भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Lecture Note - 2

Second Lecture on Calculus-I

(MA-1110)

Dr. Jyotirmoy Rana
Assistant Professor
Department of Mathematics
IIT Hyderabad

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Sequences

- Limit of a sequences
- Convergent sequences
- Null Sequences
- Some important Theorems

Limit of a Sequence

Definition: A finite real number l is said to be a limit of a sequence $\{x_n\}_n$ of real numbers if for any given positive number ε (no matter, how small) there exists a natural number N (N will usually depend on ε) such that

$$\boxed{|x_n - l| < \varepsilon \quad \forall\, n \geq N}$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

i.e, $\boxed{l-\varepsilon < x_n < l+\varepsilon, \ \forall n \geq N}$

In this case, we write:

$$\boxed{\lim_{n\to\infty} x_n = l \quad \text{or} \quad x_n \to l \text{ as } n \to \infty}$$

Note:

$$\lim_{n \to \infty} x_n = l$$

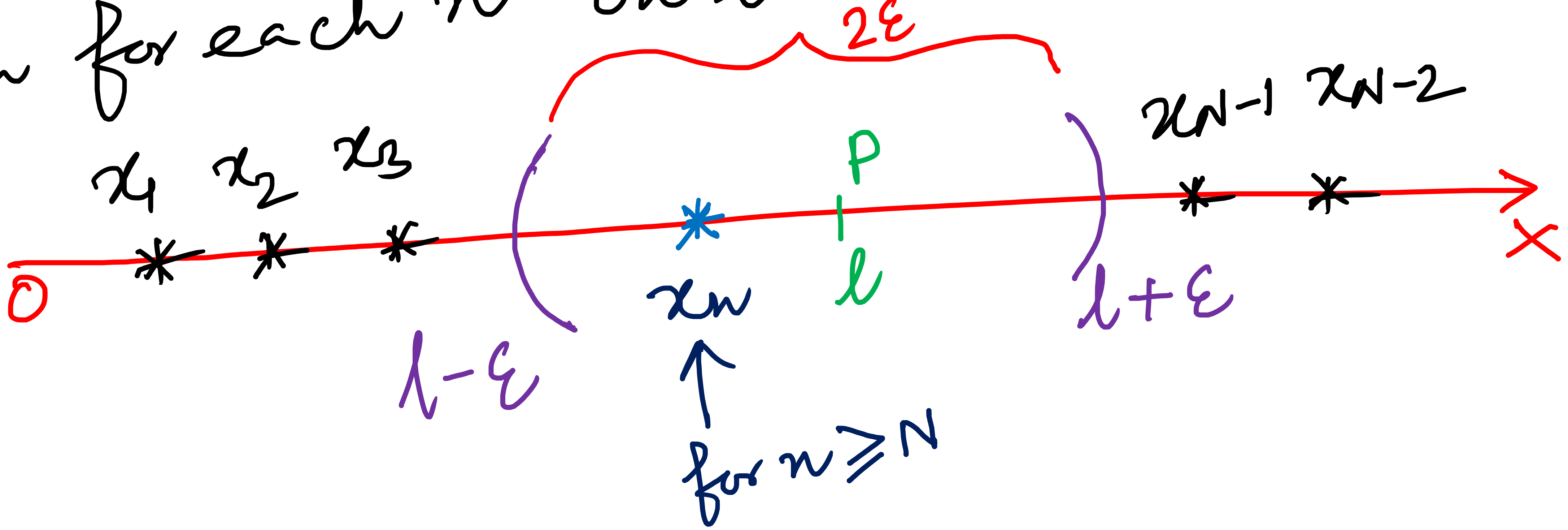
$\Rightarrow$ (i) $\lim_{n \to \infty} x_n$ exists

(ii) the limit is l.

P-3

Geometrical representations:

Let OX be a real axis. We mark points x_n for each n on the real axis OX.



Let P be the point corresponding to real number l.

$x_n \to l$ as $n \to \infty$ means after attaining the values of n to N, all member of $\{x_n\}_{n \geq N}$ belongs to the open interval $(l-\varepsilon, l+\varepsilon)$, i.e, all the terms

$x_N, x_{N+1}, x_{N+2}, \ldots$ of the Sequence $\{x_n\}_n$ belong to the open interval $(l-\varepsilon, l+\varepsilon)$.

