Random Numbers

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***Abstract*—This manual provides solutions for random numbers assignment.**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files.

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/uniform.c
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/coeffs.h
```

Then use the following commands in linux terminal,

```
cc uniform.c -lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

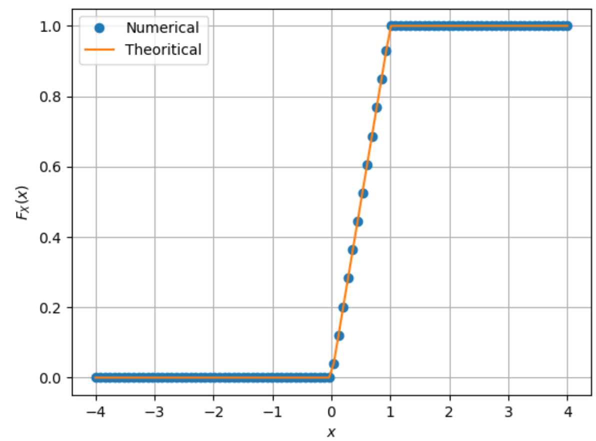

Fig. 1.1: The CDF of U

Solution: Use the following code to plot Fig. 1.1

```
wget https://github.com/Charanyash/Random-Numbers-/blob/main/codes/Q1/uniform_cdf_plot.py
```

Run the following command in the linux terminal,

```
python3 uniform_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given that, random variable U is uniformly distributed in interval $(0,1)$. So we can write that, the probability density function

$$f_U(x) = \frac{1}{1-0} \tag{1.2}$$
$$= 1 \tag{1.3}$$

So for $x \in (0,1)$,the probability distribution function $F_U(x)$ can be calculated as,

$$F_U(x) = \int_0^x f_x(x)\,dx \tag{1.4}$$
$$= \int_0^x 1dx \tag{1.5}$$
$$= x \tag{1.6}$$

For $x < 0$,

$$F_U(x) = \Pr(U \leq x) \quad = 0\,(\because f_U(x) = 0) \tag{1.7}$$

And for $x > 1$

$$F_U(x) = \Pr(U \leq x) \quad = 1\,(\because f_U(x) = 0) \tag{1.8}$$

Overall,

$$F_U(x) = \begin{cases} 0 & ,x \leq 0 \\ x & ,0 < x < 1 \\ 1 & ,x \geq 1 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.
Solution: Download the following code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q1/
    mean_var_uniform.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q1/coeffs.h
```

Run the following command,

```
cc mean_var_uniform.c −lm
./a.out
```

We will get output as,

$$mean = 0.500007 \tag{1.11}$$
$$variance = 0.083301 \tag{1.12}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.13}$$

Solution: Already we know that,

$$F_U(x) = \begin{cases} 0 & ,x \leq 0 \\ x & ,0 < x < 1 \\ 1 & ,x \geq 1 \end{cases}$$

So, the given integral solves down to,

$$E\left[U^k\right] = \int_0^1 x^k dx \tag{1.14}$$

Since $F_U(x)$ is constant w.r.t x for $x \geq 1$ amd $x \leq 0$. For mean,

$$E[U] = \int_0^1 xdx \tag{1.15}$$
$$= \left\{\frac{x^2}{2}\right\}_0^1 \tag{1.16}$$
$$= 0.5 \tag{1.17}$$

Now for variance,we know that

$$Var[U] = E[U - E[U]]^2 \tag{1.18}$$
$$= E\left[U^2 + E[U]^2 - 2E[U]U\right] \tag{1.19}$$
$$\tag{1.20}$$

Since expected value is a linear operator, we can write

$$= E\left[U^2\right] + E[U]^2 - 2E[U]^2 \tag{1.21}$$
$$= E\left[U^2\right] - E[U]^2 \tag{1.22}$$

To get variance we will find,

$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.23}$$
$$= \left\{x^3/3\right\}_0^1 \tag{1.24}$$
$$= \frac{1}{3} \tag{1.25}$$

Therefore,

$$Var[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.26}$$
$$= \frac{1}{3} - \frac{1}{4} \tag{1.27}$$
$$= \frac{1}{12} \tag{1.28}$$
$$= 0.0833 \tag{1.29}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/gaussian.c
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/coeffs.h
```

Run the following command

```
cc gaussian.c
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the below code,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    gaussian_cdf_plot.py
```

Run the following command to get CDF plot,

```
python3 gaussian_cdf_plot.py
```

The CDF of X is plotted in Fig. 2.1.

Properties Of CDF:

- CDF is montonically increasing from $-\infty < x < \infty$
- Let us define the $Q(x)$ function as, $Q(x) = \Pr(X > x)$
- The CDF,$F_X(x) = 1 - Q(x) = Q(-x)$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    gaussian_pdf_plot.py
```

