भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

# Eleventh Lecture on Calculus-I

(MA-1110)

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# Differential Calculus

- Limit of function of a Single Real Variable
- Continuity and Differentiability
- Rolle’s Theorem
- Lagrange’s Mean Value theorem

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# Today’s Class Lecture

- Differentiability of the function
- Rolle’s Theorem
- Lagrange’s Mean Value theorem

# Differentiability of function

- Differentiability of a function means we are talking about the derivative of this function at a particular point.

- Differentiability comes from differentiation which is the process of finding derivative.

## What is the derivative?

⇒ The derivative of a function at a point describes the rate of change of the function near that point.

Geometrically, the derivative of the function at a point is the slope of the tangent line to the graph of the function at that point.

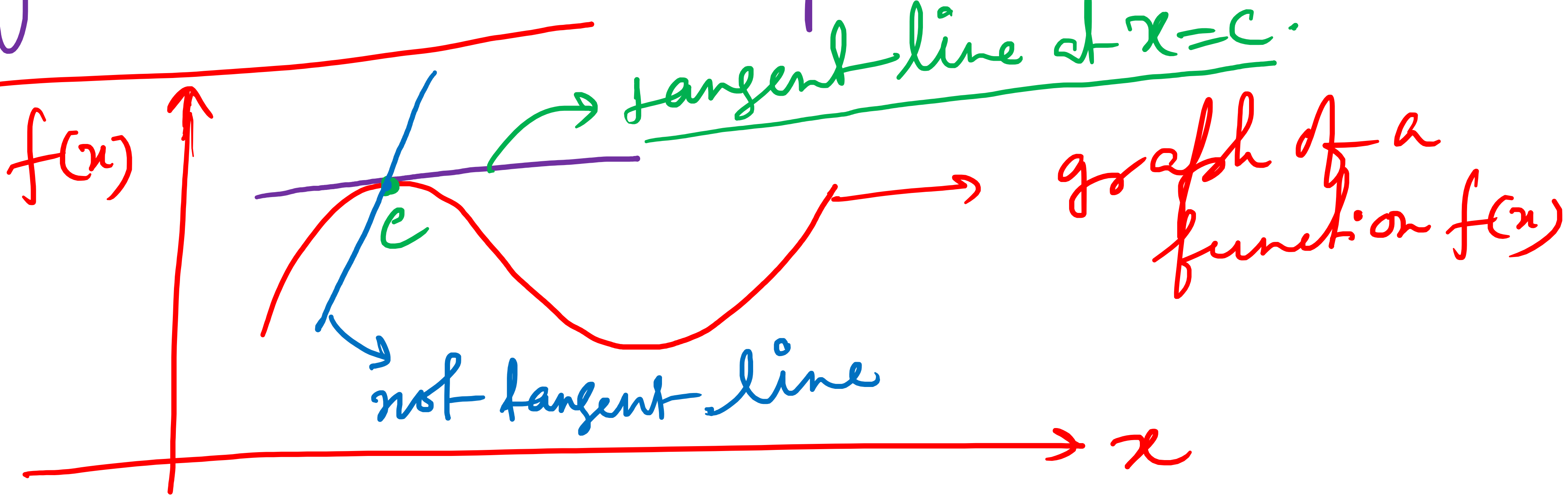
The derivative at a point exist means there is an unique tangent line to the graph of the function at that point.

If multiple tangent lines exist, then the derivative does not exist.

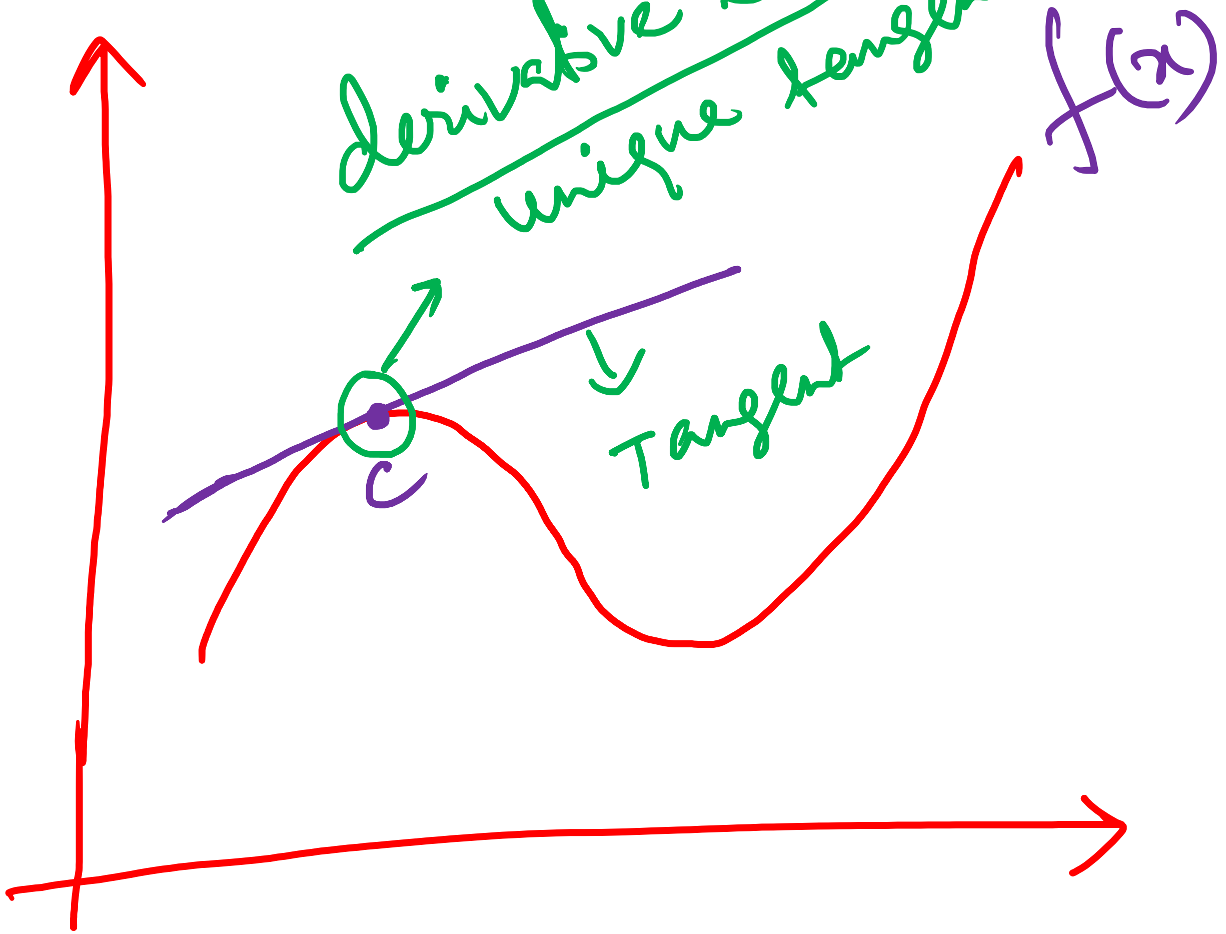
↳ example: $f(x) = |x|$ at $x = 0$.

