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Lecture Note - 4

Fourth Lecture on Calculus-I

(MA-1110)

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Sequences

- Some important theorems
- Monotone Sequences

Since $x_1 = 2$, we can write it

$$\boxed{2 \leq x_n < 3, \quad \forall n \in \mathbb{N}}$$

$\Rightarrow$ The sequence $\{x_n\}$ is bounded.

$\Rightarrow$ By theorem, the sequence $\{x_n\}$ is convergent and the limit of the sequence is e.

i.e. $\boxed{\lim_{n\to\infty} x_n = e}$

note that
$2 \leq \lim_{n\to\infty} x_n \leq 3$

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Some Important Theorems

Theorem: Let $\{x_n\}_n$ be a sequence of positive real numbers such that

$$\boxed{\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l.}$$

Also one can consider $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = l$ for a sequence of positive and negative real numbers.

(i) If $0 \leq l < 1$, then $\lim_{n\to\infty} x_n = 0$

(ii) If $l > 1$, then $\lim_{n\to\infty} x_n = \infty$.

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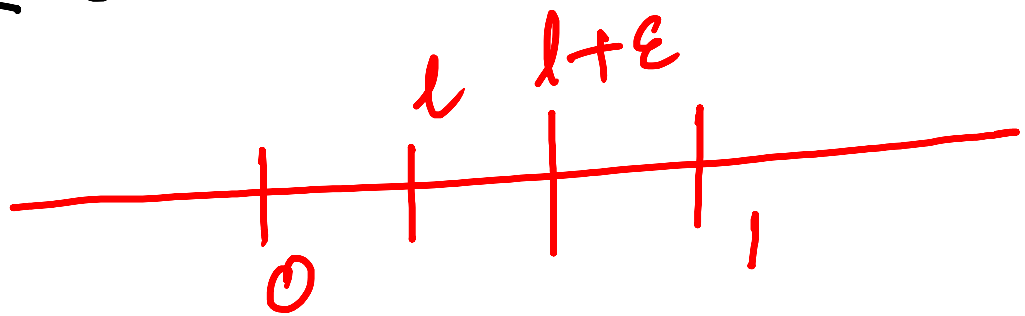
Proof:

Let us consider a positive ε such that $l+\varepsilon<1$

Since $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = l$, $\exists$ a natural number N such that

$$\left| \frac{x_{n+1}}{x_n} - l \right| < \varepsilon \quad \forall\, n \geq N$$



$$\Rightarrow l-\varepsilon < \frac{x_{n+1}}{x_n} < l+\varepsilon \quad \forall\, n \geq N$$

Let $m = l+\varepsilon$.

Here given that $0 \leq l < 1$.

So $0 < m < 1$.

Therefore $\frac{x_{n+1}}{x_n} < m \quad \forall\, n \geq N$

$$\Rightarrow x_{n+1} < x_n \cdot m \quad \forall\, n \geq N$$

For $n=N+1$, $x_{N+1} < x_N \cdot m$

For $n=N+2$, $x_{N+2} < x_{N+1} \cdot m < x_N \cdot m^2$

For $n=N+3$, $x_{N+3} < x_{N+2} \cdot m < x_N \cdot m^3$

$\cdots \qquad \cdots \qquad \cdots$

For n $\quad x_n < x_{n-1} \cdot m < x_N \cdot m^{n-N}, \forall n \geq N+1$

$x_n \to x_{N+(n-N)}$

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So we have

$$0 < x_n < \frac{x_N}{m^N} \cdot m^n, \ \forall \ n \geq N+1$$

$$\Rightarrow \ 0 < x_n < A \cdot m^n \quad \forall \ n \geq N+1$$

where $A = \frac{x_N}{m^N}$.

Here $\lim_{n \to \infty} A \cdot m^n = 0$ Since $0 < m < 1$

[we know from ealier results]

So by Sandwich theorem, we have

$$\boxed{\lim_{n \to \infty} x_n = 0 \quad \text{as } n \to \infty.}$$

(Proved)

(ii) Similarly try to prove this part (left as home task).

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(iii) **Note:** If $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$, no definite conclusion can be made about the nature of sequence.

For example: (i) choose $x_n = \frac{n+1}{n}$.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

And $\lim_{n\to\infty} x_n = 1$

(ii) However if we choose $x_n = \frac{1}{n}$, where

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$$

then $\lim_{n\to\infty} x_n = 0$

So for these sequences, we are not sure about the limit value of $\{x_n\}_n$, those are different.

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Theorem: Let $\{x_n\}_n$ be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n^{1/n} = l$.

one can consider $\lim_{n\to\infty} |x_n^{1/n}| = l$ for any sequence of positive and negative real numbers.

