Fig. 1.2

```
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/Q1/uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given that, random variable U is uniformly distributed in interval $(0,1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \tag{1.2}$$
$$= 1 \tag{1.3}$$

So for $x \in (0,1)$,the probability distribution function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x)\,dx \tag{1.4}$$
$$= \int_0^x 1dx \tag{1.5}$$
$$= x \tag{1.6}$$

For $x < 0$,

$$F_U(x) = \Pr(U \le x) \quad = 0\,(\because f_U(x) = 0) \tag{1.7}$$

And for $x > 1$

$$F_U(x) = \Pr(U \le x) \quad = 1\,(\because f_U(x) = 0) \tag{1.8}$$

Overall,

$$F_U(x) = \begin{cases} 0 & , x \le 0 \\ x & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.
Solution: Download the following code,

```
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/Q1/mean_var_uniform.c
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/Q1/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c −lm
./a.out
```

We will get output as,

$$mean = 0.500007 \tag{1.11}$$
$$variance = 0.083301 \tag{1.12}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.13}$$

Solution: Already we know that,

$$F_U(x) = \begin{cases} 0 & , x \le 0 \\ x & , 0 < x < 1 \\ 1 & , x \ge 1 \end{cases}$$

So, the given integral solves down to,

$$E\left[U^k\right] = \int_0^1 x^k dx \tag{1.14}$$

Since $F_U(x)$ is constant w.r.t x for $x \ge 1$ amd $x \le 0$. For mean,

$$E[U] = \int_0^1 xdx \tag{1.15}$$
$$= \left\{\frac{x^2}{2}\right\}_0^1 \tag{1.16}$$
$$= 0.5 \tag{1.17}$$

Now for variance,we know that

$$Var[U] = E[U - E[U]]^2 \tag{1.18}$$
$$= E\left[U^2 + E[U]^2 - 2E[U]U\right] \tag{1.19}$$
$$\tag{1.20}$$

Since expected value is a linear operator, we can write

$$= E\left[U^2\right] + E[U]^2 - 2E[U]^2 \tag{1.21}$$
$$= E\left[U^2\right] - E[U]^2 \tag{1.22}$$

To get variance we will find,

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.23}$$
$$= \left\{x^3/3\right\}_0^1 \tag{1.24}$$
$$= \frac{1}{3} \tag{1.25}$$

Therefore,

$$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.26}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.27}$$
$$= \frac{1}{12} \tag{1.28}$$
$$= 0.0833 \tag{1.29}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: Download the code below

```
wget https://github.com/Charanyash/Random–
    Numbers–/blob/main/codes/Q2/gaussian.c
wget https://github.com/Charanyash/Random–
    Numbers–/blob/main/codes/Q2/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the below code,

```
wget https://github.com/Charanyash/Random–
    Numbers–/blob/main/codes/Q2/
    gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.2.
Properties Of CDF:

- CDF is montonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as, $Q(x) = \Pr(X > x)$
- The CDF,$F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Charanyash/Random–
    Numbers–/blob/main/codes/Q2/
    gaussian_pdf_plot.py
```

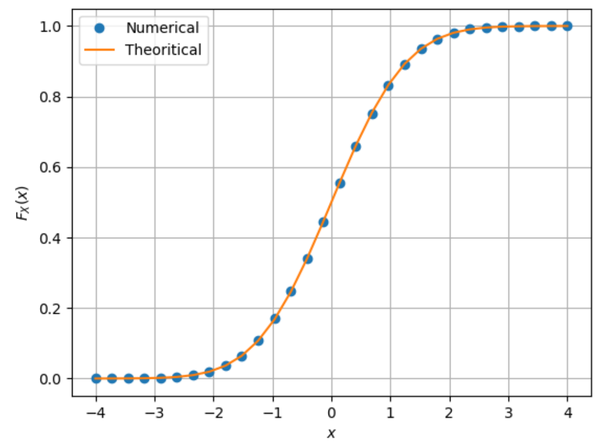

Fig. 2.2: The CDF of X

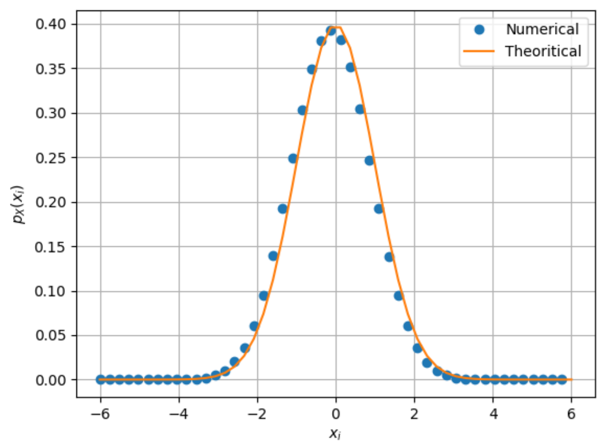

Fig. 2.3: The PDF of X

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) PDF is symmetric about the mean, in this case at $x = 0$
c) The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random–
    Numbers–/blob/main/codes/Q2/
    mean_var_gauss.c
```

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/coeffs.h
```