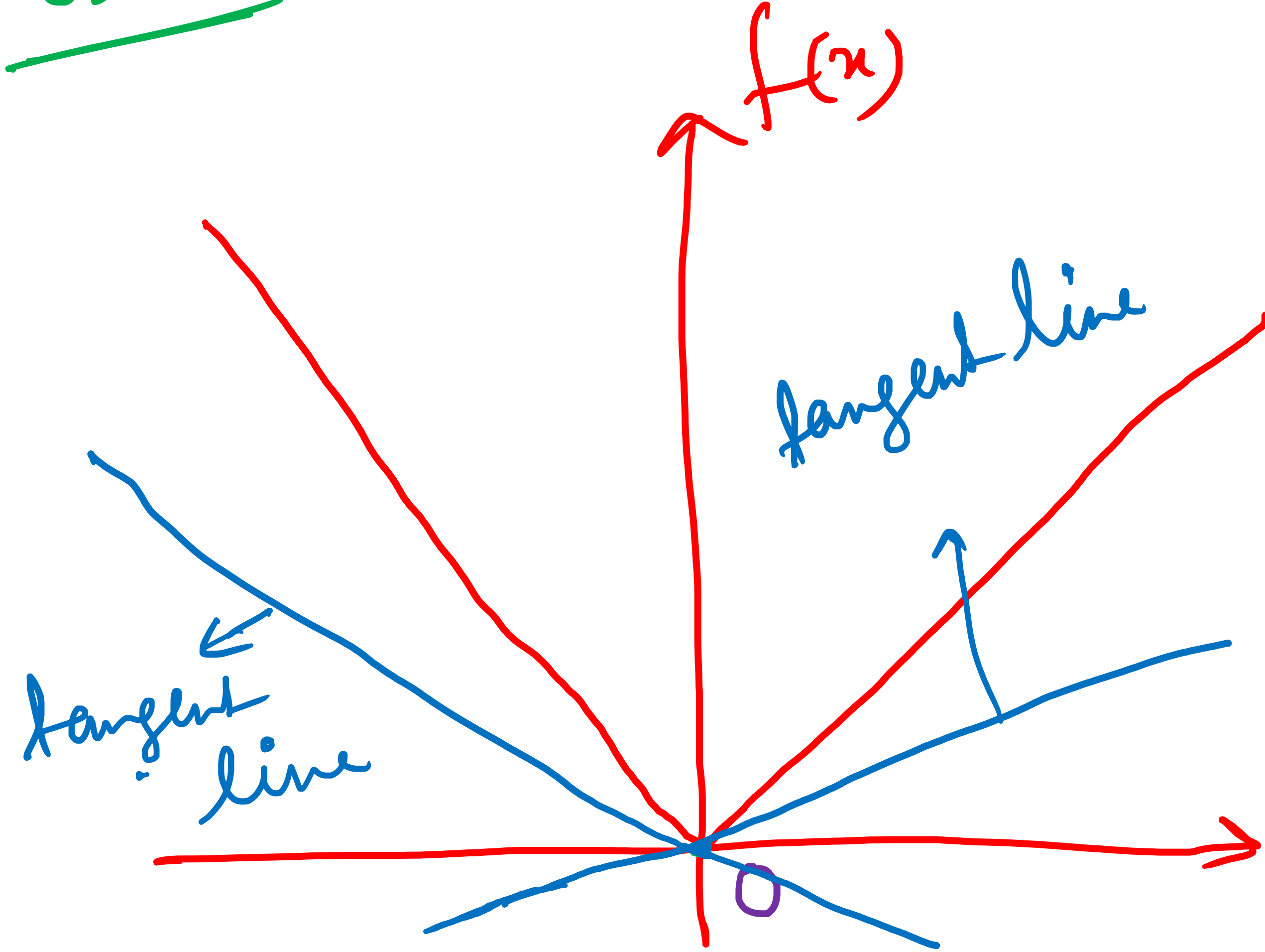
## What is tangent line?

A tangent line is a line which 'just touches' a particular point.



derivative exist
unique tangent can be obtained




derivative doesn't exist at $x=0$, since many tangent lines can be drawn at $x=0$.

# What is the slope of a function?

The equation of the straight line or the linear equation is given by

$$\boxed{y = mx + a}$$

Here $m$ is the slope of the eqn.