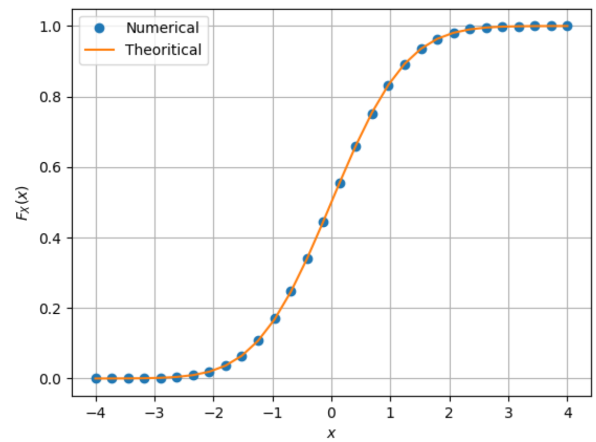

Fig. 2.1: The CDF of X

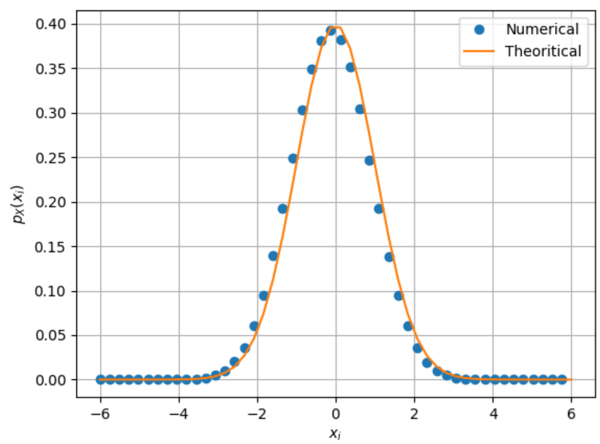

Fig. 2.2: The PDF of X

Run the following command,

```
python3 gaussian_pdf_plot.py
```

Properties of PDF:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) PDF is symmetric about the mean, in this case at $x = 0$
c) The maxima of the curve is observed at mean of distribution.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the C code from the links below,

```
wget https://github.com/Charanyash/Random−
    Numbers−/blob/main/codes/Q2/
    mean_var_gauss.c
```

```
wget https://github.com/Charanyash/Random-
    Numbers-/blob/main/codes/Q2/coeffs.h
```

Then run the following command in linux terminal

```
cc mean_var_gauss.c -lm
./a.out
```

we will get $mean = 0.000326, variance = 1.000906$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: Given that,

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

We know,

$$E[x] = \int_{-\infty}^{\infty} x p_X[x]\, dx \quad (2.5)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Since $\frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right)$ is an odd function.We can write,

$$E[x] = 0 \quad (2.7)$$

Consider the following expression,

$$E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p_X[x]\, dx \quad (2.8)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

To solve the above integral, we will use integration by parts, i.e,

$$\int uv dx = u \int v dx - \int u' \left(\int v dx\right) dx \quad (2.10)$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x e^{-\frac{x^2}{2}}\right) dx \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int x e^{-\frac{x^2}{2}} dx - \int \left(\int x e^{-\frac{x^2}{2}} dx\right)\right) \quad (2.12)$$

For the integral $\int x \exp\left(-\frac{x^2}{2}\right) dx$ let us take,

$$t = \frac{x^2}{2} \quad (2.13)$$

$$dt = x dx \quad (2.14)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t)\, dt \quad (2.15)$$

$$= -\exp(-t) + c \quad (2.16)$$

$$\implies = -\exp\left(-\frac{x^2}{2}\right) + c \quad (2.17)$$

Using (2.17),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{x^2}{2}} + \int e^{\frac{-x^2}{2}} dx\right) \quad (2.18)$$

And we know that,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} = 1 \quad (2.19)$$

Now putting limits and using (2.7),(2.19),

$$E\left[x^2\right] = 1 \quad (2.20)$$

Using (1.22) we can write,

$$Var[x] = 1 - 0 \quad (2.21)$$

$$= 1 \quad (2.22)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the C code from the link below to generate samples of V from uni.dat file

```
wget https://github.com/Charanyash/Random-
    Numbers-/blob/main/codes/Q3/V.c
```

Run the following command,

```
cc V.c -lm
./a.out
```

Then download the below python file to get CDF

```
wget https://github.com/Charanyash/Random-
    Numbers-/blob/main/codes/Q3/
    V_cdf_plot.py
```