(i) If $0 \leq l < 1$, then $\lim_{n\to\infty} x_n = 0$

(ii) If $l > 1$, then $\lim_{n\to\infty} \cancel{x_n} = \infty$.

Proof: Let us choose a positive number ε such that $l+\varepsilon<1$.

Since $\lim_{n\to\infty} x_n^{1/n} = l$; for $\varepsilon>0$ $\exists$ a natural number N such that

$$|x_n^{1/n} - l| < \varepsilon, \ \forall\, n \geq N$$

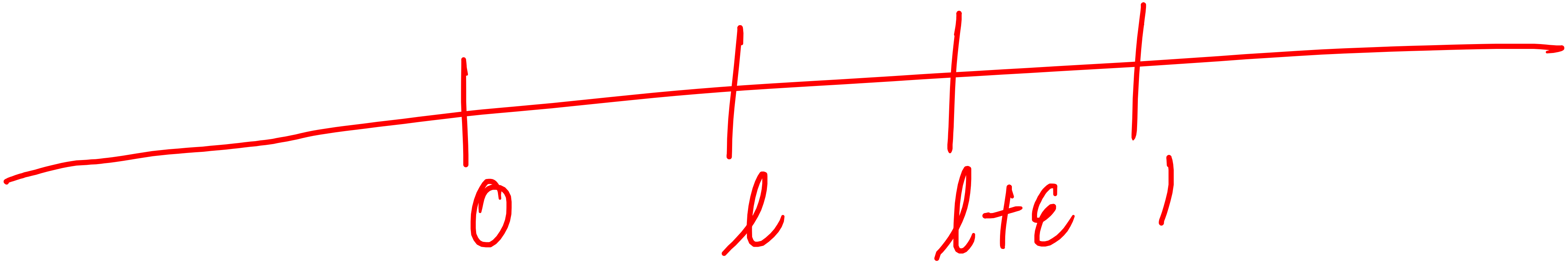
$$\Rightarrow l-\varepsilon < x_n^{1/n} < l+\varepsilon, \ \forall\, n\geq N.$$

Let $m = l + \varepsilon$.

Here given that $0 \leq l < 1$

$\Rightarrow 0 < l + \varepsilon < 1$

$\Rightarrow 0 < m < 1$



We have

$$0 < x_n < m^n , \quad \forall n \geq N.$$

Since $0 < m < 1$, $\lim_{n\to\infty} m^n = 0$.

By Sandwich Theorem, we have

$$\boxed{\lim_{n\to\infty} x_n = 0}$$

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(ii) You try this part. (Homework).

Note: If $\lim_{n\to\infty} x_n^{1/n} = 1$, then we are not sure about the nature of sequence.

For example, (i) choose $x_n = \frac{n+1}{n}$ then

$$\lim_{n\to\infty} (x_n)^{1/n} = 1$$

and $\lim_{n\to\infty} x_n = 1$.

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(ii) When we choose $x_n = \frac{n+1}{2n}$,

then $\lim_{n\to\infty} x_n^{1/n} = 1$ and

$$\lim_{n\to\infty} x_n = 1/2.$$

So we are getting different limit values of the sequence when $\lim_{n\to\infty} x_n^{1/n} = 1$.

Therefore we can't conclude about the nature of the limit of the sequence.

Examples: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real x.

Sol^n: Let $u_n = \frac{x^n}{n!}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)!}$

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$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{x/n}{(1+1/n)}$$

$$= 0/1 = 0$$

[Since $\frac{1}{n} \to 0$ as $n \to \infty$]

So. $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0$

$\Rightarrow \lim_{n\to\infty} u_n = 0$ [By the theorem]

Example: $\lim_{n\to\infty} \frac{n^p}{x^n} = 0$ provided $x>1$ and $p>0$

Sol^n: Let $u_n = \frac{n^p}{x^n}$.

$u_{n+1} = \frac{(n+1)^p}{x^{n+1}}$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{x^{n+1}} \cdot \frac{x^n}{n^p}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{x}$$

$$= \frac{1}{x} < 1 \quad \text{if } x > 1 \text{ and } p > 0$$

Therefore $\lim_{n\to\infty} u_n = 0$ if $x > 1$ and $p > 0$

Example: $\lim_{n\to\infty} \frac{x^n}{n} = 0$ if $x \le 1$

(Try it) (Home work).

Example: If $x_n = \frac{3.5.7\ldots(2n+1)}{2.5.8\ldots(3n-1)}$

then $x_n \to 0$ as $n \to \infty$

(Try it).