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}.$$

This slope can be obtained by choosing any two points $(x_1, y_1)$ and $(x_2, y_2)$ on the graph of a function.

i.e, $\text{Slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

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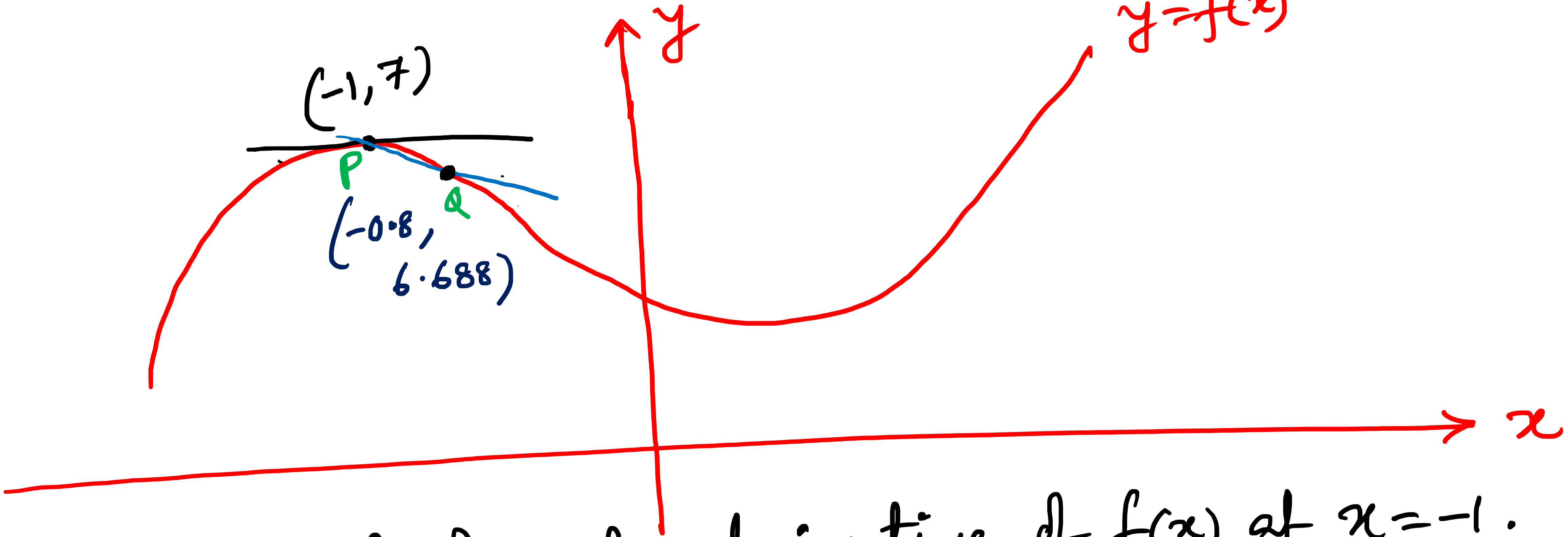
**Example:** $y = f(x) = x^3 - 4x + 4$




we have to find the derivative of $f(x)$ at $x = -1$.

Now we start to choose a nearby point Q (-0.8)

Therefore, we can obtain the slope of PQ line which is approximate to the slope of the tangent at P.

Slope of PQ is $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(-0.8) - f(-1)}{-0.8 + 1} = -2.08$$

P $(x_1, y_1)$
and Q $(x_2, y_2)$.

Similarly, we pick another point which is more closer to P than Q and calculate the slope.

| $x$ | $y$ | slope |
| --- | --- | --- |
| $-0.8$ | $6.688$ | $-2.08$ |
| $-0.9$ | $6.871$ | $-1.57$ |
| $-0.95$ | $6.94263$ | $-1.2925$ |
| $-0.98$ | $6.97881$ | $-1.1188$ |
| $-0.99$ | $6.9897$ | $-1.0597$ |
| $-0.999$ | $6.999$ | $-1.006$ |
| $-0.9999$ | $6.9999$ | $-1.0006$ |
| $-1$ | $7$ | $-1$ |

$$\frac{f(-0.9)-f(-1)}{-0.9+1}$$

These slopes are approximate values

These are making a sequence

This is its limit value.

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So we can see that as the point Q approaches to P, we can get more accurate approximate value to the slope of the tangent at P, i.e, the derivative of the function at the point P. Here limit concept is used to find the derivative.

So, slope of the curve at $(-1, 7)$ is $-1$

$\Rightarrow$ derivative of $f(x)$ at $x = -1$ is $-1$.

