भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Last Lecture on Calculus-I

(MA-1110)

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Lagrange's Mean Value Theorem

Theorem: Let a real valued function $f : [a,b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a,b]$

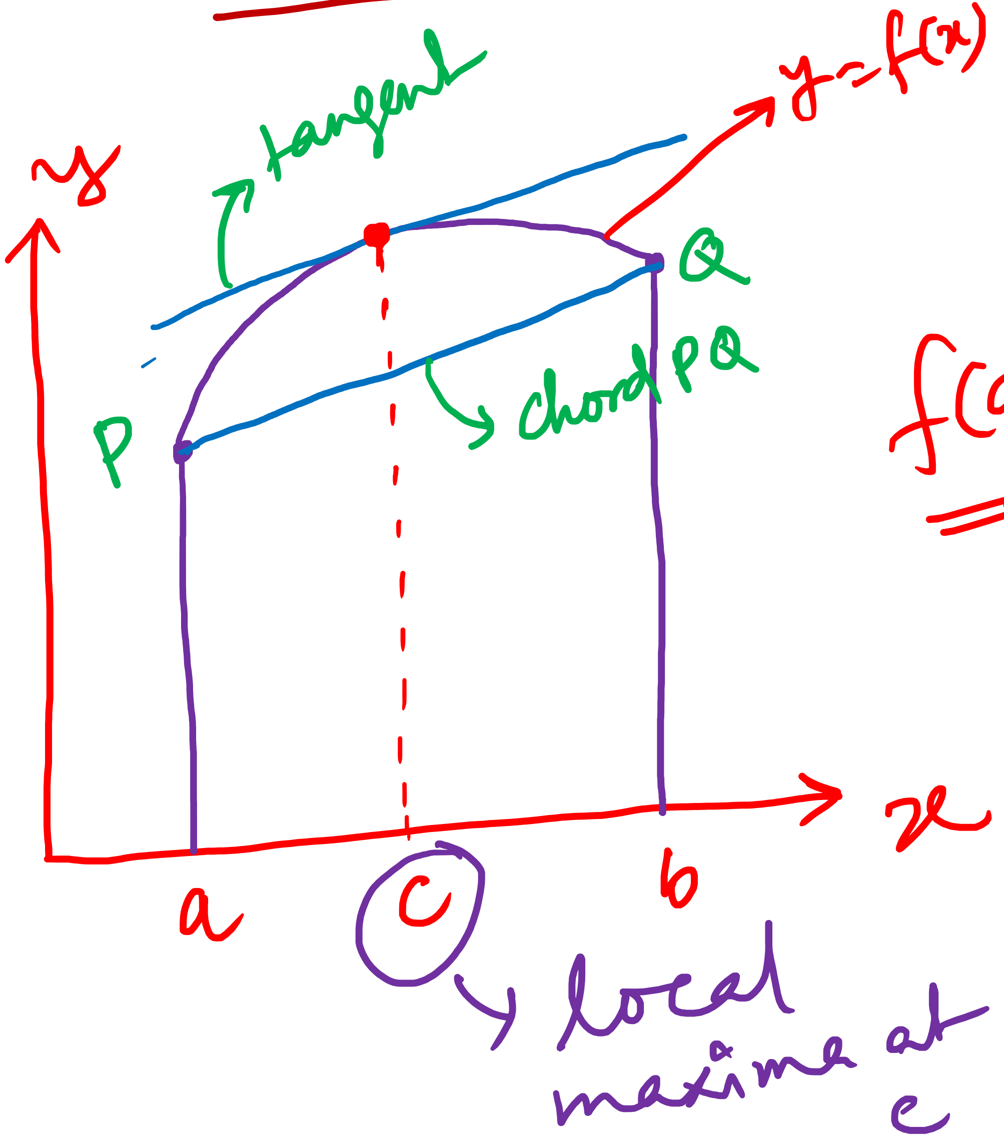
(ii) f is differentiable in (a,b)