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Example: A Sequence $\{x_n\}$ is defined by $x_{n+1} = \sqrt{2x_n}$ for $n \geq 1$ and $x_1 = \sqrt{2}$. Prove that $\lim_{n\to\infty} x_n = 2$.

Sol^n: $x_1 = \sqrt{2}$, $\quad x_2 = \sqrt{2x_1} = \sqrt{2\sqrt{2}}$

$\Rightarrow x_1 = 2^{1/2}$ $\qquad = 2^{\frac{1}{2}+\frac{1}{2^2}} = 2^{1-\frac{1}{2^2}}$

$\therefore x_1 = 2^{1/2}, \quad x_2 = 2^{1-\frac{1}{2^2}}$

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$$x_3 = \sqrt{2x_2} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}} = 2^{1-\frac{1}{2^3}}$$

$$\cdots \qquad \cdots$$

$$x_n = \sqrt{2x_{n-1}} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}}$$

$$= 2^{1-\frac{1}{2^n}}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2^{\left(1-\frac{1}{2^n}\right)} = 2^1 = 2$$

Since $\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1$ as $\lim_{n\to\infty}\frac{1}{2^n} = 0$.

Monotone Sequences

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Definition:

$\Rightarrow$ A real sequence $\{x_n\}_n$ is said to be a monotone increasing sequence if

$$\boxed{x_{n+1} \geq x_n \quad \forall n \in \mathbb{N}}$$

$\Rightarrow$ A real sequence $\{x_n\}_n$ is said to be a monotone decreasing sequence if $\boxed{x_{n+1} \leq x_n, \forall n \in \mathbb{N}}$

$\Rightarrow$ A real sequence $\{x_n\}_n$ is said to be a monotone sequence if it is either a monotone increasing sequence or a monotone decreasing sequence.

Note: $\forall n \in \mathbb{N}$, $x_{n+1} > x_n \Rightarrow$ Monotone strictly increasing sequence

$\forall n \in \mathbb{N}$, $x_{n+1} < x_n \Rightarrow$ Montone strictly decreasing sequence.

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Examples:

(i) $\left\{\frac{n}{n+1}\right\}$ is monotone increasing sequence

Since $x_{n+1} > x_n$ where $x_n = \frac{n}{n+1}$.

It is also strictly monotone increasing.

(ii) $\left\{\frac{1}{n}\right\}$ is monotone decreasing sequence

Since $x_{n+1} < x_n$ where $x_n = \frac{1}{n}$.

It is also strictly monotone decreasing sequence.

(iii) $\{2^n\}$ is monotone increasing sequence since $x_{n+1} > x_n$, $\forall n \geq N$, where $x_n = 2^n$. It is also strictly monotone increasing sequence.

(iv) $\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right\}$, $n \geq 1$ is monotone increasing sequence and it is also strictly monotone.

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(v) The Sequence $\{(-2)^n\}$ is neither a monotone increasing Sequence nor a monotone decreasing Sequence. So it is not a monotone sequence.

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Monotone Convergence Theorems

Theorem: If a monotone increasing sequence is bounded above as well as convergent then it converges to the least upper bound (lub).

Proof: Let $\{x_n\}$ be a monotone increasing sequence and bounded above.
Let M be its least upper bound.

Then we have by the defn.

(i) $x_n \le M \quad \forall n \in \mathbb{N}$

and (ii) for every $\varepsilon > 0$, $\exists$ a natural number N such that

$$x_N > M - \varepsilon.$$

Now since $\{x_n\}$ is a monotone increasing sequence,

$$x_n \leq x_{n+1}, \quad \forall n$$

$$\Rightarrow \quad M-\varepsilon < x_N \leq x_{N+1} \leq x_{N+2} \cdots \leq M < M+\varepsilon$$

Therefore $M-\varepsilon < x_n < M+\varepsilon, \; \forall n \geq N$

$$\Rightarrow |x_n - M| < \varepsilon, \; \forall n \geq N$$

$\Rightarrow \{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = M$ (Proved)

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Theorem: If a monotone decreasing sequence is bounded below as well as convergent then it converges to greatest lower bound (glb).

Proof: Similar proof. (Home work).

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Theorems:

(a) A monotone increasing sequence which is unbounded above, diverges to ∞.

(b) A monotone decreasing sequence which is unbounded below, diverges to $-\infty$.

Proof: (a) Let $\{x_n\}$ be a monotone increasing sequence and not bounded above.