Then run the following command

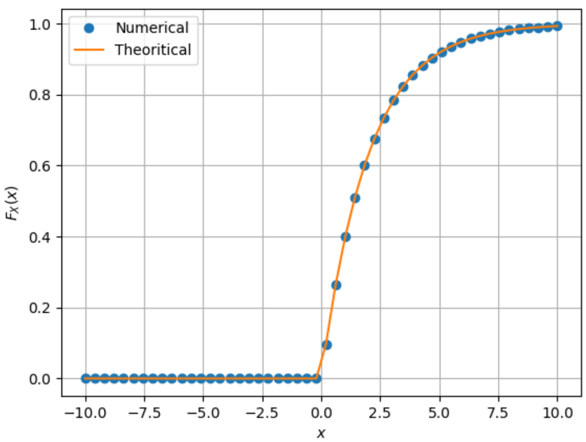

Fig. 3.1: The CDF of V

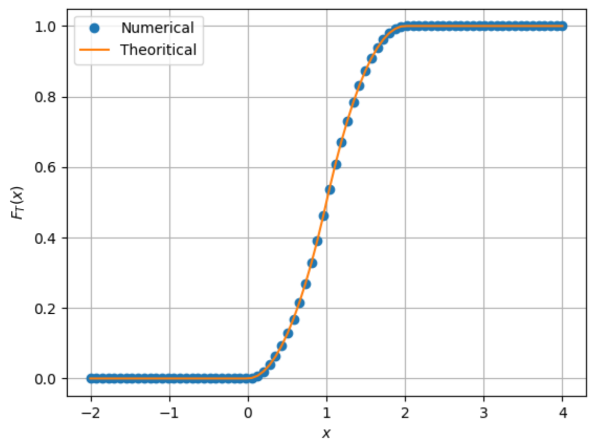

Fig. 4.1: The CDF of T

```
python3 V_cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Given

$$V = -2\ln(1-U) \tag{3.2}$$
$$F_V(x) = \Pr(V \le x) \tag{3.3}$$

we will use (3.3)

$$\begin{aligned} F_V(x) &= \Pr(-2\ln(1-U) \le x) && (3.4)\\ &= \Pr\left(\ln(1-U) \ge \frac{-x}{2}\right) && (3.5)\\ &= \Pr\left(1-U \ge \exp\left(\frac{-x}{2}\right)\right) && (3.6)\\ &= \Pr\left(U \le 1-\exp\left(\frac{-x}{2}\right)\right) && (3.7)\\ &= F_U\left(1-\exp\left(\frac{-x}{2}\right)\right) && (3.8) \end{aligned}$$

For$x > 0, 1-e^{\frac{-x}{2}} < 1$ and $x < 0, 1-e^{\frac{-x}{2}} < 0$

$$F_V(x) = \begin{cases} 0 & x \le 0 \\ 1-\exp\left(\frac{-x}{2}\right) & x > 0 \end{cases} \tag{3.9}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the below code,

```
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/codes/
    Q4/coeffs.h
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/codes/
    Q4/triangular.c
```

and run the following command,

```
cc triangular.c -lm
./a.out
```

You will get required generated random numbers in tri.dat file.

4.2 Find the CDF of T.
Solution: Download the below files,

```
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/
    codes/Q4/tri.dat
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/
    codes/Q4/tri_cdf_plot.py
```

Run the following command,

```
python3 tri_cdf_plot.py
```

4.3 Find the PDF of T.
Solution: Download the below files,

```
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/
    codes/Q4/tri.dat
wget https://github.com/Charanyash/
    Random-Numbers-/blob/main/
    codes/Q4/tri_pdf_plot.py
```

Run the following command,

