भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Ninth Lecture on Calculus-I

(MA-1110)

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Differential Calculus

- Limit of function of a Single Real Variable
- Continuity and Differentiability
- Rolle's Theorem
- Lagrange's Mean Value theorem

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Today’s Class Lecture

- Limit of function of a Single Real Variable
- Definition of Limit
- Sequential Criterion on Limit
- Sandwich Theorem
- One Sided Limit
- Theorem on existence of Limit

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Limit of a function

- The limit of a function is a fundamental concept. It provides an idea to understand the behaviour of the function near a particular point.

⇒ The limit of a function is more important to describe the concept continuity and differentiability of this function at a particular point.

Let consider a real single valued function $f(x)$ which is defined on a domain $D \subseteq \mathbb{R}$. $\mathbb{R} \to$ Set of real numbers

So we have $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$

Let c be a limit point of D in ~~[illegible]~~ $\mathbb{R}$.

Let l be a limit ~~point~~ of the function $f(x)$ at a point c. Here $l \in \mathbb{R}$.

The function $f(x)$ has a limit l at a point c, i.e., $f(x) \rightarrow l$ as $x \rightarrow c$ means

as x moves closer and closer to c, $f(x)$ gets closer and closer to l.

Important:

The limit of a function f at a point c

$\Rightarrow$ The function need not be defined at the point c, but it should be defined at the points near c, i.e., at enough points close to c.

See one example for this statement

Example: The function $f(x) = \frac{\sin x}{x}$ has a limit 1 as x tends to 0. But the function $f(x)$ is not defined at $x=0$. The value of $f(x)$ becomes close to 1 as x goes close to 0.

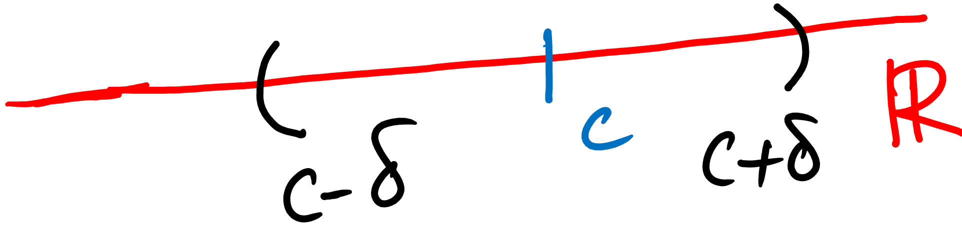
- In the definition of limit of $f(x)$ at a point c, we consider the concept of x close to c. This concept of closedness of x to c is defined by the concept of δ-neighbourhood of point c.

- Let δ be a positive real number.

  See the definition of the δ-neighbourhood of a point c.

Definition: (δ-nbd of a point)

The δ-neighbourhood of a point c in $\mathbb{R}$ is defined to be



$$N(c,\delta) = (c-\delta,\ c+\delta)$$

$$= \{x \in \mathbb{R} : |x-c| < \delta\}$$

$$= \{x \in \mathbb{R} : c-\delta < x < c+\delta\}$$

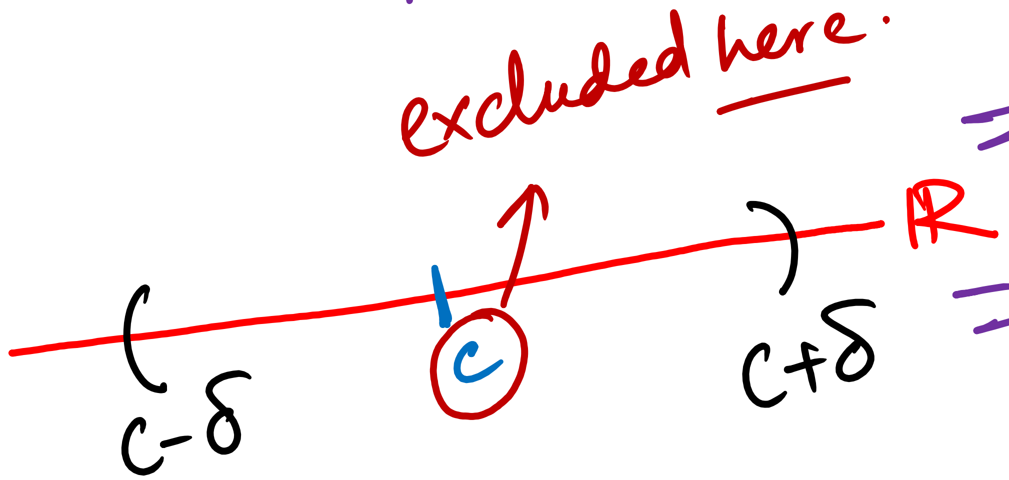
Definition (deleted δ-nbd):

The deleted δ-nbd of a point c in $\mathbb{R}$ is defined to be

$$N'(c,\delta) = (c-\delta,\, c+\delta)\setminus\{c\}$$

$$= \{x\in\mathbb{R} : 0<|x-c|<\delta\}$$

$$= \{x\in\mathbb{R} : c-\delta<x<c+\delta \text{ and } x\neq c\}$$



Definition of limit point:

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

Then c is a limit point of D if for every $\delta > 0$ $\exists$ atleast one point $x \in D - \{c\}$ such that $|x - c| < \delta$.

i.e, every deleted δ-nbd of c contains atleast one point. $N'(c,\delta) \cap D \neq \phi$

$\Rightarrow x \in N'(c,\delta) \cap D$.