Since the sequence is unbounded above for any given positive number m, $\exists$ a natural number N such that

$$\boxed{x_N > m.}$$

Since the sequence $\{x_n\}$ is monotone increasing, we have

$$\boxed{x_n \leq x_{n+1}, \quad \forall n}$$

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$$\Rightarrow \quad m < x_N \leq x_{N+1} \leq x_{N+2} \leq \cdots$$

$$\Rightarrow \quad m < x_n, \quad \forall n \geq N$$

$$\boxed{\therefore \ \forall n \geq N, \quad x_n > m}$$

$\Rightarrow \{x_n\}$ diverges to ∞.

(b) Similar Proof. (Home work).

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Example: Show that the sequence $\{(1+\frac{1}{n})^n\}$ is convergent.

Sol^n: Our aim is to show that the sequence $\{(1+\frac{1}{n})^n\}$ is monotone increasing and bounded above. Then we can easily say by the theorem that the given sequence is convergent.

Let $x_n = \left(1+\frac{1}{n}\right)^n$.

Then $x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$.

ⓐ The sequence is monotone increasing:

Proof:

$$x_n = \left(1+\frac{1}{n}\right)^n$$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}$$

$$+ \cdots + \frac{n(n-1)\cdots 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Then

$$x_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

It is easily noted that x_{n+1} contains one more term than x_n.

we know

$$1-\frac{1}{n} < 1-\frac{1}{n+1}$$

and $1-\frac{2}{n} < 1-\frac{2}{n+1}$, So on.

Therefore we can write that

$$x_{n+1} > x_n \quad \forall n.$$

$\Rightarrow \{x_n\}$ is monotone increasing sequence (proved)

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Alternative:

Let $x_n = \left(1+\frac{1}{n}\right)^n$. Then $x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$

Let us consider $(n+1)$ positive numbers

$1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}$ (n times)

and 1.

Applying A.M. > G.M.

we have $\dfrac{n\left(1+\frac{1}{n}\right)+1}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$

as, $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}$

$$\boxed{\text{i.e., } x_{n+1} > x_n, \ \forall n \in \mathbb{N}}$$

$\Rightarrow$ $\{x_n\}$ is a monotone increasing sequence.

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(b) The Sequence $\{x_n\}$ is bounded.

Proof:

$$x_n = 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2.1}{n!}\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\frac{2}{n}\cdot\frac{1}{n}$$

$$<1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} \quad \forall\ n\geq 2$$

Now we have $n! > 2^{n-1}$ $\forall\ n>2$.

Therefore,

$$x_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}, \text{ for } n > 2$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right) < 3, \quad \forall n \in \mathbb{N}$$

So we have $\boxed{x_n < 3 \quad \forall n \in \mathbb{N}}$

$\Rightarrow$ The sequence is bounded above

Since $x_1 = 2$, we can write it

$$\boxed{2 \leq x_n < 3, \quad \forall n \in \mathbb{N}}$$

$\Rightarrow$ The Sequence $\{x_n\}$ is bounded.

$\Rightarrow$ By theorem, the sequence $\{x_n\}$ is convergent and the limit of the sequence is e.

i.e. $\boxed{\lim_{n\to\infty} x_n = e}$

note that

$$2 \leq \lim_{n\to\infty} x_n \leq 3$$

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Example: If $x_n = \dfrac{3n-1}{n+2}$, prove that $\{x_n\}$ is convergent.

Proof:

$$x_n = \frac{3n-1}{n+2}, \quad x_{n+1} = \frac{3(n+1)-1}{n+1+2}$$

$$= \frac{3n+2}{n+3}$$

$$x_{n+1} - x_n = \frac{3n+2}{n+3} - \frac{3n-1}{n+2} = \frac{7}{n^2+5n+6}.$$

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$x_{n+1} - x_n = \frac{7}{n^2+5n+6}$, which is positive $\forall n \geq 1$

$\therefore x_{n+1} - x_n > 0$

$\Rightarrow \boxed{x_{n+1} > x_n}$

$\Rightarrow \{x_n\}$ is strictly monotone increasing.

Now $x_n = \frac{3n-1}{n+2} = \frac{3(n+2)-7}{n+2} = 3 - \frac{7}{n+2}$

$< 3 \ \forall n \geq 1$

Thus $\{x_n\}$ is monotone increasing and bounded above.

Here $\{x_n\}$ is also bounded below since

$$x_n > x_1 = 2/3 .$$

Therefore $\{x_n\}$ is monotone increasing and bounded. Hence $\{x_n\}$ is convergent.

Proved

Note that 3 is upper bound. So

$$\lim_{n\to\infty} x_n \leq 3 .$$

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Example: Prove that $\{x_n\}$ where

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n},$$

is a Convergent Sequence.

Try it (Home work).