भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

First Lecture on Calculus-I

(MA-1110)

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Calculus-I (MA-1110) Syllabus

- Sequence and Series:
  - Limit of a sequence
  - Convergent sequence
  - Monotone and Cauchy sequences
  - Infinite series
  - Test for convergence and divergence
  - Integral test
  - Alternating series
  - Leibnitz test
- Differential Calculus
  - Continuity and Differentiability
  - Rolle’s Theorem
  - Lagrange’s Mean Value theorem

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Suggested Books

- Robert G. Bartle, Donald R. Sherbert: Introduction to real analysis.
- Kenneth A. Ross: Elementary Analysis: The Theory of Calculus, 2nd Edition, Springer Science, Business Media, New York 2013.
- Richard R. Goldberg: Methods of real analysis, 2nd Edition, John Wiley and Sons Inc, New York, 1976.
- Sterlin K Berberian: A First Course in Real Analysis
- Wilian R. Wade: Introduction to Analysis, 3rd Edition Pearson Education, New Jersey, 2004.

Introduction

What is Calculus?

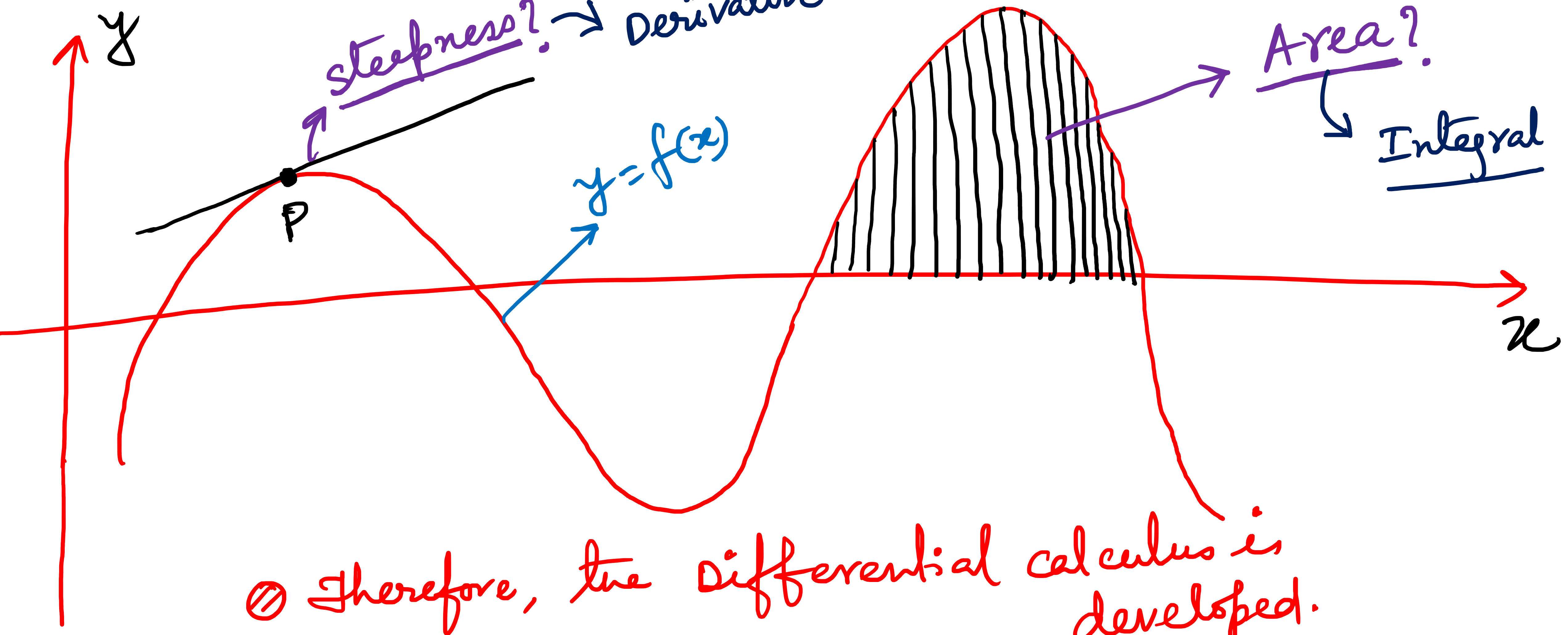
Calculus is a branch of mathematics mainly focused on derivatives and integrals. Also, it is focused on limits, continuity, sequences, and series.