So, we get the idea of derivative as

the slope of tangent to the curve $y=f(x)$ at a point $P(x_1, f(x_1))$

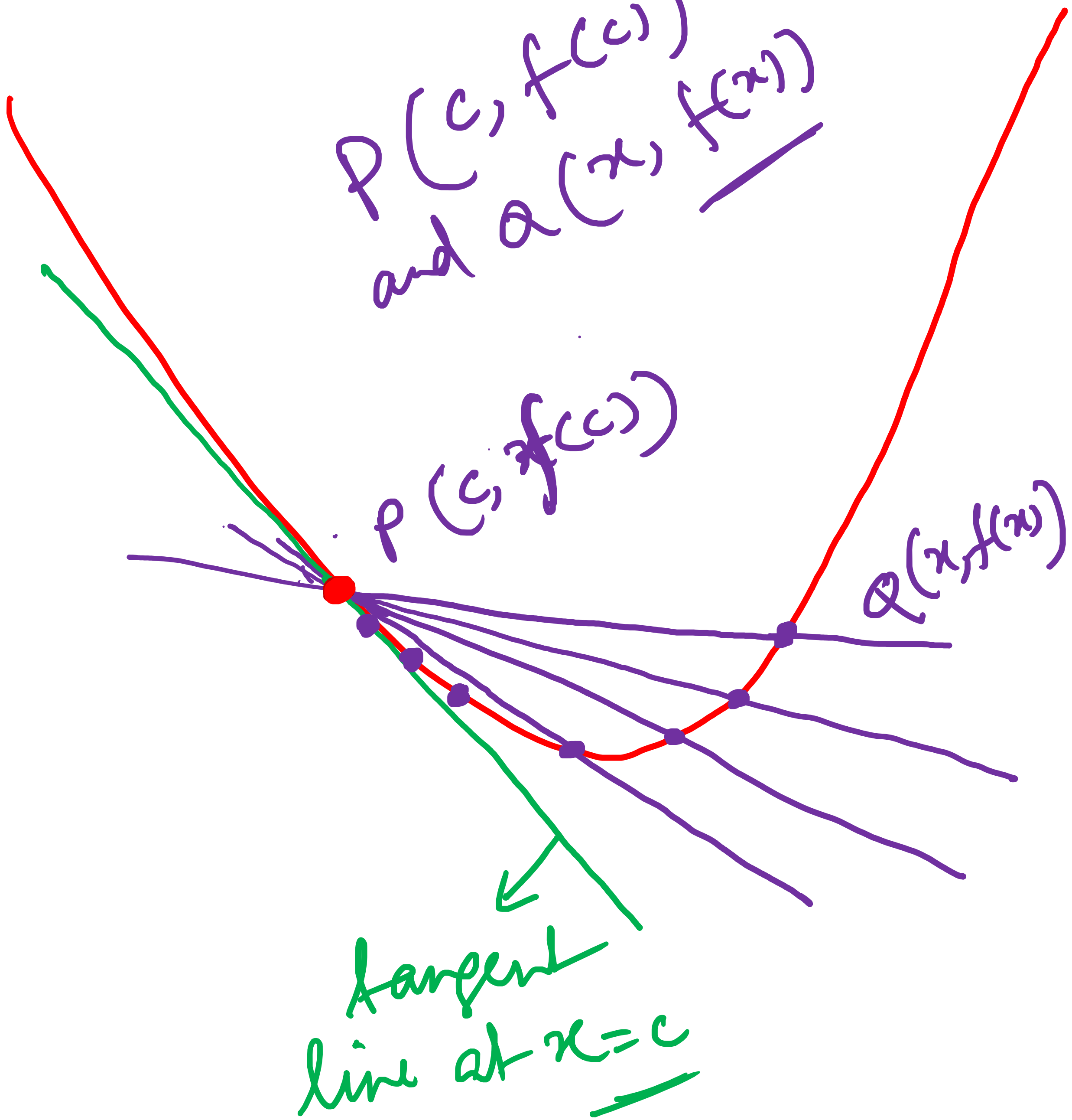
$=$ derivative of $f(x)$ at a point $x_1$

$$= \lim_{x \to x_1} \frac{f(x)-f(x_1)}{x-x_1}$$

rate of change of the function $f(x)$.

$P(c, f(c))$ and $Q(x, f(x))$

$y = f(x)$




Here Q point approaches to P and as it gets closer closer to P, we can get the slope of the tangent line at P..

So, the function is differentiable at $x=c$ if the derivative exists at $x=c$ and

Definition of differentiability

$$f'(c) = \left.\frac{df}{dx}\right|_{x=c}$$

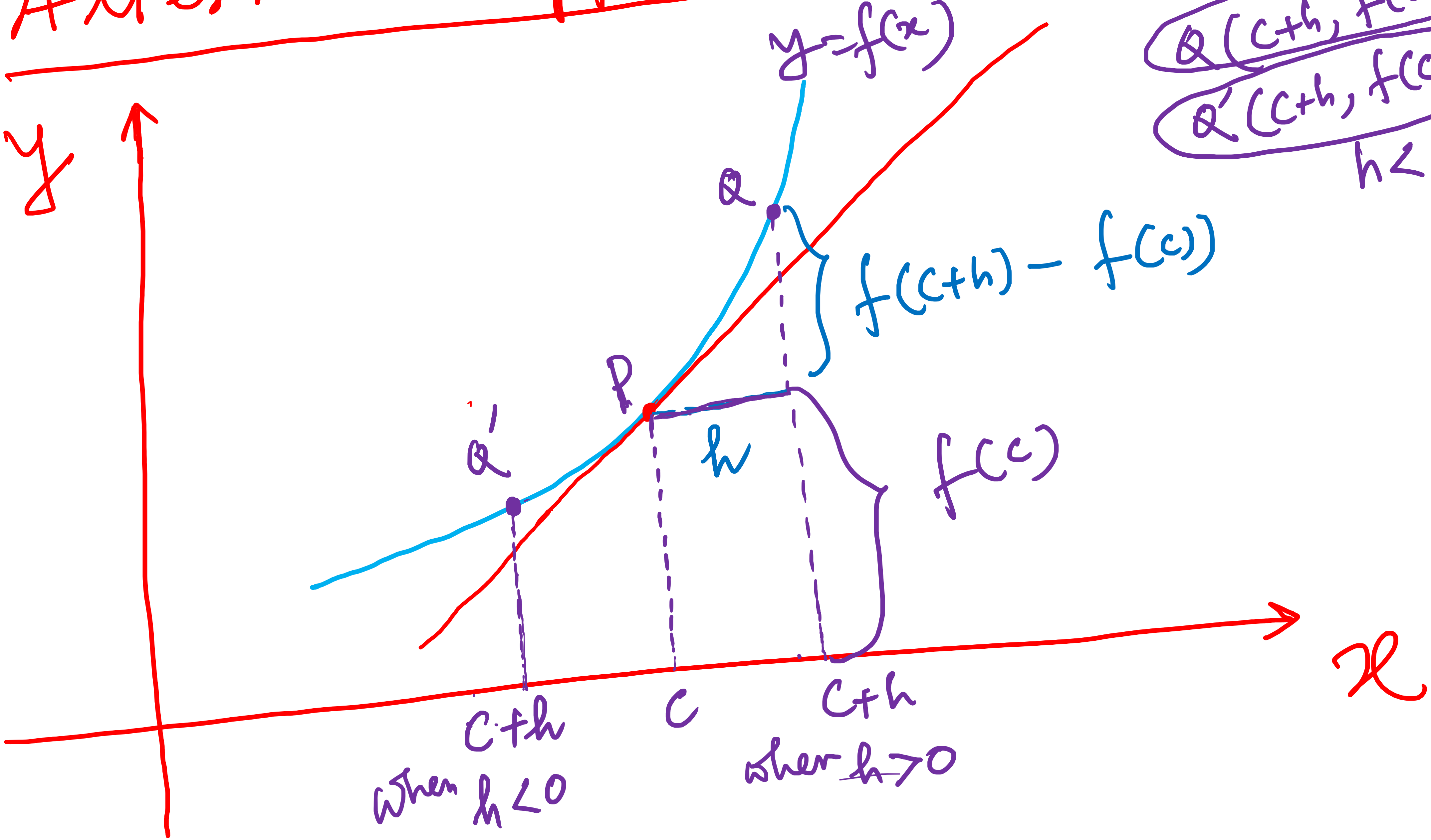
$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