Then run the following command in linux terminal

```
cc mean_var_gauss.c −lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x]\, dx \quad (2.5)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.We can write,

$$E[x] = 0 \quad (2.7)$$

Consider the following expression,

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p_X[x]\, dx \quad (2.8)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

To solve the above integral, we will use integration by parts, i.e,

$$\int uv dx = u \int v dx - \int u' \left(\int v dx\right) dx \quad (2.10)$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}}\right) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx\right)\right) \quad (2.12)$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \quad (2.13)$$

$$dt = x dx \quad (2.14)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t)\, dt \quad (2.15)$$

$$= -\exp(-t) + c \quad (2.16)$$

$$\implies = -\exp\left(-\frac{x^2}{2}\right) + c \quad (2.17)$$

Using (2.17),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{\frac{-x^2}{2}} dx\right) \quad (2.18)$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} = 1 \quad (2.19)$$

Now putting limits and using (2.7),(2.19),

$$E\left[x^2\right] = 1 \quad (2.20)$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \quad (2.21)$$

$$= 1 \quad (2.22)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the C code from the link below to generate samples of V from uni.dat file

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q3/V.c
```

Run the following command,

```
cc V.c −lm
./a.out
```

Then download the below python file to get CDF

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q3/
    V_cdf_plot.py
```

Then run the following command

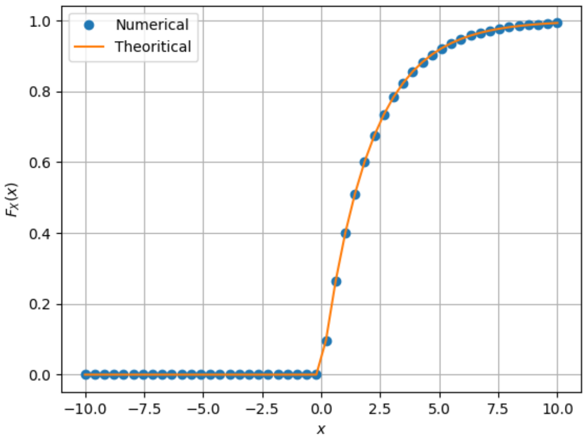

Fig. 3.1: The CDF of V

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Given

$$V = -2\ln(1-U) \tag{3.2}$$
$$F_V(x) = \Pr(V \le x) \tag{3.3}$$

we will use (3.3)

$$F_V(x) = \Pr(-2\ln(1-U) \le x) \tag{3.4}$$
$$= \Pr\left(\ln(1-U) \ge \frac{-x}{2}\right) \tag{3.5}$$
$$= \Pr\left(1 - U \ge \exp\left(\frac{-x}{2}\right)\right) \tag{3.6}$$
$$= \Pr\left(U \le 1 - \exp\left(\frac{-x}{2}\right)\right) \tag{3.7}$$
$$= F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \tag{3.8}$$

For$x > 0, 1 - e^{\frac{-x}{2}} < 1$ and $x < 0, 1 - e^{\frac{-x}{2}} < 0$

$$F_V(x) = \begin{cases} 0 & x \le 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \tag{3.9}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the below code,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/codes/
    Q4/coeffs.h
```

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/codes/
    Q4/triangular.c
```

and run the following command,

```
cc triangular.c −lm
./a.out
```

You will get required generated random numbers in tri.dat file.

4.2 Find the CDF of T.
Solution: Given that,

$$T = U1 + U2 \tag{4.2}$$

where, $U1, U2$ are uniform random variables $\in \{0, 1\}$.
The CDF of T is defined as,

$$F_T(t) = \Pr(T \le t) \tag{4.3}$$

Now from (4.2) we can write,

$$F_T(t) = \Pr(U1 + U2 \le t) \tag{4.4}$$

Now when $t > 2$ the $\Pr(U1 + U2 \le t) = 1$ because $\forall u1 \in U1, u2 \in U2, u1 + u2 < 2$.

$$\implies F_T(t) = 1\,(t > 2) \tag{4.5}$$

And for $t < 0$ the $\Pr(U1 + U2 \le t) = 0$ because $\forall u1 \in U1, u2 \in U2, u1 + u2 > 0$.

$$\implies F_T(t) = 0\,(t < 0) \tag{4.6}$$

Now we will see the case when $t \in \{0, 2\}$.We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U1 = x$ where $x \in (0, t)$. So in this case CDF will be,

$$F_T(t) = \Pr(U1 + U2 \le t) \quad = \Pr(U1 = x, U2 \le t - x) = \tag{4.7}$$

Now note that$x$ is a variable and varies in between$(0, t)$, so we have to take integral over x to evaluate the $\Pr(U1 = x)$,

$$F_T(t) = \int_0^t f_U(x)\Pr(U2 \le t - x)\,dx \quad = \int_0^t f_U(x) F_U(\tag{4.8}$$

Case - 1 For $t\{0, 1\}$,

4.3 Find the PDF of T.

4.4 Find the theoretical expressions for the PDF and CDF of T.

4.5 Verify your results through a plot. **Solution:** Download the below files,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri.dat
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q4/tri_cdf_plot.py
```

and the run the following command

```
python3 tri_cdf_plot.py
```

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

5.3 Plot Y using a scatter plot.

5.4 Guess how to estimate X from Y.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.6 Find P_e assuming that X has equiprobable symbols.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.4}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.7.2 and 7.7.2 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.