Ex: Let $D = \{1, 2, 3\}$. Then the point 1 is not a limit point.

Take $\delta = \frac{1}{2}$. Then $N'(1, \frac{1}{2}) = (1 - \frac{1}{2}, 1 + \frac{1}{2})$ does not contain any point of D other than 1. $\therefore N'(1, \frac{1}{2}) \cap D = \phi$

Similarly for 2 and 3, So the set D has no limit point.

Definition of Limit

- A real number l is said to be a limit of a function $f(x)$ at a point c if for every positive number ε, $\exists$ a positive number δ such that

$$|f(x)-l| < \varepsilon, \ \forall x \in N'(c,\delta) \cap D$$

$$\Rightarrow \boxed{l-\varepsilon < f(x) < l+\varepsilon, \ \forall\, 0<|x-c|<\delta.}$$

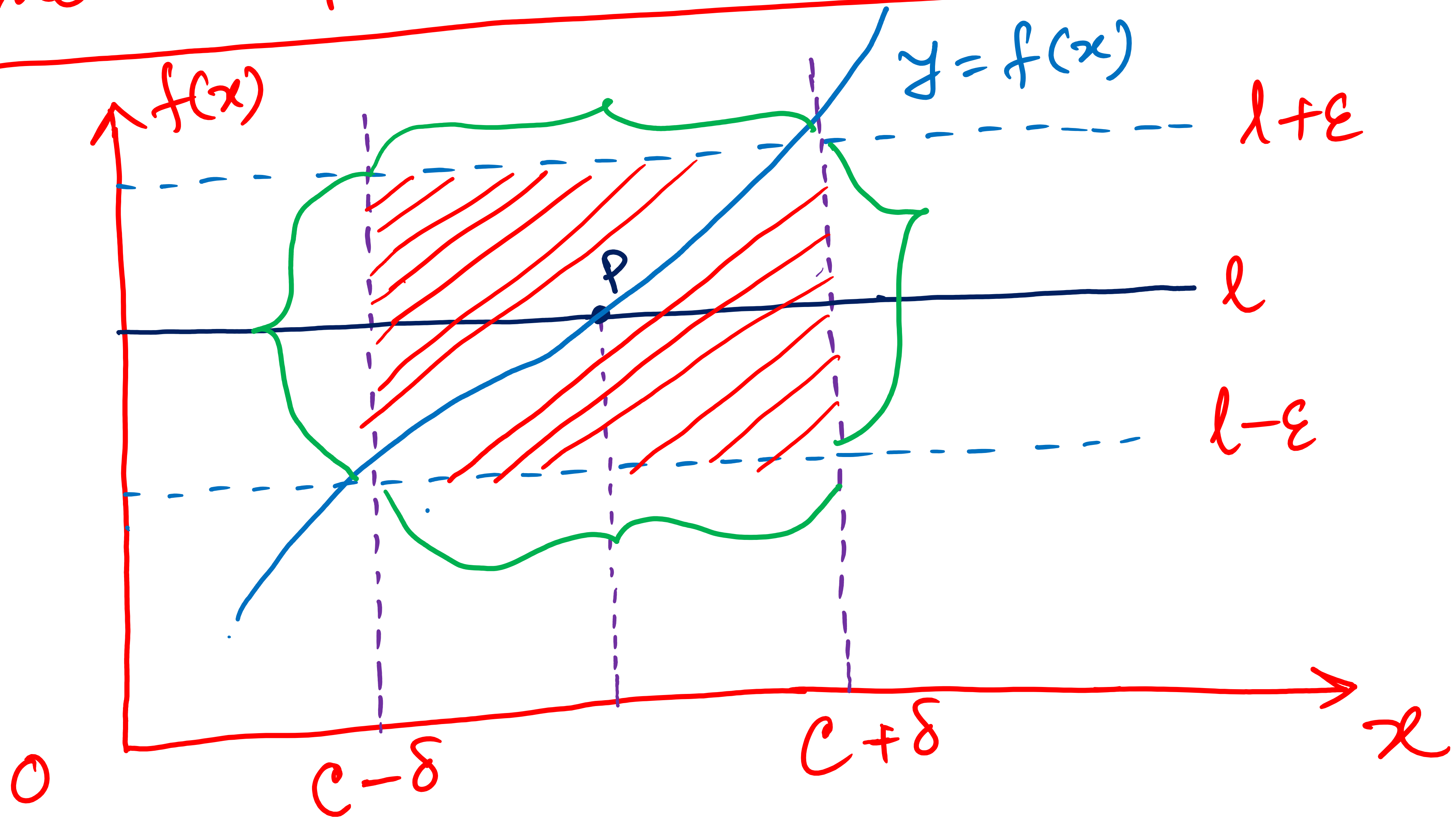
$\Rightarrow f(x) \longrightarrow l$ as $x \to c$