then there exists atleast a point c in (a,b) such that

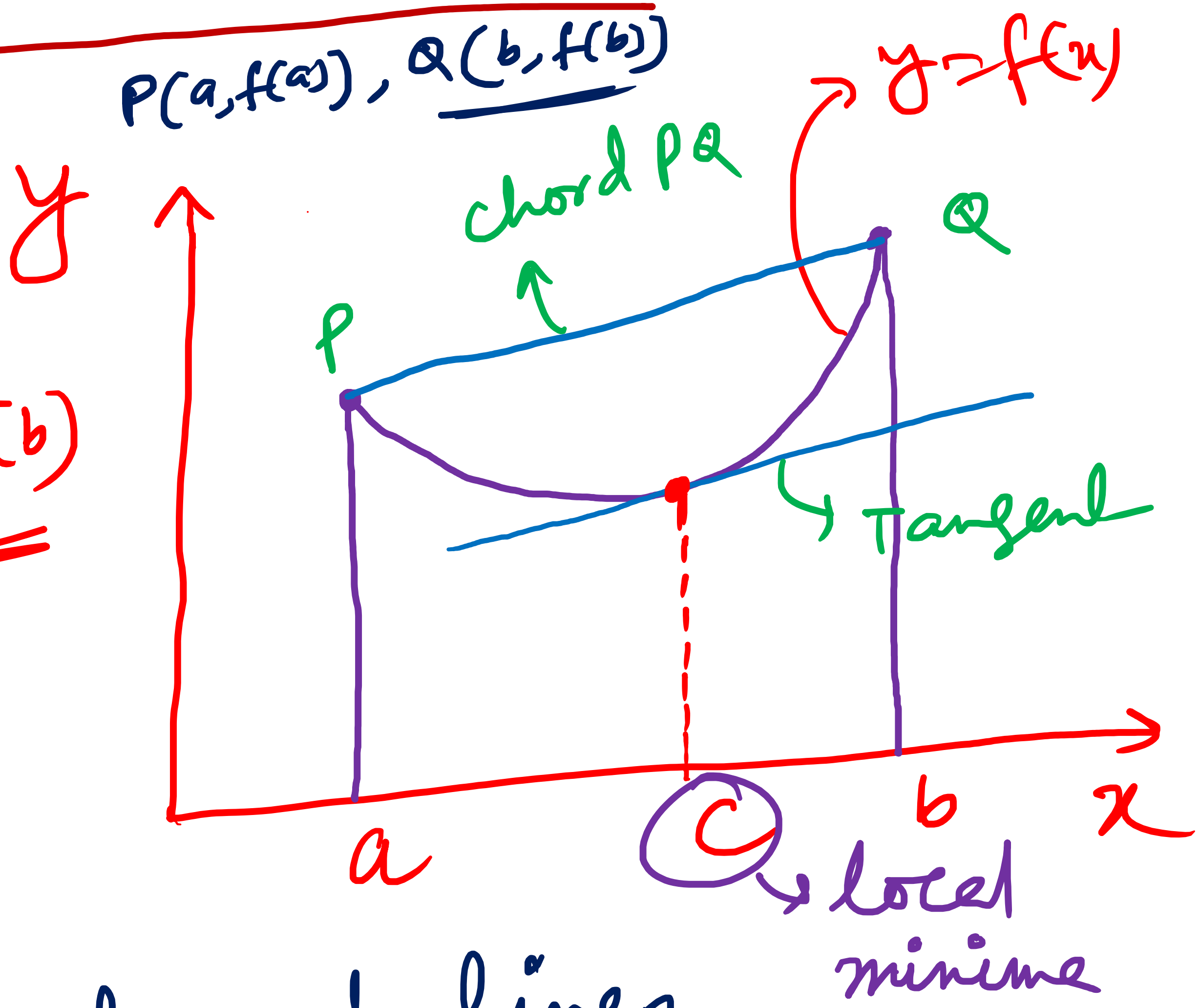
$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

This theorem gives an idea on a local minima or local maxima when $f(a) \neq f(b)$.