```
python3 tri_pdf_plot.py
```

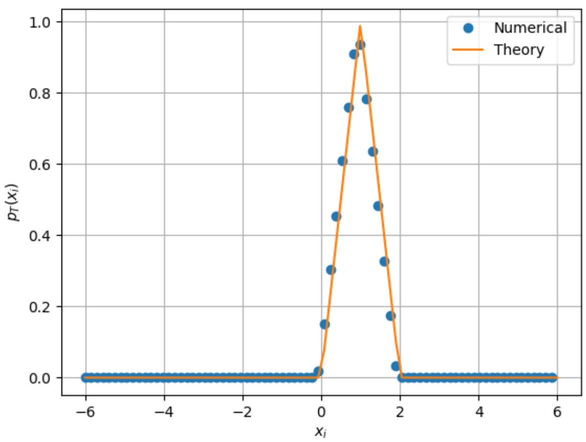

Fig. 4.2: The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Given that,

$$T = U1 + U2 \tag{4.2}$$

where$U1, U2$ are uniform random variables $\in (0, 1)$.
Calculation of CDF The CDF of T is defined as,

$$F_T(t) = \Pr(T \leq t) \tag{4.3}$$

Now from (4.2) we can write,

$$F_T(t) = \Pr(U1 + U2 \leq t) \tag{4.4}$$

Case -1 : For $t > 2$.
The $\Pr(U1 + U2 \leq t) = 1$, because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 < 2$,

$$\implies F_T(t) = 1 \tag{4.5}$$

Case -2 : For $t < 0$.
The $\Pr(U1 + U2 \leq t) = 0$ because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 > 0$,

$$\implies F_T(t) = 0 \tag{4.6}$$

Case - 3: For, $t \in (0, 2)$.
We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U1 = x$ where $x \in (0, t)$.
So in this case CDF will be,

$$F_T(t) = \Pr(U1 + U2 \leq t) \tag{4.7}$$
$$= \Pr(U1 = x, U2 \leq t - x) \tag{4.8}$$
$$= \Pr(U1 = x)\Pr(U2 \leq t - x) \tag{4.9}$$

Since U1,U2 are i.i.d.
Now note that x is a variable and varies in $(0, t)$, so we have to take integral over x to evaluate the $\Pr(U1 = x)$,

$$F_T(t) = \int_0^t f_U(x)\Pr(U2 \leq t - x)\,dx \tag{4.10}$$
$$= \int_0^t f_U(x)F_U(t - x)\,dx \tag{4.11}$$

Case - 1 For $t \in (0, 1)$,we know $f_U(x) = 1$ so,

$$F_T(t) = \int_0^t 1.F_U(t - x)\,dx \tag{4.12}$$

As, $x < t$, $0 < t - x < t < 1$ using (1),we can write

$$F_T(t) = \int_0^t (t - x)\,dx \tag{4.13}$$
$$= \left\{tx - \frac{x^2}{2}\right\}_0^t \tag{4.14}$$
$$= \frac{t^2}{2} \tag{4.15}$$

Case -2 For $t \in (1, 2)$, we know $f_U(x) = 0$ at $x > 1$,so the integral solves down to,

$$F_T(t) = \int_0^1 f_U(x)F_U(t - x)\,dx \tag{4.16}$$
$$= \int_0^1 1.F_U(t - x)\,dx \tag{4.17}$$
$$\tag{4.18}$$

To solve the above integral we will use integration by substitution,

$$k = t - x \tag{4.19}$$
$$dk = -dx \tag{4.20}$$
$$F_T(t) = \int_t^{t-1} F_U(k)(-dk) \tag{4.21}$$
$$= \int_{t-1}^t F_U(k)\,dk \tag{4.22}$$

As $1 \leq t \leq 2, 0 \leq t - 1 \leq 1$ we will break integral at 1 because $F_U(k)$ changes at 1.Using (1),

$$F_T(t) = \int_{t-1}^{1} F_U(k)\,dk + \int_{1}^{t} F_U(k)\,dk \tag{4.23}$$

$$= \int_{t-1}^{1} kdk + \int_{1}^{t} 1dk \tag{4.24}$$

$$= \left\{\frac{k^2}{2}\right\}_{t-1}^{1} + t - 1 \tag{4.25}$$

$$= \frac{1}{2} - \left(\frac{(t-1)^2}{2}\right) + t - 1 \tag{4.26}$$

$$= 2t - \frac{t^2}{2} - 1 \tag{4.27}$$

Overall we can write the CDF of $F_T(x)$ as,

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \le x \le 1 \\ 2t - \frac{t^2}{2} - 1 & , 1 \le x \le 2 \\ 1 & , x > 2 \end{cases} \tag{4.28}$$

Calculation of PDF Now we will find PDF of T,
As,

$$T = U1 + U2 \tag{4.29}$$

We will use method of convolution to get PDF of T as $U1$ and $U2$ are i.i.d.

$$f_T(t) = \int_{-\infty}^{\infty} f_{U1}(x)\, f_{U2}(t-x)\,dx \tag{4.30}$$

Since $U1, U2$ are of same distribution we can write,

$$f_{U1}(x) = f_{U2}(x) = f_U(x) \tag{4.31}$$

$$\implies f_T(t) = \int_{-\infty}^{\infty} f_U(x)\, f_U(t-x)\,dx \tag{4.32}$$

From the PDF of U,we can write

$$f_T(t) = \int_{0}^{1} f_U(x)\, f_U(t-x)\,dx \tag{4.33}$$

$$= \int_{0}^{1} 1.f_U(t-x)\,dx \tag{4.34}$$

$$\tag{4.35}$$

we will solve the above integral using substitution.

$$z = t - x \tag{4.36}$$

$$dz = -dx \tag{4.37}$$

$$\implies f_T(t) = \int_{t}^{t-1} f_U(z)(-dz) \tag{4.38}$$

$$= \int_{t-1}^{t} f_U(z)\,dz \tag{4.39}$$

Case -1 For $t < 0$ as $z < t$,the PDF $f_U(z) = 0$. So,

$$f_T(t) = 0 \tag{4.40}$$

Case -2 For $0 \le t \le 1$, we will break the integral at $z = 0$, since $f_U(z)$ changes at 0.

$$f_T(t) = \int_{t-1}^{0} f_U(z)\,dz + \int_{0}^{t} f_U(z)\,dz \tag{4.41}$$

$$= 0 + \int_{0}^{t} 1dz \tag{4.42}$$

$$= t \tag{4.43}$$

Case-3 Similarly for $1 \le t \le 2$, we will break the integral at $z = 1$,

$$f_T(t) = \int_{t-1}^{1} f_U(z)\,dz + \int_{1}^{t} f_U(z) \tag{4.44}$$

$$= \int_{t-1}^{1} 1.dz + 0 \tag{4.45}$$

$$= 2 - t \tag{4.46}$$

Case-4 For $t > 2$,as $z > t - 1 > 1$,the PDF $f_U(z) = 0$. So,

$$f_T(z) = 0 \tag{4.47}$$

Overall, the PDF of T will be,

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x \le 1 \\ 2 - x & , 1 \le x \le 2 \\ 0 & , x > 2 \end{cases} \tag{4.48}$$

4.5 Verify your results through a plot.
Solution: This is already done in 4.1 ,4.2.

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: The generating X or bernoulie random variable (X) is done by using uni.dat file. Download the below files

