

Random Numbers

Mannem Charan - AI21BTECH11019

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Abstract—This manual provides solutions for random numbers assignment.

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files.

```
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/uniform.c
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

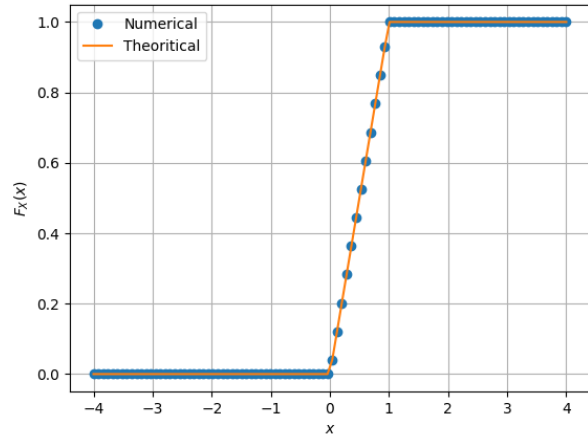


Fig. 1.2: The CDF of U

Solution: Use the following code to plot Fig. 1.2

```
wget https://github.com/Charanyash/Random-
Numbers-/blob/main/codes/Q1/
uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given that, random variable U is uniformly distributed in interval $(0, 1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \quad (1.2)$$

$$= 1 \quad (1.3)$$

So for $x \in (0, 1)$, the probability distribution

function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x) dx \quad (1.4)$$

$$= \int_0^x 1 dx \quad (1.5)$$

$$= x \quad (1.6)$$

For $x < 0$,

$$F_U(x) = \Pr(U \leq x) = 0 (\because f_U(x) = 0) \quad (1.7)$$

And for $x > 1$

$$F_U(x) = \Pr(U \leq x) = 1 (\because f_U(x) = 0) \quad (1.8)$$

Overall,

$$F_U(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.9)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.10)$$

Write a C program to find the mean and variance of U .

Solution: Download the following code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/mean_var_uniform.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c -lm
./a.out
```

We will get output as,

$$\text{mean} = 0.500007 \quad (1.11)$$

$$\text{variance} = 0.083301 \quad (1.12)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.13)$$

Solution: Already we know that,

$$F_x(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

So, the given integral solves down to,

$$E[U^k] = \int_0^1 x^k dx \quad (1.14)$$

Since $F_x(x)$ is constant w.r.t x for $x \geq 1$ and $x \leq 0$. For mean,

$$E[U] = \int_0^1 x dx \quad (1.15)$$

$$= \left\{ \frac{x^2}{2} \right\}_0^1 \quad (1.16)$$

$$= 0.5 \quad (1.17)$$

Now for variance, we know that

$$\text{Var}[U] = E[U - E[U]]^2 \quad (1.18)$$

$$= E[U^2 + E[U]^2 - 2E[U]U] \quad (1.19)$$

$$(1.20)$$

Since expected value is a linear operator, we can write

$$= E[U^2] + E[U]^2 - 2E[U]U \quad (1.21)$$

$$= E[U^2] - E[U]^2 \quad (1.22)$$

To get variance we will find,

$$E[U^2] = \int_0^1 x^2 dx \quad (1.23)$$

$$= \left\{ x^3/3 \right\}_0^1 \quad (1.24)$$

$$= \frac{1}{3} \quad (1.25)$$

Therefore,

$$\text{Var}[U] = \frac{1}{3} - \left(\frac{1}{2} \right)^2 \quad (1.26)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.27)$$

$$= \frac{1}{12} \quad (1.28)$$

$$= 0.0833 \quad (1.29)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the code below

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the below code,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.2.

Properties Of CDF:

- CDF is monotonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as,
 $Q(x) = \Pr(X > x)$
- The CDF, $F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/gaussian_pdf_plot.py
```