$\Rightarrow \boxed{\lim_{x \to c} f(x) = l.}$

The definition says that for any ε-nbd of l $\exists$ a δ-nbd of c such that $f(x)$ belongs to ε-nbd of l for all x belongs to deleted δ-nbd of c.

It does not matter whether c belongs to D or not. Even if $c \in D$, $f(c)$ need not lie in the ε-nbd of l.

Geometric point of view on limit:



$f(x) \to l$ as $x \to c$

$\Rightarrow$ we can find a deleted nbd of $x = c$ such that all points of $f(x)$ corresponding to points x of the deleted nbd of c lie within the strip bounded by the lines $f(x) = l \pm \varepsilon$ and $x = c \pm \delta$.

Note that the point P may or may not belong to the graph of $y = f(x)$.

we are simply not concerned on it.

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let $c \in \mathbb{R}$. Then $f(x)$ can have atmost one limit at c.

Proof: Suppose l_1 and l_2 be two limits of $f(x)$ at c. Then for any $\varepsilon > 0$ $\exists$ $\delta' > 0$ such that $|f(x) - l_1| < \varepsilon/2$, $\forall\ 0 < |x - c| < \delta'$

Also for any $\varepsilon > 0$ $\exists$ $\delta'' > 0$ such that

$$|f(x) - l_2| < \varepsilon/2, \ \forall\ 0 < |x-c| < \delta'$$

Now take $\delta = \min\{\delta', \delta''\}$.

Therefore we have $\forall\ 0 < |x-c| < \delta$,

$$\begin{aligned}
|l_1 - l_2| &= |l_1 - f(x) + f(x) - l_2| \\
&\le |f(x) - l_1| + |f(x) - l_2| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

$\Rightarrow \underline{\underline{l_1 = l_2.}}$ ($\underline{\underline{\text{proved}}}$)

<u>Example :</u>

Verify the validity of the Statement:

$$\lim_{x \to 2} 5x = 10.$$

<u>Soln :</u> This Statement is true if for any given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|5x-10| < \varepsilon, \quad \forall\ 0<|x-2|<\delta.$$

Here we need to find the existence of such δ.

$$|5x-10| = 5|x-2| < \varepsilon$$

$$\Rightarrow |x-2| < \varepsilon/5$$

choose $\delta = \varepsilon/5$. Then we have

for $\varepsilon > 0$ $\exists$ $\delta = \varepsilon/5$ such that

$|5x - 10| < \varepsilon$, $\forall$ $0 < |x-2| < \delta$

$\Rightarrow$ The statement $\lim_{x \to 2} 5x = 10$

is true.

$x \in N'(2,\delta) \cap D$.

where $N'(2,\delta)$ is the deleted nbd of 2.

Ex. Show that $\lim_{x \to 2} f(x) = 4$,

where $f(x) = \dfrac{x^2 - 4}{x - 2}$

Sol: Here the domain D of $f(x)$ is $\mathbb{R} - \{2\}$. 2 is a limit point of D. Let us choose $\varepsilon > 0$.

$$|f(x)-4| = \left|\frac{x^2-4}{x-2} - 4\right| = |x-2|$$

$$|f(x)-4| = |x-2| < \varepsilon$$

choose $\delta = \varepsilon$.