```
wget https://github.com/
    Charanyash/Random−
    Numbers−/blob/main/codes/
    Q5/coeffs.h
wget https://github.com/
    Charanyash/Random−
    Numbers−/blob/main/codes/
    Q5/bernoulie.c
```

Run the following command

```
cc bernoulie.c −lm
./a.out
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

Solution: To generate distribution of Y random variable we will need previously generated bernoulie distribution and gaussian distribution.Download the below files

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/coeffs.h
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.c
```

Then run the following command,

```
cc Y.c −lm
./a.out
```

5.3 Plot Y using a scatter plot.

Solution: Download the below files

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.py
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.dat
```

Then run the following command,

```
python3 Y.py
```

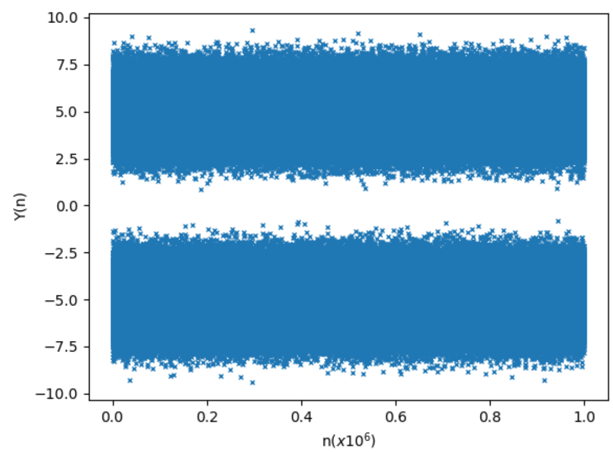

Fig. 5.1: The scatter plot of Y

5.4 Guess how to estimate X from Y.

Solution: When $Y > 0$, we can more probably say that $X = 1$ as X can take values from $[-1, 1]$.As A increases the signal contribution will increase compared to noise.The scatter plot will not be intermixed as A increases. So in this case, the scatter plot of Y is seperated with decision boundary as 0. So we can more probably say that,

$$X = \begin{cases} 1 & , Y > 0 \\ -1 & , Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: The $\hat{X}$ is defined as,

$$\hat{X} = \begin{cases} 1 & , Y > 0 \\ 0 & , Y \leq 0 \end{cases} \tag{5.5}$$

The error probability, when the actual signal is $X = 1$ but transmitted as $\hat{X} = -1$ is,

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) && (5.6) \\ &= \Pr\left(Y \leq 0 | X = 1\right) && (5.7) \\ &= \Pr\left(AX + N \leq 0 | X = 1\right) && (5.8) \\ &= \Pr\left(A + N \leq 0\right) && (5.9) \\ &= \Pr\left(N \leq -A\right) && (5.10) \\ &= F_N\left(-A\right) && (5.11) \\ &= 1 - Q\left(-A\right) && (5.12) \\ &= 2.866515718791946e - 07 && (5.13) \end{aligned}$$

And for the case when actual signal is $X = -1$ but transmitted as $\hat{X} = 1$ the error probability will be,