Note: (i) The limit of a sequence may or may not exist.

(ii) The value N depends on ε and if ε is small, N will have to be large.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Theorem: If the limit of a real Sequence $\{x_n\}_n$ exists, then it is unique

i.e, a Sequence $\{x_n\}_n$ of real numbers Can have atmost one limit.

Proof: Let us Consider $\lim_{n\to\infty} x_n = l_1$

and $\lim_{n\to\infty} x_n = l_2$.

Then we will prove that

$$l_1 = l_2.$$

Consider, $\varepsilon > 0$. Then by the definition of limit, $\exists N_1$ so that

$$|x_n - l_1| < \varepsilon/2, \quad \forall n \geq N_1$$

and also $\exists N_2$ so that

$$|x_n - l_2| < \varepsilon/2, \quad \forall n \geq N_2.$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Let $N = \max(N_1, N_2)$ = greater of the two integers N_1 and N_2

Then $\forall\ n \geq N$, we have

$$|l_1 - l_2| = |(l_1 - x_n) + (x_n - l_2)|$$
$$\leq |x_n - l_1| + |x_n - l_2|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow |l_1 - l_2| = 0 \Rightarrow \underline{l_1 = l_2}$. (Proved)

So, we have $\forall \varepsilon > 0$, $|l_1 - l_2| < \varepsilon$

$$\Rightarrow |l_1 - l_2| = 0$$

$$\Rightarrow l_1 = l_2$$

Hence it is proved.

Indian Institute of Technology Hyderabad

Convergent Sequence

* Definition:

A sequence $\{x_n\}_n$ of real numbers is said to be a convergent sequence if it has a limit $l \in \mathbb{R}$. Then we can say that the sequence $\{x_n\}_n$ converges to l.

Thus $\{x_n\}_n$ converges to l implies that

P-12

for given $\varepsilon > 0$, $\exists$ a natural number N such that $|x_n - l| < \varepsilon$, $\forall\ n \geq N$.

So, we can write

$$\lim_{n \to \infty} x_n = l$$

or simply $\lim x_n = l$.

Examples

(1) Show that $\lim_{n\to\infty} \frac{1}{n^2} = 0$ (By the definition only)

Sol^n: Let $\varepsilon > 0$ be given.
Now our aim is to find such natural number N such that the inequality

$$\left|\frac{1}{n^2} - 0\right| < \varepsilon \text{ holds } \forall\, n \geq N.$$

Then $\left|\frac{1}{n^2} - 0\right| < \varepsilon$

$$\Rightarrow \frac{1}{n^2} < \varepsilon \Rightarrow n > \frac{1}{\sqrt{\varepsilon}}.$$

So for a given $\varepsilon > 0$, the last inequality holds whenever $n > \frac{1}{\sqrt{\varepsilon}}$.

Set $N = \left[\frac{1}{\sqrt{\varepsilon}}\right] + 1$ [For example if $\varepsilon = 0.01$ then $N = 11$; if $\varepsilon = 0.001$ then $N = 32$]

Indian Institute of Technology Hyderabad

P-15

Then $\forall\ n \geq N$, $\left|\frac{1}{n^2} - 0\right| < \epsilon$.

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^2} = 0. \quad \text{(Proved)}$$

So, the sequence $\left\{\frac{1}{n^2}\right\}$ has a limit 0

i.e., the sequence $\left\{\frac{1}{n^2}\right\}$ converges to 0.

Try the similar problem $\lim \frac{n^2+1}{n^2} = 1$.

