भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Lecture Note - 3

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Third Lecture on Calculus-I

(MA-1110)

Dr. Jyotirmoy Rana
Assistant Professor
Department of Mathematics
IIT Hyderabad

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Sequences

- Sandwich Theorem
- Divergent Sequences
- Oscillatory Sequences
- Some important theorems

Sandwich Theorem

Theorem: Let $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ be three sequences of real numbers and $\exists$ a natural number N such that

$$\boxed{x_n < y_n < z_n \quad \forall n \geq N.}$$

If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$, then $\{y_n\}$ is convergent and $\lim_{n\to\infty} y_n = l$.

Proof: Let $\varepsilon > 0$ be given.

Given that $\lim\limits_{n\to\infty} x_n = l$ and $\lim\limits_{n\to\infty} z_n = l$

For $\varepsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon, \ \forall\, n \geq N_1$$

and $\exists\, N_2 \in \mathbb{N}$ such that $|z_n - l| < \varepsilon \ \forall\, n \geq N_2$

So $|x_n - l| < \varepsilon, \ \forall\, n \geq N_1$

$$\Rightarrow \boxed{l - \varepsilon < x_n < l + \varepsilon, \ \forall\, n \geq N_1}$$

and $|z_n - l| < \varepsilon, \ \forall\, n \geq N_2$

$$\Rightarrow \boxed{l - \varepsilon < z_n < l + \varepsilon, \ \forall\, n \geq N_2.}$$

Given that $x_n < y_n < z_n, \ \forall\, n \geq N.$

Choose $N_3 = \max\{N_1, N_2\}$.

Then we have

$$\boxed{\begin{gathered} l-\varepsilon < x_n < l+\varepsilon \quad \text{and} \quad l-\varepsilon < z_n < l+\varepsilon \\ \forall\, n \geq N_3 \end{gathered}}$$

Choose $N' = \max\{N_3, N\}$.

Then

$$l-\varepsilon < x_n < y_n < z_n < l+\varepsilon, \quad \forall n \geq N'$$

$$\Rightarrow \quad l-\varepsilon < y_n < l+\varepsilon, \ \forall n \geq N'$$

Now we can write

$$|y_n - l| < \varepsilon \quad \forall n \geq N'$$

$$\Rightarrow \lim_{n\to\infty} y_n = l \quad \text{(Proved)}.$$

Note: If $x_n \leq y_n \leq z_n \ \forall n \geq N$

and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$, then

$$\boxed{\lim_{n\to\infty} y_n = l}$$

P-6

Example: $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$

Solⁿ: Let $x_n = \dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}$

we have $\quad n^2+2 > n^2+1$

$$\Rightarrow \frac{1}{n^2+2} < \frac{1}{n^2+1}$$

$$\Rightarrow \frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$$

Similarly,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$$

$$\cdots \qquad \cdots$$

$$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}$$

P-9

Therefore

$$x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

$$< \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}$$

$$= \frac{n}{\sqrt{n^2+1}}$$

P-10

$$\boxed{x_n < \frac{n}{\sqrt{n^2+1}} \quad \forall\ n \geq 2}$$

Again, $\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

Since $\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2+2}}$

Similarly

$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$$

$$\cdots$$

$$\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}} > \frac{n}{\sqrt{n^2+n}}$$

$$\Rightarrow \boxed{x_n > \frac{n}{\sqrt{n^2+n}}, \ \forall n \geq 2.}$$

So we have

$$\frac{n}{\sqrt{n^2+n}} < x_n < \frac{n}{\sqrt{n^2+1}}, \quad \forall\, n \geq 2$$

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{n^2+n}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = \frac{1}{\sqrt{1+0}}$$

$$= 1.$$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{\frac{n^2+1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}}$$

$$= \frac{1}{\sqrt{1+0}} = 1.$$

P-13

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1 = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$

and $\frac{n}{\sqrt{n^2+n}} < x_n < \frac{n}{\sqrt{n^2+1}}$

$\forall\, n \geq 2$

by Sandwich theorem, we have

$$\lim_{n\to\infty} x_n = 1.$$

Example: $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0.$

Sol^n: we know that for $x \in [0, \pi/2]$

$$0 \le \sin x \le x$$

Therefore $\forall n \in \mathbb{N}$, we have

$$0 \le \sin\left(\frac{1}{n}\right) \le \frac{1}{n} \qquad \left[0 < \frac{1}{n} < 1\right]$$

By Sandwich Theorem, we have $\lim_{n\to\infty} \sin\frac{1}{n} = 0.$

Divergent Sequence

Definition:

A Sequence $\{x_n\}_n$ of real numbers is said to be a divergent Sequence if the Sequence $\{x_n\}_n$ has no limit $l \in \mathbb{R}$.

i.e. A Sequence $\{x_n\}_n$ is said to be properly divergent Sequence if it either diverges to ∞ or diverges to $-\infty$.

