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Lecture Note - 5

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Fifth Lecture on Calculus-I

(MA-1110)

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Sequences

- Subsequences
- Subsequential Limit
- Upper and Lower Limit
- Bolzano-Weierstrass Theorem
- Cauchy Sequences

Subsequence

Definition: Let $X = \{x_n\}_n$ be a sequence of real numbers and let $\{n_k\}$ be a strictly increasing sequence of natural numbers, i.e.,

$$n_1 < n_2 < \dots < n_{k-1} < n_k < \dots$$

Then the sequence $X' = \{x_{n_k}\}_{k=1}^{\infty} = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ is called a subsequence of X.

Example: Let $X = \{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of real numbers.

(i) Therefore $X' = \{x_{2n}\} = \{x_2, x_4, x_6, \ldots\}$ is a subsequence of X.

(ii) $X'' = \{x_{2n+1}\} = \{x_3, x_5, x_7, \ldots\}$ is a subsequence of X.

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iii $X''' = \{x_{n^2}\} = \{x_1, x_4, x_9, \cdots\}$

is a subsequence of X.

iv $X'''' = \{x_{2^n-1}\} = \{x_1, x_3, x_7, \cdots\}$

is a subsequence of X.

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Example: Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$

$X = \{x_n\} = \{\frac{1}{n}\}_{n \in \mathbb{N}}$

Here $X' = \{x_{2n}\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$

is a subsequence of X.

$X'' = \{x_{2n-1}\} = \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}, \ldots\right\}$

is a subsequence of X.

$X''' = \{x_{n!}\} = \left\{1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots, \frac{1}{n!} \ldots\right\}$

is a Subsequence of X.

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But $\left\{\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots\right\}$

Remember

or $\left\{\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right\}$

are not subsequences of X.

Example: $X = \{x_n\} = \{n\} = \{1, 2, 3, \ldots\}$

is a real sequence.

$X' = \{x_{2n-1}\} = \{1, 3, 5, 7, \ldots\}$

and $X'' = \{x_{2n}\} = \{2, 4, 6, 8, \ldots\}$ are

two subsequences of X.

But $\{4, 2, 8, 6, 12, 10, \ldots\}$ is not a subsequence of X.

Subsequential Limit

Definition: Let $\{x_n\}_n$ be a sequence of real numbers. Then a real number l is said to be a subsequential limit of the sequence $\{x_n\}$ if $\exists$ a subsequence of $\{x_n\}$ that converges to l.

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Upper Limit and Lower Limit

Definition:

$\Rightarrow$ **Upper Limit or Limit Superior:**

The greatest subsequential limit of $\{x_n\}_n$ is said to be the upper limit or the limit superior of $\{x_n\}$.

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This is denoted by

$\overline{\lim}\, x_n$ or $\lim_{n\to\infty} \sup x_n$

$\Rightarrow$ **Lower Limit or Limit Inferior:**

The least subsequential limit of the sequence $\{x_n\}_n$ is said to be the lower limit or limit inferior of $\{x_n\}_n$.

This is denoted by

$$\underline{\lim}\, x_n \text{ or } \liminf_{n\to\infty} x_n.$$

Remark: (i) The upper limit is the greatest value among all the subsequential limits of $\{x_n\}$

(ii) The lower limit is the least value among all subsequential limits of $\{x_n\}$.

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Note:

(i) If $\{x_n\}$ is unbounded above, then $\overline{\lim}\, x_n = \infty$.

(ii) If $\{x_n\}$ is unbounded below, then $\underline{\lim}\, x_n = -\infty$.

(iii) If $\{x_n\}$ is unbounded above but bounded below, then $\underline{\lim}\, x_n$ is defined to be the least subsequential limit. If there is no subsequential limit, we can write $\boxed{\underline{\lim}\, x_n = \infty} = \overline{\lim}\, x_n$.