# Alternative approach:

$P(c, f(c))$

$Q(c+h, f(c+h))$ $h>0$

$Q'(c+h, f(c+h))$ $h<0$



Our aim is to obtain the derivative at the point $x=c$. Let the point on function at $x=c$ be $P(c, f(c))$.

Consider a point $Q(c+h, f(c+h))$ which is very close to $P(c, f(c))$, where $h$ is very small real number.

So, slope of the line $PQ$ is

$$= \frac{f(c+h) - f(c)}{c+h-c}$$

$$= \boxed{\frac{f(c+h) - f(c)}{h}}$$ rate of change in $f(x)$.

Now we consider the limit concept to find the derivative.

Since $h$ tends to 0, the slope of the line PQ gets closer and closer to the slope of the tangent line to the function at c.

So, it is written as –

$$\boxed{f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}}.$$

# Definition of Differentiability

**Definition:** Let $I$ be an interval.

Let $f: I \to \mathbb{R}$ be a function. Let $c \in I$.

It is differentiable at $x = c$,

if the limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ or

$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists and it is denoted by

$$f'(c) = \left.\frac{dy}{dx}\right|_{x=c} = \left.\frac{df}{dx}\right|_{x=c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} .$$

**Remark:** The limit $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

exists mean

(i) $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$

exist and it is equal.

Right hand derivative
$Rf'(c)$

Left hand derivative
$Lf'(c)$.

**Theorem:** If $f: I \to \mathbb{R}$ has a derivative at a point $c \in I$, then $f$ is continuous at $c$.

So, $f$ is differentiable at a point
$\Rightarrow f$ is continuous at that point.

**Proof:** Let $x \in I$, $x \neq c$.

Since $f$ is differentiable at $x=c$

$f'(c)$ exists and it is defined by

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}}$$

we have, $f(x)-f(c) = \dfrac{f(x)-f(c)}{(x-c)} (x-c)$

Taking the limit both side -

$$\lim_{x \to c} \left[ f(x) - f(c) \right] = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} (x - c)$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c} (x - c)$$

$$= f'(c) \cdot 0 \quad = 0.$$

or, $$\lim_{x \to c} \left[ f(x) - f(c) \right] = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c).$$

$\Rightarrow$ $f$ is continuous at $x = c$.

**Note:** The converse is not true.

See one example.

**Ex.** $f(x) = |x|$ is continuous at $x=0$ but it is not differentiable at $x=0$.

**Proof:**

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0. \end{cases}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

So, this limit does not exist, we have seen earlier.

$$\lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = 1$$

and $$\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

not equal

$\therefore f'(0)$ does not exist.

**Note:** The function $|x|$ is differentiable at any point $\underline{x \neq 0}$.

**Ex.:** 1 Consider a constant function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = K \ \forall x \in \mathbb{R}$, where $K \in \mathbb{R}$ is some constant.
Then $f'(c) = 0$ for every $c \in \mathbb{R}$.

**Proof:** Let $c \in \mathbb{R}$.

Then $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

**OR** $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{K - K}{h} = 0$$

$$= \lim_{x \to c} \frac{K - K}{x - c}$$

$$= 0.$$

**Ex. 2** Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ $\forall x \in \mathbb{R}$, where $n \geq 1$ is some positive integer. Then $f'(c) = nc^{n-1}$ for every $c \in \mathbb{R}$.

**Proof:** Let $c \in \mathbb{R}$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0} \frac{(c+h)^m - c^m}{h}$$

$$= \lim_{h \to 0} \frac{\left(c^m + mc^{m-1}h + \dots + h^m\right) - c^m}{h}$$

$$= \lim_{h \to 0} \left(mc^{m-1} + \binom{m}{2} c^{m-2} \cdot h + \dots + h^{m-1}\right)$$

$$= mc^{m-1}$$

**Theorem:** Let $I$ be an interval and $c \in I$. Let $f, g : I \to \mathbb{R}$ be differentiable at $c$. Then

(i) $f \pm g$ is differentiable at $c$ and $(f \pm g)'(c) = f'(c) \pm g'(c)$

(ii) if $K \in \mathbb{R}$, $Kf$ is differentiable at $c$ and $(Kf)'(c) = Kf'(c)$.

(iii) $f \cdot g$ is differentiable at $c$ and $(f \cdot g)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$

(iv) if $g(c) \neq 0$, $f/g$ is differentiable at $c$ and $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{\{g(c)\}^2}$

**Proof:** (iii) Let $h = f \cdot g$. Then for $x \in I$, $x \neq c$.

$$\frac{h(x) - h(c)}{x - c} = \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c}$$

$$= \frac{\{f(x) - f(c)\}g(x) + f(c)\{g(x) - g(c)\}}{(x - c)}$$

$$= \frac{f(x)-f(c)}{x-c} \cdot g(x) + f(c) \cdot \frac{g(x)-g(c)}{x-c}$$

Since $f$ and $g$ are differentiable at $c$,

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ and } g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}}$$

Therefore,

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} g(x)$$

$$+ f(c) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= f'(c)\,g(c) + f(c)\,g'(c). \quad \text{(Proved)}$$

(iv) Let $h = f/g$ and $g(c) \neq 0$.