Indian Institute of Technology Hyderabad

P-16

② Verify that $\lim_{n\to\infty} \frac{3n+1}{7n-4} = 3/7$

Solⁿ: For a given $\varepsilon > 0$,

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| = \left| \frac{21n+7-21n+12}{7(7n-4)} \right|$$

$$< \varepsilon .$$

P-17

Indian Institute of Technology Hyderabad

$$\Rightarrow \left| \frac{19}{7(7n-4)} \right| < \varepsilon$$

$$\Rightarrow \frac{19}{7(7n-4)} < \varepsilon \quad \left[\text{Since } 7n-4>0 \ \forall\ n \geq 1 \right]$$

$$\Rightarrow 7n-4 > \frac{19}{7\varepsilon}$$

$$\Rightarrow 7n > \frac{19}{7\varepsilon} + 4$$

$$\Rightarrow n > \frac{19}{49\varepsilon} + \frac{4}{7}.$$

So for given $\varepsilon > 0$, choose

$$N = \left[\frac{19}{49\varepsilon} + \frac{4}{7} \right] + 1.$$

Therefore we have

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon, \quad \forall\, n \geq N$$

$$\Rightarrow \lim_{n \to \infty} \frac{3n+1}{7n-4} = \frac{3}{7}.$$

(Proved)

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

(3) Verify a constant sequence $\{c, c, \ldots\}$ converges to a constant c.

Sol^n:

Here $x_n = c,\ \forall n \in \mathbb{N}$. Let us choose a positive ε.

Now $|x_n - c| = |c - c| = 0 < \varepsilon,\ \forall n \geq 1$

Therefore $\lim x_n = c$

④ Verify $\lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$

Sol^n:

$$\left|\frac{n^2+1}{2n^2+3} - \frac{1}{2}\right| = \left|-\frac{1}{4n^2+6}\right|$$

$$= \frac{1}{4n^2+6} < \text{any given positive number } \varepsilon.$$

whenever $4n^2+6 > \frac{1}{\varepsilon}$, i.e., whenever $n > \sqrt{\frac{1}{4\varepsilon} - \frac{3}{2}}$.

Choose $N = \left[\sqrt{\frac{1}{4\varepsilon} - \frac{3}{2}}\right] + 1$

Therefore for $n \geq N$,

$$\left|\frac{n^2+1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon$$

Hence by def^n, $x_n \to \frac{1}{2}$ as $n \to \infty$,

where $x_n = \frac{n^2+1}{2n^2+3}$.

5. $\lim_{n\to\infty} \frac{n^2-1}{n^2+n+1}$. Guess the limit and then verify its truth by definition test.

Sol^n:

$$\frac{n^2-1}{n^2+n+1} = \frac{1-\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}} \longrightarrow 1 \quad \text{as } n\to\infty$$

Since for large values of n, $\frac{1}{n}$ and $\frac{1}{n^2}$ approach to zero.

Now we guess that the limit may be 1.

We now verify $\lim_{n\to\infty} \frac{n^2-1}{n^2+n+1} = 1$

$$\left| \frac{n^2-1}{n^2+n+1} - 1 \right| = \left| \frac{-n-2}{n^2+n+1} \right|$$

$$= \frac{n+2}{n^2+n+1}$$

P-23

P-24

For $n > 2$, $\quad n+n > 2+n$

or, $n+2 < n+n$

and $n^2+n+1 > n^2$

or, $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$

So we have $\dfrac{n+2}{n^2+n+1} < \dfrac{2n}{n^2} = \dfrac{2}{n}$

$$\left|\frac{n^2-1}{n^2+n+1}-1\right| = \frac{n+2}{n^2+n+1} < \frac{2}{n} < \text{ any given positive number } \varepsilon,$$

whenever $n > \frac{2}{\varepsilon}$.

Now choose $N = \left[\frac{2}{\varepsilon}\right]+1$.

Therefore for $n \geq N$, $\left|\frac{n^2-1}{n^2+n+1}-1\right| < \varepsilon$

Try a similar problem

$$\lim_{n \to \infty} \frac{2n^2+1}{n^2+3n} = 2.$$

Null Sequence

Definition:

A Sequence $\{x_n\}_n$ is called a null Sequence if and only if $\boxed{\lim_{n\to\infty} x_n = 0.}$

i.e., for given $\varepsilon > 0$, $\exists$ a natural number N such that $\boxed{|x_n| < \varepsilon, \ \forall\ n \geq N}$

Examples:

(1) Prove that the sequence $\left\{\frac{1}{n^p}\right\}$, where $p>0$, is a null sequence.

Proof: Let $\varepsilon>0$ be given.

Then $\left|\frac{1}{n^p}-0\right| = \frac{1}{n^p} < \varepsilon$,

whenever $n^p > \frac{1}{\varepsilon}$, i.e., whenever $n > \left(\frac{1}{\varepsilon}\right)^{1/p}$.

