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Indian Institute of Technology Hyderabad

Seventh Lecture on Calculus-I

(MA-1110)

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Infinite Series

- Series of arbitrary terms
- Absolutely Convergent
- Conditionally Convergent
- Tests for convergence
    - Comparison Tests for positive series
    - Ratio Test for arbitrary series
    - Root Test for arbitrary series
    - Cauchy's Condensation Test for positive series
    - Raabe's Test for positive series
    - Integral Test for positive series
- Alternating Series
    - Leibnitz's test

Series of Positive and Negative Terms

- Let $\sum x_n$ be a Series of positive and negative real numbers. This type of Series may be Absolutely Convergent or Conditional Convergent or Divergent.

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Absolutely Convergent

Definition: An infinite series $\sum_{n\geq 1}^{\infty} x_n$ of real constants is said to be absolutely convergent when the series $\sum_{n\geq 1}^{\infty} |x_n|$ i.e., when

$|x_1| + |x_2| + \cdots + |x_n| + \cdots$ is convergent.

Theorem:

An absolutely convergent series is convergent. But the converse theorem is not true [Ex. $\sum (-1)^{n+1} \cdot \frac{1}{n}$ is convergent but not absolutely]

Proof: Let $\sum x_n$ be a series of positive and negative real numbers. Let it be absolutely convergent. Then $\sum |x_n|$ is a convergent series of positive terms.

Choose $\varepsilon > 0$.

Then for $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$$\boxed{\Big|\, |x_{n+1}| + |x_{n+2}| + \cdots + |x_{n+p}| \,\Big| < \varepsilon \quad \forall\, n \geq N \text{ and } p \in \mathbb{N}}$$

We know that

$$\left| x_{n+1} + x_{n+2} + \cdots + x_{n+p} \right|$$

$$\leq |x_{n+1}| + |x_{n+2}| + \cdots + |x_{n+p}|$$

$$< \varepsilon, \quad \forall\, n \geq N \text{ and } p \in \mathbb{N}$$

By Cauchy's principle of Convergence,
$\sum x_n$ is Convergent. (proved)

Examples:

$$1-\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}-\frac{1}{6^2}+\cdots$$

Solⁿ: The given Series is convergent since it is absolutely Convergent.

It is easily checked that

$$\sum \frac{1}{n^2}$$ is convergent

by the test of p-series

we know that the p-series $\sum \frac{1}{n^p}$ is convergent for $p > 1$. Here $p = 2$.

Example:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots$$

Sol^n:

You try this. (similar).

Example: Prove that for a fixed value of x, the series $\sum \frac{\sin nx}{n^2}$ is an absolutely convergent.

Proof:

Consider $\sum x_n$, where $x_n = \frac{\sin nx}{n^2}$.

$$|x_n| = \left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ is convergent by comparison test. So, $\sum |x_n|$ is convergent

$\Rightarrow \sum x_n$ is absolutely convergent

Home work.

This test is discussed later.

Theorem:

If the series $\sum x_n$ be absolutely convergent and $\{y_n\}$ be a bounded sequence, then the series $\sum x_n y_n$ is absolutely convergent.

Example:

$1.\ \frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \ldots$

Sol^n: Consider this infinite series

$\sum x_n$, where $x_n = (-1)^{n+1} \cdot \frac{n+1}{n^3}$

$\therefore x_n = \frac{(-1)^{n+1}}{n^2} \cdot \left(1+\frac{1}{n}\right).$

$$x_n = \frac{(-1)^{n+1}}{n^2}\left(1+\frac{1}{n}\right)$$

$$= a_n b_n, \quad \text{where } a_n = \frac{(-1)^{n+1}}{n^2}$$

and $b_n = \left(1+\frac{1}{n}\right)$

The series $\sum a_n$ is absolutely convergent and $\{b_n\}$ is bounded sequence.

Therefore, $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

Conditionally Convergent

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Definition:

An infinite Series $\sum x_n$ of real constants is said to be Conditionally Convergent if the series $\sum x_n$ converges but not absolutely.

i.e., $\sum x_n$ Converges but $\sum |x_n|$ does not converge.

Example:

1. The Series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent but the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is divergent.

Therefore the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent.

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Comparison Test (First Type) for Positive Series

Theorem: (Comparison Test) (First type)

Let $\sum x_n$ and $\sum y_n$ be two series of positive terms. There exists a natural number N such that $\boxed{x_n \leq K y_n}$ $\forall n \geq N$, K being a fixed positive number.

Then (i) $\sum x_n$ is convergent if $\sum y_n$ is convergent.