A Sequence $\{x_n\}_n$ is said to diverge to ∞ if for any positive number M, $\exists$ a natural number N such that

$$\boxed{x_n > M \quad \forall\, n \geq N}$$

$$\Rightarrow \lim_{n \to \infty} x_n = \infty \text{ (diverge)}$$

or $x_n \to \infty$ as $n \to \infty$.

A Sequence $\{x_n\}_n$ is said to diverge to $-\infty$ if for any positive number M $\exists$ a natural number N such that

$$\boxed{x_n < -M \quad \forall\, n \geq N}$$

$$\Rightarrow \lim_{n\to\infty} x_n = -\infty$$

or $x_n \to -\infty$ as $n \to \infty$.

P-18

Theorem: A Sequence which diverges to ∞ is unbounded above but bounded below.

Theorem: A sequence which diverges to $-\infty$ is unbounded below but bounded above.

Note: The converse of above two theorems may not be true.

For example, $\left\{ n^{(-1)^n} \right\}$ does not diverge to ∞ but it is unbounded above and bounded below. lower bound is 0.

Examples :

(i) $\lim_{n \to \infty} (\sqrt{n} + 10) = \infty$

Sol^n : Let us consider a positive real number M.

Now $\sqrt{n} + 10 > M$

$\Rightarrow \sqrt{n} > M - 10$

$\Rightarrow n > (M - 10)^2$

Choose $N = [(M-10)^2] + 1$

Then for $M > 0$ $\exists N \in \mathbb{N}$ such that

$$\boxed{\sqrt{n} + 10 > M \quad \forall n \geq N}$$

$$\Rightarrow \lim_{n \to \infty} (\sqrt{n} + 10) = \infty.$$

(ii) $\lim_{n\to\infty} n^2 = \infty$

(iii) $\lim_{n\to\infty} 2^n = \infty$

(iv) $\lim_{n\to\infty} \sqrt{n} = \infty$

(v) $\lim_{n\to\infty} -n^2 = -\infty$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Oscillatory Sequence

- **Definition:** A bounded sequence which is not convergent is said to be an oscillatory sequence of finite oscillation.

- **Examples:** The sequence $\{(-1)^n\}$ is a bounded sequence but it is not convergent. It is an oscillatory sequence of finite oscillation.

Example: To show that the sequence $\{x_n\}_n$ oscillates finitely where $x_n = 1 + (-1)^n$, $\forall n \in \mathbb{N}$.

Sol^n: The sequence $\{x_n\}$ is

$$\{0, 2, 0, 2, \ldots\}$$

First we have to show that the given sequence does not converge to a finite limit.

P-25

Let us consider $\{x_n\}$ converge to a finite limit l.

Therefore, for $\varepsilon > 0$ $\exists$ a natural number N such that $|x_n - l| < 1/2$, $\forall n \geq N$

where $\varepsilon = \frac{1}{2}$.

$$\Rightarrow l - 1/2 < x_n < l + 1/2, \quad \forall n \geq N$$

$$\Rightarrow l - 1/2 < 1 + (-1)^n < l + 1/2, \quad \forall n \geq N.$$

P-26

$$-\frac{1}{2} < 1+(-1)^n - l < \frac{1}{2}, \quad \forall\ n \geq N.$$

When $n \geq N$ and n is even, we get

$$\frac{3}{2} < l < \frac{5}{2}$$

When $n \geq N$ and n is odd, we get

$$-\frac{1}{2} < l < \frac{1}{2}.$$

This arguments lead to contradiction.

Therefore, $\{x_n\}$ does not converge to a finite limit.

Indian Institute of Technology Hyderabad

Def^n: An unbounded sequence which is not properly divergent (i.e., does not diverge to either $+\infty$ or $-\infty$) is said to be an oscillatory sequence of infinite oscillation.

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Examples:

(i) The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is an unbounded sequence and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

(ii) $\left\{n^{(-1)^n}\right\}$ is also an oscillatory sequence of infinite oscillation.

Note: Many Sequences do not diverge to either $+\infty$ or $-\infty$ even if they are unbounded.

Oscillatory Sequence of infinite oscillation.

Important Conclusions

Sequences

- **Convergent** ($\to$ a finite limit)
- **Non-Convergent** ($\not\to$ a finite limit)
  - **Properly divergent**
    - Diverges to $+\infty$
    - Diverges to $-\infty$
  - **Improperly divergent**
    - oscillates finitely
    - oscillates infinitely