- In the late 17th Century, the calculus was developed independently by Issac Newton and Gottfried wilhelm Leibniz.

- In calculus, we first start with two big questions about the functions.

- First question: How steep is a function at a point? So, here we are talking about the steepness of a function at a point.

- This is answered by describing derivatives.



- Therefore, the differential calculus is developed.

What is derivative?

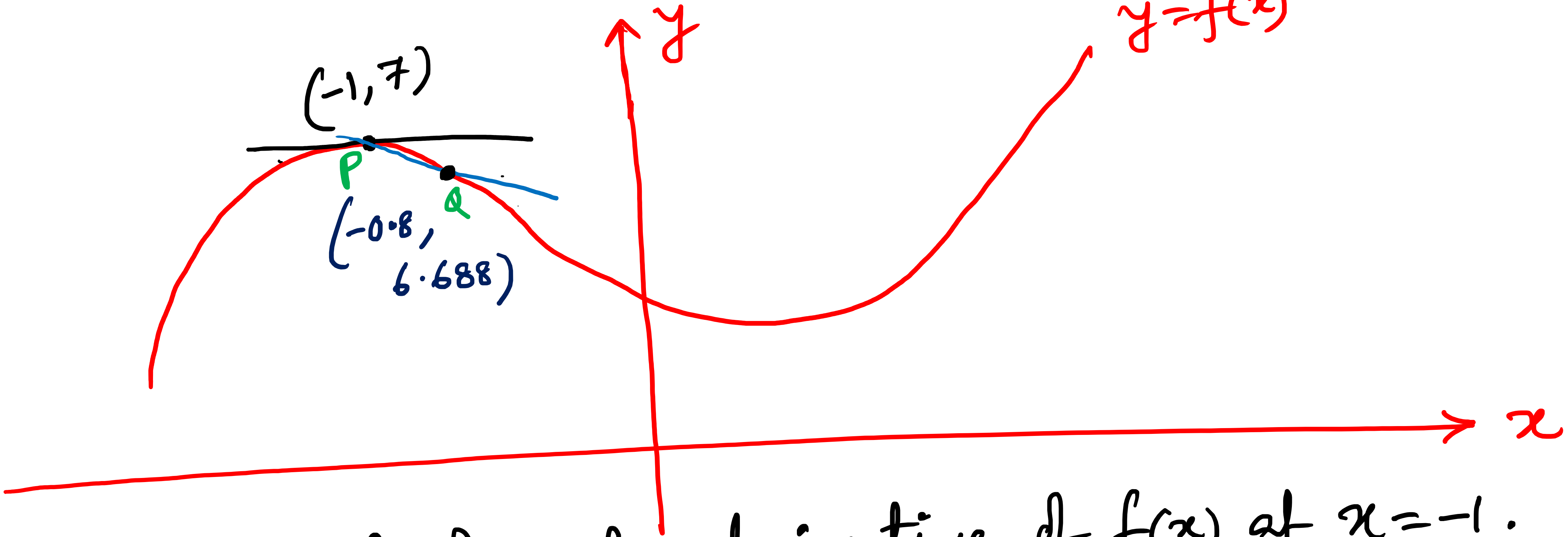
This says the steepness of a function at a point.

Geometrically, it is the slope of the tangent to the curve.

It describes the rate of change of a function at a point.

