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Indian Institute of Technology Hyderabad

Sixth Lecture on Calculus-I

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Infinite Series

- Series of real numbers
- Sequence of partial sums
- Convergence and divergence
- Two important series
- Cauchy’s Principle of Convergence
- Series of Positive terms

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Series of Real Numbers

- **Definition:** Let $\{a_n\}$ be a sequence of real numbers.

Then the symbolic expression

1st term of the series.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

nth term of the series

or $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$

is called as infinite series of real numbers.

i.e., a Series means we are talking about Series of real numbers.

This series is a infinite summation of real numbers.

Now a question arises that what is the quantity of this infinite sum of real numbers?

Can we get the finite quantity always from the series?

Ans: No. we can't get the finite quantity always.

$\Rightarrow$ If the infinite series gives the finite quantity, then we can say that the series is convergent.

For this case, we consider the limit of the infinite series to find out the finite quantity.

That's mean, we can get the finite limit for convergent series.

Otherwise if the infinite summation of the real numbers does not exists, that's mean the limit of this series is not finite (∞ or $-\infty$), then this series is not convergent, it is divergent.

we will discuss more on limit concept for convergent or divergent series later.

Sequence of partial sums of series

Definition:

Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers.

The sum of the first n terms of the series $\sum_{n=1}^{\infty} x_n$ is called as the nth partial sum of a series.

It is denoted by

$$\boxed{S_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \quad \forall n \in \mathbb{N}}$$

The sequence $\{S_n\}_{n=1}^{\infty}$ is called as the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$.

Convergence and divergence of Series

Definition :

Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers.

Now as discussed earlier, to interpret the convergence and divergence of series, the limit concept has been arised.

The Sum of the infinite Series is obtained by the limit of the Sequence $\{S_n\}$ of partial Sum.

So, the Sum of the infinite Series is

$$\boxed{\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} x_k\right)}$$

Convergent Series

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The infinite series $\sum x_n$ is convergent if

(i) $\lim_{n\to\infty} S_n$ exists

(ii) and $\lim_{n\to\infty} S_n = S$ (a finite real number)

Therefore we can say that

the Sequence $\{S_n\}$ of partial sums

Converges to S.

$\Rightarrow$ The Series $\sum_{n=1}^{\infty} x_n$ Converges

to S. we can write

$$\boxed{\sum_{n=1}^{\infty} x_n = S.}$$

Examples :

(1) Let us consider the Series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$$

Sol^n: Let the given Series be $\sum_{n\geq 1}^{\infty} x_n$.

Then we can easily see that the nth term of the series is $\frac{1}{n(n+1)}$ and it is considered as x_n.

So, $x_n = \frac{1}{n(n+1)}$ and we have the series as $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Let S_n (sum of the first n terms of the series) $= \sum_{K=1}^{n} x_K$

$$= x_1 + x_2 + \cdots + x_n$$

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$\therefore \quad S_n = 1 - \dfrac{1}{n+1}$

Now $\lim\limits_{n \to \infty} S_n = 1$ (finite)

$\Rightarrow$ The Series $\sum x_n$ is Convergent and the Sum of the Series is 1.

② Let us consider the Series

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots$$

Sol^n: Let us choose the given Series as $\sum_{n=1}^{\infty} x_n$,

where $S_n = x_1+x_2+\cdots+x_n$

$$= \sum_{k=1}^{n} x_k$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$= 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}$$

and $\lim\limits_{n\to\infty} S_n = 2$, since

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n-1} = 0.$$

Therefore the given Series is convergent and the Sum of the Series is 2.

Divergent Series

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- The infinite Series $\sum x_n$ is said to be divergent if the limit of the sequence $\{x_n\}$ of partial sums, i.e., $\lim_{n\to\infty} S_n$ does not exist.

- Series
  - Convergent
  - Not Convergent
    - Properly Divergent
      - Diverges to ∞
      - Diverges to $-\infty$
    - Improperly Divergent
      - oscillates finitely
      - oscillates infinitely.