Therefore we have

$$|f(x)-4| < \varepsilon, \ \forall \ 0 < |x-2| < \delta$$

$x \in N'(2,\delta) \cap D$

Ex, Show that $\lim_{x \to 2} f(x) = 4$,

where $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$

$= 10$, $x = 2$

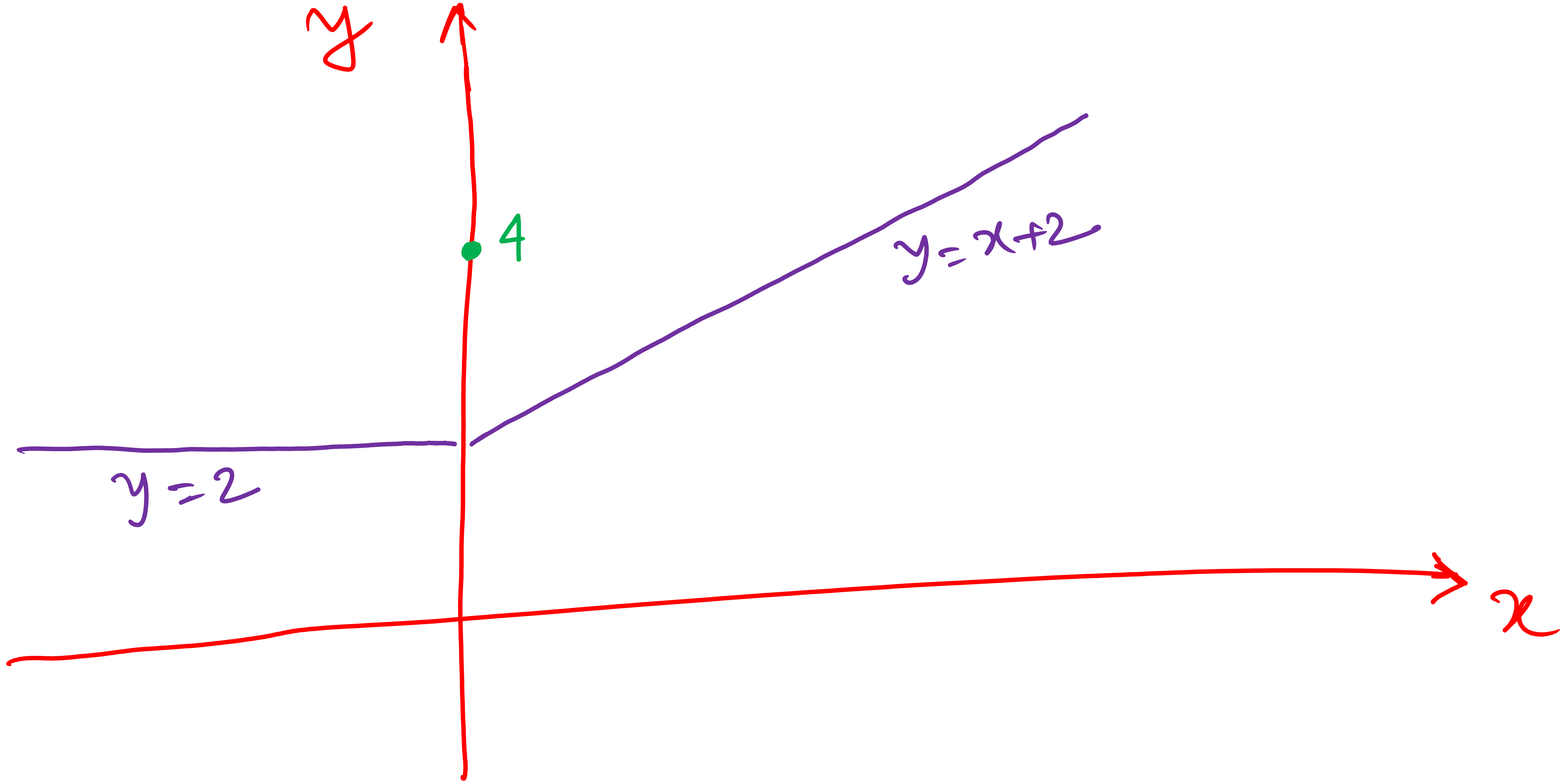
We will use the same procedure to show that $\lim_{x \to 2} f(x) = 4$, Here we will not bother about the function value 10 defined at $x = 2$.

Ex. Consider the function

$$f: \mathbb{R} \to \mathbb{R}.$$

$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ x+2 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \end{cases}$$

Though $f(0) = 4$, limit of $f(x)$ at $x = 0$ is 2.


y
4
y = x+2
y = 2
x

Exercise (Homework):

① Let $D = [0,\infty) \setminus \{9\}$ and define

$f : D \to \mathbb{R}$ by $f(x) = \dfrac{x-9}{\sqrt{x}-3}$.

Prove that $\lim_{x \to 9} f(x) = 6$.

② $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$. Is this statement true?

Sequential Criterion on Limit

Theorem: Let $f: D \to \mathbb{R}$ be a function where $D \subseteq \mathbb{R}$ is domain of f. Let c be a limit point of D and $l \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = l$ if and only if for every sequence $\{x_n\}$ in $D \setminus \{c\}$ converging to c, the sequence $\{f(x_n)\}$

Converges to l.

Proof: Let $\lim_{x \to c} f(x) = l$. Then for a given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \forall\ x \in N'(c, \delta) \cap D$$

i.e., $0 < |x - c| < \delta$

$$\Rightarrow \quad l - \varepsilon < f(x) < l + \varepsilon.$$

Let $\{x_n\}$ be a sequence in $D-\{c\}$ converging to c.

Since $\lim_{n\to\infty} x_n = c$, for $\delta>0$

$\exists\ N\in\mathbb{N}$ such that

$|x_n-c|<\delta, \ \forall\ n\geq N$

$\Rightarrow c-\delta<x_n<c+\delta$.