Now choose $N = \left[\left(\frac{1}{\varepsilon}\right)^{1/p} \right] + 1.$ P-29

$$= \left[e^{\frac{1}{p} \log \frac{1}{\varepsilon}} \right] + 1$$

Therefore for all $n \geq N$

$$\left| \frac{1}{n^p} - 0 \right| < \varepsilon$$

This proves that $\lim_{n \to \infty} \frac{1}{n^p} = 0$, i.e., $\left\{ \frac{1}{n^p} \right\}_n$, $p > 0$ is a null sequence.

P-30

(2) The sequence $\{x^n\}_n$ is a null sequence if $|x|<1$. (Prove that)

Proof: We have to prove that

$$\lim_{n\to\infty} x^n = 0 \quad \text{when } |x|<1.$$

Let us choose a positive ε.

We have $|x|<1 \Rightarrow -1<x<1.$

P-21

Two cases arise.

Case 1: $x=0$. In this Case, the Sequence is $\{0,0,0,0,\ldots\}$ which converges to 0.

So $\lim_{n\to\infty} x^n = 0$ when $x=0$.

Case 2: $x\neq 0$ and $|x|<1$

$|x|<1 \Rightarrow \frac{1}{|x|} > 1$. Let $\frac{1}{|x|} = h+1$ where $h>0$

$$\frac{1}{|x|} = h+1 \Rightarrow |x| = \frac{1}{h+1}, \; h>0$$

now $\left|x^n - 0\right| = |x|^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh}$

Since $(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \dots + h^n$

$> 1+nh$, h being positive.

$> nh$

P-32

Then $|x^n - 0| < \frac{1}{nh} < \varepsilon$ (given)

whenever $n > \frac{1}{h\varepsilon}$.

choose $N = \left[\frac{1}{h\varepsilon}\right] + 1 \in$ the set of natural numbers $\mathbb{N}$

Therefore $|x^n - 0| < \varepsilon \ \forall \ n \geq N$.

Hence by definition, $x^n \to 0$ as $n \to \infty$ where $|x| < 1$.

Try Similarly

$$\lim_{n\to\infty} n x^n = 0 \quad (\text{if } |x|<1).$$

P-35

(3) To prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof: Here our aim is to prove that $\{\sqrt[n]{n} - 1\}_n$ is a null sequence i.e., $\lim_{n\to\infty} \sqrt[n]{n} - 1 = 0$.

For sake of simplicity, let $h_n = \sqrt[n]{n} - 1$

Now for $n>1$, $\sqrt[n]{n} > 1$

and $h_n = \sqrt[n]{n} - 1 > 0$

$$h_n = n^{1/n} - 1 \Rightarrow n = (1+h_n)^n$$

$$= 1 + nh_n + \frac{n(n-1)}{2!} h_n^2 + \cdots + h_n^n$$

$$> \frac{n(n-1)}{2!} h_n^2 \quad (\text{since } h_n > 0)$$

So we have $h_n^2 < \frac{2}{n-1}$ or, $h_n < \sqrt{\frac{2}{n-1}}$

Let us choose a positive ε.

$$\left|\sqrt[n]{n}-1-0\right| = |h_n - 0| = |h_n| = h_n < \sqrt{\frac{2}{n-1}} < \varepsilon$$

whenever $\frac{2}{n-1} < \varepsilon^2$ i.e., $n > \frac{2}{\varepsilon^2}+1$

Choose $N = \left[\frac{2}{\varepsilon^2}+1\right]$.

Therefore $\left|\sqrt[n]{n}-1-0\right| < \varepsilon \;\; \forall\, n \geq N$.

$\Rightarrow$ Try Similar $\lim_{n\to\infty} x^{1/n} = 1$ whenever $x>0$.

Theorems on limits of sequences

Theorem: A convergent sequence is bounded.

Proof: Let $\{x_n\}$ be a convergent sequence and $\lim_{n \to \infty} x_n = l$.

For a sake of definiteness, let us choose $\varepsilon = 1$.