(iv) Similarly, if $\{x_n\}$ is unbounded below but bounded above, then $\overline{\lim}\, x_n$ is defined to be the greatest subsequential limit. If there is no subsequential limit, we have $\overline{\lim}\, x_n = -\infty$ $\qquad \underline{\lim}\, x_n = -\infty$.

Example :

(i) The sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}^{\infty}$ is bounded sequence.

Here, $\boxed{\overline{\lim}\, \frac{1}{n} = 0 = \underline{\lim}\, \frac{1}{n}}$

Here 0 is only the subsequential limit. $\lim_{n\to\infty} \frac{1}{n} = 0.$

(ii) The sequence $\{(-1)^n\}$ is bounded sequence. Here $x_n = (-1)^n$.

The sequence is $\{-1, 1, -1, 1, \ldots\}$

$\Rightarrow \{-1, -1, -1, \ldots\}$ is a subsequence of $\{(-1)^n\}_n$.

It converges to -1.

$\{1, 1, 1, \dots\}$ is also a subsequence of

$\{(-1)^n\}_n$.

It converges to 1.

So, -1 and 1 are the subsequential limits of respective sequences.

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So $\underline{\lim}\, x_n = -1$

and $\overline{\lim}\, x_n = 1$.

Note that the sequence $\{x_n\}$ is not convergent. $\Downarrow$

oscillating finitely

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(iii) Let $x_n = (-1)^n\left(1+\frac{1}{n}\right)$, $n \geq 1$

The sequence $\{x_n\}$ is a bounded sequence.

$$\{x_n\} = \left\{-(1+1),\ \left(1+\frac{1}{2}\right),\ -\left(1+\frac{1}{3}\right), \dots\right\}$$

Consider the subsequence $\{x_{2n-1}\} = \{x_1, x_3, \dots\}$

This subsequence converges to a subsequential limit -1.

Consider the another subsequence

$$\{x_{2n}\} = \{x_2, x_4, x_6, \cdots\}.$$

This subsequence converges to a subsequential limit 1.

Therefore $\underline{\lim}\, x_n = -1$

and $\overline{\lim}\, x_n = 1$ $\Rightarrow$ oscillating finitely

Remember, the sequence $\{x_n\}$ is not convergent.

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(iv) The sequence $\{x_n\}_n = \{(-1)^n n^2\}$ is unbounded above and unbounded below.

$\underline{\lim}\, x_n = -\infty$ and $\overline{\lim}\, x_n = \infty$.

$\lim x_n$ oscillates infinitely

(v) The sequence $\{x_n\} = \{ n^{(-1)^n \cdot n} \}$ is unbounded above and bounded below.

Therefore $\overline{\lim}\, x_n = \infty$

and $\underline{\lim}\, x_n = 0$ (Try).

$\lim x_n$ oscillates infinitely.

(vi) The sequence $\{x_n\} = \{n^2\}$ is a unbounded above and bounded below.

Therefore $\overline{\lim}\, x_n = \infty$ and $\underline{\lim}\, x_n = \infty$.

$\Rightarrow \lim x_n = \infty$. $\Rightarrow$ Divergent sequence

(vii) The sequence $\{x_n\} = \{-n^2\}$ is a unbounded below and bounded above

Therefore $\overline{\lim}\, x_n = -\infty$, $\underline{\lim}\, x_n = -\infty$.

$\Rightarrow \lim x_n = -\infty$. $\Rightarrow$ Divergent sequence.

Diverges to $-\infty$.

Theorem: If a real sequence $\{x_n\}_n$ converges to a limit l, then every sub-sequence of $\{x_n\}$ must converge to a same limit and conversely.

Proof: If a sequence $\{x_n\}$ converges to l, then $\lim_{n\to\infty} x_n = l$.

So for any given $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon \quad \forall\, n \geq N$$

$$\Rightarrow l - \varepsilon < x_n < l + \varepsilon \quad \forall\, n \geq N$$

Let $\{x_{n_k}\}$ be a Subsequence of Sequence $\{x_n\}$.

