Eleventh Lecture on Calculus-I

(MA-1110)

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Today’s Class Lecture

- Continuity of function of a Single Real Variable
- Definition of Continuity
- Sequential Criterion
- Boundness Theorem
- Intermediate Value Theorem

Continuity of Function

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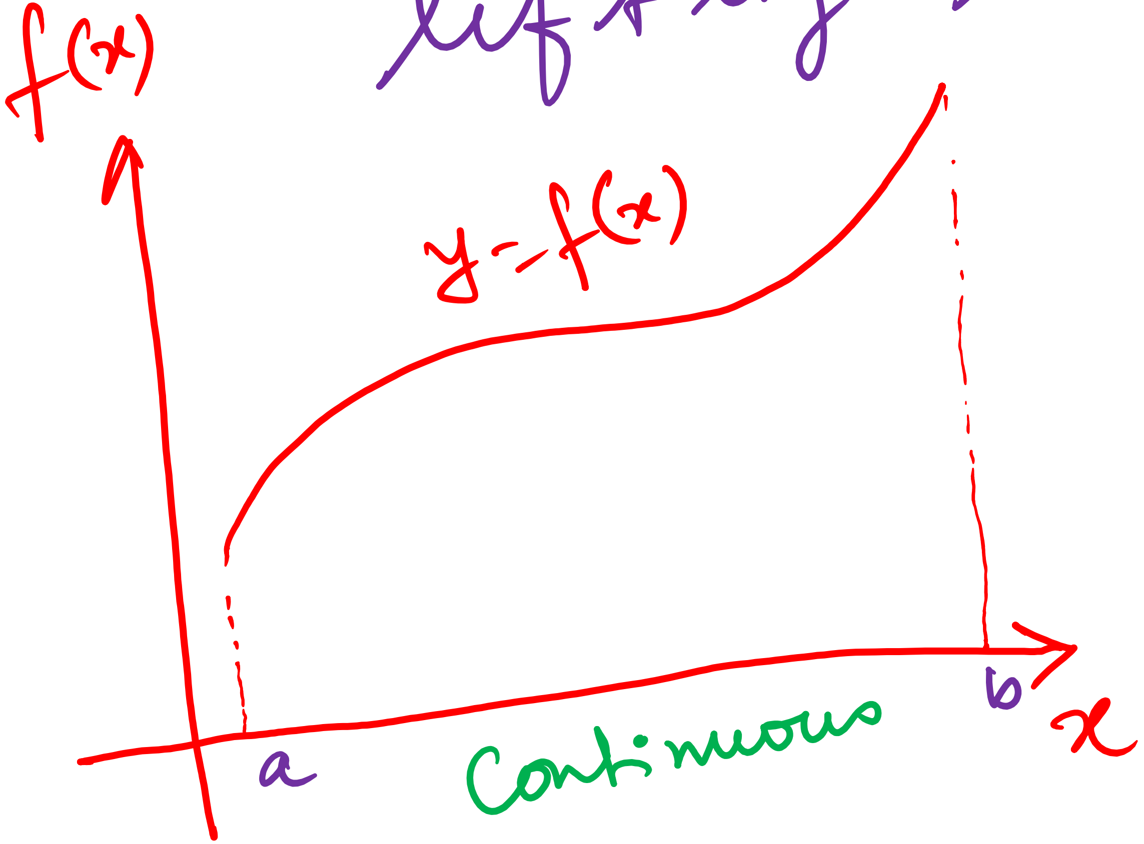
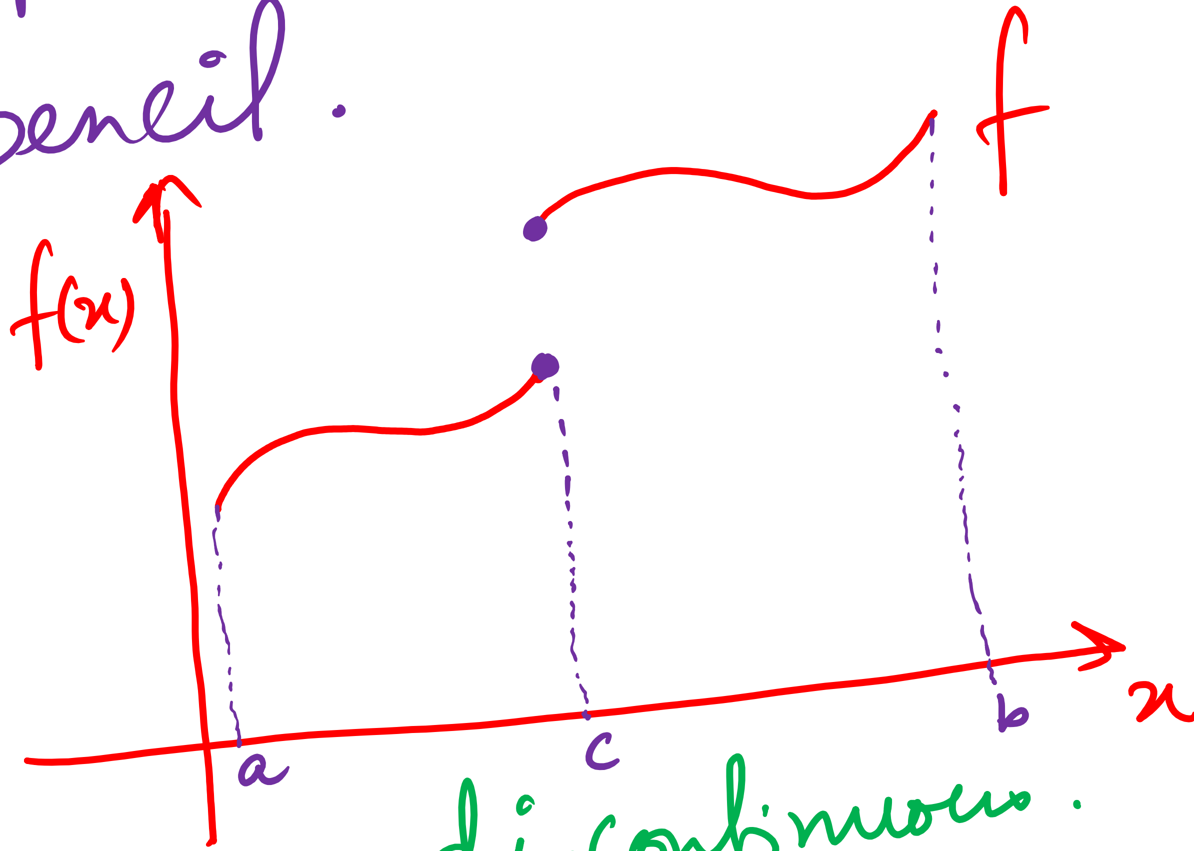
* What is continuous function?