If $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$,

then the series $\sum_{n=1}^{\infty} x_n$ is said to be **properly divergent series**

Example: Consider a series

$$1+2+3+\cdots$$

Solⁿ:

Let $S_n = 1+2+3+\dots+n$

Then $S_n = \frac{n(n+1)}{2}$

and clearly it is Seen that

$\{S_n\} \longrightarrow \infty$ as $n \to \infty$.

$\therefore \boxed{\lim_{n\to\infty} S_n = \infty.}$

Hence the Series is divergent.

- If $\{S_n\}$ is oscillating finitely then the series $\sum_{n=1}^{\infty} x_n$ is said to oscillate finitely, where

$$S_n = x_1 + x_2 + \cdots + x_n = \sum_{k=1}^{n} x_k.$$

- Example: Consider the series as

$$1-1+1-1+\cdots$$

Solⁿ: Here $S_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

So, $\{S_n\}$ oscillates between 0 and 1 and consequently the given series $\sum_{n=1}^{\infty} (-1)^{n-1}$ also oscillates between 0 and 1.

$\Rightarrow$ The given series is not convergent

* If $\{S_n\}$ oscillates infinitely then $\sum_{n=1}^{\infty} x_n$ is said to oscillate infinitely

* **Example:** Consider the series as

$$\sum_{n=1}^{\infty} x_n = 1-2+3-4+5-6+\cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n$$

Solⁿ: Here $S_n = \begin{cases} \frac{1}{2}(n+1), & \text{if } n \text{ is odd} \\ -\frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$

When n is even:

$$S_n = 1-2+3-4+5-6+\cdots+(n-1)-n$$

$$= (1-2)+(3-4)+(5-6)+\cdots+[(n-1)-n]$$

$$= -1-1-1-\cdots-1 = -n/2$$

When n is odd:

$$S_n = 1 - 2 + 3 - 4 + 5 - 6 + \cdots + (n-2) - (n-1) + n$$

$$= (1-2) + (3-4) + (5-6) + \cdots + [(n-2) - (n-1)] + n$$

$$= -1 - 1 - 1 - \cdots - 1 + n$$

$$= -\frac{n-1}{2} + n = \frac{n+1}{2}.$$

Hence $\{S_n\}$ oscillates infinitely between $-\infty$ and ∞.

So the given series $\sum_{n=1}^{\infty} x_n$ also oscillates infinitely.

$\Rightarrow$ This series is not convergent.

Two important Series

Geometric Series

The infinite Geometric Series is

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

where $a > 0$.

(i) This Series is convergent if the common ratio r lies between

-1 and 1 (i.e., $|r|<1$).

For this case, the sum of the series is

$$\frac{a}{1-r}.$$

(ii) This series is properly divergent to ∞ if $r \geq 1$.

(iii) This Series

oscillates finitely, if $r = -1$

and

oscillates infinitely if $r < -1$

Improperly divergent

Proof: Here $S_n = a + ar + ar^2 + \dots + ar^{n-1}$

$$= a \cdot \frac{r^n - 1}{r - 1} \quad (r \neq 1)$$

(i) Case I: $|r|<1$:

Now if $|r|<1$ then $r^n \to 0$ as $n\to\infty$.

$$\boxed{\therefore\ S=\lim_{n\to\infty} S_n=\frac{a}{1-r}.}$$

$\therefore$ The Geometric Series is Convergent if $|r|<1$ and its sum is $\frac{a}{1-r}$.

(ii) Case II: $r > 1$:

If $r > 1$, $r^n \to \infty$ as $n \to \infty$

and then $S_n \to \infty$ as $n \to \infty$.

Hence the Geometric Series properly diverges to ∞ if $r > 1$.

Case-III: $r=1$:

If $r=1$, $S_n = a + a + \cdots + a$

$= na \to \infty$ as $n \to \infty$

So, the series properly diverges to ∞.