Therefore,

$$\left| f(x_n) - l \right| < \varepsilon, \quad \forall \; n \geq N.$$

and $x_n \in (c-\delta, c+\delta) \setminus \{c\}$

$$\Rightarrow \lim_{n \to \infty} f(x_n) = l.$$

(proved)

Remark:

This theorem is very useful to show that (i) a certain number is not a limit of a function at a point

OR

(ii) the function $f(x)$ does not have a limit at a point.

Note: The function $f(x)$ does not have a limit at c iff there exists a sequence $\{x_n\}$ in $D \setminus \{c\}$ $\forall n \in \mathbb{N}$ such that $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge to l.

$$\therefore \boxed{x_n \to c \text{ but } f(x_n) \not\to l \text{ as } n \to \infty}$$

Ex. Let $f : \mathbb{R}-\{0\} \to \mathbb{R}$ defined as $f(x) = \sin\frac{1}{x}$.

Prove that $\lim_{x\to 0} f(x)$ doesn't exist.

Sol^n: $f(x) = \sin\frac{1}{x}$, $x \neq 0$.

Take $x_n = \frac{1}{2\pi n} \Rightarrow x_n \to 0$ as $n \to \infty$.

$$f(x_n) = \sin(2n\pi) = 0 \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = 0$$

Again take $y_n = \dfrac{1}{2n\pi + \pi/2}$.

So, $y_n \to 0$ as $n \to \infty$

$$f(y_n) = \sin\left(2n\pi + \pi/2\right) = 1, \ \forall\, n \in \mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} f(y_n) = 1.$$

Thus we have two sequence $\{x_n\}$ and $\{y_n\}$ both converging to 0 but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to two different limits.

Therefore $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

<u>Ex.</u> Prove $\lim_{x\to 0} \frac{1}{x}$ does not exist.

(Try)

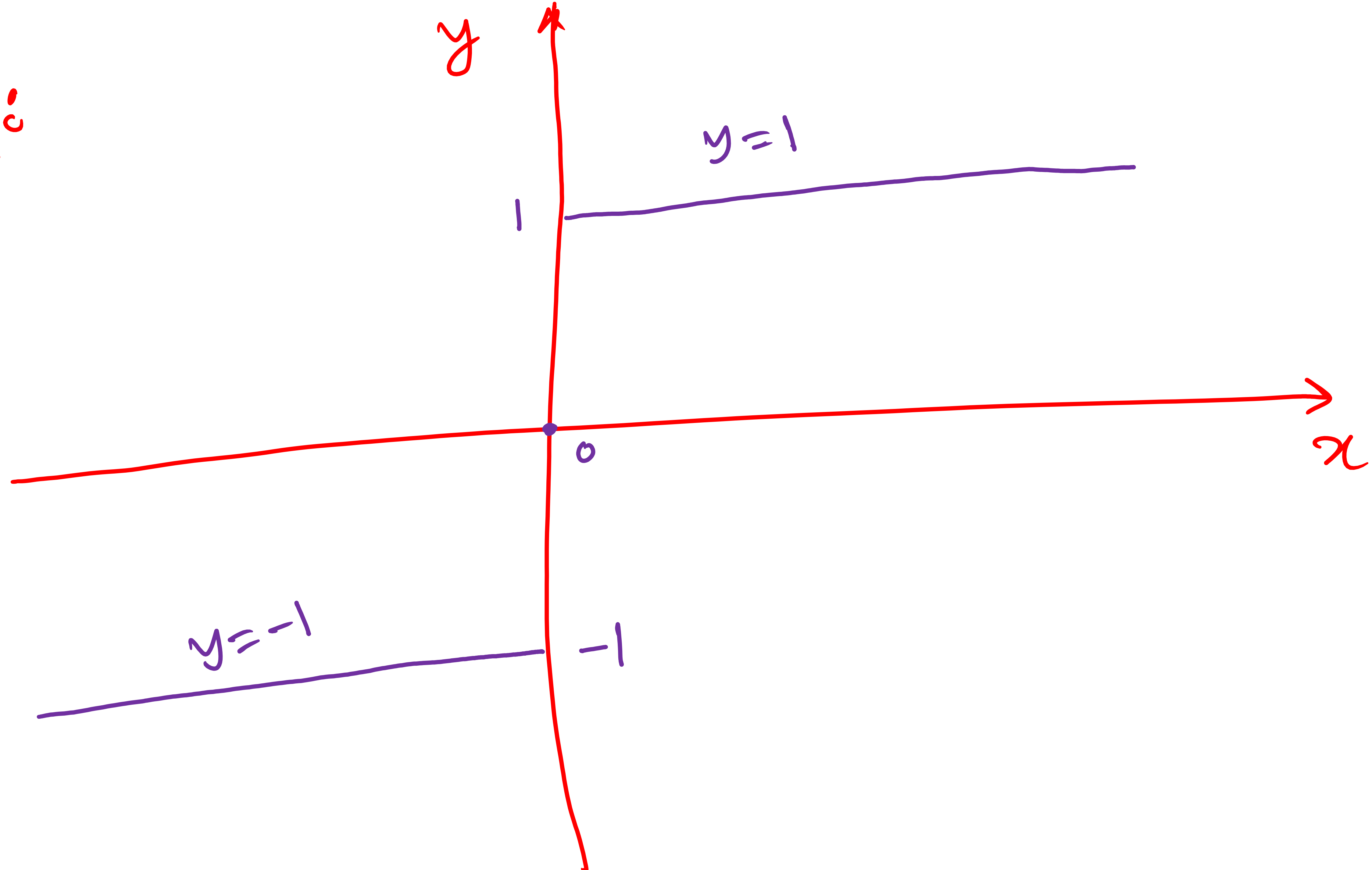
Ex. Let $D = \mathbb{R} - \{0\}$ and consider a function $f : D \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here $f(x) = \text{Sgn } x$. → This is Sign function or Signum function