Roughly speaking, a function f defined by $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ is said to be a continuous function if its graph has no breaks. Or one can say like that continuous functions are such functions whose graphs can be drawn on a paper without lifting the pencil.



continuous



discontinuous.
at c

Now we want to get the more clear knowledge of continuity by the concept of limit.

* We know from the limit concept, if we have $\lim_{x \to c} f(x) = l$, then $f(x)$ may or may not defined at $x = c$.

Though $f(x)$ is defined at $x = c$,

the functional value at $x=c$ i.e, $f(c)$ need not be equal to the limit value l.

we know this from earlier study of limit.

** Now when $f(c)=l$, i.e, $\lim\limits_{x\to c} f(x)=f(c)$

then we can say that f is continuous at $x=c$.

Note: For studying the continuity, we can consider the domain D of f as closed or open interval $I \subseteq \mathbb{R}$.

$$I = [a,b] \text{ or } (a,b)$$

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ — **Closed Interval**

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$. — **Open Interval**

Definition of Continuity

Definition: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let $c \in D$. f is said to be continuous at c if for a given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \forall\ \underline{x \in N(c, \delta) \cap D},$$

i.e., $\underline{0 < |x - c| < \delta}$.

which implies that

$\Rightarrow \lim_{x \to c} f(x)$ exists and it is equal to $f(c)$.