(iii) Case IV : $r < -1$:

If $r < -1$, $\{r^n\}$ oscillates infinitely.

So, $\{S_n\}$ oscillates infinitely and the series also oscillates infinitely.

Case V: $r = -1$:

If $r = -1$, we have

$S_n = a - a + a - a + \dots$ to n terms

$\therefore S_n = \begin{cases} a, & \text{if } n \text{ is odd.} \\ 0, & \text{if } n \text{ is even.} \end{cases}$

The Sequence $\{S_n\}$ oscillates finitely.

$\Rightarrow$ The Series oscillates finitely.

Thus the Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$ Converges only when $|r| < 1$.

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The p-Series

The infinite p-Series is given by

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(i) This series converges if $p > 1$.

(ii) it diverges if $p \leq 1$.

In particular, for $p=1$, we will have

Harmonic Series.

Harmonic Series:

For $p=1$, the p-Series becomes to Harmonic Series

$$\sum_{n\geq 1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

This Series is divergent.
we can easily check it by using the
Concept of subsequence of Sequence
$\{S_n\}$.

Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ be the
Sequence of nth partial Sum of the
given Series.

Now $S_{n+1} - S_n = \frac{1}{n+1} > 0 \ \forall n.$

which implies that the sequence $\{S_n\}$ is monotone increasing.

Consider a subsequence $\{S_{2^n}\}$ of sequence $\{S_n\}$, where $\{2^n\}$ is a strictly increasing sequence of natural numbers.

Now,

$$S_{2^n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right)$$

$$+ \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots$$

$$+ \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \cdots + 2^{n-1} \cdot \frac{1}{2^n}$$

$$= 1 + n/2$$

$$\therefore \boxed{S_{2^n} > 1 + n/2, \ \forall \ n > 2}$$

$$\Rightarrow \lim_{n \to \infty} S_{2^n} = \infty.$$

Therefore the Sequence $\{S_n\}$ is a

monotone increasing sequence having a properly divergent subsequence $\{S_{2^n}\}$ and hence by the theorem, the sequence $\{S_n\}$ is properly divergent.

$\therefore \boxed{\lim_{n\to\infty} S_n = \infty}$

$\Rightarrow$ the given series $\sum_{n=1}^{\infty} \frac{1}{n}$ is properly divergent to ∞.

Theorem: If $\sum x_n$ and $\sum y_n$ are two convergent series with sums x and y, respectively, then

(i) the series $\sum (x_n + y_n)$ converges to the sum $(x+y)$.

(ii) the series $\sum K x_n$, K being a real number, converges to the sum Kx.

Cauchy's Principle of Convergence

Theorem: The infinite Series $\sum_{n=1}^{\infty} x_n$ Converges if and only if for given any $\varepsilon > 0$, $\exists$ a natural number N (depending on ε), such that

$$|S_{n+p} - S_n| < \varepsilon, \quad \forall\, n \geq N \text{ and } p \in \mathbb{N}$$

i.e., $\left| x_{n+1} + x_{n+2} + \cdots + x_{n+p} \right| < \varepsilon,$

$\forall\ n \geq N$ and $p \in \mathbb{N}$

This is the necessary and Sufficient Condition for the Convergence of a Series.

Alternative:

An infinite Series $\sum x_n$ Converges, if and only if for any $\varepsilon > 0$, $\exists$ a natural number N such that

$$\boxed{|S_m - S_n| < \varepsilon, \ \forall\, n, m \geq N}$$

i.e., $|x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon$, $\forall\, n, m \geq N$.

Proof: Let $\sum_{n=1}^{\infty} x_n$ be an infinite series.

Let $S_n = x_1 + x_2 + \cdots + x_n$.

Consider $\sum_{n=1}^{\infty} x_n$ be convergent.

Then $\{S_n\}$ is also convergent.