Example :

$$y = f(x) = x^3 - 4x + 4$$



we have to find the derivative of $f(x)$ at $x = -1$.

Now we start to choose a nearby point Q (-0.8)

Therefore, we can obtain the slope of PQ line which is approximate to the slope of the tangent at P.

Slope of PQ is $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(-0.8) - f(-1)}{-0.8 + 1} = -2.08$$

Similarly, we pick another point which is more closer to P than Q and calculate the slope.

x	y	slope
-0.8	6.688	-2.08
-0.9	6.871	-1.57
-0.95	6.94263	-1.2925
-0.98	6.97881	-1.1188
-0.99	6.9897	-1.0597
-0.999	6.999	-1.006
-0.9999	6.9999	-1.0006
-1	7	-1

These slopes are approximate values

These are making a sequence

This is it's limit value.

So we can see that as the point Q approaches to P, we can get more accurate approximate value to the slope of the tangent at P, i.e, the derivative of the function at the point P.

So, slope of the curve at $(-1, 7)$ is -1

$\Rightarrow$ derivative of $f(x)$ at $x = -1$ is -1.

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So, we get the idea of derivative as

the slope of tangent to the curve $y=f(x)$ at a point $P(x_1, f(x_1))$

= derivative of $f(x)$ at a point x_1

$$= \lim_{x \to x_1} \frac{f(x)-f(x_1)}{x-x_1}$$

→ rate of change of the function $f(x)$.

we will discuss more on derivatives later.

- **Second Question:**

What is the area under or between the curves over some region?

This question is answered by describing the Integral.

Therefore, the Integral calculus developed.

Calculus

- Differential Calculus
- Integral Calculus

(i) Differential calculus and Integral calculus are connected by the fundamental theorem of Calculus which state that Differentiation is the reverse process of integration.

(ii) However, in this calculus-I course, we will only focus on the Differential calculus.

Sequences

Definition :

A Sequence of real numbers is a function or mapping $f : N \to R$, where N is the set of Natural numbers and R is the set of Real numbers.

N Domain $\qquad\qquad$ R co-domain

Pre-images $\qquad\qquad\qquad\qquad$ Images

$1 \xrightarrow{f} f(1)$ or f_1

$2 \xrightarrow{f} f(2)$ or f_2

$\vdots \qquad\qquad \vdots$

$n \xrightarrow{f} f(n)$ or f_n

- Sequence means we are usually talking about the sequence of real numbers. It is called as Real Sequence.

- A real sequence is generally denoted by $\{f_n\}_{n=1}^{\infty}$ or $\{f(n)\}_{n=1}^{\infty}$ or $\{f(n)\}_{n\in\mathbb{N}}$ or $\{f_n\}_{n\in\mathbb{N}}$.

So, we can write a sequence as $\{f(1), f(2), \dots\}$ or $\{f_1, f_2 \dots\}$

Note: (i) The images $f(1), f(2), f(3), \ldots f(n), \ldots$ are the real numbers.

(ii) $f(n)$ or f_n which is the image of the nth element, is called as the nth term of the real Sequence $\{f(n)\}$ or $\{f_n\}$.

(iii) The range of a sequence is a subset $\{f(n) : n \in \mathbb{N}\}$ of $\mathbb{R}$.

- To express a real sequence, anyone can use the symbol like $\{x_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, etc. instead of $\{f_n\}_{n\in\mathbb{N}}$.

- **Examples:** (i) Let $x : \mathbb{N} \to \mathbb{R}$ be defined by $x_n = n$, $n \in \mathbb{N}$. Then $x_1 = 1$, $x_2 = 2$, ... so on. Therefore $\{x_n\}_{n\in\mathbb{N}}$ i.e., $\{1, 2, 3, \ldots, n, \ldots\}$ is a sequence of positive integers.

(ii) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ where $x_n = 2n$. Then the sequence $\{2, 4, 6, \ldots, 2n, \ldots\}$ is the sequence of even integers.