Therefore, for given $\varepsilon > 0$ $(\varepsilon = 1)$,

$\exists\ N \in \mathbb{N}$ such that

$$|x_n - l| < 1 \quad \forall\ n \geq N,$$

[Since $x_n \to l$ as $n \to \infty$]

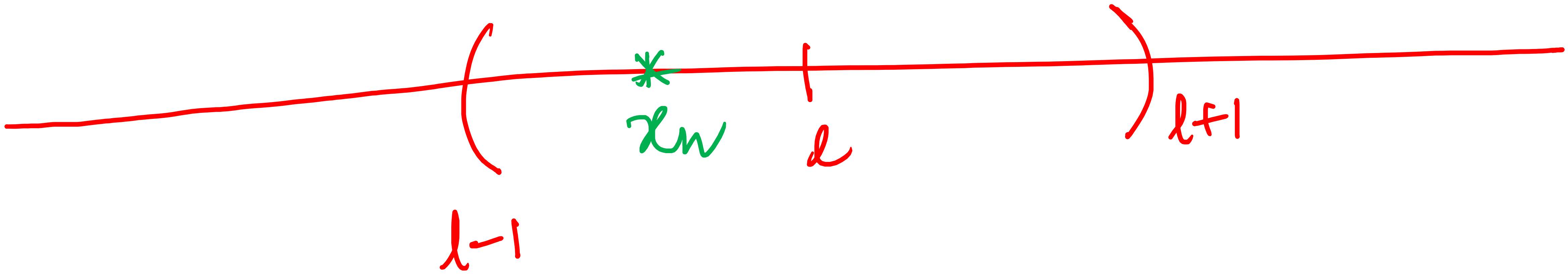
Now our aim is to prove that the sequence $\{x_n\}$ is bounded.

P-39

So $\forall\, n \geq N$, $|x_n - l| < 1$

$$\Rightarrow -1 < x_n - l < 1$$

$$\Rightarrow l-1 < x_n < l+1$$



Let $B = \max\{x_1, x_2, \ldots, x_{N-1}, l+1\}$

and $b = \min\{x_1, x_2, \ldots, x_{N-1}, l-1\}$

Then $b \leq x_n \leq B \quad \forall n \in \mathbb{N}$

This implies that $\{x_n\}_n$ is a bounded sequence.

P-4)

Remarks:

$\{x_n\}$ Converges
$\Rightarrow \{x_n\}$ is bounded.

Remember

Also, $\{x_n\}$ is not bounded
$\Rightarrow \{x_n\}$ is not convergent

i.e., An unbounded Sequence is not Convergent.

P-43

Note: The converse of this theorem is not true always.

$\Rightarrow$ Every bounded sequence is not convergent.

For example : $\{x_n\}$ where $x_n = 1+(-1)^n$

or $x_n = (-1)^n$

Two sequences are oscillatory sequences of finite oscillation.

The sequences $\{1+(-1)^n\}$ and $\{(-1)^n\}$ both are bounded but they are not convergent.

The sequence $\{1+(-1)^n\}$ oscillates between 0 and 2 without converging to a unique limit.

Similarly for $\{(-1)^n\}$, it oscillates between -1 and 1

without converging a unique limit.

Theorem: If a sequence $\{x_n\}$ converges to a finite number l, then the sequence $\{|x_n|\}$ must converge to $|l|$, i.e., $x_n \to l$ as $n \to \infty \Rightarrow |x_n| \to |l|$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given.

Since $x_n \to l$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon, \forall n \geq N$

now $\big|\,|x_n| - |\ell|\,\big| \leq |x_n - \ell| < \varepsilon$,

whenever $n \geq N$.

This proves that $\lim_{n\to\infty} |x_n| = |\ell|$.

Note: The converse of this theorem is not true.

For example, the sequence $\{|(-1)^n|\}_n$ is convergent but $\{(-1)^n\}_n$ is not convergent.

Sum, Difference, Product and Quotient on Two Convergent Sequences

P-42

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Theorem: Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences.

Let $\{x_n\} \to x$ as $n \to \infty$ and $y_n \to y$ as $n \to \infty$.

