भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

# Eight Lecture on Calculus-I

(MA-1110)

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# Cauchy's Condensation Test for Positive Series

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**Theorem:** Let $\{x_n\}_{n\geq 1}^{\infty}$ be a monotone decreasing sequence of positive real numbers and $a$ be a positive integer $> 1$. Then the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} a^n x_{a^n}$ converge or diverge together.

$x(n)$

$x(a^n)$

Example

(1) Test the convergence of the series $\sum \frac{1}{n}$.

Solⁿ: Let $x_n = \frac{1}{n}$.

Then $\{\frac{1}{n}\}$ is monotone decreasing sequence of positive real numbers.

By Cauchy condensation test, two series $\sum x_n$ and $\sum 2^n x_{2^n}$ converge or diverge together.

$$2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{2^n} = 1.$$

Therefore $\sum 2^n x_{2^n}$ is divergent.

It implies that $\sum x_n$ is divergent

**Ex:** Examine the convergence of the

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0.$$

**Sol$^n$:** Let $x_n = \frac{1}{n^p}$. As $p>0$, the sequence $\left\{\frac{1}{n^p}\right\}$ is a monotone decreasing sequence of positive real numbers.

By Cauchy Condensation test, two series $\sum x_n$ and $\sum 2^n x_{2^n}$ converge or diverge together.

$$2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{2^{np}} = \frac{1}{2^{n(p-1)}}$$

$\sum \left(\frac{1}{2^{p-1}}\right)^n$ is a geometric Series

and it Converges if $p>1$ and diverge if $p \leq 1$. Therefore, $\sum_{n\geq 1} \frac{1}{n^p}$ is Convergent when $p>1$ and divergent when $0<p\leq 1$.

**Ex.** Try this Series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad p>0$$

# Raabe's Test for Positive Series

**Note:** If the D'Alembert Ratio Test and Cauchy's Root test fail, then one can easily go for Rabbe's Test.

(You can skip also)

**Theorem:** Let $\sum x_n$ be a series of positive real numbers and $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = l$.

Then $\sum x_n$ is convergent if $l<1$.

$l>1$, $\sum x_n$ is divergent if

**Example:** Test the convergence of the following series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}+\frac{1}{2}\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}+\cdots$$

**Solⁿ:** Let $\sum x_n$ be a given Series.

Then $x_n = \dfrac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)} \cdot \dfrac{1}{2n-1} \quad \forall n \geq 2$

$$\frac{x_{n+1}}{x_n} = \frac{(2n-1)^2}{2n(2n+1)} \text{ and } \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1,$$

D'Alembert Ratio test gives no decision. So go for Rabbe's Test.

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2 - n}{(2n-1)^2} = 3/2 > 1$$

So, by Rabbe's Test,

$\sum x_n$ is Convergent.

# Integral Test for Positive Series

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This is Improper integral. It will be taught in calculus-2.

**Theorem:**

If $f(x)$ is a positive monotone decreasing function defined on the interval $[1,\infty)$ with $f(n) = x_n \ \forall n \in \mathbb{N}$. Then the series $\sum_{n\geq 1}^{\infty} x_n$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ is finite.

Example:

For a real number $p$, the $p$-Series $\sum_{n\geq 1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p>1$.

Proof: Consider the Series as $\sum_{n\geq 1}^{\infty} x_n$ where $f(x) = x_n = \frac{1}{n^p}$.

Consider $f(x) = \frac{1}{x^p}$, $\forall x \in [1, \infty)$

Now use the integral test.

If $\int_1^\infty \frac{1}{x^p} dx$ is finite then $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent.

Now we have to check the integral.

## Case I : For $p=1$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx$$

$$= \lim_{a\to\infty}\left[\ln a - \ln 1\right]$$

$$= \infty.$$

So, $\sum \frac{1}{n}$ is divergent.