(iii) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ where $x_n = \frac{1}{n}$. This is well known harmonic sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$.

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(iv) Consider a sequence $\{x_n\}$ where $x_n = 2n-1$, $n \in \mathbb{N}$. This sequence $\{1, 3, 5, \ldots, 2n-1, \ldots\}$ is the sequence of odd positive integers.

(v) Consider a sequence $\{x_n\}_n$, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \in \mathbb{N}$. First term is $x_1 = 1$, the second term is $x_2 = 1 + \frac{1}{2} = 3/2$.

third term is $x_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$, so on.

So, the sequence $\{x_n\}$ can be written as

$$\{1, 3/2, 11/6, \ldots\}.$$

vi Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = 4$.

So every term of the sequence is 4.

Therefore, this sequence is called as constant sequence. It is $\{4, 4, 4, \ldots\}$

(vii) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ where $x_n = (-1)^{n-1}, n\in\mathbb{N}$. Then the sequence is written as $\{1,-1,1,-1,1,-1,\ldots\}$. The range/trace is the set $\{1,-1\}$, containing only two elements 1 and -1.

(viii) Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ where $x_n = 1/n^2, n\in\mathbb{N}$. Then the sequence is $\{1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots\}$.

ix) Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = \sin \frac{n\pi}{2}$, $n \in \mathbb{N}$. The sequence is $\{1, 0, -1, 0, 1, 0, \dots\}$. Therefore the range of sequence is $\{-1, 0, 1\}$.

Note: Sometimes we can not get the explicit analytical formula to describe x_n; but the terms of the sequence may be well-determined.

(x) Consider a sequence $\{x_n\}_n$ where $x_n = n$th prime number. This gives the sequence $\{2, 3, 5, 7, 11, 13, \ldots\}$

(xi) This is the example for recursive definition of a sequence.

Consider a sequence $\{x_n\}_{n \in N}$ where $x_n = x_{n-1} + x_{n-2}$ $(n \in N, n \geq 3)$
$x_1 = 1$ and $x_2 = 1$.

P-22

Therefore the sequence is written as

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence is known as Fibonacci's sequence.

Xii

Consider a sequence $\{x_n\}_n$ defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$, $n \geq 1$, $n \in \mathbb{N}$

The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$

xiii $\{x_n\}_{n\in\mathbb{N}}$ where $x_n = \sec\left(\frac{\pi\sqrt{n}}{2}\right)$

does not define a sequence, since x_n is not defined for infinite number of values of n (whenever n is the square of an odd positive integer).

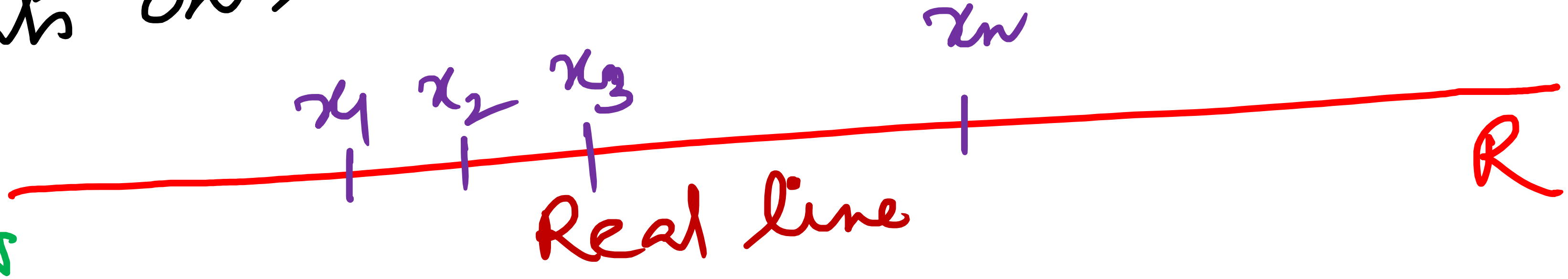
For example, we can't get the value of x_1, x_9, x_{25}, x_{49}, ... so on

These terms of the sequences don't exist.