$$\frac{h(x) - h(c)}{x - c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \frac{f(x)\,g(c) - f(c)\,g(x)}{g(x)\,g(c)\,(x - c)}$$

$$= \frac{1}{g(x)\,g(c)}\left[\frac{f(x)-f(c)}{x-c}\,g(c) - f(c)\cdot\frac{g(x)-g(c)}{x-c}\right]$$

Since f and g are differentiable

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

and $$g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}.$$

Since $g(x)$ is differentiable, it is also continuous and we have $\lim_{x \to c} g(x) = g(c)$

Therefore $\lim_{x \to c} \dfrac{h(x) - h(c)}{x - c}$

$$= \frac{f'(c)\, g(c) - f(c)\, g'(c)}{\{g(c)\}^2}$$

## Chain Rule:

Let $I$ and $J$ be two intervals. Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ be two functions such that $f(I) \subseteq J$. Let $c \in I$ and $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$. Then the composite function $(g \circ f)$ is differentiable at $c$ and

$$\boxed{(g \circ f)'(c) = g'(f(c)) \cdot f'(c)}$$

**Ex**  we know that $\frac{d}{dx}(\sin x) = \cos x$ $\forall x \in \mathbb{R}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\forall x \neq 0$. using these, one can obtain

$$\frac{d}{dx}\left(\sin\frac{1}{x}\right) = -\frac{1}{x^2}\cos\frac{1}{x} \quad \forall x \neq 0.$$

Here $f(x) = \frac{1}{x}$ and $g(x) = \sin x$.

**Ex.** $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is not differentiable at $x = 0$.

**Solⁿ:** $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x}$

$= \lim_{x\to 0} \dfrac{x\sin\frac{1}{x}}{x}$

$= \lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

**Ex.** $f(x) = x^2 \sin\frac{1}{x}, \quad x \neq 0$

$= 0, \quad \text{if } x = 0$

Show that $f(x)$ is differentiable at 0 but $f'$ is not continuous at 0.

**Sol$^n$** Home work.

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# Rolle's Theorem

Before going to discuss on Rolle's Theorem, I want to discribe the local maxima or local minima.

- **Definition of Local maxima:**

Let $f: I \to \mathbb{R}$ be a function and $I$ be an interval. A point $x_0 \in I$ is called as local maxima of $f$ if there exists a $\delta > 0$ such that

$\boxed{f(x) \leq f(x_0)}$, whenever $\underbrace{x \in N(x_0, \delta) \cap I}$

$x \in (x_0 - \delta, x + \delta) \cap I$

## Def$^n$ of local minima:

Let $f: I \to \mathbb{R}$ be a function and $I$ be an interval. Let $y_0 \in I$.

A point $y_0$ is called as local minima of $f$ if $\exists$

a $\delta > 0$ such that

$$\boxed{f(x) \geq f(y_0) \text{ whenever } x \in N(y_0, \delta) \cap I}$$

**Theorem:** Let $f : [a, b] \to \mathbb{R}$ be a real single valued function.
Suppose $f$ has either a local maxima or a local minima at $x_0$

and if $f$ is differentiable at $x_0$

then $\boxed{f'(x_0) = 0.}$

**Remark:** The previous theorem is not valid if $x_0$ is one of the end points of the interval $[a,b]$ i.e, a or b.

For example ______

Let $f: [0,1] \to \mathbb{R}$ such that

$$\boxed{f(x) = x}$$

Then it is clearly seen that the maxima of $f$ is at $x = 1$, i.e. the end point of the interval $[0,1]$.

but $f'(x) = 1$ not $0$ $\forall x \in [0,1]$.

* now See the following theorem which is an application of previous theorem on local maxima or local minima.

This theorem is known as

**Rolle's Theorem**

# Rolle's Theorem

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**Theorem:** Let $f : [a,b] \to \mathbb{R}$ be a real-valued function. If the function $f$ is continuous on $[a,b]$, $f$ is differentiable in $(a,b)$ and $f(a) = f(b)$, then $\exists$ a point $c \in (a,b)$ such that

$$\boxed{f'(c)=0.}$$

This theorem says that $f$ has a local maxima or minima at one point $c$ in the open interval $(a,b)$ if the function $f$ is continuous on $[a,b]$, differentiable in $(a,b)$ and $f(a)=f(b)$.

**Proof:** Since $f$ is continuous on $[a,b]$

So $f$ is bounded on $[a,b]$ and

also there exists $x_0, y_0$ in $[a,b]$

such that

$$\boxed{f(y_0) \le f(x) \le f(x_0)}$$

Now if $x_0$ and $y_0$ are both end points of $[a,b]$, then $f$ is a constant function since $f(a)=f(b)$