Show that $\lim_{x \to 0} f(x)$ does not exist.

Sol^n:



Proof: Consider two sequences $\left\{x_n = \frac{1}{n}\right\}$ and $\left\{y_n = -\frac{1}{n}\right\}$, both converge to 0.

But $f(x_n) \to 1$ and $f(y_n) \to -1$

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

Remark: For the sign function,

$\lim_{x \to c} f(x)$ exists for any $c \neq 0$.

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D. If f has a limit $l \in \mathbb{R}$ at c then f is bounded on $N(c,\delta) \cap D$ for some δ-nbd $N(c,\delta)$ of c.

Corollary: If f is not bounded on $N(c,\delta)\cap D$ for some δ-nbd $N(c,\delta)$ of c, then $\lim_{x\to c} f(x)$ does not exist in $\mathbb{R}$.

For example, $\lim_{x\to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$. Here $D = \mathbb{R}-\{0\}$ and let $f(x)=\frac{1}{x}$, $x\in D$. $0\in D'$. f is unbounded on every nbd of 0. Therefore $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Algebraic Properties:

Theorem:

Let $c \in \mathbb{R}$. Let $f, g : D \to \mathbb{R}$ and $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$

Then (i) $\lim_{x \to c} \left(f(x) \pm g(x) \right) = l_1 \pm l_2$

(ii) $\lim_{x \to c} K f(x) = K \cdot l_1$, where $K \in \mathbb{R}$.

(iii) $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = l_1 l_2$

(iv) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{l_1}{l_2}$ (Provided $l_2 \neq 0$)

Proof: (i) Let $\{x_n\}$ be a sequence in $D-\{c\}$ converging to c.

Since $\lim_{x\to c} f(x)=l_1$ and $\lim_{x\to c} g(x)=l_2$

we have $\lim_{n\to\infty} f(x_n)=l_1$

and $\lim_{n\to\infty} g(x_n)=l_2$

by the sequential criterion.

we have from algebraic operations on sequences,

$$\lim_{n\to\infty}\left(f(x_n) \pm g(x_n)\right) = l_1 \pm l_2 .$$

Since $\{x_n\}$ is an arbitrary sequence in $D\setminus\{c\}$ converging to c, it follows from the sequential criterion for limits that $\lim_{x\to c}\left(f(x)\pm g(x)\right) = l_1 \pm l_2$. (proved)

Example:

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, (a polynomial of degree n). For every point $c \in \mathbb{R}$, the limit $\lim_{x \to c} f(x)$ exists and

$$\lim_{x \to c} f(x) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$$

by algebraic properties on limit.

Theorem: Let $D \subseteq \mathbb{R}$ and $f, g : D \to \mathbb{R}$ be two functions. Let c be a limit point of D. If $f(x)$ is bounded on $N'(c,\delta) \cap D$ for some deleted nbd $N'(c,\delta)$ of c and $\lim_{x \to c} g(x) = 0$ then $\lim_{x \to c} \left(f(x)\, g(x) \right) = 0$.

(You can skip)

Example: Prove that $\lim_{x\to 0} x \sin\frac{1}{x^2} = 0.$

Sol^n: Here $\lim_{x\to 0} x = 0$ and $\sin\frac{1}{x^2}$ is bounded in some deleted nbd of 0.

Therefore $\lim_{x\to 0} x \sin\frac{1}{x^2} = 0.$

Example: Try this: $\lim_{x\to 0} \sqrt{x} \sin\frac{1}{x} = 0.$

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D.