(i) Geometrical Representation:

Each term of sequence $\{x_n\}_n$ of real numbers corresponds to a point on the real axis. Therefore the collection of such points is called as a sequence of points on the real line.

Sequence $\{x_n\}_{n \in \mathbb{N}}$



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Bounded Sequences

Bounded above Sequence:

A Sequence $\{x_n\}_n$ is said to be bounded above if $\exists$ a real number U such that $\boxed{x_n \le U}$ $\forall n \in \mathbb{N}$. The real number U is called as an upper bound of $\{x_n\}_n$.

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Bounded below Sequence:

A Sequence $\{x_n\}_n$ is said to be **bounded below** if $\exists$ a real number u such that $\boxed{x_n \geq u}$, $\forall n \in \mathbb{N}$. Therefore the real number u is called as a **lower bound** of $\{x_n\}_n$

Bounded Sequence:

A sequence $\{x_n\}_n$ is said to be bounded if it is bounded both above and below, i.e., $\exists$ real numbers u and U such that

$$\boxed{u \le x_n \le U \quad \forall n \in \mathbb{N}}$$

$\Rightarrow$ For a bounded sequence, the sequence is bounded above as well as bounded below.

P-28

Remark: The real sequence $\{x_n\}_n$ is bounded if and only if it is bounded above as well as bounded below.

Alternative Definition:

A real sequence $\{x_n\}_n$ is said to be bounded if $\exists$ a positive real number B such that

$$\boxed{\begin{aligned} &|x_n| \leq B, \quad \forall n \in \mathbb{N} \\ &\Rightarrow -B \leq x_n \leq B, \quad \forall n \in \mathbb{N}. \end{aligned}}$$

Supremum or least upper bound:

The supremum or least upper bound of a real sequence $\{x_n\}_n$ is a real number M satisfying the following properties -

(i) $\forall n \in \mathbb{N}$, $\boxed{x_n \leq M}$ Bounded above sequence

(ii) Given any $\varepsilon > 0$ (no matter, how small) $\exists$ a natural number K such that $\boxed{x_K > M - \varepsilon}$

Infimum or greatest lower bound:

The infimum or greatest lower bound of $\{x_n\}_n$ is a real number m satisfying the following properties

(i) $\forall n \in \mathbb{N}$, $\boxed{x_n \geq m}$ ← Bounded below sequence

(ii) For given any $\varepsilon > 0$ (no matter how small) $\exists$ a natural number K such that

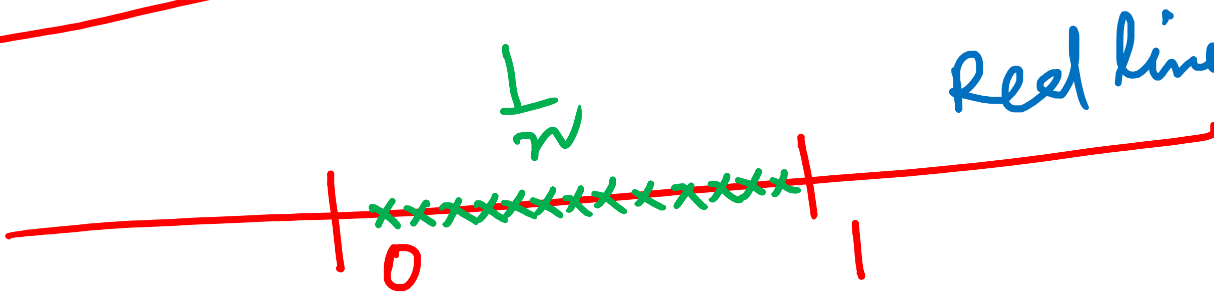
$$\boxed{x_K < m + \varepsilon}$$

Note: For most the problems, it is required to know whether the sequence is bounded or not (i.e., we may not require to find the supremum or infimum for a bounded sequence)

Examples:

(i) The sequence $\{\frac{1}{n} : n \in \mathbb{N}\}$ is a bounded sequence since 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

$$\boxed{0 < \frac{1}{n} < 1}$$



(ii) The real sequence $\{n^2\}$ is bounded below since $\boxed{n^2 \geq 1, \forall n}$. However it is not bounded above i.e, unbounded above. So, the sequence $\{n^2\}$ is not bounded i.e, unbounded.