Since $\{n_k\}$ is strictly increasing sequence of natural numbers $\exists$ a natural number N_0 such that $n_k > N$ $\forall n \geq N_0$

Therefore $l - \varepsilon < x_{n_k} < l + \varepsilon$ $\forall n \geq N_0$

This shows that

$$\lim_{n \to \infty} x_{n_k} = l.$$

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Note: If two subsequences of the sequence $\{x_n\}$ converge to two different real numbers, then the sequence does not converge.

Examples we have lots of examples.
Some examples of oscillatory sequence have been discussed earlier. See one example-

Q. Prove that the sequence $\left\{\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}\right\}$ is not convergent.

Proof: To prove this, we need to show that two subsequences of a given sequence $\{x_n\}_n$ have different limits.

(i) Consider the subsequence

$$\{x_{2n}\} = \left\{\left(1-\frac{1}{2}\right)\sin\frac{2\pi}{2}, \left(1-\frac{1}{4}\right)\sin\frac{4\pi}{2}, \dots\right\}$$

$$= \left\{\frac{1}{2}\sin\pi, \frac{3}{4}\sin 2\pi, \dots\right\}$$

$$= \{0, 0, \dots\}$$

This subsequence converges to 0.

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(ii) Consider a subsequence

$$\{x_{4n+1}\}_{n\geq 1}^{\infty} = \{x_5, x_9, x_{13}, \ldots\}$$

$$= \left\{\left(1-\frac{1}{5}\right)\sin\frac{5\pi}{2}, \left(1-\frac{1}{9}\right)\sin\frac{9\pi}{2}, \ldots\right\}$$

$$= \left\{\frac{4}{5}, \frac{8}{9}, \frac{12}{13}, \ldots\right\}$$ which converge to 1.

iii

Consider a subsequence

$$\{x_{4n-1}\} = \{x_3, x_7, x_{11}, \ldots\}$$

$$= \left\{-\frac{2}{3}, -\frac{8}{7}, -\frac{10}{11}, \ldots\right\}$$

which converges to -1.

Since two subsequences have two different subsequential limits, the given Sequence $\{x_n\}$ is not convergent.

$\overline{\lim}\, x_n = 1$ and $\underline{\lim}\, x_n = -1$.

This Sequence is oscillatory Sequence of finite oscillation.

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(7) **Remark:** A monotone sequence of real numbers having a divergent subsequence is properly divergent.

$\Rightarrow$ Prove that $\{x_n\}$ is divergent sequence where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

(Home work).

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Remark: Every sequence of real numbers has a monotone subsequence.

Bolzano-Weierstrass Theorem

Theorem: Every bounded sequence of real numbers has a convergent subsequence.

Note: Another way to state this theorem as —

Every bounded sequence of real numbers has a subsequential limit.

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Example:

(a) $\left\{ \sin \frac{n\pi}{2} \right\}$ is bounded sequence. Let $x_n = \sin \frac{n\pi}{2}$, $n \geq 1$

(i) The subsequence $\{x_{2n}\}$ is convergent with the subsequential limit 0.

(ii) The subsequence $\{x_{4n-3}\}$ is a convergent subsequence with the subsequential limit 1.

iii The subsequence $\{x_{2n-1}\}$ is a divergent subsequence.

Example: The sequence $\{x_n = n^{(-1)^n}\}$ is an unbounded sequence. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \dots\}$

i The subsequence $\{x_{2n}\}$ is a properly divergent since $\lim_{n\to\infty} x_{2n} = \infty$.

(ii) The subsequence $\{x_{2n-1}\}$ i.e. $\{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$ is a convergent subsequence since its subsequential limit is 0.

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Note: From the last example, it is very easy to tell that the converse of the Bolzano weierstrass theorem may or may not be true always.

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Note:

An unbounded Sequence may have a Convergent Subsequence.

This statement is true from the last example.

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Theorem: A bounded sequence $\{x_n\}_n$ is convergent if and only if

$$\boxed{\underline{\lim}\, x_n = \overline{\lim}\, x_n.}$$

Proof: Homework (Try).