(ii) $\sum y_n$ is divergent if $\sum x_n$ is divergent.

Proof: Let $S_n = x_1 + x_2 + \cdots + x_n$

and $T_n = y_1 + y_2 + \cdots + y_n$

Therefore, $S_n - S_m = x_{m+1} + x_{m+2} + \cdots + x_n$

$\leq K(y_{m+1} + y_{m+2} + \cdots + y_n)$

$= K(T_n - T_m)$

$$\boxed{S_n - S_m \leq K(T_n - T_m)}$$

$\Rightarrow \boxed{S_n \leq K T_n + h,}$ where $h = S_m - K T_m$, a finite number.

(i) Let $\sum y_n$ be convergent. Then the sequence $\{T_n\}$ is bounded.

Let M be a upper bound.

Then $T_n < M, \forall n \in \mathbb{N}$.

Therefore, $\boxed{S_n < KM + h, \forall n \geq m}$

This shows that the sequence $\{S_n\}$ is bounded above.

we know that $\{S_n\}$ is monotone increasing sequence.

So, it is convergent.

Therefore, $\sum x_n$ is convergent. (Proved)

(ii) Let $\sum x_n$ is divergent.

Then the sequence $\{S_n\}$ is not bounded above.

Since $\boxed{S_n \leq K T_n + h}$, the sequence $\{T_n\}$ is not bounded above.

Therefore $\sum y_n$ is divergent.

Example:

Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent for $p \leq 1$.

Proof: The given series is p-series. We can easily check that

$$\boxed{n^p \leq n, \text{ for } p \leq 1.}$$

Hence $\boxed{\frac{1}{n} \leq \frac{1}{n^p}, \forall n.}$

By Comparison test, Since $\sum_{n\geq 1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n\geq 1}^{\infty} \frac{1}{n^p}$ is also divergent.

Since $\sum \frac{1}{n^p}$ is a Series of positive terms, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty .$$ (Proved)

Example: Check the convergence of $\sum \frac{3}{n^2+10}$.

Sol^n: we have $\forall n$,

$$\boxed{\frac{3}{n^2+10} \leq \frac{3}{n^2}}$$

Since $\sum \frac{1}{n^2}$ converges, by comparison test, $\sum \frac{3}{n^2+10}$ converges.

Example:

check the convergence of $\sum \frac{1}{\sqrt{n} - 3/2}$

Solⁿ: Homework. (try it)

Limit Comparison Test for Positive Series

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Theorem:

Let $\sum x_n$ and $\sum y_n$ be two series of positive real numbers and $\boxed{\lim_{n\to\infty} \frac{x_n}{y_n} = l,}$ where l is a nonzero finite number. Then two series $\sum x_n$ and $\sum y_n$ converge or diverge together.

Here $0 < l < \infty$.

Proof: Consider $l > 0$.

Let us choose $\varepsilon > 0$ such that

$l - \varepsilon > 0$.

Since $\lim_{n \to \infty} \frac{x_n}{y_n} = l$, for $\varepsilon > 0$,

$\exists\, N \in \mathbb{N}$ such that

$$\boxed{l - \varepsilon < \frac{x_n}{y_n} < l + \varepsilon,} \quad \forall\, n \geq N.$$

Therefore $\boxed{x_n < K y_n}$ $\forall n \geq N$ — (i)

where $K = l + \varepsilon > 0$.

and $\boxed{y_n < K' x_n}$ where $K' = \frac{1}{l-\varepsilon} > 0$ — (ii)

By comparison test, $\sum x_n$ is convergent if $\sum y_n$ is convergent and $\sum y_n$ is divergent if $\sum x_n$ is divergent. [From (i)]

By comparison test,

$\sum y_n$ is convergent if $\sum x_n$ is convergent

and $\sum x_n$ is divergent if $\sum y_n$ is divergent. [From (ii)]

Therefore, two series $\sum x_n$ and $\sum y_n$ converge or diverge together.

(Proved).

Examples:

1. Test the convergence of the series

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots$$

Sol^n: Let $\sum x_n$ be given series.

So, $x_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{(n+2)}{2(n+1)^2}$

Let $y_n = \frac{1}{n}$.

Then $\frac{x_n}{y_n} = \frac{n(n+2)}{2(n+1)^2}$

$$\boxed{\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{2} \; (>0)}$$

Since $\sum y_n$ is divergent, $\sum x_n$ is divergent by comparison test.

② Test the Series

$$\frac{1}{1\cdot 2^2}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 4^2}+\cdots$$

Solⁿ: Let $\sum_{n=1}^{\infty} x_n$ be the given Series. Then $x_n=\frac{1}{n(n+1)^2}$.