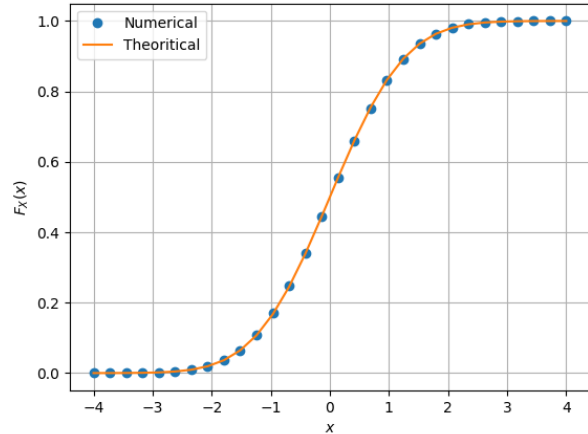


Fig. 2.2: The CDF of X

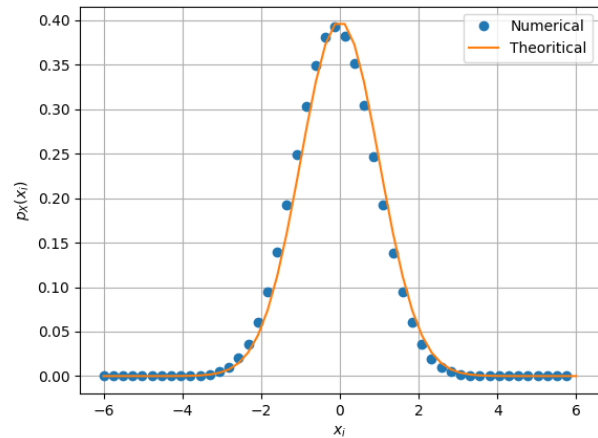


Fig. 2.3: The PDF of X

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

- $\forall x \in \mathbb{R}, p_X(x) \geq 0$
- PDF is symmetric about the mean, in this case at $x = 0$
- The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/mean_var_gauss.c
```

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q2/coeffs.h

Then run the following command in linux terminal

```
cc mean_var_gauss.c -lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x] dx \quad (2.5)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function. We can write,

$$E[x] = 0 \quad (2.7)$$

Consider the following expression,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X[x] dx \quad (2.8)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

To solve the above integral, we will use integration by parts, i.e.,

$$\int u v dx = u \int v dx - \int u' \left(\int v dx \right) dx \quad (2.10)$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}} \right) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx \right) \right) \quad (2.12)$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \quad (2.13)$$

$$dt = x dx \quad (2.14)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \quad (2.15)$$

$$= -\exp(-t) + c \quad (2.16)$$

$$\Rightarrow = -\exp\left(-\frac{x^2}{2}\right) + c \quad (2.17)$$

Using (2.17), we can write

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \right) \quad (2.18)$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = 1 \quad (2.19)$$

Now putting limits and using (2.7), (2.19),

$$E[x^2] = 1 \quad (2.20)$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \quad (2.21)$$

$$= 1 \quad (2.22)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the C code from the link below to generate samples of V from uni.dat file

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q3/V.c

Run the following command,

```
cc V.c -lm
./a.out
```

Then download the below python file to get CDF

wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q3/V_cdf_plot.py

Then run the following command

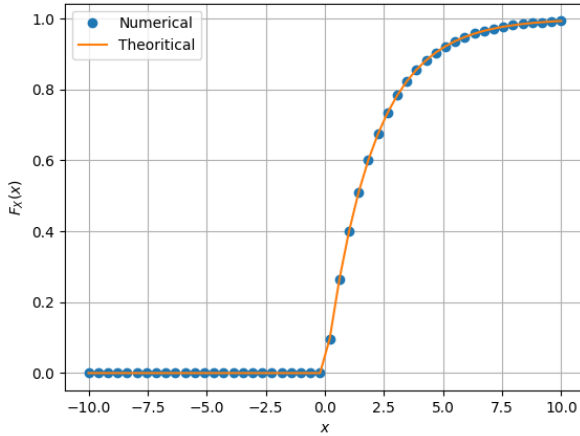


Fig. 3.1: The CDF of V

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: Given

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$F_V(x) = \Pr(V \leq x) \quad (3.3)$$