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Note: This theorem can be restated as

A bounded sequence is convergent if and only if it has only one subsequential limit.

See the table :

Sequence $\{x_n\}_n$	Lower Limit $\lambda = \underline{\lim}\, x_n$	Upper Limit $\mu = \overline{\lim}\, x_n$	Convergent, Divergent or oscillating
(i) $\{1+(-1)^n\}$	0	2	$\lambda \neq \mu$, λ, μ finite oscillating finitely
(ii) $\frac{(-1)^n}{n^2}$	0	0	$\lambda = \mu$, λ, μ finite Convergent
(iii) $n(1+(-1)^n)$	0	∞	$\lambda \neq \mu$, λ finite but μ not oscillating infinitely
(iv) $(-1)^n \cdot n$	$-\infty$	∞ →	oscillating infinitely
(v) n^2	∞	∞	Diverges to ∞, i.e. properly divergent.

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Theorem: Let $\{x_n\}$ and $\{y_n\}$ be bounded sequence. Then

(i) $\overline{\lim}\, x_n + \overline{\lim}\, y_n \geq \overline{\lim}\,(x_n+y_n)$

(ii) $\underline{\lim}\, x_n + \underline{\lim}\, y_n \leq \underline{\lim}\,(x_n+y_n)$

For example: Let $x_n = \sin\frac{n\pi}{2}$, $n\in\mathbb{N}$

and $y_n = \cos\frac{n\pi}{2}$, $n\in\mathbb{N}$

Then $\underline{\lim}\,(x_n+y_n) = -1$, $\underline{\lim}\, x_n = -1$

and $\underline{\lim}\, y_n = -1$.

$\overline{\lim}\, x_n = 1$ and $\overline{\lim}\, y_n = 1$

$\overline{\lim}\,(x_n + y_n) = 1$

So, for this problem, we have

$\underline{\lim}\, x_n + \underline{\lim}\, y_n < \underline{\lim}\,(x_n + y_n)$

and $\overline{\lim}\, x_n + \overline{\lim}\, y_n > \overline{\lim}\,(x_n + y_n)$.

Remark: Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences and $u_n > 0$, $v_n > 0$ $\forall n \in \mathbb{N}$.

Then

(i) $\overline{\lim}\, x_n \cdot \overline{\lim}\, y_n \geq \overline{\lim}\,(x_n \cdot y_n)$

(ii) $\underline{\lim}\, x_n \cdot \underline{\lim}\, y_n \leq \underline{\lim}\,(x_n \cdot y_n)$

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Cauchy Sequence

- **Definition:** A Sequence $\{x_n\}_n$ of real numbers is said to be a Cauchy Sequence if for any $\varepsilon > 0$ $\exists$ a natural number N such that

$$\Rightarrow \boxed{|x_m - x_n| < \varepsilon \quad \forall\, m,n \geq N}$$

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replacing m by $n+p$, we may write the inequality as

$$\boxed{|x_{n+p} - x_n| < \varepsilon, \ \forall\, n \geq N \text{ and } \forall\, p \in \mathbb{N}}$$

The inequality of this definition is referred as Cauchy criterion.

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Example: The sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Sol^n: Let us choose natural numbers m and n such that $m > n$

Let $\varepsilon > 0$ be given.

Then $|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right|$

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$$|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| = \frac{m-n}{mn}$$

$$= \frac{1}{n}\left(1 - \frac{n}{m}\right)$$

$$< \frac{1}{n} < \varepsilon$$

Since $m > n$.

$\therefore |x_m - x_n| < \varepsilon$ whenever

$n > \frac{1}{\varepsilon}$ and $m > n$.

choose $N = \left[\frac{1}{\varepsilon}\right] + 1$.

For any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$|x_m - x_n| < \varepsilon \;\; \forall\, m, n \geq N$.

This shows the sequence $\{\frac{1}{n}\}$ is Cauchy sequence.

Example: The sequence $\{n^2\}$ is not Cauchy sequence.