Then (i) $\lim_{n \to \infty} (x_n \pm y_n) = x \pm y$

$= \lim_{n \to \infty} x_n \pm \lim_{n \to \infty} y_n$

Sum & Difference:

(ii) if $c \in \mathbb{R}$, $\lim_{n\to\infty} c \cdot x_n = c \cdot x$

Scalar multiplication: $= c \cdot \lim_{n\to\infty} x_n$

(iii) $\lim_{n\to\infty} (x_n \cdot y_n) = x \cdot y$

Multiplication: $= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$

(iv) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

(Provided $\lim_{n\to\infty} y_n = y \,(\neq 0)$)

Quotient:

Proof: (i) Sum: Let $\varepsilon > 0$ given.

Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$,

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$\exists$ two natural numbers N_1 and N_2 P-50

such that $|x_n - x| < \varepsilon/2 \quad \forall\, n \geq N_1$

$\& \quad |y_n - y| < \varepsilon/2 \quad \forall\, n \geq N_2$

Let $N = \max\{N_1, N_2\}$. Then $\forall\, n \geq N$,

we have $|(x_n + y_n) - (x + y)|$

$= |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

So $\left|(x_n+y_n)-(x+y)\right|<\varepsilon,\ \forall\ n\geq N$

$$\therefore \lim_{n\to\infty}(x_n+y_n)=x+y$$
$$=\lim_{n\to\infty}x_n+\lim_{n\to\infty}y_n$$

(Proved)

In similar way, you try to prove

$$\lim_{n\to\infty}(x_n-y_n)=x-y=\lim_{n\to\infty}x_n-\lim_{n\to\infty}y_n$$

(ii) Scalar Product:

Let us consider $c \neq 0$.

For $c=0$, the theorem is obvious.

Let $\varepsilon > 0$ be given.

Since $\{x_n\} \to x$ as $n \to \infty$, $\exists N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon'$, $\forall n \geq N$.

where $\varepsilon' = \frac{\varepsilon}{|c|} > 0$.

$$\Rightarrow \quad |x_n - x| < \frac{\varepsilon}{|c|}$$

$$\Rightarrow |c|\,|x_n - x| < \varepsilon$$

$$\Rightarrow |cx_n - cx| < \varepsilon, \quad \forall n \geq N.$$

$$\Rightarrow \{cx_n\} \longrightarrow cx \text{ as } n \to \infty.$$

(Proved)

(iii) **Product:**

Let $\varepsilon > 0$ be given.

Consider $|x_n y_n - xy|$

$$= |x_n y_n - x_n y + x_n y - xy|$$

$$= |x_n(y_n - y) + y(x_n - x)|$$

$$\leq |x_n(y_n - y)| + |y(x_n - x)|$$

$$= |x_n|\,|y_n - y| + |y|\,|x_n - x|$$

Since $\{x_n\}$ is convergent, $\{x_n\}$ is bounded, i.e., $\exists$ a positive number M such that $|x_n| \leq M$, $\forall n$.

Since $\lim\limits_{n\to\infty} x_n = x$, $\exists N_1 \in \mathbb{N}$, such that $|x_n - x| < \dfrac{\varepsilon/2}{(1+|y|)}$, $\forall n \geq N_1$, Since $\lim_{n \to \infty} y_n = y$, $\exists\, N_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\varepsilon/2}{M}, \quad \forall\, n \geq N_2$$

Now $$|x_n y_n - xy| \leq |x_n|\,|y_n - y| + |y|\,|x_n - x|$$

$$< M \cdot \frac{\varepsilon/2}{M} + |y| \cdot \frac{\varepsilon/2}{1+|y|}$$

$$< \underbrace{\varepsilon/2 + \varepsilon/2}_{=\varepsilon} \quad \left[\text{Since } \frac{|y|}{1+|y|} < 1\right]$$

P-56)

Therefore, $|x_n y_n - xy| < \varepsilon$, $\forall n \geq N$

where $N = \max\{N_1, N_2\}$.

This proves that

$$\boxed{\lim_{n\to\infty} x_n y_n = xy = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n}$$

P-57

Indian Institute of Technology Hyderabad

Note that $|x_n - x| < \dfrac{\varepsilon/2}{1+|y|}$ is considered instead of

$$|x_n - x| < \frac{\varepsilon/2}{|y|},$$

because $|y|$ may be zero.

Corollary: $\{x_n\}_n \to 0$ as $n \to \infty$

and $\{y_n\}_n$ is bounded

$\Rightarrow \{x_n y_n\} \to 0$ as $n \to \infty$

(Try this).