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \quad (5.14)$$
$$= \Pr(Y > 0|X = -1) \quad (5.15)$$
$$= \Pr(AX + N > 0|X = 1) \quad (5.16)$$
$$= \Pr(N - A > 0) \quad (5.17)$$
$$= \Pr(N > A) \quad (5.18)$$
$$= 1 - F_N(A) \quad (5.19)$$
$$= Q(A) \quad (5.20)$$
$$= 2.866515719235352e - 07 \quad (5.21)$$

The above calculations are coded in below python file,

```
wget https://github.com/Charanyash/
    Random−Numbers−/tree/main/
    codes/Q5/5.5.py
```

Run the following command

```
python3 5.5.py
```

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: Given that X has equiprobable symbols so,

$$\Pr(X = 1) = \frac{1}{2} \quad (5.22)$$
$$\Pr(X = -1) = \frac{1}{2} \quad (5.23)$$

From total probability theorem,

$$P_e = \Pr(e|1)\Pr(X = -1) + \Pr(e|0)\Pr(X = 1) \quad (5.24)$$
$$= \frac{1}{2}(\Pr(e|1) + \Pr(e|0)) \quad (5.25)$$

From (5.13),(5.21)

$$\Pr(e) = 2.866515719013649e - 07 \quad (5.26)$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: We know,

$$P_e = P_{e|1}\Pr(X = -1) + P_{e|0}\Pr(X = 1) \quad (5.27)$$
$$= \frac{1}{2}(1 - Q(-A)) + \frac{1}{2}(Q(A)) \quad (5.28)$$

The above mentioned is the theoritical expression of P_e w.r.t to A, it is plotted in

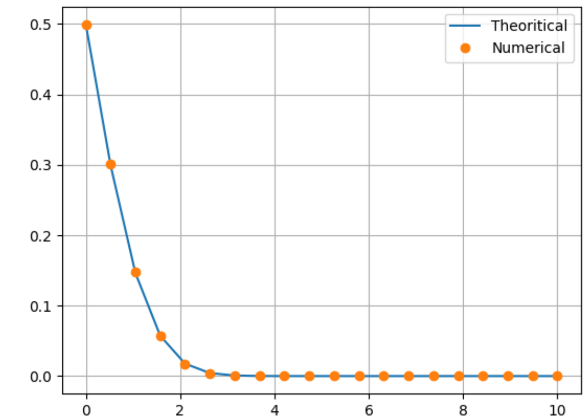

Fig. 5.2: P_e vs A

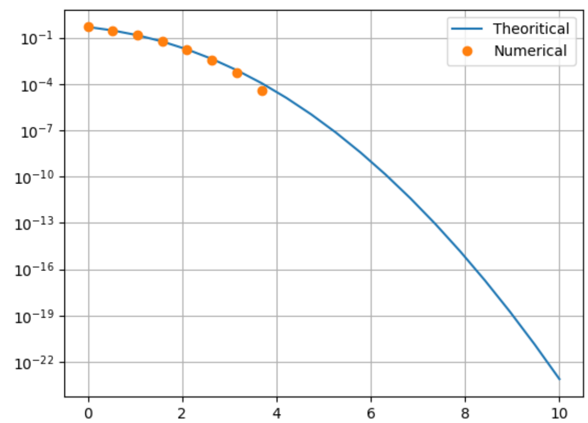

Fig. 5.3: P_e vs A(semi-log)

rectangular axes and semi-log y axes using the below python codes,

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/P_e_A.py
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/P_e_A_semilog.py
```

Then the following commands

```
python3 P_e_A.py
python3 P_e_A_semilog.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.29}$$
$$= \Pr\left(Y \le \delta|X = 1\right) \tag{5.30}$$
$$= \Pr\left(AX + N \le \delta|X = 1\right) \tag{5.31}$$
$$= \Pr\left(A + N \le \delta\right) \tag{5.32}$$
$$= \Pr\left(N \le \delta - A\right) \tag{5.33}$$
$$= F_N\left(\delta - A\right) \tag{5.34}$$
$$= 1 - Q\left(\delta - A\right) \tag{5.35}$$

And,

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.36}$$
$$= \Pr\left(Y > \delta|X = -1\right) \tag{5.37}$$
$$= \Pr\left(AX + N > \delta|X = -1\right) \tag{5.38}$$
$$= \Pr\left(N - A > \delta\right) \tag{5.39}$$
$$= \Pr\left(N > \delta + A\right) \tag{5.40}$$
$$= Q\left(\delta + A\right) \tag{5.41}$$

So we can write,

$$P_e = P_{e|1}\Pr\left(X = -1\right) + P_{e|0}\Pr\left(X = 1\right) \tag{5.42}$$
$$= \frac{1}{2}\left(1 - Q\left(\delta - A\right) + Q\left(\delta + A\right)\right) \tag{5.43}$$

Now to maximize the P_e, we will differentiate the expression w.r.t δ and equate it to 0.

