Random Numbers

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Contents

***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/uniform.c
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Use the following code to plot Fig. 1.2

```
wget https://github.com/Charanyash/Random−Numbers−/blob/main/codes/uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

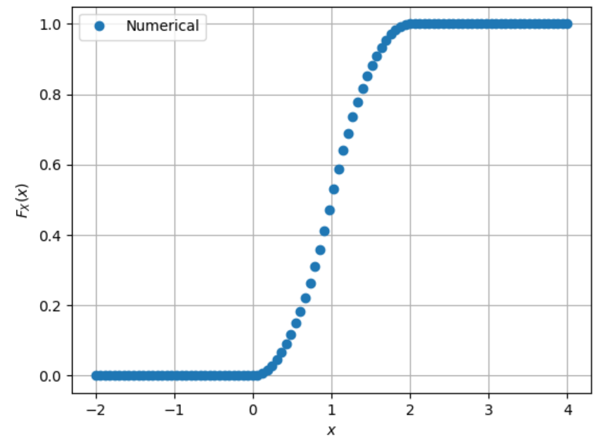

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given that, random variable U is uniformly distributed in interval $(0, 1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \tag{1.2}$$
$$= 1 \tag{1.3}$$

So for $x \in (0, 1)$,the probability distribution function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x)\,dx \tag{1.4}$$
$$= \int_0^x 1dx \tag{1.5}$$
$$= x \tag{1.6}$$

For $x < 0$,

$$F_U(x) = \Pr(U \leq x) \quad = 0\,(\because f_U = 0) \tag{1.7}$$

And for $x > 1$

$$F_U(x) = \Pr(U \leq x) \quad = 1\,(\because f_U = 0) \tag{1.8}$$

Overall,

$$F_U(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.
Solution: Download the following code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    mean_var_uniform.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c −lm
./a.out
```

We will get output as,

$$mean = 0.500007 \tag{1.11}$$
$$variance = 0.083301 \tag{1.12}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.13}$$

Solution: Already we know that,

$$F_x(x) = \begin{cases} 0 & , x \leq 0 \\ x & , 0 < x < 1 \\ 1 & , x \geq 1 \end{cases}$$

So, the given integral solves down to,

$$E\left[U^k\right] = \int_0^1 x^k dx \tag{1.14}$$

Since $F_x(x)$ is constant w.r.t x for $x \geq 1$ amd $x \leq 0$.
For mean,

$$E[U] = \int_0^1 x dx \tag{1.15}$$
$$= \left\{\frac{x^2}{2}\right\}_0^1 \tag{1.16}$$
$$= 0.5 \tag{1.17}$$

Now for variance,we know that

$$Var[U] = E[U - E[U]]^2 \tag{1.18}$$
$$= E\left[U^2 + E[U]^2 - 2E[U]U\right] \tag{1.19}$$
$$\tag{1.20}$$

Since expected value is a linear operator, we can write

$$= E\left[U^2\right] + E[U]^2 - 2E[U]^2 \tag{1.21}$$
$$= E\left[U^2\right] - E[U]^2 \tag{1.22}$$

To get variance we will find,

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.23}$$
$$= \left\{x^3/3\right\}_0^1 \tag{1.24}$$
$$= \frac{1}{3} \tag{1.25}$$