⊘ Equivalently, one can say that the function $f(x)$ is continuous at a point c if the following conditions are satisfied —

(i) f is defined at $x = c$,
i.e., $f(c)$ exist

(ii) $\lim\limits_{x \to c} f(x)$ exists i.e.,
both right hand limit $\lim\limits_{x \to c+} f(x)$
and left hand limit $\lim\limits_{x \to c-} f(x)$ are
exist and equal, i.e, $\boxed{\lim\limits_{x \to c+} f(x) = \lim\limits_{x \to c-} f(x)}$

(iii) $\lim_{x \to c} f(x) = f(c).$

i.e, the limit value of $f(x)$ at $x = c$ is equal to $f(c)$.

If any one condition is violated, then $f(x)$ is said to be discontinuous.

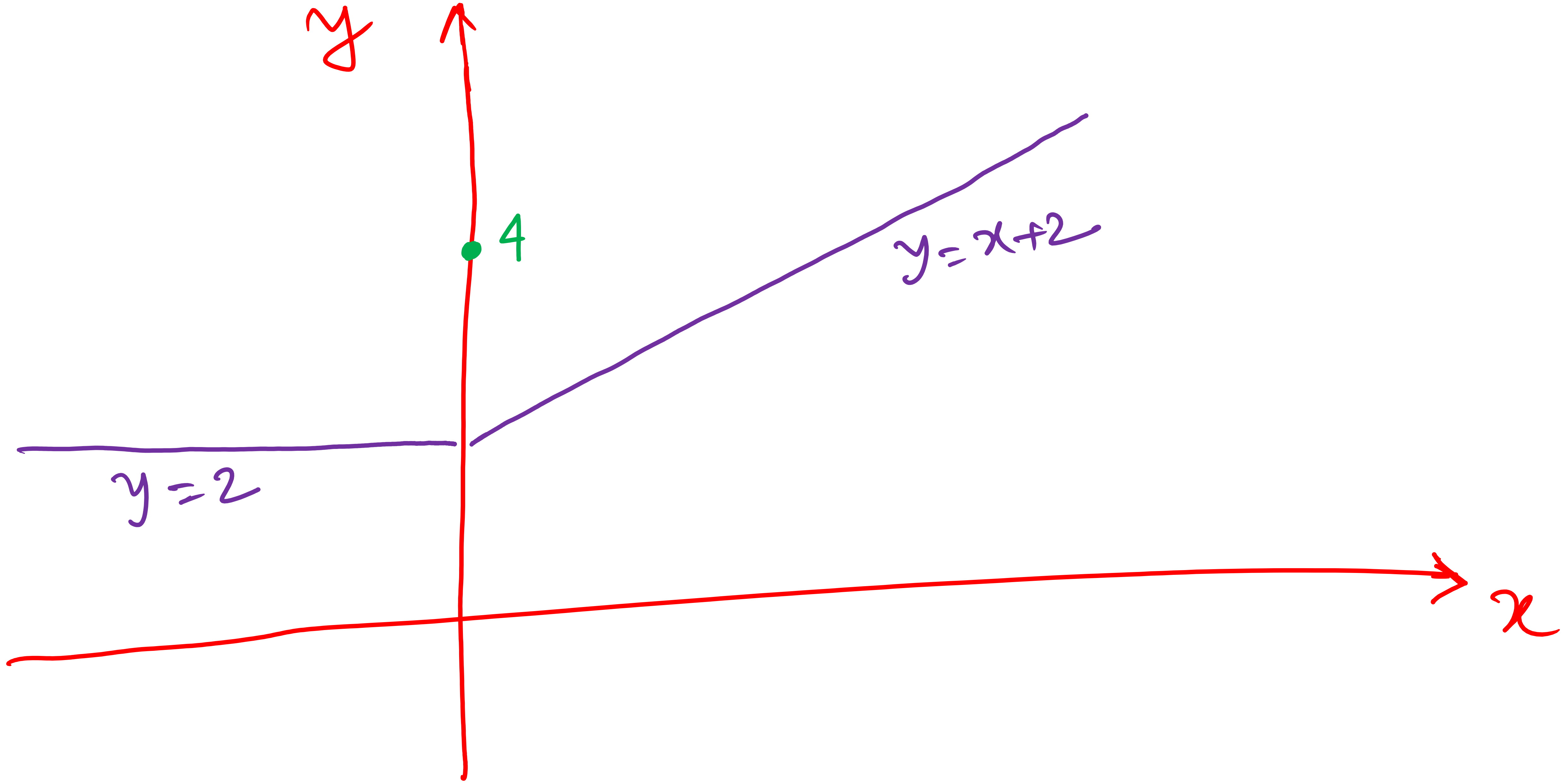
If (i) and (ii) conditions are satisfied but (iii) condition is not satisfied this type of discontinuity is called as removable discontinuity.