$$\frac{dP_e}{d\delta} = \frac{d}{d\delta}\left(1 - Q\left(\delta - A\right) + Q\left(\delta + A\right)\right) = 0 \tag{5.44}$$
$$\implies \frac{d}{d\delta}\left(F_N\left(\delta - A\right) + 1 - F_N\left(\delta + A\right)\right) = 0 \tag{5.45}$$
$$\implies p_N\left(\delta - A\right) - p_N\left(\delta + A\right) = 0 \tag{5.46}$$
$$\exp\left(-\frac{(\delta - A)^2}{2}\right) - \exp\left(-\frac{(\delta + A)^2}{2}\right) = 0$$

Since e^x is one - one function, we can write,

$$-\frac{(\delta - A)^2}{2} = -\frac{(\delta + A)^2}{2} \tag{5.47}$$
$$(\delta - A)^2 = (\delta + A)^2 \tag{5.48}$$
$$\delta = 0 \tag{5.49}$$

Now we will find whether P_e attains maxima

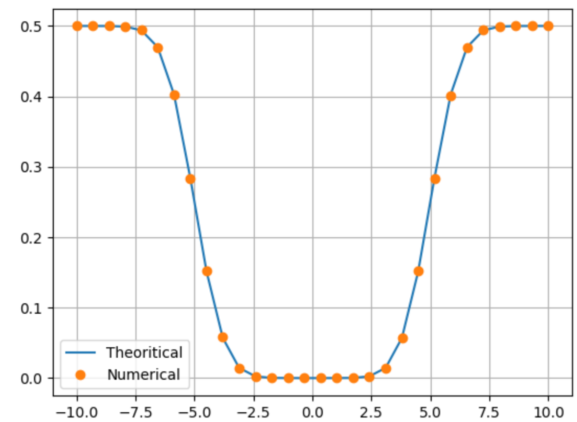

Fig. 5.4: P_e vs δ

or minima at $\delta = 0$

$$\frac{d^2P_e}{d\delta^2}|_{\delta=0} = 2A\exp\left(\frac{-A^2}{2}\right) > 0 \tag{5.50}$$

For $A > 0$ the above expression is positive.So at $\delta = 0$, P_e attains minimum. Plot of P_e w.r.t δ for $A = 5$ is shown in plotted using below python code.

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/P_e_delta.py
```

Then run the following command,

```
python3 P_e_delta.py
```

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.51}$$

Solution: Given that,

$$p_X\left(0\right) = p \tag{5.52}$$

So,

$$\Pr\left(X = 1\right) = p_X\left(0\right) = p \tag{5.53}$$
$$\Pr\left(X = -1\right) = 1 - p \tag{5.54}$$

From (5.42) we can write

$$P_e = P_{e|1}\Pr\left(X = -1\right) + P_{e|0}\Pr\left(X = 1\right) \tag{5.55}$$
$$= (1 - p)\,Q\left(\delta + A\right) + p\left(1 - Q\left(\delta - A\right)\right) \tag{5.56}$$
$$= (1 - p)\,Q\left(\delta + A\right) + pQ\left(A - \delta\right) \tag{5.57}$$

Now to maximize the P_e, we will differentiate the expression w.r.t δ and equate it to 0.

$$\frac{d}{d\delta}P_e = p\frac{d}{d\delta}Q\left(A-\delta\right)$$
$$+\left(1-p\right)\frac{d}{d\delta}Q_N\left(A+\delta\right)=0 \tag{5.58}$$
$$p\frac{d}{d\delta}F_N\left(-A+\delta\right)$$
$$+\left(1-p\right)\frac{d}{d\delta}\left(1-F_N\left(A+\delta\right)\right)=0 \tag{5.59}$$
$$\implies p\times p_N\left(-A+\delta\right) \tag{5.60}$$
$$-\left(1-p\right)p_N\left(A+\delta\right)=0 \tag{5.61}$$

From the PDF of gaussian, we will get

$$\delta = \frac{\ln\left(\frac{1}{p}-1\right)}{2A} \tag{5.62}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: From the Bayes theorem, we can write

$$\Pr\left(X=1|Y=y\right) \tag{5.63}$$
$$=\frac{\Pr\left(X=1,Y=y\right)}{\Pr\left(Y=y\right)} \tag{5.64}$$
$$=\frac{p\Pr\left(N=y-A\right)}{pP_{Y|X}\left(y|1\right)+\left(1-p\right)P_{Y|X}\left(y|-1\right)} \tag{5.65}$$
$$=\frac{pp_N\left(y-A\right)}{pp_N\left(y-A\right)+\left(1-p\right)p_N\left(y+A\right)} \tag{5.66}$$
$$=\frac{p}{p+\left(1-p\right)\exp\left(-2yA\right)} \tag{5.67}$$

And similarly for,

$$\Pr\left(X=-1|Y=y\right) \tag{5.68}$$
$$=\frac{\Pr\left(X=-1,Y=y\right)}{\Pr\left(Y=y\right)} \tag{5.69}$$
$$=\frac{\left(1-p\right)\Pr\left(N=y+A\right)}{pP_{Y|X}\left(y|1\right)+\left(1-p\right)P_{Y|X}\left(y|-1\right)} \tag{5.70}$$
$$=\frac{\left(1-p\right)p_N\left(y+A\right)}{pp_N\left(y-A\right)+\left(1-p\right)p_N\left(y+A\right)} \tag{5.71}$$
$$=\frac{1-p}{1-p+p\exp 2yA} \tag{5.72}$$

Now for a particular y, to make $X=1$ more likely than $X=-1$,

$$\Pr\left(X=1|Y=y\right)>\Pr\left(X=-1|Y=y\right) \tag{5.73}$$
$$\frac{p}{p+\left(1-p\right)\exp\left(-2yA\right)}>\frac{1-p}{1-p+p\exp\left(2yA\right)} \tag{5.74}$$
$$p^2e^{2yA}>\left(1-p\right)^2e^{-2yA} \tag{5.75}$$
$$e^{2yA}>\frac{1-p}{p} \tag{5.76}$$
$$y>\frac{\ln\left(\frac{1-p}{p}\right)}{2A} \tag{5.77}$$

And similarly for a particular y, to make $X=-1$ more likely than $X=1$, we need

$$y<\frac{\ln\left(\frac{1-p}{p}\right)}{2A} \tag{5.78}$$

So to minimise the P_e we need a threshold of

$$\delta=\frac{\ln\left(\frac{1-p}{p}\right)}{2A} \tag{5.79}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}\left(0,1\right)$ and $X_2 \sim \mathcal{N}\left(0,1\right)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download the below files to generate the random variable V,