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

(iv) Try this. It is your home work.

Also try to Solve this theorem

$\{x_n\}_n \to x\ (\neq 0)$ as $n \to \infty$

$\Rightarrow \left\{\frac{1}{x_n}\right\}_n \to \frac{1}{x}$, as $n \to \infty$

Home work.

P-60

Theorem: If $\{x_n\}_n \to x$ as $n \to \infty$ and $x_n > 0$, $\forall\, n \geq N$, where $N \in \mathbb{N}$,

then $x \geq 0$

i.e., $\lim_{n \to \infty} x_n \geq 0$.

P-G

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Theorem: If $\{x_n\} \to x$ as $n \to \infty$ and $\{y_n\} \to y$ as $n \to \infty$ and $\forall n \geq N$, $x_n > y_n$, where $N \in \mathbb{N}$, then

$$x \geq y$$

i.e., $\lim\limits_{n\to\infty} x_n \geq \lim\limits_{n\to\infty} y_n$.

P-62

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Examples:

(1) $\lim_{n\to\infty} \dfrac{n^2+3n}{2n^2+n-1}$ (Find the limit value)

Sol^n:

$$\lim_{n\to\infty} \frac{n^2+3n}{2n^2+n-1}$$

$$= \lim_{n\to\infty} \frac{\dfrac{n^2+3n}{n^2}}{\dfrac{2n^2+n-1}{n^2}}$$

$$= \lim_{n \to \infty} \frac{1 + 3/n}{2 + \frac{1}{n} - \frac{1}{n^2}}$$

$$= \lim_{n \to \infty} \frac{x_n}{y_n}, \quad \text{where } x_n = 1 + 3/n \text{ and } y_n = 2 + \frac{1}{n} - \frac{1}{n^2}$$

We know by limit theorems on quotient

$$\lim_{n \to \infty} \frac{x_n}{y_n} = \frac{x}{y} \quad (y \neq 0), \text{ where}$$

$x = \lim_{n \to \infty} x_n$ and

$y = \lim_{n \to \infty} y_n$.

P-65

Now $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 + 3/n$

$= 1 + 0 = 1$ $\therefore \underline{x = 1.}$

Since as $n \to \infty$, $3/n \to 0$

and the constant sequence $\{1\}_n$ has same constant 1 as a limit.

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} 2 + \frac{1}{n} - \frac{1}{n^2} = 2 + 0 - 0$

$= 2.$

$\therefore \underline{y = 2.}$

P-66

Since, as $n \to \infty$, $\frac{1}{n} \to 0$

and $\frac{1}{n^2} \to 0$

and $\{2\}_n \to 2$.

$$\therefore \lim_{n \to \infty} \frac{n^2 + 3n}{2n^2 + n - 1} = \lim_{n \to \infty} \frac{x_n}{y_n}$$

$$= \frac{x}{y} = \frac{1}{2}.$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Try this : $\lim_{n\to\infty} \frac{3n^2+2n+1}{n^2+1}$ (Home work).

Example : $\lim_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)$

Sol^n : $\lim_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)}$

P-67

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

p-68

$$= \lim_{n\to\infty} \frac{1/\sqrt{n}}{\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{1}{1+\sqrt{1+1/n}}$$

Let $x_n = \frac{1}{\sqrt{n}}$ and $y_n = \frac{1}{1+\sqrt{1+1/n}}$

P-69

$$= \lim_{n\to\infty} x_n \cdot y_n$$

$$= x \cdot y \quad \text{where } x = \lim_{n\to\infty} x_n$$

$$\text{and } y = \lim_{n\to\infty} y_n.$$

[By limit theorem on Product]

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0. \quad \therefore x = 0.$$

$$\text{and } \lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+1/n}}$$

$$\lim_{n\to\infty} \frac{1}{1+\sqrt{1+1/n}} = \frac{1}{1+1+0}$$

Since $\frac{1}{n}\to 0$ as $n\to\infty$. $= \frac{1}{2}$.

$\Rightarrow y = 1/2$.

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty} x_n y_n$$

$$= x \cdot y = 0 \cdot \frac{1}{2} = 0.$$

$\underline{P-70}$