(iii) The sequence $\{-n^2\}$ is not bounded since it is bounded above but unbounded below.

(iii) $\left\{ \dfrac{3n+1}{n+2} \right\}_{n \in \mathbb{N}}$ is a bounded sequence.

The sequence is $\left\{ \dfrac{4}{3}, \dfrac{7}{4}, \dfrac{10}{5}, \dfrac{13}{6}, \ldots \right\}$

We can see $\dfrac{3n+1}{n+2} = \dfrac{3 + 1/n}{1 + 2/n}$, $\forall n$.

$\dfrac{3n+1}{n+2}$ cannot exceed 3.

So, we have $\boxed{\dfrac{4}{3} \leq \dfrac{3n+1}{n+2} \leq 3}$

Hence it is bounded above and bounded below. Therefore $\left\{\frac{3n+1}{n+2}\right\}$ is bounded sequence.

(iv) The sequence $\{(-1)^n n\}$ is neither bounded above nor bounded below. So, it is unbounded sequence.

(v) The sequence $\{(-1)^{n-1}\}$ is bounded since it bounded above and bounded below.

The sequence is $\{1, -1, 1, -1, \ldots\}$.

So, we have $\boxed{-1 \leq (-1)^{n-1} \leq 1}$, $\forall n \in \mathbb{N}$.

$$\Rightarrow \left|(-1)^{n-1}\right| \leq 1, \forall n \in \mathbb{N}$$

Operations on Sequences

P-37

- The algebraic operations can be performed on the sequences.
These algebraic operations are namely addition, subtraction, multiplication, scalar multiplication, division.

Note: Element by element operations are performed.

(i) **Addition:** For two real sequences $\{u_n\}$ and $\{v_n\}$, $n \in \mathbb{N}$, we have

$$\{u_n\} + \{v_n\} := \{u_n + v_n\}$$

Example:

$$\{1\} + \left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} = \left\{1 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$$

(ii) Subtraction:

For any two sequences $\{u_n\}$ and $\{v_n\}$,

$$\boxed{\{u_n\} - \{v_n\} := \{u_n - v_n\}}$$

Example:

$$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} - \left\{\frac{1}{n^2}\right\}_{n\in\mathbb{N}} = \left\{\frac{1}{n} - \frac{1}{n^2}\right\}_{n\in\mathbb{N}}$$

$$= \left\{\frac{n-1}{n^2}\right\}_{n\in\mathbb{N}}$$

(iii) **Multiplication:**

For any two sequences $\{u_n\}$ and $\{v_n\}$,

$$\boxed{\{u_n\} \cdot \{v_n\} := \{u_n \cdot v_n\}}$$

(iv) **Scalar multiplication:**

$$\boxed{c \cdot \{u_n\} := \{c \cdot u_n\}}$$

where c is a scalar.