$\boxed{\text{Removable discontinuity}}$

Ex. Consider the function $f: \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} 2 & \text{if } x < 0 \\ x+2 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \end{cases}$$

Though $f(0) = 4$, limit of $f(x)$ at $x=0$ is 2.



y
4
y = x+2
y = 2
x

we see that

(i) $f(0)$ is defined and it is 4.

(ii) $\lim_{x \to 0} f(x)$ exists since

$$\lim_{x \to 0^+} f(x) = 2 = \lim_{x \to 0^-} f(x)$$

(iii) but $\lim_{x \to 0} f(x) = 2 \neq f(0) = 4$

So this removable discontinuity.

Ex.

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

Sol^n:

(i) $f(0) = 0$ is defined

(ii) $\lim\limits_{x \to 0} f(x) = 1$

(iii) but $\lim\limits_{x \to 0} f(x) \neq f(0)$

So, it is removable discontinuity

However, this function $f(x)$ defined by

$$f(x) = \frac{\sin x}{x}, \quad x \neq 0$$
$$= 1, \quad x = 0$$

is continuous at $x = 0$. (check)

Also $f(x) = 2, \quad x < 0$

$= x+2, \quad x > 0$

$= 2, \quad x = 0$

is continuous at $x = 0$ (check).

Ex. Investigate the continuity of the function:

$$f(x) = \begin{cases} |x|/x, & \text{when } x \neq 0 \\ -1 & \text{when } x = 0 \end{cases}$$

Sol^n: (i) The function is defined at $x = 0$.

(ii) $\lim_{x \to 0+} f(x) = 1$

and $\lim_{x \to 0^-} f(x) = -1$

$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0^-} f(x)$

So, $\lim_{x \to 0} f(x)$ does not exist.

Hence it is not continuous.

Ex. Show that the function $f(x)$ is continuous at $x=0$ and $x=1$, where

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & 0 < x < 1 \\ 2-x, & x \ge 1 \end{cases}$$

Try it (Home **work**)

Ex. Show that the function f defined by $f(x) = 5x + 3$ is continuous at $x = 1$. (by ε-δ definition).

Proof: Here $f(1) = 8$.

Let $\varepsilon > 0$ be given.

we have $|f(x) - f(1)| < \varepsilon$

$\Rightarrow |5x + 3 - 8| < \varepsilon$

or, $|x-1| < \varepsilon/5$.

Taking $\delta = \varepsilon/5$, we can get the δ-nbd of 1 easily.

So, $f(x)$ is continuous at $x=1$.

Ex, $f(x) = \sqrt{x}, \quad \forall x \geq 0.$

Show that $f(x)$ is continuous at $x = 0$. (by $\varepsilon - \delta$ **def^n**)

(**Home work**)

Sequential Criterion for Continuity

Theorem: A function $f: D \to \mathbb{R}$ is continuous at the point $c \in D$ if and only if for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

The function $f(x)$ is not continuous means —

for every sequence $\{x_n\} \to c$,

$$f(x_n) \not\to f(c)$$

Ex: $f(x) = \frac{1}{x}$ is not continuous

at $x = 0$.

Theorem: Let $f: D \to \mathbb{R}$ be a continuous function at $x = c$, then $|f(x)|$ and $Kf(x)$, $K \in \mathbb{R}$ are continuous at c.

② **Theorem:** Let $f, g : D \to \mathbb{R}$ be continuous functions at $x = c$, then

(i) $f \pm g$, fg are continuous at c.

(ii) if $g(x) \neq 0 \ \forall x \in D$, f/g is continuous at c.

Ex.

(i) $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ polynomial function.

is continuous on $\mathbb{R}$

(ii) $\sin x$, $\cos x$ are continuous on $\mathbb{R}$

(iii) $\tan x$, $\cot x$, $\sec x$ are continuous where they are defined only. They are not continuous on $\mathbb{R}$.