we will use (3.3)

$$F_V(x) = \Pr(-2 \ln(1 - U) \leq x) \quad (3.4)$$

$$= \Pr\left(\ln(1 - U) \geq \frac{-x}{2}\right) \quad (3.5)$$

$$= \Pr\left(1 - U \geq \exp\left(\frac{-x}{2}\right)\right) \quad (3.6)$$

$$= \Pr\left(U \leq 1 - \exp\left(\frac{-x}{2}\right)\right) \quad (3.7)$$

$$= F_U\left(1 - \exp\left(\frac{-x}{2}\right)\right) \quad (3.8)$$

For $x > 0$, $1 - e^{\frac{-x}{2}} < 1$ and $x < 0$, $1 - e^{\frac{-x}{2}} < 0$

$$F_V(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \quad (3.9)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the below code,

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/codes/
Q4/coeffs.h
```

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/codes/
Q4/triangular.c
```

and run the following command,

```
cc triangular.c -lm
./a.out
```

You will get required generated random numbers in tri.dat file.

4.2 Find the CDF of T .

Solution: Given that,

$$T = U_1 + U_2 \quad (4.2)$$

where, U_1, U_2 are uniform random variables $\in [0, 1]$.

The CDF of T is defined as,

$$F_T(t) = \Pr(T \leq t) \quad (4.3)$$

Now from (4.2) we can write,

$$F_T(t) = \Pr(U_1 + U_2 \leq t) \quad (4.4)$$

Now when $t > 2$ the $\Pr(U_1 + U_2 \leq t) = 1$ because $\forall u_1 \in U_1, u_2 \in U_2, u_1 + u_2 < 2$.

$$\Rightarrow F_T(t) = 1 (t > 2) \quad (4.5)$$

And for $t < 0$ the $\Pr(U_1 + U_2 \leq t) = 0$ because $\forall u_1 \in U_1, u_2 \in U_2, u_1 + u_2 > 0$.

$$\Rightarrow F_T(t) = 0 (t < 0) \quad (4.6)$$

Now we will see the case when $t \in \{0, 2\}$. We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U_1 = x$ where $x \in (0, t)$. So in this case CDF will be,

$$F_T(t) = \Pr(U_1 + U_2 \leq t) = \Pr(U_1 = x, U_2 \leq t - x) \quad (4.7)$$

Now note that x is a variable and varies in between $(0, t)$, so we have to take integral over x to evaluate the $\Pr(U_1 = x)$,

$$F_T(t) = \int_0^t f_U(x) \Pr(U_2 \leq t - x) dx = \int_0^t f_U(x) F_U(t - x) dx \quad (4.8)$$

Case - 1 For $t \in \{0, 1\}$,

4.3 Find the PDF of T .

4.4 Find the theoretical expressions for the PDF and CDF of T .

- 4.5 Verify your results through a plot. **Solution:**
Download the below files,

```
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri.dat
wget https://github.com/Charanyash/
Random-Numbers-/blob/main/
codes/Q4/tri_cdf_plot.py
```

and then run the following command

```
python3 tri_cdf_plot.py
```

5 MAXIMUM LIKELIHOOD

- 5.1 Generate equiprobable $X \in \{1, -1\}$.
5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

- 5.3 Plot Y using a scatter plot.
5.4 Guess how to estimate X from Y .
5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

- 5.6 Find P_e assuming that X has equiprobable symbols.
5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
5.8 Now, consider a threshold δ while estimating X from Y . Find the value of δ that maximizes the theoretical P_e .
5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.4)$$

- 5.10 Repeat the above exercise using the MAP criterion.

6 GAUSSIAN TO OTHER

- 6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

- 6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

- 6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.3)$$

7 CONDITIONAL PROBABILITY

- 7.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

- 7.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$.
7.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

- 7.4 Plot P_e in problems 7.7.2 and 7.7.2 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

- 8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

- 8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .
8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

- 8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.