$$\Rightarrow f(x) = \text{constant}$$

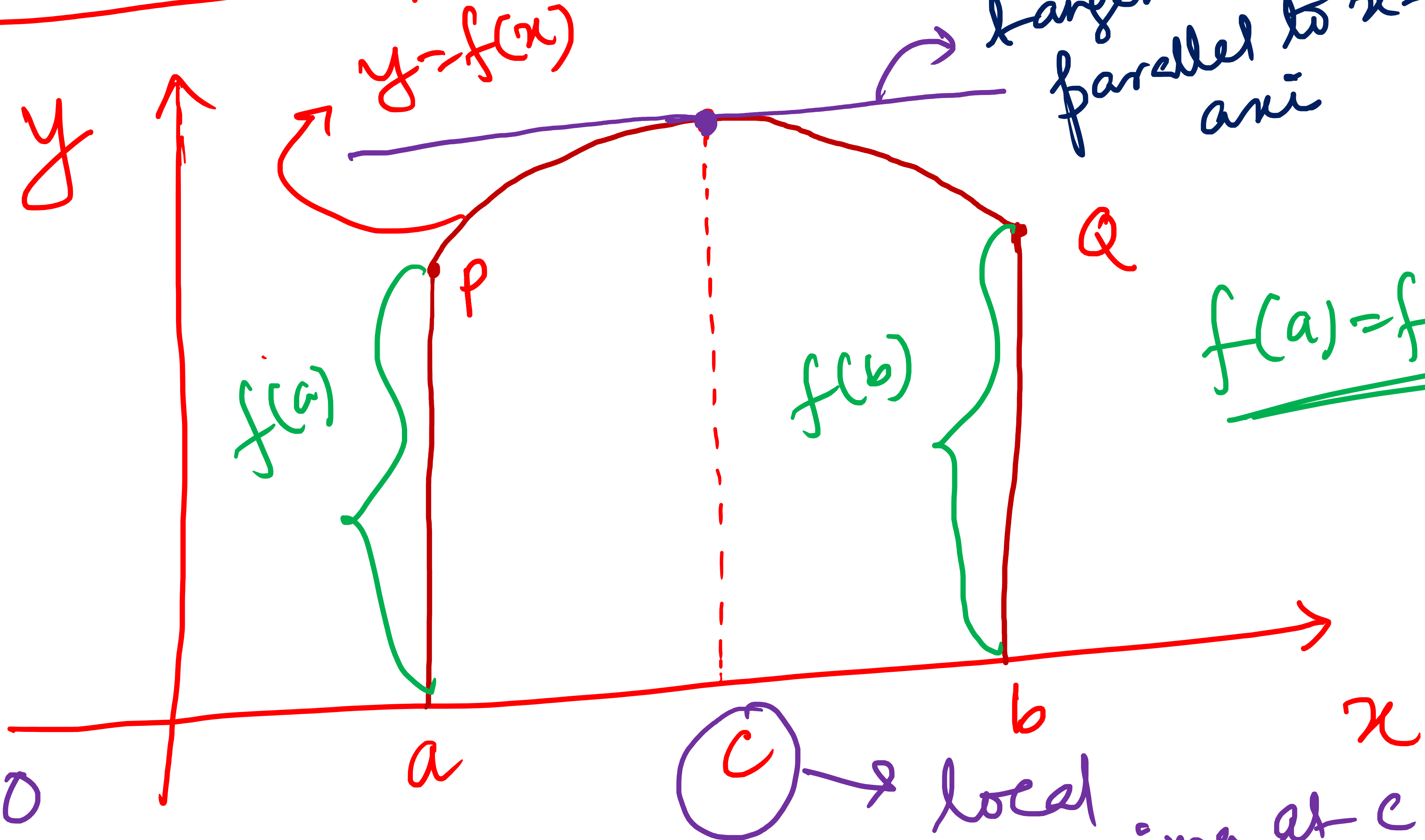
$$\Rightarrow f'(x) = 0 \quad \forall x \in (a,b).$$

otherwise $f$ assume either a maximum or a minimum at a point $x$ in $(a,b)$

So, $f'(x)=0$, $x\in(a,b)$ by the theorem on local maxima or minima.