(iv) The exponential function $f(x) = e^x$, $x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

(v) Logarithmic function $f(x) = \log x$ is continuous on $(0, \infty)$.

⊘ Theorem: Let $f: D \to \mathbb{R}$ and $g: D' \to \mathbb{R}$

such that $f(D) \subseteq D'$.

If f is continuous at c and g is continuous at $f(c)$, then the composite function $g \circ f : D \to \mathbb{R}$ is continuous at $x = c$.

$g \circ f \Rightarrow g(f(x))$.

$\Rightarrow$ The composite of continuous function is continuous.

Ex: The function $h(x) = \sqrt{x^2+2}$ $\forall x \in \mathbb{R}$ is continuous.

Sol^n: Let $f(x) = x^2+2$, $\forall x \in \mathbb{R}$

and $g(x) = \sqrt{x}$, $\forall x \geq 0$

Then $h(x) = g(f(x)) = \sqrt{x^2+2}$ $\forall x \in \mathbb{R}$.

Since f and g are continuous, h is also continuous.

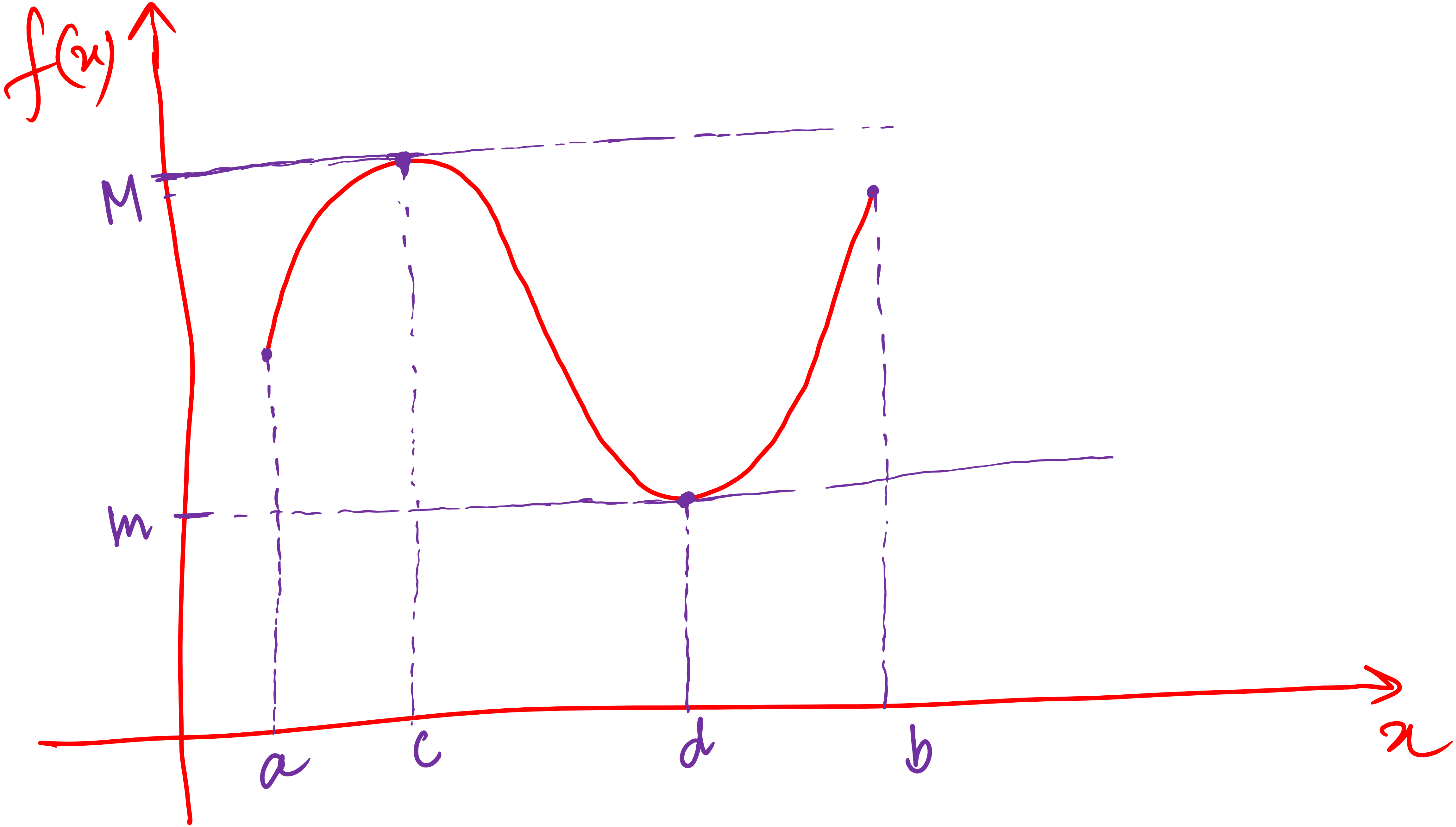
Now see some important properties of continuous function