By Cauchy's principle of convergence for the sequence, for given $\varepsilon > 0$, $\exists$ a natural number N such that

$$|S_{n+p} - S_n| < \varepsilon, \ \forall \ n \geq N \text{ and } p \in \mathbb{N}.$$

$$\Rightarrow |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon, \ \forall \ n \geq N \text{ and } p \in \mathbb{N}$$

Conversely, Consider that for any $\varepsilon > 0$, $\exists$ a natural number N such that

$$\boxed{|S_{n+p} - S_n| < \varepsilon, \ \forall \ n \geq N \text{ and } p \in \mathbb{N}}$$

This implies that the sequence $\{S_n\}$ is convergent by Cauchy's principle of convergence.

Hence $\sum x_n$ is convergent.

This completes the proof.

Examples:

① Prove that the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$ is convergent.

Proof: Let the series be $\sum_{n=1}^{\infty} x_n$,

where $x_n = (-1)^{n+1} \cdot \frac{1}{n}$.

Let $S_n = x_1 + x_2 + \cdots + x_n$

$= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}\frac{1}{n}$

$$|S_{n+p} - S_n| = \left| \sum_{k=n+1}^{n+p} (-1)^{k+1}\frac{1}{k} \right|$$

$$= \left| (-1)^{n+2}\frac{1}{n+1} + (-1)^{n+3}\frac{1}{n+2} + \cdots + (-1)^{n+p+1}\frac{1}{n+p} \right|$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{p-1} \frac{1}{n+p} \right|$$

$$= \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) - \cdots \right|$$

$$< \frac{1}{n+1} .$$

Let $\varepsilon > 0$.

Then $|S_{n+p} - S_n| < \varepsilon$ holds

if $n > \frac{1}{\varepsilon} - 1$.

Choose $N = \left[\frac{1}{\varepsilon} - 1\right] + 2$.

Therefore $|S_{n+p} - S_n| < \varepsilon$, $\forall\, n \geq N$ and $p \in \mathbb{N}$

$\Rightarrow \{S_n\}$ is convergent $\Rightarrow \sum x_n$ is <u>convergent</u>.

(2) Prove that by using Cauchy Criterion, the Series

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$ Converges.

Proof: Let $\sum x_n$ be the Series where $x_n = \frac{1}{n!}$.

Let $S_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

$$\left| S_{n+p} - S_n \right| = \left| \sum_{k=n+1}^{n+p} \frac{1}{k!} \right|$$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+p)!}$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p-1}}$$

$$= \frac{1}{2^n} \cdot \frac{1-(1/2)^p}{1-1/2}$$

$$< \frac{1}{2^n} \cdot \frac{1}{1-1/2} \quad \forall p$$

$$= \frac{1}{2^{n-1}}$$

Let $\varepsilon > 0$.

$|S_{n+p} - S_n| < \varepsilon$ if

$$\frac{1}{2^{n-1}} < \varepsilon \Rightarrow 2^{n-1} > \frac{1}{\varepsilon}$$

$$\text{a, } n > 1 - \frac{\log \varepsilon}{\log 2}$$

choose $N = \left[1 - \frac{\log \varepsilon}{\log 2}\right] + 1$.

Therefore $|S_{n+p} - S_n| < \varepsilon \quad \forall\, n \geq N$ and $p \in \mathbb{N}$

So $\{S_n\}$ is convergent

which implies that

$\sum_{n=1}^{\infty} x_n$ is also convergent.

(3) Prove that using Cauchy criterion the series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

is convergent.

Proof: Here $|S_{n+p}-S_n|$

$$=\left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right|$$

Choose $p = n$.

$$= \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right|$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$= n \cdot \frac{1}{2n} = \frac{1}{2}$$

i.e., $\left| S_{n+p} - S_n \right| > \frac{1}{2}$ $\forall$ n and $p = n$.

$\Rightarrow$ Cauchy condition is not Satisfied.

$\Rightarrow$ The Sequence $\{S_n\}$ is not convergent

$\Rightarrow$ The given series is also convergent.