Sol^n: Let us choose two natural numbers m and n such that $m > n$.

Let $\varepsilon > 0$ be given.

Choose $\varepsilon = \underline{1}$.

Now $|x_m - x_n| = |m^2 - n^2|$

$= (m+n)(m-n) > 2n > 1$

for any $n \in \mathbb{N}$

So, for $\varepsilon = 1$, $\nexists$ no such $N \in \mathbb{N}$

such that

$|x_m - x_n| < \varepsilon, \forall m, n \geq N.$

Hence by def^n, $\{n^2\}$ is not a Cauchy Sequence.

Example: $\left\{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right\}_n$ is not a Cauchy Sequence.

Home work (Try it).

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Theorem: A Convergent Sequence is a Cauchy Sequence.

Proof: Let $\{x_n\}$ be a convergent sequence and let $\lim_{n\to\infty} x_n = l$.

So for any given $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that $\boxed{|x_n - l| < \varepsilon/2, \forall n \geq N.}$

Let m and n be natural numbers considered to be greater than equal to natural number N.

i.e. $\boxed{m, n \geq N.}$

Then $|x_m - l| < \varepsilon/2$ and $|x_n - l| < \varepsilon/2$

Now $|x_m - x_n| = |x_m - l + l - x_n| \leq |x_m - l| + |x_n - l|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall\, m, n \geq N$

Therefore $|x_m - x_n| < \varepsilon, \ \forall \ m, n \geq N$

which proves that the sequence $\{x_n\}_n$ is a Cauchy Sequence.

(Proved)

Theorem: A Cauchy sequence of real numbers is convergent.

Proof: Home work (Try to prove it).

Use the Bolzano Weierstrass theorem.

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Note: Every Cauchy Sequence is bounded.

Cauchy's General Principle of Convergence

Cauchy's general principle of convergence or Cauchy's convergence criterion:

⇒ A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

This the necessary and sufficient condition for the convergence of a sequence.

Cauchy's Limit Theorems

Theorem (Cauchy's First theorem):

If the sequence $\{x_n\}$ has a limit l then the sequence $\left\{\frac{x_1+x_2+\cdots+x_n}{n}\right\}$ has the same limit l.

So $\lim_{n\to\infty} x_n = l \Rightarrow \lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = l.$

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- The converse of this theorem is not true. See example

$$x_n = (-1)^n \ldots$$

- **Corollary:** If $\lim_{n\to\infty} x_n = l$ where $x_n > 0$, $\forall n$ and $l \neq 0$, then

$$\lim \sqrt[n]{x_1 \cdot x_2 \cdots x_n} = l.$$

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Example: Prove that

$$\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$$

Sol^n: Let $x_n = \frac{1}{n}$. Then $\lim_{n\to\infty} x_n = 0$.

By Cauchy's theorem

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0.$$

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Example: Prove that

$$\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$$

Try this (Home work).

Cauchy's Second Theorem:

Theorem: Let $\{x_n\}$ be a real sequence such that $x_n > 0 \ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$ (finite or infinite)

Then

$$\lim_{n\to\infty} x_n^{1/n} = l.$$

The converse of this theorem is not true.

Example: Prove that $\lim_{n\to\infty} n^{1/n} = 1$.

Sol^n: Let $x_n = n$. Then $x_n > 0 \ \forall n \in \mathbb{N}$

and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 > 0$.

By theorem we have $\lim_{n\to\infty} x_n^{1/n} = \lim_{n\to\infty} n^{1/n} = \underline{\underline{1}}$

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Example: Prove that $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}$.

Sol^n: Let $x_n = \frac{n!}{n^n}$. Then $x_n > 0$, $\forall n \in \mathbb{N}$

and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{e} > 0$.

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By theorem, we have

$$\lim_{n\to\infty} x_n^{1/n} = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.$$

Example: Prove that

$$\lim_{n\to\infty} \frac{\{(n+1)(n+2)\cdots 2n\}^{1/n}}{n} = \frac{4}{e}.$$

(Home work).