Geometrical Representation :



$f(a) \neq f(b)$



The tangent lines are parallel to the line joining the end points i.e. chord PQ.

These two graphs clearly say that if f is continuous on $[a,b]$ and differentiable in (a,b), then $\exists$ atleast one point $c \in (a,b)$ such that the tangent line at c is parallel to the line segment joining the end points $P(a, f(a))$ and $Q(b, f(b))$.

Proof: Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right]$

$\left[\because \text{ The eq}^n \text{ of the line joining } P \text{ and } Q \text{ is}\right.$

$$\frac{y - f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$\left.\Rightarrow \boxed{y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)}\right]$$

Now (i) $g(x)$ is continuous on $[a,b]$

(ii) $g(x)$ is differentiable (a,b)

(iii) $g(a) = 0 = g(b)$

Then by Rolle's Theorem we have

$\exists\ c \in (a,b)$ such that

$g'(c) = 0 \implies \boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}.}$

Proved.

Theorem: Let a function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x)=0\ \forall x\in(a,b)$ then f is a constant on $[a,b]$.

Proof: Let $x_1, x_2 \in [a,b]$ and $a \leq x_1 < x_2 \leq b$.

Then f is continuous on $[x_1, x_2]$ and differentiable in (x_1, x_2).

By the Mean Value theorem (MVT), $\exists$ a point $c \in (x_1, x_2)$ such that

$$\boxed{f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}.$$

But $f'(x) = 0 \ \forall x \in (a,b)$

$\Rightarrow f'(c) = 0 \Rightarrow f(x_1) = f(x_2)$

x_1 and x_2 are arbitrary points in (a,b).

So f is constant on (a,b).

Ex: Prove that between any two real roots of $e^x \sin x = 1$, there exists at least one real root of $e^x \cos x + 1 = 0$

Solⁿ: Let $g(x) = e^x \sin x - 1$. Let a, b be any two roots of $g(x)$ & $a < b$.

Then $g(a) = g(b) = 0$.

Define $f(x) = e^{-x} g(x)$
$= \sin x - e^{-x}$

& $f(a) = f(b) = 0$

Also $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). So by Rolle's theorem —

$\exists\, c \in (a,b)$ such that $f'(c) = 0$.

$\Rightarrow f'(c) = \cos c + e^{-c} = 0$

$\Rightarrow e^{c} \cos c + 1 = 0$ for $c \in (a,b)$.

$\Rightarrow h(c) = 0$, for $c \in (a,b)$

where $h(x) = e^{x} \cos x + 1$.

$\Rightarrow c$ is a root of the eq^n $e^{x} \cos x + 1 = 0$.

(Proved)

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Cauchy Mean Value Theorem

Theorem: Let the functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be such that

(i) f and g are both continuous on $[a,b]$

(ii) f and g are both differentiable in (a,b)

(iii) $g'(x) \neq 0 \;\forall x \in (a,b)$.

Then there exists a point $c \in (a, b)$ such that

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

Proof:

Define

$$h(x) = \left(f(x) - f(a)\right)\left(g(b) - g(a)\right) - \left(g(x) - g(a)\right)\left(f(b) - f(a)\right)$$

And apply Rolle's Theorem on $\underline{h(x)}$

Try this / otherwise you can

Skip it

Definition: Let $f: I \to \mathbb{R}$ be a given function. Then

(i) f is monotonically increasing on I if $\boxed{f(x) \le f(y)}$, $\forall\, x, y \in I$ with $\boxed{x < y}$.

(ii) f is monotonically decreasing on I if $\boxed{f(x) \geq f(y)}$, $\forall\, x, y \in I$ with $\boxed{x < y.}$

(iii) f is strictly increasing on I if $\boxed{f(x) < f(y)}$, $\forall\, x, y \in I$ with $\boxed{x < y.}$

(iv) Strictly decreasing if $\boxed{f(x) > f(y)}$ $\forall\, x, y \in I$ with $\boxed{x < y.}$

Theorem: Let f be differentiable function on an interval (a, b). Then we have the following –

(i) If $f'(x) > 0$ on (a,b), then f is strictly increasing on (a,b).