```
wget https://github.com/
    Charanyash/Random-
    Numbers-/blob/main/codes/
    Q6/coeffs.h
wget https://github.com/
    Charanyash/Random-
    Numbers-/blob/main/codes/
    Q6/chi_square.c
```

Then run the following command,

```
cc chi_square.c -lm
./a.out
```

For CDF download the below python file,

```
wget https://github.com/
    Charanyash/Random-
    Numbers-/blob/main/codes/
    Q6/chi_square_cdf_plot.py
```

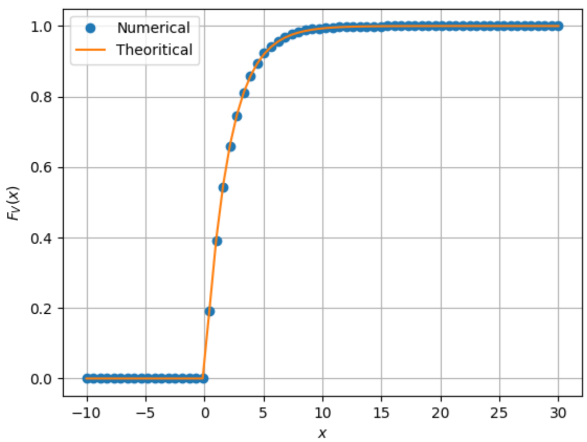

Fig. 6.1: The CDF of V

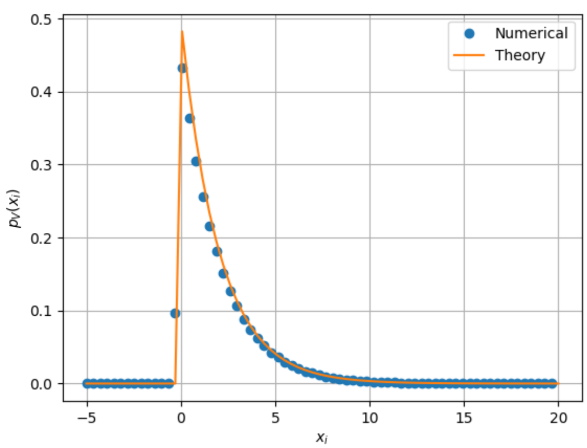

Fig. 6.2: The PDF of V

Then run the following command,

```
python3 chi_square_cdf_plot.
  py
```

For PDF download the below pyhon file,

```
wget https://github.com/
    Charanyash/Random−Numbers
    −/blob/main/codes/Q6/
    chi_square_pdf_plot.py
```

Then run the follwing terminal in terminal,

```
python3 chi_square_pdf_plot.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E[P_e(N)]$.

7.4 Plot P_e in problems 7.7.2 and 7.7.2 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.