Therefore,

$$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.26}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.27}$$
$$= \frac{1}{12} \tag{1.28}$$
$$= 0.0833 \tag{1.29}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: Download the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/gaussian.c
```

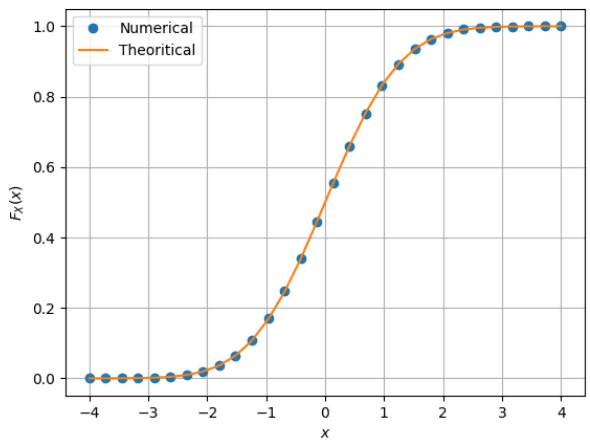

Fig. 2.2: The CDF of X

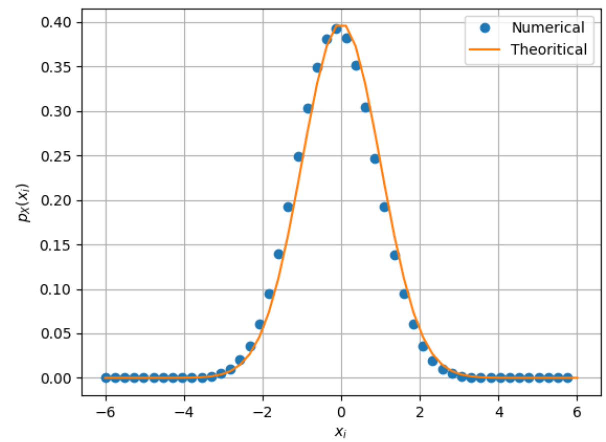

Fig. 2.3: The PDF of X

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: Download the below code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.2.
Properties Of CDF:

- CDF is montonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as, $Q(x) = \Pr(X > x)$
- The CDF,$F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    gaussian_pdf_plot.py
```

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) PDF is symmetric about the mean, in this case at $x = 0$
c) The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/
    mean_var_gauss.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/coeffs.h
```

Then run the following command in linux terminal

```
cc mean_var_gauss.c −lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x]\, dx \quad (2.5)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.We can write,

$$E[x] = 0 \quad (2.7)$$

Consider the following expression,

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p_X[x]\, dx \quad (2.8)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

To solve the above integral, we will use integration by parts, i.e,

$$\int uv dx = u \int v dx - \int u' \left(\int v dx \right) dx \quad (2.10)$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}} \right) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx \right) \right) \quad (2.12)$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \quad (2.13)$$

$$dt = x dx \quad (2.14)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t)\, dt \quad (2.15)$$

$$= -\exp(-t) + c \quad (2.16)$$

$$\implies = -\exp\left(-\frac{x^2}{2}\right) + c \quad (2.17)$$

Using (2.17),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{\frac{-x^2}{2}} dx \right) \quad (2.18)$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} = 1 \quad (2.19)$$

Now putting limits and using (2.7),(2.19),

$$E\left[x^2\right] = 1 \quad (2.20)$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \quad (2.21)$$

$$= 1 \quad (2.22)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.
Solution: Download the C code from the link below to generate samples of V from uni.dat file

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/V.c
```

Run the following command,

```
cc V.c -lm
./a.out
```

Then download the below python file to get CDF

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/V_cdf_plot.py
```

Then run the following command

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Given

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$F_V(x) = \Pr(V \le x) \quad (3.3)$$

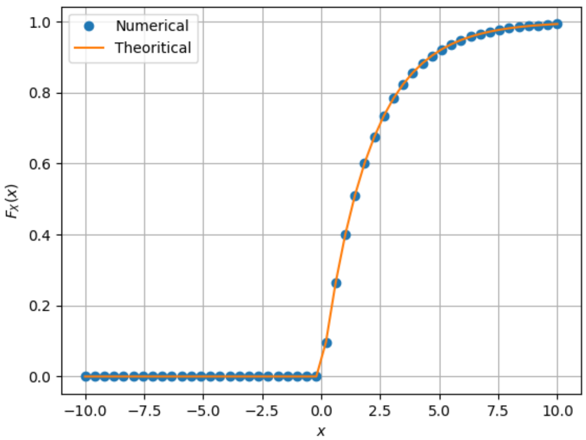

Fig. 3.1: The CDF of V

we will use (3.3)

$$\begin{aligned} F_V(x) &= \Pr(-2\ln(1-U) \le x) && (3.4)\\ &= \Pr\left(\ln(1-U) \ge \frac{-x}{2}\right) && (3.5)\\ &= \Pr\left(1-U \ge \exp\left(\frac{-x}{2}\right)\right) && (3.6)\\ &= \Pr\left(U \le 1-\exp\left(\frac{-x}{2}\right)\right) && (3.7)\\ &= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) && (3.8) \end{aligned}$$

$$F_V(x) = \begin{cases} 0 & x \le 0 \\ 1-\exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \tag{3.9}$$