Let $y_n=\frac{1}{n^3}$. Then $\lim_{n\to\infty}\frac{x_n}{y_n}=1\ (>0)$

Since $\sum y_n$ is convergent, by comparison test $\sum x_n$ is also convergent.

③ Test the convergence of the series

$$\sqrt{x^4+1} - \sqrt{x^4-1}.$$

(Homework)

Comparison Test (Second Type) for Positive Series

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Theorem: (You can skip this).

Let $\sum x_n$ and $\sum y_n$ be two series of positive real numbers and there is a natural number N such that

$$\boxed{\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n} \quad \forall\, n \geq N.}$$

Then

(i) $\sum x_n$ is convergent if $\sum y_n$ is convergent.

(ii) $\sum y_n$ is divergent if $\sum x_n$ is divergent.

Proof: we have

$$\boxed{\frac{x_{n+1}}{x_n} \le \frac{y_{n+1}}{y_n} \quad \forall\, n \ge N}$$

$$\Rightarrow \quad \frac{x_{N+1}}{x_N} \le \frac{y_{N+1}}{y_N},$$

$$\frac{x_{N+2}}{x_{N+1}} \le \frac{y_{N+2}}{y_{N+1}}, \ldots$$

Therefore $\dfrac{x_n}{x_N} \leq \dfrac{y_n}{y_N}, \forall n > N$

or, $x_n \leq \dfrac{x_N}{y_N} \cdot y_n \quad \forall n > N$

or, $\boxed{x_n \leq K y_n, \quad \forall n > N}$

where $K = \dfrac{x_N}{y_N}$ is a positive number.

Therefore, by Comparison test (first type), $\sum x_n$ is Convergent if $\sum y_n$ is Convergent and $\sum y_n$ is divergent if $\sum x_n$ is divergent.

(Proved)

There are other tests for Convergence of positive Series —

(i) D'Alembert's Ratio Test

(ii) Cauchy's Root Test

(iii) Cauchy's Condensation Test

(iv) Raabe's Test (You may skip)

(v) Integral Test, etc.

D'Alembert's Ratio Test for both Positive and Negative Series

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② **Theorem:** Let $\sum x_n$ be an infinite series of arbitrary terms, i.e. both positive and negative terms.

Let $\lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = l$.

For positive series only one can consider $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$.

This is applicable for positive and negative terms. $\therefore \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = l$

Then

(i) $\sum x_n$ is absolutely convergent if $l<1$

(ii) $\sum x_n$ is divergent if $l>1$.

Proof: Let us choose $\varepsilon>0$ such that $l+\varepsilon<1$.

$\exists$ a natural number N such that

$$l-\varepsilon < \left|\frac{x_{n+1}}{x_n}\right| < l+\varepsilon \quad \forall n \geq N$$

Let $l+\varepsilon = r$. Then $0<r<1$

and $\left|\frac{x_{n+1}}{x_n}\right| < r, \quad \forall n \geq N.$

$\therefore \left|\frac{x_{N+1}}{x_N}\right| < r, \quad \left|\frac{x_{N+2}}{x_{N+1}}\right| < r, \quad \left|\frac{x_{N+3}}{x_{N+2}}\right| < r$

$\ldots$

$$\Rightarrow \left|\frac{x_n}{x_m}\right| < r^{n-m}, \text{ for all } n > m$$

$$\Rightarrow |x_n| < \frac{|x_m|}{r^m} \cdot r^n \quad \forall\, n > m$$

But $\sum r^n$ is a convergent series since $0 < r < 1$.

So, by comparison test, the series $\sum |x_n|$ is convergent.

Therefore, the Series $\sum x_n$ is absolutely Convergent.

(ii) Similar proof as discussed in the 'Sequence''s lecture.

<u>Try it.</u>

Otherwise,

Try this proof in another way

Let $\varepsilon > 0$ such that $l - \varepsilon > 1$.

Then $\exists\, N \in \mathbb{N}$ such that

$$l - \varepsilon < \left| \frac{x_{n+1}}{x_n} \right| < l + \varepsilon, \quad \forall\, n \geq N.$$

Therefore, $\left|\dfrac{x_{n+1}}{x_n}\right| > l - \varepsilon > 1$

$\forall\, n \geq N.$

Hence the Sequence $\{|x_{n+1}|\}$ is a monotone increasing sequence of positive real numbers. So $\lim_{n\to\infty} |x_n| \neq 0$

$\Rightarrow \lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum x_n$ is divergent.