(ii) If $f'(x) < 0$ on (a,b), then f is strictly decreasing on (a,b).

(iii) If $f'(x) \geq 0$ on (a,b), then f is increasing on (a,b).

(iv) If $f'(x) \le 0$ on (a,b), then f is decreasing on (a,b).

Proof: (i) Let $a < x_1 < x_2 < b$. Then by MVT $\exists\, x \in (x_1, x_2)$ such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

Since $x_2 - x_1 > 0$

$$f(x_2) - f(x_1) > 0$$

$\Rightarrow \boxed{f(x_2) > f(x_1)}$ when $x_2 > x_1$

$\Rightarrow f(x)$ is strictly increasing on (a, b) (Proved)

Try (ii), (iii), and (iv) similarly.

You can skip this also.

Theorem: Suppose $f : [a,b] \to \mathbb{R}$ and f has either a local minima or a local maxima at $x_0 \in (a,b)$ and if the function f is differentiable, then $\exists\, x_0 \in (a,b)$ such that

$$\boxed{f'(x_0) = 0}, \quad x_0 \in (a,b).$$

You have already discussed theorem earlier. Here I have discussed again to prove it. See the proof.

Proof: Suppose f has a local maxima at x_0. Then $\exists\ \delta > 0$ s.t.

$$\boxed{f(x) \leq f(x_0)} \quad \forall\, x \in (x_0 - \delta, x_0 + \delta) \cap I$$

First, Considering the points to the left of x_0, we have

$$\frac{f(x)-f(x_0)}{x-x_0} \geq 0 \quad \text{for } x_0-\delta < x < x_0$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = Lf'(x_0) \geq 0.$$

Next considering points to the right of x_0, we have

$$\frac{f(x)-f(x_0)}{x-x_0} \le 0 \quad \text{for } x_0 < x < x_0+\delta$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \underline{Rf'(x_0) \le 0}$$

Since $f'(x)$ exists at x_0, So

$$Rf'(x_0) = Lf'(x_0) = \underline{f'(x_0)}$$

$$\Rightarrow f'(x_0) \geq 0 \;\&\; f'(x_0) \leq 0$$

$$\Rightarrow \boxed{f'(x_0) = 0.}$$

Remark: The function $f(x) = |x|$ has a local minimum at 0, although $f(x)$ is not differentiable at 0.

This shows that a function may have local extrema at a point without being differentiable at that point.

Remark: Consider $f(x) = x^3$, $x \in [-1, 1]$,

This function f doesn't have a local minima or maxima at 0, although $f'(0) = 0$.

So this example shows that

$f'(x_0) = 0$ does not imply

that $f(x)$ has an extremum

at the point x_0.

So, the Converse of the previous theorem is not true.

Def^n of critical point:

A point x_0 is called a critical point if either f is not differentiable at x_0 or if it is $f'(x_0)=0$.

Saddle point:

A critical point that is not a local extremum is called a Saddle point.

⊘ Second order derivative to test for local extrema:

Let f be a function defined in an open interval containing x_0 such that $f'(x_0) = 0$. Then we have the following —

(i) If $f''(x_0) > 0$ then $f(x_0)$ is a local minima for $f(x)$

(ii) If $f''(x_0) < 0$, then $f(x_0)$ is a local maxima for f.

(iii) If $f''(x_0) = 0$, then the test fails, we can't get any conclusion on maxima or minima.

For this case we have to look on the higher order derivatives.

Also we say the local extremum as Relative extremum.

Example:

$$f(x) = 2x^3 + 3x^2 + 1.$$

Then $f'(x) = 6x(x+1)$, $f''(x) = 6(2x+1)$

Since polynomial function is continuous and differentiable

$f'(x) = 0 \Rightarrow x = 0$ and $x = 1$ are the critical values of f.

$f''(0) = 6 > 0$

and $f''(-1) = -6 < 0$

So, $x = -1$ is a point of local maximum and $x = 0$ is a point of local minimum.