Case 2: $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx$$

$$= \lim_{a\to\infty} \int_1^a \frac{1}{x^p}\,dx$$

$$= \lim_{a\to\infty} \left[ \frac{x^{-p+1}}{-p+1} \Bigg|_1^a \right]$$

$$= \frac{1}{-p+1} \cdot \lim_{a \to \infty} \left[ a^{1-p} - 1 \right]$$

This integral is finite if $(1-p) < 0$

$\Rightarrow p > 1$

and is infinite $(+\infty)$ if

$(1-p) > 0$

$\Rightarrow p < 1.$

which implies that

$\sum \frac{1}{n^p}$ is convergent if $p>1$

is divergent if $p<1$.

Also $\sum \frac{1}{n^p}$ is divergent if $p=1$.

## Example :

Prove by Integral test, that the Series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ Converges if $p>1$, diverges if $p \leq 1$.

(Try it)

(Home work)

# Alternating Series

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Definition:

If $x_n > 0 \ \forall n$, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot x_n$ is called as an Alternating Series.

$$\sum (-1)^{n+1} \cdot x_n = x_1 - x_2 + x_3 - x_4 + \cdots$$

# Leibnitz's Test for alternating Series

**Theorem:**

If $\{x_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim_{n\to\infty} x_n = 0$, then the alternating series $\sum (-1)^{n+1} \cdot x_n = x_1 - x_2 + \cdots$ is convergent.

**Proof:** Let $\{S_n\}$ be a sequence of $n$th partial sum of given alternating Series,

where $S_n = x_1 - x_2 + x_3 - x_4 + \cdots + (-1)^{n+1} x_n$

Here consider two subsequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ of $\{S_n\}$.

Then $S_{2n+2} - S_{2n}$

$$= x_{2n+1} - x_{2n+2} \geq 0 \ \forall n \in \mathbb{N}$$

Since $\{x_n\}$ is a monotone decreasing sequence.

$\therefore \ S_{2n+2} \geq S_{2n} \Rightarrow \{S_{2n}\}$ is monotone increasing sequence.

$$S_{2n+1} - S_{2n-1} = -x_{2n} + x_{2n+1} \leq 0 \quad \forall n \in \mathbb{N}$$

The Sequence $\{S_{2n+1}\}$ is a monotone decreasing sequence.

Again, $S_{2n} = x_1 - x_2 + x_3 - x_4 + \dots - x_{2n}$

$$= x_1 - (x_2 - x_3) - (x_4 - x_5) - \dots - x_{2n} < x_1$$

The Sequence $\{S_{2n}\}$ is bounded above.

$$S_{2n+1} = x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n+1}$$

$$= (x_1 - x_2) + (x_3 - x_4) + \cdots + x_{2n+1}$$

$$> (x_1 - x_2)$$

The sequence $\{S_{2n+1}\}$ is bounded below.

Therefore both subsequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are convergent.

Since $\lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} x_{2n+1} = 0.$$

$$\Rightarrow \boxed{\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}}$$

Both subsequences $\{S_{2n+1}\}$ and $\{S_{2n}\}$ have same sequential limit. So $\{S_n\}$ is convergent and hence $\sum (-1)^{n+1} x_n$ is convergent.

## Examples:

The Series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum (-1)^{n+1} \cdot \frac{1}{n}$

is Convergent by Leibnitz's test.

Since $\left\{ \frac{1}{n} \right\}$ is monotone decreasing Sequence and $\lim_{n \to \infty} \frac{1}{n} = 0$.

② $\sum (-1)^{n+1} \cdot \frac{1}{n+a^2}$

(Try this). Home work.

③ $1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \cdots$

(Try this). Home work.

Also there exist several tests for infinite series such as

**For Positive Series**

(A) Logarithmic Test
(B) Kummer's Test
(C) Gauss's Test

**For Alternating Series**

(A) Abel's Test
(B) Dirichlet's Test

This is for your reference only.