Boundness Theorem

Theorem: Let $I = [a,b]$ be a closed interval and let $f : I \to \mathbb{R}$ be continuous on I, then f is bounded on I.

i.e, we have $m, M \in \mathbb{R}$ s.t.

$$\boxed{m \leq f(x) \leq M, \quad \forall x \in [a,b]}$$


f(x)
M
m
a
c
d
b
x

Note: But this theorem is not true for open interval, for example

$f:(0,1)\rightarrow \mathbb{R}$, $f(x)=\frac{1}{x}$, $x\in(0,1)$

Here $f(x)$ is not bounded but $f(x)$ is continuous function.

Maximum-Minimum Theorem

Theorem (Extreme Value Theorem):

Let f be a continuous on $I=[a,b]$. Then f assumes its maximum and minimum on I i.e. $\exists\ x^*$ and x_* in I such that

$$\boxed{f(x_*) \le f(x) \le f(x^*) \quad \forall\, x \in I}$$

This is also not true for

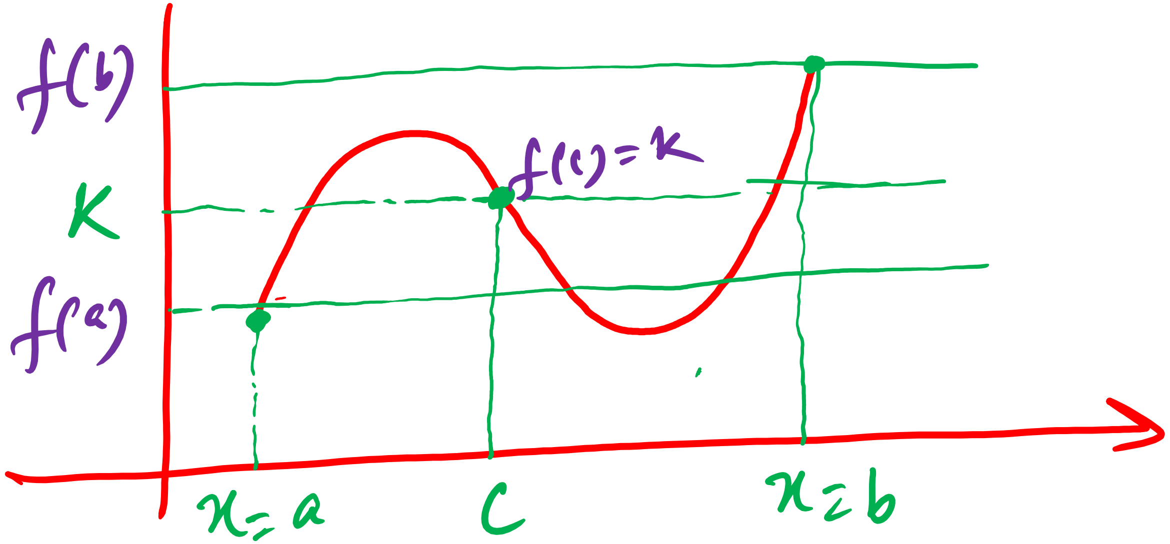
open interval

Intermediate Value Theorem

- This is the one of the most important properties of continuous functions.

- **Theorem**: Let $I = [a,b]$ be a closed bounded interval and $f: I \to R$ be continuous function on I. If $f(a) \neq f(b)$

and K be a real number lying in between $f(a)$ and $f(b)$ i.e. $f(a) < K < f(b)$, then there exists a point $c \in I$ such that $f(c) = K$.



