1.1 Model Assumptions

We work with models verifying **(S1)–(S10)** below:

(S1) Finite segmentation with stable proportions. The number of segments k is fixed (does not depend on n), and $\alpha_j \to \bar{\alpha}_j \in]0,1]$ for all $j = 1, \dots, k+1$.

(S2) Segmentwise strict stationarity and ergodicity. For each segment j, the array $\{X_t : \omega_j(t) = 1\}$ is strictly stationary and ergodic under $\mathbb{P}^{(n)}_{\varpi_0}$. Hence, for any integrable $g : \mathcal{X} \to \mathbb{R}$,

$$\frac{1}{n_j} \sum_{t=\tau_{j-1}}^{\tau_j - 1} g(X_t) \xrightarrow{\text{a.s.}} \mathbb{E}_j[g(X_1)],$$

where $\mathbb{E}_j$ denotes expectation under the invariant law of segment j.

(S3) Identifiability and local parameterization. There exists a local direction $h \in \mathbb{R}^d$ such that local alternatives are encoded as

$$H_0 : \ \varpi = \varpi_0, \qquad H_{1,n} : \ \varpi = \varpi_0 + \frac{h}{\sqrt{n}},$$

and $\varpi \mapsto \mathbb{P}^{(n)}_{\varpi}$ is identifiable at ϖ_0.

(S4) Central sequence and LAN-type expansion. There exists a *central sequence* $\Pi_n \in \mathbb{R}^d$ and a symmetric, nonnegative definite matrix $\Gamma \in \mathbb{R}^{d \times d}$ such that, under H_0,

$$\log \frac{d\mathbb{P}^{(n)}_{\varpi_0 + h/\sqrt{n}}}{d\mathbb{P}^{(n)}_{\varpi_0}} \ = \ h^\top \Pi_n \ - \ \tfrac{1}{2} h^\top \Gamma\, h \ + \ o_{\mathbb{P}}(1),$$

with the expansion understood uniformly for h in compact sets.

(S5) Martingale/array CLT construction of Π_n. There exist $\mathbb{R}^d$-valued, $\mathcal{F}_t$-adapted increments $\psi_{n,t}$ with $\mathbb{E}[\psi_{n,t} \mid \mathcal{F}_{t-1}] = 0$ such that

$$\Pi_n \ = \ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \psi_{n,t},$$

and

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\big[\psi_{n,t} \psi_{n,t}^\top \mid \mathcal{F}_{t-1}\big] \ \xrightarrow{\mathbb{P}} \ \Gamma,$$

together with a Lindeberg condition

$$\forall \varepsilon > 0, \quad \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\Big[\|\psi_{n,t}\|^2 \, \mathbb{1}\{\|\psi_{n,t}\| > \varepsilon\sqrt{n}\} \,\Big|\, \mathcal{F}_{t-1} \Big] \ \xrightarrow{\mathbb{P}} 0.$$

These conditions are used to derive the asymptotic normality in (S6).

(S6) Asymptotic normality. The central sequence Π_n is asymptotically Gaussian with covariance matrix Γ. More precisely,

$$\Pi_n \;\to\; \mathcal{N}(0,\Gamma) \quad \text{under } H_0,$$

and, under local alternatives $H_{1,n} : \varpi = \varpi_0 + h/\sqrt{n}$,

$$\Pi_n \;\to\; \mathcal{N}(\Gamma h,\ \Gamma).$$

(S7) Regular handling of nuisance. If ν denotes nuisance parameters, assume that replacing ν_0 by an estimator $\widehat{\nu}$ has a negligible first-order effect:

$$\Pi_n(\widehat{\nu}) = \Pi_n(\nu_0) + o_{\mathbb{P}}(1),$$

where ν_0 is the true nuisance configuration under the data-generating process.

(S8) Non-degeneracy of information. There exists $c > 0$ such that

$$\inf_{\|u\|=1} u^\top \Gamma u \;\geq\; c,$$

or, more generally, $u^\top \Gamma u > 0$ for all nonzero u in the span of admissible local directions.

(S9) Basic measurability and integrability. All objects above are measurable, and there exists $\delta > 0$ with

$$\sup_{n,t} \mathbb{E}\|\psi_{n,t}\|^{2+\delta} < \infty.$$

(S10) ρ-mixing. Under H_0 and $H_{1,n}$, (X_t) is independent or weakly dependent; this is only used later for indicator-type functionals.