If $f(x) \leq b$ $\forall$ all $x \in D - \{c\}$ and

$\lim_{x \to c} f(x) = l$, then $\boxed{l \leq b}$

Proof: Let $\{x_n\}$ be a sequence in $D - \{c\}$ converging to c. Since $\lim_{x \to c} f(x) = l$

$$\lim_{n\to\infty} f(x_n) = l.$$

Let us define a sequence $\{u_n\}$ by $u_n = b \quad \forall\, n \in \mathbb{N}$

Then $f(x_n) \leq u_n, \quad \forall\, n \in \mathbb{N}$

Since $\lim_{n\to\infty} f(x_n) = l$ and $\lim_{n\to\infty} u_n = b$

we have $\boxed{l \leq b}$ (by the theorem on sequence).

Theorem: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D. If $f(x) \geq a$ $\forall x \in D \setminus \{c\}$ and $\lim_{x \to c} f(x) = l$, then $l \geq a$.

Proof: Similar

⊘ Theorem :

Let $D \subseteq \mathbb{R}$ and c be a limit point of D.
Let $f: D \to \mathbb{R}$ a function such that

$$\lim_{x \to c} f(x) = l.$$

we can use the Sequential criterion to prove this

Then $\lim_{x \to c} |f(x)|$ exists and

$$\lim_{x \to c} |f(x)| = |l|$$

Sandwich Theorem

Theorem: Let $D \subseteq \mathbb{R}$ and $f, g, h : D \to \mathbb{R}$ be functions. Let c be the limit point of D. If $\boxed{f(x) \leq g(x) \leq h(x)}$ $\forall\, x \in D \setminus \{c\}$ and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$, then

$$\lim_{x \to c} g(x) = l.$$

Proof: Let $\{x_n\}$ be a sequence in $D \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.

Then $\boxed{f(x_n) \le g(x_n) \le h(x_n)}$ $\forall n \in \mathbb{N}$

and $\lim_{n\to\infty} f(x_n) = l = \lim_{n\to\infty} h(x_n)$

By Sandwich theorem on sequences,

$\lim_{n\to\infty} g(x_n) = l \Rightarrow \lim_{x\to c} g(x) = l.$

Example: Show that $\lim_{x \to 0} x \cos\frac{1}{x} = 0$.

Solⁿ: Let $f(x) = \cos\frac{1}{x}$, $x \in D \setminus \{0\}$

we have $-1 \leq f(x) \leq 1$, $\forall x \in D \setminus \{0\}$

Hence $-x \leq x f(x) \leq x$, $\forall x > 0$

$x \leq x f(x) \leq -x$, $\forall x < 0$

Therefore $-|x| \leq x f(x) \leq |x|$, $\forall x \neq 0$.

Since $\lim_{x \to 0} |x| = \lim_{x \to 0} -|x| = 0$

By Sandwich theorem,

$$\lim_{x \to 0} x f(x) = 0$$

i.e., $\lim_{x \to 0} x \cos \frac{1}{x} = 0$.

Ex. Prove that $\lim_{x \to 0} x \sin \frac{1}{x^2} = 0$

Ex Prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

(Try).

Theorem:

Let $c \in \mathbb{R}$. Let $f, g : D \to \mathbb{R}$ be functions such that

$$\boxed{f(x) \leq g(x)} \quad \forall x \in D.$$

Let $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$.

Then $\boxed{l_1 \leq l_2}$.

Theorem:

Let $c \in \mathbb{R}$. Let $f, g : D \to \mathbb{R}$ be functions such that $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$. If $\boxed{l_1 < l_2}$, then $\exists\, \delta > 0$ such that $\boxed{f(x) < g(x)}$, $\forall x \in N'(c, \delta) \cap D$

One Sided Limit

- **Right hand limit:**

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function.
Let c be a limit point of $D_1 = D \cap (c, \infty)$
$= \{x \in D : x > c\}$.
Then f is said to have a right hand limit l_1 $(\in \mathbb{R})$ at c if for any

given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - l_1| < \varepsilon, \quad \forall\, x \in N'(c,\delta) \cap D_1$$

i.e., $l_1 - \varepsilon < f(x) < l_1 + \varepsilon$, $\forall\, x$ satisfying $c < x < c+\delta$

In this case we write

$$\lim_{x \to c+} f(x) = l_1 \quad \text{or} \quad f(c+0) = l_1$$

i.e., as x approaches from right, f approaches to l_1

Sequential criterion on Right handed limit:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of $D_1 = D \cap (c, \infty)$. Then $\lim_{x \to c+} f(x) = l_1$ if and only if for every sequence $\{x_n\}$ in D_1 converging to c, the sequence $\{f(x_n)\}$ converges to l_1.

Left hand limit:

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of $D_2 = D \cap (-\infty, c)$ $= \{x \in D : x < c\}$.