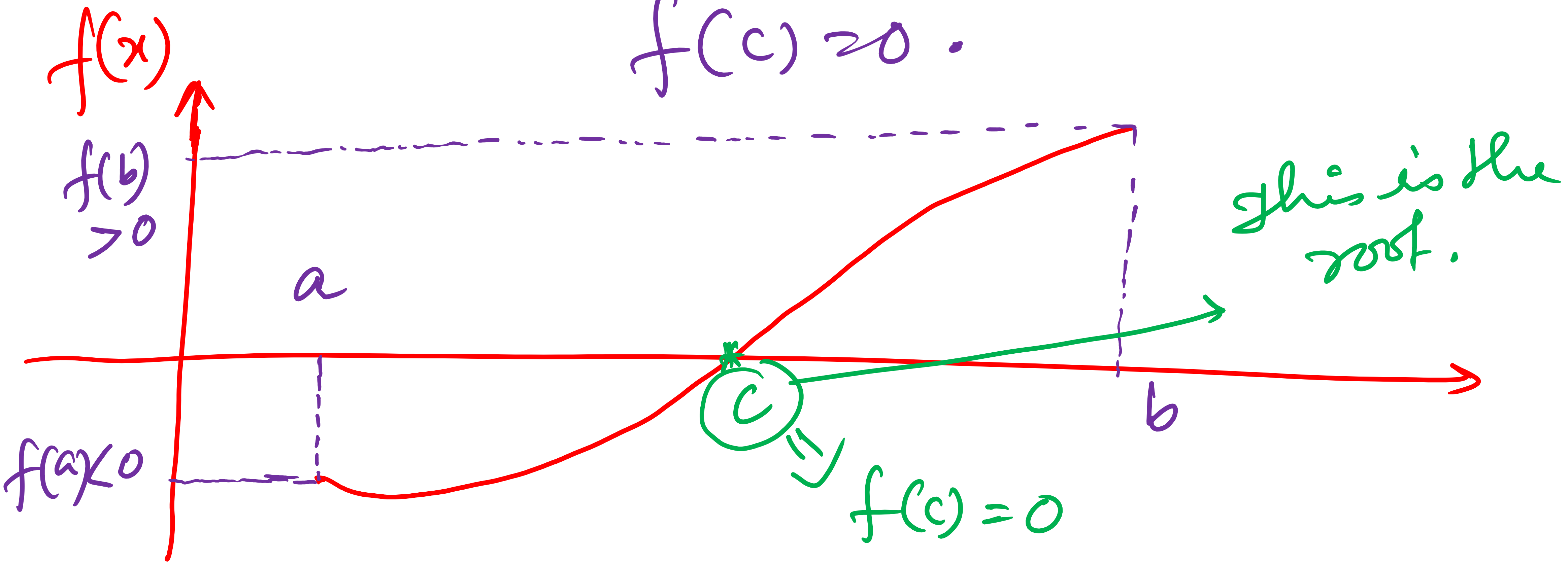
The intermediate value theorem is very important to find the roots of a polynomial or to find the zeros a function. i.e, to find the points $x \in R$ such that $f(x) = 0$.

③ **Corollary (Bolzano Theorem):**

Let $f: I \to \mathbb{R}$ be a continuous function, where $I = [a, b]$ is closed and bounded interval. If $f(a)$ and $f(b)$ have the opposite signs (i.e., $f(a) < 0$ and $f(b) > 0$), then f has a root in (a,b).

i.e. $\exists\, c \in (a,b)$ such that

$$f(c) = 0.$$



Example : Show that the equation $x^2 = x\sin x + \cos x$ has atleast two real roots.

Proof: $f(x) = x^2 - x\sin x - \cos x$ is continuous.

and $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$

and $f(\pi) = \pi^2 + 1 > 0$

Since $f(-\pi) > 0$ and $f(0) < 0$

there is one real root

in $(-\pi, 0)$ by Bolzano's theorem

and similarly since $f(0) < 0$ and

$f(\pi) > 0$, there is another

real root in $(0, \pi)$.

So, two real roots exist

in $(-\pi, \pi)$.

Note: Every polynomial of odd degree has atleast one real root.

Ex. Try for $f(x) = x^3 - 5x + 1$ in $[0,1]$.