Theorem:

Let $\sum_{n=1}^{\infty} x_n$ be an infinite series.

If $\sum_{n=1}^{\infty} x_n$ converges,

then $\lim_{n\to\infty} x_n = 0$.

Proof: Let the infinite series $\sum x_n$ be convergent. Then by the Cauchy's principle of convergence, for any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$|S_{n+p} - S_n| < \varepsilon, \quad \forall\, n \geq N \text{ and } p \in \mathbb{N}$$

$$\Rightarrow |x_{n+1} + x_{n+2} + \dots + x_{n+p}| < \varepsilon \quad \forall\, n \geq N \text{ and } p \in \mathbb{N}$$

Taking $p=1$, we have

$$|x_{n+1}| < \varepsilon, \ \forall\, n \geq N$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0 \quad \text{(Proved)}$$

Note: This theorem is the necessary condition for convergence of the series but it is not sufficient condition, i.e., the converse of this theorem is not true.

Example: Let us consider the series $\sum_{n=1}^{\infty} x_n$ where $x_n = 1/n$.

Here $\lim_{n \to \infty} x_n = 0$.

However, $\sum x_n$ is a divergent series.

The last theorem also gives the contra-positive statement as

If $\lim_{n\to\infty} x_n \not\to 0$, then the series $\sum_{n\geq 1}^{\infty} x_n$ does not converge.

Example:

(1) Prove that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

Proof: Here $\lim_{n\to\infty} \frac{n}{n+1} = 1 \; (\neq 0)$

$\Rightarrow \sum \frac{n}{n+1}$ is divergent.

② Prove that $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

Proof:

$$x_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} > 1$$

whenever $n > 1$

$\therefore x_n$ cannot tend to zero as $n \to \infty \Rightarrow \sum_{n=1}^{\infty} x_n$ is not convergent.

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Series of Positive Terms

- **Definition:**

A series $\sum x_n$ is said to be a series of positive terms if x_n is a positive real number for all $n \in \mathbb{N}$.

Remark: A positive series can never oscillate, either it will converge or it will diverge to $+\infty$.

Theorem:

An infinite series $\sum_{n=1}^{\infty} x_n$ of positive real numbers is convergent if and only if the sequence $\{S_n\}$ of partial sums is bounded above.

Proof:

Let $S_n = x_1 + x_2 + \dots + x_n = \sum_{k=1}^{n} x_k$

Let $\{S_n\}$ be bounded above.

Then $S_{n+1} - S_n = x_{n+1} > 0 \ \forall n \in \mathbb{N}$.

Hence the sequence $\{S_n\}$ is a monotone increasing sequence.

Therefore we know that if the sequence $\{S_n\}$ is a monotone increasing and bounded above, $\{S_n\}$ is Convergent.

$\Rightarrow$ The series $\sum x_n$ is Convergent.

Similarly we can easily prove that if the series $\sum x_n$ is convergent, the sequence of partial sums $\{S_n\}$ is bounded above.

Note: If $\{S_n\}$ is not bounded above, the sequence $\{S_n\}$ which is monotone increasing diverges to ∞.

unbounded above

which implies that $\sum x_n$ diverges to ∞.

Examples:

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Proof: Set $S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

Then $\{S_n\}$ is the sequence of partial sums of the given series.

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

$$< \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n}$$

$$\left(\because\ 2^2 = 2 \cdot 2 > 1 \cdot 2,\quad 3^2 = 3 \cdot 3 > 2 \cdot 3, \right.$$
$$\left. n^2 = n \cdot n > (n-1) \cdot n \right)$$

$$S_n < \frac{1}{1} + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$
$$+ \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

or, $S_n < 1 + 1 - \frac{1}{n} < 2, \ \forall\, n \geq 1$

$\therefore \{S_n\}$ is bounded above.

Therefore $\{S_n\}$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

The sum of this series is $\pi^2/6$.