# Geometrical Representation :

$P(a, f(a))$ & $Q(b, f(b))$



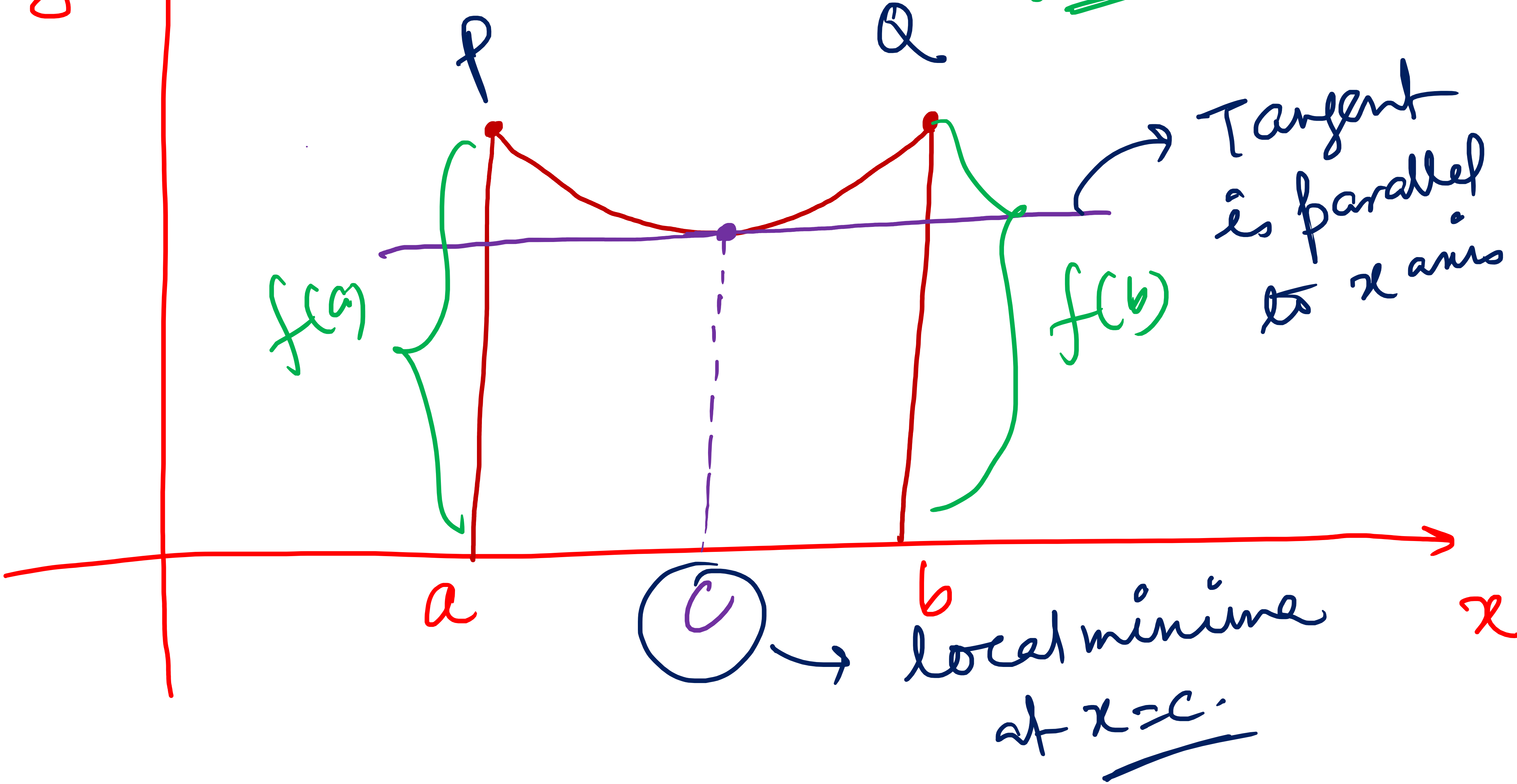

tangent line is parallel to x-axis

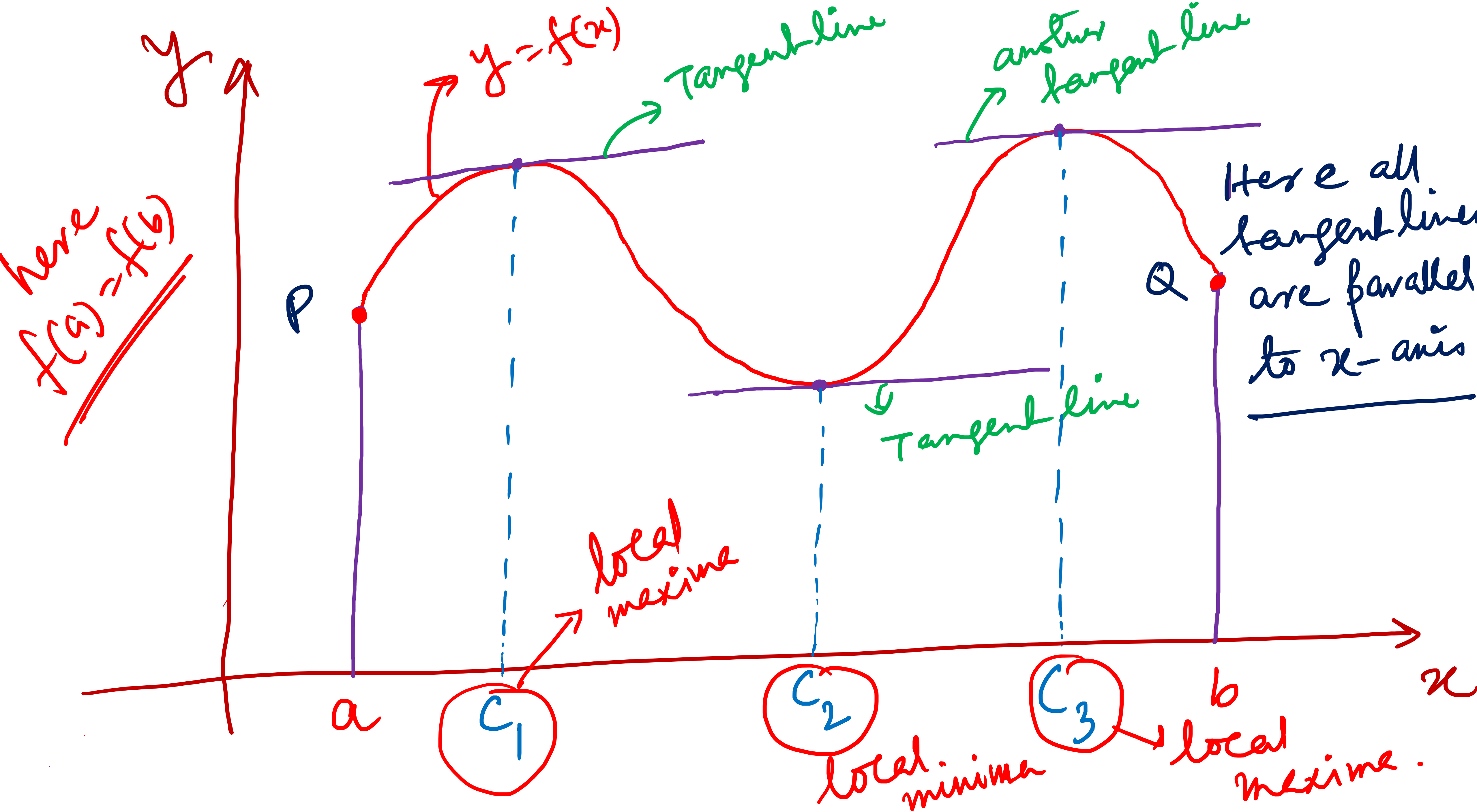
$f(a) = f(b)$

$c \rightarrow$ local maxima at c.

$P(a, f(a))$, $Q(b, f(b))$

$f(a) = f(b)$





y
y = f(x)
Tangent line
another tangent line
here f(a) = f(b)
P
Q
Here all tangent lines are parallel to x-axis
Tangent line
local maxima
local minima
local maxima.
a
c1
c2
c3
b
x

From these three graphs it is clear that if the function $f$ is continuous on $[a,b]$, differentiable in $(a,b)$ and functional values are equal at the end points i.e, $f(a)=f(b)$. Then there exists atleast one point $c$ in $(a,b)$ such that the tangent line at $c$ is parallel to $x$-axis.

**Example:** Let $f$ and $g$ be two functions which are continuous on $[a,b]$ and differentiable in $(a,b)$ and $\boxed{f(a)=f(b)=0.}$ Prove that there is a point $c\in(a,b)$ such that $\boxed{g'(c)f(c)+f'(c)=0}$

**Proof:** Define $h(x)=f(x)e^{g(x)}$.
Then $h(x)$ is continuous on $[a,b]$ and differentiable on $(a,b)$

Also $h(a) = 0 = h(b)$. So by Rolle's theorem $\exists\ c \in (a,b)$ s.t. $h'(c) = 0$.

$$\Rightarrow \boxed{f'(c) + f(c)\, g'(c) = 0 \qquad \left[\because e^{g(c)} \neq 0\right].}$$