(v) **Division:** For any two any real sequence $\{u_n\}$ and $\{v_n\}$,

$$\boxed{\frac{\{u_n\}}{\{v_n\}} := \left\{\frac{u_n}{v_n}\right\} \text{ if } v_n \neq 0 \ \forall n \in \mathbb{N}}$$

Limit of a Sequence

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Definition:

A finite real number l is said to be a limit of a sequence $\{x_n\}_n$ of real numbers if for any given positive number ε (no matter, how small) there exists a natural number N (N will usually depend on ε) such that

$$\boxed{|x_n - l| < \varepsilon \quad \forall\, n \geq N}$$

P-43

i.e, $\boxed{l-\varepsilon < x_n < l+\varepsilon, \ \forall n \geq N}$

In this case, we write :

$\boxed{\lim_{n \to \infty} x_n = l \quad \text{or} \quad x_n \to l \text{ as } n \to \infty}$

Note:

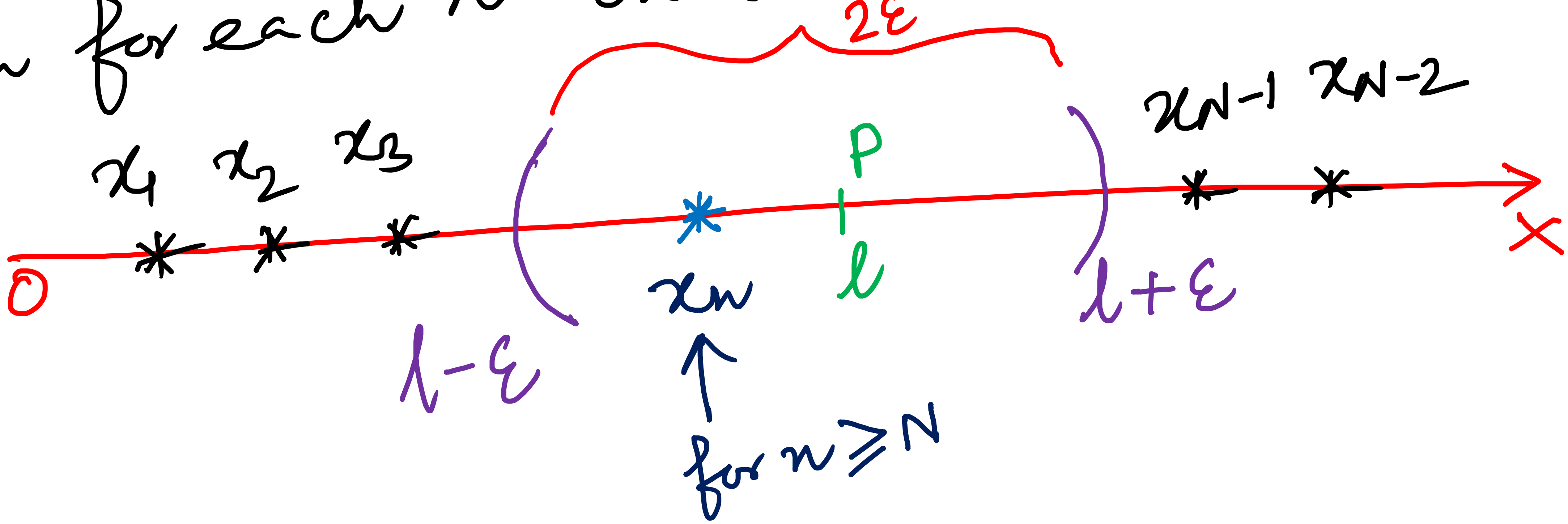
$$\lim_{n\to\infty} x_n = l$$

$\Rightarrow$ i $\lim_{n\to\infty} x_n$ exists

ii the limit is l.

Geometrical representations:

Let OX be a real axis. We mark points x_n for each n on the real axis OX.



Let P be the point corresponding to real number l.

$x_n \to l$ as $n \to \infty$ means after attaining the values of n to N, all member of $\{x_n\}_{n \geq N}$ belongs to the open interval $(l-\varepsilon, l+\varepsilon)$, i.e, all the terms

$x_N, x_{N+1}, x_{N+2}, \ldots$ of the sequence $\{x_n\}_n$ belong to the open interval $(l-\varepsilon, l+\varepsilon)$.

Note: (i) The limit of a sequence may or may not exist.

(ii) The value N depends on ε and if ε is small, N will have to be large.