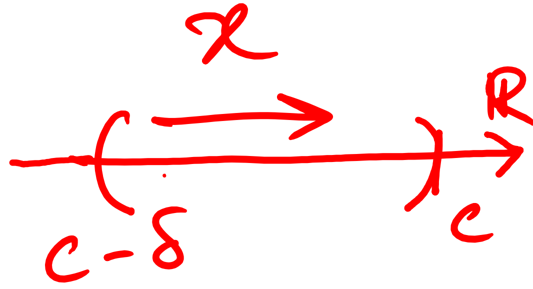
Then f is said to have a left hand limit $l_2 (\in \mathbb{R})$ at c if for any given $\varepsilon > 0$ $\exists\ \delta > 0$ such that

$$|f(x)-l_2| < \varepsilon, \quad \forall x \in N'(c,\delta) \cap D_2$$

i.e., $l_2-\varepsilon < f(x) < l_2+\varepsilon$, $\forall x$ satisfying $c-\delta < x < c$.

In this case $\lim\limits_{x \to c-} f(x) = l_2$ or $f(c-0) = l_2$

i.e., as x approaches from left $f(x)$ approaches to l_2



Sequential criterion for Left handed limit:

Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of $D_2 = D \cap (-\infty, c)$. Then $\lim_{x \to c-} f(x) = l_2$ if and only if for every sequence $\{x_n\}$ in D_2 converging to c, the sequence $\{f(x_n)\}$ converges to l_2.

Examples:

① Let $f(x) = \operatorname{sgn} x$. Examine if $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ exist.

Soln: Here the domain D of f is $\mathbb{R}$.

check $\lim_{x \to 0+} f(x)$ exist or not

Since $x > 0$ for the right hand limit, we have $D_1 = D \cap (0, \infty)$

$= \{x \in \mathbb{R} : x > 0\}$

0 is the limit point of D_1.

$f(x) = 1 \quad \forall x \in D_1$. Therefore

$$\lim_{x \to 0^+} f(x) = 1.$$

OR. Consider any sequence $\{x_n\}$ with $x_n > 0$ which converge to 0.

Then $f(x_n) \to 1$ as $n \to \infty$

Therefore $\lim_{x \to 0+} f(x) = 1$

So, it exist.

Now check the left hand limit

$$\lim_{x \to 0^-} f(x).$$

Consider any sequence $\{x_n\}$ with $x_n < 0$ which converges to 0. Then

$$f(x_n) \longrightarrow -1 \text{ as } n \to \infty$$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \underline{\underline{-1}}, \quad \underline{\underline{\text{It exist.}}}$$

OR

Let $D_2 = D \cap (-\infty, 0)$.

Then $D_2 = \{x \in \mathbb{R} : x < 0\}$. 0 is a limit point of D_2.

$f(x) = -1 \ \forall \ x \in D_2$. Therefore

$$\lim_{x \to 0^-} f(x) = -1$$

Important theorem on existence of Limit

Theorem: Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let c be a limit point of both sets $D_1 \left(= D \cap (c, \infty)\right)$ and $D_2 \left(= D \cap (-\infty, c)\right)$.

Then $\lim_{x \to c} f(x) = l \,(\in \mathbb{R})$ if and only if $\boxed{l_1 = l_2}$ i.e, $\boxed{\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)}$

i.e, $\lim_{x \to c} f(x)$ exists if and only if both $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exists and are equal.

Example:

Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 2 & \text{if } x<0 \\ x+2 & \text{if } x>0 \\ 4 & \text{if } x=0 \end{cases}$.

Solⁿ:

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x+2) = 2.$$

Since $f(x) = 2$, $\forall x<0$

$$\lim_{x\to 0-} f(x) = 2,$$

$\therefore \lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) \Rightarrow$ So $\lim_{x\to 0} f(x) = 2.$

Check $\lim_{x \to 0} \frac{1}{x}$.

Sol^n:

for $\lim_{x \to 0+} \frac{1}{x} = +\infty$

and $\lim_{x \to 0-} \frac{1}{x} = -\infty$.

So, $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Ⓐ Check $\lim_{x\to 0} \frac{1}{1+e^{1/x}}$.

Sol^n:

$$\lim_{x\to 0+} \frac{1}{1+e^{1/x}} = 0 \quad \left(\text{as } x\to 0+,\ 1/x \to +\infty\right)$$

$$\lim_{x\to 0-} \frac{1}{1+e^{1/x}} = 1 \quad \left(\text{as } x\to 0-,\ e^{1/x} \to 0\right)$$

The two limits are not equal

$\Rightarrow$ limit does not exist.

Example:

Find $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Sol^n: For $0 < x < \pi/2$, $\sin x < x < \tan x$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ in $0 < x < \pi/2$

As $\lim_{x\to 0+} \frac{1}{\cos x} = 1$, we have

$\lim_{x\to 0+} \frac{x}{\sin x} = 1$ by Sandwich theorem

For $-\pi/2 < x < 0$,

$$\tan x < x < \sin x,$$

$$\Rightarrow \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$ by Sandwich theorem. (similarly).

$$\lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{\sin x}{x} = 1.$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$