Remark:

This Test is applicable for both Positive and negative Series.

If $l = 1$, the Ratio test fails to give the decision on Convergence of the infinite Series.

(A) For $\sum \frac{1}{n}$, $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = 1$, So Ratio test fails here. However, we know that $\sum \frac{1}{n}$ is divergent.

(B) For $\sum \frac{1}{n^2}$, $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = 1$, So Ratio test fails here. However we know that $\sum \frac{1}{n^2}$ is convergent.

Examples :

① Test the Convergence of the positive Series –

$$1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\cdots$$

Solⁿ: Let $\sum x_n$ be the given Series where $x_n = \frac{2n-1}{n!}$.

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{2n+1}{(n+1)(2n-1)}$$

$$= 0 < 1$$

By D' Alembert Ratio test, or simply Ratio test, $\sum x_n$ is Convergent.

② Examine the convergence of the positive series $x+\frac{x^2}{2}+\frac{x^3}{3}+\dots, \; x>0.$

Soln:

Let $\sum x_n$ be the series such that $x_n = \frac{x^n}{n}, \; x>0.$

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{nx}{n+1} = x$$

By D'Alembert's ratio test

$\sum x_n$ is convergent if $x < 1$

and $\sum x_n$ divergent if $x > 1$

(3) Test the Convergence of the Positive Series $1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots$

Try This (Home work).

④ Examine the Convergence of the arbitrary Series —

$$1 - \frac{2^2}{2!} + \frac{3^3}{3!} - \frac{4^4}{4!} + \cdots$$

Solⁿ: Let $\sum_{n=1}^{\infty} x_n$ be the given Series, Such that $x_n = (-1)^{n+1} \cdot \frac{n^n}{n!}$

Now $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$= e > 1$

By Ratio test, the given series $\sum x_n$ is divergent.

(5) Examine the convergence of the arbitrary series

$$1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots$$

Try this. (Home work).

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Cauchy Root Test for both Positive and Negative Series

Theorem (Root Test):

Let $\sum x_n$ be a Series of arbitrary terms and let $\lim_{n\to\infty} |x_n|^{1/n} = l$.

For Positive series only one can consider $\lim_{n\to\infty} x_n^{1/n} = l$.

Then (i) $\sum x_n$ is absolutely convergent if $l < 1$.

(ii) $\sum x_n$ is divergent if $l>1$.

(iii) If $l=1$, this test fails.

Proof: Let $\varepsilon>0$ such that $l+\varepsilon<1$

Then $\exists\, N\in\mathbb{N}$ such that

$$l-\varepsilon<|x_n|^{1/n}<l+\varepsilon,\ \forall\, n\geq N.$$

Let $l+\varepsilon=r$. Then $0<r<1$.

we have $|x_n|^{1/n}<r \quad \forall\, n\geq N$.

or, $|x_n|<r^n, \quad \forall\, n\geq N$

Since $\sum r^n$ is convergent series for $0<r<1$, by Comparison test the series $\sum|x_n|$ is convergent

Therefore the Series $\sum x_n$ is absolutely Convergent.

(ii) Let $\varepsilon > 0$ such that $l - \varepsilon > 1$.

Then $\exists\, N \in \mathbb{N}$ such that

$$l - \varepsilon < |x_n|^{1/n} < l + \varepsilon \quad \forall\, n \geq N$$

$$\Rightarrow \quad |x_n| > 1, \quad \forall\, n \geq N.$$

So, $\lim\limits_{n\to\infty} |x_n| \neq 0$, and

this implies that

$\lim\limits_{n\to\infty} x_n \neq 0$,

Hence $\sum x_n$ is divergent.

(proved)

Note: This test is applicable for the series of both positive and negative terms.

Example :

1. Test the Convergence of the positive Series

$$1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\cdots$$

Solⁿ: Here $x_n = \dfrac{1}{2^{n+(-1)^n}}$

$$\lim_{n\to\infty} x_n^{1/n} = \lim_{n\to\infty}\left\{2^{1+\frac{(-1)^n}{n}}\right\}^{-1}$$

$$= \frac{1}{2}$$

Therefore the Series is Convergent.

Note: Here $\frac{x_{n+1}}{x_n} = \frac{1}{8}$ if n is odd

$= 2$ if n is even

So $\lim_{n \to \infty} \frac{x_{n+1}}{x_n}$ does not exist.

Therefore, Ratio test is not applicable here.

2. Investigate the convergence of the series — (positive)

$$\sum \frac{n^n}{n!}.$$

Homework.